CENTER FOR ADVANCED RESEARCH IN FINANCE

C A R F W o r k i n g P a p e r

CARF-F-102

An Asymptotic Expansion Approach in Finance

Akihiko Takahashi
University of Tokyo

First Version: August, 2007
Current Version: April, 2009

CARF is presently supported by AIG, Bank of Tokyo-Mitsubishi UFJ, Ltd., Citigroup, Dai-ichi Mutual Life Insurance Company, Meiji Yasuda Life Insurance Company, Mizuho Financial Group, Inc., Nippon Life Insurance Company, Nomura Holdings, Inc. and Sumitomo Mitsui Banking Corporation (in alphabetical order). This financial support enables us to issue CARF Working Papers.

CARF Working Papers can be downloaded without charge from:
http://www.carf.e.u-tokyo.ac.jp/workingpaper/index.cgi

An Asymptotic Expansion Approach in Finance *

Akihiko Takahashi
Graduate School of Economics, the University of Tokyo

August, 2007

Abstract

This paper reviews an asymptotic expansion approach to numerical problems for pricing financial assets and securities.

1 Introduction

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ denote a probability space with filtration, on which a m-dimensional standard Wiener process w is defined where P is the equivalent martingale measure (the risk neutral measure) in finance, and T denotes some positive constant. Now, let $F(\omega)$ be a Wiener functional, and then $\mathbf{V}$, the value of a portfolio or a security can be expressed as $\mathbf{V} = \mathbf{E}[F(\omega)]$ under certain conditions. Evaluating this expectation is one of the main problems in finance. Moreover, if F depends on the parameter θ, evaluation of $\frac{\partial \mathbf{V}}{\partial \theta} = \frac{\partial}{\partial \theta}\mathbf{E}[F(\omega;\theta)]$, the change of a security value caused by a minimal change of this parameter is also an important task in practice. As an example, consider a d-dimensional diffusion process $X^{(\epsilon)}$ which is obtained as a strong solution to the stochastic differential equation;

$$dX_t^{(\epsilon)} = V_0(X_t^{(\epsilon)}, \epsilon)dt + V(X_t^{(\epsilon)}, \epsilon)dw_t,\ t \in [0,T];\ X_0^{(\epsilon)} = x_0,$$

where $\epsilon \in [0,1]$ is a known parameter. Here, the coefficients are assumed to be smooth and to satisfy some regularity conditions. In finance, the problems of evaluating the present value of derivatives or the portfolio value in investment theory are mostly reduced to the problems of computing

*The main part of this paper is translation of [35].

$\mathbf{E}[f(X_T^{(\epsilon)})]$, the expectation of $f(X_T^{(\epsilon)})$, a function of $X_T^{(\epsilon)}$. In financial applications, it is important to deal with the case not only the function $f(x)$ is smooth but also the case it is not. For example, when various options are evaluated, it is expressed as $f = T \circ g$, where $T(x) = \max\{x, 0\}$ and g denotes a smooth function of $\mathbf{R}^d \mapsto \mathbf{R}$. In general, it is difficult to represent this expectation explicitly except for special cases. Therefore, methods such as Monte Carlo simulation or numerical solutions of partial differential equations are applied and various speeding up techniques are developed, since fast computation is required as well as accuracy for practical purposes. As another approach, an approximation of the expectation by an asymptotic expansion of the stochastic differential equation around $\epsilon = 0$ can be considered. Further, because $\frac{\partial}{\partial x_0}\mathbf{E}[f(X_T^{(\epsilon)})]$ and $\frac{\partial}{\partial \epsilon}\mathbf{E}[f(X_T^{(\epsilon)})]$, the changes of the security value caused by the minimal changes of the initial value x_0 and the parameter ϵ, are important indicators for practical purposes, obtaining the approximations with high accuracy are very useful. Moreover, some methods which combines Monte Carlo simulation and an asymptotic expansion with low orders are developed since the asymptotic expansion up to the first or second order can be easily evaluated. By this way, the efficiency of Monte Carlo simulation or the accuracy of approximation obtained by the asymptotic expansion can be improved.

The application of the asymptotic expansion to finance is closely related to the study in mathematical statistics, which is on asymptotic expansions of statistics for stochastic differential equations, though it seems not to be so relevant to finance. In particular, it is based on the asymptotic expansions of estimators for unknown parameters in small diffusions developed by Yoshida [51], [52], [53], which applies Watanabe theory(Watanabe [50]) in Malliavin calculus. For further readings on asymptotic expansions of small diffusions in mathematical statistics, see Dermoune-Kutoyants [3], Kutoyants [18], [19], Masuda-Yoshida [22], Sakamoto-Yoshida [30], [31], Sakamoto-Takada-Yoshida [29], Taniguchi-Kakizawa [45], Uchida-Yoshida [46], [47], and Yoshida [54], [55], [56].

To my knowledge, the asymptotic expansion is first applied to finance for evaluation of an average option that is a popular derivative in commodity markets. [12] and [33] derive the approximation formulas for an average option by an asymptotic method based on log normal approximations of an average price distribution when the underlying asset price follows a geometric Brownian motion. [52] applies a formula derived more generally by the asymptotic expansion of small diffusion processes. Thereafter, the asymptotic expansion is applied to a broad class of problems in finance. For basic theory in finance, see for instance Björk [1] and Karatzas-Shreve [6], and for the general relationship between finance and the asymptotic expansion, see [15]. In what follows, more concrete applications of the asymptotic expansion to numerical problems in finance are introduced.

2 Foundation of an Asymptotic Expansion

This section provides basic methodology to approximate the values of financial assets or securities after a summary of the framework of the asymptotic expansion approach based on [51], [52] and [14] in Section 2.1. As for details on Watanabe theory([50]) in Malliavin calculus that is a core theory of this method, as well as the asymptotic expansion, see Takanobu [43], Takanobu-Watanabe [44] and Uemura [48] for instance besides the literatures on mathematical statistics mentioned above. Kusuoka-Strook [17] also derives an asymptotic expansion of a certain Wiener functional by Malliavin calculus. Further, see lecture notes or textbooks such as Ikeda-Watanabe [5], Malliavin [21], Nualart [26], Shigekawa [32] and Watanabe [49] for general references for Malliavin calculus.

2.1 The Framework of an Asymptotic Expansion

First, I consider a d-dimensional diffusion process $X^{(\epsilon)}$, which is the strong solution to the following stochastic differential equation:

$$dX_t^{(\epsilon)} = V_0(X_t^{(\epsilon)})dt + \epsilon V(X_t^{(\epsilon)})dw_t;\ X_0^{(\epsilon)} = x_0,\ t \in [0,T], \tag{1}$$

where w denotes a m-dimensional standard Wiener process and $\epsilon \in [0,1]$ is a known parameter. Suppose that coefficients $V_0 : \mathbf{R}^d \mapsto \mathbf{R}^d,\ V : \mathbf{R}^d \mapsto \mathbf{R}^d \otimes \mathbf{R}^m$ are smooth and satisfies regularity conditions. Let V_i denote the i-th column of the V and a $\mathbf{R}^d \otimes \mathbf{R}^d$-valued stochastic process Y denote the solution to the stochastic differential equation;

$$dY_t^{(\epsilon)} = \partial V_0(X_t^{(\epsilon)})Y_t^{(\epsilon)}dt + \epsilon \sum_{i=1}^{m} \partial V_i(X_t^{(\epsilon)})Y_t^{(\epsilon)}dw_{it};\ Y_0^{(\epsilon)} = I_d,$$

where $\partial_k = \frac{\partial}{\partial x_k}$, and $\partial V_i (i = 0, 1, \cdots, m)$ denotes the $d \times d$ matrix whose (j,k)-element is $\partial_k V_{ji}$. I_d denotes the $d \times d$ identity matrix. Moreover, define Y_t as $Y_t = Y_t^{(0)}$.

Next, suppose that a function $g : \mathbf{R}^d \mapsto \mathbf{R}$ to be smooth and all derivatives have polynomial growth orders. Then, for $\epsilon \downarrow 0$, $g(X_T^{(\epsilon)})$ has its asymptotic expansion;

$$g(X_T^{(\epsilon)}) \sim g_{0T} + \epsilon g_{1T} + \epsilon^2 g_{2T} + \epsilon^3 g_{3T} \cdots \quad in\ \mathbf{D}_\infty, \tag{2}$$

where $g_{0T}, g_{1T}, g_{2T}, g_{3T} \cdots \in \mathbf{D}_\infty$. For any $k \in \mathbf{N}$, $q \in (1,\infty)$ and $s > 0$, this expansion means that

$$\frac{1}{\epsilon^k}\|g(X_T^{(\epsilon)}) - (g_{0T} + \epsilon g_{1T} + \cdots + \epsilon^{k-1} g_{k-1,T})\|_{q,s} < \infty\ (as\ \epsilon \downarrow 0),$$

where $\|G\|_{q,s}$ represents the sum of the L^q -norms of Malliavin derivatives of a Wiener functional G up to the s-th order. Further, a Banach space $\mathbf{D}_{q,s} = \mathbf{D}_{q,s}(\mathbf{R})$ can be regarded as the totality of random variables bounded with respect to (q,s)-norm $\|\cdot\|_{q,s}$, and $\mathbf{D}_\infty = \cap_{s>0} \cap_{1<q<\infty} D_{q,s}$.

The coefficients in the expansion, $g_{0T}, g_{1T}, g_{2T} \cdots$ can be obtained by Taylor's formula and represented as multiple Wiener-Ito integrals. In particular, let $D_t = \frac{\partial X_t^{(\epsilon)}}{\partial \epsilon}|_{\epsilon=0}$, $E_t = \frac{\partial^2 X_t^{(\epsilon)}}{\partial \epsilon^2}|_{\epsilon=0}$ and $F_t = \frac{\partial^3 X_t^{(\epsilon)}}{\partial \epsilon^3}|_{\epsilon=0}$ then g_{0T}, g_{1T}, g_{2T} and g_{3T} can be written as

$$\begin{aligned} g_{0T} &= g(X_T^{(0)}),\ g_{1T} = \sum_{i=1}^{d} \partial_i g(X_T^{(0)}) D_{iT}, \\ g_{2T} &= \frac{1}{2} \sum_{i,j=1}^{d} \partial_i \partial_j g(X_T^{(0)}) D_{iT} D_{jT} + \frac{1}{2} \sum_{i=1}^{d} \partial_i g(X_T^{(0)}) E_{iT}, \end{aligned}$$

and,

$$g_{3T} = \frac{1}{6} \sum_{i,j,k=1}^{d} \partial_i \partial_j \partial_k g(X_T^{(0)}) D_{iT} D_{jT} D_{kT} + \frac{1}{2} \sum_{i,j=1}^{d} \partial_i \partial_j g(X_T^{(0)}) E_{iT} D_{jT} + \frac{1}{6} \sum_{i=1}^{d} \partial_i g(X_T^{(0)}) F_{iT}.$$

Here, D_{it}, E_{it} and F_{it}, $i = 1, \cdots, d$, which are the elements of D_t, E_t and F_t respectively are represented by

$$\begin{aligned} D_{it} &= \int_0^t Y_t^{(i,\cdot)} Y_u^{-1} V(X_u^{(0)}) dw_u, \\ E_{it} &= \int_0^t Y_t^{(i,\cdot)} Y_u^{-1} [\sum_{j,k=1}^{d} \partial_j \partial_k V_0(X_u^{(0)}) D_{ju} D_{ku} du + 2 \sum_{j=1}^{d} \partial_j V(X_u^{(0)}) D_{ju} dw_u], \\ F_{it} &= \int_0^t Y_t^{(i,\cdot)} Y_u^{-1} [\sum_{j,k,l=1}^{d} \partial_j \partial_k \partial_l V_0(X_u^{(0)}) D_{ju} D_{ku} D_{lu} du + 3 \sum_{j,k=1}^{d} \partial_j \partial_k V_0(X_u^{(0)}) E_{ju} D_{ku} du \\ &\qquad + 3 \sum_{j,k=1}^{d} \partial_j \partial_k V(X_u^{(0)}) D_{ju} D_{ku} dw_u + 3 \sum_{j=1}^{d} \partial_j V(X_u^{(0)}) E_{ju} dw_u], \end{aligned}$$

where $Y_t^{(i,\cdot)}$ denote the i-th row of Y_t.

Next, normalize $g(X_T^{(\epsilon)})$ as

$$G^{(\epsilon)} = \frac{g(X_T^{(\epsilon)}) - g_{0T}}{\epsilon}$$

for $\epsilon \in (0, 1]$. Then, for $h \in H$(where H denotes the Cameron-Martin subspace of the m-dimensional Wiener space), the H-derivative of $G^{(\epsilon)}$ is expressed as

$$\mathbf{D}_h G^{(\epsilon)} = \frac{1}{\epsilon} \sum_{i=1}^{d} \partial_i g(X_T^{(\epsilon)}) \mathbf{D}_h X_{iT}^{(\epsilon)} = \sum_{i=1}^{d} \partial_i g(X_T^{(\epsilon)}) \int_0^T [Y_T^{(\epsilon)} (Y_t^{(\epsilon)})^{-1} V(X_t^{(\epsilon)}) \dot{h}_t]_i \, dt.$$

With a notation $a_t^{(\epsilon)}$,

$$a_t^{(\epsilon)} = (\partial g(X_T^{(\epsilon)}))' [Y_T^{(\epsilon)} (Y_t^{(\epsilon)})^{-1} V(X_t^{(\epsilon)})],$$

the Malliavin (co)variance of $G^{(\epsilon)}$ is given by

$$\sigma_{G^{(\epsilon)}} = \int_0^T a_t^{(\epsilon)} (a_t^{(\epsilon)})' \, dt, \tag{3}$$

where $x^{'}$ denotes the transpose of x. Moreover, let

$$a_t = a_t^{(0)} = (\partial g(X_T^{(0)}))^{'}[Y_T Y_t^{-1} V(X_t^{(0)})]$$

and make the following assumption:

$$\text{(Assumption 1)} \quad \Sigma_T = \int_0^T a_t a_t^{'} dt > 0. \tag{4}$$

Since Σ_T is the variance of the random variable g_{1T}, which follows a normal distribution, Assumption 1 means the condition that the distribution of g_{1T} does not degenerate. In application, it is easy to check this condition in most cases, hence it plays an important role for practical purposes. In what follows, I derive asymptotic expansions useful for finance under Assumption 1. For detailed arguments such as the proofs omitted here because of the space limitation, see [51], [52] and [14].

Under Assumption 1, $\sigma_{G^{(\epsilon)}}$ is uniformly non-degenerate for $\{|\eta_c^{(\epsilon)}| \leq 1\}$; that is, it can be shown that there exists a positive real number $c_0 > 0$ such that for any $c > c_0$ and $p > 1$,

$$\sup_{\epsilon \in (0,1]} \mathbf{E}[1_{\{|\eta_c^{(\epsilon)}| \leq 1\}} (|\sigma_{G^{(\epsilon)}}|)^{-p}] < \infty, \tag{5}$$

where $\eta_c^{(\epsilon)} = c \int_0^T |a_t^{(\epsilon)} - a_t|^2 dt$. Then, for $T : \mathbf{R} \mapsto \mathbf{R}$, $T \in S^{'}(\mathbf{R})$ where $S^{'}(\mathbf{R})$ denotes the totality of Schwartz tempered distribution on $\mathbf{R}$, a composite function $\psi(\eta_c^{(\epsilon)}) T \circ G^{(\epsilon)} = \psi(\eta_c^{(\epsilon)}) T(G^{(\epsilon)})$ is well-defined as an element of $\tilde{\mathbf{D}}_{-\infty} = \cup_{s<0} \cap_{1<p<\infty} \mathbf{D}_{p,s}$; $\psi(x)$, $x \in R$ denotes a smooth function $0 \leq \psi(x) \leq 1$, defined as $\psi(x) = 1$ for $|x| \leq 1/2$ and $\psi(x) = 0$ for $|x| \geq 1$. Here, a Banach space $\mathbf{D}_{p,s}$, $s < 0$ is the dual space of $\mathbf{D}_{q,-s}(\mathbf{R})(q = p/(p-1))$. Moreover, the coupling with the function 1 is well-defined, which is called as generalized expectation and is written as $\mathbf{E}[\psi(\eta_c^{(\epsilon)}) T \circ G^{(\epsilon)}]$. Further, $\psi(\eta_c^{(\epsilon)}) T \circ G^{(\epsilon)}$ can be expanded in $\tilde{\mathbf{D}}_{-\infty}$. Since a function T such as $T(x) = \max\{x, 0\}$ that is measurable but not smooth appears frequently in finance, the framework mentioned above is necessary for the asymptotic expansion.

It also can be shown that $\{\eta_c^{(\epsilon)}(w); \epsilon \in (0,1]\} \subset \mathbf{D}_\infty$,
and $\eta_c^{(\epsilon)}(w)$ is $O(1)$ in $\mathbf{D}_\infty$ as $\epsilon \downarrow 0$. Moreover, for any $k = 1, 2, \cdots$,

$$\lim_{\epsilon \downarrow 0} \frac{P(|\eta_c^{(\epsilon)}| > \frac{1}{2})}{\epsilon^k} < \infty.$$

This means that the probability of the events truncated by $\psi(\eta_c^{(\epsilon)})$ is smaller than any polynomial orders of ϵ. Then, in the expansion of $\psi(\eta_c^{(\epsilon)}) T \circ G^{(\epsilon)}$, the coefficients expressed as generalized Wiener functionals belonging to $\tilde{\mathbf{D}}_{-\infty}$ can be written by applying Taylor's formula to $T(g_{0T} + \epsilon g_{1T} + \epsilon^2 g_{2T} + \cdots)$. Therefore, the asymptotic expansion of the expectation $\mathbf{E}[T(G^{(\epsilon)})]$ can be obtained relatively easily.

Now let us consider a more specific case. For a smooth function $\phi^{(\epsilon)} : [0,1] \times \mathbf{R} \mapsto \mathbf{R}$, of which all derivatives with respect to x have polynomial

growth orders uniformly in ϵ, the following holds uniformly in $B \in \mathbf{B}$ for a positive large number $c > 0$:

$$\psi(\eta_c^{(\epsilon)})\phi^{\epsilon}(G^{(\epsilon)})I_B(G^{(\epsilon)}) \sim \Phi_0 + \epsilon\Phi_1 + \epsilon^2\Phi_2 \cdots \quad in\ \tilde{\mathbf{D}}_{-\infty}\ (as\ \epsilon \downarrow 0), \qquad (6)$$

where $\mathbf{B} = \mathbf{B}(\mathbf{R})$ denotes a Borel set in $\mathbf{R}$. Here, the coefficients $\Phi_0, \Phi_1, \Phi_2, \cdots$ are generalized Wiener functionals and they are obtained by applying formal Taylor's formula to $\phi^{\epsilon}(G^{(\epsilon)})I_B(G^{(\epsilon)})$; in particular, Φ_0 and Φ_1 are expressed as

$$\begin{aligned}
\Phi_0 &= \phi^{(0)}(g_{1T})I_B(g_{1T}) \\
\Phi_1 &= \{\partial_\varepsilon\phi^{(0)}(g_{1T}) + \partial\phi^{(0)}(g_{1T})g_{2T}\}I_B(g_{1T}) + \phi^{(0)}(g_{1T})g_{2T}\partial I_B(g_{1T}), \\
\Phi_2 &= \left[\partial_\varepsilon\phi^{(0)}(g_{1T}) + \partial\phi^{(0)}(g_{1T})g_{2T}\right]\partial I_B(g_{1T})g_{2T} \\
&+ \left[\frac{1}{2}\partial_\varepsilon^2\phi^{(0)}(g_{1T}) + \{\frac{\partial^2\phi^{(0)}(x)}{\partial x\partial\varepsilon}|_{x=g_{1T}}\}g_{2T} + \partial\phi^{(0)}(g_{1T})g_{3T}\right. \\
&\quad \left. + \frac{1}{2}\partial^2\phi^{(0)}(g_{1T})g_{2T}^2\right] I_B(g_{1T}) + \phi^{(0)}(g_{1T})\left\{\frac{1}{2}\partial^2 I_B(g_{1T})g_{2T}^2 + \partial I_B(g_{1T})g_{3T}\right\}.
\end{aligned}$$

where $\partial_\varepsilon^k\phi^{(0)} = \frac{\partial^k\phi^{(\epsilon)}}{\partial\epsilon^k}|_{\epsilon=0}$, $k = 1, 2$.

Therefore, the expectation $\mathbf{E}[\phi(G^{(\epsilon)})I_B(G^{(\epsilon)})]$ has its asymptotic expansion uniformly in B as

$$\mathbf{E}[\phi^{(\epsilon)}(G^{(\epsilon)})I_B(G^{(\epsilon)})] \sim \mathbf{E}[\Phi_0] + \epsilon\mathbf{E}[\Phi_1] + \epsilon^2\mathbf{E}[\Phi_2] + \cdots \ (as\ \epsilon \downarrow 0), \qquad (7)$$

where each term of the expansion can be expressed as the expectation of a multiple Wiener-Ito integral conditional on a normal random variable. For example, $\mathbf{E}[\Phi_0]$, $\mathbf{E}[\Phi_1]$ and $\mathbf{E}[\Phi_2]$ are written as

$$\begin{aligned}
\mathbf{E}[\Phi_0] &= \int_B \phi^{(0)}(x)n[x; 0, \Sigma_T]dx, \\
\mathbf{E}[\Phi_1] &= \int_B (\partial_\varepsilon\phi^{(0)}(x)n[x; 0, \Sigma_T] - \phi^{(0)}(x)\partial\{\mathbf{E}[g_{2T}|g_{1T} = x]n[x; 0, \Sigma_T]\})dx, \\
\mathbf{E}\left[\Phi_2\right] &= \int_B \Big(-\partial_\varepsilon\phi^{(0)}(x)\partial\{\mathbf{E}[g_2|g_1 = x]n[x|0, \Sigma_T]\} + \frac{1}{2}\partial_\varepsilon^2\phi^{(0)}(x)n[x|0, \Sigma_T] \\
&\quad + \frac{1}{2}\phi^{(0)}(x)\partial^2\{\mathbf{E}[g_2^2|g_{1T} = x]n[x|0, \Sigma_T]\} \\
&\quad - \phi^{(0)}(x)\partial\{\mathbf{E}[g_3|g_{1T} = x]n[x|0, \Sigma_T]\}\Big)\, dx,
\end{aligned}$$

where $n[x; 0, \Sigma_T]$ denotes the density function of the normal distribution with mean 0 and variance Σ_T. Moreover, $E[g_{2T}|g_{1T} = x]$, $E[g_{2T}^2|g_{1T} = x]$ and $E[g_{3T}|g_{1T} = x]$ are polynomial functions of x and hence the computations of the expectations become easier. It is known that precise approximations are obtained by applying asymptotic expansions up to such low orders in financial examples.

2.2 Valuation of Option Values by an Asymptotic Expansion

Now approximations for values of various derivatives can be provided by the framework explained above. First, note that $X^{(\epsilon)}$ in the previous subsection

represents the key variable such as the underlying asset's price, which is an important factor to determine values of derivatives. Next, I show the application of the asymptotic expansion to finance, specifically by using an example of a standard European vanilla option which is examined in [34]. Let w be a one-dimensional Wiener process, α be a constant, $\epsilon \in [0,1]$, and $\sigma(x,t)$ be a smooth function of x satisfying appropriate regularity conditions. Suppose that under the equivalent martingale measure, the underlying asset price of the option $S^{(\epsilon)}$ follows a stochastic process;

$$dS_t^{(\epsilon)} = \alpha S_t^{(\epsilon)} dt + \epsilon \sigma(S_t^{(\epsilon)}, t) dw_t; \ S_0^{(\epsilon)} = s(>0). \tag{8}$$

Let a strike price K be some positive constant, then the values of vanilla call and put options at the expiration date T are respectively written as

$$V_c(T) = \max\left\{S_T^{(\epsilon)} - K, 0\right\} \text{ and } V_p(T) = \max\left\{K - S_T^{(\epsilon)}, 0\right\}. \tag{9}$$

Moreover, the values at the contract date($t = 0$) are respectively expressed as

$$V_c(0) = e^{-rT}\mathbf{E}[V_c(T)] \text{ and } V_p(0) = e^{-rT}\mathbf{E}[V_p(T)], \tag{10}$$

where r, a nonnegative constant denotes the (instantaneous) short-term interest rate. In what follows, I explain the case of call options, which can be also applied to the case of a put option. First, in the setting of Section 2.1, let $d = 2$, $m = 1$, and $X_t^{(\epsilon)} = (X_{0t}^{(\epsilon)}, X_{1t}^{(\epsilon)})'$ with $X_{0t}^{(\epsilon)} = t$, and $X_{1t}^{(\epsilon)} = S_t^{(\epsilon)}$. Next, with $x = (x_0, x_1)$ and $g(x) = x_1 - K$, $G^{(\epsilon)}$ is given by

$$G^{(\epsilon)} = \frac{X_{1T}^{(\epsilon)} - X_{1T}^{(0)}}{\epsilon},$$

where $\epsilon \in (0,1]$ and $X_{1T}^{(0)} = S_T^{(0)} = e^{\alpha T}s$. Then, the value at expiration can be written as

$$\begin{aligned} V_c(T) &= \max\left\{S_T^{(\epsilon)} - K, 0\right\} = \max\left\{\epsilon G^{(\epsilon)} + X_{1T}^{(0)} - K, 0\right\} \\ &= \max\left\{\epsilon(G^{(\epsilon)} + y^{(\epsilon)}), 0\right\}, \end{aligned}$$

where $y^{(\epsilon)} = \frac{X_{1T}^{(0)} - K}{\epsilon}$. Note also that Σ_T is expressed as

$$\Sigma_T = \int_0^T e^{2\alpha(T-u)}\sigma^2(X_{1u}^{(0)}, X_{0u}^{(0)})du = \int_0^T e^{2\alpha(T-u)}\sigma^2(S_u^{(0)}, u)du. \tag{11}$$

Although $\phi^{(\epsilon)}(x)$ and B can be set as $\phi^{(\epsilon)}(x) = \epsilon x + (X_{1T}^{(0)} - K)$ and $B = \{X_{1T}^{(\epsilon)} \geq K\} = \{G^{(\epsilon)} \geq -y^{(\epsilon)}\}$ respectively, I consider this problem under the following assumption for more practical purposes:
(Assumption 2)
For arbitrary $y \in \mathbf{R}$ the strike price K is given by $K = S_T^{(0)} - \epsilon y$.

Then, $y^{(\epsilon)} = y$. Note also that in Assumption 2, $S_T^{(0)}$ is equivalent to the forward price with the same expiration date as the option's one.

Now I set

$$\phi^{(\epsilon)}(x) = \epsilon(x+y),$$

and $B = \{G^{(\epsilon)} \geq -y\}$. Then, by the equation (7), an approximation formula up to the ϵ^2-order for the call option's value at the contract date is derived as

$$\begin{aligned} V_c(0) &= e^{-rT}\{\epsilon \int_{-y}^{\infty} (x+y) n[x; 0, \Sigma_T] dx \\ &+ \epsilon^2 \int_{-y}^{\infty} (cx^2 + f) n[x; 0, \Sigma_T] dx\} + o(\epsilon^2), \end{aligned} \tag{12}$$

where the constants c and f are given by

$$\begin{aligned} c &= \frac{1}{\Sigma_T^2} \int_0^T e^{\alpha(T-u)} \sigma(S_u^{(0)}, u) \partial\sigma(S_u^{(0)}, u) \int_0^u e^{2\alpha(T-v)} \sigma^2(S_u^{(0)}, v) dv du \ , \\ f &= -c\Sigma_T. \end{aligned} \tag{13}$$

Moreover, a more concrete approximation formula can be obtained by integration by parts.

Theorem 1 *Suppose that the underlying asset price follows the stochastic process (8) under the equivalent martingale measure. Then, under Assumptions 1 and 2, the asymptotic expansion up to the ϵ^2-order of $V_c(0)$, the price of a vanilla call option at the contract date with the maturity date T and the strike price K, is given by*

$$\begin{aligned} V_c(0) &= e^{-rT}\epsilon \left\{ \Sigma_T n[y; 0, \Sigma_T] + yN\left(\frac{y}{\sqrt{\Sigma_T}}\right) \right\} \\ &+ e^{-rT}\epsilon^2 c \left\{ \Sigma_T N\left(\frac{y}{\sqrt{\Sigma_T}}\right) - y\Sigma_T n[y; 0, \Sigma_T] + fN\left(\frac{y}{\sqrt{\Sigma_T}}\right) \right\} + o(\epsilon^2), \end{aligned} \tag{14}$$

where Σ_T and c, f are respectively defined as (11) and (13), and $N(x)$ denotes the distribution function of a standard normal distribution.

Values of various option contracts can be evaluated by using the similar method. In particular, with redefinition of Σ_T, c and f in (14), almost the same approximation formula can be applied to various options. In what follows, some examples for call options are presented. For more details, see [34] and [40].

1. A Basket Option

   Suppose that $S_{it}^{(\epsilon)}$, $i = 1, \cdots, n$, the underlying asset prices of a basket option follow the stochastic processes;

   $$dS_{it}^{(\epsilon)} = \alpha_i S_{it}^{(\epsilon)} dt + \epsilon \sum_{j=1}^{n} \sigma_{ij}(S_{it}^{(\epsilon)}, t) dw_{jt}; \ S_{i0}^{(\epsilon)} = s_i (> 0),$$

   where α_i, $i = 1, \cdots, n$ are constants, and $w = (w_1, \cdots, w_n)$ is a n-dimensional Wiener process. Then, the value of the basket option at the maturity date T is given by

   $$V_c(T) = \max\left\{ \sum_{i=1}^{n} \beta_i S_{iT}^{(\epsilon)} - K, 0 \right\},$$

where the strike price K is a positive constant and β_i, $i = 1, \cdots, n$ are constants. Further, let r be a nonnegative constant. Then, the price at the contract date($t = 0$) is expressed as $V_c(0) = e^{-rT}\mathbf{E}[V_c(T)]$. In this case, it is specified that $d = n+1$, $m = n$, $X_t^{(\epsilon)} = (X_{0t}^{(\epsilon)}, X_{1t}^{(\epsilon)}, \cdots, X_{nt}^{(\epsilon)})'$ with $X_{0t}^{(\epsilon)} = t$, $X_{it}^{(\epsilon)} = S_{it}^{(\epsilon)} (i = 1, \cdots, n)$, and $g(x) = \sum_{i=1}^{n} \beta_i x_i - K$, $x = (x_0, x_1, \cdots, x_n)$ in the setting of Section 2.1.

In the following examples, α and r are set to be a constant and a nonnegative constant respectively as well as in this example, unless otherwise stated.

2. An Average Option

   The stochastic processes which describe the dynamics of the underlying variables of an average option are expressed as

$$\begin{aligned} dS_t^{(\epsilon)} &= \alpha S_t^{(\epsilon)} dt + \epsilon \sigma(S_t^{(\epsilon)}, t) dw_t;\ S_0^{(\epsilon)} = s(> 0) \\ dZ_t^{(\epsilon)} &= S_t^{(\epsilon)} dt;\ Z_0^{(\epsilon)} = 0, \end{aligned}$$

   where w denotes a one-dimensional Wiener process. Then, the maturity value of an average option is expressed as

$$V_c(T) = \max\left\{\frac{1}{T} Z_T^{(\epsilon)} - K, 0\right\}.$$

   Thus the price at the contract date is given by

   $V_c(0) = e^{-rT}\mathbf{E}[V_c(T)]$, where $d = 3$, $m = 1$, $X_t^{(\epsilon)} = (X_{0t}^{(\epsilon)}, X_{1t}^{(\epsilon)}, X_{2t}^{(\epsilon)})'$ with $X_{0t}^{(\epsilon)} = t$, $X_{1t}^{(\epsilon)} = S_t^{(\epsilon)}$, $X_{2t}^{(\epsilon)} = Z_t^{(\epsilon)}$, and $g(x) = \frac{1}{T} x_2 - K$, $x = (x_0, x_1, x_2)$ in the setting of Section 2.1.

3. A Vanilla Option with the Stochastic Volatility

   Assume that the underlying variables of a vanilla option with the stochastic volatility follow the stochastic processes below;

$$\begin{aligned} dS_t^{(\epsilon)} &= \alpha S_t^{(\epsilon)} dt + \epsilon \sum_{j=1}^{2} \sigma_{1j}(S_t^{(\epsilon)}, Z_t^{(\epsilon)}, t) dw_{jt};\ S_0^{(\epsilon)} = s(> 0) \\ dZ_t^{(\epsilon)} &= \mu(S_t^{(\epsilon)}, Z_t^{(\epsilon)}, t) dt + \epsilon \sum_{j=1}^{2} \sigma_{2j}(S_t^{(\epsilon)}, Z_t^{(\epsilon)}, t) dw_{jt};\ Z_0^{(\epsilon)} = z(> 0), \end{aligned}$$

   where $w = (w_1, w_2)$ denotes a two-dimensional Wiener process.

   The payoff at maturity is expressed as $V_c(T) = \max\{S_T^{(\epsilon)} - K, 0\}$, and the price at the contract date is evaluated by $V_c(0) = e^{-rT}\mathbf{E}[V_c(T)]$.

   In this case, it is specified that $d = 3$, $m = 2$, $X_t^{(\epsilon)} = (X_{0t}^{(\epsilon)}, X_{1t}^{(\epsilon)}, X_{2t}^{(\epsilon)})'$ with $X_{0t}^{(\epsilon)} = t$, $X_{1t}^{(\epsilon)} = S_t^{(\epsilon)}$, $X_{2t}^{(\epsilon)} = Z_t^{(\epsilon)}$, and $g(x) = x_1 - K$, $x = (x_0, x_1, x_2)$ in the setting of Section 2.1.

4. A Vanilla Option under the Stochastic Interest Rate

   Assume that the stochastic processes determining the value of the option under the stochastic interest rate follow

$$\begin{aligned} dS_t^{(\epsilon)} &= (r_t^{(\epsilon)} - \alpha) S_t^{(\epsilon)} dt + \epsilon \sum_{j=1}^{2} \sigma_{1j}(S_t^{(\epsilon)}, r_t^{(\epsilon)}, t) dw_{jt};\ S_0^{(\epsilon)} = s(>0) \\ dZ_t^{(\epsilon)} &= -r_t^{(\epsilon)} Z_t^{(\epsilon)} dt;\ Z_0^{(\epsilon)} = 1 \\ dr_t^{(\epsilon)} &= \mu(S_t^{(\epsilon)}, r_t^{(\epsilon)}, t) dt + \epsilon \sum_{j=1}^{2} \sigma_{2j}(S_t^{(\epsilon)}, r_t^{(\epsilon)}, t) dw_{jt};\ r_0^{(\epsilon)} = r(\geq 0), \end{aligned}$$

   where $w = (w_1, w_2)$ denotes a two-dimensional Wiener process.

   The value at maturity is expressed as $V_c(T) = \max\{S_T^{(\epsilon)} - K, 0\}$.

   Then the value at the contract date is evaluated by

$$V_c(0) = \mathbf{E}\left[e^{-\int_0^T r_u^{(\epsilon)} du} V_c(T) \right],$$

   where $d = 4$, $m = 2$, $X_t^{(\epsilon)} = (X_{0t}^{(\epsilon)}, \cdots, X_{3t}^{(\epsilon)})'$ with $X_{0t} = t$, $X_{1T}^{(\epsilon)} = S_t^{(\epsilon)}$, $X_{2t}^{(\epsilon)} = Z_t^{(\epsilon)}$, $X_{3t}^{(\epsilon)} = r_t^{(\epsilon)}$, and $g(x) = x_2(x_1 - K)$, $x = (x_0, x_1, x_2, x_3)$ in the setting of Section 2.1.

2.3 Approximations of the Values of Financial Assets and Securities under Diffusion Processes

The framework of the asymptotic expansion can be applied not only to the simple cases mentioned above, but also to evaluation of much broader range of assets' and securities' values. In particular, there are many cases where the asymptotic expansion can be applied to approximate that values when the underlying asset prices of financial assets or securities, cash flows and discount rates such as interest rates are expressed as a function of a random vector $X^{(\epsilon)}$ that follows a diffusion process. The method is almost the same as the one illustrated above and hence it is omitted. In this subsection, only the method to represent values of financial assets or securities will be reviewed.

First, just as in the previous subsections, I consider a d-dimensional diffusion process $X^{(\epsilon)}$ defined as the strong solution to the stochastic differential equation (1). In general, the contract value V of a financial asset which generates a cash flow at the maturity date T is represented as

$$V = \mathbf{E}\left[e^{-\int_0^T R_2(X_u^{(\epsilon)}) du} F(f(X_T^{(\epsilon)})) \right], \tag{15}$$

where f denotes the underlying asset price and F is the cash flow which characterizes the asset or security to be evaluated. Note that the dynamics of the underlying asset price f is expressed by a diffusion process, whose drift term (the coefficient of the dt term) is $R_1(X_t^{(\epsilon)})f - D(X_t^{(\epsilon)})$ under the

equivalent martingale measure. Moreover, R_1 at time $t \in [0,T]$ is represented as

$$R_1(X_t^{(\epsilon)}) = r(X_t^{(\epsilon)}) + \sum_{j=1}^{J_1} s_{1j}(X_t^{(\epsilon)}),$$

where r is the risk-free interest rate and s_{1j}, $j = 1, \cdots, J_1$ are various spreads (the differences from the risk-free rate) such as credit spreads or liquidity spreads. Suppose that all of them are expressed as functions of the variable $X^{(\epsilon)}$. Further, $D(X_t^{(\epsilon)})$ denotes a payoff generated by the underlying asset such as a dividend or an interest rate and is also represented as a function of the variable $X^{(\epsilon)}$. Meanwhile, R_2, the discount rate of the objective asset or security to be evaluated at time t is also expressed as

$$R_2(X_t^{(\epsilon)}) = r(X_t^{(\epsilon)}) + \sum_{j=1}^{J_2} s_{2j}(X_t^{(\epsilon)}),$$

where s_{2j}, $j = 1, \cdots, J_2$ are various spreads related to the objective asset or security. Suppose also that all of them are expressed as some functions of the variable $X^{(\epsilon)}$.

As an example, let $F = 1$ in (15) for a zero-coupon bond with the face value 1 and the maturity date T. Next, let V_i denote the price of the zero-coupon bond with the maturity T_i. Then, V, the value of a coupon bond with the maturity T_N and coupon payments c_i at $T_i (i = 1, \cdots, N, T_1 < \cdots < T_N)$ is represented by the equation $V = \sum_{i=1}^{N} c_i V_i$. Moreover, the present value of a call option on the coupon bond with the option maturity $T (< T_1)$ can be evaluated if I set $F(x) = (x - K)^+$ and $f(X_T^{(\epsilon)}) = \sum_{i=1}^{N} c_i f_i(X_T^{(\epsilon)})$ in the equation (15), where $f_i(X_T^{(\epsilon)})$, $i = 1, \cdots, N$ are given by

$$f_i(X_T^{(\epsilon)}) = \mathbf{E}\left[e^{-\int_T^{T_i} R_1(X_u^{(\epsilon)}) du} | X_T^{(\epsilon)} \right].$$

3 Asymptotic Expansions in an Instantaneous Forward Rates Model and a Jump-Diffusion Model

Among main stochastic models in finance, there exist models where the stochastic processes of the underlying variables do not belong to the class of diffusion processes. This section illustrates two typical examples.

3.1 An Instantaneous Forward Rates Model

Among stochastic models for evaluating the interest rate derivatives, there exists a model developed by Heath-Jarrow-Morton [4] which is formulated based on the forward rates with infinitesimal terms of the interest rates, that is the instantaneous forward rates $\{f(s,t) : 0 \leq s \leq t \leq T\}$. Here, s is the time when the forward rate is determined and t denotes the time when the forward rate starts to be applied.

The stochastic processes for the instantaneous forward rates are considered in the framework of the asymptotic expansion by introducing a parameter $\epsilon \in [0,1]$. For example, let w be a m-dimensional standard Wiener

process and let $f(0,t)$, $t \in [0,T]$ be a given Lipschitz continuous function of t. Then, under the equivalent martingale measure, the stochastic processes of $\{f^{(\epsilon)}(s,t) : 0 \le s \le t \le T\}$ are written as

$$\begin{aligned} f^{(\epsilon)}(s,t) = f(0,t) &+ \epsilon^2 \int_0^s \sum_{i=1}^m \left[\sigma_i(f^{(\epsilon)}(v,t),v,t) \int_v^t \sigma_i(f^{(\epsilon)}(v,y),v,y)dy \right] dv \\ &+ \epsilon \sum_{i=1}^m \int_0^s \sigma_i(f^{(\epsilon)}(v,t),v,t)dw_i(v) \; ; \epsilon \in [0,1], \end{aligned} \tag{16}$$

where the volatility functions $\{\sigma_i(f^{(\epsilon)}(s,t),s,t); i = 1,\cdots,m\}$ are smooth and satisfy the regularity conditions which guarantee that the equation (16) has its solution. It is to be noted that the drift term (the coefficient of the dv term) of $f^{(\epsilon)}(s,t)$ depends on $\{f^{(\epsilon)}(v,y); 0 \le v < s, v \le y < t\}$. Moreover, the stochastic process of the instantaneous short-term interest rate $r^{(\epsilon)}(t)$ is determined by the relationship, $r^{(\epsilon)}(t) = f^{(\epsilon)}(t,t)$.

Even for this model, the approximations of the values for interest rate derivatives can be still considered in a unified framework with derivation of asymptotic expansions of the instantaneous forward rates when $\epsilon \downarrow 0$ and with use of the relation between the instantaneous forward rates and a zero-coupon bond price;

$$P^{(\epsilon)}(t,T) = \exp\left\{ - \int_t^T f^{(\epsilon)}(t,u)du \right\}. \tag{17}$$

As an example, I consider a call option on a coupon bond, which is a standard interest rate derivative. The payoff at maturity of the option is given by

$$V_c(T) = \max\{\sum_{i=1}^n c_i P^{(\epsilon)}(t,T_i) - K, 0\},$$

where $0 \le T \le T_1 < \cdots < T_n$, c_i,$i = 1,\cdots,n$ are positive constants and $K(>0)$ is a strike price. Then, the contract value is evaluated by

$$V_c(0) = \mathbf{E}\left[e^{-\int_0^T r_u^{(\epsilon)} du} V_c(T) \right].$$

When $\epsilon \downarrow 0$, the forward rate $f^{(\epsilon)}(s,t)$ is expanded as

$$f^{(\epsilon)}(s,t) \sim f(0,t) + \epsilon f_1(s,t) + \epsilon^2 f_2(s,t) + \cdots \; in \; \mathbf{D}^\infty, \tag{18}$$

where $f_1(t,u), f_2(t,u) \cdots \in \mathbf{D}^\infty$. As a result, $P^{(\epsilon)}(t,T)$ and $\exp\left\{ - \int_0^T r^{(\epsilon)}(t)dt \right\}$ are expanded as

$$\begin{aligned} P^{(\epsilon)}(t,T) &\sim \frac{P(0,T)}{P(0,t)} \left[1 - \epsilon \int_t^T f_1(t,u)du - \epsilon^2 \int_t^T f_2(t,u)du \right. \\ &+ \left. \epsilon^2 \frac{1}{2} \left\{ \int_t^T f_1(t,u)du \right\}^2 \right] + \cdots \; in \; \mathbf{D}^\infty, \end{aligned}$$

$$
\begin{aligned}
e^{-\int_0^T r^{(\epsilon)}(s)ds} &\sim P(0,T)\left[1-\epsilon\int_0^T f_1(t,t)dt-\epsilon^2\int_0^T f_2(t,t)dt\right. \\
&+ \left.\epsilon^2\frac{1}{2}\left\{\int_0^T f_1(t,t)dt\right\}^2\right]+\cdots \ in\ \mathbf{D}^\infty,
\end{aligned}
$$

where $f_i(s,t)$, $i=1,2$ are given by

$$
\begin{aligned}
f_1(s,t) &= \tfrac{\partial f^{(\epsilon)}(s,t)}{\partial\epsilon}|_{\epsilon=0}=\int_0^s\sum_{i=1}^m\sigma_i^{(0)}(v,t)dw_i(v), \\
f_2(s,t) &= \tfrac{1}{2}\tfrac{\partial^2 f^{(\epsilon)}(s,t)}{\partial^2\epsilon}|_{\epsilon=0} \\
&= \int_0^s b^{(0)}(v,t)dv+\int_0^s\sum_{i=1}^m\partial\sigma_i^{(0)}(v,t)f_1(v,t)dw_i(v).
\end{aligned}
$$

Here, $\sigma_i^{(0)}(v,t)=\sigma_i(f^{(0)}(v,t),v,t)$, and $b^{(0)}(v,t)$ and $\partial\sigma_i^{(0)}(v,t)$ are defined as

$$
\begin{aligned}
b^{(0)}(v,t) &= \sum_{i=1}^n\sigma_i(f^{(0)}(v,t),v,t)\int_v^t\sigma_i(f^{(0)}(v,y),v,y)dy, \\
\partial\sigma_i^{(0)}(v,t) &= \frac{\partial\sigma_i(x,v,t)}{\partial x}|_{x=f(0,t)}.
\end{aligned}
$$

Therefore, in the framework of the previous section with definition of $X_{0t}^{(\epsilon)}$ and $X_{it}^{(\epsilon)}, i=1,\cdots,n$ as

$$
\begin{aligned}
X_{0t}^{(\epsilon)} &= \exp\left\{-\int_0^t r^{(\epsilon)}(u)du\right\} \\
X_{it}^{(\epsilon)} &= P^{(\epsilon)}(t,T_i)=\exp\left\{-\int_t^{T_i} f^{(\epsilon)}(t,u)du\right\},\ i=1,\cdots,n,
\end{aligned}
$$

the payoff at maturity of the call option on a coupon bond is written as

$$
V_c(T)=\max\left\{\sum_{i=1}^n c_iX_{it}^{(\epsilon)}-K,0\right\}.
$$

Moreover, let $x=(x_0,x_1,\cdots,x_n)$ and define $g(x)$ as

$$
g(x)=x_0\left(\sum_{i=1}^n c_ix_i-K\right).
$$

Then, a similar method can be applied as in the case of diffusion processes. Consequently, with redefinition of Σ_T, c and f, the approximation of this option can be obtained based on almost the same asymptotic expansion as for the approximation (14) given in Theorem 1. For more details, see [13], [14] and [36].

For evaluation of other various interest rate derivatives, approximations based on the asymptotic expansion can also be derived. Moreover, the approximate formula for the value of a derivative dependent on instantaneous forward rates and other variables following diffusion processes is given by [33].

3.2 Evaluation of the Values of Securities under Jump-Diffusion Processes

So far, stochastic models where uncertainty is generated by Wiener processes have been used. However, I can also apply the asymptotic expansion to stochastic processes including jumps in their sample paths. Suppose that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ is a filtered probability space with an equivalent martingale measure P and is equipped with a m-dimensional Wiener process w and a stationary Poisson random measure μ on $[0,T]\times\mathbf{E}$, which are mutually independent. Let also $(\mathbf{E}, \mathcal{E})$ be a measurable space. Moreover, suppose that the intensity measure of the Poisson random measure μ is

$$\hat{\lambda}(dt, dx) = dt \times \nu(dx),$$

where ν is a positive σ-finite measure on $(\mathbf{E}, \mathcal{E})$ and the compensated Poisson measure is represented as

$$\tilde{\mu}(dt, dx) = \mu(dt, dx) - \hat{\lambda}(dt, dx).$$

In this setting, examples for evaluation of the values of a bond and an option are provided.

Evaluation of a Bond Price

Let $X^{(\epsilon)}$ denote the R^d-valued stochastic process defined as the solution to the following stochastic differential equation:

$$dX^{(\epsilon)}(t) = A(X^{(\epsilon)}(t), \epsilon)dt + \epsilon\sigma(X^{(\epsilon)}(t))dw(t) + \epsilon\int_E C(X^{(\epsilon)}(t-), x)\tilde{\mu}(dt, dx), \tag{19}$$

where $\epsilon \in [0,1]$ and assume that the coefficients $A : \mathbf{R}^d \times [0,1] \mapsto \mathbf{R}^d$, $\sigma : \mathbf{R}^d \mapsto \mathbf{R}^d \otimes \mathbf{R}^m$, and $C : \mathbf{R}^d \times \mathbf{R}^d \mapsto \mathbf{R}^d$ are smooth and satisfy regularity conditions. Define also $\mathbf{R}^d$-valued random vectors D and E as

$$D(t) = \frac{\partial X^{(\epsilon)}(t)}{\partial\epsilon}|_{\epsilon=0} \text{ and } E(t) = \frac{\partial^2 X^{(\epsilon)}(t)}{\partial\epsilon^2}|_{\epsilon=0}.$$

In this case, $D(t)$ satisfies the following stochastic differential equation:

$$\begin{aligned} dD(t) &= \partial_\epsilon A(X^{(0)}(t), 0)dt + \partial A(X^{(0)}(t), 0)D(t)dt + \sigma(X^{(0)}(t))dw(t) \\ &\quad + \textstyle\int_E C(X^{(0)}(t-), x)\tilde{\mu}(dt, dx);\ D(0) = 0. \end{aligned}$$

Therefore, the random vector $D(t)$ can be expressed as

$$D(t) = Y_t \int_0^t Y_s^{-1}[\partial_\epsilon A(X^{(0)}(s), 0)ds + \sigma(X_s^{(0)})dw(s) + \int_E C(X^{(0)}(s-), x)\tilde{\mu}(ds, dx)],$$

where Y is a $\mathbf{R}^d \otimes \mathbf{R}^d$-valued function and is the solution to the following ordinary differential equation:

$$dY_t = \partial A(X^{(0)}(t), 0)Y_t dt;\ Y_0 = I_d.$$

Similarly, E satisfies the stochastic differential equation;

$$\begin{aligned} dE(t) &= \partial A(X_t^{(0)},0)E(t)dt + 2\partial_\epsilon \partial A(X_t^{(0)},0)D(t)dt + \partial_\epsilon^2 A(X_t^{(0)},0)dt \\ &\quad +\partial\partial A(X_t^{(0)},0)D(t)D(t)dt + 2\partial\sigma(X_t^{(0)})D(t)dw(t) \\ &\quad +2\textstyle\int_E \partial C(X_{t-}^{(0)},x)D_{t-}\tilde{\mu}(dt,dx);\ E(0)=0, \end{aligned}$$

where $\partial\partial A(X_t^{(0)},0)D(t)D(t)$ is a d-dimensional vector and is given by

$$\begin{aligned} &\partial\partial A(X_t^{(0)},0)D(t)D(t) \\ &= \left[D'\{\partial_i\partial_j A_1(X_t^{(0)},0)\}_{(i,j)}D,\cdots,D'\{\partial_i\partial_j A_d(X_t^{(0)},0)\}_{(i,j)}D\right]' \quad (i,j=1,2,\cdots,d). \end{aligned}$$

Here, $\left\{\partial_i\partial_j A_k(X_t^{(0)},0)\right\}_{(i,j)}$ denotes a $d\times d$ matrix defined for each $k=1,\cdots,d$. Further, $\partial\sigma(X_t^{(0)})D_t$ denotes a $d\times m$ matrix and is represented by

$$\partial\sigma(X_t^{(0)})D(t) = \left\{\sum_{k=1}^d \partial_k\sigma_{ij}(X_t^{(0)})D_k(t)\right\}_{i,j} \quad (i=1,\cdots,d, j=1,\cdots,m).$$

Moreover, a notation $\partial C(X_{t-}^{(0)},x)D_{t-}$ denotes a d-dimensional vector whose i-th element is $\sum_{k=1}^d \partial_k C_i(X_{t-}^{(0)},x)D_k(t-)$. Therefore, $E(t)$ is given by

$$\begin{aligned} & E(t) = Y_t\textstyle\int_0^t Y_s^{-1}\left[2\partial_\epsilon\partial A(X_s^{(0)},0)D(s)ds + \partial_\epsilon^2 A(X_s^{(0)},0)ds\right. \\ +\ & \partial\partial A(X_s^{(0)},0)D(s)D(s)ds + 2\partial\sigma(X_s^{(0)})D(s)dw(s) \\ +\ & \left. 2\textstyle\int_E \partial C(X^{(0)}(s-),x)D_{s-}\tilde{\mu}(ds,dx)\right]. \end{aligned}$$

On the other hand, it is known that the price of a zero-coupon bond is generally expressed as

$$P(0,T) = \mathbf{E}\left[e^{-\int_0^T g(X_s^{(\epsilon)})ds}\right].$$

Here, g denotes the sum of the risk-free rate and the spreads determined by risk factors such as the credit risk and the liquidity risk, and is supposed to be represented by a function of $X^{(\epsilon)}(t)$. Then, an asymptotic expansion up to the ϵ^2-order of $P(0,T)$, the zero-coupon bond price is given by

$$\begin{aligned} & P(0,T) = e^{-\int_0^T g_s^{(0)}ds}\left(1-\epsilon\int_0^T \mathbf{E}\left[(\partial g_s^{(0)})'D(s)\right]ds\right. \\ +\ & \epsilon^2\left\{-\frac{1}{2}\int_0^T\sum_{i,j=1}^n \partial_i\partial_j g_s^{(0)}\mathbf{E}[D_i(s)D_j(s)]ds - \frac{1}{2}\int_0^T \partial g_s^{(0)}\mathbf{E}[E(s)]ds\right. \\ & \left.\left. +\frac{1}{2}\ \mathbf{E}[(\int_0^T \partial g_s^{(0)}D(s)ds)^2]\right\}\right) + o(\epsilon^2). \end{aligned} \tag{20}$$

With definition of $\partial_\epsilon A(\tau)$ and $C(t)$ as $\partial_\epsilon A(\tau) = \partial_\epsilon A(X^{(0)}(\tau),0)$ and $C(t) = C(X^{(0)}(t-),x)$, $\mathbf{E}[D_i(s)D_j(u)]$ is represented as the following:

$$\mathbf{E}[D_i(s)D_j(u)] \tag{21}$$

$$
\begin{aligned}
= \quad & \left[Y^{(i,\cdot)}(s)\int_0^s Y^{-1}(\tau)\partial_\epsilon A(\tau)d\tau\right]\left[Y^{(j,\cdot)}(u)\int_0^u Y^{-1}(\tau)\partial_\epsilon A(\tau)d\tau\right] \\
& +Y^{(i,\cdot)}(s)\left(\int_0^{s\wedge u} Y^{-1}(\tau)\sigma(\tau)\sigma(\tau)'\left(Y^{-1}(\tau)\right)' d\tau\right)\left(Y^{(j,\cdot)}(u)\right)' \\
& +Y^{(i,\cdot)}(s)\left(\int_0^{s\wedge u} Y^{-1}(\tau)\left\{\int_E C(\tau)C(\tau)'\nu(dx)\right\}\left(Y^{-1}(\tau)\right)' d\tau\right)\left(Y^{(j,\cdot)}(u)\right)'.
\end{aligned}
$$

A more specific expression of (20) can be obtained based on the equations such as (21).

Evaluation of an Option Value

Here, I consider a more specific example for evaluating the value of an option. Let $m = d = 1$, $\epsilon \in [0,1]$ and α be a constant. Further, under an equivalent martingale measure, suppose that the underlying asset price of an option follows a stochastic process;

$$
dS_t^{(\epsilon)} = \alpha S_t^{(\epsilon)}dt + \epsilon\sigma(S_t^{(\epsilon)},t)dw_t + \int_{\mathbf{R}} S_{t-}^{(\epsilon)}(e^{\epsilon x}-1)\tilde{\mu}(dtdx);\ S_0^{\epsilon} = s(>0), \quad (22)
$$

where $\sigma(x,t)$ is a function satisfying appropriate regularity conditions. Here, $\mu([0,t]\times A)$ is given by

$$
\mu([0,t]\times A) = \sum_{j=1}^{N_t} I_A(\xi_j^{(\epsilon)}),
$$

where $A \in B(\mathbf{R})$, and N_t denotes a Poisson process with a constant intensity $\lambda(>0)$. Define random variables $(\xi_j^{(\epsilon)})_{j\geq 1}$ determining jump sizes as

$$
\xi_j^{(\epsilon)} = e^{\epsilon\eta_j} - 1,\ j \geq 1,
$$

where $(\eta_j)_{j\geq 1}$ denotes a sequence of random variables which follow independent identical distributions $(i.i.d.)$ and its probability law is ν. Suppose also that $v = \mathbf{E}[\eta_j] < \infty$ and $\mathbf{E}[\xi_j^{(\epsilon)}] < \infty$. Note that $\xi_j^{(\epsilon)}$ is defined so that the underlying asset price does not become negative by its jumps. In this case, the compensated Poisson measure $\tilde{\mu}(dt,dx)$ is given by

$$
\tilde{\mu}(dt,dx) = \mu(dt,dx) - \lambda dt \times \nu(dx).
$$

Note that this model is an extension of Merton[24].

Here, if I assume r to be a nonnegative constant, V, the contract price of a vanilla call option with the maturity T and the strike price $K(>0)$ is represented as $V = e^{-rT}\mathbf{E}[(S_T^{(\epsilon)} - K)_+]$; $(x)_+$ denotes $\max\{x,0\}$. Here, in order to apply an asymptotic expansion, I define $X^{(\epsilon)}$ as

$$
X_t^{(\epsilon)} = \frac{S_t^{(\epsilon)} - S_t^{(0)}}{\epsilon},
$$

where $S_t^{(0)}$ is given by $S_t^{(0)} = S_0 e^{\alpha t}$ and $X^{(\epsilon)}$ follows the stochastic differential equation;

$$
\begin{aligned}
dX_t^{(\epsilon)} \quad = \quad & \alpha X_t^{(\epsilon)}dt + \sigma(\epsilon X_t^{(\epsilon)} + S_t^{(0)})dw_t \\
& + \textstyle\int_{\mathbf{R}}(\epsilon X_{t-}^{(\epsilon)} + S_{t-}^0)\left(\frac{e^{\epsilon x}-1}{\epsilon}\right)\tilde{\mu}(dt,dx);\ X_0^{(\epsilon)} = 0.
\end{aligned}
$$

With this $X^{(\epsilon)}$, the option value V is expressed as

$$V = e^{-rT}\epsilon\mathbf{E}[(X_T^{(\epsilon)} + k^{(\epsilon)})_+],$$

where $k^{(\epsilon)} = \frac{S_T^{(0)} - K}{\epsilon}$. As in the Assumption 2 in Section 2.2, assume that $k^{(\epsilon)}$ is represented as $k^{(\epsilon)} = k + O(\epsilon)$ by a real number k. Then, the asymptotic expansion up to the ϵ order of the call option value V is expressed as

$$V = e^{-rT}\epsilon\mathbf{E}[(X_T^{(0)} + k)_+] + o(\epsilon).$$

$X^{(0)}$ in this equation follows the stochastic differential equation:

$$dX_t^{(0)} = \alpha X_t^{(0)}dt + \sigma(S_t^{(0)}, t)dw_t + \int_{\mathbf{R}} S_{t-}^{(0)} x\tilde{\mu}(dt, dx);\ X_0^{(0)} = 0.$$

Moreover, by further calculation, $\mathbf{E}[(X_T^{(0)} + k)_+]$ is expressed as

$$\begin{aligned}\mathbf{E}[(X_T^{(0)} + k)_+] &= \sum_{j=0}^{\infty} \mathbf{E}\left[\Sigma_T\, n\left(k_2 + S_0 e^{\alpha T}\sum_{i=1}^{j}\eta_i\right)\right. \\ &+ \left.(k_2 + S_0 e^{\alpha T}\sum_{i=1}^{j}\eta_i)N\left(\frac{k_2}{\sqrt{\Sigma_T}} + \frac{S_0 e^{\alpha T}}{\sqrt{\Sigma_T}}\sum_{i=1}^{j}\eta_i\right)\right] e^{-\lambda T}\frac{(\lambda T)^j}{j!},\end{aligned}$$

where $\sum_{i=1}^{0}\eta_i = 0$. k_2, $n(x)$ and Σ_T are given by

$$\begin{aligned}k_2 &= k - v(\lambda T)S_0 e^{\alpha T},\ n(x) = \frac{1}{\sqrt{2\pi\Sigma_T}}\exp\left(\frac{-x^2}{2\Sigma_T}\right), \\ \Sigma_T &= \int_0^T e^{2\alpha(T-s)}\sigma^2(S_s^0, s)ds. \qquad (23)\end{aligned}$$

Suppose also that $\Sigma_T > 0$. Moreover, with an assumption that the random variables $(\eta_i)_{i\geq 1}$ which determine the distributions of the jump sizes follow a normal one, a more concrete expression can be obtained.

Theorem 2 *Suppose that the underlying asset price follows the stochastic process in (22) and $(\eta_i)_{i\geq 1}$ follow a normal distribution, $\eta_i \sim N(v, \sigma^2)$. Then, under the assumption that $\Sigma_T > 0$, an asymptotic expansion up to the ϵ-order of V, the contract price of a vanilla call option with the maturity date T and the strike price K, is given by*

$$\begin{aligned}V &= e^{-rT}\epsilon\left\{\sum_{j=0}^{\infty}\left[\sqrt{\frac{\Sigma_T}{2\pi(c_{4j}^2+1)}}\quad\exp\left(\frac{-c_{3j}^2}{2(c_{4j}^2+1)}\right) + c_{1j}N\left(\frac{c_{3j}}{\sqrt{c_{4j}^2+1}}\right)\right.\right. \\ &+ \left.\left.\frac{c_{2j}c_{4j}}{\sqrt{2\pi(c_{4j}^2+1)}}\exp\left(\frac{-c_{3j}^2}{2(c_{4j}^2+1)}\right)\right] e^{-\lambda T}\frac{(\lambda T)^j}{j!})\right\} + o(\epsilon), \qquad (24)\end{aligned}$$

where k_2 and Σ_T are defined as (23), and the constant coefficients c_{1j}, c_{2j}, c_{3j} and c_{4j} are defined as

$$\begin{aligned}c_{1j} &\equiv k_2 + S_0 e^{\alpha T}(vj)\ , c_{2j} \equiv S_0 e^{\alpha T}(\sigma\sqrt{j})\ , \\ c_{3j} &\equiv \frac{c_{1j}}{\sqrt{\Sigma_T}}\ , c_{4j} \equiv \frac{c_{2j}}{\sqrt{\Sigma_T}}.\end{aligned}$$

For the details such as numerical examples, see [15] and [16].

4 Conclusion

Finally, I briefly review other applications of the asymptotic expansion technique to numerical problems in finance, which can not be introduced in the paper because of the space limitation.

[41] applies an asymptotic expansion to a dynamic investment problem with utility maximization for the asset at the end of the investment period, and derives an approximation formula for evaluating the optimal portfolio. Although the optimal portfolio has been numerically evaluated as a function of derivatives of the solution to some Bellman equation except for special cases, it is a hard task to compute it when the number of assets is large. [41] provides its approximation based on the representation which Ocone-Karatzas [27] derives by using the Clark-Ocone formula. Moreover, [10] applies this method to a dynamic bond portfolio problem.

In evaluation of the expectation of a Wiener functional by Monte Carlo simulation, [42] proposes a new estimator using a random variable that has its expectation explicitly obtained by an asymptotic expansion and has a high correlation with the objective Wiener functional. The convergence of the simulation based on this estimator becomes faster and the approximation error due to the asymptotic expansion up to a low order is decreased. As for the extension of this method, see [16], [36] and [40].

[37] extends the decomposition formula for an American option value by Carr-Jarrow-Myneni [2] and proposes an approximation of the value using the fact that the density function of the underlying asset can be approximated by the asymptotic expansion.

[23] provides approximations for the risk indicators of options by asymptotic expansions of the derivatives of a stochastic differential equation with respect to parameters as well as by extending the variance reduction method of Monte Carlo simulation in [42]. [47] studies the bias correction when unknown parameters in the representation of an option value are substituted by their estimators.

Moreover, because of its generality, the asymptotic expansion technique has been applied to broad class of valuation models which have become popular recently in practice. The following works are examples: Kawai [8] for the market models of interest rates. Muroi [25] for default risk models. Osajima [28], Kawai and Jäckel [9], and [38], [39] for cross-currency models with stochastic interest rates.

This paper illustrates the cases where the diffusion terms of the stochastic processes for the asset prices are 0 when $\epsilon = 0$. However, expansions for the cases where the diffusion processes are not 0 when $\epsilon = 0$, but become the processes that are relatively easy to be computed are also considered. For more details, see Kashiwakura-Yoshida [7], Kunitomo-Kim [11], Lütkebohmert [20], and [39], [41] for instance.

As introduced above, various applications of the asymptotic expansion method are developed for numerical problems in finance.

Acknowledgement: I thank two referees for their valuable comments on

the previous version of [35]. I am also grateful to Professor Koichiro Takaoka for his extensive help in preparing this paper. Finally, I thank Mr. Kohta Takehara, Mr. Keita Nakayama and Mr. Fumihiro Endo for their assistance in this research.

References

[1] Björk, T. [2004] "Arbitrage Theory in Continuous Time(second ed.)", Oxford University Press, 2004.

[2] Carr, P., Jarrow, R. and Myneni, R. [1992], "Alternative characterizations of American put options," *Mathematical Finance*, Vol. 2, 87-106.

[3] Dermoune A. and Kutoyants, Y. [1995], "Expansion of distribution of Maximum Likelihood Estimate for Misspecified Diffusion Type Observation," *Stochastics and Stochastic Reports*, Vol. 52, no. 1-2, 121-145.

[4] Heath, D. Jarrow, R. and Morton, A. [1992], "Bond Pricing and the Term Structure of Interest Rates : A New Methodology for Contingent Claims Valuation," *Econometrica*, Vol. 60, 77-105.

[5] Ikeda, N. and Watanabe, S. [1989], *Stochastic Differential Equations and Diffusion Processes*, Second Edition, North-Holland/Kodansha, Tokyo.

[6] Karatzas, I. and Shreve, S. [1998], "Methods of Mathematical Finance," Springer.

[7] Kashiwakura, K. and Yoshida, N. [2001], "Numerical Experiment for Hybrid Expansion," Preprint, Graduate School of Mathematical Sciences, the University of Tokyo.

[8] Kawai, A. [2003], "A New Approximate Swaption Formula in the LIBOR Market Model: An Asymptotic Expansion Approach," *Applied Mathematical Finance*, Vol.10, 49-74.

[9] Kawai, A. and Jäckel, P. [2007], "An Asymptotic FX Option Formula in the Cross Currency Libor Market Model," Willmot Magazine, March, 74-84.

[10] Kobayashi, T., Takahashi, A. and Tokioka, N. [2005], "Dynamic Optimality of Yield Curve Strategies", *International Review of Finance*, Vol.4, 49-78.

[11] Kunitomo, N. and Kim, Y-J.[2005], "Effects of Stochastic Interest Rates and Volatility on Contingent Claims," *Japanese Economic Review*, Vol.58, 71-106.

[12] Kunitomo, N. and Takahashi, A. [1992], "Pricing Average Options," *Japan Financial Review*, Vol. 14, 1-20. (in Japanese).

[13] Kunitomo, N. and Takahashi, A. [2001], "The Asymptotic Expansion Approach to the Valuation of Interest Rate Contingent Claims," *Mathematical Finance*, Vol. 11, 117-151.

[14] Kunitomo, N. and Takahashi, A. [2003a], "On Validity of the Asymptotic Expansion Approach in Contingent Claim Analysis," *Annals of Applied Probability* Vol. 13-3, 914-952.

[15] Kunitomo, N. and Takahashi, A. [2003b], "Foundation of Mathematical Finance -Application of Malliavin Calculus and Asymptotic Expansion Method-," Toyo-Keizai (in Japanese).

[16] Kunitomo, N. and Takahashi, A. [2004], "Applications of the Asymptotic Expansion Approach based on Malliavin-Watanabe Calculus in Financial Problems," Stochastic Processes and Applications to Mathematical Finance, World Scientific, 195-232.

[17] Kusuoka S. and Strook, D. [1991], "Precise Asymptotics of Certain Wiener Functionals," *Journal of Functional Analysis*, Vol.99, 1-74.

[18] Kutoyants, Y. [1994], "Identification of Dynamical Systems with Small Noise," *Mathematics and its Applications*, 300, Kluwer Academic Publishers Group.

[19] Kutoyants, Y. [1998], "Semiparametric Estimation for Dynamic Systems with Small Noise," *Mathematical Methods of Statistics*, 7, no. 4, 457-465.

[20] Lütkebohmert, K. [2004], "An Asymptotic Expansion for a Black-Scholes Type Model," Bull. Sci. math., 128, 661-685.

[21] Malliavin, P. [1997], "Stochastic Analysis," Springer.

[22] Masuda, H. and Yoshida, N. [2004], "An Application of the Double Edgeworth Expansion to a Filtering Model with Gaussian Limit," *Statistics and Probability Letters*, 70, no. 1, 37-48.

[23] Matsuoka, R. Takahashi, A. and Uchida, Y. [2004], "A New Computational Scheme for Computing Greeks by the Asymptotic Expansion Approach," *Asia-Pacific Financial Markets*, Vol.11, 393-430.

[24] Merton, R.C. [1976], "Option Pricing When Underlying Stock Returns are Discontinuous," *Journal of Financial Economics*, Vol. 3, 125-144.

[25] Muroi, Y. [2005], "Pricing Contingent Claims with Credit Risk: Asymptotic Expansion Approach," *Finance and Stochastics*, Vol 9(3), 415-427.

[26] Nualart, D. [1995], "The Malliavin Calculus and Related Topics," Springer.

[27] Ocone, D. and Karatzas, I. [1991], "A Generalized Clark Representation Formula, with Application to Optimal Portfolios," *Stochastics and Stochastics Reports*, Vol. 34, 187-220.

[28] Osajima, Y. [2006], "The Asymptotic Expansion Formula of Implied Volatility for Dynamic SABR Model and FX Hybrid Model," Preprint, Graduate School of Mathematical Sciences, the University of Tokyo,

[29] Sakamoto, Y., Takada, Y. and Yoshida, N. [2004], "Expansions of the Coverage Probabilities of Prediction Region Based on a Shrinkage Estimator," *Statistics*, Vol. 38, no. 5, 381-390.

[30] Sakamoto, Y. and Yoshida, N. [1996], "Expansion of Perturbed Random Variables Based on Generalized Wiener Functionals," *Journal of Multivariate Analysis*, Vol. 59, no. 1, 34-59.

[31] Sakamoto, Y. and Yoshida, N. [1998], "Asymptotic Expansion of M-estimator over Wiener Space," *Statistical Inference for Stochastic Processes*, Vol. 1-1, 85-103.

[32] Shigekawa, I. [1998], "Stochastic Analysis," Iwanami(in Japanese).

[33] Takahashi, A. [1995], "Essays on the Valuation Problems of Contingent Claims," Unpublished Ph.D. Dissertation, Haas School of Business, University of California, Berkeley.

[34] Takahashi, A. [1999], "An Asymptotic Expansion Approach to Pricing Contingent Claims," *Asia-Pacific Financial Markets*, Vol. 6, 115-151.

[35] Takahashi, A. [2007], "On an Asymptotic Expansion Approach to Numerical Problems in Finance," Sugaku, Mathematical Society of Japan, Vol.59-1, 75-91. (in Japanese).

[36] Takahashi, A. and Matsushima, S. [2004], "Monte Carlo Simulation with an Asymptotic Expansion in HJM Framework," *FSA Research Review 2004*, 82-103, Financial Services Agency (in Japanese).

[37] Takahashi, A. and Saito, T. [2003], "An Asymptotic Expansion Approach to Pricing American Options," *Monetary and Economic Studies*, Vol. 22, 35-87. (in Japanese).

[38] Takahashi, A. and Takehara, K. [2006], "An Asymptotic Expansion Approach to Currency Options with a Market Model of Interest Rates under Stochastic Volatility Processes of Spot Exchange Rates," Working paper, CARF-F-092, the University of Tokyo, http://www.carf.e.u-tokyo.ac.jp/workingpaper/

[39] Takahashi, A. and Takehara, K. [2007], "Fourier Transform Method with an Asymptotic Expansion Approach: an Application to Currency Options," Working paper, CARF-F-097, the University of Tokyo, http://www.carf.e.u-tokyo.ac.jp/workingpaper/

[40] Takahashi, A. and Uchida, Y. [2006], "New Acceleration Schemes with the Asymptotic Expansion in Monte Carlo Simulation," *Advances in Mathematical Economics*, Vol. 8, 411-431.

[41] Takahashi, A. and Yoshida, N. [2004], "An Asymptotic Expansion Scheme for Optimal Investment Problems," *Statistical Inference for Stochastic Processes*, Vol. 7-2, 153-188.

[42] Takahashi, A. and Yoshida, N. [2005], "Monte Carlo Simulation with Asymptotic Method," *The Journal of Japan Statistical Society*, Vol. 35-2, 171-203.

[43] Takanobu, S. [1988], "Diagonal Short Time Asymptotics of Heat Kernels for Certain Degenerate Second Order Differential Operators of Hormander Type," *Publications of Research Insitute for Mathematical Sciences, Kyoto University*, Vol. 24, 169-203.

[44] Takanobu, S. and Watanabe, S. [1990], "Asymptotic Expansion formula of the Schilder Type for a Class of Conditional Wiener Functional Integrations," In Asymptotic Expansion Problems in Probability Theory: Proceedings of the Taniguchi International Symposium, Sanda and Kyoto, 1990, Longman, 194-241.

[45] Taniguchi, M. and Kakizawa, Y. [2000], "Asymptotic Theory of Statistical Inference for Time Series," Springer Series in Statistics, Springer-Verlag, New York.

[46] Uchida, M. and Yoshida, N. [2004a], "Information Criteria for Small Diffusions via the Theory of Malliavin-Watanabe," *Statistical Inference for Stochastic Processes*, Vol. 7-1, 35-67.

[47] Uchida, M. and Yoshida, N. [2004b], "Asymptotic Expansion for Small Diffusions Applied to Option Pricing," *Statistical Inference for Stochastic Processes*, Vol. 7-3, 189-223.

[48] Uemura, H. [1987], "On a Short Time Expansion of the Fundamental Solution of the Heat Equations by the Method of Wiener Functionals," *Journal of Mathematics of Kyoto University*, Vol. 27-3, 417-431.

[49] Watanabe, S. [1984], *Lectures on Stochastic Differential Equations and Malliavin Calculus*, Tata Institute of Fundamental Research, Springer-Verlag.

[50] Watanabe, S. [1987], "Analysis of Wiener Functionals (Malliavin Calculus) and its Applications to Heat Kernels," *The Annals of Probability*, Vol. 15, 1-39.

[51] Yoshida, N. [1992a], "Asymptotic Expansion for Small Diffusions via the Theory of Malliavin-Watanabe," *Probability Theory and Related Fields*, Vol. 92, 275-311.

[52] Yoshida, N. [1992b], "Asymptotic Expansions for Statistics Related to Small Diffusions," *The Journal of Japan Statistical Society*, Vol. 22, 139-159.

[53] Yoshida, N. [1993], "Asymptotic Expansion of Bayes Estimators for Small Diffusions," *Probability Theory and Related Fields*, Vol. Vol. 95, no. 4, 429-450.

[54] Yoshida, N. [1996], "Asymptotic Expansion for Perturbed Systems on Wiener Space: Maximum Likelihood Estimators," *Journal of Multivariate Analysis*, Vol. 57, no. 1, 1-36.

[55] Yoshida, N. [2003], "Conditional Expansions and their Applications," *Stochastic Processes and their Applications*, Vol. 107, no. 1, 53-81.

[56] Yoshida, N. [2003], "Malliavin Calculus and Mathematical Statistics," *Sugaku*, Vol. 55, no. 3, 225-244, (in Japanese).