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Abstract

This paper studies the probability distribution and option pricing for drawdown in a stochastic volatility environment. Their analytical approximation formulas are derived by the application of a singular perturbation method (Fouque *et al.* [7]). The mathematical validity of the approximation is also proven. Then, numerical examples show that the instantaneous correlation between the asset value and the volatility state crucially affects the probability distribution and option prices for drawdown.

1 Introduction

In asset management business, drawdown related risk measures, such as maximum drawdown, are considered very important for the risk investigation of mutual or hedge funds. Drawdown related risk measures are defined in a dynamic setting. Let $\{S_t\}_{0\leq t\leq T}$ be a stochastic process that represents the net asset value of a fund. Drawdown of $\{S_t\}_{0\leq t\leq T}$ at time t is defined by $D_t = M_t - S_t$, where $M_t = \max_{0\leq u\leq t} S_u$. In other words, drawdown is the lost wealth of investors from the record high level. Maximum drawdown is its historical maximum. (See Fig. 1.) Drawdown related risk measures fit hedge fund managers as well as investors. Most hedge funds set high water mark provision in the fee structures. It means that hedge fund managers receive a fixed rate performance fee of exceeding the high water mark, or record high level. In other words, they cannot get performance fee during suffering drawdown. Therefore, the risk for fund managers is exactly drawdown.

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Methods of portfolio optimization with controlling drawdown have been developed so far. Grossman and Zhou [11] proposed an portfolio optimization of a risk-free asset and a risky asset under drawdown constraints in the Black-Scholes economy. It solved the optimization problem by dynamic programming. Cvitanic and Karatzas [3] extended it to multi-risky asset and more general settings, and solved the optimization problem by martingale method. Chekhlov *et al.* [2] introduced a risk measure conditional drawdown (CDD) and proposed a portfolio optimization method with controlling CDD. Hakamada *et al.* [12] and Krokhmal *et al.* [15] applied the method to portfolio construction of hedge funds.

Properties of (maximum) drawdowns have also been studied. Magdon-Ismail *et al.* [18] and Magdon-Ismail and Atiya [16] researched the probability distribution of maximum drawdowns for Brownian motion and geometric Brownian motion, respectively. Belentepe [1] examined the probability distribution drawdown for geometric Brownian motion, and then considered how portfolio diversification reduced the expected drawdown. Vecer [22] studied relation between directional trade and maximum drawdown (and drawup), and Vecer [23] considered pricing and hedging contingent claims on maximum drawdown. These two research papers implemented the analysis by Monte Carlo simulation under the assumption that the underlying asset followed geometric Brownian motion. Pospisil and Vecer [20] analyzed it by a PDE method under the same assumption.

This article studies the probability distribution and option pricing for drawdown in a stochastic volatility environment by an analytical approach. The option for drawdown can be a powerful risk management tool. Their analytical approximation formulas are derived by applying a singular perturbation method (Fouque *et al.* [7]). Fouque *et al.* [7] argues the method for option pricing in detail. The accuracy of the approximation is examined in Yamamoto and Takahashi [25]. In this paper, it is shown that the first order stochastic volatility term is linearly related to the instantaneous correlation between asset value and volatility state. The mathematical validity of the approximation for European option is shown by Fouque *et al.* [8]. This article proves that the validity is also held for the analysis of drawdown. Our numerical examples clarified that the correlation affects the probability distribution and option prices for drawdown. If asset value and volatility state are positively correlated, the expected drawdown is higher than those for uncorrelated case or Black-Scholes economy, and the standard deviation of drawdown is lower than those cases. Due to the effect of the correlation on the probability distribution, the option prices for drawdowns are also affected by the correlation.

The organization of the paper is as follows. The next section studies the probability distribution and pricing options of drawdowns in the Black-Scholes economy. In section 3, they are considered in a stochastic volatility environment. Section 4 presents numerical examples. Section 5 concludes. In appendix A, the singular perturbation method for our problem is explained. Appendix B proves the convergence result of the approximation.

2 Drawdown in the Black-Scholes Economy

First, probability distribution and option pricing for drawdown are considered in the Black-Scholes economy. Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{0 \le t \le T < \infty})$ be a complete probability space with a filtration satisfying the usual conditions. There are a risk-free asset with a constant risk-free rate r, and a risky asset. In $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\})$, it is assumed that the risky asset price $\{S_t\}$ follows the stochastic differential equation (SDE)

$$dS_t = \mu S_t dt + \sigma S_t dW_t^1,$$

where $\{W_t^1\}$ is a standard Brownian motion, and μ and σ is a constant. Defining $M_t = \max_{0 \le u \le t} S_u$, the drawdown from time 0 to t is given by

$$D_t = M_t - S_t.$$

In other words, drawdown is the lost wealth of investors from the record high level. For the purpose of convenience, we calculate the joint probability of

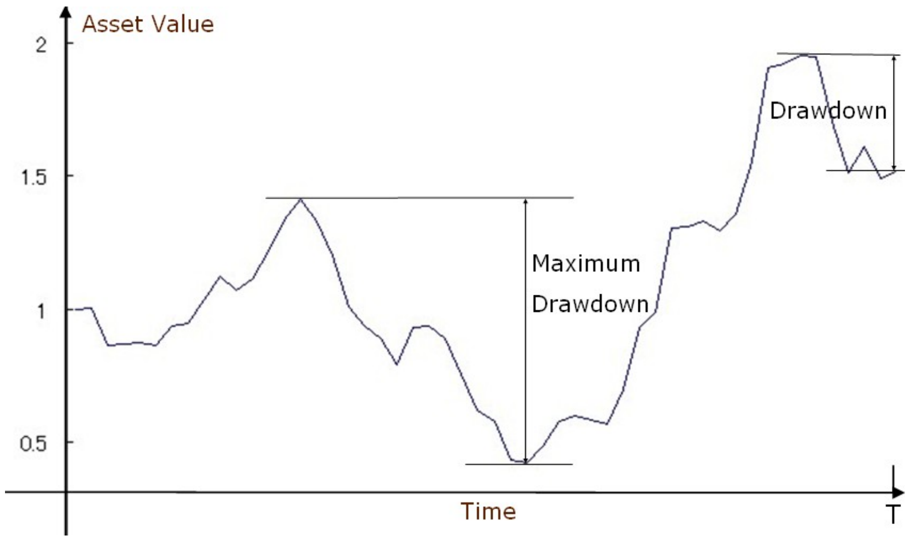

Figure 1: Drawdown and Maximum Drawdown

(S_T, M_T) through their logarithm. Let $X_t = \log S_t$ and $Z_t = \log M_t$. Then,

$$dX_t = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dW_t^1, \quad X_0 = x_0,$$

and $Z_t = \max_{0 \le u \le t} X_u$ We first calculate the simultaneous probability density function of (X_T, Z_T). For $Z_t < b$, let

$$P_{BS}(t, x; a, b) = \mathbb{P}(X_T \le a, Z_T \le b \mid X_t = x, Z_t < b).$$

Then, by Feynman-Kac's theorem, $P_{BS}(t,x;a,b)$ is the solution of the boundary value problem

$$\begin{cases} \mathscr{L}_{BS} P_{BS}(t,x;a,b) = 0, \\ P_{BS}(t,b;a,b) = 0, \\ P_{BS}(T,x;z,b) = 1_{\{x \le a\}} \end{cases}$$

where

$$\mathscr{L}_{BS} = \frac{\partial}{\partial t} + \left(\mu - \frac{\sigma^2}{2}\right)\frac{\partial}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}.$$

By method of images (See, for example, Wilmott *et al.* [24].), P_{BS} is obtained by

$$P_{BS}(t,x;a,b) = N(d_1(T-t,x)) - \exp\{(2\mu/\sigma^2 - 1)(b-x)\}N(d_2(T-t,x)), \quad (1)$$

where

$$d_1(s,x) = \frac{a - x - (\mu - \sigma^2/2)s}{\sigma\sqrt{s}}, \quad d_2(s,x) = \frac{a + x - 2b - (\mu - \sigma^2/2)s}{\sigma\sqrt{s}}.$$

Differentiating (1) with respect to a and b, we get the simultaneous density function of (X_T, Z_T). Then, for any function g of (S_T, M_T), $E[g(S_T, M_T)]$ is evaluated by

$$E[g(S_T, M_T)] = \int_{x_0}^{\infty}\int_{-\infty}^{b} g(e^a, e^b)\frac{\partial^2 P_{BS}^0}{\partial a \partial b}(0,x;a,b)dadb.$$

For example, the distribution function $F(c) = P(D_T \le c)$ and nth moment of drawdown is obtained by setting $g(S_T, M_T) = 1_{\{M_T - S_T \le a\}}$ and $g(S_T, M_T) = (M_T - S_T)^n$, respectively.

Next, proceed to the calculation of option prices for drawdown. In the economy, the risk neutral measure is defined by

$$\mathbb{P}^*(A) = E[\exp(-\theta W_T^1 - \theta^2 T/2)1_A] \quad \text{for} \quad A \in \mathcal{F},$$

where $\theta = \frac{\mu - r}{\sigma}$. By Maruyama-Girsanov's theorem, when $\{W_t^{1*}\}$ is defined by

$$W_t^{1*} = W_t^1 + \theta t,$$

it is a standard Brownian motion under $\mathbb{P}^*$. Let

$$P_{BS}^*(t,x;a,b) = \mathbb{P}^*(X_T \le a, Z_T \le b \mid X_t = x, Z_t < b).$$

Since X_t follows the SDE

$$dX_t = \left(r - \frac{\sigma^2}{2}\right)dt + \sigma dW_t^{1*},$$

the previous argument under the risk neutral measure shows that $P_{BS}^*(t,x;a,b)$ is given by

$$P^*_{BS}(t,x;a,b) = N(d^*_1(T-t,x)) - \exp\{(2r/\sigma^2-1)(b-x)\}N(d^*_2(T-t,x)), \quad (2)$$

where

$$d^*_1(s,x) = \frac{a-x-(r-\sigma^2/2)s}{\sigma\sqrt{s}}, \quad d^*_2(s,x) = \frac{a+x-2b-(r-\sigma^2/2)s}{\sigma\sqrt{s}}.$$

Therefore, the call option prices for drawdown with strike K and maturity T at time 0 is given by

$$C(0,x) = E^*[e^{-rT}(M_T-S_T-K)_+] = e^{-rT}\int_{x_0}^{\infty}\int_{-\infty}^{b}(e^b-e^a-K)_+\frac{\partial^2 P^*_{BS}}{\partial a\partial b}(0,x;a,b)dadb.$$

If we have asset S and this option, the drawdown of S exceeding K is covered by the option. Therefore, this option can be a powerful risk management tool against drawdowns.

3 Drawdown in a Stochastic Volatility Environment

Next, the argument of previous section is extended to a stochastic volatility circumstance. In $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\})$, it is assumed that the risky asset price $\{S_t\}$ follows the SDE

$$dS_t = \mu S_t dt + \sigma_t S_t dW^1_t,$$

where $\{W^1_t\}$ is a standard Brownian motion, and μ is a constant. The volatility σ_t is the stochastic process expressed as follows by using Ornstein-Uhlenbeck (OU) process $\{Y_t\}$.

$$\sigma_t = f(Y_t),$$

$$dY_t = \frac{1}{\epsilon}(m-Y_t)dt + \nu\sqrt{\frac{2}{\epsilon}}dW^2_t, \qquad Y_0 = y_0,$$

where f is a positive increasing function, and $\{W^2_t\}$ is a standard Brownian motion that have instantaneous correlation $\rho \in (-1,1)$ with $\{W^1_t\}$,

$$d\langle W^1, W^2\rangle = \rho dt.$$

It is assumed that f and $\frac{1}{f}$ are bounded: there are constants l_1 and l_2 such that

$$0 < l_1 \leq f(y) \leq l_2 < \infty \quad \text{for any} \quad y \in \mathbb{R}.$$

Explanations for the parameters are given shortly. m is a constant, and ϵ and ν are positive constants. In accordance with [7], the fast mean-reverting stochastic volatility is supposed, and consequently ϵ is a positive small number. As shown in Fouque *et al.* [7], $\{Y_t\}$ has the normal invariant distribution $N(m, \nu^2)$. Finally, ρ is a constant that expresses the instantaneous correlation between $\{S_t\}$ and $\{Y_t\}$.

We calculate the simultaneous probability density function of (X_T, Z_T). For $a < b$, let

$$P_{SV}(t, x, y; a, b) = \mathbb{P}(X_T \leq a, Z_T \leq b \mid X_t = x, Y_t = y, Z_t < b). \tag{3}$$

Then, by Feynman-Kac's theorem, $P_{SV}(t, x, y; a, b)$ is the solution of the boundary value problem

$$\begin{cases} \mathscr{L}_{SV}^{\epsilon} P_{SV}(t, x, y; a, b) = 0, \\ P_{SV}(t, b, y; a, b) = 0, \\ P_{SV}(T, x, y; a, b) = 1_{\{x \leq a\}}, \end{cases}$$

where

$$\mathscr{L}_{SV}^{\epsilon} = \frac{1}{\epsilon}\mathscr{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathscr{L}_1 + \mathscr{L}_2,$$

$$\begin{cases} \mathscr{L}_0 = \nu^2 \frac{\partial^2}{\partial y^2} + (m - y)\frac{\partial}{\partial y}, \\ \mathscr{L}_1 = \sqrt{2}\nu\rho f(y)\frac{\partial^2}{\partial x \partial y}, \\ \mathscr{L}_2 = \frac{\partial}{\partial t} + \left(\mu - \frac{1}{2}f(y)^2\right)\frac{\partial}{\partial x} + \frac{1}{2}f(y)^2\frac{\partial^2}{\partial x^2}. \end{cases}$$

The value of P_{SV} is approximated up to the order of $\sqrt{\epsilon}$ by singular perturbation method. From Appendix A, the approximation up to the first order stochastic volatility correction is given by

$$P_{SV}(t, x, y; a, b) \approx P_{SV}^0(t, x; a, b) + \sqrt{\epsilon}P_{SV}^1(t, x; a, b), \tag{4}$$

where $P_{SV}^0(t, x; a, b)$ is equal to the value of P_{BS} with constant volatility parameter $\bar{\sigma}$, which is defined by $\bar{\sigma} = \sqrt{\langle f^2 \rangle}$, where $\langle \cdot \rangle$ represents the expectation under the invariant distribution of Y: $N(m, \nu^2)$. $P_{SV}^1(t, x; a, b)$ is the first order stochastic volatility correction term, which is of order $\sqrt{\epsilon}$. The first two terms of the expansion do not depend on y. According to A.2, P_{SV}^0 is the solution of the boundary value problem

$$\begin{cases} \langle \mathscr{L}_2 \rangle P_{SV}^0(t, x; a, b) = 0 \quad \text{in} \quad 0 < x < b \quad \text{and} \quad t < T, \\ P_{SV}^0(t, b; a, b) = 0, \\ P_{SV}^0(T, x; a, b) = 1_{\{x \leq a\}}, \end{cases}$$

where $\langle \mathscr{L}_2 \rangle = \frac{\partial}{\partial t} + \left(\mu - \frac{\bar{\sigma}^2}{2}\right)\frac{\partial}{\partial x} + \frac{\bar{\sigma}^2}{2}\frac{\partial^2}{\partial x^2}$. By method of images, P_{SV}^0 is given by

$$P_{SV}^0(t, x; a, b) = N(d_1(T - t, x)) - \exp\{(2\mu/\bar{\sigma}^2 - 1)(b - x)\}N(d_2(T - t, x)), \tag{5}$$

where

$$d_1(s, x) = \frac{a - x - (\mu - \bar{\sigma}^2/2)s}{\bar{\sigma}\sqrt{s}}, \quad d_2(s, x) = \frac{a + x - 2b - (\mu - \bar{\sigma}^2/2)s}{\bar{\sigma}\sqrt{s}}.$$

As described in A.3, the first order stochastic volatility correction term $P^1_{SV}(t,x;a,b)$ is the solution of the boundary value problem

$$\begin{cases} \langle \mathscr{L}_2 \rangle P^1_{SV}(t,x;a,b) = V\left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2}\right) P^0_{SV}(t,x;a,b) \quad \text{in } \ 0<x<b \ \text{ and } \ t<T, \\ P^1_{SV}(t,b;a,b) = 0, \\ P^1_{SV}(T,x;a,b) = 0. \end{cases} \tag{6}$$

V is a constant defined by

$$V = \frac{\rho\nu}{\sqrt{2}} \langle f\phi' \rangle, \tag{7}$$

where ϕ' is a function of y defined in (16).

Defining

$$\hat{P}_{SV}(t,x;a,b) = \frac{1}{V} P^1_{SV}(t,x;a,b) + (T-t)\Big(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2}\Big) P^0_{SV}(t,x;a,b),$$

the inhomogeneous boundary value problem (6) is transformed into the following homogeneous problem;

$$\begin{cases} \langle \mathscr{L}_2 \rangle \hat{P}_{SV}(t,x;a,b) = 0 \quad \text{in } \ 0<x<b \ \text{ and } \ t<T, \\ \hat{P}_{SV}(t,b;a,b) = (T-t)\Big(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2}\Big) P^0_{SV}(t,b;a,b), \\ \hat{P}_{SV}(T,x;a,b) = 0. \end{cases}$$

$\hat{P}_{SV}$ is obtained by numerical integration as follows. The probabilistic representation of $\hat{P}_{SV}$ is

$$\hat{P}_{SV}(t,x;a,b) = E\left[\ (T-\tau)\Big(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2}\Big) P^0_{SV}(\tau,b;a,b) 1_{\{\tau \le T\}} \Big| X^0_t = x \right],$$

where X^0_t is the stochastic process that satisfies SDE

$$dX^0_t = \Big(\mu - \frac{\bar{\sigma}^2}{2}\Big) dt + \bar{\sigma} dW^1_t,$$

and τ is the first time after t that X^0 hits b. Changing a variable and using the distribution of the first hitting time of Brownian motion (see e.g. Karatzas and Shreve [14], Chapter 2, Proposition 8.5),

$$\hat{P}_{SV}(t,x;a,b) = \int_t^T (T-s)\Big(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2}\Big) P^0_{SV}(s,b;a,b) h(s;x,b) ds, \tag{8}$$

where

$$h(s;x,b) = \frac{b-x}{\bar{\sigma}\sqrt{2\pi(s-t)^3}} \exp\left[- \frac{\{b-x-(\mu-\bar{\sigma}^2/2)(s-t)\}^2}{2\bar{\sigma}^2(s-t)} \right].$$

If the integration in (8) is evaluated numerically, $P^1_{SV}(t,x;a,b)$ is obtained by

$$P^1_{SV}(t,x;a,b) = V\Big\{\hat{P}_{SV}(t,x;a,b) - (T-t)\Big(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2}\Big)P^0_{SV}(t,x;a,b)\Big\}.$$

Consequently, the approximation of P_{SV} is obtained by

$$P_{SV}(t,x;a,b) \approx P^0_{SV}(t,x;a,b) + \sqrt{\epsilon}P^1_{SV}(t,x;a,b). \tag{9}$$

The next theorem confirms the validity of the approximation.

Theorem 1 *Under the assumption that f and $\frac{1}{f}$ is bounded, at a fixed point $t<T$, $x,y\in\mathbb{R}$,*

$$P_{SV}(t,x,y;a,b) = P^0_{SV}(t,x;a,b) + \sqrt{\epsilon}P^1_{SV}(t,x;a,b) + O(\epsilon).$$

Proof. *See Appendix B.*

Differentiating this with respect to a and b, the approximate simultaneous probability density function of (X_T, Z_T) is obtained. Stochastic volatility affects the simultaneous probability distribution of (X_T, Z_T) through V defined in (7). V depends on ρ, which represents the instantaneous correlation between the asset value and the volatility, and ν, which scales the volatility of volatility. For any function g of (S_T, M_T), $E[g(S_T,M_T)]$ is approximately evaluated by

$$\begin{aligned} E[g(S_T,M_T)] \quad = \quad & \int_{x_0}^{\infty}\int_{-\infty}^{b} g(e^a,e^b)\frac{\partial^2 P^0_{SV}}{\partial a\partial b}(0,x;a,b)dadb \\ & +\sqrt{\epsilon}V\int_{x_0}^{\infty}\int_{-\infty}^{b} g(e^a,e^b)\frac{\partial^2 \hat{P}_{SV}}{\partial a\partial b}(0,x;a,b)dadb \\ & -\sqrt{\epsilon}VT\int_{x_0}^{\infty}\int_{-\infty}^{b} g(e^a,e^b)\Big(\frac{\partial^5 P^0_{SV}}{\partial x^3\partial a\partial b} - \frac{\partial^4 P^0_{SV}}{\partial x^2\partial a\partial b}\Big)(0,x;a,b)dadb. \end{aligned}$$

The first term is the Black-Scholes part, and the second and third terms are the first order stochastic volatility correction part. Note that the first order correction term is linearly related to V, and therefore linearly related to ρ.

Next, proceed to the calculation of option prices for drawdown. While risk-neutral measure is uniquely determined in the Black-Scholes economy, there are infinitely many risk-neutral measures in this economy, because the market is incomplete. The risk-neutral measure depends on the market price of volatility risk. For simplicity, it is assumed that the stochastic process of S_t under risk-neutral measure $\mathbb{P}^*$ is described as follows.

$$dS_t = rS_tdt + \sigma_tS_tdW^{1*}_t, \qquad S_0 = e^{x_0},$$

where $\{W^{1*}_t\}$ is a standard Brownian motion under $\mathbb{P}^*$. The volatility σ_t is the stochastic process expressed as follows by using Ornstein-Uhlenbeck (OU) process $\{Y_t\}$.

$$\sigma_t = f(Y_t),$$

$$dY_t = \frac{1}{\epsilon}(m - Y_t)dt + \nu\sqrt{\frac{2}{\epsilon}}dW_t^{2*}, \qquad Y_0 = y_0,$$

where $\{W_t^{2*}\}$ is a standard Brownian motion that have instantaneous correlation $\rho \in (-1, 1)$ with $\{W_t^{1*}\}$. Let

$$P^*_{SV}(t, x; a, b) = \mathbb{P}^*(X_T \leq z, Z_T \leq b \mid X_t = x, Z_t < b).$$

The previous argument under the risk neutral measure shows that the approximate value of P^*_{SV} is obtained by changing μ appeared in (9) to r. Then, the approximate prices of call option for drawdown with strike K and maturity T at time 0 is given by

$$C(0, x) = E^*[e^{-rT}(M_T - S_T - K)_+] = e^{-rT}\int_{x_0}^{\infty}\int_0^b (e^b - e^a - K)_+ \frac{\partial^2 P^*_{SV}}{\partial a \partial b}(0, x; a, b)dadb.$$

Sure, it can be calculated under other risk-neutral measures. In other words, we can allow for market price of volatility risk as Fouque *et al.* [7]. Parameters $\bar{\sigma}$ and $\sqrt{\epsilon}V$ can be calibrated from the implied volatilities of liquid European call options (see e.g. Fouque *et al.* [7]).

4 Numerical Examples

This section presents some numerical examples. First, expectation and standard deviation of drawdown are studied. We set $S_0 = 100$, and assign

$$f(y) = \begin{cases} 10^{-10} + e^y & (y < 0), \\ 10^{-10} + 2 - e^{-y} & (y \geq 0). \end{cases}$$

Then, f and $1/f$ are bounded. The parameter settings are as follows. Since Fouque *et al.* [7] found fast mean-reverting volatility, this paper also considers the case; $\epsilon = 1/200$, for example. We set $\mu = 0.15$, $m = -1.89$, $\nu = 0.40$, and $T = 1/12$. Then, $\bar{\sigma} = 0.2$. Since stochastic volatility affects the distribution of drawdown through parameter ρ, we consider three patterns: $\rho = -0.75,\ 0,\ 0.75$. As previously mentioned, when $\rho = 0$, the first order stochastic volatility correction term is 0. Therefore, the statistics for $\rho = 0$ is equal to those in the Black-Scholes economy with $\sigma = 0.2$.

	$\rho = -0.75$	$\rho = 0$	$\rho = 0.75$
Expectation	3.68	4.10	4.52
Standard Deviation	3.83	3.14	2.16

Table 1: Expactations and standard deviations of drawdowns

Table 1 exhibits the expectations and standard deviations of drawdowns. Positive instantaneous correlation between the asset and the volatility state

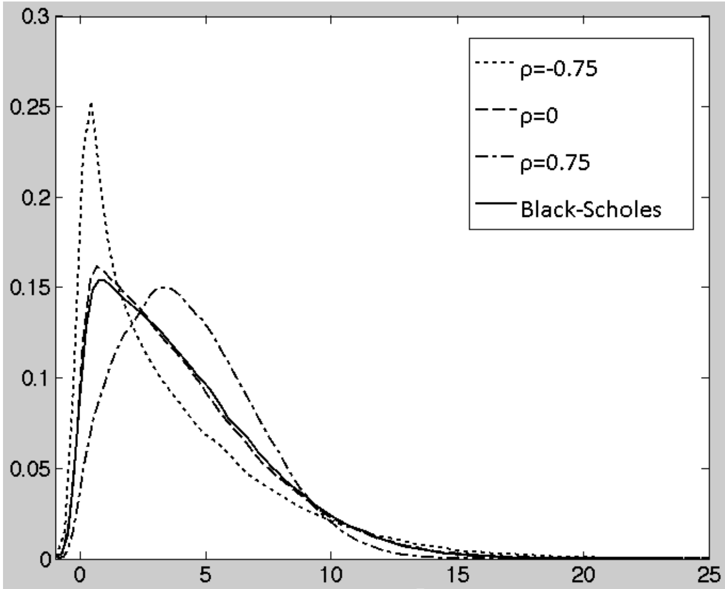

Figure 2: Probability Density of Drawdown

increases expected drawdown, while negative correlation decreases. And the positive correlation decreases standard deviation of drawdown, while negative correlation increases it.

Fig. 2 shows the probability density function estimated by Monte Carlo simulation. This graph enables us to observe the effect of correlations on the probability distribution of drawdown visually. The density functions of drawdowns for the cases of $\rho = 0$ and Black-Scholes look very similar. From this figure, it is confirmed again that a positive correlation increases expected drawdown, while a negative correlation decreases it.

Next, proceed to numerical studies of options for drawdown. The parameter settings are same as the above analysis, and we set the risk-free rate as $r = 0.02$. For determination of strike level, expectations and standard deviations under the risk-neutral measure are also calculated. In the analysis, approximation accuracies are also studied.

In order to obtain the estimate value of the options for the two cases, Monte Carlo simulations with antithetic variables method are conducted. The number of the simulation is 1,000,000. Since the volatility of Y is very high, the time step should be very small in order to converge the simulations of Y. Time step is determined in the following way. For the case of $f(y) = e^y$, the distribution of Y at the terminal date is known analytically. In order to match the distribution of simulations and analytic one, $\Delta t = 1/100,000$ is needed. Therefore, this time step is used in our analysis. Table 2 exhibits the numerical results. The statistics calculated by the approximation method and Monte Carlo simulations are reported. In addition, we exhibit difference and difference rate between the approximation value and Monte Carlo value, which are given by $(Approximate\ value - Monte\ Carlo\ value)$ and $(Approximate\ value - Monte\ Carlo\ vaue)/(Monte\ Carlo\ value)$ respectively. We note that there are other ways than Monte Carlo methods. For example, Pospisil and Vecer [20] applied a PDE method in the analysis of maximum drawdown.

	$\rho = -0.75$	$\rho = 0$	$\rho = 0.75$
Expectations			
Approximation	4.049	4.574	5.099
Monte Carlo	4.056	4.502	4.943
Difference	-0.007	0.072	0.156
Difference rate	-0.16 Å?	1.60 Å?	3.16 Å?
Standard Deviations			
Approximation	4.014	3.347	2.395
Monte Carlo	3.847	3.390	2.794
Difference	0.167	0.047	-0.39
Difference rate	4.35 Å?	-1.28 Å?	-14.27 Å?

Table 2: Expectations and standard deviations under risk-neutral measure

Table 2 shows that the expected drawdowns can be calculated with some accuracy by our approximation method. The relation between ρ and expected drawdowns is also confirmed by Monte Carlo simulations. As for standard deviations, for the case of $\rho =$ -0.75 and 0.75, the differences result in relatively high compared to those for expectations. However, the relation between ρ and standard deviations can be also found by Monte Carlo simulations.

Next, calculate option prices for drawdown. In practice, strike levels vary among option buyers according to their risk attitudes. Hence, three different strikes are chosen based on the empirical statistics of drawdown. First, the strike is set to the expected drawdown $E[D_T]$. The other two strikes are above and below 1 standard deviation from expected drawdown. Note that since the statistics are different by ρ, strikes vary according to ρ in this analysis.

Reading across the rows of the table, the option prices decrease in ρ. This is because dispersion of D_T decrease in ρ as shown in Table 2, where dispersion levels are measured by standard deviation.

Next, the approximation accuracy of our method is discussed. For the case of $\rho = 0$, the errors of the approximation method are about 2% for all strikes. As for $\rho = 0.75$ and $\rho = -0.75$, the errors for the options with strikes $E[D_T] - SD[D_T]$ are relatively small compared to other strikes. Finally, note that the difference rates for the options with strikes $E[D_T] - SD[D_T]$ are relatively high for the case of $\rho = 0.75$ and $\rho = -0.75$. This strike corresponds to out-of-the-money (OTM) for the plain vanilla option. Yamamoto and Takahashi [25]

	$\rho = -0.75$	$\rho = 0$	$\rho = 0.75$
Strike: $E[D_T]$			
Approximation	1.605	1.382	1.105
Monte Carlo	1.494	1.353	1.137
Difference	0.111	0.029	-0.032
Difference rate	7.41 Å?	2.12 Å?	-2.79 Å?
Strike: $E[D_T] - SD[D_T]$			
Approximation	3.843	3.543	2.972
Monte Carlo	3.850	3.474	2.951
Difference	-0.007	0.069	0.022
Difference rate	-0.19 Å?	1.98 Å?	0.73 Å?
Strike: $E[D_T] + SD[D_T]$			
Approximation	0.577	0.400	0.246
Monte Carlo	0.505	0.407	0.287
Difference	0.072	-0.008	-0.041
Difference rate	14.27 Å?	-1.94 Å?	-14.18 Å?

Table 3: Option prices for drawdown

reported the result that the difference rates of this method for plain vanilla European call OTM options are also relatively high. Our result agrees with that evidence.

5 Conclusion

This article studied the probability distribution and option pricing for drawdown in a stochastic volatility environment. Their analytical approximation formulas were derived by the application of a singular perturbation method (Fouque *et al.* [7]), and it showed that the first order stochastic volatility term is linearly related to the instantaneous correlation between asset value and volatility state. The mathematical validity of the approximation was also shown. Then, numerical examples clarified that the correlation crucially affects the probability distribution and option prices for drawdown. If they are positively correlated, the expected drawdown is higher than those for uncorrelated case or Black-Scholes economy, and standard deviation of drawdown is lower than those cases. Due to the effect of the correlation on the probability distribution of drawdown, the option prices for drawdowns are also affected by the correlation.

Appendix

A Singular Perturbations

This appendix presents singular perturbation method (Fouque *et al.* [7]) for our problem. First, the general framework is described. Then, the Black-Scholes term and the first order stochastic volatility correction term are derived. Further details of this method for option pricing are argued in Fouque *et al.* [7].

A.1 Formal Expansion

First, a formal asymptotic expansion is conducted. The economical setting is the same as section 3. Consider the drawdown of the underlying asset S frome time 0 to T. Let $P_{SV}(t, x, y; a, b)$ be defined by (3). By Feynman-Kac's theorem, P_{SV} satisfies the following partial differential equation (PDE);

$$\mathscr{L}_{SV}^{\epsilon} P_{SV} = 0 \quad \text{in} \ \ (0, T) \times (-\infty, b) \times \mathbb{R} \tag{10}$$

where

$$\mathscr{L}_{SV}^{\epsilon} = \frac{1}{\epsilon}\mathscr{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathscr{L}_1 + \mathscr{L}_2,$$

$$\begin{cases} \mathscr{L}_0 = \nu^2 \frac{\partial^2}{\partial y^2} + (m - y)\frac{\partial}{\partial y}, \\ \mathscr{L}_1 = \sqrt{2}\nu\rho f(y)\frac{\partial^2}{\partial x \partial y} \\ \mathscr{L}_2 = \frac{\partial}{\partial t} + \left(\mu - \frac{1}{2}f(y)^2\right)\frac{\partial}{\partial x} + \frac{1}{2}f(y)^2\frac{\partial^2}{\partial x^2}. \end{cases}$$

P_{SV} can be obtained by solving PDE with the boundary condition and the terminnal condition. It is assumed that P_{SV} has an asymptotic expansion

$$P_{SV} = P_{SV}^0 + \sqrt{\epsilon}P_{SV}^1 + \epsilon P_{SV}^2 + \epsilon\sqrt{\epsilon}P_{SV}^3 + \cdots . \tag{11}$$

Singular perturbation method inserts this formal expantion into (10). Then, it derives the PDE that each coefficient of $\sqrt{\epsilon}$ power satisfies, and solves the PDEs one after another.

A.2 Black-Scholes term

P_{SV}^0 is calculated first. Inserting the formal expansion (11) into (10) and comparing the coefficients of ϵ^{-1} gives $\mathscr{L}_0 P_{SV}^0 = 0$. $\mathscr{L}_0$ is the generator of an ergodic Markov process and acts only on y. Therefore, P_{SV}^0 must be a constant with respect to y, which implies that we can write

$$P_{SV}^0 = P_{SV}^0(t, x; a, b).$$

Similarly, comparing the terms of order $\epsilon^{-1/2}$, it can be seen that P_{SV}^1 also does not depend on y.

Comparing the constant (with respect to ϵ) terms gives

$$\mathscr{L}_0 P_{SV}^2 + \mathscr{L}_2 P_{SV}^0 = 0, \tag{12}$$

which is a Poisson equation for P^2_{SV} with respect to the operator $\mathscr{L}_0$ in the variable y. The necessary condition for (12) to admit a solution is

$$\langle \mathscr{L}_2 P^0_{SV} \rangle = \langle \mathscr{L}_2 \rangle P^0_{SV} = 0, \tag{13}$$

which is referred to as centering condition in Fouque *et al.* [7]. $\langle \cdot \rangle$ represents the expectation with respect to the invariant measure of Y, $N(m, \nu^2)$. Since P^0_{SV} does not depend on y, P^0_{SV} gets outside the bracket in the first equality. $\langle \mathscr{L}_2 \rangle$ is represented as

$$\langle \mathscr{L}_2 \rangle = \frac{\partial}{\partial t} + \Big(\mu - \frac{\bar{\sigma}^2}{2}\Big)\frac{\partial}{\partial x} + \frac{1}{2}\bar{\sigma}^2 \frac{\partial^2}{\partial x^2},$$

where $\bar{\sigma}^2 = \langle f^2 \rangle$. P^0_{SV} is obtained by solving this PDE with boundary and terminal condition

$$\begin{cases} P^0_{SV}(t, b; a, b) = 0, \\ P^0_{SV}(T, x; a, b) = 1_{\{x \le a\}}. \end{cases}$$

Therefore, P^0_{SV} is equal to P^0_{BS} under volatility $\bar{\sigma}$, whose square is equal to the expected instantaneous variance of X under the invariant measure of Y.

A.3 First order term

Next, proceed to the calculation for the first order stochastic volatility correction term. As centering condition (13) is satisfied, we can write

$$\mathscr{L}_2 P^0_{SV} = \mathscr{L}_2 P^0_{SV} - \langle \mathscr{L}_2 \rangle P^0_{SV} = \frac{1}{2}(f(y)^2 - \bar{\sigma}^2)\Big(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\Big)P^0_{SV}.$$

Then, from (12), we have

$$\mathscr{L}_0 P^2_{SV} = -\frac{1}{2}(f(y)^2 - \bar{\sigma}^2)\Big(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\Big)P^0_{SV}.$$

Let $\phi(y)$ is a solution of the Poisson equation

$$\mathscr{L}_0 \phi = (f(y)^2 - \bar{\sigma}^2), \tag{14}$$

P^2_{SV} is given by

$$P^2_{SV}(t, x, y; a, b) = -\frac{1}{2}\phi(y)\Big(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\Big)P^0_{SV} + c(t, x), \tag{15}$$

where $c(t, x)$ is a function of (t, x) that does not depend on y. Solving the Poisson equation (14),

$$\phi'(y) = \frac{1}{\nu^2 \Phi(y)} \int_{-\infty}^{y} (f(z)^2 - \bar{\sigma}^2)\Phi(z) dz, \tag{16}$$

where $\Phi(y)$ is the probability density function of $N(m, \nu^2)$.

Comparing the coefficients of $\epsilon^{1/2}$,

$$\mathscr{L}_0 P^3_{SV} + \mathscr{L}_1 P^2_{SV} + \mathscr{L}_2 P^1_{SV} = 0,$$

which is again a Poisson equation for P^3_{SV} with respect to $\mathscr{L}_0$. The centering condition is

$$\langle \mathscr{L}_1 P^2_{SV} + \mathscr{L}_2 P^1_{SV} \rangle = 0.$$

Since P^1_{SV} does not depend on y,

$$\langle \mathscr{L}_2 \rangle P^1_{SV} = -\langle \mathscr{L}_1 P^2_{SV} \rangle.$$

Inserting (15),

$$\langle \mathscr{L}_2 \rangle P^1_{SV} = \frac{\langle \mathscr{L}_1 \phi \rangle}{2} \Big(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} \Big) P^0_{SV}.$$

Since

$$\langle \mathscr{L}_1 \phi \rangle = \sqrt{2} \rho \nu \langle f \phi' \rangle \frac{\partial}{\partial x},$$

P^1_{SV} satisfies

$$\langle \mathscr{L}_2 \rangle P^1_{SV} = V \Big(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2} \Big) P^0_{SV}, \tag{17}$$

where

$$V = \frac{\rho \nu}{\sqrt{2}} \langle f \phi' \rangle.$$

Then, P^1_{SV} is obtained by solving the PDE (17) with terminal condition and boundary condition

$$\begin{cases} P^1_{SV}(t, b; a, b) = 0, \\ P^1_{SV}(T, x; a, b) = 0. \end{cases}$$

B Proof of Theorem 1

Theorem 1 can be also shown by the similar argument in Fouque *et al.* [8] that proved the validity of the approximation of the singular perturbation method for European call option.

The outline of the proof is as follows. We first introduce the regularized value P_δ, whose terminal condition is slightly smoothed by a (small) smoothing parameter δ. The associated price approximation is obtained by $P_\delta \approx P^0_\delta + \sqrt{\epsilon} P^1_\delta$ just like the approximation of P_{SV}. Lemma B. 1, 2, and 3 in the following show the convergence of (i) $P_{SV} \approx P_\delta$, (ii) $P^0_{SV} + \sqrt{\epsilon} P^1_{SV} \approx P^0_\delta + \sqrt{\epsilon} P^1_\delta$, and (iii) $P_\delta \approx P^0_\delta + \sqrt{\epsilon} P^1_\delta$, respectively.

First, $P_{SV}(T, x, y; a, b)$ is smoothly regularized by replacing it with its Black-Sholes value with volatility $\bar{\sigma}$, with time to maturity δ, and without knock-out barrier. In other words, $P_\delta(T, x, y; a, b)$ is defined by

$$P_\delta(T, x, y; a, b) = N(d_1(\delta, x)),$$

By the same argument in section 3, P_δ^0 and P_δ^1 are given by

$$P_\delta^0(t,x;a,b) = N(d_1(T-t+\delta,x)) - \exp\{(2\mu/\bar{\sigma}^2-1)(b-x)\}N(d_2(T-t+\delta,x)), \tag{18}$$

$$P_\delta^1(t,x;a,b) = V\Big\{\hat{P}_\delta(t,x;a,b) - (T-t)\Big(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2}\Big)P_\delta^0(t,x;a,b)\Big\},$$

where

$$\hat{P}_\delta(t,x;a,b) = \int_t^T (T-s)\Big(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2}\Big)P_\delta^0(s,b;a,b)h(s;x,b)ds,$$

$$h(s;x,b) = \frac{b-x}{\bar{\sigma}\sqrt{2\pi(s-t)^3}}\exp\left[-\frac{\{b-x-(\mu-\bar{\sigma}^2/2)(s-t)\}^2}{2\bar{\sigma}^2(s-t)}\right].$$

To establish the proof of the theorem, we use the following lemmas.

Lemma 1 *For fixed $t<T$, $x,y\in\mathbb{R}$, there exist constants $\delta_1>0$, $\epsilon_1>0$, and $c_1>0$ such that*

$$|P_{SV} - P_\delta| \le c_1\sqrt{\delta}$$

for any $0<\delta<\delta_1$ and $0<\epsilon<\epsilon_1$.

Lemma 2 *For fixed $t<T$, $x,y\in\mathbb{R}$, there exist constants $\delta_2>0$, $\epsilon_2>0$, and $c_2>0$ such that*

$$|(P_{SV}^0 + \sqrt{\epsilon}P_{SV}^1) - (P_\delta^0 + \sqrt{\epsilon}P_\delta^1)| \le c_2\delta$$

for any $0<\delta<\delta_2$ and $0<\epsilon<\epsilon_2$.

Lemma 3 *For fixed $t<T$, $x,y\in\mathbb{R}$, there exist constants $\delta_3>0$, $\epsilon_3>0$, and $c_3>0$ such that*

$$|P_\delta - P_\delta^0 - \sqrt{\epsilon}P_\delta^1| \le c_3\epsilon$$

for $0<\delta<\delta_3$ and $0<\epsilon<\epsilon_3$.

Proofs of the lemmas are given in the following subsections.

Because of the lemmas, there exist constants $\bar{\delta}>0$, $\bar{\epsilon}>0$, and c_1, c_2, $c_3>0$ such that

$$\begin{aligned} & |P_{SV} - P_{SV}^0 - \sqrt{\epsilon}P_{SV}^1| \\ \le\ & |P_{SV} - P_\delta| + |(P_{SV}^0 + \sqrt{\epsilon}P_{SV}^1) - (P_\delta^0 + \sqrt{\epsilon}P_\delta^1)| + |P_\delta - P_\delta^0 - \sqrt{\epsilon}P_\delta^1| \\ \le\ & c_1\delta^{1/2} + c_2\delta + c_3\epsilon \end{aligned}$$

for $0<\delta<\bar{\delta}$ and $0<\epsilon<\bar{\epsilon}$. Setting $\delta=\epsilon^2$,

$$|P_{SV} - P_{SV}^0 - \sqrt{\epsilon}P_{SV}^1| \le c_1\epsilon + c_2\epsilon^2 + c_3\epsilon.$$

Consequently, there exist constants $\bar{\epsilon}>0$, and $c>0$ such that

$$|P_{SV} - P_{SV}^0 - \sqrt{\epsilon}P_{SV}^1| \le c\epsilon$$

for $0<\epsilon<\bar{\epsilon}$.

B.1 Proof of Lemma B.1

From the definitions,

$$P_{SV}(t,x,y;a,b) = E_t[1_{\{X_T \le a\}} 1_{\{Z_T \le b\}}],$$

$$P_\delta(t,x,y;a,b) = E_t[N(d_1(\delta, X_T)) 1_{\{Z_T \le b\}}],$$

where $E_t[\cdot]$ represents the conditional expectation under $X_t = x, Y_t = y, Z_t < b$. Then, we have

$$\begin{aligned}
|P_{SV} - P_\delta| &= |E_t[\{1_{\{X_T \le a\}} - N(d_1(\delta, X_T))\} 1_{\{Z_T \le b\}}]| \\
&\le E_t[|1_{\{X_T \le a\}} - N(d_1(\delta, X_T))|] \\
&= E_t[(1 - N(d_1(\delta, X_T)) 1_{\{X_T \le a\}}] + E_t[N(d_1(\delta, X_T)) 1_{\{X_T > a\}}] \\
&= E_t[E_t[N(-d_1(\delta, X_T)) 1_{\{X_T \le a\}} |\ Y_u :\ t \le u \le T\ ]] \\
&\quad + E_t[E_t[N(d_1(\delta, X_T)) 1_{\{X_T > a\}} |\ Y_u :\ t \le u \le T\ ]]
\end{aligned}$$

Under the condition $\{Y_u\}_{t \le u \le T}$ is observed, the conditional distribution of X_T is $N(\mathfrak{m}, \mathfrak{v})$, where

$$\mathfrak{m} = x + \Big(\mu - \frac{\bar\sigma^2}{2}\Big)(T-t) + \rho \int_t^T f(Y_s) dW_s^2, \quad \mathfrak{v} = (1-\rho^2) \int_t^T f(Y_s)^2 ds.$$

The first term is evaluated as

$$\begin{aligned}
& E_t[N(-d_1(\delta, X_T)) 1_{\{X_T \le a\}} |\ Y_u :\ t \le u \le T\ ] \\
= \ & \frac{1}{\sqrt{2\pi\mathfrak{v}}} \int_{-\infty}^a N(-d_1(\delta, z)) e^{-\frac{(z-\mathfrak{m})^2}{2\mathfrak{v}}} dz \\
= \ & \frac{1}{\sqrt{2\pi\mathfrak{v}}} \int_{-\infty}^a \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{z-a+(\mu-\frac{\bar\sigma^2}{2})\delta}{\sigma\sqrt{\delta}}} e^{-\frac{w^2}{2}} e^{-\frac{(z-\mathfrak{m})^2}{2\mathfrak{v}}} dw dz \\
= \ & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(\mu/\bar\sigma - \bar\sigma/2)\sqrt{\delta}} e^{-\frac{w^2}{2}} \frac{1}{\sqrt{2\pi\mathfrak{v}}} \int_{a+\bar\sigma\sqrt{\delta}z-(\mu-\bar\sigma^2/2)\delta}^a e^{-\frac{(z-\mathfrak{m})^2}{2\mathfrak{v}}} dz dw \\
\le \ & c(\delta + \sqrt{\delta}) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(\mu/\bar\sigma - \bar\sigma/2)\sqrt{\delta}} e^{-\frac{w^2}{2}} dw \\
\le \ & c(\delta + \sqrt{\delta}).
\end{aligned}$$

for some $c > 0$. The first inequality is followed from the boundedness of $\frac{1}{f}$. The same argument also gives

$$E_t[N(d_1(\delta, X_T)) 1_{\{X_T > a\}} |\ Y_u :\ t \le u \le T\ ] \le c(\delta + \sqrt{\delta}).$$

Consequently, there exist constants $\delta_1 > 0$, and $c_1 > 0$ such that

$$|P_{SV} - P_\delta| \le c_1 \sqrt{\delta}$$

for any $0 < \delta < \delta_1$.

B.2 Proof of Lemma B.2

From the definitions, we can see that

$$\begin{aligned}
& (P^0_{SV}+\sqrt{\epsilon}P^1_{SV})-(P^0_\delta+\sqrt{\epsilon}P^1_\delta) \\
= \; & P^0_{SV}(t,x;a,b)-P^0_\delta(t,x;a,b) \\
& +\sqrt{\epsilon}V\int_t^T (T-s)\Big(\frac{\partial^3}{\partial x^3}-\frac{\partial^2}{\partial x^2}\Big)(P^0_{SV}-P^0_\delta)(s,b;a,b)h(s;x,b)ds \\
& -\sqrt{\epsilon}V(T-t)\Big(\frac{\partial^3}{\partial x^3}-\frac{\partial^2}{\partial x^2}\Big)(P^0_{SV}-P^0_\delta)(t,x;a,b).
\end{aligned}$$

Notice that we can write

$$P^0_\delta(t,x;a,b)=P^0_{SV}(t-\delta,x;a,b).$$

P^0_{SV}, P^0_δ and their successive derivatives with respect to x are differentiable in t. Therefore, we conclude that for $t<T$, $x\in\mathbb{R}$, there exist $\delta_2>0$, $\epsilon_2>0$, and $c_2>0$ such that

$$|(P^0_{SV}+\sqrt{\epsilon}P^1_{SV})-(P^0_\delta+\sqrt{\epsilon}P^1_\delta)|\le c_2\delta$$

for any $0<\delta<\delta_2$ and $0<\epsilon<\epsilon_2$.

B.3 Proof of Lemma B.3

To evaluate the residual of the approximation, we first define the residual term by

$$P_\delta=P^0_\delta+\sqrt{\epsilon}P^1_\delta-R_\delta.$$

In the argument for pricing of barrier option, Ilhan *et al.* [13] showed that if the payoff at the terminal time is smooth,

$$|P_\delta-P^0_\delta-\sqrt{\epsilon}P^1_\delta|=O(\epsilon).$$

In other words, there exist constants $\epsilon_3>0$ and c_δ (c_δ depends on δ.) such that

$$|R_\delta|\le c_\delta\epsilon$$

for any $0<\epsilon<\epsilon_3$.

Next, we give a concrete expression of c_δ. In $t<T$ and $x<b$, R_δ satisfies

$$\begin{aligned}
\mathscr{L}^\epsilon_{SV}R_\delta &= \mathscr{L}^\epsilon_{SV}(P^0_\delta+\sqrt{\epsilon}P^1_\delta-P_\delta) \\
&= \frac{1}{\epsilon}\mathscr{L}_0P^0_\delta+\frac{1}{\sqrt{\epsilon}}(\mathscr{L}_0P^1_\delta+\mathscr{L}_1P^0_\delta)+(\mathscr{L}_1P^1_\delta+\mathscr{L}_2P^0_\delta) \\
&= \mathscr{L}_2P^0_\delta,
\end{aligned}$$

because $\mathscr{L}^\epsilon_{SV}P_\delta=0$, and P^0_δ and P^1_δ do not depend on y. At the terminal time T, R_δ has

$$\begin{aligned}
R_\delta(T,x,y;a,b) &= P^0_\delta(T,x;a,b)+\sqrt{\epsilon}P^1_\delta(T,x;a,b)-P_\delta(T,x,y;a,b) \\
&= -e^{(2\nu/\bar{\sigma}^2-1)(b-x)}N(d_2(\delta,x)) \\
&\equiv H^\delta(x,y).
\end{aligned}$$

At the boundary $x = b$,

$$R_\delta(t, b, y; a, b) = 0,$$

because $P_\delta(t, b, y; a, b) = P^0_\delta(t, b; a, b) = P^1_\delta(t, b; a, b) = 0$.

Therefore, the probabilistic representation of R_δ is given by

$$\begin{aligned}
R_\delta(t, x, y; a, b) &= E_t\left[H^\delta(X_T, Y_T)1_{\{\tau>T\}} - \int_t^\tau \mathscr{L}_2 P^0_\delta(s, X_s; a, b)ds\right] \\
&= E_t[H^\delta(X_T, Y_T)1_{\{\tau>T\}}] \\
&\quad +\frac{1}{2}E_t\left[\int_t^\tau (f(Y_s)^2 - \bar{\sigma}^2)\Big(\frac{\partial}{\partial x} - \frac{\partial^2}{\partial x^2}\Big)P^0_\delta(s, X_s; a, b)ds\right],
\end{aligned}$$

where τ represents the first time after t that X_t hits b. We can easily see that the first term is bounded. Let us evaluate the second term. Since f is bounded,

$$\begin{aligned}
&\left|E_t\left[\int_t^\tau (f(Y_s)^2 - \bar{\sigma}^2)\Big(\frac{\partial}{\partial x} - \frac{\partial^2}{\partial x^2}\Big)P^0_\delta(s, X_s; a, b)ds\right]\right| \\
\le\ & cE_t\left[\int_t^T \left|\Big(\frac{\partial}{\partial x} - \frac{\partial^2}{\partial x^2}\Big)P^0_\delta(s, X_s; a, b)\right|ds\right]
\end{aligned}$$

for some $\delta > 0$. Note that we have

$$\begin{aligned}
&E_t\left[\int_t^T \left|\Big(\frac{\partial}{\partial x} - \frac{\partial^2}{\partial x^2}\Big)P^0_\delta(s, X_s; a, b)\right|ds\right] \\
=\ & E_t\left[\int_t^T E_t\left[\left|\Big(\frac{\partial}{\partial x} - \frac{\partial^2}{\partial x^2}\Big)P^0_\delta(s, X_s; a, b)\right|\ \Big|\ Y_u : t \le u \le s\right]ds\right].
\end{aligned}$$

Since $P^0_\delta(s, x; a, b)$ is given by (18),

$$\begin{aligned}
\frac{\partial P^0_\delta}{\partial x}(s, x; a, b) &= -\frac{n(d_1(T - s + \delta, x))}{\bar{\sigma}\sqrt{T - s + \delta}} \\
&\quad +\Big(\frac{2\mu}{\bar{\sigma}^2} - 1\Big)e^{(2\mu/\bar{\sigma}^2-1)(b-x)}N(d_2(T - s + \delta, x)) \\
&\quad -e^{(2\mu/\bar{\sigma}^2-1)(b-x)}\frac{n(d_2(T - s + \delta, x))}{\bar{\sigma}\sqrt{T - s + \delta}},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 P^0_\delta}{\partial x^2}(s, x; a, b) &= -\frac{n(d_1(T - s + \delta, x))d_1(T - s + \delta, x)}{\bar{\sigma}^2(T - s + \delta)} \\
&\quad -\Big(\frac{2\mu}{\bar{\sigma}^2} - 1\Big)^2 e^{(2\mu/\bar{\sigma}^2-1)(b-x)}N(d_2(T - s + \delta, x)) \\
&\quad +2\Big(\frac{2\mu}{\bar{\sigma}^2} - 1\Big)e^{(2\mu/\bar{\sigma}^2-1)(b-x)}\frac{n(d_2(T - s + \delta, x))}{\bar{\sigma}\sqrt{T - s + \delta}} \\
&\quad +e^{(2\mu/\bar{\sigma}^2-1)(b-x)}\frac{n(d_2(T - s + \delta, x))d_2(T - s + \delta, x)}{\bar{\sigma}^2(T - s + \delta)},
\end{aligned}$$

where $n(\cdot)$ is the density function of standard normal distribution: $n(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$.

By the same argument with the proof of Lemma 5.2. in Fouque *et al.* [8] (pp. 1664), we can show that

$$E_t \left[\left| \left(\frac{\partial}{\partial x} - \frac{\partial^2}{\partial x^2} \right) P_\delta^0(s, X_s; a, b) \right| \, Y_u : t \le u \le s \right] \le c(1 + (T - s + \delta)^{-1/2}),$$

for some $c > 0$. Integrating the above inequality with respect to s, we have

$$\begin{aligned} & E_t \left[\int_t^T E_t \left[\left| \left(\frac{\partial}{\partial x} - \frac{\partial^2}{\partial x^2} \right) P_\delta^0(s, X_s; a, b) \right| \, Y_u : t \le u \le s \right] ds \right] \\ \le \; & c(1 + \delta^{1/2}). \end{aligned}$$

Therefore, c_δ can be written as $c_\delta = c(1 + \delta^{1/2})$. Consequently, there exist constants $\delta_3 > 0$, $\epsilon_3 > 0$, and $c_3 > 0$ such that

$$|R_\delta| \le c_3 \epsilon$$

for $0 < \delta < \delta_3$ and $0 < \epsilon < \epsilon_3$.

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