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A Strategic Theory of Markets

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A STRATEGIC THEORY OF A MARKET[1]

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Eiichiro Kazumori

This paper investigates the asymptotic behavior of a Bayesian Nash equilibrium in uniform price double auctions among buyers and sellers with a unit demand and supply who receive private signals independently and identically distributed conditional on the unknown state with the monotone likelihood ratio condition and have interdependent values with a strictly private value element. Every nontrivial mixed strategy Bayesian Nash equilibrium converges to the fully revealing rational expectation equilibrium as the number of buyers and sellers increases and the bid step size goes to zero. A monotone pure strategy Bayesian Nash equilibrium exists in sufficiently large finite economies, and provides a consistent and asymptotically normal estimator of the unknown value when the set of possible bids is continuous.

Keywords: Double auction, information aggregation.

1. INTRODUCTION

The role of market prices in an economy with dispersed information has been a central question in modern economic theory. In a seminal paper, Hayek (1945) argued that the price mechanism is an effective method of communicating information and utilizing knowledge initially dispersed in the economy. Examples of applications of the idea include the efficient market hypothesis in asset pricing and the use of prediction markets to estimate uncertain parameters in the economy[1].

One approach to study the properties of the market price is a rational expectation equilibrium which extends the notion of the Walrasian equilibrium to the economy under uncertainty. Grossman (1981) showed that a fully revealing rational expectation equilibrium price in an Arrow-Debreu complete market is the sufficient statistic for all of the economy's information. But Grossman and Stiglitz (1980) and others pointed out that the rational expectation equilibrium does not formulate the process by which players incorporates the private information into the price.

This observation prompted an investigation of the properties of the market price formed through auction and bidding processes. In a pioneering paper, Wilson (1977) claimed that a Bayesian Nash equilibrium price of the first price auction among symmetric buyers with common values could be a consistent estimator of the unknown value. Subsequently, fundamental papers by Milgrom (1979, 81) and Pesendorfer and Swinkels (1997) extended the result to one-sided uniform price auctions with common values and obtained necessary and sufficient conditions for the consistency.

These papers developed the asymptotic theory of a Bayesian Nash equilibrium in one-sided auctions with common values. In view of the Wilson (1977)'s conjecture that

[1]This version: December 21, 2008.

[1]See Arrow et al. (2008) for a recent proposal to facilitate the development of prediction markets. Segal (2006) pointed out the need for a theoretical foundation of the prediction market.

a theory of price formation can be consistent with a theory of value, and of the fact that many financial markets are organized as large double auctions, it is of significance to study the asymptotic properties of a Bayesian Nash equilibrium in double auctions with interdependent values.

In an important contribution, Reny and Perry (2006) developed the key model of uniform price double auctions among buyers and sellers with interdependent values. Reny and Perry (2006) offered a through analysis of the single crossing conditions (Athey (2001)) of the double auction game and demonstrated that, generically, there exists a nontrivial monotone pure strategy Bayesian Nash equilibrium in sufficiently large finite economies, and that the Bayesian Nash equilibrium converges to the fully revealing rational expectation equilibrium in the large economy as the number of buyers and sellers increases and the bid step size goes to zero.

The goal of this paper is to extend and strengthen the universality of the information aggregation result of Wilson (1977), Milgrom (1979, 81), Pesendorfer and Swinkels (1997), and Reny and Perry (2006), among others, by studying the asymptotic property of a Bayesian Nash equilibrium of uniform price double auctions among buyers and sellers who receive private signals independently and identically distributed conditional on the unknown state with the monotone likelihood ratio condition and have interdependent values with a strictly private value element. The main results are:

(a) when the bid grid sizes is sufficiently small, a nontrivial mixed strategy equilibrium exists in the finite economy, and it is asymptotically equivalent to the fully revealing rational expectation equilibrium in the large economy as the bid grid size goes to zero
(b) when the set of possible bids is continuum, a monotone pure strategy equilibrium exists in the large finite economies, and it provides a consistent and asymptotically normal estimator of the unknown value with the rate of convergence $O(1/n)$.

In short, the results generalize the results of Reny and Perry (2006) by showing that information aggregation takes place not only in a monotone pure strategy equilibrium, but also in a more robust way, in a larger class of mixed Bayesian Nash equilibria, and that a Bayesian Nash equilibrium price is not only a consistent estimator of the unknown value, but also an asymptotically normal estimator with the rate of convergence $O(1/n)$.

An intuition is as follows. In the finite economy, the structures of one-sided auctions and double auctions differ since buyers and sellers want to influence the market clearing price in a different way and employ asymmetric strategies. But in a large economy, buyers and sellers are price takers. Consequently, ex ante buyers and sellers have a symmetric best response even when the strategies to which they choose the best responses are asymmetric. Thus, the structure of double auction game among ex ante symmetric buyers and sellers is closely related to the structure of one-sided uniform price auction game among symmetric buyers as the number of buyers and sellers increase. This *asymptotic equivalence* between one-sided uniform price auction and uniform price double auctions allows us to extend the monotonic structure of one-sided uniform price auction to the two-sided uniform price auction in the large economy.

The key structure is that the double auction game in the large economy satisfies *a best response strict single crossing property to a Bayesian Nash equilibrium strategy*[2]. That is, incremental returns from a low bid to a high bid, which is a best response, cross zero at most once, only from below, as a function of a player's signal. In other words, if a player with some signal prefers a high bid to a low bid as a best response to a Bayesian Nash equilibrium strategy, then a player with a higher signal strictly prefers this high bid to this low bid. To show that the best reply strict single crossing property holds in the double auction game in the large economy, we first start with a local comparison when the distance between the high bid and the low bid is small. Since a signal is affiliated with the state, a player with the higher signal has a more favorable estimate of the value of the good. When the distance between the high bid and the low bid is small, the increase in the estimate of the value of the good outweighs the possible increase in the expected price. Thus, the high bid is preferred to the low bid for a high signal. Even when the distance between the high bid and the low bid is large, it is still strictly preferable for a player with the high signal to increase a bid incrementally.

There are two important consequences of the strict single crossing property. The first consequence is that a player's equilibrium strategy has *monotone supports*. When a player's decision problem satisfies the strict single crossing condition, every selection from a best response is monotone nondecreasing in a player's signal[3]. Therefore, the smallest bid of the support of the best response strategy of a player with a higher signal is at least equal to the largest bid of the support of the best response of a player with a lower signal.

The second consequence is that the single crossing property implies an existence of *winner's curse*. When a player uses a strategy whose supports are monotone in signals, losing the good at the tie actually conveys a good news compared with winning the good at the tie at an equilibrium, since losing the good at the tie indicates there are more higher bids than the case of winning the good at the tie. An implication is that,

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[2]This single crossing property is different than the single crossing conditions (SCC) for games of incomplete information introduced by Athey (2001), and throughly analyzed by Reny and Perry (2006) for the double auction game. Athey (2001)'s condition requires that a best response satisfies single crossing property to every monotone strategy of other players. In contrast, the condition here requires that a best response satisfies the strict single crossing property to every equilibrium strategy of other players. This condition is satisfied in private value models and versions of this condition are studied in the interdependent value models of Pesendorfer and Swinkels (1997) and Reny and Zamir (2004).

[3]See Milgrom and Shannon (1994), Theorem 4'. This monotonicity is stronger than monotonicity in a strong set order commonly used in monotone comparative statics. It is a consequence of the fact that the best response in this model satisfies a strict single crossing property, stronger than the standard single crossing property in the sense that it requires that the incremental return to a best response crosses zero at most at a single point.

for a sufficiently small bid grid size, when two players are tied for some bid, a player with a higher signal prefers to change the bid to break the tie.

Using these ideas, the results of the paper can be summarized as follows. We begin with the double auction game when the set of possible bids and the set of signals are finite in the finite economy. By introducing a possibility of nonstrategic bids and then taking the probability to zero, we can show that there exists a nontrivial mixed Bayesian Nash equilibrium. We then increase the number of buyers and sellers in the economy. When the best response satisfies the strict single crossing condition, a player's best response has a monotone support. We then take the bid grid size go to zero. Winner's curse effect then implies that players with distinct signals will place distinct bids. It follows that the limit equilibrium as the bid grid size goes to zero is separating and does not involve a tie. This implies that the equilibrium is characterized by the limit of the first order condition of the double auction game with a positive bid grid size. It follows that a bid is equal to the expected value of the good conditional on the bidder being on the margin. Consequently, the limit equilibrium is symmetric and increasing, and equivalent to the fully revealing rational expectation equilibrium.

We then consider a behavior of a Bayesian Nash equilibrium in the large finite economy. Since the payoff changes continuously from the finite economy to the large economy, the best response strict single crossing property holds in the large finite economy. Thus a Bayesian Nash equilibrium in the large finite economy has monotone supports. We then take the signal grid size go to zero. Since every Bayesian Nash equilibrium with a finite set of signals has monotone supports, a Bayesian Nash equilibrium when the set of signals is continuous has also monotone supports. When the set of possible bids is finite and the set of possible signals is continuous, the Bayesian Nash equilibrium has to be pure and monotone almost everywhere. As the bid grid size becomes small, winner's curse effect and a strictly private value element in the value implies that the limit strategy profile does not involve tie, thus we have a monotone pure strategy equilibrium in the double auction game with a continuous set of bids and a continuous set of signal in the large finite economy.

We next study asymptotic distributions, which will be useful when comparing asymptotic efficiency of exchange mechanisms, in constructing confidence intervals, and in conducting hypothesis testing. The asymptotic behavior of a Bayesian Nash equilibrium price consists of the sample size effect which deals with the asymptotic behavior of order statistics and the strategic effect which deals with players' misrepresentation of bids from price-taking behavior. In contrast to a case of one-sided auctions (Hong and Shum (2004)), we need to deal with not only a buyer's misrepresentation but also a seller's misrepresentation. But both buyers' and sellers' misrepresentation will vanish at the rate of $O(1/n)$, since, for every possible state, due to the conditional independence of the signal distribution, a room that a player can manipulate the market price without changing allocation vanishes at the rate of $O(1/n)$. It follows that the strategic effect is asymptotically negligible and that the asymptotic behavior of a Bayesian Nash equilibrium price can be evaluated based on the asymptotic theory of an order statistics.

The organization of the paper is as follows. Section 2 defines the model. Section 3 contains the main results of the paper. Section 4 provides a detailed sketch of the proof. Section 4.1 discusses the existence of a nontrivial mixed strategy Bayesian Nash equilibrium of the double auction game in the finite economy. Section 4.2 studies the asymptotic behavior of a nontrivial mixed strategy Bayesian Nash equilibrium. Section 4.3 deals with the existence of a monotone pure strategy equilibrium in the large finite market with a continuous set of signals and a finite set of bids. Section 4.4 examines asymptotic normality. The supplement to the present paper contains the proof.

1.1. *Related literature*

This paper puts together three strands of the literature: one-sided uniform price auctions with common values, uniform price double auctions with private values, and uniform price double auction with interdependent values.

One-sided uniform price auction with common values. Milgrom (1981) developed the canonical model of a one-sided uniform price auction of a fixed number of units among buyers with unit demand, symmetric values, signals distributed iid conditional on the state, and the monotone likelihood ratio condition. Milgrom (1981) showed that a symmetric monotone pure strategy Bayesian-Nash equilibrium converges to the true value of the good when the distinguishability condition (Milgrom (1979)) is satisfied. The equilibrium identified by Milgrom (1981) is also the unique equilibrium of the uniform price double auction game among ex ante symmetric buyers and sellers in the large economy.

Pesendorfer and Swinkels (1997) showed that the symmetric monotone pure strategy Bayesian-Nash equilibrium of Milgrom (1981) is indeed unique among symmetric strategies, and that when the signal conveys only a limited amount of information, the equilibrium price converges to the true value of the good if and only if the double largeness condition (i.e. both the number of units of the good and the number of bidders who do not receive the good grow large) holds. In our double auction setting, buyers and sellers are asymmetric in the finite economy. But since buyers and sellers will be symmetric in the large economy, their uniqueness result extends to the double auction in the large market.

Hong and Shum (2004) established the rate of convergence and asymptotic distribution of a monotone pure strategy equilibrium in uniform price and English auctions under Wilson (1977) and Pesendorfer and Swinkels (1997) assumptions on the information structure. In our double auction setting, buyers and sellers are asymmetric and there is not any closed form representation of the equilibrium strategies. But since misrepresentations by buyers and sellers vanish sufficiently fast, it is still possible to derive the asymptotic distributions.

Uniform price double auction with private values. Rustichini, Satterthwaite, and Williams (1994) considered k-double auctions among buyers and sellers with unit demand and supply and independently distributed types. They showed that the equilibrium bid converges to the truthful bidding at the rate of $O(1/n)$. In our setting, players have interdependent values. But since the distribution of signals is iid conditional on the state, the probability that there is a bid in an interval increases at the rate of $O(1/n)$ at each state, thus it is possible to extend their results.

Jackson and Swinkels (2005) showed existence of a nontrivial mixed strategy equilibrium in the large class of private value auctions. We extend their result to an interdependent value setting by introducing asymptotic expansion of payoffs around the limit where the probability of perturbation is zero.

Cripps and Swinkels (2006) showed that in uniform price double auctions among buyers and sellers with multiple units of demand and supply and much weaker assumptions of distributions (no asymptotic atoms, no asymptotic gaps, and z-independence conditions), every nontrivial mixed strategy equilibria are asymptotically unique and efficient. In our setting, the values are nonprivate. But in our model, the strictly increasing private value element of the values, ex ante symmetries and monotone likelihood ratio conditions provide a set of conditions under which their asymptotic uniqueness and efficiency can be extended, and they serve a first step to generalize their results to a nonprivate value setting.

Uniform price double auction with interdependent values. In Reny and Perry (2006)'s analysis, they first showed existence of a monotone pure strategy Bayesian Nash equilibrium in the large economy and its convergence to a fully revealing rational expectation equilibrium as the grid size goes to zero. Then they constructed a monotone pure strategy Bayesian Nash equilibrium in large finite economies by showing that the strict single crossing condition holds in large finite economies. In contrast, we first show existence of a nontrivial mixed strategy Bayesian Nash equilibrium in the finite economy and then show that it converges to the monotone pure strategy equilibrium of the double auction game in the large economy. A difference between the approach by Reny and Perry (2006) and ours is that they established a strict single crossing condition as a response to every monotone strategy of other players, and that we need to show a strict single crossing property only for (possibly mixed) equilibrium strategies.

2. THE MODEL

2.1. *Players*

We first define the information structure which generates the state and the signals.

1. The state variable. Let $\theta_0 \in (0{,}1)$ be the true state of the world. θ_0 is unobservable, and each player considers it as a realization of a random variable θ. Each player has a correct and common prior that θ takes a value in [0,1] with the distribution function F_θ and the density function f_θ.

ASSUMPTION 1 *(a). There exists $\overline{f}_0 < \infty$ such that for every θ, $0 < f_\theta(\theta) < \overline{f}_0 < \infty$. (b). f_θ is continuous.*

2. The signal. Let X_i be a random variable which represents player i's private information about θ. X_i takes values in [0,1] according to the distribution $F_{X_i|\theta}(x_i|\theta)$ and the density function $f_{X_i|\theta}(x_i|\theta)$.

We now define order statistics and quantile functions which play important roles in analysis of auctions.

3. Order statistics and quantile functions. If the random variables $X_1,\dots,X_n$ are arranged in the order of magnitude, we write $X_{1:n} \geq \dots \geq X_{n:n}$. For example, $X_{m:n}$ implies the mth highest out of n random variables.

For a given α, for each θ, let

$$x_i(\theta) = \sup\{x_i : F_{X_i|\theta}(x_i(\theta)|\theta) \leq \alpha\}.$$

That is, given θ, $x_i(\theta)$ is the largest signal such that there are more than $1-\alpha$ of players whose signal is equal or higher than $x_i(\theta)$.

For each x_i, let

$$\theta(x_i) = \{\theta \in [0,1] : F_{X_i|\theta}(x_i|\theta(x_i)) = \alpha\}^{4}.$$

That is, when the signal is x_i, $\theta(x_i)$ is the state under which there are $1-\alpha$ of players whose signal is equal or above x_i. When X_i is continuous, for each x_i, $x_i(\theta(x_i)) = x_i$ and for each θ, $\theta(x_i(\theta)) = \theta$.

4. Assumptions on the conditional distribution X_i given θ. We first distinguish between two assumptions on the common support of a signal X_i.

The first assumption is that the support of the signal is finite with the signal grid size $\gamma > 0$[5].

[4]When there are multiple x_0 which will give x_i as the αth percentile of the distribution, we choose an arbitrary one and fix it.

[5]The assumption of a discrete set of signals is used, for example, in Pesendorfer and Swinkels (1995).

Assumption 2 *The common support of* X_i *is* $\mathcal{X}_\gamma = \{0,\gamma...,1\}$.

The second assumption is that the signals are continuum.

Assumption 3 *The common support of* X_i *is* $\mathcal{X} = [0,1]$.

We consider the following conditions on the behavior of $f_{X_i|\theta}(x_i|\theta)$.

Assumption 4 *(a). The signals* $\{X_i\}$ *are independent and identically distributed conditional on* θ *with the distribution function* $F_{X_i|\theta}(x_i|\theta)$*. (b). For each* x_i*,* $f_{X_i|\theta}(x_i|\theta)$ *is continuous in* $\theta \in [0,1]$*. (c). When* $X = [0,1]$*, for every* θ*,*$f_{X_i|\theta}(x_i|\theta)$ *is continuous in* x_i*. (d). There exists* $\overline{f}_1 < \infty$ *such that for every* $\theta \in [0,1]$ *and* $x_i \in X$*,* $0 < f_{X_i|\theta}(x_i|\theta) < \overline{f}_1$*. (e). The family of conditional density functions* $f_{X_i|\theta}(x_i|\theta)$*, indexed by* θ*, satisfies the monotone likelihood ratio property. (f).* $x_i(0) = 0$ *and* $x_i(1) = 1$.

Assumption 4 (a) and (e) imply that random variables θ,X_1,..., are affiliated (Milgrom and Weber (1982a)). Assumption 4 (e) ensures that $x_i(\theta)$ is a nondecreasing function of θ and that $\theta(x_i)$ is a nondecreasing functions of x_i. Assumption 4 (g) implies that every signal can be on the margin. This assumption was used in Proposition 1(d).

We next define players' objective functions.

5. The value function. Let $v(\theta,x_i)$ denote the actual value of a unit of good to player i. Player i puts zero value on further units of the good. We assume following conditions.

Assumption 5 *(a). There exists* $0 < \underline{v} \leq \overline{v} < \infty$ *such that, for every* θ *and* x_i*,* $0 < \underline{v} < \frac{\partial v(\theta,x_i)}{\partial \theta} < \overline{v}$*. (b). There exists* $0 < \underline{\lambda} \leq \overline{\lambda} < \infty$ *such that for every* θ *and* x_i*,* $\underline{\lambda} < \frac{\partial v(\theta,x_i)}{\partial x_i} < \overline{\lambda}$*. (c). There exists* $0 < \overline{v} < \infty$ *such that, for every* $\theta \in [0,1]$*,* $v(\theta,0) = 0$ *and* $v(\theta,1) = \overline{v}$.

Assumption 5 (c) means that, when the signals are extreme, values are private[6][7]. Its role is to ensure that a player with a lower signal will have a low estimate of the value of the object regardless of the state[8].

As explained in Introduction, a player's objectives are different depending on whether a player is a buyer or a seller.

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[6]Reny and Perry (2006) also make an assumption about the behavior of the value function at the boundary.

[7]The probabilities that a player has these exterme signal can be arbitary small.

[8]As a concrete example, consider a political prediction market where X_0 takes a value of 1 if a candidate is elected and 0 otherwise, and X_i is the impression of the candidate held by a player. When a player has a good impression of the candidate (that is, the signal x_i is high), it will not only provide a favorable assessment about electability of a candidate and but also a higher possibility that the player will be able to work with the candidate for a future project, regardless of whether the candidate will be elected to this particular pubic office. Thus, a player with a higher signal will value the candidate higher than a player with a lower signal regardless of whether the candidate will get elected or not. Furthermore, if a player has really a good impression of a candidate, the player will value the candidate highly regardless of the outcome of a particular election.

6. Endowments. Let $e_i \in \{0,1\}$ denote player i's endowment. Players with $e_i = 0$ are potential buyers and players with $e_i = 1$ are potential sellers.

7. Ex post utility function. Let $u_i(\theta,x_i,p,q_i)$ be player i's ex post utility function when the state is θ, the signal is x_i, the market clearing price is $p \in \mathbf{R}$, and the allocation is $q_i \in \{0,1\}$[9]. We assume

$$u_i(\theta, x_i, p, q_i) = \begin{cases} (v(\theta, x_i) - p)q_i \text{ if } i \text{ is a buyer} \\ (p - v(\theta, x_i))(1 - q_i) \text{ if } i \text{ is a seller} \end{cases}$$

We assume that every player is risk neutral.

2.2. *Rational expectation equilibrium in the large economy*

We now consider the rational expectation equilibrium in the large economy.

8. The large economy. The large economy has a unit mass of players of whom $\alpha \in (1/2,1)$ are buyers and $1 - \alpha$ are sellers.

9. The rational expectation equilibrium. A rational expectation equilibrium in the large economy is a price function $p^{REE}(\theta)$ and an allocation $q_i(x_i,p^{REE}(\theta))$ such that

(a) a player chooses the excess demand taking his signal, the price, and the information contained in the price as given[10].

(b) The demand and the supply balance at each state.

In the large economy, the actual distribution of players' signals is equal to the prior distribution of signals. Thus, given the state, there are no uncertainty about the distribution of players' signals. It follows that the rational expectation equilibrium in the large economy is defined as a function of the state variable θ.

10. Characterization of the rational expectation equilibrium. Reny and Perry (2006), Proposition 3.1 (i) derived a fully revealing rational expectation equilibrium in the large economy.

LEMMA 2.1 (RENY AND PERRY (2006)). *There is a unique fully revealing rational expectation equilibrium in the large economy with $p^{REE}(\theta_0) = v(\theta_0,x_i(\theta_0))$.*

In this equilibrium, for each state, the fully revealing rational expectation equilibrium price will be the expected value of the good of the player who is on the margin[11].

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[9]We sometimes denote u_b to be an ex post payoff function of a buyer and u_s to be an ex post payoff function of a seller.

[10]We assume that when a player is indifferent between $q_i = 1$ and 0, a tie will be broken in a way so that the resulting allocation is consistent with the market clearing condition.

[11]Even when the distribution of the signal is discrete, the argument in Reny and Perry (2006) can be applied and Lemma 1 holds. To see this, first note that $v(\theta, x_i(\theta))$ is a strictly increasing function of θ since even when $x_i(\theta)$ is nondecreasing and constant, v is still strictly increasing in the first argument θ. Thus, given the price $v(\theta_0, x_i(\theta_0))$, a player can infer the state θ_0. Then, a player strictly prefers to be assigned a good if the value is strictly higher than $v(\theta_0, x_i(\theta_0))$, which takes place if the signal is strictly above $x_i(\theta_0)$. A player does not want to be assigned a good if the signal is strictly less than

2.3. *Bayesian Nash equilibrium in uniform price double auctions*

We now specify a Bayesian Nash equilibrium of the uniform price double auction.

11. Consider a sequence of auctions where the nth auctions has n players of whom n_B players are potential buyers and n_S players are potential sellers. That is $n = n_B + n_S$. Let N_n, $N_{n,B}$, and $N_{n,S}$ denote the set of players, buyers and sellers in the nth auction. It follows $N_n = N_{n,B} \cup N_{n,S}$.

We consider central rank case auctions (Hong and Shum (2004)) where the double largeness condition (Pesendorfer and Swinkels (1997)) holds and the supply grows proportionally to demand.

Assumption 6 A *sequence* $\{n_S, n_B\}$ *satisfies the following conditions: (a).* $0 < \frac{n_S}{n} < \frac{1}{2}$. *(b).* $\lim_{n\to\infty} \frac{n_S}{n} = 1 - \alpha$.

We suppose that the prices are formed through a uniform price double auction (Rustichini, Satterthwaite, and Williams (1994), Wilson (1995), Cripps and Swinkels (2004), and Reny and Perry (2006)).

12. Bids. Each buyer submits a bid and each seller submits an offer. Let b_i denote the bid or the offer submitted by player i and let $b = (b_1,...,b_n)$.

We distinguish between two assumptions about the set of possible bids. The first assumption is that the set of possible bids is finite and the bid grid size is Δ. Let $B_\Delta = \{0,\Delta,2\Delta,...,\bar{b}\} \cup \underline{b}$ denote the set of possible bids when the bid step size Δ is positive where $v(1,1) < \bar{b} < \infty$ is the largest possible bid and $\underline{b}$ is a nonparticipation option with zero expected payoff regardless of the behavior of other players.

The second assumption is that the bids are continuum. Let $B_0 = [0,\bar{b}] \cup \underline{b}$ denote the set of possible bids in this case.

13. The price function. Let $p_n(b)$ denote the market price $p_n(b)$ in the nth auction when players submits bids $b = (b_1, ..., b_n)$. We assume that the price is determined by the k-double auction pricing rule. That is, for a fixed $k \in [0,1]$,

$$p_n(b) = (1-k)b_{n_S+\mathbf{1}:n} + kb_{n_S:n}$$

14. The allocation function. Let $q_n(b_i,b_{-i})$ be the allocation of player i in the nth auction when player i bids b_i and other players' bids are b_{-i}. When a player's bid or offer b_i is higher than $b_{n_S:n}$, then the player is assigned a unit of the good. When b_i is lower than $b_{n_S:n}$, then the player is not assigned a unit of good. The ties will be

$x_i(\theta_0)$. When the signal is equal to $x_i(\theta_0)$, the player is indifferent whether a good is assigned or not. By breaking ties in a way consistent with the market clearing condition, the price function $p^{REE}(\theta_0) = v(\theta_0, x_i(\theta_0))$ satisfies the conditions for the rational expectation equilibrium.

broken symmetrically among buyers and sellers[12], independently from other events in the game. That is, $q_n(b_i,b_{-i})$ satisfies the following condition:

$$q_n(b_i,b_{-i}) = \begin{cases} 1 \text{ if } b_i > b_{n_S:n}. \\ 1 \text{ with probability } \frac{n_S - |j:b_j > b_{n_S:n}|}{|j:b_j = b_{n_S:n}|} \text{if } b_i = b_{n_S:n}. \\ 0 \text{ else.} \end{cases}$$

Let $\mathcal{G}(\gamma,f,\Delta,n)$ be a non-cooperative game induced by a double auction when the support of the signal is $\mathcal{X}_\gamma$, the information structure is f, the size of the economy is n, and the bid step size is Δ.

15. Distributional strategies. Player i's distributional strategy (Milgrom and Weber (1985)) in $\mathcal{G}(\gamma,f,\Delta,n)$ is a probability measure $\mu_{\gamma,\Delta,n,i}$ on $B_\Delta \times \mathcal{X}_\gamma$ with its distribution function $H_{\gamma,\Delta,n,i}(b_i,x_i)$ and the probability mass function $h_{\gamma,\Delta,n,i}(b_i,x_i)$ such that the marginal distribution on $\mathcal{X}_\gamma$ is equal to $F_{X_i}(x_i)$.

Given a distributional strategy $\mu_{\gamma,\Delta,n,i}$, let $\beta_{\gamma,\Delta,n,i}$ be a corresponding behavioral strategy in the sense that $\beta_{\gamma,\Delta,n,i}(x_i)$ is a random variable whose distribution over B_Δ corresponds to the distribution of bids conditional on x_i according to $H_{\gamma,\Delta,n,i}(b_i|x_i)$.

Let supp$H_{\gamma,\Delta,n,i}(b_i|x_i)$ denote the support of $\beta_{\gamma,\Delta,n,i}(x_i)$.

Let $\mathcal{B}_\Delta$ be the set of possible behavioral strategies when the set of possible bids is B_Δ.

We say that a strategy is pure if there exists a function $\beta_{\gamma,\Delta,n,i}(x_i)$ such that $H_{\gamma,\Delta,n,i}(b_i,x_i)$ puts a probability 1 on the set $\{(x_i,\beta_{\gamma,\Delta,n,i}(x_i))\}_{x_i\in\mathcal{X}_\gamma}$. A pure strategy is monotone if $\beta_{\gamma,\Delta,n}(x_i)$ is a nondecreasing function of x_i.

Given a set of strategies by players other than i, we can write down player i's objective function as follows.

16. Expected payoffs. Let $U_{\gamma,\Delta,n}(x_i,b_i,\beta^*_{\gamma,\Delta,n,-i})$ be player i's interim expected payoff in $\mathcal{G}(\gamma,f,\Delta,n)$ when player i's signal is x_i, a bid is b_i, and other players place bids b_{-i} according to strategies $\beta^*_{\gamma,\Delta,n,-i}$. That is,

$$U_{\gamma,\Delta,n}(x_i,b_i,\beta^*_{\gamma,\Delta,n,-i}) = \mathbf{E}_{\theta,X_{-i}|X_i}[u_i(\theta,x_i,p_n(b_i,b_{-i}),q_n(b_i,b_{-i}))].$$

Let $\pi_{\gamma,\Delta,n,i}(\beta_i,\beta^*_{\gamma,\Delta,n,-i})$ be player i's ex ante expected payoff function in $\mathcal{G}(\gamma,f,\Delta,n)$ when the strategy profile is $\beta_{\gamma,\Delta,n}$.

We are now in a position to specify a Bayesian Nash equilibrium of the double auction game.

17. Bayesian Nash equilibrium. A Bayesian Nash equilibrium of $\mathcal{G}(\gamma,f,\Delta,n)$ is a profile of distributional strategies $\beta^*_{\gamma,\Delta,n}$ where (a) every buyer chooses the symmetric strat-

[12] See Reny and Zamir (2004) and Reny and Perry (2006) for a discussion of the implication of symmetric tie-breaking rules.

egy and every seller chooses the symmetric strategy[13] and (b) each player' s strategy satisfies the best response property. That is, for every i,

$$\beta^*_{\gamma,\Delta,n,i} \in \arg \max_{\beta'_{\Delta,i} \in \mathcal{B}_\Delta} \pi_{\gamma,\Delta,n,i}(\beta'_{\Delta,i}, \beta^*_{\gamma,\Delta,n,-i}).$$

18. A nontrivial Bayesian Nash equilibrium. We say that an outcome according to a strategy profile is nontrivial if trade occurs between buyers and sellers. We say that a Bayesian Nash equilibrium is nontrivial if the probability the outcome according to the equilibrium strategy profile is nontrivial is positive. Let $\tau(\beta^*_{\gamma,\Delta,n})$ denote the probability that an outcome according to $\beta^*_{\gamma,\Delta,n}$ is nontrivial. A Bayesian-Nash equilibrium $\beta^*_{\Delta,n}$ is nontrivial if $\tau(\beta^*_{\gamma,\Delta,n}) > 0$.

19. A Bayesian Nash equilibrium price. Let $P_{\gamma,\Delta n}(\beta^*_{\gamma,\Delta,n})$ be a random variable which represents the price generated by the equilibrium strategy profile $\beta^*_{\gamma,\Delta,n}$ of the double auction game $\mathcal{G}(\gamma,f,\Delta,n)$.

3. THE MAIN RESULT

20. The main proposition of the paper states asymptotic properties of a nontrivial Bayesian-Nash equilibrium of the uniform price double auction game.

(a) *Given Assumption 1, 2, 4, 5 and 6, there exists a signal grid size* $\overline{\gamma} > 0$ and a bid grid size $\overline{\Delta} > 0$ *such that for every* $0 < \gamma < \overline{\gamma}$ *and* $0 < \Delta < \overline{\Delta}$, *there exists a nontrivial Bayesian-Nash equilibrium* $\beta^*_{\gamma,\Delta,n}$ *of the double auction game* $\mathcal{G}(\gamma,f,\Delta,n)$.

(b) *Given Assumption 1, 2, 4, 5, 6, and with conditions in Proposition 1(a), every sequence of nontrivial Bayesian-Nash equilibrium* $\beta^*_{\gamma,\Delta,n}$ *is asymptotically outcome equivalent to the fully revealing rational expectation equilibrium in the large economy identified in Lemma 1 as the number of players* $n \to \infty$ *and the bid step size* $\Delta \to 0$.

(c) *Given Assumption 1, 3, 4, 5, 6, and conditions of Proposition 1(a), there exists* $\overline{n}$ such that for every $n > \overline{n}$, *there exists a nontrivial monotone pure strategy equilibrium* $\beta^*_{\Delta,n}$ *of the double auction game* $\mathcal{G}(f,\Delta,n)$.

(d) *Given Assumption 1, 3, 4, 5, 6 and condition of Proposition 1(a) and 1(c),* there exists a nontrivial monotone pure strategy equilibrium β^*_n of the double auction game $\mathcal{G}(f,n)$ and it satisfies the following relationship

$$\sqrt{n}(P_n(\beta^*_n) - v(\theta_0, x_i(\theta_0))$$
$$\xrightarrow{d} N(0, \frac{\alpha(1-\alpha)}{f^2_{X_i|\theta}(x_i(\theta_0)|\theta_0)} \cdot$$
$$(\frac{\partial v(\theta_0, x_i(\theta_0))}{\partial \theta} \frac{\partial \theta(x_i(\theta_0))}{\partial x_i} + \frac{\partial v(\theta_0, x_i(\theta_0))}{\partial x_i})^2) \text{ as } n \to \infty.$$

[13]The assumption that equilibrium strategies are symmetric among buyers and also among sellers can be relaxed at the cost of additional notations.

We now offer an interpretation of Proposition 1 and its relation to the previous results.

21. Part (a) says that there exists a mixed nontrivial Bayesian-Nash equilibrium of the double auction game with a finite set of signals and a finite set of bids in the finite market. It extends the existence result of Jackson and Swinkels (2005) in the private value environment to an interdependent value environment[14]. It also complements the result of Reny and Perry (2006) by showing that there exists a mixed strategy equilibrium in the small finite market, even when Reny and Perry (2006)'s Theorem 6.1. (i) does not ensure the existence of a monotone pure strategy Bayesian Nash equilibrium[15].

22. Part (b) says that every nontrivial mixed strategy equilibrium in the finite economy converges to the fully revealing rational expectation equilibrium in the large economy. It extends Theorem 6.1. (ii)-(iv) of Reny and Perry (2006) by showing that information aggregation can take place in the larger class of nontrivial mixed strategy Bayesian Nash equilibria. It also extends Theorem 1 of Cripps and Swinkels (2006) by showing that asymptotic uniqueness and efficiency of mixed strategy equilibria in private value uniform price double auctions can be extended to a class of interdependent value uniform price double auctions.

23. Part (c) says that there exists a monotone pure strategy equilibrium of the double auction in the sufficiently large finite economy. This result was first proved by Reny and Perry (2006) as Theorem 6.1 (i).

Part (a), (b) and (c), taken together, generalize Theorem 6.1 of Reny and Perry (2006) in the following sense. Reny and Perry (2006) Theorem 6.1 showed that the uniform price auction among sufficiently large number of buyers and sellers has a monotone pure strategy equilibrium in the sufficiently large finite economies where bidding behavior is arbitrary close to a price taking behavior, the market clearing price is arbitrary close to the rational expectation equilibrium price, and the allocation is arbitrary close to efficient. This paper shows that every nontrivial mixed strategy equilibrium, including the monotone pure strategy equilibrium identified in Reny and Perry (2006), satisfies these properties[16].

[14]To be precise, Jackson and Swinkels (2005) assumed a continuum set of bids and signals, while the result here assumes a finite set of bids and signals. It is straightforward to extend the existence result of Part (a) to the case of a continuum set of signals. But an assumption of a finite set of bids is somewhat necessary since it is harder to deal with discontinuities due to ties in the interdependent value models without monotonicity structure of equilibrium strategies.

[15]Reny and Perry (2006) considered a continuum set of signals and the finite set of bids, while the result here assumes a finite set of bids and signals. It is straightforward to extend the existence result of Part (a) to the case of a continuum set of signals.

[16]To be precise, the first result (asymptotic uniqueness and efficiency of nontrivial mixed strategy Bayesian Nash equilibria) assumes the finite set of signals and the second result (existence of a monotone pure strategy Bayesian Nash equilibrium in the sufficienlty large economies) assumes the contiuum of signals, while Reny and Perry (2006) assumes the continuum of signals.

24. Part (d) says when the signal is continuum, the Bayesian Nash equilibrium price of the double auction is not only a consistent and asymptotically normal estimator of the value of the good. It follows that the Bayesian Nash equilibrium price provides a consistent and asymptotically normal estimator of the uncertain state of the world. This result extends the result of Reny and Perry (2006) by showing that a Bayesian Nash equilibrium price is not only a consistent estimator of the value but also an asymptotically normal estimator. Also this result generalizes the asymptotic normality result of Hong and Shum (2004) in the one-sided auctions to a uniform price double auction.

We now provide a short sketch of the proof. We elaborate a detailed sketch in the next subsections.

25. Proof of Part (a). We first define a modified game $\mathcal{G}(\gamma,f,\Delta,n,\delta)$ where a nonstrategic bidding takes place with probability δ. When the set of possible bids is finite, the mixed extension of the game is continuous. When there is a possibility of nonstrategic bidding, buyers and sellers place a serious bid to trade. Thus, there exists a nontrivial Bayesian-Nash equilibrium $\beta^*_{\gamma,\Delta,n,\delta}$ of $\mathcal{G}(\gamma,f,\Delta,n,\delta)$. Furthermore, since buyers and sellers prefer to bid seriously even with small δ, the limit strategy profile $\beta^*_{\gamma,\Delta,n}$ as $\delta \to 0$ is a nontrivial equilibrium of $\mathcal{G}(\gamma,f,\Delta,n)$.

26. Proof of Part (b). Consider a limit $\beta^*_{\gamma,\Delta}$ of a sequence of nontrivial Bayesian Nash equilibria $\{\beta^*_{\gamma,\Delta,n}\}$ of $\mathcal{G}(\gamma,f,\Delta,n)$. Since the payoff function of $\mathcal{G}(\gamma,f,\Delta,n)$ converges to the payoff function in the large economy $\mathcal{G}(\gamma,f,\Delta)$ as $n \to \infty$, $\beta^*_{\gamma,\Delta}$ is a nontrivial Bayesian Nash equilibrium of $\mathcal{G}(\gamma,f,\Delta)$. Then we consider the behavior of $\beta^*_{\gamma,\Delta}$ for small bid grid size Δ. In the large economy, a player's bid cannot affect the market clearing price and every player faces a symmetric distribution of strategies of other players. Thus, buyers and sellers' decision problems are symmetric. Then, for sufficiently small Δ, a best response to $\beta^*_{\gamma,\Delta}$ satisfies the strict single crossing property. It follows that β^*_{Δ} has monotone supports. When we take the bid grid size $\Delta \to 0$, winner's curse effect implies $\beta^*_{\gamma,\Delta}$ will be separating. Thus the limit β^* is characterized by the limit of first order conditions for a Bayesian-Nash equilibrium in $\mathcal{G}(\gamma,f,\Delta)$. It follows that the limit equilibrium is outcome equivalent to the rational expectation equilibrium.

27. Proof of Part (c). From Part (b), we know that the best response to $\beta^*_{\gamma,\Delta}$ satisfies the strict single crossing property for $\mathcal{G}(\gamma,f,\Delta)$. Since the convergence of the expected payoff of $\mathcal{G}(\gamma,f,\Delta,n)$ to that of $\mathcal{G}(\gamma,f,\Delta)$ is uniform when the set of signals and the set of bids are finite, for sufficiently large n, the best response to $\beta^*_{\gamma,\Delta,n}$ satisfies strict single crossing property. It follows that both buyer's and seller's strategies have monotone supports. We now set the signal grid size $\gamma \to 0$ and consider $\mathcal{G}(f,\Delta,n)$. Then, since equilibrium strategies have monotone supports for every $\gamma > 0$, the limit equilibrium $\beta^*_{\Delta,n}$ also has monotone supports. Since the set of signals is continuum and the set of possible bids is finite in $\mathcal{G}(f,\Delta,n)$, $\beta^*_{\Delta,n}$ has to be pure and monotone almost everywhere.

28. Proof of Part (d). We first extend the result of Part (c) to $\mathcal{G}(f,n)$ by taking the bid grid size $\Delta \to 0$. Since the equilibrium strategies $\beta^*_{\Delta,n}$ is monotone, winner's curse effect implies that the limit strategy profile β^*_n does not involve ties and thus a monotone pure strategy Bayesian Nash equilibrium. We can now derive the asymptotic distribution of β^*_n by decomposing into the difference between $v(\theta(x_{n_S:n}),x_{n_S:n})$ and $v(\theta, x_i(\theta))$ and the difference between $v(x_0(x_{n:n_S}),x_{n:n_S})$ and β^*_n. The first difference can be evaluated by the delta method. The second difference goes to zero because β^*_n converges to $v(\theta(x_i), x_i)$ at the rate of $O(1/n)$.

4. PROOF

4.1. *Proof of Part (a)*

We begin with the definition of the modified game.

29. The modified game $\mathcal{G}(\gamma,f,\Delta,n,\delta)$. Given $\mathcal{G}(\gamma,f,\Delta,n)$, let $\mathcal{G}(\gamma,f,\Delta,n,\delta)$ denote the game where, in addition to players in N_n, there is a nonstrategic player $\widehat{i}$ who participates in the auction as a buyer with probability $\delta/2$ and participates as a seller with probability $\delta/2$. Either as a buyer or a seller, $\widehat{i}$ bids uniformly over $\underline{b} \cup \{0,\Delta,2\Delta,...,\overline{b}\}$, independent of other events in the game. Let $H_{\widehat{i},\Delta}$ denote its distribution function and let $h_{\widehat{i},\Delta}$ denote the probability mass function.

The auction mechanism treats players in N_n and a nonstrategic player $\widehat{i}$ symmetrically. Let $p_n(b_i,b_{-i},b_{\widehat{i}})$ and $q_n(b_i,b_{-i},b_{\widehat{i}})$ be the market clearing price and the allocation of player i when players N_n submitted bids (b_i,b_{-i}) and a nonstrategic player $\widehat{i}$ submitted a bid $b_{\widehat{i}}$.

30. We now introduce notations to describe strategies and equilibrium of the modified game $\mathcal{G}(\gamma,f,\Delta,n,\delta)$.

Let $\beta_{\Delta,n,\delta,i}$ be player i's distributional strategy in $\mathcal{G}(\gamma,f,\Delta,n,\delta)$. Let $H_{\Delta,n,\delta,i}$ be its distribution function and let $h_{\Delta,n,\delta,i}(b,x_i)$ be its probability mass function.

Let $U_{\gamma,\Delta,n,\delta,i}(x_i,b_i,\beta_{\gamma,\Delta,n,\delta,-i})$ denote player i's interim expected utility given x_i, b_i, and $\beta_{\gamma,\Delta,n,\delta,-i}$ where the expectation is taken over the participation and the distribution of bids by nonstrategic bidders, in addition to the distribution of the state and other players' signals conditional on x_i. Let $\pi_{\gamma,\Delta,n,\delta,i}(\beta_{\gamma,\Delta,n,\delta,i},\beta_{\gamma,\Delta,n,\delta,-i})$ denote the ex ante expected utility given players' strategies $\beta_{\gamma,\Delta,n,\delta,i}$ and $\beta_{\gamma,\Delta,n,\delta,-i}$.

31. Continuity of ex ante payoff functions $\pi_{\gamma,\Delta,n,\delta}(\beta_{\gamma,\Delta,n,\delta,i},\beta_{\gamma,\Delta,n,\delta,-i})$. We first note that the payoff function $\pi_{\gamma,\Delta,n,\delta}$ is continuous, since the set of possible bids B_Δ is finite and we consider distributional strategies.

Lemma 4.1 $\pi_{\gamma,\Delta,n,\delta}(\beta_{\gamma,\Delta,n,\delta,i},\beta_{\gamma,\Delta,n,\delta,-i})$ *is continuous in* $(\beta_{\gamma,\Delta,n,\delta,i},\beta_{\gamma,\Delta,n,\delta,-i})$.

32. Definition of a best response correspondence. Let $\Phi : \mathcal{B}_\Delta \times \mathcal{B}_\Delta \to \mathcal{B}_\Delta \times \mathcal{B}_\Delta$ denote a correspondence defined by

$$\Phi(\beta_{\gamma,\Delta,n,\delta,b},\beta_{\gamma,\Delta,n,\delta,s}) = \begin{pmatrix} \arg\max_{\beta'_b\in\mathcal{B}_\Delta} \pi_{\gamma,\Delta,n,b}(\beta'_b, \underbrace{\beta_{\gamma,\Delta,n,\delta,b},\dots,\beta_{\gamma,\Delta,n,\delta,b}}_{n_B-1 \text{ buyers}}, \\ \underbrace{\beta_{\gamma,\Delta,n,\delta,s},\dots,\beta_{\gamma,\Delta,n,\delta,s}}_{n_S \text{ sellers}}) \\ \arg\max_{\beta'_s\in\mathcal{B}_\Delta} \pi_{\gamma,\Delta,n,s}(\underbrace{\beta_{\gamma,\Delta,n,\delta,b},\dots\beta_{\gamma,\Delta,n,\delta,b}}_{n_B \text{ buyers}}, \\ \beta'_s, \underbrace{\beta_{\gamma,\Delta,n,\delta,s},\dots,\beta_{\gamma,\Delta,n,\delta,s}}_{n_S-1 \text{ sellers}}) \end{pmatrix}$$

This correspondence takes bidding strategies of a buyer $\beta_{\gamma,\Delta,n,\delta,b}$, and a seller $\beta_{\gamma,\Delta,n,\delta,s}$, and returns a best response of a buyer when all other buyers follow $\beta_{\gamma,\Delta,n,\delta,b}$ and every seller follows $\beta_{\gamma,\Delta,n,\delta,s}$, and a best response of a seller when every buyer follows $\beta_{\gamma,\Delta,n,\delta,b}$ and all other sellers follow $\beta_{\gamma,\Delta,n,\delta,s}$.

33. Existence of a nontrivial Bayesian Nash equilibrium $\beta^*_{\gamma,\Delta,n,\delta}$. Since the set of distributional strategies is compact and ex ante payoff function $\pi_{\gamma,\Delta,n,\delta}$ is continuous, there exists a fixed point of Φ, which is a symmetric Bayesian equilibrium. Furthermore, since a nonstrategic player offers a serious bid with a positive probability, this equilibrium is nontrivial.

Lemma 4.2 *There exists* $\overline{\Delta} > 0$ *such that for every* $0 < \Delta < \overline{\Delta}$ *and* $\delta > 0$, *there exists a nontrivial Bayesian-Nash equilibrium* $\beta^*_{\gamma,\Delta,n,\delta}$ *of* $\mathcal{G}(\gamma, f, \Delta, n, \delta)$.

We now take the probability of nonstrategic bidding $\delta \to 0$.

34. Construction of an equilibrium of $\mathcal{G}(\gamma, f, \Delta, n, \delta)$. Let $\{\beta^*_{\gamma,\Delta,n,\delta}\}_\delta$ be a sequence of a nontrivial Bayesian-Nash equilibrium $\mathcal{G}(\gamma, f, \Delta, n, \delta)$. Then there is a subsequence δ and a subsequence limit $\beta^*_{\gamma,\Delta,n}$ such that $\beta^*_{\gamma,\Delta,n,\delta} \to \beta^*_{\gamma,\Delta,n}$ in the sense that, for each x_i and b_i, the probability that a player with signal x_i chooses a bid b_i converges as $\delta \to 0$. Since the expected utility function $\pi_{\gamma,\Delta,n,\delta}$ is continuous, $\beta^*_{\gamma,\Delta,n}$ is an equilibrium of $\mathcal{G}(\gamma, f, \Delta, n)$. Since the probability of trade τ is a continuous function of equilibrium strategies, the probability of trade also converges.

Lemma 4.3

(a) $\beta^*_{\gamma,\Delta,n}$ *is a Bayesian Nash equilibrium of the double auction game* $\mathcal{G}(\gamma, f, \Delta, n)$.

(b) *For each* x_0,

$$\lim_{\delta \to 0} \tau_{\gamma,\Delta,n,\delta}(\beta^*_{\gamma,\Delta,n,\delta}|x_0) = \tau_{\gamma,\Delta,n}(\beta^*_{\gamma,\Delta,n}|x_0).$$

We now suppose that $\beta^*_{\gamma,\Delta,n}$is trivial and derive an estimate of the distribution of bids.

35. Estimates of the distribution of bids in $\mathcal{G}(\gamma, f, \Delta, n, \delta)$ *for small* δ. Suppose that $\beta^*_{\gamma,\Delta,n}$ is trivial. That is, $\tau_{\gamma,\Delta,n}(\beta^*_{\gamma,\Delta,n}) = 0$. The only way that $\beta^*_{\gamma,\Delta,n}$ can be trivial is that sell bids will be very high and buy bids will be very low. Consequently, when δ is small, the probability of a high buy bid and a low sell bid need to be very small.

LEMMA 4.4 *Suppose* $\beta^*_{\gamma,\Delta,n}$ *is trivial. Then, for every* x_i, $0 < b_i \leq \bar{b}$,

$$\lim_{\delta \to 0} h^*_{\gamma,\Delta,n,\delta,b}(b_i|x_i) = 0$$

and $0 \leq b_i < \bar{b}$,

$$\lim_{\delta \to 0} h^*_{\gamma,\Delta,n,\delta,s}(b_i|x_i) = 0.$$

We now consider whether a seller with a low signal might prefer to lower the offer rather than the high offer required to sustain a trivial equilibrium.

36. An alternative strategy. By Assumption 5 (c), $v(x_0,0) = 0$ for every x_0. Then, for sufficiently small $\Delta > 0$,

$$\Pr\{x_i \in \mathcal{X}_\gamma : v(1,1) - 6\Delta - v(1,x_i) > 0\} > 0.$$

We note that this Δ are independent of other structure of the economy.

Consider an alternative strategy for seller i whose signal x_i satisfies the above condition such that whenever the equilibrium sell bid is at least $v(1,1)$, seller i lowers the bid to $v(1,1) - 2\Delta$.

37. The change in payoffs from the alternative strategy. Let $\overline{W}_{i,n}$ be the n_S-th highest bid among bids by players other than i, that is, $\overline{W}_{i,n} = b_{n_S:|N_n \cup \{\hat{i}\} - \{i\}|}$. There are four cases to be considered.

(a) Consider the case $\overline{W}_{i,n} = v(1,1)$. An offer of $v(1,1)$ can be accepted (subject to ties). An offer of $v(1,1) - 2\Delta$ will be accepted. But the sales price will be lower with an offer of $v(1,1) - 2\Delta$. An upper bound of the loss from changing to an offer of $v(1,1) - 2\Delta$ can be obtained by assuming that both offers will be accepted for sure. The k double auction pricing rule implies that the loss is bounded above by 2Δ.
(b) Consider the case $\overline{W}_{i,n} = v(1,1) - \Delta$. An offer of $v(1,1)$ is above $\overline{W}_{i,n}$, so it will not be accepted. An offer of $v(1,1) - 2\Delta$ is below $\overline{W}_{i,n}$, so it will be accepted.

With an offer of $v(1{,}1) - 2\Delta$, the sales price is between $\overline{W}_{i,n} = v(1{,}1) - \Delta$ and $v(1{,}1) - 2\Delta$. The lowest possible price is $v(1{,}1) - 2\Delta$. On the other hand, the largest possible value is of the good is $v(1{,}x_i)$. Thus the lower bound of payoff from sales is $v(1{,}1) - 2\Delta - v(1{,}x_i)$.

(c) Consider the case $W_{i,n} = v(1{,}1) - 2\Delta$. An offer of $v(1{,}1)$ is above $\overline{W}_{i,n}$, so it will not be accepted. An offer of $v(1{,}1) - 2\Delta$ is equal to $\overline{W}_{i,n}$, so it can be accepted. When there is a sale, the lower bound of payoff from sales price is $v(1{,}1) - 2\Delta - v(1{,}x_i)$.

(d) When $\overline{W}_{i,n} < v(1{,}1) - 2\Delta$, both offers of $v(1{,}1)$ and $v(1{,}1) - 2\Delta$ is above $\overline{W}_{i,n}$. Consequently, neither offers will be accepted and payoffs are zero.

The gain from the sales at case (c) is larger than 4Δ and the loss of sales at case (a) is at most 2Δ.

It remains to evaluate the probabilities of case (a) and case (c). We use an asymptotic expansion method to evaluate the distribution of the equilibrium bids at small δ by ignoring the higher order terms in terms of δ. For sufficiently small δ, the probability of event (a) and (c) can be approximated by the probability that a nonstrategic player $\widehat{i}$ chooses a bid of $v(1{,}1)$ and $v(1{,}1) - \Delta$, respectively. Since $H_{\widehat{i},\Delta}$ is uniform, these probabilities are approximately equal. Consequently, the expected benefit from new sales outweighs the expected cost of lower prices from existing sales.

LEMMA 4.5 *Suppose that $\beta^*_{\gamma,\Delta,n}$ is trivial. Then there exists $\overline{\delta} > 0$ and $\overline{\Delta} > 0$ such that for every $0 < \delta < \overline{\delta}$ and $\Delta < \overline{\Delta}$, the set of signals such that every seller i with the signal in that set prefers to deviate from $\beta_{\gamma,\Delta,n,\delta,i}$ has a positive measure.*

This is a contradiction to the assumption that $\beta^*_{\gamma,\Delta,n,\delta}$ is an equilibrium of $\mathcal{G}(\gamma,f,\Delta,n,\delta)$. Consequently, it has to be that a Bayesian Nash equilibrium $\beta^*_{\gamma,\Delta,n}$ is nontrivial. This establishes Part (a) of Proposition 1.

4.2. *Proof of Part (b)*

Having established existence of a nontrivial Bayesian Nash equilibrium of the double auction game in the finite economy, we now study its asymptotic properties.

38. Convergence to a Bayesian-Nash equilibrium of double auction in the large economy. It follows from Part (a) that for sufficiently small Δ, there exists a sequence of nontrivial mixed strategy equilibria $\{\beta^*_{\gamma,\Delta,n}\}_n$. Let $\beta^*_{\gamma,\Delta} = (\beta^*_{\gamma,\Delta,b}, \beta^*_{\gamma,\Delta,s})$ be its subsequence limit.

We need to show that $\beta^*_{\gamma,\Delta}$ is a nontrivial equilibrium of the double auction game in the large economy $\mathcal{G}(\gamma,\mathrm{f},\Delta)$. For that purpose, we show that the ex ante payoff function $\pi_{\gamma,\Delta,n,i}(\beta^*_{\gamma,\Delta,n,i}, \beta^*_{\gamma,\Delta,n,-i})$ of $\mathcal{G}(\gamma,f,\Delta,n)$ converges to the ex ante payoff function $\pi_{\gamma,\Delta,i}(\beta^*_{\gamma,\Delta,i}, \beta^*_{\gamma,\Delta,i})$ of $\mathcal{G}(\gamma,\mathrm{f},\Delta)$. The change in the ex ante payoffs is the sum of two effects:

(a) The first effect concerns the effect of change of strategies from $\beta^*_{\gamma,\Delta,n}$ to $\beta^*_{\gamma,\Delta}$ while keeping the size of the economy n constant. This term $\pi_{\gamma,\Delta,n,i}(\beta^*_{\gamma,\Delta,n,i},\beta^*_{\gamma,\Delta,n,-i}) - \pi_{\gamma,\Delta,n,i}(\beta^*_{\gamma,\Delta,i},\beta^*_{\gamma,\Delta,i})$ converges to zero since $\beta^*_{\gamma,\Delta,n}$ converges to $\beta^*_{\gamma,\Delta}$.

(b) The second effect concerns the effect of the change of the size of the economy n while keeping the strategy $\beta^*_{\gamma,\Delta,n}$ constant. This term $\pi_{\gamma,\Delta,n,i}(\beta^*_{\gamma,\Delta,n,i},\beta^*_{\gamma,\Delta,n,-i}) - \pi_{\gamma,\Delta,i}(\beta^*_{\gamma,\Delta,n,i},\beta^*_{\gamma,\Delta,n,-i})$ converges to zero because of convergence of the empirical distribution of bids generated by $\beta^*_{\gamma,\Delta,n}$ converges to the distribution of bids generated by $\beta^*_{\gamma,\Delta,n}$.

It remains to show that $\beta^*_{\gamma,\Delta}$ is nontrivial. To see this, suppose $\beta^*_{\gamma,\Delta}$ is trivial. Then, it follows that for sufficiently large n, the probability of trade under $\beta^*_{\gamma,\Delta,n}$ has to be very small. But an argument similar to a previous lemma implies that it cannot be the case. Thus we have

LEMMA 4.6 *$\beta^*_{\gamma,\Delta}$ is a nontrivial Bayesian Nash equilibrium of $\mathcal{G}(\gamma,f,\Delta)$.*

We now examine the structure of a best response to $\beta^*_{\gamma,\Delta}$. We first begin with its symmetry.

39. Symmetry of interim expected payoffs and best responses. Let $BR_{\gamma,\Delta,i}(x_i,\beta^*_{\gamma,\Delta,b},\beta^*_{\gamma,\Delta,s})$ be the set of best responses for player i when, except for player i, all buyers use $\beta^*_{\gamma,\Delta,b}$ and all sellers use $\beta^*_{\gamma,\Delta,s}$. Then, since buyers and sellers are symmetric in values and informations, their interim expected payoff functions are symmetric. That is, and since they face a symmetric distribution of bidding strategies of other players, their best responses are symmetric. That is, there exists a best response correspondence $BR_{\gamma,\Delta}(x_i,\beta^*_{\gamma,\Delta,b},\beta^*_{\gamma,\Delta,s})$ common to every player i, such that $BR_{\gamma,\Delta}(x_i,\beta^*_{\gamma,\Delta,b},\beta^*_{\gamma,\Delta,s}) = BR_{\gamma,\Delta,i}(x_i,\beta^*_{\gamma,\Delta,b},\beta^*_{\gamma,\Delta,s})$ for every i. That is, although buyers and sellers' best responses are asymmetric in the finite economy, players' best responses will be symmetric in the double auction in the large economy.

We now proceed to examine its monotonicity structure.

40. The strict single crossing property. Let $b_i \in BR_{\gamma,\Delta}(x_i,\beta^*_{\gamma,\Delta,b},\beta^*_{\gamma,\Delta,s})$. We now want to show that the best response to β^*_{Δ} satisfies the best reply strict single crossing property[17]. That is,

If b_i is a best response for x_i to $(\beta^*_{\gamma,\Delta,b},\beta^*_{\gamma,\Delta,s})$,
then, for $b_i > \underline{b}_i$, $\overline{x}_i > x_i$,

$$\underbrace{U_{\gamma,\Delta}(x_i,b_i,\beta^*_{\gamma,\Delta,b},\beta^*_{\gamma,\Delta,s}) \geq U_{\gamma,\Delta}(x_i,\underline{b}_i,\beta^*_{\gamma,\Delta,b},\beta^*_{\gamma,\Delta,s})}_{\text{if } b_i \text{ is preferred to a lower bid by a player with the signal } x_i}$$

$$\rightarrow \underbrace{U_{\gamma,\Delta}(\overline{x}_i,b_i,\beta^*_{\gamma,\Delta,b},\beta^*_{\gamma,\Delta,s}) > U_{\gamma,\Delta}(\overline{x}_i,\underline{b}_i,\beta^*_{\gamma,\Delta,b},\beta^*_{\gamma,\Delta,s})}_{\text{then } b_i \text{ is still, and strictly preferred to the lower bid by a player with the higher signal } \overline{x}_i}$$

[17]Reny and Zamir (2004) first introduced a notion of the best reply single crossign condition (BR-SCC).

$$\text{and, for } b_i < \overline{b}_i,\ x_i > \underline{x}_i,$$
$$\underbrace{U_{\gamma,\Delta}(x_i, b_i, \beta^*_{\gamma,\Delta,b}, \beta^*_{\gamma,\Delta,s}) \geq U_{\gamma,\Delta}(x_i, \overline{b}_i, \beta^*_{\gamma,\Delta,b}, \beta^*_{\gamma,\Delta,s})}_{\text{if } b_i \text{ is preferred to a higher bid } \overline{b}_i \text{ by a player with signal } x_i}$$
$$\rightarrow \underbrace{U_{\gamma,\Delta}(\underline{x}_i, b_i, \beta^*_{\gamma,\Delta,b}, \beta^*_{\gamma,\Delta,s}) > U_{\gamma,\Delta}(\underline{x}_i, \overline{b}_i, \beta^*_{\gamma,\Delta,b}, \beta^*_{\gamma,\Delta,s})}_{\text{then } b_i \text{ is still, and strictly preferred to the higher bid by a player with a lower signal } \underline{x}_i}.$$

Intuitively, the strict single crossing property holds when the incremental return from $\underline{b}_i$ to $b_i \in BR_{\gamma,\Delta}(\beta^*_{\gamma,\Delta,b}, \beta^*_{\gamma,\Delta,s})$ crosses zero only once, only from below, and at most at a single point.

We now proceed to show that the best response correspondence $BR_{\gamma,\Delta}$ satisfies the strict single crossing properties. We proceed in the following steps.

41. We first estimate the effect of a change in the signal from x_i to $\overline{x}_i$ on the estimate of the value of the good. The change is the sum of the following two effects:

- Private value effect: By Assumption 5, a player with a signal $\overline{x}_i$ increases the estimate of the value of the good at least by $\underline{\lambda}(\overline{x}_i - x_i)$.
- Affiliation effect: By Assumption 4 and 5, the estimated value is nondecreasing in the signal.

Thus, in total, the value estimate increases at least by $\underline{\lambda}(\overline{x}_i - \underline{x}_i)$.

Next we establish the strict single crossing property holds locally.

42. Local strict single crossing property. Suppose that $\underline{b}_i$ and $\overline{b}_i$ are one grid size apart ($\overline{b}_i - \underline{b}_i = \Delta$).

- The change in the estimated value is at least $\underline{\lambda}\gamma$.
- On the other hand, the possible increase in the payment is bounded above by $\overline{b}_i - \underline{b}_i = \Delta$.

Thus, for sufficiently small Δ, the increase in the expected value of the good outweighs the possible increase in the payment, and the strict single crossing holds.

Lastly, we extend the local result to a more general case.

43. Strict single crossing property in a general case. Suppose that the difference $\overline{b}_i - \underline{b}_i$ is more than one grid size apart. We sketch the argument in three cases.

- Suppose that no other players place a bid between $\overline{b}_i$ and $\underline{b}_i$. It follows that the player with signal x_i should have a nonnegative payoff from winning with the price $\overline{b}_i$, otherwise the player could have lowered the bid. Then, a player with the high signal $\overline{x}_i$ should have a positive expected payoff from winning with the price $\overline{b}_i$. Thus, even when the probability that the price is $\overline{b}_i$ increases with signal $\overline{x}_i$, the player with signal $\overline{x}_i$ strictly prefers $\overline{b}_i$ over $\underline{b}_i$.
- Suppose that there is another player, with signal lower than $\overline{x}_i$, who places a bid

between $\overline{b}_i$ and $\underline{b}_i$. It follows from the previous local argument, the player with signal $\overline{x}_i$ will prefer to increase a bid incrementally from $\underline{b}_i$ to b_i.

- The remaining case is that the player who places a bid between $\overline{b}_i$ and $\underline{b}_i$ has a signal strictly higher than $\overline{x}_i$. But in this case, this player should have strictly preferred $\overline{b}_i$ over the bid between $\overline{b}_i$ and $\underline{b}_i$ in the first place, which is a contradiction.

Therefore, we have

Lemma 4.7 *There exists $\overline{\Delta} > 0$ such that for all $0 < \Delta < \overline{\Delta}$, in the double auction game in the large economy $\mathcal{G}(\gamma, f, \Delta)$, a best response to an equilibrium strategy $\beta^*_{\gamma,\Delta}$ satisfies the strict single crossing property for bids in the range of the equilibrium prices.*

The first important consequence of the strict single property is that the distribution of bids according to $\beta^*_{\gamma,\Delta}$ has supports nondecreasing in signals.

44. Monotone supports. Since the player's best response satisfies the strict single crossing property, a Monotone selection theorem (Milgrom and Shannon (1994), Milgrom (2004)) implies that every selection from a best response correspondence is nondecreasing in the player's signal. A consequence is that an equilibrium strategy $\beta^*_{\gamma,\Delta}$ has monotone supports in the sense that every selection from supp$\beta^*_{\gamma,\Delta}(x_i)$ is nondecreasing in x_i. That is, for the minimum bid placed by a player with a high signal is no less than the maximum bid placed by a player with a low signal.

Lemma 4.8 *$\beta^*_{\gamma,\Delta}(x_i)$ has monotone supports in the range of the equilibrium prices.*

Strict single crossing property in a decision problem itself still does not imply that the optimizer is strictly increasing[18]. That is, it is still possible that the minimum bid placed by a player with a high signal is equal to the maximum bid placed by a player with a high signal so that $\beta^*_{\gamma,\Delta}$ is not separating. But in affiliated value auctions, monotonicity of bidding strategies can affect a player's inference, and thus choice, through winner's curse, and it will lead to a separation of $\beta^*_{\gamma,\Delta}$ for Δ sufficiently small.

45. Winner's curse and separation at $\beta^*_{\gamma,\Delta}$. The next step is to show that players with distinct signals will place distinct bids. Suppose otherwise that players with two distinct signals $\overline{x}_i > \underline{x}_i$ place the same bid in b_i according to β^*_Δ with positive probability.

First, a previous lemma implies that winner's curse is present at the tie. That is, when the state is higher, the bids will be higher, and a player is more likely to lose the tie. It follows that the expected value of the good conditional on the event of losing the tie at b_i is higher than the expected value of the good conditional on the event of winning at the tie at b_i.

[18] See Athey, Milgrom, and Roberts (forthcoming) for a counterexample.

On the other hand, since bidders with distinct signals bid the same amount, it has to be that the player with signal $\overline{x}_i$ should have a negative expected payoff from losing the tie at b_i, otherwise the player will increase the bid to outbid the player with the low signal. Also, the player with signal $\underline{x}_i$ should have a positive expected payoff from winning the tie at b_i, otherwise the player will decrease the bid so that the player will not be tied with the player with the high signal.
These three conditions are mutually exclusive. Therefore,

LEMMA 4.9 *There exists $\overline{\Delta} > 0$ such that for all $\Delta < \overline{\Delta}$, for every player $i \neq j$, signal $x_i \neq x_j$, and a bid b which is in the range of the equilibrium price,* $\Pr(\beta^*_{\Delta,i}(x_i) = b) \cdot \Pr(\beta^*_{\Delta,j}(x_j) = b) = 0$.

We now proceed to prove Proposition 1 part (b).

46. Asymptotic equivalence of $\beta^*_{\gamma,\Delta}$ to a fully revealing rational expectation equilibrium as $\Delta \to 0$.

We now take $\Delta \to 0$ and consider the property of a limit strategy profile β^*_γ. Winner's curse effect will imply that there would not be a tie at β^*_γ, thus β^*_γ is a Bayesian Nash equilibrium of $\mathcal{G}(\gamma f)$.

We now consider the first order condition at $\mathcal{G}(\gamma,f,\Delta)$. Let b_i be a bid which is in the support of $\beta^*_{\gamma,\Delta}(x_i)$. Then,

$$\begin{aligned}
&\int \sum_{\mathcal{X}_\gamma : p(b_i,b_{-i})=b_i+\Delta} (v(\theta,x_i) - (b_i+\Delta))q(b_i+\Delta,b_{-i})h_\Delta(b_{-i}|x_{-i}) \\
&f_{\theta|X_i}(\theta|x_i)dx_i \\
&+\int \sum_{\mathcal{X}_\gamma\times\ldots\times\mathcal{X}_\gamma : p(b_i,b_{-i})=b_i} (v(\theta,x_i) - b_i)(1-q(b_i,b_{-i}))h_\Delta(b_{-i}|x_{-i}) \\
&f_{\theta|X_i}(\theta|x_i)dx_i \\
\leq\ & 0.
\end{aligned}$$

The first term deals with the case where a bid $b_i+\Delta$ wins the good at the price $b_i+\Delta$. The second term deals with the case when the bid of b_i previously lost the tie but now a bid $b_i+\Delta$ wins the tie. Similar condition holds for a change in bid from b_i to $b_i-\Delta$.

The previous lemma implies that there will not be a tie between bids by players with different signals for sufficiently small Δ. Thus, for Δ small, a changing a bid a little bit will not affect the allocation. Furthermore, winner's curse effect implies that there would not be a tie at the limit strategy profile $\beta^*_\gamma(x_i)$. It follows that, $q(b_i+\Delta,b_{-i}) \to q(b_i,b_{-i})$ as $\Delta \to 0$. Therefore, the first order condition implies that $\mathbf{E}[(v(\theta,X_i) - b_i)|X_i = x_i, p(b_i,b_{-i}) = b_i] = 0$. In words, the bid is equal to the expected value of the good conditional on the bid being on the margin.

Given the monotone likelihood ratio condition, the bidding strategy is strictly increasing in x_i. This implies that the limit strategy $\beta^*_{\Delta,b}$ is outcome equivalent to the fully revealing rational expectation equilibrium. Thus we have:

LEMMA 4.10 *The limit strategy profile β^*_γ is outcome equivalent to the fully revealing rational expectation equilibrium identified in Lemma 2.1.*

This lemma concludes the proof of Proposition Part (b).

4.3. *Proof of Part (c)*

This subsection studies the property of a Bayesian Nash equilibrium in the double auction game in the large finite economies. We begin with understanding the relations between the double auction games in the large economy and in the large finite economies.

47. Uniform convergence of the interim expected payoff functions from $\mathcal{G}(\gamma,f,\Delta,n)$ to $\mathcal{G}(\gamma,f,\cdot)$. We saw in Lemma 4.6 that the ex ante payoff functions in $\mathcal{G}(\gamma,f,\Delta,n)$ as the size of the economy increases. Since the set of possible bids and the set of possible signals are both finite, the convergence is uniform.

LEMMA 4.11 *Suppose $\beta^*_{\gamma,\Delta,n} \to \beta^*_{\gamma,\Delta}$. Then,*

$$U_{\gamma,\Delta,n,i}(x_i, b_i, \beta^*_{\gamma,\Delta,n,b}, \beta^*_{\gamma,\Delta,n,s}) \to U_{\gamma,\Delta}(x_i, b_i, \beta^*_{\gamma,\Delta,b}, \beta^*_{\gamma,\Delta,s}) \text{ as } n \to \infty$$

uniformly for player i, signal x_i, and a bid b_i.

Lemma 4.11 implies that, when the economy is sufficiently large, (a) the difference of the expected payoffs between the large finite economy and the large economy is small, and (b) the difference of payoffs between a buyer and a seller is small.

48. Best responses. It follows from the previous lemma that for sufficiently large n, a player's expected payoff in $\mathcal{G}(\gamma,f,\Delta,n)$ will be very close to the player's expected payoff in $\mathcal{G}(\gamma,f,\Delta)$. Since the set of signals and the bids are finite, for sufficiently large n, a best response to $\beta^*_{\gamma,\Delta,n}$ in $\mathcal{G}(\gamma,f,\Delta,n)$ is also a best response to $\beta^*_{\gamma,\Delta}$ of $\mathcal{G}(\gamma,f,\Delta)$. To see this, suppose that $b_{\gamma,n,i}$ is a best response for a player with signal x_i to $\beta^*_{\gamma,\Delta,n}$ in $\mathcal{G}(\gamma,f,\Delta,n)$ but not to $\beta^*_{\gamma,\Delta}$ of $\mathcal{G}(\gamma,f,\Delta)$. Then, there should be a best response b'_i which does better than $b_{\gamma,n,i}$ to $\beta^*_{\gamma,\Delta}$. That is, $U_{\gamma,\Delta}(x_i,b'_i,\beta^*_{\gamma,\Delta,b},\beta^*_{\Delta,s}) > U_{\gamma,\Delta}(x_i,b_{n,i},\beta^*_{\Delta,b},\beta^*_{\Delta,s})$.But it follows from Lemma 4.11 that, for sufficiently large n, b'_i does better than $b_{i,n}$ in $\mathcal{G}(\gamma,f,\Delta,n)$. It is a contradiction to the fact that $b_{i,n}$ is a best response to $(\beta^*_{\Delta,n,b},\beta^*_{\Delta,n,s})$ in $\mathcal{G}(\gamma,f,\Delta,n)$. Since the number of possible combinations of x_i and b_i is finite, it is possible to find a finite lower bound $\underline{n}$.

LEMMA 4.12 *There exists $\underline{n}$ such that for every $n > \underline{n}$, for every player i and for every signal x_i,*

$$BR_{\gamma,\Delta,n,i}(x_i, \beta^*_{\Delta,n,b}, \beta^*_{\Delta,n,s}) \subseteq BR_{\gamma,\Delta}(x_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}).$$

This relation of best responses between double auction games in the large finite economy and in the large economy allows us to extend the best response strict single crossing properties to the large finite economy.

49. Strict single crossing properties in the large finite economy. Consider a buyer with signal $x_{n,i}$ and a bid $b_{\gamma,\Delta,n,i} \in BR_{\gamma,\Delta,n,b}(x_{n,i},\beta^*_{\gamma,\Delta,n,b},\beta^*_{\gamma,\Delta,n,s})$. We now increase the signal from x_i to $\overline{x}_i$ and we would like to see that a buyer with signal $\overline{x}_i$ still and strictly prefers $b_{\Delta,n,i}$ to a lower bid $\underline{b}_i < b_{\Delta,n,i}$. That is,

$$U_{\gamma,\Delta,n,b}(\overline{x}_i, b_{\Delta,n,i}, \beta^*_{\Delta,n,b}, \beta^*_{\Delta,n,s}) > U_{\gamma,\Delta,n,b}(\overline{x}_i, \underline{b}_i, \beta^*_{\Delta,n,b}, \beta^*_{\Delta,n,s}).$$

From Lemma 4.12, $b_{\gamma,\Delta,n,i}$ is also a best response of a buyer with signal $x_{n,i}$ to $\beta^*_{\gamma,\Delta}$ in $\mathcal{G}(\gamma,f,\cdot)$. It follows from the strict single crossing condition of the double auction in the large economy (Lemma 4.7) that

$$U_{\gamma,\Delta}(\overline{x}_i, b_{\Delta,n,i}, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) > U_{\gamma,\Delta}(\overline{x}_i, \underline{b}_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}).$$

From Lemma 4.11, for sufficiently large n, the distance between $U_{\gamma,\Delta,n,b}(\overline{x}_i,b_{\Delta,n,i},\beta^*_{\Delta,n,b},\beta^*_{\Delta,n}$ and $U_{\gamma,\Delta,n}(\overline{x}_i,b_{\Delta,n,i},\beta^*_{\gamma,\Delta,b},\beta^*_{\gamma,\Delta,s})$ and the distance between $U_{\gamma,\Delta,n,b}(\overline{x}_i,\underline{b}_i,\beta^*_{\gamma,\Delta,n,b},\beta^*_{\gamma,\Delta,n,s})$ and $U_{\gamma,\Delta}(\overline{x}_i,\underline{b}_i,\beta^*_{\gamma,\Delta,b},\beta^*_{\gamma,\Delta,s})$ will be sufficiently small. Therefore, the best response strict single crossing condition holds for sufficiently large finite economies.

LEMMA 4.13 *There exists $\overline{\Delta} > 0$ and $\underline{n} < \infty$ such that for all $0 < \Delta < \overline{\Delta}$ and $n > \underline{n}$, the uniform price auction game in the large finite economy $\mathcal{G}(\gamma,f,\Delta,n)$ satisfies a best response strict single crossing property for bids in the range of the equilibrium prices.*

50. Monotone and separating supports in the large finite economy. It follows from a reasoning similar to Lemma 4.9 that both the support of equilibrium strategies of buyers and sellers increase monotonically in signals and the supports are separating for sufficiently small Δ.

LEMMA 4.14 *There exists $\overline{\Delta} > 0$ and $\underline{n} < \infty$ such that for all $\Delta < \overline{\Delta}$ and $n > \underline{n}$, (a). $\beta^*_{\gamma,\Delta,n,i}$ has a monotone supports. (b). for each player i, signal $x_i \neq x_j$, and a bid b_i which is in the range of the equilibrium price,* $\Pr(\beta^*_{\gamma,\Delta,n,i}(x_i) = b_i) \cdot \Pr(\beta^*_{\gamma,\Delta,n,i}(x_j) = b_i) = 0$.

Reny and Perry (2006) considered the setting where the set of possible bids is finite and the set of possible signals is continuous. We now take the signal grid size $\gamma \to 0$ and characterize a Bayesian-Nash equilibrium $\beta^*_{\Delta,n}$ of the double auction with a finite set of possible bids in the large finite economy with continuum signals $\mathcal{G}(f,\Delta,n)$.

51. We approximate $\mathcal{G}(f,\Delta,n)$ by a sequence of double auction games with a finite set of signals $\{\mathcal{G}(\gamma,f_{\gamma},\Delta,n)\}_{\gamma}$ by approximating the posterior distribution $f_{X_i|\theta}(x_i|\theta)$ with $\{f_{X_i|\theta,\gamma}(x_i|\theta)\}_{\gamma}$. Specifically, for each θ and $\gamma>0$, let

$$f_{X_i|\theta,\gamma}(x_i|\theta)=\begin{cases}\int_{x_i-\gamma/2}^{x_i+\gamma/2} f_{X_i|\theta}(x_i'|\theta)dx_i' \text{ if } x_i=k\gamma\in[0,1] \text{ for some } k\in\mathbf{N}.\\ 0 \text{ else}\end{cases}$$

It follows from Lemma 4.15 that $\mathcal{G}(\gamma,f_{\gamma},\Delta,n)$ has a nontrivial Bayesian-Nash equilibrium $\beta^*_{\gamma,\Delta,n}$ with monotone and separating supports for each $\gamma>0$.

52. Bayesian-Nash equilibrium of $\mathcal{G}(f,\Delta,n)$.Let $\beta^*_{\Delta,n}$ be a limit strategy profile of $\{\beta^*_{\gamma,\Delta,n}\}_{\gamma}$ as $\gamma\to 0$. Since $f_{\theta|X_i,\gamma}(\theta|x_i)$ converges weakly to $f_{\theta|X_i}$, $\beta^*_{\Delta,n}$ is a Bayesian-Nash equilibrium of $\mathcal{G}(f,\Delta,n)$.

Furthermore, since every $\beta^*_{\gamma,\Delta,n}$ has monotone and separating supports and $\beta^*_{\gamma,\Delta,n}$ changes continuously in n,$\beta^*_{\Delta,n}$ also has monotone supports. Since the set of possible bids is finite and signals is continuum, $\beta^*_{\Delta,n}$ is monotone and pure almost everywhere.

LEMMA 4.15 *There exists* $\underline{n}<\infty$ *such that for* $n>\underline{n}$, *there exists a monotone pure Bayesian Nash equilibrium* $\beta^*_{\Delta,n}$ *of the double auction game in the large finite market* $\mathcal{G}(f,\Delta,n)$.

Lemma 4.15 corresponds to Theorem 6.1 (i) of Reny and Perry (2006). This concludes the proof of part (c).

4.4. *Proof of Part (d)*

In this subsection we establish the asymptotic normality of a monotone pure strategy Bayesian Nash equilibrium price. We first show that a monotone pure strategy Bayesian Nash equilibrium exists of the double auction game with a continuum set of bids in the large finite economy.

53. Bayesian-Nash equilibrium of the double auction game in the large finite economy $\mathcal{G}(f,n)$. Let $\beta^*_{\Delta,n}$ be a sequence of monotone pure strategy equilibria in $\mathcal{G}(f,\Delta,n)$. Let β^*_n be a limit of $\{\beta^*_{\Delta,n}\}_{\Delta}$ as $\Delta\to 0$. β^*_n is a strategy profile in the double auction game with a continuum of bids $\mathcal{G}(f,n)$. Due to winner's curse effect, when there are ties in the limit, a buyer with a high signal prefers to increase the bid in the sufficiently large finite economy. Analogous relation holds for a seller. Therefore, the limit strategy profile does not involve a tie among players. Therefore, β^*_n does not involve a tie and is a monotone pure strategy equilibrium of $\mathcal{G}(f,n)$.

LEMMA 4.16 *There exists* $\underline{n}<\infty$ *such that for each* $n>\underline{n}$, *there exists a nontrivial monotone pure strategy equilibrium* β^*_n *in the double auction game in the finite market* $\mathcal{G}(f,n)$.

We now study the limit behavior of β_n^* as $n \to \infty$.

54. Asymptotic outcome equivalence to the fully revealing rational expectation equilibrium in the large economy. Let β^* denote a limit of a sequence of monotone pure strategy equilibria β_n^* in $\{\mathcal{G}(f,n)\}_n$. Since β_n^* does not involve a tie due to winner's curse effect and the outcomes in the finite economies converge to the outcome in the large economy. it is possible to extend Lemma 4.11 to show that β^* is a nontrivial monotone pure strategy equilibrium of $\mathcal{G}(f)$. Furthermore, analysis of the first order conditions show that, similar to Lemma 4.10, β^* is outcome equivalent to the fully revealing rational expectation equilibrium in the large economy.

LEMMA 4.17 *As $n \to \infty$, a nontrivial monotone pure strategy equilibrium β_n^* in the double auction game $\mathcal{G}(f,n)$ is asymptotically outcome equivalent to the fully revealing rational expectation equilibrium identified in Lemma 2.1.*

We now study the asymptotic behavior of $P_n(\beta_n^*) - v(\theta_0,x_i(\theta_0))$.

55. Decomposition of $\sqrt{n}(P_n(\beta_n^*) - v(\theta_0,x_i(\theta_0)))$. Let $P_n(\theta_0)$ be the transaction price where each player uses a bidding strategy in the rational expectation equilibrium $v(\theta(x_i),x_i)$. Since buyers and sellers can bid asymmetrically in the finite market, the transaction price $P_n^{BNE}(\beta_n^*)$ can very well be different from $P_n(\theta_0)$. Then we split the difference as follows:

$$\begin{aligned} & \sqrt{n}(P_n(\beta_n^*) - v(\theta_0, x_i(\theta_0))) \\ = \; & \sqrt{n}\underbrace{(P_n(\theta_0) - v(\theta_0, x_i(\theta_0)))}_{\text{the sample size effect}} + \sqrt{n}\underbrace{(P_n(\beta_n^*) - P_n(\theta_0))}_{\text{the strategic effect}}. \end{aligned}$$

56. The sample size effect. Our analysis is based on a standard result of the asymptotic distribution order statistics of a continuous random variable (David and Nagaraja (2003)):

$$\sqrt{n}(X_{n_S;n} - x_i(\theta_0)) \to_d N\left(0, \frac{\alpha(1-\alpha)}{f_{X_i|\theta}^2(x_i(\theta_0)|\theta_0)}\right)$$

Since $P_n(\theta_0) = v(\theta(X_{n_S;n}),X_{n_S;n})$, $\sqrt{n}(P_n(\theta_0) - v(\theta_0, x_i(\theta_0)))$ can be evaluated by applying the delta method (e.g. van der Vaart (2000)).

57. The strategic effect. We first derive the condition that the strategic effect will be asymptotically negligible. Let

$$\eta_n(x) = P_r(\sqrt{n}(P_n(\beta^*) - v(\theta(x_i), x_0)) \leq x) - P_r(\sqrt{n}(P_n(\theta_0) - v(\theta(x_i), \theta_0)) \leq x).$$

Suppose, hypothetically, that every player use the same strategy. In this case, $\eta_n(x)$

can be evaluated as

$$\eta_n(x) = \frac{1}{\text{Beta}(n_S, n - n_S + 1)} \int_{H_n^*(v(\theta(x_i),\theta_0)+\frac{x}{\sqrt{n}})}^{H_n(v(\theta(x_i),\theta_0)+\frac{x}{\sqrt{n}})} t^{n_S-1}(1-t)^{n-n_S}$$

where H_n is the distribution of $P_n(\theta_0)$, H_n^* is the distribution of $P_n(\beta_n^*)$, and Beta(a,b) denotes a beta function. By evaluating this integral, we can show that it is suffice that the distribution of bidding strategies H_n^{BNE} converges H at a rate faster than $O(1/\sqrt{n})$.

By extending the argument to the case where buyers and sellers use asymmetric strategies, we can see that it is suffice that both $H_{n,b}^*$ and $H_{n,s}^*$ converge at the rate faster than $O(1/\sqrt{n})$.

58. The rate of convergence of $\beta_{n,b}^*$ and $\beta_{n,s}^*$. It now remains to show that $\beta_{n,b}^*$ and $\beta_{n,s}^*$ converge to $v(\theta(x_i),x_i)$ at a rate faster than $O(1/\sqrt{n})$.

We first show that the size of misrepresentation $v(\theta(x_i),x_i) - \beta_{n,i}^*(x_i)$ converges to 0 at the rate of $O(1/n)$. To see this, consider a buyer who bids below $v(\theta(x_i),x_i)$ and increases a bid from b_i by a small amount ε. This change in the payoff is the sum of the following two effects:

- If the bid b_i is on the margin and losing and if there is a bid between b_i and $b_i + \varepsilon$, then the bid $b_i + \varepsilon$ wins.
- If the bid b_i is on the margin and winning, then increasing the bid from b_i to $b_i + \varepsilon$ will increase the payment.

As the number of buyers and sellers increases, the probability that a buyer or a seller have placed a bid in the interval $[b_i,b_i + \varepsilon]$ increases at the rate of $O(n)$. On the other hand, a probability that a buyer wins at bid b_i will not drift as the n increases. That is, as $n \to \infty$, even if a buyer increases a bid just a small amount of ε, the buyer can win the good additionally with a significant probability while the cost from this increased bid is small. The rate by which the buyer can profitably increases a bid is $O(1/n)$, corresponding to the rate of increase in the probability that there will be a buyer and seller in the interval.

Since the same argument holds for the seller, the asymmetry between the buyer and the seller will vanish at the rate of $O(1/n)$ and their equilibrium bidding strategies will approach to the rational expectation equilibrium demand and supply at the rate of $O(1/n)$.

Thus we have,

LEMMA 4.18 *Let* $P_n(\beta_n^*)$ *be a Bayesian-Nash equilibrium price in an equilibrium* β_n^*

in the large finite economy $\mathcal{G}(n)$. Then,

$$\sqrt{n}(P_n(\beta_n^*) - v(\theta_0, x_i(\theta_0)) \\ \to {}_dN(0, \frac{\alpha(1-\alpha)}{f^2_{X_i|\theta}(x_i(\theta_0)|\theta_0)} \\ (\frac{\partial v(\theta_0, x_i(\theta_0))}{\partial \theta}\frac{\partial \theta(x_i(\theta_0))}{\partial x_i} + \frac{\partial v(\theta_0, x_i(\theta_0))}{\partial x_i})^2) \text{ as } n \to \infty.$$

If we hypothetically set the value function $v(\theta_0,x_i) = \theta_0$, the limit distribution in this case will be the same with the limit distribution of the one-sided uniform price auction of Pesendorfer and Swinkels (1997) derived by Hong and Shum (2004), due to the asymptotic equivalence of one-sided uniform price auctions and uniform price double auctions. Lemma 4.18 concludes the proof of part (d).

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SUPPLEMENT TO "A STRATEGIC THEORY OF A MARKET"[1] *Preliminary and Incomplete. Please do not circulate.*

Eiichiro Kazumori

This supplement presents the proof of Proposition 1 in the main paper.

5. INTRODUCTION

In this supplement we proceed as follows. In section 2, We explain the proof of Proposition 1(a). Section 3 contains the proof of Proposition 1(b). Section 4 the proof of Proposition 1(c). Section 5 is devoted to the proof of Proposition 1(d). All symbols, definitions, and assumptions are as given in the text.

6. PROOF OF PROPOSITION 1(A)

6.1. *Lemma 2.1*

LEMMA 4.1. *In* $\mathcal{G}(\gamma, f, \Delta, n, \delta)$ and $\mathcal{G}(\gamma, \text{f}, \Delta, n)$, *for each player i, for each strategy profile of other players* $\beta_{-i,\Delta,n,\delta}$, *the ex ante payoff function* $\pi_{n,i}(\beta_{\Delta,n,\delta,i}, \beta_{\Delta,n,\delta,-i})$ *is continuous in player* $i's$ *strategy* $\beta_{i,\Delta,n}$.

PROOF: Before proving the lemma, we first transform the seller's payoff by a term which is independent of the seller's choice variable[1]. Then we show that the ex ante expected payoff function $\pi_{i,n}(\beta_{i,\Delta,n}, \beta_{-i,\Delta,n,\delta})$ is continuous in $\mathcal{G}(\gamma, f, \Delta, n)$. The proof for $\mathcal{G}(\gamma, f, \Delta, n, \delta)$ is analogous.

60. Definition of a modified seller's payoff function in $\mathcal{G}(\gamma, f, \Delta, n)$. Let $p_n(b_{-i})$ denote the market clearing price calculated hypothetically assuming that every bidder but i submits the bid b_{-i}. For each $i \in N_{\mathcal{X}}$, define a modified interim expected payoff

[1]This device was first introduced by Reny and Perry (2006).

$U'_{i,n}(x_i,b_i,\beta_{\triangle,n,-i})$ by

$$
\begin{aligned}
& U'_{n,i}(x_i, b_i, \beta_{\triangle,n,-i}) \\
= & \int_{[0,1]} \sum_{\underbrace{B_\Delta \times \ldots \times B_\Delta}_{n-1} \times \underbrace{\mathcal{X} \times \ldots \times \mathcal{X}}_{n_S - 1}} \\
& \underbrace{(p_n(b_i, b_{-i}) - v(\theta_0, x_i))}_{\text{the profit when the seller sells the good}} \\
& \underbrace{(1 - q_n(b_i, b_{-i}))}_{\text{the probability that an offer } b_i \text{ sells}} \\
& \underbrace{h_{\Delta,n,-i}(b_{-i}|x_{-i}) f_{\theta,X_{-i}|X_i}(\theta_0, x_{-i}|x_i) d\theta_0}_{\text{the density of the state and the other player's bids given } x_i} \\
& - \int_{[0,1]} \sum_{\underbrace{B_\Delta \times \ldots \times B_\Delta}_{n-1} \times \underbrace{\mathcal{X} \times \ldots \times \mathcal{X}}_{n_S - 1}} \\
& \underbrace{(p_n(b_{-i}) - v(\theta_0, x_i))}_{\text{the adjustment term intended to approximate } p_n(b_i,b_{-i}) - v(\theta_0,x_i)} \\
& h_{\Delta,n,-i}(b_{-i}|x_{-i}) f_{\theta,X_{-i}|X_i}(x_0, x_{-i}|x_i) dx_0.
\end{aligned}
$$

This device was first introduced by Reny and Perry (2006). $U'_{n,i}(x_i,b_i,\beta_{\triangle,n,-i})$ is adjusted by the term which is independent of seller i's choice variable b_i.As the size of the economy n grows large, the payoff function of a seller converges to the payoff of a buyer with the same signal and the bid given other players' strategies.

61. Definition of the modified game. Suppose that, in $\mathcal{G}(\gamma,f,\Delta,n)$**,**the seller's payoff function is replaced from $U_{n,i}$ to $U'_{n,i}$. Then, for each seller i, a bid b_i is a best response to $\beta_{-i,\triangle,n}$ in a game where the seller's payoff function is $U_{n,i}(x_i,b_i,\beta_{\triangle,n,-i})$ if and only if it will be so in a game where the seller's payoff function is $U'_{n,i}(x_i,b_i,\beta_{\triangle,n,-i})$. Therefore, these two games are strategically equivalent. Therefore, it is without loss of generality to consider the game where the seller's payoff is $U'_{i,n}(x_i,b_i,\beta_{-i,\triangle,n,\delta})$. Hereafter, to simplify the notation, we denote $U_{i,n}(x_i,b_i,\beta_{-i,\triangle,n})$ by $U'_{i,n}(x_i,b_i,\beta_{-i,\triangle,n})$.

We now show that the ex ante payoff function $\pi_{i,n,\delta}(\beta_{i,\Delta,n},\beta_{-i,\Delta,n,\delta})$ of $\mathcal{G}(\gamma,f,\Delta,n,\delta)$ is jointly continuous in players' strategies.

62. Expansion of $\pi_{i,n,\delta}(\beta_{i,\Delta,n},\beta_{-i,\Delta,n,\delta,m})$. We note

$$\begin{aligned} & \pi_{i,n,\delta}(\beta_{i,\Delta,n,\delta,m},\beta_{-i,\Delta,n,\delta,m}) \\ = & \sum_{B_\Delta\times\mathcal{X}} U_{i,n,\delta}(x_i,b_i,\beta_{-i,\Delta,n,\delta,m})h_{i,\Delta,n,\delta,m}(b_i,x_i) \\ = & \underbrace{\delta}_{\text{this is the probability that there is a nonstrategic bid}} \\ & \sum_{B_\Delta\times\mathcal{X}}[(\int_{[0,1]}\sum_{\underbrace{B_\Delta\times ...\times B_\Delta}_{n-1}\times\underbrace{\mathcal{X}...\times\mathcal{X}}_{n-1}} \\ & \underbrace{u_i(x_0,x_i,p_n(b_i,b_{-i},b_{\widehat{i}}),q_n(b_i,b_{-i},b_{\widehat{i}}))}_{\text{the ex post expected payoff with a nonstrategic bid}} \\ & \underbrace{h_{-i,\Delta,n,\delta,m}(b_{-i}|x_{-i})f_{\theta,X_{-i}|X_i}(x_0,x_{-i}|x_i)}_{\text{the conditional density of the state and the other players' bid given the signal of player } i} \\ & \cdot \underbrace{h_{\widehat{i}}(b_{\widehat{i}})}_{\text{the density of a nonstrategic bid}}] \\ & \underbrace{h_{i,\Delta,n,\delta,m}(b_i|x_i)f_{X_i}(x_i)}_{\text{the density of a player's bid and a signal}} \\ & + \underbrace{(1-\delta)}_{\text{this is th e case where there is no nonstrategic bid}} \\ & \sum_{B_\Delta\times\mathcal{X}}[\int_{[0,1]}\sum_{\underbrace{B_\Delta\times ...\times B_\Delta}_{n-1}\times\underbrace{\mathcal{X}...\times\mathcal{X}}_{n-1}} \\ & \underbrace{u_i(x_0,x_i,p(b_i,b_{-i}),q(b_i,b_{-i}))}_{\text{the ex post payoff without a nonstrategic bid}} \\ & \underbrace{h_{-i,\Delta,n,\delta,m}(b_{-i}|x_{-i})f_{\theta,X_{-i}|X_i}(x_0,x_{-i}|x_i)}_{\text{the conditional density of the state and the other players' bid given the signal } x_i}] \\ & h_{i,\Delta,n,\delta,m}(b_i|x_i)f_{X_i}(x_i). \end{aligned} \tag{1}$$

That is, a player's ex ante payoff $\pi_{i,n,\delta}(\beta_{i,\Delta,n,\delta,m},\beta_{-i,\Delta,n,\delta,m})$ is the sum of payoffs when there is a nonstrategic bidder and when there is no strategic bidder. When there is a nonstrategic bidder, the expected payoff is obtained by taking expectations in terms of other players' signal and bids, the state, and the behavior of a nonstrategic bidder given the player's signal.

63. Convergence of strategies in $\mathcal{G}(\gamma,f,\Delta,n,\delta)$. We note, in $\mathcal{G}(\gamma,f,\Delta,n,\delta)$, both the set of possible signals and the set of possible bids are finite in $\mathcal{G}(\gamma,f,\Delta,n,\delta)$, a distributional strategy $\beta_{i,\Delta,n,\delta}$ is defined by a $(\bar{b}/\Delta+1)\cdot(1/\gamma+1)$ dimensional finite dimensional vector $\{h_{i,\Delta,n,\delta}(b_i|x_i)\}_{b_i\in B_\Delta,x_i\in\mathcal{X}}$. Therefore, we can define a topology on a set of strategies in a standard way. That is, a sequence of distributional strategies

$\beta_{i,\Delta,n,\delta,m}, m = 1,2,\ldots,$converges to $\beta_{i,\Delta,n,\delta}$ as $m \to \infty$, if and only if, for each b_i and x_i,

$$\lim_{m' \to \infty} h_{i,\Delta,n,\delta,m'}(b_i|x_i) = h_{i,\Delta,n,\delta}(b_i|x_i). \tag{2}$$

That is, for each signal x_i, for every bid b_i, the probability that a player uses a bid b_i given signal x_i converges.

64. Joint continuity of $\pi_{i,n,\delta}(\beta_{i,\Delta,n,\delta},\beta_{-i,\Delta,n,\delta})$ in $\mathcal{G}(\gamma,f,\Delta,n,\delta)$. Suppose that By substituting (2) into (1), we obtain

$$\lim_{m \to \infty} \pi_{i,n,\delta}(\beta_{i,\Delta,n,\delta,m}, \beta_{-i,\Delta,n,\delta,m}) = \pi_{i,n,\delta}(\beta_{i,\Delta,n,\delta}, \beta_{-i,\Delta,n,\delta}). \tag{3}$$

6.2. *Lemma 2.2*

LEMMA. *There exists* $\overline{\Delta} > 0$ *such that for every* $0 < \Delta < \overline{\Delta}$ and $\delta > 0$, *there exists a nontrivial Bayesian-Nash equilibrium* $\beta^*_{\Delta,n,\delta}$ *of* $\mathcal{G}(\gamma,f,\Delta,n,\delta)$.

PROOF: We first show existence of a Bayesian Nash equilibrium and then show that this equilibrium is nontrivial.

65. Definition of a best response correspondence. Recall the definition of $\Phi_{\Delta,n,\delta} : \mathcal{B}_\Delta \times \mathcal{B}_\Delta \to \mathcal{B}_\Delta \times \mathcal{B}_\Delta$ from the text:

$$\Phi_{\Delta,n,\delta}(\beta_{\Delta,n,\delta,b}, \beta_{\Delta,n,\delta,s}) = \begin{pmatrix} \arg\max_{\beta'_b \in \mathcal{B}_\Delta} \pi_{n,\Delta,\delta,b}(\beta'_b, \underbrace{\beta_{\Delta,n,\delta,b}, \ldots, \beta_{\Delta,n,\delta,b}}_{n_B - 1 \text{ buyers}}, \underbrace{\beta_{\Delta,n,\delta,s}, \ldots, \beta_{\Delta,n,\delta,s}}_{n_S \text{ sellers}}) \\ \arg\max_{\beta'_s \in \mathcal{B}_\Delta} \pi_{n,\Delta,\delta,s}(\underbrace{\beta_{\Delta,n,\delta,b}, \ldots \beta_{\Delta,n,\delta,b}}_{n_B \text{ buyers}}, \beta'_s, \underbrace{\beta_{\Delta,n,\delta,s}, \ldots, \beta_{\Delta,n,\delta,s}}_{n_S - 1 \text{ sellers}}) \end{pmatrix}.$$

That is, $\Phi_{\Delta,n,\delta}$ returns a best response of a buyer when all other buyers use $\beta_{\Delta,n,\delta,b}$ and every seller uses $\beta_{\Delta,n,\delta,s}$ and a best response of a seller when every buyer uses $\beta_{\Delta,n,\delta,b}$ and all other sellers use $\beta_{\Delta,n,\delta,s}$.

66. Existence of a fixed point of $\Phi_{\Delta,n,\delta}$. We will show existence of a fixed point of $\Phi_{\Delta,n,\delta}$ by showing that $\Phi_{\Delta,n,\delta}$ satisfies the conditions for Kakutani fixed point theorem.

- Nonemptiness. Follows from continuity of $\pi_{\Delta,n,\delta}$, established in Lemma 4.1.
- Closed graph. It follows from continuity of $\pi_{\Delta,n,\delta}$ and the Maximum theorem.
- Convex-valued. It follows from (1) that $\pi_{n,\Delta,\delta}$ is a linear function of $h_{i,\Delta,n,\delta}(b_i|x_i)$. Therefore, $\Phi_{\Delta,n,\delta}$ is convex-valued.

Since $\mathcal{B}_\Delta$ is a compact, convex subset of $((1/\gamma)+1)\cdot((\overline{b}/\Delta)+1)$ dimensional Euclidean space, and since $\Phi_{\Delta,n,\delta}$ is nonempty, has a closed graph, and is convex-valued, then it follows from Kakutani's fixed point theorem that there exists a fixed point $\beta^*_{\Delta,n,\delta}$ of $\Phi_{\Delta,n,\delta}$.

It remains to show that $\beta^*_{\Delta,n,\delta}$ is nontrivial.

67. Assumption of triviality. Contrary suppose that $\beta^*_{\Delta,n,\delta}$ is trivial. Then, by definition in the text, the probability of trade between buyers and sellers is zero. By the definition of trading rule, almost surely, the highest buy bid in the support of the buyer's strategy is strictly less than the lowest sell bid in the support of the seller's strategy. It is because, otherwise, a buyer's buy bid would be at least equal to a seller's sell offer with a positive probability, and it would lead to a trade with a positive probability.

68. Conditions for the support of seller's trading strategy. We now claim that, in order for $\beta^*_{\Delta,n,\delta}$ to be trivial, it has to be that every sell offer in the support of the seller's bidding strategy, including the one of a nonstrategic seller, is at least $\overline{v} = v(1,1)$.

Suppose not and that the seller will choose an offer strictly less than $\overline{v}$ with some positive probability. Then, with the positive grid size $\Delta > 0$, a seller's offer will be equal or less than $\overline{v} - \Delta$ with some positive probability. In order for this equilibrium to be trivial, it has to be that all buy bids have to be strictly less than $\overline{v} - \Delta$ almost surely.

We now show that then there will be some buyers who wants to trade with these offers. By Assumption 5 (c), when $x_i = 1$, $v(x_0,1) = \overline{v}$ for every x_0. By Assumption 4 (c), for every x_0, the probability that $x_i = 1$ conditional on x_0 is positive. Suppose that a buyer with $x_i = 1$ bids $\overline{v} - \Delta$ instead of an equilibrium strategy which will ensure the triviality. Then, this bid $\overline{v} - \Delta$ wins with some positive probability, since the sell offer is equal or less than $\overline{v} - \Delta$ with positive probability and all buy bids are less than $\overline{v} - \Delta$ almost surely. When the buyer trades, from the k-double auction pricing rule, the transaction price is equal or less than $\overline{v} - \Delta$. Therefore, the buyer gets a positive expected payoff for every $x_0 \in [0,1]$. Since the buyer's payoff at any trivial equilibrium is zero, the buyer prefers to deviate. It is a contradiction. Thus it follows that every sell offer in the support of the seller's bidding strategy is at least $\overline{v}$.

69. Derivation of contradiction. But in $\mathcal{G}(\gamma,f,\Delta,n,\delta)$, there is a positive probability of a sell offer less than $v(1,1)$ because of a nonstrategic bidder. It follows that $\beta^*_{\Delta,n,\delta}$ cannot be trivial.

6.3. *Lemma 2.3*

LEMMA. (a). *$\beta^*_{\Delta,n}$ is a Bayesian equilibrium of $\mathcal{G}(\gamma,f,\Delta,n)$*; (b).

$$\lim_{\delta\to 0} \tau_{\Delta,n,\delta}(\beta^*_{\Delta,n,\delta}|x_0) = \tau_{\Delta,n}(\beta^*_{\Delta,n}|x_0). \tag{4}$$

PROOF: We proceed in three steps. We first prove the statement (a). Then we derive a formula for $\tau_{\Delta,n,\delta}(\beta^*_{,\Delta,n,\delta}|x_0)$ and show it converges to $\tau_{\Delta,n}(\beta^*_{\Delta,n}|x_0)$ as $\delta \to 0$.

70. Equilibrium conditions for $\beta^*_{\Delta,n}$. By definition, $\beta^*_{\Delta,n}$ is a Bayesian equilibrium if, for every player i, every possible strategies $\beta_{i,\Delta,n}$,

$$\pi_{n,i}(\beta^*_{\Delta,n,i},\beta^*_{\Delta,n,-i}) \geq \pi_{n,i}(\beta_{\Delta,n,i},\beta^*_{\Delta,n,-i}) \tag{5}$$

Let $\{\beta_{\Delta,n,\delta}\}_\delta$ denote a sequence of distributional strategies such that $\beta_{\Delta,n,\delta} \to \beta_{\Delta,n}$. Since $\beta^*_{\Delta,n,\delta}$ is an equilibrium,

$$\pi_{n,\delta,i}(\beta^*_{\Delta,n,\delta,i},\beta^*_{\Delta,n,\delta,-i}) \geq \pi_{n,\delta,i}(\beta_{\Delta,n,\delta,i},\beta^*_{\Delta,n,\delta,-i}). \tag{6}$$

Therefore, in order to establish (5), it is suffice to show that

$$\lim_{\delta\to 0} \pi_{n,\delta,i}(\beta^*_{\Delta,n,\delta,i},\beta^*_{\Delta,n,\delta,-i}) = \pi_{n,i}(\beta^*_{\Delta,n,i},\beta^*_{\Delta,n,-i}) \tag{7}$$

and

$$\lim_{\delta\to 0} \pi_{n,\delta,i}(\beta_{\Delta,n,\delta,i},\beta^*_{\Delta,n,\delta,-i}) = \pi_{n,i}(\beta_{\Delta,n,i},\beta^*_{\Delta,n,-i}) \tag{8}$$

That is, it is suffice to show that the expected payoffs in $\mathcal{G}(\gamma,f,\Delta,n,\delta)$ converge to the expected payoff in $\mathcal{G}(\gamma,f,\Delta,n)$.

We now show (8). An argument for (7) is similar.

71. Decomposition of $\pi_{n,\delta,i}(\beta_{\Delta,n,\delta,i},\beta^*_{\Delta,n,\delta,-i}) - \pi_{n,i}(\beta_{\Delta,n,i},\beta^*_{\Delta,n,-i})$. We note

$$\begin{aligned} & \pi_{n,\delta,i}(\beta_{\Delta,n,\delta,i},\beta^*_{\Delta,n,\delta,-i}) - \pi_{n,i}(\beta_{\Delta,n,i},\beta^*_{\Delta,n,-i}) \\ = & \underbrace{\left[\pi_{n,\delta,i}(\beta_{\Delta,n,\delta,i},\beta^*_{\Delta,n,\delta,-i}) - \pi_{n,i}(\beta_{\Delta,n,\delta,i},\beta^*_{\Delta,n,\delta,-i})\right]}_{\substack{\text{change in the probability of participation by a nonstrategic player} \\ \text{while strategies fixed}}} \\ & + \underbrace{\left[\pi_{n,i}(\beta_{\Delta,n,\delta,i},\beta^*_{\Delta,n,\delta,-i}) - \pi_{n,i}(\beta_{\Delta,n,i},\beta^*_{\Delta,n,-i})\right]}_{\substack{\text{change in player's strategies} \\ \text{without taking into account of a nonstrategic player}}} \end{aligned} \tag{9}$$

That is, the change in expected payoff can be decomposed into the change caused by a change in the probability of participation by a nonstrategic bidder and the effect caused by a change in players' strategies without taking into account of a change in the probability of a nonstrategic bidding.

We now evaluate these two terms one by one.

72. Convergence of the first term of (9). We note that, for the first term of (1),

$$\begin{aligned} & \delta \sum_{B_\Delta\times\mathcal{X}} \Big[\int_{[0,1]} \Big(\sum_{\underbrace{B_\Delta\times\ldots\times B_\Delta}_{n-1}\times\underbrace{\mathcal{X}\ldots\times\mathcal{X}}_{n-1}} u_i(x_0,x_i,p(b_i,b_{-i},b_{\hat{i}}),q(b_i,b_{-i},b_{\hat{i}})) \\ & h_{\Delta,n,\delta,-i}(b_{-i}|x_{-i}) f_{\theta,X_{-i}|X_i}(x_0,x_{-i}|x_i) h_{\hat{i}}(b_{\hat{i}})\Big] h_{\Delta,n,\delta,i}(b_i|x_i) f_{X_i}(x_i) \\ \to\ & 0 \text{ as } \delta\to 0 \end{aligned}$$

Thus, from similar calculation for the second term of (1) that $\pi_{n,\delta,i}(\beta_{\Delta,n,\delta,i},\beta^*_{\Delta,n,\delta,-i}) - \pi_{n,i}(\beta_{\Delta,n,\delta,i},\beta^*_{\Delta,n,\delta,-i}) \to 0$

73. Convergence of the second term of (9). It follows from (1) that

$$\begin{aligned}
&\pi_{n,i}(\beta_{i,\Delta,n,\delta},\beta_{-i,\Delta,n,\delta})-\pi_{n,i}(\beta_{i,\Delta,n},\beta_{-i,\Delta,n}) \qquad (10)\\
=&\sum_{B_\Delta\times\mathcal{X}}[\int_{[0,1]}\sum_{\underbrace{B_\Delta\times...\times B_\Delta}_{n-1}\times\underbrace{\mathcal{X}...\times\mathcal{X}}_{n-1}} u_i(x_0,x_i,p(b_i,b_{-i}),q(b_i,b_{-i}))\\
&h_{\Delta,n,\delta,-i}(b_{-i}|x_{-i})f_{X_{-i}|X_i}(x_{-i}|x_i)]h_{\Delta,n,\delta,i}(b_i|x_i)f_{X_i}(x_i)\\
&-\sum_{B_\Delta\times\mathcal{X}}[\int_{[0,1]}\sum_{\underbrace{B_\Delta\times...\times B_\Delta}_{n-1}\times\underbrace{\mathcal{X}...\times\mathcal{X}}_{n-1}} u_i(x_0,x_i,p(b_i,b_{-i}),q(b_i,b_{-i}))\\
&h_{\Delta,n,-i}(b_{-i}|x_{-i})f_{X_{-i}|X_i}(x_{-i}|x_i)]h_{\Delta,n,i}(b_i|x_i)f_{X_i}(x_i).
\end{aligned}$$

Then, we have

$$\lim_{\delta\to 0}\pi_{i,n}(\beta_{i,\Delta,n,\delta},\beta_{-i,\Delta,n,\delta})=\pi_{i,n}(\beta_{i,\Delta,n},\beta_{-i,\Delta,n}). \qquad (11)$$

75. Conclusion of the proof. It follows from (9) and (11), that (8) holds. Similarly, (7) holds. Therefore, (5) holds.

Having established convergence of expected payoffs, we now show that the probability that an equilibrium is nontrivial converges from $\mathcal{G}(\gamma,f,\Delta,n,\delta)$ to $\mathcal{G}(\gamma,f,\Delta,n)$. For that purpose, we first derive the probability that an equilibrium is nontrivial at $\mathcal{G}(\gamma,f,\Delta,n,\delta)$.

76. Calculation of the probability that an equilibrium is nontrivial. Let $\beta_{\Delta,n,\delta}$ be a strategy profile of $\mathcal{G}(\gamma,f,\Delta,n,\delta)$. We note that

$$\begin{aligned}
&\text{an outcome is nontrivial}\\
\Longleftrightarrow\ &\text{the highest buy bid is equal or higher than the lowest sell offers}
\end{aligned}$$

Thus,

$$\begin{aligned}
&\Pr(\text{an outcome is nontrivial})\\
=\ &\Pr(\text{the highest buy bid is equal or higher than the lowest sell offer})
\end{aligned}$$

Thus we need to calculate this probability.

78. Decomposition into cases depending on the behavior of a nonstrategic player. The probability that the highest buy bid is equal or higher than the lowest sell offer can be calculated by conditioning on the behavior of a nonstrategic player.

- (a) the nonstrategic player $\widehat{i}$ does not participate
- (b) $\widehat{i}$ participates as a buyer
- (c) $\widehat{i}$ participates as a seller.

That is, the probability that a strategic profile is nontrivial is

$$\begin{aligned}&\tau_{\Delta,n,\delta}(\beta_{\Delta,n,\delta}|x_0) \\ =\;& (1-\delta)\tau_{\Delta,n,\delta}(\beta_{\Delta,n,\delta}|x_0, \widehat{i} \text{ is not active}) \\ &+\frac{\delta}{2}\tau_{\Delta,n,\delta}(\beta_{\Delta,n,\delta}|x_0, \widehat{i} \text{ participates as a buyer}) \\ &+\frac{\delta}{2}\tau_{\Delta,n,\delta}(\beta_{\Delta,n,\delta}|x_0, \widehat{i} \text{ participates as a seller}).\end{aligned} \tag{12}$$

We now calculate the first term.

79. The case when $\widehat{i}$ is nonactive. Let $b_{1:n_B}$ be a random variable which indicates the highest buy bid. Let $b_{n_S:n_S}$ be a random variable which indicates the lowest sell bid. Then trade takes place if and only if $b_{1:n_B} \geq b_{n_S:n_S}$. Thus we next calculate these probability distribution function of $b_{1:n_B}$ and $b_{n_S:n_S}$.

80. The probability distribution function of $b_{1:n_B}$. For each x_i, each buyer chooses a bid according to a behavioral strategy $h_{\Delta,n,\delta,b}(b_i|x_i)$. Consequently, a probability that a buyer chooses a bid b_i in state x_0 is

$$\sum_{x_i\in\mathcal{X}_\gamma} h_{\Delta,n,\delta,b}(b_i|x_i) f_{X_i|\theta}(x_i|x_0). \tag{13}$$

Thus the probability that a buyer will choose a bid equal or less than b_i is

$$\sum_{b_i'\leq b_i}\sum_{x_i\in\mathcal{X}} h_{\Delta,n,\delta,b}(b_i'|x_i) f_{X_i|\theta}(x_i|x_0). \tag{14}$$

Since players' signals are independently distributed conditional on x_0, the conditional probability distribution function of $b_{1:n_B}$ is, from (14),

$$\Pr\left(b_{1:n_B}\leq b_i|x_0\right) = \left(\sum_{b_i'\leq b_i}\sum_{x_i\in\mathcal{X}} h_{\Delta,n,\delta,b}(b_i'|x_i) f_{X_i|\theta}(x_i|x_0)\right)^{n_B}. \tag{15}$$

That is, the probability that the maximum buy bid is less than b_i conditional on x_0 is given by the probability that every buy bid is less than b_0 conditional on x_0.

81. The probability mass function of $b_{1:n_B}$. It follows from (15) that the probability that $b_{1:n_B}$ takes a value of b_i conditional on x_0 is

$$\begin{aligned}&\Pr(b_{1:n_B}=b_i|x_0) \\ =\;& \Pr\left(b_{1:n_B}\leq b_i|x_0\right) - \Pr\left(b_{1:n_B}\leq b_i-\Delta|x_0\right) \\ =\;& \left(\sum_{b_i'\leq b_i}\sum_{x_i\in\mathcal{X}} h_{\Delta,n,\delta,b}(b_i'|x_i) f_{X_i|\theta}(x_i|x_0)\right)^{n_B} \\ &-\left(\sum_{b_i'\leq b_i-\Delta}\sum_{x_i\in\mathcal{X}} h_{\Delta,n,\delta,b}(b_i'|x_i) f_{X_i|\theta}(x_i|x_0)\right)^{n_B}.\end{aligned} \tag{16}$$

82. The probability distribution function of $b_{n_S:n_S}$. Similarly, the probability that a seller chooses a bid b_i in state x_0 is $\sum_{x_i \in \mathcal{X}_\gamma} h_{\Delta,n,\delta,s}(b_i|x_i) f_{X_i|\theta}(x_i|x_0)$ and the probability that seller i will choose a bid equal or less than b_i is

$$\sum_{b_i' \leq b_i} \sum_{x_i \in \mathcal{X}} h_{s,\Delta,n,\delta}(b_i'|x_i) f_{X_i|\theta}(x_i|x_0). \tag{17}$$

Since a seller's signal is independently distributed conditional on x_0, the probability that the minimum sell bid $b_{n_S:n_S}$ is less than b_i is given by, from (17),

$$\begin{aligned} & \Pr\left(b_{n_S:n_S} \leq b_i|x_0\right) \\ = \; & 1 - \left(1 - \sum_{b_i' \leq b_i} \sum_{x_i \in \mathcal{X}} h_{s,\Delta,n,\delta}(b_i'|x_i) f_{X_i|\theta}(x_i|x_0)\right)^{n_S}. \end{aligned} \tag{18}$$

That is, the event that a minimum bid is less than b_i is a complement of the event that there is a bid above b_i.

83. The probability mass function of $b_{n_S:n_S}$. It follows from (18) that the probability mass function of $b_{n_S:n_S}$ is

$$\begin{aligned} & \Pr(b_{n_S:n_S} = b_i|x_0) \\ = \; & \Pr\left(b_{n_S:n_S} \leq b_i|x_0\right) - \Pr\left(b_{n_S:n_S} \leq b_i - \Delta|x_0\right) \\ = \; & \left[1 - \left(1 - \sum_{b_i' \leq b_i} \sum_{x_i \in \mathcal{X}} h_{s,\Delta,n,\delta}(b_i'|x_i) f_{X_i|\theta}(x_i|x_0)\right)^{n_S}\right] \\ & - \left[1 - \left(1 - \sum_{b_i' \leq b_i - \Delta} \sum_{x_i \in \mathcal{X}} h_{s,\Delta,n,\delta}(b_i'|x_i) f_{X_i|\theta}(x_i|x_0)\right)^{n_S}\right] \\ = \; & \left(1 - \sum_{b_i' \leq b_i - \Delta} \sum_{x_i \in \mathcal{X}} h_{s,\Delta,n,\delta}(b_i'|x_i) f_{X_i|\theta}(x_i|x_0)\right)^{n_S} \\ & - \left(1 - \sum_{b_i' \leq b_i} \sum_{x_i \in \mathcal{X}} h_{s,\Delta,n,\delta}(b_i'|x_i) f_{X_i|\theta}(x_i|x_0)\right)^{n_S}. \end{aligned} \tag{19}$$

84. Calculation of the probability of trade. For each value of the minimum sell bid $h_{n_S:n_S} = b_i$, $0 \leq b_i \leq \bar{b}$, trade takes place if the maximum buy bid $b_{1:n_B} \geq b_i$. Since the minimum sell bids $h_{n_S:n_S}$ and the maximum buy bids $b_{1:n_B}$ are distributed independently conditional on x_0, the probability that trade takes place given $h_{n_S:n_S} = b_i$ is

$$\sum_{b_i' \geq b_i} \Pr(b_{1:n_B} = b_i'|x_0) \Pr(b_{n_S:n_S} = b_i|x_0). \tag{20}$$

By summing up, the probability that trade takes place conditional on the state x_0 and that $\widehat{i}$ being not active is, from (20),

$$\begin{aligned} & \tau_{\Delta,n,\delta}(\beta_{\Delta,n,\delta}|x_0, \widehat{i} \text{ is not active}) \\ = & \sum_{b_i \in B_\Delta} \sum_{b_i' \geq b_i} \Pr(b_{1:n_B} = b_i'|x_0) \Pr(b_{n_S:n_S} = b_i|x_0). \end{aligned} \tag{21}$$

We now consider the second case of (12).

85. The case where $\widehat{i}$ participates in the auction as a buyer. Let $b_{1:|N_{n,B}\cup\widehat{i}|}$ be a random variable which indicates the highest buy bid among the bids by $N_{n,B} \cup \widehat{i}$. Then trade takes place if and only if $b_{1:|N_{n,B}\cup\widehat{i}|} \geq b_{n_S:n_S}$. Since we already calculated the probability distribution of $b_{n_S:n_S}$, we need to understand the probability distribution of $b_{1:|N_{n,B}\cup\widehat{i}|}$.

86. Calculation of the probability distribution function of $b_{1:|N_{n,B}\cup\widehat{i}|}$. Since bids by buyers in $N_{n,B}$ and a bid by a nonstrategic buyer are independently distributed conditional on x_0, the probability distribution function of the maximum bid among bids by $i \in N_{n,B} \cup \widehat{i}$ is

$$\begin{aligned} & \Pr\left(b_{1:|N_{n,B}\cup\widehat{i}|} \leq b_i|x_0\right) \\ = & \left(\sum_{b_i' \leq b_i} \sum_{x_i \in \mathcal{X}} h_{b,\Delta,n,\delta}(b_i'|x_i) f_{X_i|\theta}(x_i|x_0)\right)^{n_B} \left(\sum_{b_i' \leq b_i} h_{\widehat{i}}(b_i')\right). \end{aligned} \tag{22}$$

87. The probability mass function of $b_{1:|N_{n,B}\cup\widehat{i}|}$. Thus, the probability mass function of $b_{1:n_B}$ conditional on x_0 is

$$\begin{aligned} & \Pr(b_{1:|N_{n,B}\cup\widehat{i}|} = b_i|x_0) \\ = & \left(\sum_{b_i' \leq b_i} \sum_{x_i \in \mathcal{X}} h_{b,\Delta,n,\delta}(b_i'|x_i) f_{X_i|\theta}(x_i|x_0) dx_i\right)^{n_B} \left(\sum_{b_i' \leq b_i} h_{\widehat{i}}(b_i')\right) \\ & - \left(\sum_{b_i' \leq b_i - \Delta} \sum_{x_i \in \mathcal{X}} h_{b,\Delta,n,\delta}(b_i'|x_i) f_{X_i|\theta}(x_i|x_0) dx_i\right)^{n_B} \left(\sum_{b_i' \leq b_i - \Delta} h_{\widehat{i}}(b_i')\right). \end{aligned} \tag{23}$$

88. The probability that $\beta_{\Delta,n,\delta}$ is nontrivial when $\widehat{i}$ participates in the auction as a buyer. Similarly to (21),

$$\begin{aligned} & \tau_{\Delta,n,\delta}(\beta_{\Delta,n,\delta}|x_0, \widehat{i} \text{ is a buyer}) \\ = & \sum_{b_i \in B_\Delta} \sum_{b_i' \geq b_i} \Pr(b_{i:|N_{n,B}\cup\widehat{i}|} = b_i'|x_0) \Pr(b_{n_S:n_S} = b_i|x_0). \end{aligned} \tag{24}$$

We now consider the third case of (12).

89. The case where $\widehat{i}$ is a seller. Let $b_{|N_{n,S}\cup\widehat{i}|:|N_{n,S}\cup\widehat{i}|}$ be a random variable which indicates the lowest sell bid among sell bids by $i \in N_{n,S} \cup \widehat{i}$. Then trade takes place if

$b_{1:n_B} \geq b_{|N_{n,S}\cup\hat{i}|:|N_{n,S}\cup\hat{i}|}$.

90. Calculation of the probability mass function of $b_{|N_{n,S}\cup\hat{i}|:|N_{n,S}\cup\hat{i}|}$. Following a similar reasoning in the case $\hat{i}$ is a buyer, the probability that the maximum bid among bids by $i \in N_{n,S} \cup \hat{i}$ is less than b_i is given by

$$\begin{aligned} &\Pr\left(b_{n_S\cup\hat{i}:n_S\cup\hat{i}} \leq b_i|x_0\right) \\ = \;& 1 - \left(1 - \sum_{b_i'\leq b_i}\sum_{x_i\in\mathcal{X}} h_{s,\Delta,n,\delta}(b_i'|x_i) f_{X_i|\theta}(x_i|x_0)dx_i\right)^{n_S}\left(1 - \sum_{b_i'\leq b_i} h_{\hat{i}}(b_i')\right). \end{aligned} \tag{25}$$

Thus, the probability mass function of $b_{n_S\cup\hat{i}:n_S\cup\hat{i}}$ conditional on x_0 is

$$\begin{aligned} &\Pr(b_{|N_{n,S}\cup\hat{i}|:|N_{n,S}\cup\hat{i}|} = b_i|x_0) \\ = \;& \left(1 - \sum_{b_i'\leq b_i-\Delta}\sum_{x_i\in\mathcal{X}} h_{s,\Delta,n,\delta}(b_i'|x_i) f_{X_i|\theta}(x_i|x_0)dx_i\right)^{n_S}\left(1 - \sum_{b_i'\leq b_i-\Delta} h_{\hat{i}}(b_i')\right) \\ & - \left(1 - \sum_{b_i'\leq b_i}\sum_{x_i\in\mathcal{X}} h_{s,\Delta,n,\delta}(b_i'|x_i) f_{X_i|\theta}(x_i|x_0)dx_i\right)^{n_S}\left(1 - \sum_{b_i'\leq b_i} h_{\hat{i}}(b_i')\right) \end{aligned} \tag{26}$$

91. Calculation of the probability of nontriviality when $\hat{i}$ is a seller. Similar to (21),

$$\begin{aligned} &\tau_{\Delta,n,\delta}(\beta_{\Delta,n,\delta}|x_0, \hat{i} \text{ is a seller}) \\ = \;& \sum_{b_i\in B_\Delta-\{\underline{b}\}}\sum_{b_i'\geq b_i} \Pr(b_{i:n_B} = b_i'|x_0)\Pr(b_{n_S\cup\hat{i}:n_S\cup\hat{i}} = b_i|x_0). \end{aligned} \tag{27}$$

The trading probability $\tau_{\Delta,n,\delta}(\beta_{\Delta,n,\delta}|x_0)$ is obtained from substituting (21), (24), and (27) into (12). We now proceed to show Part (b) of the lemma. We first decompose the change into two terms:

92. Decomposition of $\tau_{\Delta,n,\delta}(\beta^*_{\Delta,n,\delta}|x_0) - \tau_{\Delta,n}(\beta^*_{\Delta,n}|x_0)$. We now decompose the change in trading probabilities as follows:

$$\begin{aligned} &\tau_{\Delta,n,\delta}(\beta^*_{\Delta,n,\delta}|x_0) - \tau_{\Delta,n}(\beta^*_{\Delta,n}|x_0) \\ = \;& \left[\tau_{\Delta,n,\delta}(\beta^*_{\Delta,n,\delta}|x_0) - \tau_{\Delta,n,\delta}(\beta^*_{\Delta,n}|x_0)\right] \\ & + \left[\tau_{\Delta,n,\delta}(\beta^*_{\Delta,n}|x_0) - \tau_{\Delta,n}(\beta^*_{\Delta,n}|x_0)\right]. \end{aligned} \tag{28}$$

That is, the change in the probabilities that an equilibrium is nontrivial is decomposed into the effect of changes in $\beta^*_{\Delta,n,\delta}$ while keeping the probability of nonstrategic bidding constant and the effect of changes in δ while keeping the strategy constant.

We now deal with these two cases one by one.

93. Convergence of $\tau_{\Delta,n,\delta}(\beta^*_{\Delta,n,\delta}|x_0) - \tau_{\Delta,n,\delta}(\beta^*_{\Delta,n}|x_0)$. We first consider the case that $\hat{i}$ is not active and show that, as $\beta^*_{\Delta,n,\delta} \to \beta^*_{\Delta,n}$

$$\tau_{\Delta,n,\delta}(\beta^*_{\Delta,n,\delta}|x_0, \hat{i} \text{ not active}) \to \tau_{\Delta,n,\delta}(\beta^*_{\Delta,n}|x_0, \hat{i} \text{ not active}) \tag{29}$$

In view of (21), it is suffice to show $\Pr(b_{1:n_B} = b_i|x_0)$ and $\Pr(b_{n_S:n_S} = b_i|x_0)$ calculated when players follow $\beta^*_{\Delta,n,\delta}$ converges to $\Pr(b_{1:n_B} = b_i|x_0)$ and $\Pr(b_{n_S:n_S} = b_i|x_0)$ calculated when players follow $\beta^*_{\Delta,n}$ for each $b_i \in B_\Delta$. For $\Pr(b_{1:n_B} = b_i|x_0)$, we recall, from (16),

$$\begin{aligned} &\Pr(b_{1:n_B} = b_i|x_0) \\ = &\left(\sum_{b_i' \le b_i} \sum_{x_i \in \mathcal{X}} h_{\Delta,n,\delta,b}(b_i|x_i) f_{X_i|\theta}(x_i|x_0)\right)^{n_B} \\ &- \left(\sum_{b_i' \le b_i - \Delta} \sum_{x_i \in \mathcal{X}} h_{\Delta,n,\delta,b}(b_i|x_i) f_{X_i|\theta}(x_i|x_0) dx_i\right)^{n_B} \end{aligned} \tag{30}$$

Then, $\Pr(b_{1:n_B} = b_i|x_0)$ under $\beta^*_{\Delta,n,\delta}$ converges to $\Pr(b_{1:n_B} = b_i|x_0)$ under $\beta^*_{\Delta,n}$. Similarly, $\Pr(b_{n_S:n_S} = b_i|x_0)$ converges as $\beta^*_{\Delta,n,\delta}$ to $\beta^*_{\Delta,n}$. Thus (29) holds. Following similar reasoning, it is straightforward to show the other terms of (12) also converge. Thus, the convergence of the first term of (28) holds.

94. Convergence of $\tau_{\Delta,n,\delta}(\beta^*_{\Delta,n}|x_0) - \tau_{\Delta,n}(\beta^*_{\Delta,n}|x_0)$. For the second term of (28), when the strategies $\beta^*_{\Delta,n}$ are fixed and δ goes to 0, the convergence follows from the fact that $\tau_{\Delta,n,\delta}(\beta^*_{\Delta,n}|x_0)$ is a continuous function of δ.

It follows that, from (28), $\tau_{\Delta,n,\delta}(\beta^*_{\Delta,n,\delta}|x_0) \to \tau_{\Delta,n}(\beta^*_{\Delta,n}|x_0)$ as $\delta \to 0$.

6.4. *Lemma 2.4*

LEMMA. *Suppose $\beta^*_{\Delta,n}$ is trivial. Then, for every x_i, $0 < b_i \le \overline{v}$,*

$$\underbrace{\lim_{\delta \to 0} h^*_{\Delta,n,\delta,b}(b_i|x_i) = 0}_{\substack{\text{the probability that a buyer will bid anything other than lowest possible bid of 0,} \\ \text{will go to zero}}} \tag{31}$$

and $0 \le b_i < \overline{v}$,

$$\underbrace{\lim_{\delta \to 0} h^*_{\Delta,n,\delta,s}(b_i|x_i) = 0.}_{\substack{\text{the probability that a seller will bid less than the highest possible value} \\ \text{will go to zero}}} \tag{32}$$

PROOF. Suppose that the conclusion does not hold. Suppose that, although (31) holds, there exists $b_i < \overline{v}$ and a signal x_i such that

$$\lim_{\delta \to 0} h^*_{\Delta,n,\delta,s}(b_i|x_i) > 0. \tag{33}$$

Then, there exists a signal x_i' such that for every x_0,

$$v(x_0, x_i') > b_i. \tag{34}$$

Suppose a buyer with a signal x_i' chooses to bid b_i,although (31) requires that the player with signal $x_i^{'}$ will not bid b_i. It follows from (31) that buyers with other signals will not place a competing bid, that is, a bid higher than b_i. It follows that a buyer i will get the good with a positive probability. From (34), the expected payoff is positive. It is a contradiction to the assumption that $\beta^*_{\Delta,n}$ is an equilibrium.

6.5. *Lemma 2.5*

LEMMA. *Suppose that β^*_Δ is trivial. Then there exists $\overline{\delta} > 0$ and $\overline{\Delta} > 0$ such that for every $\delta < \overline{\delta}$ and $\Delta < \overline{\Delta}$, the set of signals such that sellers with a signal in that set prefer to deviate from $\beta^*_{i,\Delta,n,\delta}$ has a positive measure.*

PROOF:

95. Introduction. We first recall from the text that the definition of the alternative strategy from the text. We first evaluate the change in expected payoffs, by evaluating the gains from an additional sales and the losses from a lower price for existing sales. We then evaluate the likelihood of these events by evaluating the probability of these events using an asymptotic expansion.

Jackson and Swinkels (2005) provided an ingenious proof about existence of a nontrivial mixed strategy equilibrium in a large class of private value double auctions using a device of a nonstrategic bidder. This proof builds on their argument to show existence of a nontrivial mixed strategy equilibrium in an uniform price double auction with a finite set of possible bids in an interdependent value setting. The argument here assumes a strictly private value element and the private value at the boundary to avoid nontrade equilibrium seen in, for example, in lemon's market. Also the argument here uses asymptotic expansion to simply the estimation of trading probability when the probability of nonstrategic bid is sufficiently small and close to zero.

96. Definition of alternative strategies. From the text, consider a signal x_i which satisfies the condition of

$$v(1,1) - 6\Delta - v(1,x_i) > 0 \tag{35}$$

For sufficiently small Δ, the probability that a player gets the signal which satisfies the condition of (35) is positive. Now consider an alternative strategy for a seller that whenever the equilibrium sell bid is at least $v(1,1)$,the seller lowers the bid to $v(1,1) - 2\Delta$.

We first introduce notations on the order statistics by other players' bids.

97. Notations. Let $\overline{W}_{i,n}$ be the n_S-th highest bid among bids by players other than i, that is, $\overline{W}_{i,n} = b_{n_S:|N_n \cup \{\widehat{i}\} - \{i\}|}$. Let $\underline{W}_{i,n}$ be the n_S+1st highest bid among bids by players other than i. That is, $\underline{W}_{i,n} = b_{n_S+1:|N_n \cup \{\widehat{i}\} - \{i\}|}$.

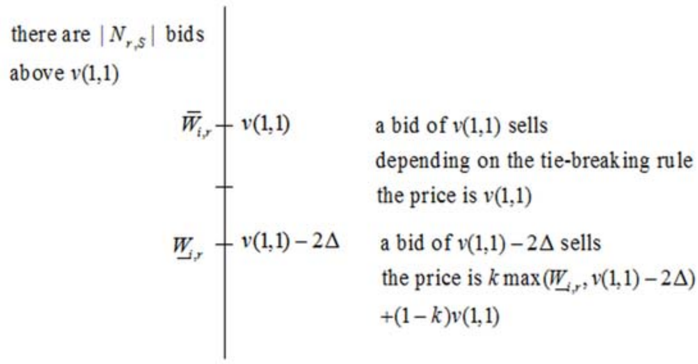

FIGURE 1.—

Using this notation, we can estimate the change in the expected payoffs.

98. Change in expected payoffs for seller i. We would like to estimate

$$\underbrace{U_{\Delta,n,\delta,s}(x_i, v(1,1)-2\Delta, \beta^*_{-i,\Delta,n,\delta})}_{\text{the seller's payoff from bidding } v(1,1)-2\Delta} - \underbrace{U_{\Delta,n,\delta,s}(x_i, v(1,1), \beta^*_{-i,\Delta,n,\delta})}_{\text{the seller's payoff from bidding } v(1,1)}. \tag{36}$$

99. Decomposition into cases depending on $\overline{W}_{i,n}$. The change in outcomes and payoffs will depend on the behavior of the highest bid among bid made players other than i.

- Case 1. $\overline{W}_{i,n} = v(1,1)$.
- Case 2 $\overline{W}_{i,n} = v(1,1) - \Delta$.
- Case 3. $\overline{W}_{i,n} = v(1,1) - 2\Delta$.
- Case 4. $\overline{W}_{i,n} < v(1,1) - 2\Delta$.

We consider each case one by one.

100. Case 1: $\overline{W}_{i,n} = v(1,1)$. In this case,

- An offer of $v(1,1)$ may be accepted (subject to ties). The price is $v(1,1)$.
- An offer of $v(1,1) - 2\Delta$ will be accepted. The price is $k(\max(v(1,1) - 2\Delta,, \underline{W}_{i,n}) + (1-k)v(1,1)$.

The maximum loss can be obtained by making the assumptions that

- An offer of $v(1,1)$ will be accepted for sure.
- The sales price is $v(1,1) - 2\Delta$.

With these assumptions, the maximum loss in payoff is

$$\mathbf{E}[v(1,1) - 2\Delta - v(1,1)|x_i, \overline{W}_{i,n} = v(1,1)] \geq -2\Delta. \tag{37}$$

We now turn to the second case.

101. Case 2. $\overline{W}_{i,n} = v(1,1) - \Delta$. In this case,

- An offer $v(1,1)$ will not be accepted.
- An offer $v(1,1) - 2\Delta$ will be accepted. The sales price is $k(v(1,1) - 2\Delta) + (1 - k)(v(1,1) - \Delta)$.

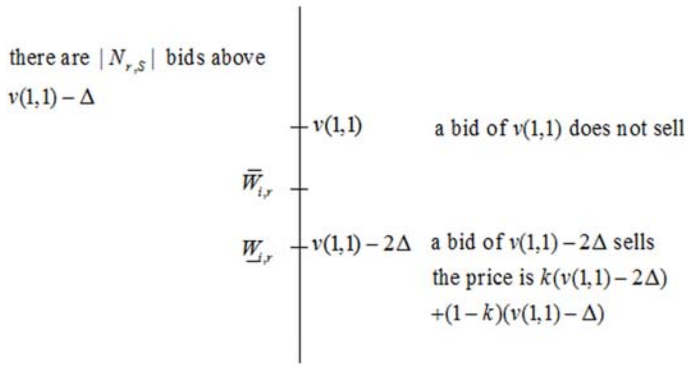

FIGURE 2.—

Assuming that the sales price is $v(1,1) - 2\Delta$, the minimum gain is

$$\begin{aligned} & \mathbf{E}[v(1,1) - 2\Delta - v(1,x_i))|x_i, \overline{W}_{i,n} = v(1,1) - \Delta] \qquad (38) \\ \geq\ & v(1,1) - 2\Delta - v(1,x_i) \\ >\ & 4\Delta \text{ from } (35). \end{aligned}$$

We now turn to the third case.

102. Case 3. $\overline{W}_{i,n} = v(1,1) - 2\Delta$**.** In this case,

- An offer $v(1,1)$ will not be accepted.
- An offer $v(1,1)-2\Delta$ may be accepted (subject to ties). When an offer of $v(1,1)-2\Delta$ is accepted, the price is $v(1,1) - 2\Delta$.

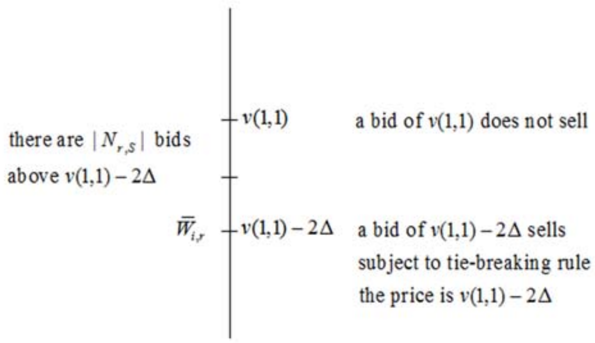

FIGURE 3.—

The payoffs from sales is at least

$$v(1,1) - 2\Delta - v(1,x_i) > 0 \text{ by } (35). \qquad (39)$$

The probability of sales from an offer of $v(1,1) - 2\Delta$ is positive depending on the number of bidders who is tied at $v(1,1) - 2\Delta$. In order to obtain the lower bound of the benefit from the alternative strategy, we assume that the probability to be zero.

103. Case 4. $\overline{W}_{i,n} < v(1,1) - 2\Delta$. Neither of $v(1,1)$ and $v(1,1) - 2\Delta$ will be accepted. In this case, the change in payoff is zero.

From the above cases, we can estimate the change in expected payoff from the alternative strategy.

104. An estimate of change in expected payoffs. Collecting (36),(37), (38), and (39), the change in expected payoff is

$$\begin{aligned}&U_{i,\Delta,n,\delta}(x_i, v(1,1) - 2\Delta, \beta^*_{-i,\Delta,n,\delta}) - U_{i,\Delta,n,\delta}(x_i, v(1,1), \beta^*_{-i,\Delta,n,\delta}) \\ &> (-2\Delta)\Pr(\overline{W}_{i,n} = v(1,1)|x_i) \\ &+ 4\Delta \Pr(\overline{W}_{i,n} = v(1,1) - \Delta|x_i)\end{aligned} \tag{40}$$

Therefore,

$$\begin{aligned}&U_{i,\Delta,n,\delta}(x_i, v(1,1) - 2\Delta, \beta^*_{-i,\Delta,n,\delta}) - U_{i,\Delta,n,\delta}(x_i, v(1,1), \beta^*_{-i,\Delta,n,\delta}) > 0 \\ \Leftrightarrow\ &\Pr(\overline{W}_{i,n} = v(1,1)|x_i) < 2\Pr(\overline{W}_{i,n} = v(1,1) - \Delta|x_i).\end{aligned} \tag{41}$$

Thus it remains to show (41).

In order to show (41), we now estimate $\Pr(\overline{W}_{i,n} = b_i|X_i = x_i)$ for δ small. For that goal, we first derive the formula for $\Pr(\overline{W}_{i,n} = b_i|X_i = x_i)$. We first simply the expression by conditioning on the state x_0.

105. Conditioning $\Pr(\overline{W}_{i,n} = b_i|X_i = x_i)$ on x_0. We note

$$\begin{aligned}&\Pr(\overline{W}_{i,n} = b_i|X_i = x_i) \\ =&\int \Pr(\overline{W}_{i,n} = b_i|\theta = x_0, X_i = x_i) f_{\theta|X_i}(x_0|x_i)dx_0 \\ =&\int \Pr(\overline{W}_{i,n} = b_i|\theta = x_0) f_{\theta|X_i}(x_0|x_i)dx_0\end{aligned} \tag{42}$$ [2]

.

107. Decomposition depending on the behavior of a nonstrategic bidder.

We now decompose the

$$\begin{aligned} & \Pr(\overline{W}_{i,n} = b_i | \theta = x_0) \qquad (43) \\ = \; & \underbrace{\Pr(\overline{W}_{i,n} = b_i | \theta = x_0, \widehat{\imath} \text{ nonactive})}_{\text{A}}(1-\delta) \\ & + \underbrace{\Pr(\overline{W}_{i,n} = b_i | \theta = x_0, \widehat{\imath} \text{ buyer})}_{\text{B}}(\delta/2) \\ & + \underbrace{\Pr(\overline{W}_{i,n} = b_i | \theta = x_0, \widehat{\imath} \text{ seller})}_{\text{C}}(\delta/2). \end{aligned}$$

We now calculate A, B, and C. Before starting the calculation, we recall the distribution of bids by buyers and sellers.

108. Calculation of A. We first decompose A into cases. We observe

$$\begin{aligned} & \Pr(\overline{W}_{i,n} = b_i | \theta = x_0, \widehat{\imath} \text{ nonactive}) \qquad (44) \\ = \; & \underbrace{\Pr(\overline{W}_{i,n} = b_i | \theta = x_0, \widehat{\imath} \text{ nonactive, there is a buy bid at } b_i)}_{\text{A1}} \\ & \underbrace{\Pr(\text{there is a buy bid at } b_i | \theta = x_0, \widehat{\imath} \text{ nonactive})}_{\text{A2}} + \\ & \underbrace{\Pr(\overline{W}_{i,n} = b_i | \theta = x_0, \widehat{\imath} \text{ nonactive, there is a sell bid at } b_i)}_{\text{A3}} \\ & \underbrace{\Pr(\text{there is a sell bid at } b_i | \theta = x_0, \widehat{\imath} \text{ nonactive})}_{\text{A4}} \end{aligned}$$

We consider A1.

109. The distribution of bids in case A1. We note that

$$\begin{aligned} & \left(\overline{W}_{i,n} = b_i | \theta = x_0, \widehat{\imath} \text{ nonactive, there is a buy bid at } b_i\right) \qquad (45) \\ \Leftrightarrow & \left(\begin{array}{c} \text{out of } n-1 \text{ bids (other than a bid by seller } i) \\ \text{there are } n_S - 1 \text{ bids equal or above } b_i \\ \text{there is a buy bid equal to } b_i \\ \text{and there are } n - n_S - 1 \text{ bids strictly lower than } b_i \end{array}\right) \end{aligned}$$

Let j denote the number of sell offers equal or above b_i. We note that, there are freedoms for $n_S - 1$ sell bids after taking off one seller whom we have been working on.

Given j, we can determine the distribution of bids.

- since there are $n_S - 1$ bids in total equal or above b_i (other than a buy bid equal to b_i), there will be $n_S - j - 1$ buy bids
- since there are $n_S - 1$ sell bids (other than the seller whom we have been working on), $n - j - 1$ sell bids below b_i
- since there are $n_B - 1$ buy bids (other than a buy bid equal to b_i), $n_B - n_S + j$ buy bids below b_i

The distribution of bids is given in the following table.

bid	# of buy bids	# of sell bids	total # of bids
equal or above b_i	$n_S - j - 1$	j	$n_S - 1$
below b_i	$n_B - n_S + j$	$n_S - j - 1$	$n_B - 1$
total # of bids	$n_B - 1$	$n_S - 1$	$n_B + n_S - 2$

It follows that

$$\Pr(\overline{W}_{i,n} = b_i | \theta = x_0, \hat{i} \text{ nonactive, there is a buy bid at } b_i) \tag{46}$$

$$= \sum_{0 \le j \le n_S - 1} \Pr(\overline{W}_{i,n} = b_i,\ j \text{sell bids equal or higher than } b_i$$

$$| x_0, \hat{i} \text{ nonactive, there is a buy bid at } b_i) \tag{47}$$

$$= \sum_{0 \le j \le n_S - 1} [\frac{(n_S - 1)!}{j!(n_S - j - 1)!}$$

$$(1 - H_{\Delta,n,\delta,s}(b_i - \triangle | x_0))^j H_{\Delta,n,\delta,s}(b_i - \triangle | x_0)^{n_S - j - 1}$$

$$\frac{(n_B - 1)!}{(n_S - j - 1)!(n_B - n_S + j)!}$$

$$(1 - H_{\triangle,n,\delta,b}(b_i - \triangle | x_0))^{n - j - 1} H_{\triangle,n,\delta,b}(b_i - \triangle | x_0)^{n_B - n_S + j}].$$

We now estimate (46) around $\delta = 0$ as follows. First we estimate the distribution of bids around $\delta = 0$.

110. Asymptotic estimation of the distribution of bids. It follows from (31) and (32) that

$$\underbrace{H_{\triangle,n,\delta,s}(b_i | x_0) \longrightarrow 0 \text{ for every } 0 \le b_i < v(1,1).}_{\substack{\text{the probability that a sell bid take the value strictly less than } v(1,1) \\ \text{goes to zero}}} \tag{48}$$

and

$$\underbrace{H_{\triangle,n,\delta,b}(b_i | x_0) \longrightarrow 1 \text{for every } 0 \le b_i \le v(1,1)}_{\substack{\text{all the buy bids will concentrate on 0} \\ \text{so the probability that a buy bid take the value above 0 goes to zero}}} \tag{49}$$

Thus, for sufficiently small δ, given that the set of possible bids is finite, there exists $\epsilon > 0$ such that

$$H_{\triangle,n,\delta,s}(b_i | x_0) \le \varepsilon \text{ for every } 0 \le b_i < v(1,1), \tag{50}$$

$$H_{\triangle,n,\delta,b}(b_i | x_0) \ge 1 - \varepsilon \text{ for every } 0 \le b_i \le v(1,1), \tag{51}$$

$$\underbrace{h_{\triangle,n,\delta,s}(b_i | x_0) \le \varepsilon \text{ for every } 0 \le b_i < v(1,1),}_{\text{the probability that a sell bid will be less than } v(1,1) \text{ goes to zero}} \tag{52}$$

and

$$\underbrace{h^*_{\triangle,n,\delta,b}(b_i|x_0)\leq\varepsilon \text{ for every } 0<b_i\leq v(1,1)}_{\text{the probability that a buy bid will be more than 0 will go to zero}} . \tag{53}$$

We now estimate (46) by finding the principal term.

111. Asymptotic estimation of (46). We note, by substituting (50) and (51) into (46),

- $1-H_{\triangle,n,\delta,s}(b_i-\triangle|x_0)\approx 1-\varepsilon$. That is, a probability that a sell bid is higher than $b_i-\triangle$ is very high.
- $H_{\triangle,n,\delta,s}(b_i-\triangle|x_0)\approx\varepsilon$. The probability that a sell bid is less than $b_i-\Delta$ is very low.
- $1-H_{\triangle,n,\delta,b}(b_i-\triangle|x_0)\approx\varepsilon$. The probability that a buy bid is higher than $b_i-\Delta$ is very low.
- $H_{\triangle,n,\delta,b}(b_i-\triangle|x_0)\approx 1-\varepsilon$. The probability that a buy bid is less than $b_i-\Delta$ is very high.

Thus,(46) has the principal term from setting $j=n_S-1$. That is, the probability is highest when the number of the seller who bids highest is largest. In this case, we can evaluate each term of (46)

- $\frac{(n_S-1)!}{j!(n_S-j-1)!}=1$ when $j=n_S-1$. That is, since every seller bids equal or above b_i, there is only one possible combination.
- $(1-H_{\triangle,s,n,\delta}(b_i-\triangle|x_0))^j H_{\triangle,s,n,\delta}(b_i-\triangle|x_0)^{n_S-j-1}\approx(1-\varepsilon)^{n_S-1}\varepsilon^0\approx 1$. That is, since every seller bids equal or above b_i and it is a high probability event that a seller bids high, it is a high probability event.
- $\frac{(n_B-1)!}{(n_S-j-1)!(n_B-n_S+j)!}=1$. That is, since every buyer bids strictly less than b_i. there is only one possible combination.
- $1-H_{\triangle,b,n,\delta}(b_i-\triangle|x_0))^{n-j}H_{\triangle,b,n,\delta}(b_i-\triangle|x_0)^{n_B-n_S+j}\approx(1-\varepsilon)^{n_B-1}\approx 1$. That is, since every buyer bids less than $b_i-\Delta$ and since it is a high probability event that a buy bid is less than $b_i-\Delta$, it is a high probability event.

By combining these four terms, we get

$$(A1)\approx 1 \tag{54}$$

We then calculate A2.

112. Calculation of A2. We observe

$$\Pr(\text{there is a buy bid at } b_i|\theta{=}x_0,\hat{\imath}\text{nonactive})\approx n_B\varepsilon \tag{55}$$

113. Calculation of A3. We note, as in case of A1,

$$\left(\overline{W}_{i,n} = b_i|\theta = x_0, \hat{i} \text{ nonactive, there is a sell bid at } b_i\right) \tag{56}$$

$$\Leftrightarrow \begin{pmatrix} \text{out of } n-1 \text{ bids (other than a bid by seller } i) \\ \text{there are } n_S - 1 \text{ bids equal or above } b_i \\ \text{there is a sell bid equal to } b_i \\ \text{and there are } n - n_S - 1 \text{ bids strictly lower than } b_i \end{pmatrix}$$

Let j be the number of sell bids. There are $n_S - 2$ freedom of sell bids other than the sell bid by the seller i we have been working on and the seller who will bid b_i.

For each $0 \le j \le n_S - 2$, the distribution of bids will be determined as follows:

- since there are $n_S - 1$ bids in total equal or above b_i (other than a buy bid equal to b_i), there will be $n_S - j - 1$ buy bids equal or above b_i
- since there are $n_S - 2$ sell bids (other than the seller whom we have been working on), $n - j - 2$ sell bids below b_i
- since there are n_B buy bids, $n_B - n_S + j + 1$ buy bids below b_i

The distribution of bids is given in the following table.

bid	# of buy bids	# of sell bids	total # of bids
equal or above b_i	$n_S - j - 1$	j	$n_S - 1$
below b_i	$n_B - n_S + j + 1$	$n_S - 2 - j$	$n_B - 1$
total # of bids	n_B	$\lvert N_{n,S}\rvert - 2$	$n_B + n_S - 2$

It follows that

$$\Pr(\overline{W}_{i,n} = b_i|\theta = x_0, \hat{i} \text{ nonactive, there is a sell bid at } b_i) \tag{57}$$

$$= \sum_{0 \le j \le n_S - 1} \Pr(\overline{W}_{i,n} = b_i,\ j\text{sell bids equal or higher than } b_i$$

$$|x_0, \hat{i} \text{ nonactive, there is a sell bid at } b_i) \tag{58}$$

$$= \sum_{0 \le j \le n_S - 1} [\frac{(n_S-2)!}{j!(n_S-j-2)!}$$

$$(1 - H_{\Delta,n,\delta,s}(b_i - \triangle|x_0))^j H_{\Delta,n,\delta,s}(b_i - \triangle|x_0)^{n_S-j-2}$$

$$\frac{(n_B)!}{(n_S-j-1)!(n_B-n_S+j+1)!}$$

$$(1 - H_{\triangle,n,\delta,b}(b_i - \triangle|x_0))^{n-j-1} H_{\triangle,n,\delta,b}(b_i - \triangle|x_0)^{n_B-n_S+j+1}$$

It is maximized at $j = n_S - 2$. We now evaluate each term at

Thus,(46) has the principal term from setting $j = n_S - 2$. That is, the probability is highest when the number of the seller who bids highest is largest. In this case, we can evaluate each term of (57)

- $\frac{(n_S-2)!}{j!(n_S-j-2)!} = 1$ when $j = n_S - 2$. That is, since every seller bids equal or above b_i, there is only one possible combination.
- $(1 - H_{\triangle,s,n,\delta}(b_i - \triangle|x_0))^j H_{\triangle,s,n,\delta}(b_i - \triangle|x_0)^{n_S-j-1} \approx (1-\varepsilon)^{n_S-1}\varepsilon^0 \approx 1$. That is, since every seller bids equal or above b_i and it is a high probability event that a

seller bids high, it is a high probability event.

- $\frac{(n_B)!}{(n_S-j-1)!(n_B-n_S+j+1)!} = n_B$. That is, since there are only $n_S - 1$ sell bids equal or above b_i, it has to be one buy bid equal or above b_i,and there is n_B possible combination.
- $(1 - H_{\triangle,b,n,\delta}(b_i - \triangle|x_0))^{n-j-1} H_{\triangle,b,n,\delta}(b_i - \triangle|x_0)^{n_B-n_S+j+1} \approx \varepsilon$. That is, since one buyer has to bid above b_i, it is a probability ε event.

By combining these four terms,

$$(A3) = n_B\epsilon. \tag{59}$$

114. Calculation of A4. We recall

$$\begin{aligned} & \Pr(\text{there is a sell bid at } b_i|\theta{=}x_0,\widehat{\imath}\text{nonactive}) \\ \approx\ & n_S\varepsilon \end{aligned} \tag{60}$$

115. Calculation of A. We can now collect these terms to estimate A. That is,

$$\begin{aligned} A &= (A1)\cdot(A2) + (A3)\cdot(A4) \\ &= 1\cdot n_B\varepsilon + n_B\epsilon\cdot n_S\varepsilon \\ &\approx n_B\varepsilon. \end{aligned} \tag{61}$$

An intuition is as follows. When there is no nonstrategic bid, in order for the n_Sth order statistics out of the bids other than seller i equal to be b_i, there are two possibilities.

The first possibility is that a bid b_i is provided by a buy bid. In this case, the probability is highest when every $n_S - 1$ seller stays above b_i and every n_B buyers stay below b_i. In this case, one buyer bids b_i, and it is the event which take place with probability ε.

The second possibility is the case where a bid b_i is provided by a sell bid. In this case, even when every remaining $n_S - 2$ seller stays above b_i, there is only $n_S - 1$ bids out of the sellers which will be equal or above b_i. It implies that there has to be one buy bid equal or above b_i. In this case, one seller has to bid below $v(1,1)$ and one buyer has to bid above 0, thus it is the event which take place with probability ε^2.

Therefore, asymptotically, the first possibility dominates the second.

We now calculate B,that is, $\Pr(\overline{W}_{i,n} = b_i|\theta = x_0,\widehat{\imath}$ buyer).

116. Decomposition of B**.** We now decompose the event $\overline{W}_{i,n} = b_i|\theta = x_0,\widehat{\imath}$ buyer by conditioning on the behavior of a nonstrategic bidder. We note

$$\begin{aligned}
& \Pr(\overline{W}_{i,n} = b_i | \theta = x_0, \widehat{\imath} \text{ buyer}) \qquad (62) \\
= \; & \underbrace{\Pr(\overline{W}_{i,n} = b_i | \theta = x_0, \widehat{\imath} \text{ buyer, there is a buy bid at } b_i)}_{\text{B1}} \\
& \cdot \underbrace{\Pr(\text{there is a buy bid at } b_i | \theta = x_0, \widehat{\imath} \text{ buyer})}_{\text{B2}} + \\
& \underbrace{\Pr(\overline{W}_{i,n} = b_i | \theta = x_0, \widehat{\imath} \text{ buyer, there is a sell bid at } b_i)}_{\text{B3}} \\
& \cdot \underbrace{\Pr(\text{there is a sell bid at } b_i | \theta = x_0, \widehat{\imath} \text{ buyer})}_{\text{B4}} + \\
& \underbrace{\Pr(\overline{W}_{i,n} = b_i | \theta = x_0, \widehat{\imath} \text{ buyer, there is a bid by a nonstrategic bidder at } b_i)}_{\text{B5}} \\
& \cdot \underbrace{\Pr(\text{there is a nonstrategic bid at } b_i | \theta = x_0, \widehat{\imath} \text{ buyer})}_{\text{B6}}
\end{aligned}$$

That is, the probability that $\overline{W}_{i,n} = b_i$ takes the value b_i can be calculated from the case where there is a buyer who bids b_i,or the seller, or the nonstrategic bidder.

We now calculate B1, $\Pr(\overline{W}_{i,n} = b_i | \theta = x_0, \widehat{\imath}$ buyer, there is a buy bid at $b_i)$.

117. Decomposition of B1.In this case, the distribution of bids is

$$\begin{aligned}
& \left(\overline{W}_{i,n} = b_i | \theta = x_0, \widehat{\imath} \text{ buyer, there is a buy bid at } b_i\right) \qquad (63) \\
\Leftrightarrow \; & \left(\begin{array}{c} \text{out of } n \text{ bids (other than a bid by seller } i\text{, but with a nonstrategic bid)} \\ \text{there are } n_S - 1 \text{ bids equal or above } b_i \\ \text{there is a buy bid equal to } b_i \\ \text{and there are } n - n_S + 1 \text{ bids strictly lower than } b_i \end{array} \right)
\end{aligned}$$

B1 can be further decomposed depending on whether $\hat{\imath}$ bids equal or above b_i.

$$\begin{aligned}
& \underbrace{\Pr(\overline{W}_{i,n} = b_i | \theta = x_0, \widehat{\imath} \text{ buyer, there is a buy bid at } b_i)}_{\text{B1}} \\
= \; & \underbrace{\Pr(\overline{W}_{i,n} = b_i | x_0, \hat{\imath} \text{ buyer, there is a buy bid at } b_i, \widehat{\imath} \text{ bids strictly above } b_i)}_{\text{B1-1}} \\
& \cdot \underbrace{\Pr(\hat{\imath} \text{ bids strictly above } b_i | x_0, \hat{\imath} \text{ buyer, there is a buy bid at } b_i)}_{\text{B1-2}} \\
& + \underbrace{\Pr(\overline{W}_{i,n} = b_i | x_0, \hat{\imath} \text{ buyer, there is a buy bid at } b_i, \widehat{\imath} \text{ bids strictly below } b_i)}_{\text{B1-3}} \\
& \cdot \underbrace{\Pr(\hat{\imath} \text{ bids strictly below } b_i | x_0, \hat{\imath} \text{ buyer, there is a buy bid at } b_i)}_{\text{B1-4}}
\end{aligned}$$

118. Calculation of B1-1. We consider

$$[\overline{W}_{i,n} = b_i | \, \hat{\imath} \text{ buyer, there is a buy bid at } b_i, \widehat{i} \text{ bids equal or above } b_i]$$

$$\Leftrightarrow \left(\begin{array}{c} \text{out of } n_B \text{ buy bid, } (n_S - 1) \text{ sell bids, and a nonstrategic bid,} \\ \text{there are } \underbrace{n_S - 2}_{\substack{\text{there should be } n_S \text{ bids above or equal } b_i \\ \text{and there is a nonstrategic bid above } b_i \\ \text{there is one bid at } b_i \text{ by a buyer}}} \text{ bids equal or higher than } b_i \\ \text{and} \\ \text{there is one bid equal or above } b_i \text{ by a nonstrategic bidder } \widehat{i} \\ \text{and} \\ \text{there is one bid at } b_i \text{ by a buyer} \\ \text{and} \\ \text{there are } n - n_S + 1 \text{ bids lower than } b_i \end{array} \right.$$

Let j denote the number of sell bids above b_i. There are $n_S - 1$ freedoms of allocation of sell bids after a seller whom we have been working on.

For each $0 \leq j \leq n_S - 1$, the distribution of bids will be determined as follows:

- since there are $n_S - 2$ bids in total equal or above b_i (other than a buy bid equal to b_i and a bid by a nonstrategic bidder $\widehat{i}$), there will be $n_S - j - 2$ buy bids equal or above b_i
- since there are $n_S - 1$ sell bids (other than the seller whom we have been working on), $n - j - 1$ sell bids below b_i
- since there are $n_B - 1$ buy bids (other than the buyer who bids b_i), there are $n_B - n_S + j + 1$ buy bids below b_i

The distribution of bids is given in the following table.

bid	# of buy bids	# of sell bids	total # of bids
equal or above b_i	$n_S - j - 2$	j	$\underbrace{n_S - 2}_{\substack{\text{there is a buyer bid at } b_i \\ \widehat{i} \text{ bids equal or above } b_i}}$
below b_i	$n_B - n_S + j + 1$	$n_S - j - 1$	$n - n_S$
total # of bids	$n_B - 1$	$n_S - 1$	$n_B + n_S - 2$

It follows that the principal term is obtained by

$$
\begin{aligned}
& \text{(B1-1)} && (64)\\
= \;& \Pr(\overline{W}_{i,n} = b_i | x_0, \hat{\imath} \text{ is buyer, there is a buy bid at } b_i, \hat{\imath} \text{ bids above } b_i) && (65)\\
= \;& \sum_{0 \le j \le n_S - 1} [\frac{(n_S-1)!}{j!(n_S-j-1)!} \\
& \underbrace{(1 - H_{\mathcal{X},\Delta,n,\delta}(b_i - \triangle|x_0))}_{1-\varepsilon}{}^j \underbrace{H_{\mathcal{X},\Delta,n,\delta}(b_i - \triangle|x_0)}_{\varepsilon}{}^{n_S-j-1} \\
& \frac{(n_B-1)!}{(n_S-j-2)!(n_B-n_S+j+1)!} \\
& \underbrace{(1 - H_{\triangle,b,n,\delta}(b_i - \triangle|x_0))}_{\varepsilon}{}^{n-2-j} \underbrace{H_{\triangle,b,n,\delta}(b_i - \triangle|x_0)}_{1-\varepsilon}{}^{n_B-n_S+j+1}.
\end{aligned}
$$

We note that, from (68), it follows that the principal terms is obtained by setting $j = n_S - 2$.In this case,

- $\frac{(n_S-1)!}{j!(n_S-j-1)!} = n_S - 1$. It is because, since there is a buy bid above b_i and a nonstrategic bid equal or above b_i, in order for the number of bids equal or above b_i to be equal to n_S, it has to be that one sell bid is below b_i, and there is $n_S - 1$ possible combinations for this.
- $(1 - H_{\triangle,n,\delta,s}(b_i - \triangle|x_0))^j H_{\triangle,n,\delta,s}(b_i - \triangle|x_0)^{n_S-j-1} \approx \varepsilon$. It is because, one sell bid has to be strictly below b_i, and it is a probability ε event.
- $\frac{(n_B-1)!}{(n_S-1-j)!(n_B-n_S+j+1)!} = 1$. It is because every n_B buy bids stays strictly below b_i.
- $(1 - H_{\triangle,n,\delta,b}(b_i - \triangle|x_0))^{n-2-j} H_{\triangle,n,\delta,b}(b_i - \triangle|x_0)^{n_B-n_S+j+2} \approx 1$.

By combining these four terms, we get

$$(B1 - 1) \approx (n_S - 1)\varepsilon \qquad (66)$$

Intuitively, in order for the n_Sth bid equal to b_i, at least one sellers need to bid less than b_i, which occurs at the probability at the order of ε^2.

119. Calculation of B1-2. We note

$$
\begin{aligned}
& \underbrace{\Pr(\hat{\imath} \text{ bids equal or above } b_i | x_0, \hat{\imath} \text{ buyer, there is a buy bid at } b_i)}_{B1-2} && (67)\\
= \;& \frac{(\frac{\overline{b}-b_i}{\Delta} - 1)}{(\frac{\overline{b}}{\Delta} + 1)}
\end{aligned}
$$

120. Calculation of $B1-3$. We note

$$[\overline{W}_{i,n} = b_i \text{ conditional on } \hat{\imath} \text{ being a buyer, there is a buy bid at } b_i, \hat{\imath} \text{ bids strictly below } \hat{\imath}$$

$$\Leftrightarrow \begin{cases} \text{out of } n \text{ bids} \\ \text{by } n_B \text{ buyers, } (n_S-1) \text{ sellers, and one nonstrategic bidder} \\ \text{there are } n_S-1 \text{ bids equal or higher than } b_i \\ \text{and} \\ \text{there is one bid at } b_i \text{ by a buyer} \\ \text{and} \\ \text{there is one bid strictly below } b_i \text{ by a nonstrategic bidder } \hat{\imath} \\ \text{and} \\ \text{there are } n-n_S+1 \text{ bids lower than } b_i \end{cases}$$

Let j denote the number of sellers who bids equal or above b_i. There is a n_S-1 degrees of freedom available sell bids.

For each j, the allocation of bids will be determined as follows:

- since there are n_S-1 bids in total equal or above b_i (other than a buy bid equal to b_i), there will be n_S-j-1 buy bids equal or above b_i
- since there are n_S-1 sell bids (other than the seller whom we have been working on), $n-j-1$ sell bids below b_i
- since there are n_B-1 buy bids (other than the buyer who bids b_i), there are n_B-n_S+j buy bids below b_i

The distribution of bids is given in the following table.

bid	# of buy bids	# of sell bids	total # of bids
equal or above b_i	n_S-j-1	j	$\underbrace{n_S-1}_{\text{take off a bid by a buyer at } b_i}$
below b_i	n_B-n_S+j	n_S-j-1	$n-n_S-1$
total # of bids	n_B-1	n_S-1	n_B+n_S-2

It follows that the principal term is obtained by

$$\text{(B1-3)} \tag{68}$$

$$= \Pr(\overline{W}_{i,n} = b_i | x_0, \hat{\imath} \text{ is buyer, there is a buy bid at } b_i, \hat{\imath} \text{ bids above } b_i) \tag{69}$$

$$= \sum_{0\le j\le n_S-1} [\frac{(n_S-1)!}{j!(n_S-j-1)!}$$

$$\underbrace{(1-H_{\mathcal{X},\Delta,n,\delta}(b_i-\triangle|x_0))}_{1-\varepsilon}{}^j \underbrace{H_{\mathcal{X},\Delta,n,\delta}(b_i-\triangle|x_0)}_{\varepsilon}{}^{n_S-j-1}$$

$$\frac{(n_B-1)!}{(n_S-1-j)!(n_B-n_S+j)!}$$

$$\underbrace{(1-H_{\triangle,b,n,\delta}(b_i-\triangle|x_0))}_{\varepsilon}{}^{n-1-j} \underbrace{H_{\triangle,b,n,\delta}(b_i-\triangle|x_0)}_{1-\varepsilon}{}^{n_B-n_S+j}.$$

We note that, from (68), it follows that the principal terms is obtained by setting $j = n_S - 1$.In this case,

- $\frac{(n_S-1)!}{j!(n_S-j-1)!} = 1$. It is because every remaining sell bid stays equal or above b_i, and there is only one possible combination for this.
- $(1 - H_{\triangle,n,\delta,s}(b_i - \triangle|x_0))^j H_{\triangle,n,\delta,s}(b_i - \triangle|x_0)^{n_S-j-1} \approx 1$. It is because every remaining sell bid stays equal or above b_i is a high probability event.
- $\frac{(n_B-1)!}{(n_S-1-j)!(n_B-n_S+j+1)!} = 1$. It is because that every remaining buy bid (other than the one bidding b_i) stays below b_i.
- $(1 - H_{\triangle,n,\delta,b}(b_i - \triangle|x_0))^{n-1-j} H_{\triangle,n,\delta,b}(b_i - \triangle|x_0)^{n_B-n_S+j+1} \approx 1$.Because every remaining buy bid stays below b_i is a high probability event.

By combining these four terms, we get

$$(B1 - 3) \approx 1. \tag{70}$$

Intuitively, when there is one buy bid at b_i, it is suffice that all remaining sell $n_S - 1$ bids stay at $v(1,1)$ in order to have $\overline{W}_{i,n} = b_i$.

121. Calculation of B1-4. We note

$$\begin{aligned} & \Pr(\hat{\imath}\text{bids strictly below } b_i|x_0, \hat{\imath}\text{buyer,there is a buy bid at } b_i) \quad (71)\\ = & \frac{(\frac{b_i}{\Delta}+1)}{(\frac{\bar{b}}{\Delta}+1)} \end{aligned}$$

We now collect terms for B1.

122. Calculation of B1.We now combine these terms to calculate

$$\begin{aligned} (B1) &= (B1-1)\cdot(B1-2) \quad (72)\\ &\quad +(B2-1)\cdot(B2-2)\\ &= (n_S-1)\varepsilon\frac{(\frac{b_i}{\Delta}+1)}{(\frac{\bar{b}}{\Delta}+1)} + 1\cdot\frac{(\frac{b_i}{\Delta}+1)}{(\frac{\bar{b}}{\Delta}+1)}\\ &\approx 1\cdot\frac{(\frac{b_i}{\Delta}+1)}{(\frac{\bar{b}}{\Delta}+1)} \quad (73) \end{aligned}$$

The intuition is as follows. When $\widehat{\imath}$ is a buyer, there is a buy bid at b_i, there are two possibilities about the behavior of $b_{\widehat{\imath}}$.

The first possibility is that $\widehat{\imath}$ bids strictly above b_i. In this case, there are already 2 bids equal or above b_i. Therefore, it has to be that one sell bid has to be less than b_i. It is a probability ε event.

The second possibility is that $\widehat{\imath}$ bids strictly less b_i. In this case, there is one bid equal or above b_i by a buyer, Therefore, it is suffice that every remaining $n_S - 1$ sell bid stay equal or above b_i, and it is a high probability event.

Therefore, the probability of the second event dominates the probability of the first event.

We now consider B2.

123. Calculation of $B2$. We note

$$\underbrace{\Pr(\text{there is a buy bid at } b_i|\theta = x_0, \hat{i} \text{ buyer})}_{\text{B2}} \tag{74}$$
$$= n_B h_{\triangle,n,\delta,b}(b_i|x_0)$$
$$\approx n_B \varepsilon \tag{75}$$

We now consider B3

124. Calculation of B3. We consider $\Pr(\overline{W}_{i,n} = b_i|\theta = x_0, \hat{i}$ buyer, there is a sell bid at $b_i)$

$$\overline{W}_{i,n} = b_i \Leftrightarrow \left(\begin{array}{c} \text{out of } n \text{ bids} \\ \text{there are} n_S - 1 \text{bids equal or higher than } b_i \\ \text{there is a sell bid at } b_i \\ \text{and there are } n - n_S \text{bids strictly lower than } b_i \end{array} \right. \tag{76}$$

B3 can be further decomposed depending on whether $\hat{\imath}$ bids equal or above b_i.

$$\underbrace{\Pr(\overline{W}_{i,n} = b_i|\theta = x_0, \hat{i} \text{ seller, there is a sell bid at } b_i)}_{\text{B3}} \tag{77}$$
$$= \underbrace{\Pr(\overline{W}_{i,n} = b_i|x_0, \hat{\imath} \text{ buyer}, \text{there is a sell bid at } b_i, \hat{i} \text{ bids strictly above } b_i)}_{B3-1}$$
$$\cdot \underbrace{\Pr(\hat{\imath} \text{ bids equal or above } b_i|x_0, \hat{\imath} \text{ buyer, there is a sell bid at } b_i)}_{B3-2}$$
$$+ \underbrace{\Pr(\overline{W}_{i,n} = b_i|x_0, \hat{\imath} \text{ buyer}, \text{there is a sell bid at } b_i, \hat{i} \text{ bids strictly below } b_i)}_{B3-3}$$
$$\cdot \underbrace{\Pr(\hat{\imath} \text{ bids strictly below } b_i|x_0, \hat{\imath} \text{ buyer, there is a sell bid at } b_i)}_{B3-4}$$

We now need to calculate B3-1.

125. Calculation of $B3-1$. We consider

$$\overline{W}_{i,n} = b_i \text{conditional on } \hat{\imath} \text{ buyer, there is a sell bid at } b_i, \hat{\imath} \text{ bids strictly above } b_i$$

$$\Leftrightarrow \begin{cases} \text{out of } n \text{ bids} \\ \text{by } n_B \text{ buyers, } \underbrace{(n_S-1)}_{\text{other than the seller who is deviating}} \text{ sellers, and one nonstrategic bidder,} \\ \text{there are } \underbrace{n_S-2}_{\substack{\text{there should be } n_S \text{ bids above or equal } b_i \\ \text{and there is a nonstrategic bid above } b_i \\ \text{there is one bid at } b_i \text{ by a seller}}} \text{ bids equal or higher than } b_i \\ \text{and} \\ \text{there is one bid equal or above } b_i \text{ by a nonstrategic bidder } \hat{\imath} \\ \text{and} \\ \text{there is one bid at } b_i \text{ by a seller} \\ \text{and} \\ \text{there are } n-n_S+1 \text{ bids lower than } b_i \end{cases}$$

Let j denote the number of sellers who bid equal or above b_i. Since there are n_S-2 freedoms of sell bids other than the seller we have been working on and the bid at b_i,

For each j,the distribution of bids will be determined as follows:

- since there are n_S-2 bids in total equal or above b_i (other than a sell bid equal to b_i and a nonstrategic bid), there will be n_S-j-2 buy bids equal or above b_i
- since there are n_S-2 sell bids (other than the seller whom we have been working on and a sell bid at b_i), $n-j-2$ sell bids below b_i
- since there are n_B buy bids, there are n_B-n_S+j+2 buy bids below b_i

The distribution of bids is given in the following table.

bid	# of buy bids	# of sell bids	total # of bids
equal or above b_i	n_S-2-j	j	$\underbrace{\lvert N_{n,S}\rvert-2}_{\substack{\text{there is a sell bid at } b_i \\ \text{there is a nonstrategic bid above } b_i}}$
below b_i	$n_B-\lvert N_{n,S}\rvert+j+2$	$\lvert N_{n,S}\rvert-2-j$	$n-\lvert N_{n,S}\rvert$
total # of bids	n_B	$\lvert N_{n,S}\rvert-2$	$n_B+\lvert N_{n,S}\rvert-2$

It follows that the principal term is obtained by

$$\begin{aligned}
& \text{(B3-1)} & (78)\\
= \; & \Pr(\overline{W}_{i,n} = b_i | x_0, \hat{\imath} \text{ is buyer, there is a sell bid at } b_i, \widehat{i} \text{ bids above } b_i) & (79)\\
= \; & \sum_{0\leq j\leq n_S-1} [\frac{(n_S-2)!}{j!(n_S-j-2)!} \\
& \underbrace{(1-H_{\Delta,n,\delta,s}(b_i-\triangle|x_0))}_{1-\varepsilon}^{j} \underbrace{H_{\Delta,n,\delta,s}(b_i-\triangle|x_0)}_{\varepsilon}^{n_S-j-2} \\
& \frac{n_B!}{(n_S-2-j)!(n_B-n_S+j+2)!} \\
& \underbrace{(1-H_{\triangle,n,\delta,b}(b_i-\triangle|x_0))}_{\varepsilon}^{n-2-j} \underbrace{H_{\triangle,n,\delta,b}(b_i-\triangle|x_0)}_{1-\varepsilon}^{n_B-n_S+j+2}.
\end{aligned}$$

We note that, from (68), it follows that the principal terms is obtained by setting $j = n_S - 2$.In this case,

- $\frac{(n_S-2)!}{j!(n_S-j-2)!} = 1$. It is because every seller will stay above b_i and there is only one combination.
- $(1-H_{\triangle,n,\delta,s}(b_i-\triangle|x_0))^j H_{\triangle,n,\delta,s}(b_i-\triangle|x_0)^{n_S-j-2} \approx 1$. Because it is a high probability event that every seller will stay above b_i.
- $\frac{(n_B)!}{(n_S-2-j)!(n_B-n_S+j+2)!} = 1$. It is because every buyer will stay below b_i and there is only one combination.
- $(1-H_{\triangle,n,\delta,b}(b_i-\triangle|x_0))^{n-2-j} H_{\triangle,n,\delta,b}(b_i-\triangle|x_0)^{n_B-n_S+j+2} \approx 1$.It is because every buyer will stay below b_i and there is only one combination.

By combining these four terms, we get

$$(B3-1) \approx 1 \tag{80}$$

We now consider B3-2.

126. Calculation of B3-2. We note

$$\begin{aligned}
& \Pr(\hat{\imath} \text{ bids equal or above } b_i | x_0, \hat{\imath} \text{ buyer, there is a sell bid at } b_i) \\
= \; & \frac{\frac{\overline{b}-b_i}{\Delta}}{(\frac{\overline{b}}{\Delta}+1)}.
\end{aligned}$$

127. Calculation of $B3-3$. We note

$$\overline{W}_{i,n} = b_i \text{conditional on } \hat{\imath} \text{ buyer, there is a sell bid at } b_i, \hat{\imath} \text{ bids strictly below } b_i$$

$$\Leftrightarrow \begin{cases} \text{out of } n \text{ bids by } n_B \text{ buyers, } (n_S-1) \text{ sellers, and one nonstrategic bidder,} \\ \text{there are } n_S-1 \text{ bids equal or higher than } b_i \\ \text{and} \\ \text{there is one bid at } b_i \text{ by a seller} \\ \text{and} \\ \text{there is one bid strictly below } b_i \text{ by a nonstrategic bidder } \hat{\imath} \\ \text{and} \\ \text{there are } n-n_S+1 \text{ bids lower than } b_i \end{cases}$$

Let j denotes the number of sellers who bid equal or higher than b_i. There are $0 \le j \le n_S - 2$ freedom of the number of sell bids after the seller we have been working on and the seller who bids b_i.

The distribution of bids are

- since there are $n_S - 1$ bids in total equal or above b_i (other than a sell bid equal to b_i), there will be $n_S - j - 1$ buy bids equal or above b_i
- since there are $n_S - 2$ sell bids (other than the seller whom we have been working on and a bid at b_i), $n - j - 2$ sell bids below b_i
- since there are n_B buy bids, there are $n_B - n_S + j + 1$ buy bids below b_i

The distribution of bids is given in the following table.

bid	# of buy bids	# of sell bids	total # of bids
equal or above b_i	$n_S - 1 - j$	j	$n_S - 1$
below b_i	$n_B - n_S + j + 1$	$n_S - 2 - j$	$n - n_S - 1$
total # of bids	n_B	$\lvert N_{n,S}\rvert - 2$	$n_B + n_S - 2$

It follows that the principal term is obtained by

$$\begin{aligned} & \text{(B3-3)} && (81)\\ = \; & \Pr(\overline{W}_{i,n} = b_i | x_0, \hat{\imath} \text{ is buyer, there is a sell bid at } b_i, \hat{\imath} \text{ bids strictly below} && (82)\\ = \; & \sum_{0 \le j \le n_S - 1} [\frac{(n_S-2)!}{j!(n_S-j-2)!} \\ & \underbrace{(1 - H_{\Delta,n,\delta,s}(b_i - \triangle | x_0))}_{1-\varepsilon}{}^{j} \underbrace{H_{\Delta,n,\delta,s}(b_i - \triangle | x_0)}_{\varepsilon}{}^{n_S - j - 2} \\ & \frac{n_B!}{(n_S - j - 1)!(n_B - n_S + j + 1)!} \\ & \underbrace{(1 - H_{\triangle,n,\delta,b}(b_i - \triangle | x_0))}_{\varepsilon}{}^{n-j-1} \underbrace{H_{\triangle,n,\delta,b}(b_i - \triangle | x_0)}_{1-\varepsilon}{}^{n_B - n_S + j + 1}. \end{aligned}$$

We note that, from (68), it follows that the principal terms is obtained by setting $j = n_S - 2$.In this case,

- $\frac{(n_S-2)!}{j!(n_S-j-2)!}=1$.Since every seller's bid is equal or above b_i.
- $(1-H_{\triangle,n,\delta,s}(b_i-\triangle|x_0))^j H_{\triangle,n,\delta,s}(b_i-\triangle|x_0)^{n_S-j-2}\approx 1$. Since it is a high probability event that ever seller's bid is equal or above b_i.
- $\frac{(n_B)!}{(n_S-2-j)!(n_B-n_S+j+2)!}=n_B$. Since, in order to have n_S bids equal or above b_i, when $\hat{i}$ bids strictly below b_i, it has to be that one buy bid has to be equal or above b_i. There is n_B possible choices.
- $(1-H_{\triangle,n,\delta,b}(b_i-\triangle|x_0))^{n-2-j}H_{\triangle,n,\delta,b}(b_i-\triangle|x_0)^{n_B-n_S+j+1}\approx\varepsilon$. Since, it is an event with probability ε that one bid is equal or above b_i.

By combining these four terms, we get

$$(B3-1)\approx n_B\varepsilon. \tag{83}$$

128. Calculation of B3-4. We note

$$\begin{aligned}&\Pr(\hat{\imath}\text{ bids strictly below }b_i|x_0,\hat{\imath}\text{ buyer, there is a sell bid at }b_i)\\ =&\ \frac{(\frac{b_i}{\Delta}+1)}{(\frac{\bar{b}}{\Delta}+1)}\end{aligned}$$

129. Calculation of B3. We now combine the above calculations to get

$$\begin{aligned}B3 &= (B3-1)\cdot(B3-2)\\ &\quad+(B3-3)\cdot(B3-4)\\ &= \frac{\frac{\bar{b}-b_i}{\Delta}}{(\frac{\bar{b}}{\Delta}+1)}\cdot 1\\ &\quad+n_B\varepsilon.\frac{(\frac{b_i}{\Delta}+1)}{(\frac{\bar{b}}{\Delta}+1)}\\ &= \frac{\frac{\bar{b}-b_i}{\Delta}}{(\frac{\bar{b}}{\Delta}+1)}\cdot 1.\end{aligned}$$

Intuitively, when there is a sell bid at b_i, there are two possibilities.

The first possibility is that $\hat{i}$ bids strictly above b_i. In this case, if every seller bids equal or above b_i, there are just n_S bids equal and above b_i. Thus it is suffice that every seller bids equal or above b_i and every buyer bids below b_i. It is an event with probability $O(1)$.

The second possibility is that $\hat{i}$ bids strictly below b_i. In this case, if every seller bids equal or above b_i, there are only n_S-1 bids equal or above b_i.That is, it is needed that one buyer bids equal or strictly above b_i. It is an event with probability $O(\varepsilon)$.

Since the probability of the first possibility dominates the possibility of the second possibility, it is an event with probability $O(1)$.

130. Calculation of B4. We note

$$\Pr(\text{there is a sell bid at } b_i|\theta = x_0, \hat{\imath}\text{buyer}) \approx n_S\varepsilon. \tag{84}$$

We now move to the calculation of B5 where there is a nonstrategic bid at b_i.

131. Calculation of B5. We consider $\Pr(\overline{W}_{i,n} = b_i|\theta = x_0, \hat{\imath}$ buyer, there is a nonstrategic bid at b_i). We note that

$$\overline{W}_{i,n} = b_i \text{conditional on } \hat{\imath} \text{ buyer, there is a nonstrategic bid at } b_i$$
$$\Leftrightarrow \left(\begin{array}{c} \text{out of } n \text{ bids by } n_B \text{ buyers, } (n_S-1) \text{ sellers, and one nonstrategic bidder,} \\ \text{there are } n_S - 1 \text{ bids equal or higher than } b_i \\ \text{and} \\ \text{there is one bid at } b_i \text{ by a nonstrategic bidder} \\ \text{and} \\ \text{there are } n - n_S \text{ bids strictly below } b_i \end{array} \right.$$

Let j denote number of possible sell bids equal or above b_i. Since there are $n_S - 1$ degree of freedom about the bids by the seller, $0 \le j \le n_S - 1$.

For each j, the distribution of bids is as follows:

- since there are $n_S - 1$ bids in total equal or above b_i (other than a sell bid equal to b_i), there will be $n_S - j - 1$ buy bids equal or above b_i
- since there are $n_S - 1$ sell bids (other than the seller whom we have been working on), $n - j - 1$ sell bids below b_i
- since there are n_B buy bids, there are $n_B - n_S + j + 1$ buy bids below b_i

The distribution of bids is given in the following table.

bid	# of buy bids	# of sell bids	total # of bids
equal or above b_i	$n_S - 1 - j$	j	$n_S - \underbrace{1}_{\text{a nonstrategic bid at } b_i}$
below b_i	$n_B - n_S + j + 1$	$n_S - 1 - j$	$n - n_S$
total # of bids	n_B	$\lvert N_{n,S}\rvert - 1$	$n_B + n_S - 1$

It follows that the principal term is obtained by

$$\begin{aligned} & (B5) && (85)\\ = \; & \Pr(\overline{W}_{i,n} = b_i | x_0, \hat{\imath} \text{ is buyer}, \widehat{\imath} \text{ bids } b_i) && (86)\\ = \; & \sum_{0 \le j \le n_S - 1} [\frac{(n_S-1)!}{j!(n_S-j-1)!} \\ & \underbrace{(1 - H_{\Delta,n,\delta,s}(b_i - \triangle | x_0))^j}_{1-\varepsilon} \underbrace{H_{\Delta,n,\delta,s}(b_i - \triangle | x_0)}_{\varepsilon}{}^{n_S-j-1} \\ & \frac{n_B!}{(n_S-1-j)!(n_B-n_S+j+1)!} \\ & \underbrace{(1 - H_{\triangle,n,\delta,b}(b_i - \triangle | x_0))}_{\varepsilon}{}^{n-1-j} \underbrace{H_{\triangle,n,\delta,b}(b_i - \triangle | x_0)}_{1-\varepsilon}{}^{n_B-n_S+j+1}. \end{aligned}$$

The principal terms is obtained by setting $j = n_S - 1$.In this case,

- $\frac{(n_S-1)!}{j!(n_S-j-1)!} = 1$.Since every seller chooses to bid equal or above b_i, there is only one possibility.
- $(1-H_{\triangle,n,\delta,s}(b_i-\triangle|x_0))^j H_{\triangle,n,\delta,s}(b_i - \triangle|x_0)^{n_S-j-1} \approx 1$. Since it is a high probability event that every seller bids equal or above b_i.
- $\frac{(n_B)!}{(n_S-1-j)!(n_B-n_S+j+1)!} = 1$. Since every buyer chooses to bid strictly less than b_i, it is the only possibility.
- $(1 - H_{\triangle,n,\delta,b}(b_i - \triangle|x_0))^{n-2-j} H_{\triangle,n,\delta,b}(b_i - \triangle|x_0)^{n_B-n_S+j+1} \approx 1$. Since it is a high probability event that every buyer bids strictly below b_i.

By combining these four terms, we get $(B5) \approx 1$.

132. Calculation of B6.We observe

$$\Pr(\text{there is a nonstrategic bid at } b_i | \theta = x_0, \widehat{\imath} \text{ buyer}) = \frac{1}{(\frac{\bar{b}}{\Delta} + 1)}$$

We now calculate B.

133. Collecting terms for B. We have

$$\begin{aligned} & \Pr(\overline{W}_{i,n} = b_i | \theta = x_0, \widehat{\imath}\text{buyer}) && (87)\\ = \; & B1 \cdot B2 + B3 \cdot B4 + B5 \cdot B6 \\ \approx \; & 1 \cdot \frac{(\frac{b_i}{\Delta} + 1)}{(\frac{\bar{b}}{\Delta} + 1)} \cdot n_B \varepsilon + \frac{\frac{\bar{b} - b_i}{\Delta}}{(\frac{\bar{b}}{\Delta} + 1)} \cdot n_B \varepsilon + \frac{1}{(\frac{\bar{b}}{\Delta} + 1)} \cdot 1 \\ \approx \; & \frac{1}{(\frac{\bar{b}}{\Delta} + 1)} \cdot 1. \end{aligned}$$

Intuition is very simple. For all three cases where a buyer, a seller, or a nonstrategic bidder places a bid at b_i, when a nonstrategic bidder places a bid b_i, every seller bids equal or above b_i, and every buyer bids below b_i, the probability is highest and dominates other possibilities.

134. Calculation of C. This case is symmetric to B. Thus,

$$(C) \approx \frac{1}{(\frac{\bar{b}}{\Delta}+1)} \cdot 1 \tag{88}$$

By collecting above cases, we can derive the formula for $\Pr(\overline{W}_{i,n} = b_i|\theta = x_0)$.

135. Formula for $\Pr(\overline{W}_{i,n} = b_i|\theta = x_0)$. From (43), (46), and (88),

$$\begin{aligned} & \Pr(\overline{W}_{i,n} = b_i|\theta = x_0) \\ = & \underbrace{\Pr(\overline{W}_{i,n} = b_i|\theta = x_0, \hat{i} \text{ nonactive})}_{\text{A}}(1-\delta) \\ & + \underbrace{\Pr(\overline{W}_{i,n} = b_i|\theta = x_0, \hat{i} \text{ buyer})}_{\text{B}}(\delta/2) \\ & + \underbrace{\Pr(\overline{W}_{i,n} = b_i|\theta = x_0, \hat{i} \text{ seller})}_{\text{C}}(\delta/2) \\ \approx & \; n_B\varepsilon \cdot (1-\delta) + 2 * \frac{1}{(\frac{\bar{b}}{\Delta}+1)} \cdot 1 \cdot (\delta/2) \\ \simeq & \; n_B\varepsilon + \frac{1}{(\frac{\bar{b}}{\Delta}+1)}\delta. \end{aligned} \tag{89}$$

which is independent of b_i. Intuitively, for sufficiently small ε and δ, the most probable case is that (1) every $n_S - 1$ seller chooses a bid equal or above b_i, one buyer bids b_i, and every other buyer bids strictly below b_i (this corresponds to the first term), and (2) every $n_S - 1$ seller chooses a bid equal or above b_i, every buyer chooses a bid strictly below b_i, and a nonstrategic bidder chooses a bid b_i (this corresponds to the second term). Since a nonstrategic bidder chooses a bid uniformly over B_Δ, the probabilities is approximately uniform, independent of the specific value of b_i.

136. Verification of (41). It follows from (89) that

$$\Pr(\overline{W}_{i,n} = v(1,1)|x_i) \approx \Pr(\overline{W}_{i,n} = v(1,1) - \triangle|x_i). \tag{90}$$

It follows from (90) that (41) holds.

7. PROOF OF PROPOSITION 1(B)

7.1. *Lemma 3.1*

LEMMA. *β^*_Δ is a nontrivial equilibrium of $\mathcal{G}(\gamma$,f,$\Delta)$.*

PROOF.

137. Overview of the proof. We first derive the double outcome functions in the large economy $\mathcal{G}(\gamma,f,\Delta)$. We then show that β^*_Δ is an equilibrium of $\mathcal{G}(\gamma$,f,$\Delta)$. Finally, we show that β^*_Δ is nontrivial. We first begin with the definition of the price and the allocation function in the auction mechanism in the large economy.

138. The price function $p(b)$. Let b be a profile of bids. Given a profile of bids b, we can construct an empirical distribution function of b as follows. Let μ be an uniform measure on the unit interval. Since the set of possible bids B_Δ is finite, it is legitimate to define, for each $b_i \in B_\Delta$,

$$\mu(b_i) = \{j \in [0,1] : b_j = b\}. \tag{91}$$

$\mu(b_i)$ is the ratio of players who bid b_i. By definition, $\sum_{b_i \in B_\Delta} \mu(b_i) = 1$.

We now recall that $1-\alpha$ is the ratio of the seller in the economy. We define

$$\overline{p}(b) = \min\{b_i \in B_\Delta : \sum_{b' \geq b_i} \mu(b_i) \leq 1-\alpha\}. \tag{92}$$

and

$$\underline{p}(b) = \max\{b_i \in B_\Delta : \sum_{b' \geq b_i} \mu(b_i) > 1-\alpha\}. \tag{93}$$

That is, $\overline{p}(b)$ is the smallest bid such that the ratio of bids equal or above b_i is less than $1-\alpha$. $\underline{p}(b)$ is the largest bid such that the ratio of bids equal or above b_i exceeds $\underline{p}(b)$. Here $\overline{p}(b)$ and $\underline{p}(b)$ is an extension of $b_{n_S:n}$ and $b_{n_S+\mathbf{1}:n}$ in the finite economy to the large economy.

The market clearing price $p(b)$ is determined as $p(b) = k\ \underline{p}(b)\ +(1-k)\overline{p}(b)$[3].

139. The allocation function. A bid which is greater than $\overline{p}(b)$ is sure to be assigned a good, since the ratio of bids above $\overline{p}(b)$ is less than $1-a$. But the players who bid $\underline{p}(b)$ may need to be rationed.

$$q(b_i, b_{-i}) = \begin{cases} 1 \text{ if } b_i \geq \overline{p}(b) \\ 1 \text{ with probability } \frac{(1-\alpha)-\mu(j:b_j>p(b))}{\mu(j:b_j=\underline{p}(b))} \text{if } \overline{p}(b) > b_i = \underline{p}(b) \\ 0 \text{ else.} \end{cases} \tag{94}$$

Given these price and the allocation functions, we can define the game induced by the double auctions defined by $p(b)$ and $q(b_i,b_{-i})$.

140. Notations for the double auction game in the large market $\mathcal{G}(\boldsymbol{\gamma},f,\Delta)$. Let $U_i(x_i,b_i,\beta_{-i})$ and $\pi_i(\beta_i,\beta_{-i})$ be the interim and the ex ante expected payoff at $\mathcal{G}(\boldsymbol{\gamma},f,\Delta)$. Let $BR_i(x_i,\beta_{-i,\Delta}) = \arg\max_{b'_i} U_i(x_i,b'_i,\beta_{-i,\Delta})$ be player i's best response to other players' strategies $\beta_{-i,\Delta}$.

[3]When there are multiple possible values of $\underline{p}(b)$ and $\overline{p}(b)$ which are consistent with the definition of (92) and (93) because of negligible number of bids, we will choose the one which will be consistent with the behavior of the market clearing price in the finite economy. That is, we define $\underline{p}(b)$ and $\overline{p}(b)$ as the limit of $b_{|N_{r,S}|:|N_r|}$ and $b_{|N_{r,S}|+\mathbf{1}:|N_r|}$.

141. Definition of the limit strategy. As defined in the text, let $\beta^*_\Delta = (\beta^*_{\Delta,b},\beta^*_{\Delta,s})$ be its subsequence limit of a sequence of a nontrivial mixed strategy equilibria $\beta^*_{\Delta,n}$ as $n \to \infty$.

We now show that β^*_Δ is an equilibrium of $\mathcal{G}(\gamma,\text{f},\Delta)$. Following the discussion in the text, we show $\pi_{n,i}(\beta^*_{\Delta,n,i},\beta^*_{,\Delta,n,-i}) \to \pi_{n,i}(\beta^*_{\Delta,i},\beta^*_{\Delta,-i})$ as and $\pi_{n,i}(\beta^*_{\Delta,i},\beta^*_{\Delta,-i}) \to \pi_i(\beta^*_{\Delta,i},\beta^*_{\Delta,-i})$. We first show the convergence in terms of the buyer's payoff. The argument for the seller's payoff is similar.

142. Show $\pi_{n,i}(\beta^*_{\Delta,n,i},\beta^*_{\Delta,n,-i}) \to \pi_{n,i}(\beta^*_{\Delta,i},\beta^*_{\Delta,-i})$. By definition.

$$\begin{aligned}&\pi_{n,i}(\beta^*_{\Delta,n,i},\beta^*_{\Delta,n,-i}) \qquad (95)\\ = &\int_{[0,1]} \sum_{\mathcal{X}\times\ldots\times\mathcal{X}} \left[\sum_{B_\Delta\times\ldots\times B_\Delta} u_i(x_0,x_i,p(b_i,b_{-i}),q(b_i,b_{-i}))h^*_{i,\Delta,n}(b_i|x_i)h^*_{-i,\Delta,n}(b_{-i}|x_{-i})\right]\\ &f_{\theta,X_i,X_{-i}}(x_0,x_i,x_{-i})dx_0\end{aligned}$$

$\beta^*_{i,\Delta,n} \to \beta^*_{i,\Delta}$ implies that $h^*_{i,\Delta,n}(b_i|x_i) \to h^*_{i,\Delta}(b_i|x_i)$ for every x_i and every b_i. Thus, $\pi_{i,n}(\beta^*_{i,\Delta,n},\beta^*_{-i,\Delta,n}) \to \pi_{i,n}(\beta^*_{i,\Delta},\beta^*_{-i,\Delta})$ as $\beta^*_{i,\Delta,n} \to \beta^*_{i,\Delta}$.

Having shown the convergence of $\pi_{n,i}(\beta^*_{\Delta,n,i},\beta^*_{\Delta,n,-i}) \to \pi_{n,i}(\beta^*_{\Delta,i},\beta^*_{\Delta,-i})$, we would like to show that $\pi_{n,i}(\beta^*_{\Delta,i},\beta^*_{\Delta,-i}) \to \pi_i(\beta^*_{\Delta,i},\beta^*_{\Delta,-i})$. For that purpose, We first take a look at the ex post payoff function.

143. Decomposition of the ex ante payoff function in terms of allocation and price. By definition,

$$\begin{aligned}&\pi_{n,i}(\beta^*_{\Delta,i},\beta^*_{\Delta,-i}) \qquad (96)\\ = &\int_{[0,1]}\sum_{\mathcal{X}_\gamma}\underbrace{\sum_{\mathcal{X}_\gamma\times\ldots\times\mathcal{X}_\gamma}}_{n-1}\sum_{B_\Delta\times\ldots\times B_\Delta}[v(x_0,x_i)-p(b_i,b_{-i}))q(b_i,b_{-i})]\\ &h_{\Delta,n,i}(b_i|x_i)h_{\Delta,n,-i}(b_{-i}|x_{-i})f_{X_i|\theta}(x_i|x_0)f_{X_{-i}|\theta}(x_{-i}|x_0)f_\theta(x_0)dx_0\\ = &[\int_{[0,1]}\sum_{s_\gamma}(\underbrace{\sum_{\mathcal{X}_\gamma\times\ldots\times\mathcal{X}_\gamma}}_{n-1} v(x_0,x_i)\sum_{B_\Delta\times\ldots\times B_\Delta} q(b_i,b_{-i})h_{\Delta,n,-i}(b_{-i}|x_{-i})\\ &f_{X_{-i}|\theta}(x_{-i}|x_0))h_{\Delta,n,i}(b_i|x_i)f_{X_i|\theta}(x_i|x_0) \qquad (97)\\ &-\int_{[0,1]}\sum_{\mathcal{X}_\gamma}(\underbrace{\sum_{\mathcal{X}_\gamma\times\ldots\times\mathcal{X}_\gamma}}_{n-1}\sum_{B_\Delta\times\ldots\times B_\Delta} p(b_i,b_{-i})q(b_i,b_{-i})h_{\Delta,n,-i}(b_{-i}|x_{-i})\\ &f_{X_{-i}|\theta}(x_{-i}|x_0))h_{\Delta,n,i}(b_i|x_i)f_{X_i|\theta}(x_i|x_0)f_\theta(x_0)dx_0. \qquad (98)\end{aligned}$$

Thus, it is suffice to show weak convergence of the distribution of the price and the allocation conditional on x_0 as the size of the market increases. In order to show this, we first define the empirical distribution of bids and its convergence.

144. The empirical distribution of bids. The empirical distribution function of bids b_1,....,b_n is

$$H_n(b') = \frac{1}{n}\sum_{i=1}^{n} 1_{b_i \le b'}. \tag{99}$$

That is, the distribution function has a jump of $\frac{1}{n}$ at each bid. Then, decomposing the empirical distribution function into the buyer's and sellers' bids distribution.

$$\begin{aligned} H_n(b') &= \frac{1}{n}\left[\sum_{i\in N_{n,B}} 1_{b_i\le b'} + \sum_{i\in N_{n,S}} 1_{b_i\le b'}\right] \\ &= \frac{n_B}{n}\left[\frac{1}{n_B}\sum_{i\in N_{n,B}} 1_{b_i\le b'}\right] + \frac{n_S}{n}\left[\frac{1}{n}\sum_{i\in N_{n,S}} 1_{b_i\le b'}\right]. \end{aligned} \tag{100}$$

145. Convergence of the empirical distribution of bids. By assumption 4, conditional on x_0, players' signals are iid. Thus, as n increases, for each of buyer and seller, the empirical distribution of bids according to $\beta_{\Delta,i}$ converges uniformly to the distribution of bids under $\beta_{\Delta,i}$. Since, as is seen in (100), the distribution of bids is a convex combination of the distribution of buyers' bids and the distribution of sellers' bids, the empirical distribution of bids under the strategies $\beta_{\Delta,i}$ in $\mathcal{G}(\gamma,f,\Delta,n,\delta)$ converges uniformly to the distribution of bids in $\mathcal{G}(\gamma$,f,$\Delta)$ under the strategies $\beta_{\Delta,i}$.

146. Convergence of the distribution of the price. Since the distribution function of the order statistics is a continuous function of the distributions function of bids, by the continuous mapping theorem, the empirical distribution of order statistics also converges uniformly in the distribution of order statistics. By k-double auction rule, the price is a convex combination of the n_Sth and the n_S+1st bids. Thus the distribution function the price in the finite game $\mathcal{G}(\gamma,f,\Delta,n,\delta)$ under the strategies $\beta_{i,\Delta}$ converges uniformly to the distribution function of the price in $\mathcal{G}(\gamma$,f,$\Delta)$ under the strategies $\beta_{i,\Delta}$.

147. Convergence of the distribution of allocation. We note, by fixing β_Δ, the probability that a bid b_i win is

$$\begin{aligned} &\Pr_n(q_i(b_i,b_{-i})=1|x_0) \\ =\ &\Pr_n(b_i > W_{i,n}|x_0) + \frac{n_S - \#(j: b_j > W_{i,n})}{\#(j: b_j = W_{i,n})}\cdot \Pr_n(b_i = W_{i,n}|x_0) \\ \to\ &\Pr(b_i > W_i|x_0) + \frac{1-\Pr(b_j > W_{i,n}|x_0)}{\Pr(b_j = W_{i,n}|x_0)}\cdot \Pr(b_i = W_i|x_0) \text{ as } n\to\infty. \end{aligned} \tag{101}$$

The last line follows since the distribution of bids converges. Therefore, the distribution of allocation also converges.

148. Convergence of the expected payoffs. From the arguments above, since $\pi_{i,n}(\beta^*_{i,\Delta,n},\beta^*_{-i,\Delta,n})$ is a continuous function of the allocation q_n and the price p_n, and p_n and q_n converges in distribution for each x_0, $\pi_{i,n}(\beta^*_{i,\Delta},\beta^*_{-i,\Delta})$ converge to $\pi_i(\beta^*_{i,\Delta},\beta^*_{-i,\Delta})$.

149. Nontriviality of β^*_Δ. Suppose β^*_Δ is trivial. Then, it has to be that for very large n, the probability of trade has to be arbitrary small. This implies that even the seller with signal $x_i = 0$ will choose to bid at least $v(1{,}1)$, although there are some buy bids (in order to keep the probability of trade positive). But then the argument similar to the previous Lemma shows that these sellers will prefer to trade by decreasing the bid, and it is a contradiction to an assumption of that β^*_Δ is trivial

7.2. *Lemma 3.2*

LEMMA. *In $\mathcal{G}(\gamma,f,\Delta)$, there exists $\overline{\Delta} > 0$ such that for all $\Delta < \overline{\Delta}$, for every player i, player i's best response to $\beta^*_{i,\Delta}$ satisfies the strict single crossing condition for bids $\overline{b}_i > \underline{b}_i$ such that $\overline{b}_i$ is in the range of equilibrium prices.*

PROOF.

151. Introduction. The argument is based on the point made in the text that buyers and sellers in $\mathcal{G}(\gamma,f,\Delta)$ are symmetric. we then show that when a player has a more optimistic signal, the expected value of the good increases with a uniform lower bound of the rate of increase. Then we study strict single crossing conditions for adjacent signals and bids. Then we extend the local single crossing condition to a more general case.

Pesendorfer and Swinkels (1997) provided a strikingly beautiful proof that the single crossing condition holds for the best response to a mixed strategy equilibrium in a one-sided uniform price auction with a continuous set of bids among a finite number of symmetric bidders. The proof here extends their argument to show the strict single crossing condition holds for best response to a possibly asymmetric mixed strategy equilibrium in a large uniform price double price auction with a finite set of bids in the large economy with a finite set of signals.

152. Symmetry of payoffs functions between buyers and sellers in $\mathcal{G}(\gamma,f,\Delta)$. Let $i \in N_{n,S}$. In the large economy, a player's bid is negligible. Thus it does not affect the market clearing price of the double auction in the large economy. That is, $p_n(b_{-i})$ will converge to $p(b_i,b_{-i})$ as $n \to \infty$. Thus,

$$\begin{aligned}
&\lim_{n\to\infty} U_{n,i}(x_i, b_i, \beta_{\Delta,-i}) \qquad (102)\\
= &\int \sum_{\mathcal{X}\times\ldots\times\mathcal{X}} \sum_{B_\Delta\times\ldots\times B_\Delta} (p(b_i, b_{-i}) - v(x_0, x_i))(1 - q(b_i, b_{-i}))\\
&h(b_{-i}|x_{-i}) f_{\theta,X_{-i}|X_i}(x_0, x_{-i}|x_i) dx_0\\
&-\int \sum_{\mathcal{X}\times\ldots\times\mathcal{X}} \sum_{B_\Delta\times\ldots\times B_\Delta} (p(b_i, b_{-i}) - v(x_0, x_i)) h(b_{-i}|x_{-i})\\
&f_{\theta,X_{-i}|X_i}(x_0, x_{-i}|x_i) dx_0\\
= &\int \sum_{\mathcal{X}\times\ldots\times\mathcal{X}} \sum_{B_\Delta\times\ldots\times B_\Delta} (v(x_0, x_i) - p(b_i, b_{-i})) q(b_i, b_{-i})) h(b_{-i}|x_{-i})\\
&f_{\theta,X_{-i}|X_i}(x_0, x_{-i}|x_i) dx_0
\end{aligned}$$

which is equal to a buyer's payoff in $\mathcal{G}(\gamma,f,\Delta)$. That is, in $\mathcal{G}(\gamma,f,\Delta)$, buyers and sellers have symmetric payoffs. Let $U(x_i,b_i,\beta_{-i,\Delta})$ denote the payoff function common to buyers and sellers. Let $BR(x_i,\beta)$ denote a best response correspondence of player with signal x_i when all other players follows strategy β, which is common to buyers and sellers.

Let me now introduce notations to define strict single crossing condition.

153. Setup. Let $\overline{x}_i > \underline{x}_i$ and $\overline{b}_i \in BR(\underline{x}_i,\beta^*_{\Delta,b},\beta^*_{\Delta,s})$. Then it follows that

$$U(\underline{x}_i, \overline{b}_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) - U(\underline{x}_i, \underline{b}_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) \geq 0. \qquad (103)$$

we now would like to show

$$U(\overline{x}_i, \overline{b}_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) - U(\overline{x}_i, \underline{b}_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) > 0. \qquad (104)$$

That is, we would like to show that when a player with a signal $\underline{x}_i$ prefers $\overline{b}_i$ to $\underline{b}_i$ and a signal increases from $\underline{x}_i$ to $\overline{x}_i$, then a player with a signal $\overline{x}_i$ still prefers $\overline{b}_i$ to $\underline{b}_i$. We now focus on the argument for The other condition of strict single crossing condition

We now introduce a useful event to express (103) and (104).

154. The event of winning from increasing the bid. When a player increases a bid from $\underline{b}_i$ to $\overline{b}_i$, a player wins a tie at price $\underline{b}_i$ and will be possibly tied at price $\overline{b}_i$. Let $Y(\underline{b}_i,\overline{b}_i)$ be this event, that is, an event that a bid $\underline{b}_i$ may not lead to an assignment of the good with positive probability the but a bid $\overline{b}_i$ will lead to an assignment of the good with positive probability given other players use an equilibrium strategy β^*_Δ.

The event $Y(\underline{b}_i,\overline{b}_i)$ may be empty if bids $\underline{b}_i$ and $\overline{b}_i$ are too high so that both bids lead to the assignment for sure or too low so that both bids will not lead to an assignment for sure. To deal with this possibility, we first define the range of prices which can take place with positive probability when buyers and sellers choose strategies $\beta^*_{\Delta,b}$ and $\beta^*_{\Delta,s}$.

155. The range of equilibrium prices. Let $p^{\min}(\beta^*_{\Delta,b},\beta^*_{\Delta,s})$ and $p^{\max}(\beta^*_{\Delta,b},\beta^*_{\Delta,s})$ be the lowest and the highest prices that can arise with a positive probability as an equilibrium outcome of $(\beta^*_{\Delta,b},\beta^*_{\Delta,s})$. Let $\mathcal{P}(\beta^*_{\Delta,b},\beta^*_{\Delta,s})$ be the set of prices which take place with positive probabilities given that players use strategies $\beta^*_{\Delta,b}$ and $\beta^*_{\Delta,s}$.

Given this definition, we only consider cases where $Y(\underline{b}_i,\overline{b}_i)$ is relevant.

156. Cases when $Y(\underline{b}_i,\overline{b}_i)$ is nonempty. We note that

$$\begin{aligned} & \underline{b}_i \leq p^{\max}(\beta^*_{\Delta,b},\beta^*_{\Delta,s}) \text{ and } \overline{b}_i \geq p^{\min}(\beta^*_{\Delta,b},\beta^*_{\Delta,s}) \\ \rightarrow \quad & f(Y(\underline{b}_i,\overline{b}_i)|x_0) > 0. \end{aligned} \tag{105}$$

To see this, suppose $\underline{b}_i \leq p^{\max}(\beta^*_{\Delta,b},\beta^*_{\Delta,s})$ and $\overline{b}_i \geq p^{\min}(\beta^*_{\Delta,b},\beta^*_{\Delta,s})$. Then, with some positive probability, $\underline{b}_i$ will not be assigned a good and $\overline{b}_i$ will be assigned a good and $f(Y(\underline{b}_i,\overline{b}_i)|x_0) > 0$. On the other hand, if $\underline{b}_i > p^{\max}(\beta^*_{\Delta,b},\beta^*_{\Delta,s})$ or $\overline{b}_i < p^{\min}(\beta^*_{\Delta,b},\beta^*_{\Delta,s})$, then both bids $\underline{b}_i$ and $\overline{b}_i$ have the same outcome. By Assumption 4, $f(x_i|x_0) > 0$ for every x_i. Thus, if $f(x_i,Y(\underline{b}_i,\overline{b}_i)) > 0$ for some x_i, then $f(x_i,Y(\underline{b}_i,\overline{b}_i)) > 0$ for every x_i. Therefore, for every x_i,the support of $f(x_i,Y(\underline{b}_i,\overline{b}_i))$ is the same and equal to $\{(\underline{b}_i,\overline{b}_i) : \underline{b}_i \leq p^{\max}(\beta^*_{\Delta,b},\beta^*_{\Delta,s}) \text{ and } \overline{b}_i \geq p^{\min}(\beta^*_{\Delta,b},\beta^*_{\Delta,s})\}$. Hereafter, in the rest of the analysis, we consider $\underline{b}_i$ and $\overline{b}_i$ such that $\underline{b}_i \leq p^{\max}(\beta^*_{\Delta,b},\beta^*_{\Delta,s})$ and $\overline{b}_i \geq p^{\min}(\beta^*_{\Delta,b},\beta^*_{\Delta,s})$.

In order to evaluate the marginal change in expected payoffs when a signal changes, we decompose the change into the change which comes from the change in the expected value of the good and the change which comes from the change in the expected payment.

157. Decomposition of payoffs. We note that

$$\begin{aligned} & U(x_i,\overline{b}_i,\beta^*_{\Delta,b},\beta^*_{\Delta,s}) - U(x_i,\underline{b}_i,\beta^*_{\Delta,b},\beta^*_{\Delta,s}) \\ = \quad & \mathbf{E}[v(\theta X_i) - p(b_i,b_{-i})|X_i = x_i, Y(\underline{b}_i,\overline{b}_i)] \\ = \quad & \mathbf{E}[v(\theta X_i)|X_i = x_i, Y(\underline{b}_i,\overline{b}_i)] - \mathbf{E}[p(b_i,b_{-i})|X_i = x_i, Y(\underline{b}_i,\overline{b}_i)] \end{aligned} \tag{106}$$

where the first term denotes the change in the estimated value and the second terms denote the expected changes in the price.

We first estimate the change in the expected value of the good when a player's signal increases. For that purpose, we first consider a property of the distribution of θ conditional on X_i and $Y(\underline{b}_i,\overline{b}_i)$.

158. Monotone likelihood condition for $f_{\theta|X_i,Y}(x_0|x_i,Y(\underline{b}_i,\overline{b}_i))$. From conditional independence, it follows that

$$
\begin{aligned}
f_{\theta|X_i,Y}(x_0|x_i,Y(\underline{b}_i,\overline{b}_i)) &= \frac{f_{\theta,X_i,Y}(x_0,x_i,Y(\underline{b}_i,\overline{b}_i))}{f_{X_i,Y}(x_i,Y(\underline{b}_i,\overline{b}_i))} \\
&\quad \text{by definition of conditional expectation} \\
&= \frac{f_{X_i,Y|\theta}(x_i,Y(\underline{b}_i,\overline{b}_i)|x_0)f_\theta(x_0)}{f_{X_i,Y}(x_i,Y(\underline{b}_i,\overline{b}_i))} \\
&\quad \text{by definition of conditional expectation} \\
&= \frac{f_{X_i|\theta}(x_i|x_0)f_{Y|\theta}(Y(\underline{b}_i,\overline{b}_i)|x_0)f_\theta(x_0)}{f_{X_i,Y}(x_i,Y(\underline{b}_i,\overline{b}_i))} \\
&\quad \text{by conditional independence}
\end{aligned} \tag{107}
$$

The likelihood ratio for $\overline{x}_0 > \underline{x}_0$ is

$$
\begin{aligned}
\frac{f(\overline{x}_0|\overline{x}_i,Y(\underline{b}_i,\overline{b}_i))}{f(\overline{x}_0|\underline{x}_i,Y(\underline{b}_i,\overline{b}_i))} &= \frac{\left(\frac{f(\overline{x}_i|\overline{x}_0)}{f(\overline{x}_i,Y(\underline{b}_i,\overline{b}_i))}\right)}{\left(\frac{f(\underline{x}_i|\overline{x}_0)}{f(\underline{x}_i,Y(\underline{b}_i,\overline{b}_i))}\right)} \text{ by (107)} \\
&= \frac{f(\overline{x}_i|\overline{x}_0)}{f(\underline{x}_i|\overline{x}_0)} \cdot \frac{f(\underline{x}_i,Y(\underline{b}_i,\overline{b}_i))}{f(\overline{x}_i,Y(\underline{b}_i,\overline{b}_i))} \text{ by rewriting} \\
&\geq \frac{f(\overline{x}_i|\underline{x}_0)}{f(\underline{x}_i|\underline{x}_0)} \cdot \frac{f(\underline{x}_i,Y(\underline{b}_i,\overline{b}_i))}{f(\overline{x}_i,Y(\underline{b}_i,\overline{b}_i))} \\
&= \frac{\left(\frac{f(\overline{x}_i|\underline{x}_0)}{f(\overline{x}_i,Y(\underline{b}_i,\overline{b}_i))}\right)}{\left(\frac{f(\underline{x}_i|\underline{x}_0)}{f(\underline{x}_i,Y(\underline{b}_i,\overline{b}_i))}\right)} \text{ by rewriting} \\
&= \frac{f(\underline{x}_0|\overline{x}_i,Y(\underline{b}_i,\overline{b}_i))}{f(\underline{x}_0|\underline{x}_i,Y(\underline{b}_i,\overline{b}_i))} \text{ by (107)}
\end{aligned} \tag{108}
$$

Therefore, $f(x_0|x_i,Y(\underline{b}_i,\overline{b}_i))$ satisfies the affiliation inequality. Intuitively, $f(x_0|\underline{x}_i,Y(\underline{b}_i,\overline{b}_i))$ is considered a garbling of $f(x_0|x_i,x_{-i})$ and x_{-i} does not affect the statistical relationship between θ and X_i[4].

Building on this property of the conditional distributions, we can now estimate the change in the expected value of the good.

[4]Pesendorfer and Swinkels (1997) proposed this interpretation.

159. Lower bound of the rate of change of $E\left[v(x_0,x_i)|x_i,Y(\underline{b}_i,\overline{b}_i)\right]$. We note

$$\begin{aligned}
& E\left[v(x_0,\overline{x}_i)|\overline{x}_i,Y(\underline{b}_i,\overline{b}_i)\right]-E\left[v(x_0,\underline{x}_i)|\underline{x}_i,Y(\underline{b}_i,\overline{b}_i)\right] \qquad (109)\\
= & \underbrace{E\left[v(x_0,\overline{x}_i)|\overline{x}_i,Y(\underline{b}_i,\overline{b}_i)\right]-E\left[v(x_0,\underline{x}_i)|\overline{x}_i,Y(\underline{b}_i,\overline{b}_i)\right]}_{\text{private value effect while keeping the conditioning information constant}}\\
& +\underbrace{E\left[v(x_0,\underline{x}_i)|\overline{x}_i,Y(\underline{b}_i,\overline{b}_i)\right]-E\left[v(x_0,\underline{x}_i)|\underline{x}_i,Y(\underline{b}_i,\overline{b}_i)\right]}_{\text{common value effect while keeping the private value element constant}}\\
& \text{by adding and subtracting } E\left[v(x_0,\underline{x}_i)|\overline{x}_i,Y(\underline{b}_i,\overline{b}_i)\right]\\
\geq & \underline{\lambda}(\overline{x}_i-\underline{x}_i).
\end{aligned}$$

For the last inequality, for the private value effect, by Assumption 5, $v(x_0,x_i)$ is increasing in x_i with uniform lower bound of the rate of increase $\underline{\lambda}$. Thus, the difference is at least $\underline{\lambda}(\overline{x}_i-\underline{x}_i)$. For the common value effect is nonnegative because of the affiliation inequality (108) and Theorem 5 of Milgrom and Weber (1982a).

We prove the single crossing conditions using mathematical induction. As a first step, we consider the local case where two bids are adjacent and then extend to then case where two bids are more than one bid step size apart.

160. Strict single crossing condition for adjacent signals and bids. Suppose that two bids are adjacent ($\overline{b}_i=\underline{b}_i+\Delta$) and two signals are adjacent ($\overline{x}_i=\underline{x}_i+\gamma$). Take Δ to be sufficiently small so that

$$\underline{\lambda}\gamma>\Delta. \qquad (110)$$

Then,

$$\begin{aligned}
& U(\overline{x}_i,\overline{b}_i,\beta^*_{\Delta,b},\beta^*_{\Delta,s})-U(\overline{x}_i,\underline{b}_i,\beta^*_{\Delta,b},\beta^*_{\Delta,s}) \qquad (111)\\
= & \mathbf{E}[v(\theta X_i)|X_i=\overline{x}_i,Y(\underline{b}_i,\overline{b}_i)]-\mathbf{E}[p(b_i,b_{-i})|X_i=\overline{x}_i,Y(\underline{b}_i,\overline{b}_i)]\\
\geq & \underbrace{\mathbf{E}[v(\theta X_i)|X_i=\underline{x}_i,Y(\underline{b}_i,\overline{b}_i)]+\underline{\lambda}\gamma}_{\text{by (109)}}\\
& -\underbrace{(\mathbf{E}[p(b_i,b_{-i})|X_i=\underline{x}_i,Y(\underline{b}_i,\overline{b}_i)]+\Delta)}_{\text{upper bound of the possible increase in payment}}\\
= & \underbrace{(\mathbf{E}[v(\theta X_i)|X_i=\underline{x}_i,Y(\underline{b}_i,\overline{b}_i)]-\mathbf{E}[p(b_i,b_{-i})|X_i=\underline{x}_i,Y(\underline{b}_i,\overline{b}_i)])}_{\geq 0 \text{ by (103)}}\\
& +\underbrace{(\underline{\lambda}\gamma-\Delta)}_{>0 \text{ by (110)}}\\
> & 0.
\end{aligned}$$

That is, when $\overline{b}_i-\underline{b}_i$ is sufficiently small, the increase in the value estimate outweighs the possible change in the price and the single crossing condition holds. Finally, we note that it is immediate to extend the argument to the case where the two signals are more than γ apart.

We now move to the next step of the induction. For a simplicity of the argument, we consider the case where bids are two step size apart. We start by defining cases.

161. Strict single crossing conditions for the case of $\overline{b}_i = \underline{b}_i + 2\Delta$ and $\overline{x}_i = \underline{x}_i + \gamma$. Let

$$\tilde{x}_i = \max_{x_i \in \mathcal{X}} \{x_i : \underline{b}_i + \Delta \in BR(x_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s})\}. \tag{112}$$

That is, $\tilde{x}_i$ is the highest signal such that a player with signal $\tilde{x}_i$ will bid $\underline{b}_i + \Delta$. Then, depending on the bids at $\tilde{x}_i$, there are three possible cases.

- There is no $\tilde{x}_i$. That is, no player bids $\underline{b}_i + \Delta$.
- $\tilde{x}_i \leq \overline{x}_i$.
- $\tilde{x}_i > \overline{x}_i$.

To see this, please consult the figure below.

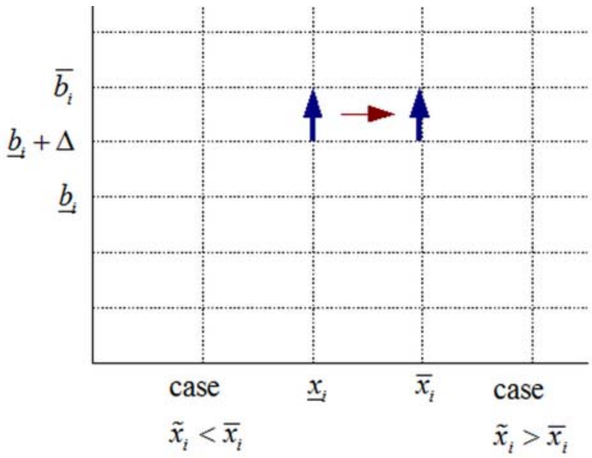

FIGURE 4.—

We now consider the first case where there is no $\tilde{x}_i$. In this case, we decompose the change into two cases where the price is $\overline{b}_i$ and where the price is $\underline{b}_i$. When there are no bids at $\underline{b}_i + \Delta$, then it has to be that player with signal $\underline{x}_i$ has nonnegative payoffs from winning at $\overline{b}_i$. Thus, even when the price $\overline{b}_i$ becomes more likely, the player with signal $\overline{x}_i$ has a positive expected payoff. We elaborate this intuition below.

162. Decomposition of payoffs. We note that 106 can be further expanded into

$$\begin{aligned} & U(x_i, \overline{b}_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) - U(x_i, \underline{b}_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) \qquad (113)\\ = \; & \Pr(p(b) = \underline{b}_i | X_i = x_i, Y(\underline{b}_i, \overline{b}_i)) \left[\mathbf{E}[v(\theta X_i) | X_i = x_i, Y(\underline{b}_i, \overline{b}_i), p(b) = \underline{b}_i] - \underline{b}_i \right] + \\ & \Pr(p(b) = \overline{b}_i | X_i = x_i, Y(\underline{b}_i, \overline{b}_i)) \left[\mathbf{E}[v(\theta X_i) | X_i = x_i, Y(\underline{b}_i, \overline{b}_i), p(b) = \overline{b}_i] - \overline{b}_i \right]. \end{aligned}$$

That is, the change in payoffs come from the case of winning the tie at the price $\underline{b}_i$ and the case of winning (may be at the tie) at the price $\overline{b}_i$.

We first estimate the expected payoff from winning at the price $\overline{b}_i$.

163. Estimation of $E[v(\theta X_i)|X_i = \overline{x}_i, Y(\underline{b}_i, \overline{b}_i), p(b) = \overline{b}_i] - \overline{b}_i$**.** By assumption, it is a best response for a player with signal $\underline{x}_i$ to bid $\overline{b}_i$. Then, it has to be that winning at the price $\overline{b}_i$ provides a nonnegative payoff, otherwise the player could bid $\underline{b}_i + \Delta$ to avoid this outcome keeping the payoffs from other events the same. That is,

$$\mathbf{E}[v(\theta X_i)|X_i = \underline{x}_i, Y(\underline{b}_i, \overline{b}_i), p(b) = \overline{b}_i] - \overline{b}_i \geq 0. \tag{114}$$

From 109 and 114, we can estimate

$$\begin{aligned} & \mathbf{E}[v(\theta X_i)|X_i = \overline{x}_i, Y(\underline{b}_i, \overline{b}_i), p(b) = \overline{b}_i] - \overline{b}_i \\ \geq\ & \mathbf{E}[v(\theta X_i)|X_i = \underline{x}_i, Y(\underline{b}_i, \overline{b}_i), p(b) = \overline{b}_i] + \underline{\lambda}\gamma - \overline{b}_i \\ >\ & 0. \end{aligned} \tag{115}$$

That is, a player with the high signal gets a positive expected payoff from winning at the high price.

We now consider two cases: (a) the probability that the price is $\overline{b}_i$ decreases with the high signal and (b) the probability that the price is $\overline{b}_i$ increases with the high signal and show that the single crossing condition holds for both cases.

164. Case (a). This is the case where

$$\Pr(p(b) = \overline{b}_i|X_i = \overline{x}_i, Y(\underline{b}_i, \overline{b}_i)) \leq \Pr(p(b) = \overline{b}_i|X_i = \underline{x}_i, Y(\underline{b}_i, \overline{b}_i)) \tag{116}$$

Since the prices can be either $\overline{b}_i$ or $\underline{b}_i$ conditional on $Y(\underline{b}_i, \overline{b}_i)$ given that no one bids $\underline{b}_i + \Delta$, it follows that

$$\Pr(p(b) = \underline{b}_i|X_i = \overline{x}_i, Y(\underline{b}_i, \overline{b}_i)) > \Pr(p(b) = \underline{b}_i|X_i = \underline{x}_i, Y(\underline{b}_i, \overline{b}_i)). \tag{117}$$

It follows from (113) that the marginal increase in the expected payoff from a higher bid is decomposed into

$$\begin{aligned} & U(\overline{x}_i, \overline{b}_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) - U(\overline{x}_i, \underline{b}_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) \\ =\ & \mathbf{E}[v(\theta X_i)|X_i = \overline{x}_i, Y(\underline{b}_i, \overline{b}_i)] - \\ & \underline{b}_i \Pr(p(b) = \underline{b}_i|X_i = \overline{x}_i, Y(\underline{b}_i, \overline{b}_i)) - \overline{b}_i \Pr(p(b) = \overline{b}_i|X_i = \overline{x}_i, Y(\underline{b}_i, \overline{b}_i)). \end{aligned} \tag{118}$$

That is, the change in the expected payoff from a higher bid is decomposed into the change in the expected value from winning at the high bid (and losing at the low bid) and the expected change in the payment.

Now from (109),

$$\mathbf{E}[v(\theta X_i)|X_i = \overline{x}_i, Y(\underline{b}_i, \overline{b}_i)] > \mathbf{E}[v(\theta X_i)|X_i = \underline{x}_i, Y(\underline{b}_i, \overline{b}_i)]. \tag{119}$$

That is, the change in the expected payoff increases when the signal increases. It remains to evaluate the change in the payment when the signal changes.

165. Change in the expected payment. From (116) and (117),

$$\begin{aligned}
& \underline{b}_i \Pr(p(b) = \underline{b}_i | X_i = \overline{x}_i, Y(\underline{b}_i, \overline{b}_i)) + \\
& \overline{b}_i \Pr(p(b) = \overline{b}_i | X_i = \overline{x}_i, Y(\underline{b}_i, \overline{b}_i)) \\
\leq\ & \underline{b}_i \Pr(p(b) = \underline{b}_i | X_i = \underline{x}_i, Y(\underline{b}_i, \overline{b}_i)) + \\
& \overline{b}_i \Pr(p(b) = \overline{b}_i | X_i = \underline{x}_i, Y(\underline{b}_i, \overline{b}_i)).
\end{aligned} \tag{120}$$

That is, since the probability of the high price ($\overline{b}_i$) is lower when the signal is high by the assumption of (116), it has to be that the expected payment decreases.

We now have the estimate of changes in the expected value of the good and the expected payment. Thus,

166. Strict single crossing for case (a). Therefore, by substituting (119) and (120) into (118), we get

$$\begin{aligned}
& U(\overline{x}_i, \overline{b}_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) - U(\overline{x}_i, \underline{b}_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) \\
>\ & U(\underline{x}_i, \overline{b}_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) - U(\underline{x}_i, \underline{b}_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) \\
\geq\ & 0 \text{ from } (103).
\end{aligned}$$

Intuitively, in this case, when the signal increases, the expected value of the good increases and the expected payment decreases. Thus the player with the high signal still prefers the high bid.

167. Case (b). This is the case where

$$\Pr(p(b) = \overline{b}_i | X_i = \overline{x}_i, Y(\underline{b}_i, \overline{b}_i)) \geq \Pr(p(b) = \overline{b}_i | X_i = \underline{x}_i, Y(\underline{b}_i, \overline{b}_i)) \tag{121}$$

It follows that

$$\Pr(p(b) = \underline{b}_i | X_i = \overline{x}_i, Y(\underline{b}_i, \overline{b}_i)) < \Pr(p(b) = \underline{b}_i | X_i = \underline{x}_i, Y(\underline{b}_i, \overline{b}_i)). \tag{122}$$

From (113), we have

$$\begin{aligned}
& U(\overline{x}_i, \overline{b}_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) - U(\overline{x}_i x, \underline{b}_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) \\
=\ & \Pr(p(b) = \underline{b}_i | X_i = \overline{x}_i, Y(\underline{b}_i, \overline{b}_i)) \\
& \left[\mathbf{E}[v(\theta X_i) | X_i = \overline{x}_i, Y(\underline{b}_i, \overline{b}_i), p(b) = \underline{b}_i] - \underline{b}_i\right] + \\
& \Pr(p(b) = \overline{b}_i | X_i = \overline{x}_i, Y(\underline{b}_i, \overline{b}_i)) \\
& \left[\mathbf{E}[v(\theta X_i) | X_i = \overline{x}_i, Y(\underline{b}_i, \overline{b}_i), p(b) = \overline{b}_i] - \overline{b}_i\right].
\end{aligned} \tag{123}$$

That is, the change in the expected payoff is the weighted sum of the payoff when the price is low ($\underline{b}_i$) and high ($\overline{b}_i$).

We first evaluate the change in the expected payoff when the price is high.

169. Expected payoff when the price is high. We recall, following the same line of calculation of (107) and (108),

$$\begin{aligned}
& \mathbf{E}[v(\theta X_i) | X_i = \overline{x}_i, Y(\underline{b}_i, \overline{b}_i), p(b) = \overline{b}_i] \\
>\ & \mathbf{E}[v(\theta X_i) | X_i = \underline{x}_i, Y(\underline{b}_i, \overline{b}_i), p(b) = \overline{b}_i].
\end{aligned} \tag{124}$$

That is, the player with the high signal has the high expected value of the good.

It follows from (115), (121), and (124), we have

$$\begin{aligned}&\Pr(p(b)=\overline{b}_i|X_i=\overline{x}_i,Y(\underline{b}_i,\overline{b}_i))\\&\left[\mathbf{E}[v(\theta X_i)|X_i=\overline{x}_i,Y(\underline{b}_i,\overline{b}_i),p(b)=\overline{b}_i]-\overline{b}_i\right]\\>\ &\Pr(p(b)=\overline{b}_i|X_i=\underline{x}_i,Y(\underline{b}_i,\overline{b}_i))\\&\left[\mathbf{E}[v(\theta X_i)|X_i=\underline{x}_i,Y(\underline{b}_i,\overline{b}_i),p(b)=\overline{b}_i]-\overline{b}_i\right].\end{aligned} \tag{125}$$

That is, the player with the high signal has the higher expected payoff from winning at the high price compared with the player with the lower signal. Since the player with the low signal has a nonnegative expected payoff from winning at the high price, the expected payoff from winning at the high price is positive for the player with the high signal. Since the probability that the price will be high increases when the signal is high in this case, the expected payoff from the high price will be higher.

Using this information, we can study single crossing condition for case (b).

171. Strict single crossing condition for case (b). It follows from (113) and (125), if there is a violation of single crossing conditions, it has to be that

$$\begin{aligned}&\Pr(p(b)=\underline{b}_i|X_i=\overline{x}_i,Y(\underline{b}_i,\overline{b}_i))\\&\left[\mathbf{E}[v(\theta,X_i)|X_i=\overline{x}_i,Y(\underline{b}_i,\overline{b}_i),p(b)=\underline{b}_i]-\underline{b}_i\right]\\<\ &\Pr(p(b)=\underline{b}_i|X_i=\underline{x}_i,Y(\underline{b}_i,\overline{b}_i))\\&\left[\mathbf{E}[v(\theta X_i)|X_i=\underline{x}_i,Y(\underline{b}_i,\overline{b}_i),p(b)=\underline{b}_i]-\underline{b}_i\right]\end{aligned} \tag{126}$$

That is, the expected payoff when the price is low should be lower with the high signal.

It follows from (122), for (126) to hold, it has to be that

$$\begin{aligned}&\mathbf{E}[v(\theta X_i)|X_i=\overline{x}_i,Y(\underline{b}_i,\overline{b}_i),p(b)=\underline{b}_i]\\<\ &\mathbf{E}[v(\theta X_i)|X_i=\underline{x}_i,Y(\underline{b}_i,\overline{b}_i),p(b)=\underline{b}_i].\end{aligned} \tag{127}$$

That is, since the probability of the low price decreases with the high signal for case (b), in order to have a lower expected payoff, it has to be that the expected value of the good decreases with the high signal.

But from a similar calculation with (124),

$$\begin{aligned}&\mathbf{E}[v(\theta X_i)|X_i=\overline{x}_i,Y(\underline{b}_i,\overline{b}_i),p(b)=\underline{b}_i]\\>\ &\mathbf{E}[v(\theta X_i)|X_i=\underline{x}_i,Y(\underline{b}_i,\overline{b}_i),p(b)=\underline{b}_i].\end{aligned} \tag{128}$$

That is, monotone likelihood ratio conditions on the distribution ensures that the player with the high signal has the higher expected value of the good than the player with the low signal.

It follows that (127) cannot happen. Therefore, the single crossing condition holds.

So far we have covered the case where there is no $\tilde{x}_i$. We next consider the case where $\tilde{x}_i \leq \overline{x}_i$ where, $\tilde{x}_i$ is the highest signal who will bid $\underline{b}_i + \Delta$ as defined in (112).

172. Decomposition of payoffs. This is the case where the highest signal who will bid $\underline{b}_i + \Delta$ is very low, lower than $\underline{x}_i$. In this case, we decompose the difference in payoff from a high bid for a high signal $\overline{b}_i$ is decomposed into the sum of incremental change in the payoffs:

$$\begin{aligned} & U(\overline{x}_i, \overline{b}_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) - U(\overline{x}_i, \underline{b}_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) \\ = & \left[U(\overline{x}_i, \overline{b}_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) - U(\overline{x}_i, \underline{b}_i + \Delta, \beta^*_{\Delta,b}, \beta^*_{\Delta,s})\right] \\ & + \left[U(\overline{x}_i, \underline{b}_i + \Delta, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) - U(\overline{x}_i, \underline{b}_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s})\right]. \end{aligned} \tag{129}$$

That is, the difference in the payoff is the sum of differences of the payoff when the player with the signal $\overline{x}_i$ increases the bid from $\underline{b}_i$ to $\underline{b}_i + \Delta$ and the payoff when the player increases the bid from $\underline{b}_i + \Delta$ to $\overline{b}_i$.

We first evaluate the first case, the payoff changes from $\underline{b}_i + \Delta$ to $\overline{b}_i$ from the result in the local case.

174. Change in the payoff from $\underline{b}_i + \Delta$ to $\overline{b}_i$. Since $\overline{b}_i \in BR(\underline{x}_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s})$,

$$U(\underline{x}_i, \overline{b}_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) - U(\underline{x}_i, \underline{b}_i + \Delta, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) \geq 0. \tag{130}$$

Then, from (111),

$$U(\overline{x}_i, \overline{b}_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) - U(\overline{x}_i, \underline{b}_i + \Delta, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) > 0. \tag{131}$$

That is, since the higher bid $\overline{b}_i$ is a best response for a player with signal $\underline{x}_i$ and $\overline{b}_i$ and $\underline{b}_i + \Delta$ are adjacent, by applying the local strict single crossing condition above, the player with the high signal $\overline{x}_i$ has the strictly prefers the high bid $\overline{b}_i$.

We now evaluate the second term.

176. Change in the payoff from $\underline{b}_i$ to $\underline{b}_i + \Delta$. It follows from the definition of $\tilde{x}_i$ that

$$U(\tilde{x}_i, \underline{b}_i + \Delta, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) - U(\tilde{x}_i, \underline{b}_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) \geq 0. \tag{132}$$

Then, it follows from the same argument of (111),

$$U(\overline{x}_i, \underline{b}_i + \Delta, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) - U(\overline{x}_i, \underline{b}_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) \geq 0. \tag{133}$$

Intuitively, the player with signal $\tilde{x}_i$ has $\underline{b}_i + \Delta$ as a best response. Since $\underline{b}_i + \Delta$ and $\underline{b}_i$ are adjacent, and since $\overline{x}_i$ is higher than $\tilde{x}_i$, by applying the local strict single crossing condition, it follows that the player with the signal $\overline{x}_i$ strictly prefers $\underline{b}_i + \Delta$ over $\underline{b}_i$.

178. Strict single crossing condition for $\tilde{x}_i \leq \overline{x}_i$. It follows from (129), (131),and (133), we get $U(\overline{x}_i,\overline{b}_i,\beta^*_{\Delta,b},\beta^*_{\Delta,s}) - U(\overline{x}_i,\underline{b}_i,\beta^*_{\Delta,b},\beta^*_{\Delta,s}) > 0$.

So far we have considered the first two cases. It remains to consider the case of $\tilde{x}_i > \overline{x}_i$.

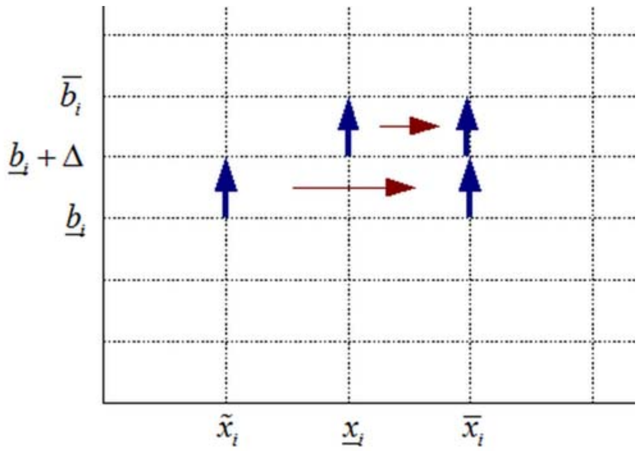

FIGURE 5.—

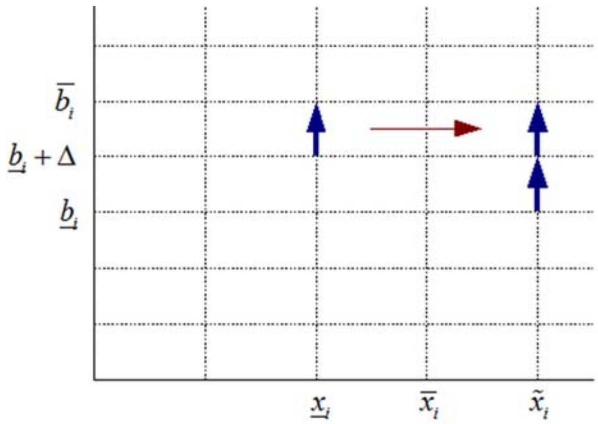

FIGURE 6.—

179. Case where $\tilde{x}_i >. \overline{x}_i$. This is the case that there is a player who bids $\underline{b}_i + \Delta$ can have a very high signal, higher than $\overline{x}_i$. In this case, from (130) and (111),

$$U(\tilde{x}_i, \overline{b}_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) - U(\tilde{x}_i, \underline{b}_i + \Delta, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) \geq 0. \tag{134}$$

That is, since a player with signal $\overline{x}_i$ prefers to bid $\overline{b}_i$ over $\underline{b}_i + \Delta$, the player with the higher signal $\tilde{x}_i$ will prefer to bid $\overline{b}_i$ over $\underline{b}_i + \Delta$ by the local strict single crossing condition. Since it is a strict condition, it is a contradiction to the assumption that $\underline{b}_i + \Delta$ is a best reply for player with signal $\tilde{x}_i$. Consequently, this case will not happen. The figure below describes the argument.

Thus we covered the case of $\overline{b}_i = \underline{b}_i + 2\Delta$ and $\overline{x}_i = \underline{x}_i + \gamma$. It follows from the argument of mathematical induction that the single crossing condition follows for the general case.

7.3. *Lemma 3.3*

Lemma. *There exists $\overline{\Delta} > 0$ such that for all $\Delta < \overline{\Delta}$, for each player i and j, signal $x_i \neq x_j$, and a bid $b_i \in B_\Delta$ which is in the range of equilibrium prices,*

$$\Pr(\beta^*_{\Delta,i}(x_i) = b_i) \cdot \Pr(\beta^*_{\Delta,j}(x_j) = b_i) = 0. \tag{135}$$

Proof

181. Introduction. We start from assuming a contradiction. We first examine the monotonic relationship about the distribution of the support of equilibrium bids and then stochastic dominance relationship of the distribution of bids. It follows that there exists a winner's curse that for each price winning a good is a bad news compared with losing a good. Its consequence is that it is not compatible that players with distinct signals choose the same bid.

The arguments that there would not be a mass point in the distribution of equilibrium bids in an interdependent value environment are presented, among others, in Pesendorfer and Swinkels (1997), Athey (2001), Reny and Zamir (2004), and Reny and Perry (2006). The proof here extends the argument to a possibly asymmetric mixed strategy equilibrium in a large uniform price double auctions with a discrete set of bids in an interdependent value environment.

182. Suppose that there is no $\overline{\Delta} > 0$ which satisfies (135). It follows that for every $\Delta > 0$, there exists player i,j and signal $x_{\Delta,i} > x_{\Delta,j}$ and a bid $b_{i,\Delta}$ such that

$$\Pr(\beta^*_{\Delta,i}(x_{\Delta,i}) = b_{i,\Delta}) \cdot \Pr(\beta^*_{\Delta,j}(x_{\Delta,j}) = b_{i,\Delta}) > 0. \tag{136}$$

That is, for every grid size $\Delta > 0$, there are two signals $x_{\Delta,i} > x_{\Delta,j}$ and a bid $b_{i,\Delta}$ such that players with these two signals will choose $b_{i,\Delta}$ with positive probability.

The strict single crossing conditions implies a monotonic relationship about the supports of the distribution of bids under $\beta^*_{\Delta,b}$.

183. The supports of the distribution of the equilibrium bids of a buyer and a seller. For each x_i, let

$$\text{supp}\beta_{\triangle,b}(x_i) = \{b_i \in B_\Delta : h_{\Delta,b}(b_i|x_i) > 0\}.$$

and

$$\text{supp}\beta_{\triangle,s}(x_i) = \{b_i \in B_\Delta : h_{\Delta,s}(b_i|x_i) > 0\}.$$

In words, $\text{supp}\beta_{\triangle,b}(x_i)$ is the set of bids that a buyer with signal x_i will choose with positive probability. For a buyer's bid, let

$$p^{\max}(\beta_{\triangle,b}(x_i)) = \max \text{supp}\beta_{\triangle,b}(x_i)$$

and

$$p^{\min}(\beta_{\triangle,b}(x_i)) = \min \operatorname{supp}\beta_{\triangle,b}(x_i).$$

That is, since the set of possible bids B_Δ is finite, $p^{\max}(\beta_{\triangle,b}(x_i))$ is the largest bid that a buyer with signal x_i will choose with a positive probability. Similarly, $p^{\min}(\beta_{\triangle,b}(x_i))$ is the smallest bid that a buyer with signal x_i will choose with a positive probability.

Similarly, for a seller, let

$$p^{\max}(\beta_{\triangle,s}(x_i)) = \max \operatorname{supp}\beta_{\triangle,s}(x_i)$$

and

$$p^{\min}(\beta_{\triangle,s}(x_i)) = \min \operatorname{supp}\beta_{\triangle,s}(x_i).$$

The figure below explains these definitions.

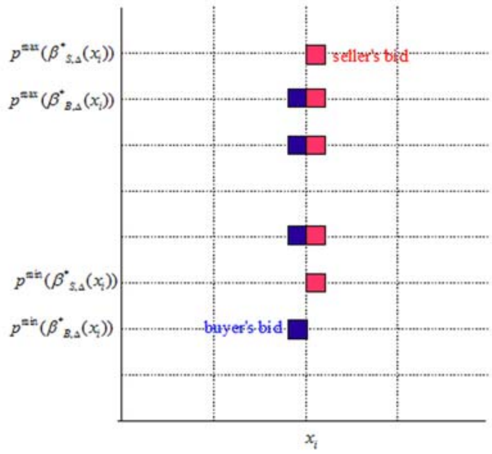

FIGURE 7.—

Since it is possible that the supports of the distribution of bids by the buyer and the seller are different even though the buyer and the seller have identical preferences, we define the union of the supports as follows:

184. The support of the equilibrium bids of buyers and sellers with the same signal. For each x_i, define $\operatorname{supp}\beta_\triangle(x_i)$ as follows:

- Suppose

$$\max(p^{\max}(\beta_{B,\triangle}(x_i)), p^{\max}(\beta_{\mathcal{X},\triangle}(x_i))) \leq p^{\max}(\beta_{B,\triangle}, \beta_{\mathcal{X},\triangle})$$

and

$$\min(p^{\max}(\beta_{B,\triangle}(x_i)), p^{\max}(\beta_{\mathcal{X},\triangle}(x_i))) \geq p^{\min}(\beta_{B,\triangle}, \beta_{\mathcal{X},\triangle}).$$

It is the case where the maximum and the minimum bids are contained in the range of the transaction price. In this case, we define

$$\operatorname{supp}\beta_\triangle(x_i) = \operatorname{supp}\beta_{\triangle,b}(x_i) \cup \operatorname{supp}\beta_{\triangle,s}(x_i).$$

- Suppose
$$\max(p^{\max}(\beta_{B,\triangle}(x_i)), p^{\max}(\beta_{\mathcal{X},\triangle}(x_i))) > p^{\max}(\beta_{B,\triangle}, \beta_{\mathcal{X},\triangle})$$
and
$$\min(p^{\max}(\beta_{B,\triangle}(x_i)), p^{\max}(\beta_{\mathcal{X},\triangle}(x_i))) \geq p^{\min}(\beta_{B,\triangle}, \beta_{\mathcal{X},\triangle}).$$
In this case, the maximum bids are above the range of the equilibrium prices. In this case, we define
$$\text{supp}\beta_{\triangle}(x_i) = \text{supp}\beta_{B,\triangle}(x_i) \cup \text{supp}\beta_{\mathcal{X},\triangle}(x_i) \ \cup [p^{\max}(\beta_{B,\triangle}, \beta_{\mathcal{X},\triangle}), \bar{b}]$$
- Suppose
$$\max(p^{\max}(\beta_{B,\triangle}(x_i)), p^{\max}(\beta_{\mathcal{X},\triangle}(x_i))) \leq p^{\max}(\beta_{B,\triangle}, \beta_{\mathcal{X},\triangle})$$
and
$$\min(p^{\max}(\beta_{B,\triangle}(x_i)), p^{\max}(\beta_{\mathcal{X},\triangle}(x_i))) < p^{\min}(\beta_{B,\triangle}, \beta_{\mathcal{X},\triangle}).$$
In this case, the minimum bids are below the range of the possible equilibrium prices. In this case, we define
$$\text{supp}\beta_{\triangle}(x_i) = \text{supp}\beta_{B,\triangle}(x_i) \cup \text{supp}\beta_{\mathcal{X},\triangle}(x_i) \ \cup [0,\ p^{\min}(\beta_{B,\triangle}, \beta_{\mathcal{X},\triangle})].$$

Suppose
$$\max(p^{\max}(\beta_{B,\triangle}(x_i)), p^{\max}(\beta_{\mathcal{X},\triangle}(x_i))) \geq p^{\max}(\beta_{B,\triangle}, \beta_{\mathcal{X},\triangle})$$
and
$$\min(p^{\max}(\beta_{B,\triangle}(x_i)), p^{\max}(\beta_{\mathcal{X},\triangle}(x_i))) < p^{\min}(\beta_{B,\triangle}, \beta_{\mathcal{X},\triangle}).$$
In this case, both the maximum bid and the minimum bids are above and below the possible equilibrium prices. In this case, we define
$$\text{supp}\beta_{\triangle}(x_i) = [0, \bar{b}].$$
This definition takes into account of the point where players can get the same outcome for the bids outside the range of the equilibrium prices. The following figure explains one construction of supp$\beta_{\triangle}(x_i)$.

We first compare the supports of the distribution of equilibrium bids for two signals where both signals place bids in the range of equilibrium prices.

185. Monotonicity of the supports of the distribution of the equilibrium bids when both signals place bids in the range of equilibrium prices. Let $\overline{x}_i > \underline{x}_i$ and $\underline{b}_i \in$supp$\beta_{\triangle}(\underline{x}_i) \cap \mathcal{P}(\beta^*_{\Delta,b}, \beta^*_{\Delta,s})$ and $\overline{b}_i \in$supp$\beta_{\triangle}(\overline{x}_i) \cap \mathcal{P}(\beta^*_{\Delta,b}, \beta^*_{\Delta,s})$. Then we claim that
$$\forall \underline{b}_i \in \text{supp}\beta_{\triangle}(\underline{x}_i) \cap P(\beta^*_{\Delta,b}, \beta^*_{\Delta,s}), \forall \overline{b}_i \in \text{supp}\beta_{\triangle}(\overline{x}_i) \cap P(\beta^*_{\Delta,b}, \beta^*_{\Delta,s}), \overline{b}_i \geq \underline{b}_i. \quad (137)$$

That is, if we take two signals, then every bid which is in the support of the bidding strategies of a buyer and a seller of a higher signal is higher than every bid which is in the support of the bidding strategy of a buyer and a seller of a lower signal.

We now present the proof of the above claim.

186. Proof of the claim. Suppose otherwise. Then there exists $\overline{b}_i \in$supp$\beta_\triangle(\overline{x}_i) \cap \mathcal{P}(\beta^*_{\Delta,b},\beta^*_{\Delta,s})$ and $\underline{b}_i \in$supp$\beta_\triangle(\underline{x}_i)\cap\mathcal{P}(\beta^*_{\Delta,b},\beta^*_{\Delta,s})$ such that $\overline{b}_i < \underline{b}_i$. Since, $\underline{b}_i \in$supp$\beta_\triangle(\underline{x}_i)$, there exists a player (buyer or seller) that will prefer $\underline{b}_i$ over other bid, including $\overline{b}_i$. That is,

$$U(\underline{x}_i, \underline{b}_i, \beta^*_{\Delta,b}, \beta^*_{\mathcal{X},\Delta}) \geq U(\underline{x}_i, \overline{b}_i, \beta^*_{\Delta,b}, \beta^*_{\mathcal{X},\Delta}). \tag{138}$$

Since $\overline{x}_i > \underline{x}_i$, the previous lemma implies that, the player with the higher signal $\overline{x}_i$ will prefer $\underline{b}_i$ as well.

$$U(\overline{x}_i, \underline{b}_i, \beta^*_{\Delta,b}, \beta^*_{\mathcal{X},\Delta}) > U(\overline{x}_i, \overline{b}_i, \beta^*_{\Delta,b}, \beta^*_{\mathcal{X},\Delta}). \tag{139}$$

This implies that both of the buyer with signal $\overline{x}_i$ and the seller with signal $\overline{x}_i$ will not choose $\overline{b}_i$ over $\underline{b}_i$. It follows that $\overline{b}_i \notin$supp$\beta_\triangle(\overline{x}_i)$. It is a contradiction. The figure below summarizes the argument.

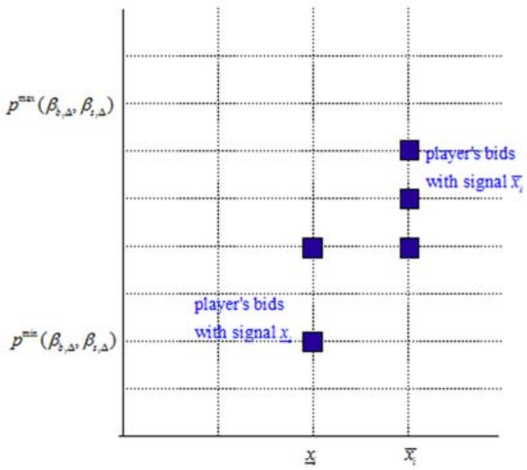

FIGURE 8.—

We now extend the argument for signals which place a bid outside the range of equilibrium prices.

187. Monotonicity of the supports of the distribution of the equilibrium bids outside the range of equilibrium prices. Let $\overline{x}_i > \underline{x}_i$ be two signals such that supp$\beta_\triangle(\underline{x}_i)\cap \mathcal{P}(\beta^*_{\Delta,b},\beta^*_{\Delta,s}) \neq \phi$ and supp$\beta_\triangle(\overline{x}_i)\cap \mathcal{P}(\beta^*_{\Delta,b},\beta^*_{\Delta,s}) \neq \phi$. That is, both signals have bids in the range of equilibrium prices. We now show that

$$\begin{aligned} \forall b_i' &\in \text{supp}\beta_{\triangle}(\underline{x}_i)\backslash\mathcal{P}(\beta^*_{\Delta,b},\beta^*_{\Delta,s}) \text{ and } b_i' \leq \min\mathcal{P}(\beta^*_{\Delta,b},\beta^*_{\Delta,s}), \\ b_i' &\notin \text{supp}\beta_{\triangle}(\overline{x}_i). \end{aligned} \tag{140}$$

That is, for two signals $\underline{x}_i$ and $\overline{x}_i$ which will place a bid in the range of equilibrium prices, if a player with signal $\underline{x}_i$ places a bid outside the range of equilibrium prices, the player with signal $\overline{x}_i$ will not choose the bid.

189. Proof of the claim. To see this, let $\underline{b}_i$ $\in$supp$\beta_{\triangle,i}(\underline{x}_i) \cap \mathcal{P}(\beta^*_{\Delta,b},\beta^*_{\Delta,s})$. Then $\underline{b}_i > b_i'$ and

$$U(\underline{x}_i,\underline{b}_i,\beta^*_{\Delta,b},\beta^*_{\Delta,s}) \geq U(\underline{x}_i,b_i',\beta^*_{\Delta,b},\beta^*_{\Delta,s}). \tag{141}$$

Thus, by the previous lemma,

$$U(\overline{x}_i,\underline{b}_i,\beta^*_{\Delta,b},\beta^*_{\Delta,s}) > U(\overline{x}_i,b_i',\beta^*_{\Delta,b},\beta^*_{\Delta,s}). \tag{142}$$

For $\overline{b}_i$ $\in$supp$\beta_{\triangle,b}(\overline{x}_i)$,

$$U(\overline{x}_i,\overline{b}_i,\beta^*_{\Delta,b},\beta^*_{\Delta,s}) \geq U(\overline{x}_i,\underline{b}_i,\beta^*_{\Delta,b},\beta^*_{\Delta,s}) \tag{143}$$

That is, from (141), (142), and (143),

$$\begin{aligned} & U(\overline{x}_i,\overline{b}_i,\beta^*_{\Delta,b},\beta^*_{\Delta,s}) - U(\underline{x}_i,b_i',\beta^*_{\Delta,b},\beta^*_{\Delta,s}) \\ = & \left(U(\overline{x}_i,\overline{b}_i,\beta^*_{\Delta,b},\beta^*_{\Delta,s}) - U(\overline{x}_i,\underline{b}_i,\beta^*_{\Delta,b},\beta^*_{\Delta,s})\right) \\ & - \left(U(\overline{x}_i,\overline{b}_i,\beta^*_{\Delta,b},\beta^*_{\Delta,s}) - U(\overline{x}_i,\underline{b}_i,\beta^*_{\Delta,b},\beta^*_{\Delta,s})\right) \\ > & \ 0 \end{aligned}$$

Therefore, b_i' $\notin$supp$\beta_{B,\triangle}(\overline{x}_i)$. This implies that only the lowest signal who will place a bid in the range of equilibrium prices will place a bid outside a range of equilibrium prices.

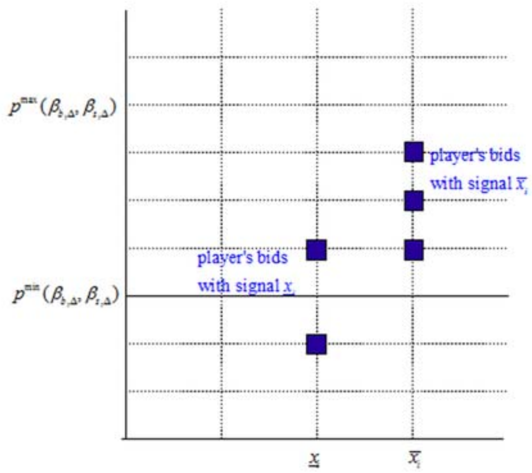

FIGURE 9.—

We further extend the result to a signal such that the player with that signal will not place a bid in the range of equilibrium prices.

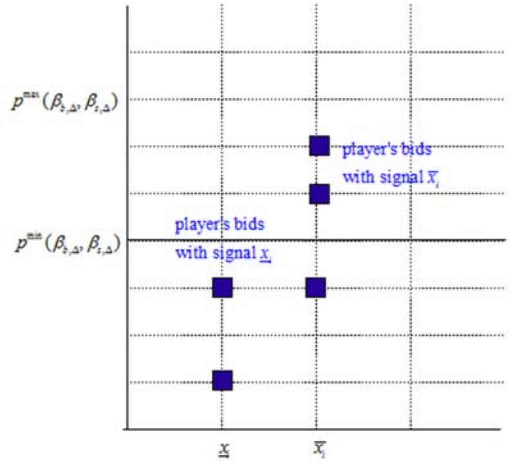

FIGURE 10.—

190. Monotonicity of the support of the equilibrium bids of buyers for signals who will not place a bid in the range of equilibrium prices. Let x_i' be such that supp$\beta_{\triangle,b}(x_i')\cap \mathcal{P}(\beta^*_{\Delta,b},\beta^*_{\Delta,s}) = \phi$ and maxsupp$\beta_{\triangle,b}(x_i') \leq \min \mathcal{P}(\beta^*_{\Delta,b},\beta^*_{\Delta,s})$. Let x_i be any signal such that supp$\beta_{\triangle,b}(x_i)\cap \mathcal{P}(\beta^*_{\Delta,b},\beta^*_{\Delta,s}) \neq \phi$. Then $x_i > x_i'$.

That is, a signal whose support of the equilibrium bid is less than the range of equilibrium prices is less than any signal such that the player with that signal places a bid in the range of equilibrium prices.

191. Proof of the claim. To see this, suppose, on the contrary that $x_i' > x_i$. Let $b_i \in$supp$\beta_{\triangle,b}(x_i)\cap \mathcal{P}(\beta^*_{\Delta,b},\beta^*_{\Delta,s})$ and $b_i \leq \min \mathcal{P}(\beta^*_{\Delta,b},\beta^*_{\Delta,s})$ and $b_i' \in$supp$\beta_{\triangle,b}(x_i')\cap \mathcal{P}(\beta^*_{\Delta,b},\beta^*_{\Delta,s})$. Then, by construction,

$$b_i > b_i'.$$

Also, since b_i is a best response for a player with signal x_i,

$$U(x_i, b_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) \geq U(x_i, b_i', \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) \tag{144}$$

Then, since the assumption is such that $x_i' > x_i$,

$$U(x_i', b_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) > U(x_i', b_i', \beta^*_{\Delta,b}, \beta^*_{\Delta,s}). \tag{145}$$

This is a contradiction to the fact that b_i' is a best response to a player with signal x_i'. The figure below summarizes the argument.

We have so far derived monotonicity properties of the supports of the distribution of equilibrium bidding strategies. That is, the support of the distribution of equilibrium strategies by a player with the higher signal is higher than the support of the distribution of equilibrium strategies by a player with the lower signal. We now derive a consequence on the distribution of equilibrium bids.

192. Distribution of the equilibrium bids. Consider two signals $\underline{x}_i < \overline{x}_i$. Let $H^*_{\triangle,i}(\cdot|\underline{x}_i)$ and $H^*_{\triangle,i}(\cdot|\overline{x}_i)$ be the distribution function of an equilibrium bidding strategy associated with signal $\underline{x}_i$ and $\overline{x}_i$. Let b_i be in the range of equilibrium prices. There are several cases.

- Case where

$$\text{supp}\beta^*_\Delta(\underline{x}_i) \subset B_\Delta \backslash \mathcal{P}(\beta^*_{\Delta,b}, \beta^*_{\Delta,s}) \text{ and } \max \text{supp}\beta^*_\Delta(\underline{x}_i) \leq \min \mathcal{P}(\beta^*_{\Delta,b}, \beta^*_{\Delta,s})$$

and

$$\text{supp}\beta^*_\Delta(\overline{x}_i) \subset B_\Delta \backslash \mathcal{P}(\beta^*_\Delta(\beta^*_{\Delta,b}, \beta^*_{\Delta,s}))) \text{ and } \max \text{supp}\beta^*_\Delta(\underline{x}_i) \leq \min \mathcal{P}(\beta^*_{\Delta,b}, \beta^*_{\Delta,s})$$

This is the case where both signals have the support of the equilibrium bids less than the range of equilibrium prices. In this case, for $b_i \in P(\beta^*_{\Delta,b},\beta^*_{\Delta,s})$,

$$H^*_{\triangle,i}(b_i|\underline{x}_i) = H^*_{\triangle,i}(b_i|\overline{x}_i) = 1.$$

- Case where

$$\text{supp}\beta^*_\Delta(\underline{x}_i) \subset B_\Delta \backslash \mathcal{P}(\beta^*_\Delta(\beta^*_{\Delta,b}, \beta^*_{\Delta,s}))) \text{ and } \max \text{supp}\beta^*_\Delta(\underline{x}_i) \leq \min \mathcal{P}(\beta^*_{\Delta,b}, \beta^*_{\Delta,s})$$

and

$$\text{supp}\beta^*_{\Delta,i}(\overline{x}_i) \cap \mathcal{P}(\beta^*_{\Delta,b}, \beta^*_{\Delta,s}) \neq \phi.$$

This is the case where the support of the distribution of a player with a high signal has a bid in the range of an equilibrium bids. In this case, for $b_i \in P(\beta^*_{\Delta,b},\beta^*_{\Delta,s})$,

$$H^*_{\triangle,i}(b_i|\underline{x}_i) = 1 \geq H^*_{\triangle,i}(b_i|\overline{x}_i).$$

- Case where

$$\text{supp}\beta^*_\Delta(\underline{x}_i) \cap \mathcal{P}(\beta^*_{\Delta,b}, \beta^*_{\Delta,s}))) \neq \phi$$

and

$$\text{supp}\beta^*_\Delta(\overline{x}_i) \cap \mathcal{P}(\beta^*_{\Delta,b}, \beta^*_{\Delta,s})) \neq \phi.$$

This is the case where the supports of the distribution of a player with a high and a low signal place a bid in the range of equilibrium prices. Suppose $b_i \in \mathcal{P}(\beta^*_{\Delta,b},\beta^*_{\Delta,s})$ and $b_i < \text{maxsupp}\beta^*_\Delta(\underline{x}_i)$. In this case, a player with a higher signal $\overline{x}_i$ will not choose to place a bid on b_i,

$$H^*_{\triangle,i}(b_i|\underline{x}_i) \geq 0 \geq H^*_{\triangle,i}(b_i|\overline{x}_i) = 0.$$

For $b_i \in \mathcal{P}(\beta^*_\Delta(\beta^*_{\Delta,b},\beta^*_{\Delta,s}))$ and $b_i \geq \text{maxsupp}\beta^*_\Delta(\underline{x}_i)$,from (137),

$$H^*_{\triangle,i}(b_i|\underline{x}_i) = 1 \geq H^*_{\triangle,i}(b_i|\overline{x}_i).$$

The argument for the cases where a support of an equilibrium bids is higher than $\max \mathcal{P}(\beta^*_{\Delta,b},\beta^*_{\Delta,s})$ is similar.

It follows from these cases that, for $b_i \in \mathcal{P}(\beta^*_{\Delta}(\beta^*_{\Delta,b},\beta^*_{\Delta,s}))$

$$H^*_{\triangle,i}(b_i|\underline{x}_i) \geq H^*_{\triangle,i}(b_i|\overline{x}_i). \tag{146}$$

Intuitively, when a strict single crossing condition holds, it cannot be that a player with a higher signal places a bid strictly lower than a bid by a player with a lower signal. Otherwise, a player with a higher signal strictly should have preferred a high bid chosen by a player with a lower signal. This implies that the support of an equilibrium bid by a low signal is less than the support of an equilibrium bid by a high signal. It will implies that the probability distribution of bids by a player with a low signal is stochastically dominated by the distribution of bids by a player with a high signal.

We now derive a stochastic dominance relationship of the distribution of equilibrium bids conditional on the state.

193. Stochastic dominance relationship of the conditional distribution of bids. Recall

$$H^*_{\triangle,b}(b_i|x_0) = \sum_{x_i \in \mathcal{X}} H^*_{\triangle,b}(b_i|x_i) f_{X_i|\theta}(x_i|x_0) \tag{147}$$

From (146), and (147),

$$\overline{x}_0 \geq \underline{x}_0 \rightarrow H^*_{\triangle,b}(b_i|\underline{x}_0) \geq H^*_{\triangle,b}(b_i|\overline{x}_0) \tag{148}$$

Similarly,

$$\overline{x}_0 \geq \underline{x}_0 \rightarrow H^*_{\triangle,s}(b_i|\underline{x}_0) \geq H^*_{\triangle,s}(b_i|\overline{x}_0) \tag{149}$$

It follows from (148) and (149) that

$$\alpha H^*_{\triangle,b}(b_i|x_0) + (1-\alpha) H^*_{\triangle,s}(b_i|x_0) \text{ is nonincreasing in } x_0. \tag{150}$$

Having established the stochastic dominance relationship about the distribution of bids, we will now start working on existence of winner's curse. As a first step, we study how the allocation probability changes as the signal changes.

194. The allocation function at the tie. Suppose there is a tie at bid b_i in the range of an equilibrium price when the state is x_0. Let $q(b_i, W = b_i|x_0)$ be the probability that a player who bid b_i will get the good when the market clearing price is b_i and the state is x_0. We note

- The ratio of goods available for buyers and sellers with bid b_i is the amount of the ratio of the good left after the goods are allocated to buyers and sellers whose bids are strictly above b_i. Noting that the set of possible bid B_Δ is finite, it is,

$$\underbrace{(1-\alpha)}_{\text{total ratio of available goods}} - \left[\alpha \underbrace{(1-H^*_{b,\triangle}(b_i|x_0))}_{\text{ratio of buyers who bid strictly above } b_i} + (1-\alpha)\underbrace{(1-H^*_{s,\triangle}(b_i|x_0))}_{\text{ratio of sellers who bid strictly above } b_i}\right] = \left[\alpha H^*_{B,\triangle}(b_i|x_0) + (1-\alpha)H^*_{\mathcal{X},\triangle}(b_i|x_0)\right] - \alpha \tag{151}$$

- The ratio of buyers and sellers who bid exactly b_i is

$$\left[\underbrace{\alpha H^*_{B,\triangle}(b_i|x_0) + (1-\alpha)H^*_{\mathcal{X},\triangle}(b_i|x_0)}_{\text{the ratio of buyers and sellers who bid equal or less than } b_i}\right] - \left[\underbrace{\alpha H^*_{B,\triangle}(b_i-\Delta|x_0) + (1-\alpha)H^*_{\mathcal{X},\triangle}(b_i-\Delta|x_0)}_{\text{the ratio of buyers and sellers who bid strictly less than } b_i}\right] \tag{152}$$

It follows from (94),(151),(152),

$$\begin{aligned}\Pr(q(b_i,W &= b_i|x_0)=1) \\ &= \frac{\left[\alpha H^*_{B,\triangle}(b_i|x_0) + (1-\alpha)H^*_{\mathcal{X},\triangle}(b_i|x_0)\right] - \alpha}{\left[\begin{array}{c}\left[\alpha H^*_{B,\triangle}(b_i|x_0) + (1-\alpha)H^*_{\mathcal{X},\triangle}(b_i|x_0)\right] \\ -\left[\alpha H^*_{B,\triangle}(b_i-\Delta|x_0) + (1-\alpha)H^*_{\mathcal{X},\triangle}(b_i-\Delta|x_0)\right]\end{array}\right]} \\ &= \frac{1-\frac{\alpha}{(\alpha H^*_{B,\triangle}(b_i|x_0)+(1-\alpha)H^*_{\mathcal{X},\triangle}(b_i|x_0))}}{1-\frac{(\alpha H^*_{B,\triangle}(b_i-\Delta|x_0)+(1-\alpha)H^*_{\mathcal{X},\triangle}(b_i-\Delta|x_0))}{(\alpha H^*_{B,\triangle}(b_i|x_0)+(1-\alpha)H^*_{\mathcal{X},\triangle}(b_i|x_0))}}.\end{aligned} \tag{153}$$

We now derive a monotonicity property of $\Pr(q(b_i,W = b_i|x_0) = 1)$.

195. Monotonicity property of $\Pr(q(b_i,W = b_i|x_0) = 1)$**.** It follows from (148) and (149) that

$$\alpha H^*_{B,\triangle}(b_i|x_0) + (1-\alpha)H^*_{\mathcal{X},\triangle}(b_i|x_0) \text{ is nonincreasing in } x_0. \tag{154}$$

and

$$\alpha H^*_{B,\triangle}(b_i-\Delta|x_0) + (1-\alpha)H^*_{\mathcal{X},\triangle}(b_i-\Delta|x_0) \text{ is nonincreasing in } x_0 \tag{155}$$

It follows from (154) and (155) that

$$\begin{aligned}&\frac{\alpha}{(\alpha H^*_{B,\triangle}(b_i|x_0) + (1-\alpha)H^*_{\mathcal{X},\triangle}(b_i|x_0))} \text{ increases faster than} \\ &\frac{(\alpha H^*_{B,\triangle}(b_i-\Delta|x_0) + (1-\alpha)H^*_{\mathcal{X},\triangle}(b_i-\Delta|x_0))}{(\alpha H^*_{B,\triangle}(b_i|x_0) + (1-\alpha)H^*_{\mathcal{X},\triangle}(b_i|x_0))} \text{ as } x_0 \text{ increases.}\end{aligned} \tag{156}$$

Consequently, from (156),the numerator of (153) decreases faster than the denominator. Therefore,

$$\begin{aligned}&\overline{x}_0 \geq \underline{x}_0 \\ \rightarrow\ & \Pr(b_i \text{ wins } |\theta = \overline{x}_0, W = b_i) \leq \Pr(b_i \text{ wins } |\theta = \underline{x}_0, W = b_i).\end{aligned} \tag{157}$$

It follows that

$$\begin{aligned}&\overline{x}_0 \geq \underline{x}_0 \\ \rightarrow\ & \Pr(b_i \text{ loses } |\theta = \overline{x}_0, W = b_i) \geq \Pr(b_i \text{ loses } |\theta = \underline{x}_0, W = b_i).\end{aligned} \tag{158}$$

That is, as x_0 increases, the distribution of bids by other players increases in the sense of stochastic dominance, so the ratio of bids equal or above at x_0 increases. Given that a bid b_i is tied with W, it implies that the ratio of bids strictly above x_0 is the same in these two cases of $x_0 = \underline{x}_0$ and $x_0 = \overline{x}_0$. This implies that the ratio of bids equal to b_i is larger with $\overline{x}_0$ compared with $\underline{x}_0$. It follows that the probability that a good is assigned conditional on being at a tie will be lower with $\overline{x}_0$ rather with $\underline{x}_0$.

(157) and (158) imply that losing at the tie is a good news compared with winning at the tie.

196. Winner's curse. Consider a player with signal x_i and a bid b_i. The expected value of the good from winning at the tie is $E[v(\theta,X_i)|X_i = x_i,W(\theta) = b_i,b_i\text{wins at the tie}]$ and the expected value of the good from losing at the tie is $E[v(\theta,X_i)|X_i = x_i,W(\theta) = b_i,b_i \text{ loses at the tie}]$.

We first examine the monotone likelihood ratio condition for $f(x_0|X_i = x_i,W(\theta) = b_i,b_i \text{ loses at the tie})$. We note that

$$\begin{aligned}
& f_{\theta|X_i,W}(x_0|x_i, W(\theta) = b_i, b_i\text{loses at the tie}) \\
= & \frac{f_{\theta,X_i,W}(x_0, x_i, W(\theta) = b_i, b_i\text{loses at the tie})}{f_{X_i,W}(x_i, W(\theta) = b_i, b_i\text{loses at the tie})} \\
= & \frac{f_{X_i,Y|\theta}(x_i, W(\theta) = b_i, b_i\text{loses at the tie}|x_0) f_\theta(x_0)}{f_{X_i,Y}(x_i, W(\theta) = b_i, b_i\text{loses at the tie})} \\
= & \frac{\left[\begin{array}{c} f_{X_i,Y|\theta}(x_i|x_0) \Pr(b_i\text{loses at the tie}|x_0, W(\theta) = b_i) \\ f(W(x_0) = b_i|x_0) f_\theta(x_0)\end{array}\right]}{f_{X_i,Y}(x_i, W(\theta) = b_i, b_i\text{loses at the tie})}
\end{aligned} \tag{159}$$

It follows from (159) that the likelihood ratio for $\overline{x}_0 > \underline{x}_0$ is

$$\frac{f_{\theta|X_i,Y}(\overline{x}_0|x_i, W(\theta) = b_i, b_i\text{loses at the tie})}{f_{\theta|X_i,Y}(\overline{x}_0|x_i, W(\theta) = b_i, b_i\text{win at the tie})} \tag{160}$$

$$= \frac{\left(\frac{\Pr(b_i\text{loses at the tie}|\overline{x}_0, W(\theta)=b_i)}{f_{X_i,Y}(x_i, W(\theta)=b_i, b_i\text{loses at the tie})}\right)}{\left(\frac{\Pr(b_i\text{wins at the tie}|x_0, W(\theta)=b_i)}{f_{X_i,Y}(x_i, W(\theta)=b_i, b_i\text{wins at the tie})}\right)}$$

by (159)

$$= \frac{\Pr(b_i\text{loses at the tie}|\overline{x}_0, W(\theta) = b_i)}{\Pr(b_i\text{wins at the tie}|\overline{x}_0, W(\theta) = b_i)} \cdot \frac{f_{X_i,Y}(x_i, W(\theta) = b_i, b_i\text{wins at the tie})}{f_{X_i,Y}(x_i, W(\theta) = b_i, b_i\text{loses at the tie})}$$

$$> \frac{\Pr(b_i\text{loses at the tie}|W(\theta) = b_i, \underline{x}_0)}{\Pr(b_i\text{wins at the tie}|W(\theta) = b_i, \underline{x}_0)}$$

by (157) and (158)

$$\cdot \frac{f_{X_i,Y}(x_i, W(\theta) = b_i, b_i\text{wins at the tie})}{f_{X_i,Y}(x_i, W(\theta) = b_i, b_i\text{loses at the tie})}$$

$$= \frac{f_{\theta|X_i,Y}(\underline{x}_0|x_i, W(\theta) = b_i, b_i\text{loses at the tie})}{f_{\theta|X_i,Y}(\underline{x}_0|x_i, W(\theta) = b_i, b_i\text{win at the tie})}$$

That is, the variable θ and the variable b_i wins or loses at the tie satisfy the monotone likelihood ratio condition. Thus, it follows from Theorem 5 of Milgrom and Weber (1982a) and Assumption 5 that the value is strictly increasing in the state with a uniform lower bound that

$$E[v(\theta, X_i)|X_i = x_i, W(\theta) = b_i, b_i\text{loses at the tie}] \tag{161}$$
$$> E[v(\theta, X_i)|X_i = x_i, W(\theta) = b_i, b_i \text{ wins at the tie}]$$

It follows that

$$E[v(\theta, X_i) - b_i|X_i = x_i, W(\theta) = b_i, b_i\text{loses at the tie}] \tag{162}$$
$$> E[v(\theta, X_i) - b_i|X_i = x_i, W(\theta) = b_i, b_i \text{ wins at the tie}]$$

Intuitively, winning at the tie suggests, compared with losing at the tie, that the state is lower, thus the value of the good is lower. Since the value function is strictly increasing in the state with a uniform lower bound, the expected payoff conditional on the information of winning the tie is strictly lower than the expected payoff conditional on the information of losing the tie.

We have deduced the existence of winner's curse. We now derive consequences of the assumption that bidders with distinct signals will submit the same bid. For that purpose, we consider a property of a mapping $Z(p)$ which gives the set of states such that p is the market clearing price given that players choose strategies β^*_Δ.

197. Monotonicity properties of $Z(p)$. Consider two prices $\underline{p} < \overline{p}$ which are in

$\mathcal{P}(\beta^*_{\Delta,b},\beta^*_{\Delta,s})$. We first show that

$$\text{for any } x_0 \in Z(\underline{p}) \text{ and } x_0' \in Z(\overline{p}), x_0 \leq x_0'. \tag{163}$$

That is, every state which will induces a lower price is lower than the state which will induce a higher price.

To see this, suppose, on the contrary, that $x_0 > x_0'$. Then, for every b_i, $\alpha H^*_{B,\triangle}(b_i|x_0)+ (1-\alpha)H^*_{\mathcal{X},\triangle}(b_i|x_0)$ is nonincreasing in x_0. It follows that $\underline{p}\geq \overline{p}$. It is a contradiction to the assumption.

Next we consider another property of $Z(p)$ such that $Z(p)$ does not have a gap.

198. $Z(p)$ **does not have a gap.** Suppose there are two signals $\underline{x}_0 < \overline{x}_0$ such that the market clearing prices under $\underline{x}_0$ and $\overline{x}_0$ are both p. From (150), the market clearing price $p(\beta^*_\Delta(x_0))$ is monotone in x_0. Thus, for any $x_0 \in [\underline{x}_0,\overline{x}_0]$, $p(\beta^*_\Delta(x_0)) = p(\beta^*_\Delta(\underline{x}_0)) = p(\beta^*_\Delta(\overline{x}_0))$. Thus, $x_0 \in Z(p)$.

From these two properties, for each $p \in \mathcal{P}(\beta^*_{\Delta,b},\beta^*_{\Delta,s})$, there is an interval $Z(p) \subset [0,1]$ which yields p as an equilibrium market clearing price. We also note that $Z(p)$ changes 'smoothly' in p in the following sense.

199. $Z(p)$ **does not jump around.** Let $\underline{p}< \overline{p}$ be two consecutive prices in $\mathcal{P}(\beta^*_{\Delta,b},\beta^*_{\Delta,s})$. Let $Z(\underline{p})$ be the set of signals such that the equilibrium outcome price is $\underline{p}$. Let $\underline{x}= \sup Z(\underline{p})$. Then, there does not exist $\varepsilon > 0$ such that the equilibrium price with $\underline{x}+\varepsilon$ is $\overline{p}$ and the equilibrium price p' with $\underline{x}+\varepsilon',\varepsilon' < \varepsilon$ is different from $\overline{p}$.

To see this, suppose otherwise. Then, if the price p' is lower than $\underline{p}$, it will contradict monotonicity property of $Z(p)$ in terms of $\underline{p}$ and p', and if p' is higher than $\overline{p}$,it will contradict monotonicity property of $Z(p)$ in terms of p' and $\overline{p}$.

Intuitively, we have already seen that the distribution of equilibrium bids is increasing in the sense of stochastic dominance as the state increases. This implies that, as the state increases, the market clearing price, which is a convex combination of the order statistics of the distribution of bids, increases monotonically.

We are now able to study the first order conditions for the bidder with the high signal $x_{i,\Delta}$.

200. First order condition for the bidder with the high signal. It follows from the first order condition for the signal $x_{i,\Delta}$ that it has to be that a player with a signal

$x_{i,\Delta}$ does not wish to increase a bid from $b_{i,\Delta}$ to $b_{i,\Delta}+\triangle$. That is,

$$\begin{aligned}&\int_{x_0\in Z(b_{i,\Delta}+\triangle)}(v(x_0,x_{i,\Delta})-(b_{i,\Delta}+\triangle))\\&q(b_{i,\Delta}+\triangle,W=b_{i,\Delta}+\triangle|x_0)f_{\theta|X_i}(x_0|x_{i,\Delta})dx_0\\&+\int_{x_0\in Z(b_{i,\Delta})}(v(x_0,x_{i,\Delta})-b_{i,\Delta})\\&\quad(1-q(b_{i,\Delta},W=b_{i,\Delta}|x_0))f_{\theta|X_i}(x_0|x_{i,\Delta})dx_0\\&\leq 0.\end{aligned} \tag{164}$$

The first term of (164) evaluates the payoff of a player when a player's new bid $b_{i,\Delta}+\Delta$ wins when the market clearing price is $b_{i,\Delta}+\Delta$ and the player wins the possible tie and the allocation is $q(b_{i,\Delta}+\triangle,W=b_{i,\Delta}+\triangle|x_0)$. The second term of (164) evaluates the payoff of a player when a player's new bid $b_{i,\Delta}+\Delta$ wins when the market clearing price is $b_{i,\Delta}$.

We have already seen the first order condition in the form of $Y(\underline{b}_i,\overline{b}_i)$ in the previous Lemma for example, in (113). The condition here, (164), explicitly deals with the information about the state contained in $Y(\underline{b}_i,\overline{b}_i)$ in the form of $Z(\underline{b}_i)$ and $Z(\overline{b}_i)$. The reason that we use this formulation here is that, since we know now more about the structure of the equilibrium strategies, we know more about the structure of Z.

202. The limit of the first order condition as $\Delta\to 0$. We now take $\triangle\to 0$. By taking subsequences, let x_i denote a subsequence limit of $x_{i,\Delta}$, b_i is a subsequence limit of $b_{i,\Delta}$, and $q(b_i,W=b_{i,\Delta}|x_0)$ is a subsequence limit of $q(b_i+\triangle,W=b_{i,\Delta}+\triangle|x_0)$,$Z_1$ be the limit of the interval of $Z(b_{i,\Delta}+\triangle)$ and Z_2 be the limit of the intervals of $Z(b_{i,\Delta})$. Then,

$$\begin{aligned}\int_{Z_1}(v(x_0,x_i)-b_i)\overline{q}(b_{i,\Delta},W&=b_i|x_0)f(x_0|x_i)dx_0\\+\int_{Z_2}(v(x_0,x_i)-b_i)(1-q(b_{i,\Delta},W&=b_i|x_0))f(x_0|x_i)dx_0\leq 0\end{aligned} \tag{165}$$

Given the monotonic structure of $Z(p)$, we can draw inference on the payoff of a high signal bidder at the tie.

204. Estimation of the payoff of a high signal player with losing at the tie. From the monotonic structure of Z, let $\overline{x}_0$ be the common point of Z_1 and Z_2.

- Suppose

$$v(\overline{x}_0,x_i)-b_i\geq 0. \tag{166}$$

  That is, at the highest state where the price is $b_{i,\Delta}$, it is profitable to own the good. Then, since, for every $x_0\in Z_1$, $x_0\geq\overline{x}_0$, it follows from (166) that

$$v(x_0,x_i)-b_i>0. \tag{167}$$

  From $\overline{q}(b_{i,\Delta},W=b_i|x_0)\geq 0$ and (167), it implies that it is profitable to win the good when the state is Z_1 :

$$\int_{Z_1}(v(x_0,x_i)-b_i)\overline{q}(b_{i,\Delta},W=b_i|x_0)f(x_0|x_i)dx_0 \geq 0 \tag{168}$$

Therefore, from (165) and (168), it has to be that

$$\int_{Z_2}(v(x_0,x_i)-b_i)(1-q(b_{i,\Delta},W=b_i|x_0))f(x_0|x_i)dx_0 \leq 0 \tag{169}$$

- Suppose

$$v(\overline{x}_0,x_i)-b_i<0. \tag{170}$$

That is, it is not profitable to own the good at the highest state where the price is b_i. For every $x_0 \in Z_1$, $x_0 \leq \overline{x}_0$. Thus, from (170),

$$v(x_0,x_i)-b_i<0. \tag{171}$$

Therefore, from $\overline{q}(b_{i,\Delta},W=b_i|x_0) \geq 0$ and (171),

$$\int_{Z_2}(v(x_0,x_i)-b_i)(1-q(b_{i,\Delta},W=b_i|x_0))f(x_0|x_i)dx_0 \leq 0 \tag{172}$$

Consequently, (169) holds for all cases. That is, losing the tie at the price b_i should not be profitable.

For intuition, suppose otherwise and that losing the tie at the price b_i is profitable. Then, winning at the price $b_i+\Delta$ when the market clearing price is $b_i+\Delta$ is a better news for the state than losing at the price b_i. It is because, in order for the market clearing price to be $b_i+\Delta$, the ratio of the bids equal or strictly above $b_i+\Delta$ is at least $1-\alpha$. In this case, it cannot be that the market clearing price is b_i. This implies that winning when the market clearing price is $b_i+\Delta$ is a better news. This implies that, if losing the tie at the price b_i is profitable, winning at the price $b_i+\Delta$ is profitable for sufficiently small Δ, and it will provide contradiction to the assumption that the player with the signal $x_{i,\Delta}$ does not wish to increase the bid to $b_i+\Delta$.

We bow consider the first order condition for the low signal.

205. First order condition for the low signal and its limit. It follows from (136), as in the case of a high signal $x_{i,\Delta}$, that a player with signal $x_{j,\Delta}$ prefers to increase the bid from $b_{i,\Delta}-\Delta$ to $b_{i\,\Delta}$. That is,

$$\begin{aligned}\int_{x_0\in Z(b_{i,\Delta}-\Delta)}(v(x_0,x_{j,\Delta})-(b_{i,\Delta}-\Delta))(1-q(b_{i,\Delta}-\Delta,W &= b_{i,\Delta}-\Delta|x_0))f_{\theta|X_i}(x_0|x_{j,\Delta})dx\\ +\int_{x_0\in Z(b_{i,\Delta})}(v(x_0,x_{j,\Delta})-b_{i,\Delta})q(b_{i,\Delta},W &= b_{i,\Delta}|x_0)f_{\theta|X_i}(x_0|x_{j,\Delta})dx_0\\ &\geq 0.\end{aligned}$$

The first term expresses the payoff when the state is such that the market clearing price is $b_{i,\Delta}-\Delta$ and a higher bid of $b_{i,\Delta}$ wins the good even when the bid of $b_{i,\Delta}-\Delta$

does not win the good. The second case expresses the payoff from when the state is such that the market clearing price is $b_{i,\Delta}$ and a higher bid of $b_{i,\Delta}$ wins the good. The first order condition implies that increasing the bid from $b_{i,\Delta} - \Delta$ to $b_{i,\Delta}$ is preferable.

We take $\Delta \to 0$. Let x_i be a subsequence limit of $x_{i,\Delta}$, b_i is a subsequence limit of $b_{i,\Delta}$, and $q(b_i,W = b_{i,\Delta}|x_0)$ is a subsequence limit of $q(b_i + \triangle,W = b_{i,\Delta} + \triangle|x_0)$, $\underline{q}(b_i,W = b_{i,\Delta}|x_0)$ is a subsequence limit of $q(b_i - \triangle,W = b_{i,\Delta} - \triangle|x_0)$ for each x_0,Z_2 be the limit of the intervals of $Z(b_{i,\Delta})$, and Z_2 be the limit of the intervals of $Z(b_{i,\Delta} - \Delta)$. Then,

$$\int_{Z_2}(v(x_0,x_i) - b_i)\overline{q}(b_{i,\Delta}, W = b_i|x_0)f(x_0|x_i)dx_0 + \int_{Z_3}(v(x_0,x_i) - b_i)(1 - \underline{q}(b_{i,\Delta}, W = b_i|x_0))f(x_0|x_i)dx_0 \geq 0 \tag{173}$$

207. Estimation of the payoff of a low signal player with winning at the tie. Let $\underline{x}_0$ be the common point of Z_2 and Z_3.

- Suppose
$$v(\underline{x}_0, x_j) - b_i \geq 0. \tag{174}$$
Then, for every $x_0 \in Z_2$,
$$v(x_0, x_j) - b_i > 0. \tag{175}$$
Therefore, from $q(b_i,W = b_i|x_0) \geq 0$,
$$\int_{Z_2}(v(x_0,x_j) - b_i)q(b_i, W = b_i|x_0)f(x_0|x_j)dx_0 \geq 0. \tag{176}$$
- Suppose
$$v(\underline{x}_0, x_j) - b_i < 0. \tag{177}$$
Then for every $x_0 \in Z_3$,
$$v(x_0, x_j) - b_i < 0. \tag{178}$$
It follows that, from $1 - \underline{q}(b_i,W = b_i|x_0) \geq 0$, by integrating,
$$\int_{Z_3}(v(x_0,x_j) - b_i)(1 - \underline{q}(b_i, W = b_i|x_0))f(x_0|x_j)dx_0 \leq 0. \tag{179}$$
It follows from (173) and (179) that
$$\int_{Z_2}(v(x_0,x_j) - b_i)q(b_i, W = b_i|x_0)f(x_0|x_j)dx_0 \geq 0. \tag{180}$$

This implies that for a player with a low signal, it has to be that winning the tie at when the market clearing price b_i is profitable. To see the intuition, suppose otherwise and assume that it is not profitable to win at the tie when the market clearing price is b_i. Then, for sufficiently small Δ, since losing the tie at $b_i - \Delta$ is a worse than winning the tie at the price b_i, it will be that losing the tie at the price $b_i - \Delta$ and winning the tie by a higher bid of b_i is still nonprofitable. This implies that the player with signal $x_{j,\Delta}$ will not prefer to increase the bid from $b_i - \Delta$ to b_i, which is a contradiction.

We now consider the relationship between winner's curse (162) and the first order condition (172) and (180).

208. Derivation of contradiction. We note that from (162), for every x_i,

$$\int_{Z_2}(v(x_0,x_i)-b_{i,\Delta})(1-q(b_{i,\Delta},W=b_{i,\Delta}|x_0))f(x_0|x_i)dx_0 \geq \int_{Z_2}(v(x_0,x_i)-b_{i,\Delta})q_i(b_i,p(b)=b_i|x_0)f(x_0|x_i)dx_0 \tag{181}$$

Therefore,

$$\begin{aligned} 0 &\geq \int_{Z_2}(v(x_0,x_i)-b_i)(1-q(b_{i,\Delta},W=b_i|x_0))f(x_0|x_i)dx_0 \\ &\geq \int_{Z_2}(v(x_0,x_i)-b_i)(1-q_i(b_i,p(b)=b_i|x_0))f(x_0|x_i)dx_0 \\ &> \int_{Z_2}(v(x_0,x_i)-b_i)q_i(b_i,p(b)=b_i|x_0)f(x_0|x_i)dx_0 \\ &> \int_{Z_2}(v(x_0,x_j)-b_i)q_i(b_i,p(b)=b_i|x_0)f(x_0|x_i)dx_0 \\ &\geq 0 \end{aligned} \tag{182}$$

The first inequality comes from (169).The second inequality comes from the affiliation inequality and the strictly private value element. The third inequality comes from (162).The fourth inequality comes from the affiliation inequality and the private value element. The last inequality comes from (180).

(182) is not consistent. Consequently, the assumption (136) is not logically consistent. Thus (135) holds.

7.4. *Lemma 3.4*

LEMMA. *The limit strategy profile β^*_Δ is an equilibrium of the limit game $\mathcal{G}(\gamma,f)$ and is outcome equivalent to the fully revealing rational expectation equilibrium identified in Lemma 1.*

PROOF.

210. Introduction. We show that the limit strategy profile $\beta^*_{\Delta,b}$ is an equilibrium of the limit game $\mathcal{G}(\gamma,f)$. Then, we show outcome equivalence to the fully revealing rational expectation equilibrium.

211. Consequence of the previous lemma. The previous lemma implies that for sufficiently small Δ, players with distinct signals will place distinct bids so that the tie with distinct signals will not take place with positive probability. Players with the same signal choose the same strategy. In the large economy, the allocation probability is symmetric among buyers and sellers with the same signal. It follows that for each x_0, as $\Delta \to 0$,

$$\lim_{\Delta\to 0}\Pr(q(b_i,\beta^*_{\triangle,b},\beta^*_{\triangle,s}|x_0)=1)=\Pr(q(b_i,\beta^*_b,\beta^*_{\mathcal{X}}|x_0)=1)). \tag{183}$$

Consequently, $q(b_i,\beta^*_{\triangle,b},\beta^*_{\triangle,s})$ converges in distribution to $q(b_i,\beta^*_b,\beta^*_s|x_0)$.

212. Equilibrium conditions for β^*_Δ. We first consider convergence of expected payoff in $\mathcal{G}(\gamma,f,\Delta)$ to expected payoff in $\mathcal{G}(\gamma,f)$.

$$\begin{aligned} & U(x_i,b_i,\beta^*_{\triangle,b},\beta^*_{\triangle,s}) - U(x_i,b_i,\beta^*_b,\beta^*_s) \\ = & \int (v(x_0,x_i) - p(b_i,\beta^*_{\triangle,b},\beta^*_{\triangle,s}|x_0))q(b_i,\beta^*_{\triangle,b},\beta^*_{\triangle,s}|x_0)f(x_0|x_i)dx_0 \\ & -\int (v(x_0,x_i) - p(b_i,\beta^*_b,\beta^*_s|x_0))q(b_i,\beta^*_b,\beta^*_s|x_0)f(x_0|x_i)dx_0 \\ = & \int v(x_0,x_i)(q(b_i,\beta^*_{\triangle,b},\beta^*_{\triangle,s}|x_0) - q(b_i,\beta^*_b,\beta^*_s|x_0))f(x_0|x_i)dx_0 \\ & -\int [p(b_i,\beta^*_{\triangle,b},\beta^*_{\triangle,s}|x_0)q(b_i,\beta^*_{\triangle,b},\beta^*_{\triangle,s}|x_0) - p(b_i,\beta^*_b,\beta^*_s|x_0)q(b_i,\beta^*_b,\beta^*_s|x_0)] \\ & f(x_0|x_i)dx_0. \end{aligned} \tag{184}$$

From (183),$q(b_i,\beta^*_{\triangle,b},\beta^*_{\triangle,s}|x_0) \to q(b_i,\beta^*_b,\beta^*_s|x_0)$ weakly, thus

$$\begin{aligned} & \lim_{\Delta\to 0} \int v(x_0,x_i)(q(b_i,\beta^*_{\triangle,b},\beta^*_{\triangle,s}|x_0) - q(b_i,\beta^*_b,\beta^*_s|x_0)) \\ & f(x_0|x_i)dx_0 = 0. \end{aligned} \tag{185}$$

For the second term, weak convergence of $\beta^*_{\triangle,b}$ and $\beta^*_{\triangle,s}$ to β^*_b and β^*_s implies that for each b_i,

$$p(b_i,\beta^*_{\triangle,b},\beta^*_{\triangle,s}|x_0) \to p(b_i,\beta^*_b,\beta^*_s|x_0) \tag{186}$$

It follows from (186) that

$$\begin{aligned} & \lim_{\Delta\to 0} \int [p(b_i,\beta^*_{\triangle,b},\beta^*_{\triangle,s}|x_0)q(b_i,\beta^*_{\triangle,b},\beta^*_{\triangle,s}|x_0) \\ & -p(b_i,\beta^*_b,\beta^*_s|x_0)q(b_i,\beta^*_b,\beta^*_s|x_0)] \\ & f(x_0|x_i)dx_0 = 0. \end{aligned} \tag{187}$$

Thus, it follows from (185) and (187) that

$$\lim_{\Delta\to 0} U(x_i,b_i,\beta^*_{\triangle,b},\beta^*_{\triangle,s}) = U(x_i,b_i,\beta^*_b,\beta^*_s). \tag{188}$$

We now characterize the property of β^*_b and β^*_s from the first order condition for β^*_b and β^*_s.

213. First order conditions for $\beta^*_{\Delta,b}$. Since $\beta^*_\triangle$ is an equilibrium of $\mathcal{G}(\gamma,f,\Delta)$, for each $b_i \in \text{supp}\beta_{\triangle,b}(x_i)$,

$$\begin{aligned} & \int_{x_0\in Z(b_i+\triangle)} (v(x_0,x_i) - (b_i+\triangle)) \\ & q(b_i+\triangle, W = b_i+\triangle|x_0)f_{\theta|X_i}(x_0|x_i)dx_0 \\ & +\int_{x_0\in Z(b_i)} (v(x_0,x_i) - b_i) \\ & \ (1 - q(b_i, W = b_i|x_0))f_{\theta|X_i}(x_0|x_i)dx_0 \\ & \leq 0. \end{aligned} \tag{189}$$

and

$$\int_{x_0 \in Z(b_i)} (v(x_0, x_i) - b_i) \quad (190)$$
$$q(b_i, W = b_i | x_0) f_{\theta|X_i}(x_0|x_i) dx_0$$
$$+ \int_{x_0 \in Z(b_i - \Delta)} (v(x_0, x_i) - b_i)$$
$$(1 - q(b_i, W = b_i - \Delta | x_0)) f_{\theta|X_i}(x_0|x_i) dx_0$$
$$\geq \ 0.$$

The first inequality says that for a player with signal x_i, it is not preferable to increase the bid from b_i to $b_i + \Delta$. The second inequality says that it is preferable to increase the bid from $b_i - \Delta$ to b_i.

214. The limit of the first order condition. Following the notation of the previous lemma, we have, from (189),

$$\int_{Z_1} (v(x_0, x_i) - b_i) \overline{q}(b_{i,\Delta}, W \ = \ b_i | x_0) f_{\theta|X_i}(x_0|x_i) dx_0 \quad (191)$$
$$+ \int_{Z_2} (v(x_0, x_i) - b_i)(1 - q(b_{i,\Delta}, W \ = \ b_i | x_0)) f_{\theta|X_i}(x_0|x_i) dx_0 \leq 0$$

and from (190),

$$\int_{Z_2} (v(x_0, x_i) - b_i) q(b_i, W \ = \ b_i | x_0) f_{\theta|X_i}(x_0|x_i) dx_0 \quad (192)$$
$$+ \int_{Z_3} (v(x_0, x_i) - b_i)(1 - \underline{q}(b_i, W \ = \ b_i | x_0)) f_{\theta|X_i}(x_0|x_i) dx_0 \geq 0.$$

216. Conditions for expected payoffs when the market clearing price is b_i. Following the steps of the previous lemma, from (191),

$$\int_{Z_2} (v(x_0, x_i) - b_i)(1 - q(b_{i,\Delta}, W = b_i | x_0)) f_{\theta|X}(x_0|x_i) dx_0 \leq 0$$

and

$$\int_{Z_2} (v(x_0, x_i) - b_i)(1 - q(b_{i,\Delta}, W = b_i | x_0)) f_{\theta|X_i}(x_0|x_i) dx_0 \geq 0, \quad (193)$$

and, from (161),

$$\int_{Z_2} (v(x_0, x_i) - b_i)(1 - q(b_i, W = b_i | x_0)) f_{\theta|X_i}(x_0|x_i) dx_0 \quad (194)$$
$$\geq \ \int_{Z_2} (v(x_0, x_i) - b_i) q_i(b_i, W = b_i | x_0) f_{\theta|X_i}(x_0|x_i) dx_0.$$

Therefore, from (193), and (194),

$$\int_{Z_2} (v(x_0, x_i) - b_i)(1 - q(b_{i,\Delta}, W = b_i | x_0)) f_{\theta|X_i}(x_0|x_i) dx_0 = 0 \quad (195)$$

and

$$\int_{Z_2} (v(x_0, x_i) - b_i)(1 - q(b_{i,\Delta}, W = b_i | x_0)) f_{\theta|X_i}(x_0|x_i) dx_0 = 0. \quad (196)$$

From (195) and (196),

$$\int_{Z_2} (v(x_0, x_i) - b_i) f_{\theta|X_i}(x_0|x_i) dx_0 = 0. \quad (197)$$

The argument is similar to the one used in the proof of 4.9. The difference is that, in the proof of the previous lemma, the assumption was that these two first order conditions about increasing the bid and decreasing the bid were derived for players with distinct signals, and the winner's curse condition was used to derive inconsistency of the assumption that players with two distinct signals chooses the same bid. Here, after the argument of the previous lemma, we already know that players with two distinct signals will not choose the same bid, and the first order conditions were applied to players with the same signal.

218. Interpretation of the conditions (197)

By rewriting (197), we get

$$\mathbf{E}\left[v(\theta, X_i) - b_i | X_i = x_i, W(\theta) = b_i\right] = 0. \tag{198}$$

That is,

$$b_i = \mathbf{E}\left[v(\theta, X_i) | X_i = x_i, W(\theta) = b_i\right]. \tag{199}$$

This condition holds for both buyers and sellers, so we conclude that $\beta^*_{\Delta,b}$ is symmetric among buyers and sellers. This is the equilibrium strategy derived in Milgrom (1981), Pesendorfer and Swinkels (1997), and Reny and Perry (2006). In the large economy, since buyers and sellers are symmetric, the equilibrium strategy in the finite one-sided uniform price auctions continues to apply in the uniform price double auctions.

We now show that this bidding strategy is monotone in x_i.

219. Monotonicity of b_i. Let $\overline{x}_i \geq \underline{x}_i$ and define, following (199),

$$\overline{b}_i = \mathbf{E}\left[v(\theta, X_i) | X_i = \overline{x}_i, W(\theta) = \overline{b}_i\right] \tag{200}$$

and

$$\underline{b}_i = \mathbf{E}\left[v(\theta, X_i) | X_i = \underline{x}_i, W(\theta) = \underline{b}_i\right]. \tag{201}$$

Suppose

$$\underline{b}_i \geq \overline{b}_i. \tag{202}$$

Then, from (200) and (201)

$$\begin{aligned} & \mathbf{E}[v(\theta, X_i) | X_i = \overline{x}_i, W(\theta) = \overline{b}_i] \\ \leq\ & \mathbf{E}\left[v(\theta, X_i) | X_i = \underline{x}_i, W(\theta) = \underline{b}_i\right]. \end{aligned} \tag{203}$$

On the other hand,

$$\begin{aligned}
&\mathbf{E}[v(\theta,\overline{x}_i)|X_i=\overline{x}_i,W(\theta)=\overline{b}_i] \\
&-\mathbf{E}\left[v(\theta,\underline{x}_i)|X_i=\underline{x}_i,W(\theta)=\underline{b}_i\right] \\
=&\underbrace{\begin{pmatrix}\mathbf{E}[v(\theta,\overline{x}_i)|X_i=\overline{x}_i,W(\theta)=\overline{b}_i] \\ -\mathbf{E}\left[v(\theta,\overline{x}_i)|X_i=\underline{x}_i,W(\theta)=\underline{b}_i\right]\end{pmatrix}}_{\text{common value effect}} \\
&+\underbrace{\begin{pmatrix}\mathbf{E}[v(\theta,\overline{x}_i)|X_i=\underline{x}_i,W(\theta)=\overline{b}_i] \\ -\mathbf{E}\left[v(\theta,\underline{x}_i)|X_i=\underline{x}_i,W(\theta)=\underline{b}_i\right]\end{pmatrix}}_{\text{private value effect}}.
\end{aligned} \tag{204}$$

The first term is positive because X_i is affiliated with θ and from $W(\theta)$ is monotone in x_0. The second term is also positive. We note that (203) and (204) are mutually exclusive. Thus we conclude that (202) does not hold. Therefore we conclude $\overline{b}_i > \underline{b}_i$.

220. Asymptotic equivalence to the fully rational expectation equilibrium. The above argument shows that the bidding strategy $\beta^*_{\Delta,b}$ is pure, symmetric between buyers and sellers, and strictly increasing in x_i Since the market clearing price is determined by the $1-\alpha$th quantile of bids (counted from above), for each realization of state x_0 it is determined by the bid of the bidder who is on the margin $x_i(x_0)$. Therefore, $p(x_0) = v(x_0,x_i(x_0))$. Thus, the equilibrium under $\beta^*_{\Delta,b}$ is outcome equivalent to the fully revealing rational expectation equilibrium identified in Lemma 1.

8. PROOF OF PROPOSITION 1(C)

8.1. *Lemma 4.1*

LEMMA. *Suppose $\beta^*_{\Delta,n} \to \beta^*_{\Delta}$ and consider the interim expected payoff functions $U_{n,i}(x_i,b_i,\beta^*_{\Delta,b,n},\beta^*_{\Delta,s,n})$ in the double auction game in the finite economy $\mathcal{G}(\gamma,f,\Delta,n)$ and the interim expected payoff function $U(x_i,b_i,\beta^*_{\Delta,b},\beta^*_{\Delta,s})$ in the double auction game in the large economy $\mathcal{G}(\gamma,f,\Delta)$. Then, as $n \to \infty$, $U_{n,i}(x_i,b_i,\beta^*_{\Delta,b,n},\beta^*_{\Delta,s,n}) \to U(x_i,b_i,\beta^*_{\Delta,b},\beta^*_{\Delta}$* uniformly for player i, signal x_i, and a bid b_i.

PROOF.

222. Introduction. We show convergence for buyer i. An argument for a seller is similar, and uniform convergence across buyers and sellers can be obtained by taking the largest bound out of the bound for buyers and sellers. First of all, the difference between $U_{n,b}(x_i,b_i,\beta^*_{\Delta,n,b},\beta^*_{\Delta,n,s})$ and $U(x_i,b_i,\beta^*_{\Delta,b},\beta^*_{\Delta,s})$ can be decomposed into the differenced caused by the difference in strategies of other players an difference from the size of the economy. We can deal with the first difference from the fact that the behavioral strategies converge for each signal and bids. We can deal with the second difference from the fact that the distribution of bids converges for each signal and bids.

223. Decomposition of the difference. We note, by adding and subtracting $U_{n,b}(x_i,b_i,\beta^*_{\Delta$

$$\begin{aligned} & U_{n,b}(x_i,b_i,\beta^*_{\Delta,n,b},\beta^*_{\Delta,n,s}) - U(x_i,b_i,\beta^*_{\Delta,b},\beta^*_{\Delta,s}) \\ = & \underbrace{\left[U_{n,b}(x_i,b_i,\beta^*_{\Delta,n,b},\beta^*_{\Delta,n,s}) - U_{n,b}(x_i,b_i,\beta^*_{\Delta,b},\beta^*_{\Delta,s})\right]}_{\text{change in other players' strategies}} \\ & + \underbrace{\left[U_{n,b}(x_i,b_i,\beta^*_{\Delta,b},\beta^*_{\Delta,s}) - U(x_i,b_i,\beta^*_{\Delta,b},\beta^*_{\Delta,s})\right]}_{\text{change in the size of the economy}}. \end{aligned} \tag{205}$$

That is, the change in the expected payoff from $U_{n,b}(x_i,b_i,\beta^*_{\Delta,n,b},\beta^*_{\Delta,n,s})$ to $U(x_i,b_i,\beta^*_{\Delta,b},\beta^*_{\Delta,s})$ is decomposed into the change in the expected payoff from (1) the change in other players' strategies while keeping the size of the economy fixed, and (2) the change in the size of the economy while keeping other players' strategies fixed.

We now consider the first term of (205).

224. Effect of the changes in other players' strategies. We note, by definition

$$\begin{aligned} & U_{n,b}(x_i,b_i,\beta_{-i,\triangle,n}) \\ = & \int_{[0,1]} \sum_{\underbrace{\mathcal{X}\times\ldots\times\mathcal{X}}_{n_S-1}} \left[\sum_{\underbrace{B_\Delta\times\ldots\times B_\Delta}_{n-1}} (v(x_0,x_i)-p_n(b_i,b_{-i}))q(b_i,b_{-i})h_{\Delta,n,-i}(b_{-i}|x_{-i}) \right] \\ & f_{\theta,X_{-i}|X_i}(x_0,x_{-i}|x_i)dx_0. \end{aligned} \tag{206}$$

That is, the interim expected payoff is obtained taking expectations in terms of the distribution of bids of other players and the distribution of the state conditional on a player's signal.

It follows from (206) that the first term of (205) is

$$\begin{aligned} & U_{n,b}(x_i,b_i,\beta^*_{\Delta,n,b},\beta^*_{\Delta,n,s}) - U_{n,b}(x_i,b_i,\beta^*_{\Delta,b,},\beta^*_{\Delta,s}) \\ = & \int_{[0,1]} \sum_{\underbrace{\mathcal{X}\times\ldots\times\mathcal{X}}_{n_S-1}} \left[\begin{array}{c} \sum_{\underbrace{B_\Delta\times\ldots\times B_\Delta}_{n-1}} (v(x_0,x_i)-p_n(b_i,b_{-i}))q(b_i,b_{-i}) \\ (h_{\Delta,n,-i}(b_{-i}|x_{-i}) - h_{\Delta,-i}(b_{-i}|x_{-i})) \end{array} \right] \\ & f_{\theta,X_{-i}|X_i}(x_0,x_{-i}|x_i)dx_0. \end{aligned} \tag{207}$$

That is, the difference of the interim expected payoff between strategies $(\beta^*_{\Delta,n,b},\beta^*_{\Delta,n,s})$ and $(\beta^*_{\Delta,b},\beta^*_{\Delta,s})$ is expressed in terms of the differences in probabilities that these strategies will assume for a bid profile b_{-i}.

It follows from the assumption of the lemma that $\beta^*_{\Delta,n} \to \beta^*_\Delta$. Since the sets of signals and bids are finite and we assume buyers use symmetric strategies and sellers use symmetric strategies, we have, for each buyer or seller i,

$$\lim_{n\to\infty} h_{\Delta,n,i}(b_i|x_i) = h_{\Delta,i}(b_i|x_i). \tag{208}$$

That is, for each buyer or seller i, b_i and x_i, for every $\varepsilon > 0$, there exists $n_{i,x_i,b_i}(\varepsilon)$ such that, for every $n > n_{i,x_i,b_i}(\varepsilon)$,

$$|h_{i,\Delta,n}(b_i|x_i) - h_{i,\Delta}(b_i|x_i)| < \varepsilon. \tag{209}$$

It follows from (209) that, for every $\varepsilon > 0$, there exists

$$n(\varepsilon) = \max_{i\in\{b,s\},x_i\in\mathcal{X}_\gamma,b_i\in B_\Delta} n_{i,x_i,b_i}(\varepsilon)$$

such that, for every $n > n(\varepsilon)$,

$$|h_{i,\Delta,n}(b_i|x_i) - h_{i,\Delta}(b_i|x_i)| < \varepsilon \text{ for every } i, b_i, \text{ and } x_i. \tag{210}$$

Now, there exists $\overline{v}$ such that $0 \le v(x_0,x_i) < \overline{v}$ for every x_0 and x_i. It follows that there exists $\overline{f}$ such that $0 < f_{\theta,X_{-i}|X_i}(x_0,x_{-i}|x_i) < \overline{f}$ for every x_0,x_i, and x_{-i}. Then, $0 \le p_n(b_i,b_{-i}) < \overline{b}$ for every b_i and b_{-i}. Then, $0 \le q(b_i,b_{-i}) \le 1$. It follows from (207) that for every $\eta > 0$, there exists $n(\eta)$ such that, for every $n > n(\eta)$,

$$|U_{n,b}(x_i, b_i, \beta^*_{\Delta,n,b}, \beta^*_{\Delta,n,s}) - U_{n,b}(x_i, b_i, \beta^*_{b\,\Delta,n}, \beta^*_{\Delta,n,s})| < \eta \tag{211}$$
$$\text{for every } i \in \{b, s\}, \text{ every } x_i, \text{ and } b_i.$$

We now examine effect of the change in the size of the economy. We note the distribution of bids conditional on signal x_i and state x_0. From Assumption 4, conditional on x_0, x_i does not affect the distribution of $\{x_j\}_{j\neq i}$. Therefore, once we condition on x_0, the conditional distribution is independent of x_i. (Of course x_i affects the conditional distribution of x_0 given x_i).

225. The distribution of bids conditional on the state. Given signal x_i, player i knows that the state x_0 is distributed according to the conditional density $f_{\theta|X_i}(x_0|x_i)$. Conditional on x_0, another buyer's signal is distributed according to the conditional density $f_{X_i|\theta}(x_i|x_0)$. A buyer's bidding behavior is characterized by the behavioral strategy $h_{\Delta,n,b}(b_j|x_j)$. Thus, the probability that another buyer's bid is b'_j conditional on x_i and x_0 is described by

$$\Pr(b'_j|x_i, x_0) = \sum_{x_j\in\mathcal{X}_\gamma} h_{\Delta,n,b}(b'_j|x_j) f_{X_i|\theta,X_i}(x_j|x_0, x_i) \tag{212}$$

By Assumption 4, we have

$$\begin{aligned} & \Pr(b'_j|x_i, x_0) \\ = & \sum_{x_j\in\mathcal{X}_\gamma} h_{b,\Delta,n}(b'_j|x_j) f_{X_i|\theta,X_i}(x_j|x_0, x_i) \\ = & \sum_{x_j\in\mathcal{X}_\gamma} h_{b,\Delta,n}(b'_j|x_j) f_{X_i|\theta,X_i}(x_j|x_0) \\ = & \Pr(b'_j|x_0) \end{aligned} \tag{213}$$

We now apply the Glivenko-Cantelli theorem to the distribution of bids conditional on the state.

226. Application of Glivenko-Cantelli theorem. By (213), conditional on x_0, buyer's bids are distributed independently and identically distributed. Let $f(b'_j|x_0)$ be the probability mass function. Let $F_{n,-i}(b'_j|x_0)$ be the empirical probability mass function of buyer's bids by buyers other than i. That is,

$$F_{n,-i}(b'_j|x_0) = \frac{1}{n_B - 1} \sum_{j \in N_{n,B} \setminus \{i\}} 1_{\{b_j = b'_j\}} \tag{214}$$

Then, by Glivenko-Cantelli theorem, for every x_0 and every $b'_j \in B_\Delta$,

$$F_{n,-i}(b'_j|x_0) \to F(b'_j|x_0) \text{ almost surely} \tag{215}$$

We now derive the convergence of the distribution of bids given x_i.

227. Convergence of the distribution of bids conditional on x_0. It follows that, for every b_j and x_i,

$$\begin{aligned} & F_{n,-i}(b_j = b'_j|x_i) && (216) \\ = \; & \int F_{n,-i}(b'_j|x_0, x_i) f(x_0|x_i) dx_0 && (217) \\ = \; & \int F_{n,-i}(b'_j|x_0) f(x_0|x_i) dx_0 \\ \to \; & \int F(b'_j|x_0) f(x_0|x_i) dx_0 \\ = \; & F(b_j = b'_j|x_i) \end{aligned}$$

That is, for each $b_j \in B_\Delta$, the probability mass function of buyers' bid by buyers other than i, conditional on a player's signal x_i, converges as $n \to 0$.

We now show that when player i chooses a bid b_i, the outcome of the auction game in $\mathcal{G}(\gamma, f, \Delta, n)$ converges to the outcome in $\mathcal{G}(\gamma, f, \Delta)$.

228. Convergence of the market clearing price. By definition,

$$\begin{aligned} & b_{n_S:n\setminus\{i\}} = b'_i && (218) \\ \Leftrightarrow & \begin{pmatrix} F_{n,-i}(b_i \leq b'_i - \Delta|x_i) < \alpha \\ F_{n,-i}(b_i > b'_i + \Delta|x_i) \geq \alpha. \end{pmatrix} \end{aligned}$$

It follows from (216) that

$$b_{n_S:n\setminus\{i\}} \to \overline{p}(b) \tag{219}$$

Similarly,

$$b_{n_S - 1:n\setminus\{i\}} \to \overline{p}(b) \tag{220}$$

and

$$b_{n+1:n\backslash\{i\}} \to \underline{p}(b) \tag{221}$$

Therefore,

$$kb_{n_S+1:n\backslash\{i\}} + (1-k)b_{n_S:n\backslash\{i\}} \to k\underline{p}(b) + (1-k)\overline{p}(b) \tag{222}$$

Now, the only way that the market clearing price can be different from $k\underline{p}(b)+(1-k)\overline{p}(b)$ is

$$b_{n_S+1:n\backslash\{i\}} < b_i < b_{n_S-1:n\backslash\{i\}}. \tag{223}$$

We consider two cases.

- If $\underline{p}(b) = \overline{p}(b)$, then, conditional on $b_{n_S+1:n\backslash\{i\}} < b_i < b_{n_S-1:n\backslash\{i\}}$,

$$kb_{n_S+1:n\backslash\{i\}} + (1-k)b_i \to k\underline{p}(b) + (1-k)\overline{p}(b) \tag{224}$$

  and

$$kb_i + (1-k)b_{n_S:n\backslash\{i\}} \to k\underline{p}(b) + (1-k)\overline{p}(b) \tag{225}$$

- If $\underline{p}(b) < \overline{p}(b)$, $\underline{p}(b)$ and $\overline{p}(b)$ are indeterminate because of b_i. In this case, by a convention adapted in the definition of $\underline{p}(b)$ and $\overline{p}(b)$, we can use choose the market clearing price which will be a limit of the market clearing price in the finite economy.

It follows from (222), (224), and (225) that under strategies $(\beta^*_{\Delta,b},\beta^*_{\Delta,s})$, for each b_i,

$$p_n(b_i, b_{-i}|x_i) \to p(b_i, b_{-i}|x_i) \tag{226}$$

229. Convergence of the allocation. Following steps similar to the one in the proof of the previous lemma,

$$q_n(b_i, b_{-i}|x_i) \to q(b_i, b_{-i}|x_i) \tag{227}$$

We are now able to show convergence of expected payoffs.

230. Convergence of expected payoffs. From (226) and (227), it follows that, for each b_i and x_i,

$$\begin{aligned} & U_{n,b}(x_i, b_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) - U_b(x_i, b_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) \\ = \; & \mathbf{E}\left[(v(x_0, x_i) - p_n(b_i, b_{-i}))q_n(b_i, b_{-i})|x_i\right] \\ & -\mathbf{E}\left[(v(x_0, x_i) - p(b_i, b_{-i}))q(b_i, b_{-i})|x_i\right] \\ \to \; & 0 \text{ as } n \to \infty. \end{aligned} \tag{228}$$

We now combine (205), (211) and (228) to get for each b_i and x_i,

$$U_{n,b}(x_i, b_i, \beta^*_{\Delta,n,b}, \beta^*_{\Delta,n,s}) - U(x_i, b_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) \to 0 \text{ as } n \to \infty. \tag{229}$$

That is, for each ε, there exists $n(x_i,b_i,\varepsilon) < \infty$ such that for every $n > n(x_i,b_i,\varepsilon)$,

$$|U_{n,b}(x_i, b_i, \beta^*_{\Delta,n,b}, \beta^*_{\Delta,n,s}) - U(x_i, b_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s})| < \epsilon. \tag{230}$$

It follows from (230) that, for each ε, there exists $n(\varepsilon) = \max_{x_i \in \mathcal{X}_\gamma, b_i \in B_\Delta} n(x_i,b_i,\varepsilon) < \infty$ such that for every $n > n(x_i,b_i,\varepsilon)$,

$$|U_{n,b}(x_i, b_i, \beta^*_{\Delta,n,b}, \beta^*_{\Delta,n,s}) - U(x_i, b_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s})| < \epsilon. \tag{231}$$

8.2. *Lemma 4.2*

LEMMA. *There exists $\underline{n}$ such that for every $n > \underline{n}$, for every player i and for every signal x_i, the set of best response $BR_{i,\Delta}(x_i,\beta^*_{\Delta,n,b},\beta^*_{\Delta,n,s})$ in $\mathcal{G}(\gamma,f,\Delta,n)$ and the set of best response in $\mathcal{G}(\gamma,f,\Delta)$,$BR_\Delta(x_i,\beta^*_{\Delta,b},\beta^*_{\Delta,s})$ satisfy the following relationship*

$$BR_{n,i}(x_i, \beta^*_{\Delta,n,b}, \beta^*_{\Delta,n,s}) \subseteq BR_\Delta(x_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) \tag{232}$$

PROOF.

232. Suppose that the conclusion does not hold. Then, there exists player i and signal x_i such that there exists a bid b_i which satisfies the following relationship:

$$b_i \in BR_{\Delta,n,i}(x_i, \beta^*_{\Delta,n,b}, \beta^*_{\Delta,n,s}) \text{ but } b_i \notin BR_\Delta(x_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}). \tag{233}$$

It follows from (233) that

$$U_{n,i}(x_i, b_i, \beta^*_{\Delta,n,b}, \beta^*_{\Delta,n,s}) \geq U_{n,i}(x_i, b'_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) \text{ for every } b'_i \in B_\Delta \tag{234}$$

and that there exists $b''_i \in B_\Delta$ such that

$$U(x_i, b''_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) > U(x_i, b_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}). \tag{235}$$

Then, there exists $\eta > 0$ such that

$$\eta = U(x_i, b''_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) - U(x_i, b_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) \tag{236}$$

From the previous Lemma, there exists $n(i,x_i,b_i) > 0$ such that for every $n > n(i,x_i,b_i)$,

$$|U(x_i, b''_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) - U_{n,i}(x_i, b''_i, \beta^*_{\Delta,n,b}, \beta^*_{\Delta,n,s})| < \eta/2 \tag{237}$$

and

$$|U(x_i, b_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) - U_{n,i}(x_i, b_i, \beta^*_{\Delta,n,b}, \beta^*_{\Delta,n,s})| < \eta/2 \tag{238}$$

It follows from (237) and (238),

$$
\begin{aligned}
& U_{n,i}(x_i, b_i'', \beta^*_{\Delta,n,b}, \beta^*_{\Delta,n,s}) - U_{n,i}(x_i, b_i, \beta^*_{\Delta,n,b}, \beta^*_{\Delta,n,s}) \\
> & \left[U(x_i, b_i'', \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) - \eta/2\right] \\
& - \left[U(x_i, b_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) + \eta/2\right] \\
= & \; U(x_i, b_i'', \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) - U(x_i, b_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) - \eta \\
> & \; 0
\end{aligned}
\tag{239}
$$

But they are mutually exclusive. That is, for x_i and for b_i, for $n > n(i,x_i,b_i)$,

$$BR_{n,i}(x_i, \beta^*_{\Delta,n,b}, \beta^*_{\Delta,n,s}) \subseteq BR_\Delta(x_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}).$$

Let $\underline{n} = \max_{i\in\{b,s\},x_i\in\mathcal{X}_\gamma,b_i\in B_\Delta} n(i,x_i,b_i)$. Then, since $\mathcal{X}_\gamma$ and B_Δ are finite, $\underline{n} < \infty$. For any $n > \underline{n}$, for every x_i and b_i, (232) holds.

8.3. *Lemma 4.3*

Lemma. *There exists $\overline{\Delta} > 0$ and $\underline{n} < \infty$ such that for all $0 < \Delta < \overline{\Delta}$ and $n > \underline{n}$, in the uniform price auction game in the finite economy $\mathcal{G}(\gamma,f,\Delta,n)$, for every player i, a best response to an equilibrium strategy $\beta^*_{\Delta,n}$ satisfies the strict single crossing condition for bids in the range of the equilibrium prices.*

Proof.
234. Definition of strict single crossing conditions. It follows from the definition that we need to prove

$$
\begin{aligned}
& \textit{If } b_i \textit{ is a best reply for } x_i \textit{ to } (\beta^*_{\Delta,n,b}, \beta^*_{\Delta,n,s}), \\
& \textit{then, for every } b_i > \underline{b}_i \textit{ and every } \overline{x}_i > x_i, \\
& \underbrace{U_{n,i}(x_i, b_i, \beta^*_{\Delta,n,b}, \beta^*_{\Delta,n,s}) \geq U_{n,i}(x_i, \underline{b}_i, \beta^*_{\Delta,n,b}, \beta^*_{\Delta,n,s})}_{\textit{if } b_i \textit{ is preferred to a lower bid by a player with the signal } x_i} \\
\rightarrow \quad & \underbrace{U_{n,i}(\overline{x}_i, b_i, \beta^*_{\Delta,n,b}, \beta^*_{\Delta,n,s}) > U_{n,i}(\overline{x}_i, \underline{b}_i, \beta^*_{\Delta,n,b}, \beta^*_{\Delta,n,s})}_{\textit{then } b_i \textit{ is still preferred to a lower bid by player with the higher signal } \overline{x}_i} \\
& \textit{and, for } b_i < \overline{b}_i,\ x_i > \underline{x}_i, \\
& \underbrace{U_{n,i}(x_i, b_i, \beta^*_{\Delta,n,b}, \beta^*_{\Delta,n,s}) \geq U_{n,i}(x_i, \overline{b}_i, \beta^*_{\Delta,n,b}, \beta^*_{\Delta,n,s})}_{\textit{if } b_i \textit{ is preferred to a higher bid } \overline{b}_i \textit{ by a player with signal } x_i} \\
\rightarrow \quad & \underbrace{U_{n,i}(\underline{x}_i, b_i, \beta^*_{\Delta,n,b}, \beta^*_{\Delta,n,s}) > U_{n,i}(\underline{x}_i, \overline{b}_i, \beta^*_{\Delta,n,b}, \beta^*_{\Delta,n,s})}_{\textit{then } b_i \textit{ is still preferred to a higher bid by a player with the lower signal } \underline{x}_i}
\end{aligned}
$$

We first choose a signal x_i, a best response b_i and consider a condition for an approximation n such that single crossing condition holds for an economy larger than n

235. Strict single crossing conditions for a single comparison of bids and signals. Let $b_i \in BR_{\Delta,n,i}(x_i,\beta^*_{\Delta,n,b},\beta^*_{\Delta,n,s})$. Let $\underline{b}_i \in B_\Delta$ and $\overline{x}_i > x_i$. It follows from the previous Lemma that, for sufficiently large n, b_i is also a best response to $(\beta^*_{\Delta,b}, \beta^*_{\Delta,s})$ in $\mathcal{G}(\gamma,f,\Delta)$. It follows that

$$U(x_i, b_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,n,s}) \geq U(x_i, \underline{b}_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) \tag{240}$$

It follows from the strict single crossing condition of best response in the large economy, for a higher signal $\overline{x}_i > x_i$, we have

$$U(\overline{x}_i, b_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) > U(\overline{x}_i, \underline{b}_i, \beta^*_{\Delta,b}, \beta^*_{\Delta,s}) \tag{241}$$

It follows that there exists $n(\overline{x}_i,b_i,\underline{b}_i)$ such that for every $n > n(\overline{x}_i,b_i,\underline{b}_i)$,

$$U_{i,n}(\overline{x}_i, b_i, \beta^*_{\Delta,n,b}, \beta^*_{\Delta,n,s}) - U_{i,n}(\overline{x}_i, \underline{b}_i, \beta^*_{\Delta,n,b}, \beta^*_{\Delta,n,s}) > 0 \tag{242}$$

That is, a strict single crossing condition holds for a comparison between b_i and $\underline{b}_i$ for a signal $\overline{x}_i$. It is because, a strict single crossing condition for the double auction game in the large economy says that it is strictly preferable to increase the bid from $\underline{b}_i$ to $\overline{b}_i$ for a player with signal $\overline{x}_i$. Since the payoff of the double auction game in the large finite economy will be very close to the payoff of the double auction game in the large economy, the strict single crossing condition extends to the large finite double auction game.

We now extend the argument to obtain strict single crossing condition for every comparison in the double auction game in the large finite economy. It is because the set of possible signals and the set of possible bids are finite.

236. Single crossing condition for all cases. Let

$$\overline{n} = \max_{\substack{\overline{x}_i \in \mathcal{X}_i, b_i \in B_\Delta \text{ such that there exists } x_i < \overline{x}_i \\ \text{such that } b_i \text{ is a best response to } x_l}} n(\overline{x}_i, b_i, \underline{b}_i) \tag{243}$$

Then, for every $n > \overline{n}$,

$$U_{n,i}(x_i, b_i, \beta^*_{\Delta,n,b}, \beta^*_{\Delta,n,s}) \geq U_{n,i}(x_i, \underline{b}_i, \beta^*_{\Delta,n,b}, \beta^*_{\Delta,n,s})$$

and

$$U_{n,i}(\overline{x}_i, b_i, \beta^*_{\Delta,n,b}, \beta^*_{\Delta,n,s}) - U_{n,i}(\overline{x}_i, \underline{b}_i, \beta^*_{\Delta,n,b}, \beta^*_{\Delta,n,s}) > 0. \tag{244}$$

From (244), the result holds.

The other direction of single crossing condition can be shown in a similar manner.

Intuitively, the previous paragraph showed that for sufficiently large n, a strict single crossing condition holds for a set of a signal and two bids. Since the set of possible bids and signals are finite, by taking the largest n which will work for all possible combinations of a signal, and bids, we can ensure that the strict single crossing condition holds of the double auction game in the large finite economy.

8.4. *Lemma 4.4*

LEMMA. *There exists* $\overline{\Delta} > 0$ *and* $\underline{n} < \infty$ *such that for all* $\Delta < \overline{\Delta}$ *and* $n > \underline{n}$, *(a)* $\beta^*_{\Delta,b}$ has a monotone supports, and (b) *for each buyer* $i \neq j$, *signal* $x_i \neq x_j$, *and a bid* b *which is in the range of the equilibrium price,*

$$\Pr(\beta^*_{\Delta,n,b}(x_i) = b) \cdot \Pr(\beta^*_{\Delta,n,b}(x_j) = b) = 0. \tag{245}$$

The similar condition holds for sellers.

PROOF.

238. Introduction. We prove the condition for buyers' strategies. The argument for the seller is similar. The argument is similar to the previous lemma extends to the large economy. There are two differences. The first difference is that, since buyers and sellers are asymmetric in the finite economy, the supports of the distribution of buyer's equilibrium strategies and seller's equilibrium strategies are monotone in their signals. The second difference is that, since a bid can affect a market clearing price of the double auction in the finite economy. But these differences will not affect the argument

239. Suppose that there is no $\overline{\Delta} > 0$ which satisfies (245). It follows that for every $\Delta > 0$, there exists buyer i,j and signal $x_{i,\Delta} > x_{j,\Delta}$ and a bid $b_{i,\Delta}$ such that

$$\Pr(\beta^*_{\Delta,b,n}(x_{i,\Delta}) = b_{i,\Delta}) \cdot \Pr(\beta^*_{\Delta,s,n}(x_{j,\Delta}) = b_{i,\Delta}) > 0. \tag{246}$$

From lemma 13, we derive some monotonic relationships of the support of the distribution of equilibrium bids for buyers. We first define the support of the distribution of equilibrium bids as we did in the previous lemma.

240. Monotonicity of the supports of the distributions of bidding strategies. Consider two signals $\underline{x}_i < \overline{x}_i$ and consider the distribution of equilibrium bids $H^*_{\triangle,n,i}(b_i|\underline{x}_i)$ and $H^*_{\triangle,n,i}(b_i|\overline{x}_i)$. Let b_i be in the range of equilibrium prices. Then, we have, for each of a buyer and a seller,

$$H^*_{\triangle,n,i}(b_i|\underline{x}_i){\ge}H^*_{\triangle,n.i}(b_i|\overline{x}_i). \tag{247}$$

Let

$$H^*_{\triangle,n,i}(b_i|x_0) = \sum_{x_i\in\mathcal{X}} H^*_{\triangle,n,i}(b_i|x_i) f_{X_i|\theta}(x_i|x_0) \tag{248}$$

From (248)

$$\overline{x}_0 \geq \underline{x}_0 \to H^*_{\triangle,n,i}(b_i|\underline{x}_0) \geq H^*_{\triangle,n,i}(b_i|\overline{x}_0) \tag{249}$$

Thus,

$$\alpha H^*_{\triangle,n,b}(b_i|x_0) + (1-\alpha)H^*_{\triangle,n,s}(b_i|x_0) \text{ is nonincreasing in } x_0. \tag{250}$$

That is, even if buyers and sellers have asymmetric payoffs, each of a buyer and a seller's best response has supports monotonically increasing in bids, thus the distribution of bids, which is a convex combination of bids by buyers and sellers, is stochastically increasing.

From (250), we have monotonicity of allocation probability at ties, as in the previous lemma. It follows that there exists a winner's curse at winning the tie. That is,

$$\begin{aligned} & E[v(\theta,X_i)|X_i = x_i, W_{n,-i}(\theta) = b_i, b_i \text{loses at the tie}] \\ > \ & E[v(\theta,X_i)|X_i = x_i, W_{n,-i}(\theta) = b_i, b_i \text{ wins at the tie}] \end{aligned} \tag{251}$$

This implies that

$$\begin{aligned} & E[v(\theta,X_i) - b_i|X_i = x_i, W_{n,-i}(\theta) = b_i, b_i \text{loses at the tie}] \\ > \ & E[v(\theta,X_i) - b_i|X_i = x_i, W_{n,-i}(\theta) = b_i, b_i \text{ wins at the tie}] \end{aligned} \tag{252}$$

8.5. *Lemma 4.5*

LEMMA. *There exists $\overline{\Delta} > 0$ and $\underline{n} < \infty$ such that for all $\Delta < \overline{\Delta}$ and $n > \underline{n}$, there exists a monotone pure strategy equilibrium $\beta^*_{\Delta,n}$ in the double auction game in the finite market $\mathcal{G}(f,\Delta,n)$.*

PROOF

242. We first extend the result of Lemma 4.4. to a setting with a continuous set of signals. Since the distribution of signals with a finite set of possible signals converge smoothly to the distribution of signals with a continuous set of signals, the limit strategy profile is a Bayesian Nash equilibrium strategy profile. Furthermore, since every Bayesian Nash equilibrium with finite set of signals has monotone and separating supports, the limit Bayesian Nash equilibrium strategy profile has also monotone supports. Given that the set of possible bids is finite and the set of possible signals is continuous, it has to be that the Bayesian Nash equilibrium is monotone and pure almost everywhere.

243. Approximation by the game with a discrete signal. Consider $\mathcal{G}(f,\Delta,n)$ be a double auction game characterized by Assumption 1, Assumption 3-6. Thus we first approximate game $\mathcal{G}(f,\Delta,n)$ by a sequence of a double auction game $\mathcal{G}(\gamma,f_\gamma,\Delta,n)$. The following figure explains the approximation.

We first verify that, given f satisfies Assumption 3 and 4, f_γ satisfies Assumption 4

- Assumption 4(a) holds.
- Assumption 4(b) holds since $f$satisfies Assumption 4(b).

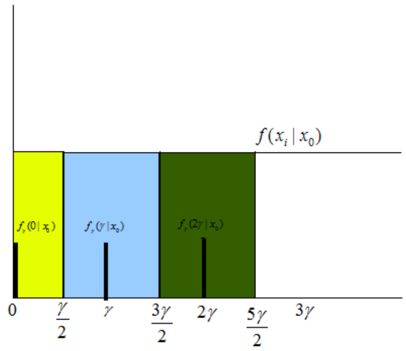

FIGURE 11.—

- Assumption 4(c) holds since $f$satisfies Assumption 4(c).

Thus it remains to show that f_γ satisfies the monotone likelihood ratio condition (Assumption 4(d)). Let $\overline{x}_0 > \underline{x}_0$ in [0,1] and $\overline{x}_i >\underline{x}_i$ in $\mathcal{X}_\gamma$. By definition, we need to show, that, for each $\gamma > 0$,

$$f_{X_i|\theta,\gamma}(\overline{x}_i|\overline{x}_0)f_{X_i|\theta,\gamma}(\underline{x}_i|\underline{x}_0) - f_{X_i|\theta,\gamma}(\overline{x}_i|\underline{x}_0)f_{X_i|\theta,\gamma}(\underline{x}_i|\underline{x}_0) \geq 0.$$

By the definition of f_γ, it is equivalent to show that

$$\begin{aligned} & \int_{\overline{x}_i-\gamma/2}^{\overline{x}_i+\gamma/2} f_{X_i|\theta}(x_i'|\overline{x}_0)dx_i' \int_{\underline{x}_i-\gamma/2}^{\underline{x}_i+\gamma/2} f_{X_i|\theta}(x_i'|\overline{x}_0)dx_i' \\ & - \int_{\overline{x}_i-\gamma/2}^{\overline{x}_i+\gamma/2} f_{X_i|\theta}(x_i'|\overline{x}_0)dx_i' \int_{\underline{x}_i-\gamma/2}^{\underline{x}_i+\gamma/2} f_{X_i|\theta}(x_i'|\overline{x}_0)dx_i' \\ \geq \; & 0. \end{aligned} \tag{253}$$

To show (253), we further approximate $f_{X_i|\theta}(x_i|x_0)$ by

$$f_{X_i|\theta,n}(x_i|x_0) = \begin{cases} f_{X_i|\theta}(x_i|x_0) \text{ if } x_i = 0, 1/n\gamma, 2/n\gamma, ..., 1 \\ 0 \text{ else} \end{cases} \tag{254}$$

The following figure explains the approximation (254). We note

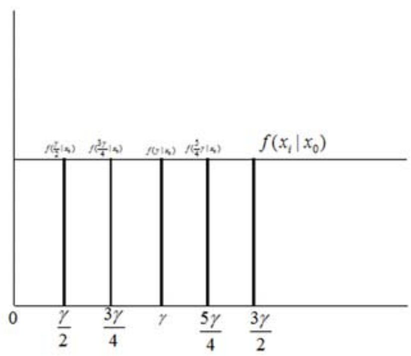

FIGURE 12.—

$$f_{X_i|\theta,n}(x_i|\overline{x}_0) \to f_{X_i|\theta}(x_i|\overline{x}_0) \text{ weakly as } n \to \infty \tag{255}$$

and

$$\frac{1}{(2n-1)} \sum_{\overline{x}_i-\gamma/2\le x_i''\le\overline{x}_i+\gamma/2} f_{X_i|\theta,n}(x_i''|\overline{x}_0) \tag{256}$$
$$\to \int_{\overline{x}_i-\gamma/2}^{\overline{x}_i+\gamma/2} f_{X_i|\theta}(x_i'|\overline{x}_0)dx_i' \text{ as } n\to\infty$$

Also, we have

$$\begin{aligned}
&\frac{1}{(2n-1)} \sum_{\overline{x}_i-\gamma/2\le x_i''\le\overline{x}_i+\gamma/2} f_{X_i|\theta,n}(x_i''|\overline{x}_0) \\
&\cdot\frac{1}{(2n-1)} \sum_{\underline{x}_i-\gamma/2\le x_i'''\le\underline{x}_i+\gamma/2} f_{X_i|\theta,n}(x_i'''|\overline{x}_0) \\
&-\frac{1}{(2n-1)} \sum_{\overline{x}_i-\gamma/2\le x_i''\le\overline{x}_i+\gamma/2} f_{X_i|\theta,n}(x_i''|\underline{x}_0) \\
&\cdot\frac{1}{(2n-1)} \sum_{\underline{x}_i-\gamma/2\le x_i'''\le\underline{x}_i+\gamma/2} f_{X_i|\theta,n}(x_i'''|\underline{x}_0) \\
=\ &\frac{1}{(2n-1)^2}\Big[\sum_{\overline{x}_i-\gamma/2\le x_i''\le\overline{x}_i+\gamma/2,\underline{x}_i-\gamma/2\le x_i'''\le\underline{x}_i+\gamma/2} f_{X_i|\theta,n}(x_i''|\overline{x}_0)f_{X_i|\theta,n}(x_i'''|\overline{x}_0) \\
&-\sum_{\overline{x}_i-\gamma/2\le x_i''\le\overline{x}_i+\gamma/2,\underline{x}_i-\gamma/2\le x_i'''\le\underline{x}_i+\gamma/2} f_{X_i|\theta,n}(x_i''|\underline{x}_0)f_{X_i|\theta,n}(x_i'''|\underline{x}_0)\Big] \\
=\ &\frac{1}{(2n-1)^2} \sum_{\substack{\overline{x}_i-\gamma/2\le x_i''\le\overline{x}_i+\gamma/2,\underline{x}_i-\gamma/2\le x_i'''\le\underline{x}_i+\gamma/2 \\ x_i''=\overline{x}-\gamma/2+n/\gamma, x_i'''=\overline{x}-\gamma/2+m/\gamma \text{ for some } n,m}} \\
&\left[f_{X_i|\theta,n}(x_i''|\overline{x}_0)f_{X_i|\theta,n}(x_i'''|\overline{x}_0)-f_{X_i|\theta,n}(x_i''|\underline{x}_0)f_{X_i|\theta,n}(x_i'''|\underline{x}_0)\right] \\
=\ &\frac{1}{(2n-1)^2} \sum_{\substack{\overline{x}_i-\gamma/2\le x_i''\le\overline{x}_i+\gamma/2,\underline{x}_i-\gamma/2\le x_i'''\le\underline{x}_i+\gamma/2 \\ x_i''=\overline{x}-\gamma/2+n/\gamma, x_i'''=\overline{x}-\gamma/2+m/\gamma \text{ for some } n,m}} \qquad (258) \\
&\left[f_{X_i|\theta}(\overline{x}-\gamma/2+n/\gamma|\overline{x}_0)f_{X_i|\theta}(\overline{x}-\gamma/2+m/\gamma|\overline{x}_0)-f_{X_i|\theta}(\overline{x}-\gamma/2+n/\gamma|\underline{x}_0)f_{X_i|\theta}(\overline{x}-\gamma/2+m/\gamma|\underline{x}_0)\right] \qquad (259) \\
\ge\ & 0
\end{aligned}$$

(257)

It follows from (256) and (257). By letting $n\to\infty$ that f_γ satisfies Assumption 4(d). It follows that there exists a nontrivial monotone pure strategy $\beta^*_{\Delta,n,\gamma}$ of the game $\mathcal{G}(\gamma,f_\gamma,\Delta,n)$.

244. Construction of a Bayesian-Nash equilibrium of $\mathcal{G}(f,\Delta,n)$. Let $\beta^*_{\Delta,n}$ be a limit of a sequence of a nontrivial Bayesian-Nash equilibrium strategy profile $\beta^*_{\Delta,n,\gamma}$ of the game $\mathcal{G}(\gamma,f_\gamma,\Delta,n)$ as $\gamma\to 0$. We need to show that $\beta^*_{\Delta,n}$ is an equilibrium of $\mathcal{G}(f,\Delta,n)$. That is, for each i, for each $\beta_{\Delta,n,i}$,

$$\pi_{n,i}(\beta^*_{\Delta,n,i},\beta^*_{\Delta,n,-i})-\pi_{n,i}(\beta_{\Delta,n,i},\beta^*_{\Delta,n,-i})\ge 0. \tag{260}$$

Let $\beta_{\Delta,n,\gamma}$ be a strategy of the game $\mathcal{G}(\gamma,f_\gamma,\Delta,n)$ such that

$$\beta_{\Delta,n,\gamma} \to \beta_{\Delta,n} \text{ weakly}^{5} \tag{262}$$

It follows from the fact that $\beta^*_{\Delta,n,\gamma}$ is an equilibrium that, for each i and for each $\beta_{\Delta,i,n,\gamma}$,

$$\pi_{n,i}(\beta^*_{\Delta,n,\gamma,i},\beta^*_{\Delta,n,\gamma,-i}) - \pi_{i,n}(\beta_{\Delta,n,\gamma,i},\beta^*_{\Delta,n,\gamma,-i}) \geq 0. \tag{263}$$

We now need to show that

$$\pi_{n,i}(\beta^*_{\Delta,n,\gamma,i},\beta^*_{\Delta,n,\gamma,-i}) \to \pi_{n,i}(\beta^*_{\Delta,n,i},\beta^*_{\Delta,n,-i}) \tag{264}$$

and

$$\pi_{n,i}(\beta_{\Delta,n,\gamma,i},\beta^*_{\Delta,n,\gamma,-i}) \to \pi_{n,i}(\beta_{\Delta,n,i},\beta^*_{\Delta,n,-i}) \text{ as } \gamma \to 0. \tag{265}$$

We show (264). The argument for (265) is similar.

$$\begin{aligned}
& \pi_{n,i}(\beta_{\Delta,n,\gamma,i},\beta_{\Delta,n,\gamma,-i}) \qquad (266)\\
= & \sum_{B_\Delta\times\mathcal{X}} U_{n,i}(x_i,b_i,\beta_{\Delta,n,\gamma,-i})h_{\Delta,n,\gamma,i}(b_i,x_i)\\
= & \sum_{B_\Delta\times\mathcal{X}}[(\sum_{\underbrace{B_\Delta\times ...\times B_\Delta}_{n-1}\times\underbrace{\mathcal{X}...\times\mathcal{X}}_{n-1}} u_i(x_0,x_i,p(b_i,b_{-i}),q(b_i,b_{-i}))\\
& h_{\Delta,n,\gamma,-i}(b_{-i}|x_{-i})f_{\theta,X_{-i}|X_i,\gamma}(x_0,x_{-i}|x_i)]h_{\Delta,n,\gamma,i}(b_i|x_i)f_{X_i,\gamma}(x_i)\\
\to & \sum_{B_\Delta\times\mathcal{X}}[(\sum_{\underbrace{B_\Delta\times ...\times B_\Delta}_{n-1}\times\underbrace{\mathcal{X}...\times\mathcal{X}}_{n-1}} u_i(x_0,x_i,p(b_i,b_{-i}),q(b_i,b_{-i}))\\
& h_{,\Delta,n,\gamma,-i}(b_{-i}|x_{-i})f_{\theta,X_{-i}|X_i}(x_0,x_{-i}|x_i)]h_{\Delta,n,\gamma,i}(b_i|x_i)f_{X_i}(x_i)
\end{aligned}$$

because $h_{\Delta,n,\gamma,-i}(b_{-i}|x_{-i}) \to h_{\Delta,n,\gamma,-i}(b_{-i}|x_{-i})$ by (261),
$f_{\theta,X_{-i}|X_i,\gamma}(x_0,x_{-i}|x_i) \to f_{\theta,X_{-i}|X_i}(x_0,x_{-i}|x_i)$
$h_{\Delta,n,\gamma,i}(b_i|x_i) \to h_{\Delta,n,i}(b_i|x_i)$ by (261)
and $f_{X_i,\gamma}(x_i) \to f_{X_i}(x_i)$ by (261)

Therefore, (264) holds. Similarly, (265) holds. Thus (260) holds.

We now show that the important property of an equilibrium, monotonicity of supports, also holds for $\beta^*_{\Delta,n}$ since $\beta^*_{\Delta,n,\gamma}$ converges to $\beta^*_{\Delta,n}$.

245. Monotonicity of supports of a Bayesian-Nash equilibrium as $\gamma \to \infty$. For each i, let $\overline{x}_i > \underline{x}_i$ and let $\overline{b}_i$ be a minimum of the support of $\beta^*_{\Delta,n,i}(\overline{x}_i)$ and let $\underline{b}_i$ be the maximum of the support of $\beta^*_{\Delta,n,i}(\underline{x}_i)$. Then we claim that $\overline{b}_i \geq \underline{b}_i$.

That is, a Bayesian Nash equilibrium of the double auction game in the large finite economy with a continuous set of signals satisfies the monotone support property.

246. Proof. We first consider the case that there exists some $\underline{\gamma}$ such that both $\underline{x}_i$ and $\overline{x}_i$ are contained in $B_{\underline{\gamma}}$. Then, by the construction of $\mathcal{X}_\gamma$, for every $\gamma >\underline{\gamma}$, $\underline{x}_i$ and $\overline{x}_i$ are contained in $\mathcal{X}_\gamma$. Let $\overline{b}_{\gamma,i}$ be the minimum of the support of the distribution of bids according to $\beta^*_{\Delta,n,\gamma,i}(\overline{x}_i)$ and $\overline{b}_{\gamma,i}$ be the maximum of the support of the distribution of bids according to $\beta^*_{\Delta,n,i,\gamma}(\underline{x}_i)$. Then the previous lemma implies that $\overline{b}_{\gamma,i} > \underline{b}_{\gamma,i}$.

By definition $\beta^*_{\Delta,n,\gamma,i} \to \beta^*_{\Delta,n,i}$. This implies that, since the set of possible bids is finite, for each $\underline{x}_i$ and $\overline{x}_i$, for each $b_i \in B_\Delta$, the probability that $\beta^*_{\Delta,n,i,\gamma}$ chooses a bid b_i converges to the probability that $\beta^*_{\Delta,n,i}$ chooses a bid b_i. That is, therefore, the supports of $\beta^*_{\Delta,n,\gamma,i}$ under $\underline{x}_i$ and $\overline{x}_i$ converge to the supports of $\beta^*_{\Delta,n,i}$ under $\underline{x}_i$ and $\overline{x}_i$. Therefore, it has to be that $\overline{b}_i \geq \underline{b}_i$.

It remains to consider the case where either $\underline{x}_i$ and $\overline{x}_i$ are not contained in $\mathcal{X}_\gamma$ for any γ.In this case, since $\{\mathcal{X}_\gamma\}_\gamma$ is dense in [0,1], there exists a sequence of signals $\{\underline{x}_{i,\gamma}\}$ and $\{\overline{x}_{i,\gamma}\}$which will be contained in some γ, such that $\underline{x}_{i,\gamma} \to \underline{x}_i$ and $\overline{x}_{i,\gamma} \to \overline{x}_i$. Since the set of possible bids is finite and the payoff function is continuous in x_i in the mixed extension, the equilibrium strategies under $\{\underline{x}_{i,\gamma}\}$ and $\{\overline{x}_{i,\gamma}\}$also converges to equilibrium strategies under $\{\underline{x}_i\}$ and $\{\overline{x}_i\}$ by the maximum theorem. Since $\overline{b}_{i,\gamma} \geq \underline{b}_{i,\gamma}$ holds for each γ, $\overline{b}_i \geq \underline{b}_i$ follows.

The above result implies that the supports of equilibrium strategies $\beta^*_{\Delta,n,\gamma,i}$ have supports increasing in the signal, and separating except at the boundary of the support. Since the set of possible bids is finite and the set of possible signals is continuous, it has to be that the support of the equilibrium strategies $\beta^*_{\Delta,n,\gamma,i}$ has to be singleton except for a finite number of signals. It implies that $\beta^*_{\Delta,n,\gamma,i}$ is monotone and pure almost everywhere.

9. PROOF OF PROPOSITION 1(D)

9.1. *Lemma 5.1*

LEMMA. *There exists* $\underline{n}< \infty$ *such that for each* $n >\underline{n}$, *there exists a nontrivial monotone pure strategy equilibrium* β^*_n *of the double auction game in the finite market* $\mathcal{G}(f,n)$.

PROOF.

248.Introduction. We first construct a strategy profile β^*_Δ from a sequence of a monotone pure strategy profile $\{\beta^*_{\Delta,n}\}$. If the probability that a tie occurs among players in the β^*_Δ is zero, then an expected payoff of a player under $\beta^*_{\Delta,n}$ converges to an expected payoff under β^*_Δ, therefore β^*_Δ is an equilibrium. Therefore, we need to consider whether a tie occurs among players in β^*_Δ. Contrary suppose that there are players with distinct signals who will choose the same bid with a positive probability. It implies that, for sufficiently large finite game n, their bids are sufficiently close to each other. This implies two conditions. A player with a high signal does not want to extend the distance between two bids by increasing the bid and a player with a low signal does not want to extend the distance between two bids by lowering the bid. But a player with a low signal. But the winner's curse effect implies that a player with a higher signal prefers to increase the bid sufficiently higher than a player with a lower signal. These conditions imply that players with distinct signals will bid distinct bids.

In contrast to a previous lemma, we need to show that players with distinct bids will place distinct bids with some distance. It is possible because the strict private value element and a uniform lower bound on the rate of increase in the value as a function of signals provides a lower bound of the increase in the expected value of the good. Consequently, when the distance between two bids is smaller than this lower bound, a player with a higher signal prefers to increase the bid further.

249.Construction of a limit strategy profile. From the previous lemma, there exists a monotone pure strategy equilibrium $\{\beta^*_{\Delta,n}\}$ of the double auction game in the large finite market $\mathcal{G}(f,\Delta,n)$ for sufficiently small Δ and sufficiently large n. Consider a sequence of a monotone pure strategy equilibrium $\beta^*_{\Delta,n}$ for $\Delta \to 0$. Since every bid in $\beta^*_{\Delta,n}$ is bounded above by $\overline{b}$ and bounded below by $\underline{b}$, by Helly selection theorem, it is without loss of generality to assume that there exists a monotone pure strategy profile β^* such that $\beta^*_n \to \beta^*$ almost everywhere.

We now argue that β^* is a monotone pure strategy equilibrium.

250.Conditions for β^* to be an equilibrium. We note that β^* is an equilibrium if for every β^*,

$$U(\beta^*_i, \beta^*_{-i}) \geq U(\beta'_i, \beta^*_{-i}).$$

On the other hand, If there are no ties in β^*_Δ, it follows from an argument similar to the previous lemma, that β^*_Δ is an equilibrium of the double auction game in the large finite market $\mathcal{G}(f,\Delta)$. Thus it now remains to show that β^*_Δ does not involve a tie.

251. Suppose, in order to derive contradiction, that β^*_Δ involves a tie with a positive probability. Then, there exists two distinct signal $\overline{x}_i$ and $\underline{x}_i$ and a bid b_i such that players with signal $\overline{x}_i$ and $\underline{x}_i$ choose a bid b_i under β^*_Δ.

It follows that, for every $d > 0$, there exists $\overline{\Delta}$ such that, for every $\Delta < \overline{\Delta}$, there exists two bids $\underline{b}_{\Delta,i}$ and $\overline{b}_{\Delta,i}$ such that (a) a player with signal $\underline{x}_i$ chooses a bid $\underline{b}_{\Delta,i}$,(b) a player with signal $\overline{x}_i$ chooses a bid $\overline{b}_{\Delta,i}$, and (c) $\underline{b}_{\Delta,i}$ and $\overline{b}_{\Delta,i}$ are less than d apart. Since $\beta^*_{\Delta,n}$ is a monotone pure strategy equilibrium, it implies that

252.First order conditions for a player with signal $\overline{x}_i$. A consequence of the above condition is that a player with a signal $\overline{x}_i$ does not prefer to bid above $\overline{b}_{\Delta,i}$, otherwise the distance between two bids will be strictly more than d apart.

It follows that, similar to (189)

$$\int_{\{x_0:W_{-i,n}=\overline{b}_{i,\Delta}+\Delta\}}(v(x_0,x_i)-p_n(\overline{b}_{i,\Delta}+\Delta,b_{-i}))$$
$$q_n(\overline{b}_{i,\Delta}+\Delta,W_{-i,n}(\beta^*_{\Delta,n,-i}(X_{-i})=\overline{b}_{i,\Delta}+\Delta|x_0)f_{\theta|X_i}(x_0|x_i)dx_0+$$
$$+\int_{\{x_0:W_{-i,n}=\overline{b}_{i,\Delta}\}}(v(x_0,x_i)-p_n(\overline{b}_{i,\Delta},b_{-i}))$$
$$(1-q_n(\overline{b}_{i,\Delta},W_{-i,n}(\beta^*_{\Delta,n,-i}(X_{-i}))=\overline{b}_{i,\Delta}|x_0))f_{\theta|X_i}(x_0|x_i)dx_0\leq 0. \tag{267}$$

The first term deals with the case where a new bid $\overline{b}_{i,\Delta}+\Delta$ will win when the marginal bid by other players is $\overline{b}_{i,\Delta}+\Delta$. The second term deals withe the case where a new bid $\overline{b}_{i,\Delta}+\Delta$ will win and an old bid of $\overline{b}_{i,\Delta}$ will not win when the marginal bid by other players is $\overline{b}_{i,\Delta}$. The first order condition says that it is not preferable for a player with a signal $\overline{x}_i$ to increase the bid from $\overline{b}_{i,\Delta}$ to $\overline{b}_{i,\Delta}+\Delta$.

253.The first order conditions of a player with a signal $\overline{x}_i$ for sufficiently small Δ. In (167), we have seen that it has to be that the expected payoff from losing at the lower price has to be nonpositive in the double auction game in the large economy $\mathcal{G}(\gamma,f,\Delta)$. We will show that the same conclusion holds for $\mathcal{G}(f,\Delta,n)$. That is,

$$\int_{\{x_0:W_{-i,n}=\overline{b}_{i,\Delta}\}}(v(x_0,\overline{x}_i)-p_n(\overline{b}_{i,\Delta},b_{-i}))$$
$$(1-q_n(\overline{b}_{i,\Delta},W_{-i,n}(\beta^*_{\Delta,n,-i}(X_{-i}))=\overline{b}_{i,\Delta}|x_0))f_{\theta|X_i}(x_0|x_i)dx_0\leq 0. \tag{268}$$

In other words,

$$E[v(\theta,X_i)-\overline{b}_{i,\Delta}|X_i=x_i,W_{n,-i}(\beta^*_{\Delta,n,-i}(X_{-i}))=\overline{b}_{i,\Delta},\overline{b}_{i,\Delta}\text{loses at the tie}]\leq 0. \tag{269}$$

254. Proof of the claim

To see this, suppose that, contrary to (268),

$$\int_{\{x_0:W_{-i,n}=\overline{b}_{i,\Delta}\}}(v(x_0,\overline{x}_i)-p_n(\overline{b}_{i,\Delta},b_{-i}))$$
$$(1-q_n(\overline{b}_{i,\Delta},W_{-i,n}(\beta^*_{\Delta,n,-i}(X_{-i}))=\overline{b}_{i,\Delta}|x_0))f_{\theta|X_i}(x_0|x_i)dx_0>0. \tag{270}$$

This implies that,

$$\int_{\{x_0:W_{-i,n}=\overline{b}_{i,\Delta}\}}(v(x_0,x_i)-p_n(\overline{b}_{i,\Delta},b_{-i}))(1-q_n(\overline{b}_{i,\Delta},W_{-i,n}(\beta^*_{\Delta,n,-i}(X_{-i}))=\overline{b}_{i,\Delta}|x_0)$$
$$f_{\theta|X_i}(x_0|x_i)dx_0>0. \tag{271}$$

Then, since players use a monotone strategy, the event that a player wins a tie when the marginal bid by other players is $\overline{b}_{i,\Delta}+\Delta$ is a good news compared with the event that a player lose a tie when the marginal bid by other players is $\overline{b}_{i,\Delta}$. Therefore, it follows that

$$\int_{\{x_0:W_{-i,n}=\overline{b}_{i,\Delta}+\Delta\}}(v(x_0,x_i)-p_n(\overline{b}_{i,\Delta},b_{-i}))$$
$$q_n(\overline{b}_{i,\Delta}+\Delta,W_{-i,n}(\beta^*_{\Delta,n,-i}(X_{-i})=\overline{b}_{i,\Delta}+\Delta)f_{\theta|X_i}(x_0|x_i)dx_0>0. \tag{272}$$

Also, from the pricing rule of k double auctions, it follows that

$$p_n(\bar{b}_{i,\Delta} + \Delta, b_{-i}) - p_n(\bar{b}_{i,\Delta}, b_{-i}) \leq \Delta. \tag{273}$$

Therefore, for sufficiently small Δ, it follows that

$$\begin{aligned} &\int_{\{x_0 : W_{-i,n} = \bar{b}_{i,\Delta} + \Delta\}} (v(x_0, x_i) - p_n(\bar{b}_{i,\Delta} + \Delta, b_{-i})) \\ &q_n(\bar{b}_{i,\Delta} + \Delta, W_{-i,n}(\beta^*_{\Delta,n,-i}(X_{-i}) = \bar{b}_{i,\Delta} + \Delta) f_{\theta|X_i}(x_0|x_i) dx_0 > 0. \end{aligned} \tag{274}$$

Now, (270) and (274) imply that, it is strictly preferable for a player with signal x_i to increase a bid from $\bar{b}_{i,\Delta}$ to $\bar{b}_{i,\Delta} + \Delta$. It is a contradiction to the first order condition (267).Therefore, it is not the case that (270) holds.

255.The first order conditions of a player with a signal $\underline{x}_i$ for sufficiently small Δ. We can run a similar argument for a player with signal $\underline{x}_i$ where it is not preferable for a player with signal $\underline{x}_i$ to decrease a bid from $\underline{b}_{\Delta,i}$ to $\underline{b}_{\Delta,i} - \Delta$. This implies that, similar to (268)

$$E[v(\theta, X_i) - \underline{b}_{i,\Delta} | X_i = \underline{x}_i, W(\beta^*_{\Delta,n,-i}(X_{-i}) = \underline{b}_{i,\Delta}, \underline{b}_{i,\Delta} \text{wins at the tie}] \geq 0. \tag{275}$$

That is, it is preferable for a player with signal $\underline{x}_i$ to win the good when the marginal bid by other players is $\underline{b}_{i,\Delta}$. Otherwise, if it is not preferable for a player with signal $\underline{x}_i$ to win the good when the marginal bid by other players is $\underline{b}_{i,\Delta}$, for sufficiently small Δ, it is not preferable to win the good with a slightly lower price of $\underline{b}_{i,\Delta} - \Delta$ even by winning the tie lost by a smaller bid, thus the player will want to decrease the bid to $\underline{b}_{i,\Delta} - \Delta$.

256.Winner's curse. We have seen that, in the double auction game in the large economy $\mathcal{G}(\gamma,f,\Delta)$ that a winner's curse holds in the sense that losing a tie is a good news compared with winning a tie when players use monotone strategies. In the similar argument, we have,

$$\begin{aligned} &E[v(\theta, X_i) - \bar{b}_{i,\Delta} | X_i = \bar{x}_i, W_{n,-i}(\beta^*_{\Delta,n,-i}(X_{-i})) = \bar{b}_{i,\Delta}, \bar{b}_{i,\Delta} \text{loses at the tie}] \\ > \ &E[v(\theta, X_i) - \bar{b}_{i,\Delta} | X_i = \bar{x}_i, W(\beta^*_{\Delta,n,-i}(X_{-i}) = \bar{b}_{i,\Delta}, \bar{b}_{i,\Delta} \text{ wins at the tie}] \end{aligned} \tag{276}$$

Intuitively, when other players use a monotone strategy, conditional on that the marginal bid by other players is $\bar{b}_{i,\Delta}$, a bid $\bar{b}_{i,\Delta}$ is more likely win when the number of competing bids is smaller, and it is the case when the state is lower. Since this inference relation holds whether it is a large economy or a finite economy, the winner's curse relation holds.

257.Moving from $\bar{x}_i$ to $\underline{x}_i$. It follows that, since X_i,X_{-i} and θ are affiliated, $v(x_0$,$x_i)$ is strictly increasing in x_i with a uniform lower bound of the rate of increase $\underline{\lambda}$, and it follows that

$$\begin{aligned} &E[v(\theta, X_i) - \bar{b}_{i,\Delta} | X_i = \bar{x}_i, W(\beta^*_{\Delta,n,-i}(X_{-i}) = \bar{b}_{i,\Delta}, \bar{b}_{i,\Delta} \text{ wins at the tie}] \\ > \ &E[v(\theta, X_i) - \bar{b}_{i,\Delta} | X_i = \underline{x}_i, W(\beta^*_{\Delta,n,-i}(X_{-i}) = \underline{b}_{i,\Delta}, \underline{b}_{i,\Delta} \text{ wins at the tie}] \\ &+\underline{\lambda}(\bar{x}_i - \bar{x}_i). \end{aligned} \tag{277}$$

Therefore, for sufficiently small d, since $\bar{b}_{i,\Delta} - \underline{b}_{i,\Delta} < d$, from (277),

$$\begin{aligned} & E[v(\theta, X_i) - \bar{b}_{i,\Delta}|X_i = \bar{x}_i, W(\beta^*_{\Delta,n,-i}(X_{-i}) = \bar{b}_{i,\Delta}, \bar{b}_{i,\Delta} \text{ wins at the tie}] \\ > \;& E[v(\theta, X_i) - \underline{b}_{i,\Delta}|X_i = \underline{x}, W(\beta^*_{\Delta,n,-i}(X_{-i}) = \underline{b}_{i,\Delta}, \underline{b}_{i,\Delta} \text{ wins at the tie}]. \end{aligned} \tag{278}$$

258.Putting it all together. It now follows from (269), (276), and (278) that,

$$\begin{aligned} 0 \;\geq\;& E[v(\theta, X_i) - \bar{b}_{i,\Delta}|X_i = \bar{x}_i, W_{n,-i}(\beta^*_{\Delta,n,-i}(X_{-i})) = \bar{b}_{i,\Delta}, \bar{b}_{i,\Delta} \text{loses at the tie}] \\ >\;& E[v(\theta, X_i) - \bar{b}_{i,\Delta}|X_i = \bar{x}_i, W(\beta^*_{\Delta,n,-i}(X_{-i}) = \bar{b}_{i,\Delta}, \bar{b}_{i,\Delta} \text{ wins at the tie}] \\ >\;& E[v(\theta, X_i) - \underline{b}_{i,\Delta}|X_i = \underline{x}, W(\beta^*_{\Delta,n,-i}(X_{-i}) = \underline{b}_{i,\Delta}, \underline{b}_{i,\Delta} \text{ wins at the tie}] \\ \geq\;& 0. \end{aligned}$$

It is a contradiction, therefore, it cannot be that $\bar{x}_i$ and $\underline{x}_i$ and a bid b_i such that players with signal $\bar{x}_i$ and $\underline{x}_i$ choose a bid b_i under β^*_Δ. It follows that β^*_Δ is a monotone pure strategy equilibrium.

9.2. *Lemma 5.2*

LEMMA. *A nontrivial monotone pure strategy equilibrium β^*_n in the double auction game $\mathcal{G}(f,n)$ is asymptotically equivalent to a fully revealing rational expectation equilibrium.*

PROOF.
258. Introduction. We proceed in two steps. First, we show that there exists a monotone pure strategy equilibrium β^* of a limit strategy profile β^* of the double auction game in the large economy $\mathcal{G}(f)$. Second, we show that β^* is outcome equivalent to the fully revealing rational expectation equilibrium.

First we construct the limit strategy profile β^*.

259. Definition of a limit strategy profile β^*. From Proposition 1(c), there exists a nontrivial monotone pure strategy equilibrium β^*_n of the double auction game in the large finite economy $\mathcal{G}(f,n)$. Let $\{\beta^*_n\}_n$ be a sequence of pure strategy equilibria of $\mathcal{G}(f,n)$. Since for every bid in β^*_n is bounded above by $\bar{b}$ and bounded below by $\underline{b}$, by Helly's selection theorem, it is without loss of generality to assume that there exists a monotone pure strategy profile β^* such that $\beta^*_n \to \beta^*$.

We now argue that β^* is a monotone pure strategy equilibrium of $\mathcal{G}(f)$.

260. Conditions for β^* to be an equilibrium. We not that β^* is an equilibrium of $\mathcal{G}(f)$ if

$$U(\beta^*_i, \beta^*_{-i}) \geq U(\beta_i, \beta^*_{-i}) \tag{279}$$

for every strategy β_i. Let $\{\beta_{n,i}\}_n$ be a sequence of strategies which will converge to β_i.

Since β_n^* is an equilibrium of $\mathcal{G}(f,n)$, it follows that

$$U_n(\beta_{n,i}^*,\beta_{n.-i}^*) \geq U_n(\beta_{n,i},\beta_{n,-i}^*) \tag{280}$$

for every strategy $\beta_{n,i}$. We first show that

$$U_n(\beta_{n,i}^*,\beta_{n.-i}^*) \to U(\beta_i^*,\beta_{-i}^*) \tag{281}$$

261. Conditions for (281). We note

$$\begin{aligned} & U_n(\beta_{n,i}^*,\beta_{n.-i}^*) - U(\beta_i^*,\beta_{-i}^*) \\ = \; & \left[U_n(\beta_{n,i}^*,\beta_{n.-i}^*) - U_n(\beta_i^*,\beta_{-i}^*)\right] + \left[U_n(\beta_i^*,\beta_{-i}^*) - U(\beta_i^*,\beta_{-i}^*)\right] \end{aligned} \tag{282}$$

That is, $U_n(\beta_{n,i}^*,\beta_{n.-i}^*) - U(\beta_i^*,\beta_{-i}^*)$ is expressed as the sum of the two terms where the first term $U_n(\beta_{n,i}^*,\beta_{n.-i}^*) - U_n(\beta_i^*,\beta_{-i}^*)$ deals with the change in the strategies from $\beta_{n,i}^*$ to β_i^* while keeping the size of the economy n constant, and the second term $U_n(\beta_i^*,\beta_{-i}^*) - U(\beta_i^*,\beta_{-i}^*)$ deals with the change in the size of the economy n while keeping the strategy profile β_i^* constant.

262. Convergence of $U_n(\beta_{n,i}^*,\beta_{n.-i}^*) - U_n(\beta_i^*,\beta_{-i}^*)$. Since the discontinuity in payoffs takes place only when ties occur with positive probabilities, and since β_n^* does not involve ties, it is suffice to show that the limit strategy profile β^* does not involve ties.

For that purpose, we can apply the argument of the previous lemma to β^*. The previous lemma shows that the limit strategy β_n^* obtained as a limit of a sequence of equilibrium strategies $\beta_{n,\Delta}^*$ as $\Delta \to 0$ profile does not involve ties since a player. Similarly, we can show that the limit strategy β^* does not involve ties, since a player with two distinct signals will place distinct bids.

Therefore, we have

$$U_n(\beta_{n,i}^*,\beta_{n.-i}^*) - U_n(\beta_i^*,\beta_{-i}^*) \to 0 \text{ as } n \to \infty. \tag{283}$$

263. Convergence of $U_n(\beta_i^*,\beta_{-i}^*) - U(\beta_i^*,\beta_{-i}^*)$. We have seen from above that β^* does not involve ties. Then, following an argument similar to that of the previous lemma, since the empirical distribution of bids generated by β^* converges in distribution to the distribution of bids generated by β^* and payoffs are continuous at β^*, we have

$$U_n(\beta_i^*,\beta_{-i}^*) - U(\beta_i^*,\beta_{-i}^*) \to 0 \text{ as } n \to \infty. \tag{284}$$

It follows from (283) and (284) that **(281)** holds. We now consider the relation between $U_n(\beta_{n,i},\beta_{n,-i}^*)$ and $U_n(\beta_i,\beta_{-i}^*)$.We consider three cases.

264. Cases for (β_i,β_{-i}^*). We can consider two cases:

(a) (β_i,β_{-i}^*) does not involve a tie with positive probability.

(b) (β_i,β^*_{-i}) involves a tie with a positive probability

We first consider the first possibility.

265. Convergence of $U_n(\beta_{n,i},\beta^*_{n,-i}) \to U(\beta_i,\beta^*_{-i})$ when (β_i,β^*_{-i}) does not involve a tie. Suppose player i is a buyer (a case where player i is a seller is similar.) Suppose (β_i,β^*_{-i}) does not involve a tie. Then, even if $(\beta_{n,i},\beta^*_{n,-i})$ involves a tie, there exists another sequence of strategies $\{\beta^{'}_{n,i}\}_n$ such that $\beta^{'}_{n,i} \to \beta_i$ as $n \to \infty$ and $(\beta^{'}_{n,i},\beta^*_{n,-i})$ does not involve a tie. Then, we can show that

$$U_n(\beta_{n,i},\beta^*_{n,-i}) \to U(\beta_i,\beta^*_{-i}). \tag{285}$$

Therefore, from (280) and (285), we have

$$U(\beta^*_i,\beta^*_{-i}) \geq U(\beta_i,\beta^*_{-i}). \tag{286}$$

We now consider the second case.

266. Case (β_i,β^*_{-i}) involves a tie with a positive probability. Suppose a player i is a buyer. Let x_i be a signal such that player i with signal x_i will bid $\beta_i(x_i)$ which will involve a tie with $\beta_{-i}(x_i)$ with a positive probability.

We first construct an alternative strategy which does not involve a tie with $\beta_{-i}(x_i)$ and still does at least equally well with $\beta_{-i}(x_i)$ as $\beta_i(x_i)$. Suppose that player i with signal x_i has a nonnegative expected payoff from winning a tie at a bid $\beta_i(x_i)$. Then, winner's curse argument shows that it is preferable to increase a bid a little bit from $\beta_i(x_i)$ to win the tie. Therefore, there exists a bid $\beta'_i(x_i)$ such that a player with signal x_i prefers over $\beta_i(x_i)$. Similarly, when player i with signal x_i has a negative payoff from winning a tie at a bid $\beta_i(x_i)$, there exists a bid $\beta'_i(x_i)$, which decrease a bid a little bit from $\beta_i(x_i)$ to lose a nonprofitable tie. Thus we can define a strategy $\tilde{\beta}_i(x_i)$ such that $\tilde{\beta}_i(x_i)$ is equal to $\beta_i(x_i)$ when $\beta_i(x_i)$ does not involve a tie with β^*_{-i} and equal to $\beta'_i(x_i)$ when $\beta_i(x_i)$ involves a tie with β^*_{-i} with positive probability. Then

$$U(\tilde{\beta}_i,\beta_{-i}) \geq U(\beta_i,\beta_{-i}). \tag{287}$$

It follows that there exists a sequence of strategies $\{\tilde{\beta}_{n,i}\}_n$ such that (a) $\tilde{\beta}_{n,i}(x_i) \to \tilde{\beta}_i(x_i)$ for each x_i and (b) $\tilde{\beta}_{n,i}(x_i)$ does not involve $\beta^*_{n,-i}(x_i)$ with positive probability. Then, the equilibrium condition for β^*_n implies that

$$U_n(\beta^*_{n,i},\beta^*_{n,-i}) \geq U(\tilde{\beta}_{n,i},\beta_{n,-i}) \tag{288}$$

It follows from (281) and applying the argument used to establish (281) to $\tilde{\beta}_{n,i}$ that

$$U(\beta^*_i,\beta^*_{-i}) \geq U(\tilde{\beta}_i,\beta^*_{-i}) \tag{289}$$

It now follows from (287) and (288) that

$$U(\beta_i^*, \beta_{-i}^*) \geq U(\beta_i, \beta_{-i}). \tag{290}$$

We have now shown that β^* is an equilibrium strategy. We now argue that β^* is outcome equivalent to the fully revealing rational expectation equilibrium.

267. First order conditions for x_i at $\beta^*(x_i)$. We consider the first order condition for x_i that a player with signal x_i does not prefer to increase a bid to $\beta^*(x_i) + \Delta$ or decrease a bid to $\beta^*(x_i) - \Delta$. The first condition is

$$\begin{aligned} & \int (v(x_0, x_i) - \beta^*(x_i))(1 - q(\beta^*(x_i), W(x_0) = \beta^*(x_i)|x_0)) \\ & f(x_0|x_i)dx_0dx_{-i} + \\ & \int (v(x_0, x_i) - \beta^*(x_i))q(\beta^*(x_i) + \Delta, W(x_0) = \beta^*(x_i) + \Delta|x_0)) \\ & f(x_0|x_i)dx_0dx_{-i} \\ \leq \ & 0. \end{aligned} \tag{291}$$

Here the first term represents the expected payoff of winning the tie that a bid $\beta^*(x_i)$ used to lose, and the second term represents the expected payoff from winning the tie when the market clearing price is $\beta^*(x_i) + \Delta$.

The second condition is

$$\begin{aligned} & \int (v(x_0, x_i) - \beta^*(x_i))(1 - q(\beta^*(x_i) - \Delta, W(x_0) = \beta^*(x_i) - \Delta|x_0)) \\ & f(x_0|x_i)dx_0dx_{-i} + \\ & \int (v(x_0, x_i) - \beta^*(x_i))q(\beta^*(x_i), W(x_0) = \beta^*(x_i)|x_0)) \\ & f(x_0|x_i)dx_0dx_{-i} \\ \geq \ & 0. \end{aligned} \tag{292}$$

Here the first term represents the expected payoff of winning the tie that a bid $\beta^*(x_i) - \Delta$ used to lose, and the second term represents the expected payoff from winning the tie when the market clearing price is $\beta^*(x_i)$.

268. The limit of the first order condition as $\Delta \to 0$. It follows from the above argument that the distribution of bids according to $\beta^*(x_i)$ does not involve a tie with positive probability. It follows that

$$\begin{aligned} & q(\beta^*(x_i) + \Delta, W(x_0) = \beta^*(x_i) + \Delta|x_0)) \\ \to \ & q(\beta^*(x_i), W(x_0) = \beta^*(x_i)|x_0) \text{ as } \Delta \to 0. \end{aligned} \tag{293}$$

It follows from (291) and (292) that

$$\int_{x_0: W(x_0) = \beta^*(x_i)} (v(x_0, x_i) - \beta^*(x_i)) f(x_0|x_i) dx_0 dx_{-i} = 0. \tag{294}$$

That is,

$$\beta^*(x_i) = E[v(x_0, x_i)|x_i, W(x_0) = \beta^*(x_i)] \tag{295}$$

269. Outcome equivalence to the fully revealing rational expectation equilibrium. It now follows from an argument used in the previous lemma that

$$\beta^*(x_i) = v(x_0(x_i), x_i) \tag{296}$$

and the outcome is equivalent to the fully revealing rational expectation equilibrium.

9.3. *Lemma 5.3*

LEMMA. *Let $P_n(\beta_n^*)$ be a price of a Bayesian Nash equilibrium β_n^** of *the double auction game in the large finite economy $\mathcal{G}(f,n)$. Then,*

$$\begin{aligned}&\sqrt{n}(P_n(\beta_n^*) - v(x_0, x_i(x_0)) \\ &\xrightarrow{p} N(0, \frac{\alpha(1-\alpha)}{f^2_{X_i|\theta}(x_i(x_0))} \\ &(\frac{\partial v(x_0, x_i(x_0))}{\partial x_0}\frac{\partial x_0(x_i(x_0))}{\partial x_i} + \frac{\partial v(x_0, x_i(x_0))}{\partial x_i})^2)\end{aligned} \tag{297}$$

PROOF.

270. Overview of the proof. We begin by noting that

$$\begin{aligned}&\sqrt{n}(P_n(\beta_n^*) - v(x_0, x_i(x_0)) \\ = \ &\sqrt{n}(P_n(x_0) - v(x_0, x_i(x_0))) \\ &+\sqrt{n}(P_n(\beta_n^*) - P_n(x_0))\end{aligned} \tag{298}$$

where $P_n(x_0)$ is the price formed from the bids when every player $i \in N_n$ bids $v(x_0(x_i),x_i)$. The first term deals with the sample size effect. The second term deals with the strategic effect which considers misrepresentation from the price taking behavior. The proof proceeds in two steps. First we show that $\sqrt{n}(P_n(x_0) - v(x_0,x_i(x_0))) \xrightarrow{d} N(0,\frac{\alpha(1-\alpha)}{f^2_{X_i|\theta}(x_i(x_0))}(\frac{\partial v(x_0,x_i(x_0))}{\partial x_0}\frac{\partial x_0(x_i(x_0))}{\partial x_i}+\frac{\partial v(x_0,x_i(x_0))}{\partial x_i})^2)$ and second we show that $\sqrt{n}(P_n(\beta_n^*)-P_n(x_0)) \xrightarrow{p} 0$.

We show them in case of $k = 1$ in the double auction pricing rule. The case for $k = 0$ is similar. Then we can extend the result for a general $k \in (0,1)$ by sandwiching arguments.

271. Evaluation of the sample size effect. By definition

$$P_n(x_0) = v(x_0(X_{n:n_S}), X_{n:n_S}) \tag{299}$$

That is, $P_n(x_0)$ is the expected value of the good conditional being on the margin by a player with signal $X_{n:n_S}$.

We recall, from David and Nagaraja (2004, Theorem 10.3) that, for each x_0,

$$\sqrt{n}(X_{n_S;n} - x_i(x_0)) \to_d N\left(0, \frac{\alpha(1-\alpha)}{f^2_{X_i|\theta}(x_i(x_0))}\right) \tag{300}$$

From Assumption 4, $x_0(X_{n:n_S})$ is well-defined and $x_0'(X_{n:n_S}) \geq 0$.

Thus, we are now able to apply the delta method. According to van der Varrt (2000), Theorem 3.1, when ϕ is differentiable at x, and $\sqrt{n}(T_n - \theta) \xrightarrow{d} N(0,\sigma^2)$, then $\sqrt{n}(\phi(T_n) - \phi(\theta)) \xrightarrow{d} N(0,\phi(\theta)^2\sigma^2)$. In the statement of the delta method, we set

$$\phi(X_{n:n_S}) = v(x_0(X_{n:n_S}), X_{n:n_S}). \tag{301}$$

Then,

$$\begin{aligned} \phi'(X_{n:n_S}) &= \frac{\partial v(x_0(X_{n:n_S}), X_{n:n_S})}{\partial x_0} \frac{\partial x_0(X_{n:n_S})}{\partial X_{n:n_S}} \\ &+ \frac{\partial v(x_0(X_{n:n_S}), X_{n:n_S})}{\partial X_{n:n_S}} \end{aligned} \tag{302}$$

It follows that

$$\begin{aligned} \phi'(x_i(x_0)) &= \frac{\partial v(x_0, x_i(x_0))}{\partial x_0} \frac{\partial x_0(x_i(x_0))}{\partial x_i} \\ &+ \frac{\partial v(x_0, x_i(x_0))}{\partial x_i} \end{aligned} \tag{303}$$

Thus,

$$\begin{aligned} &\sqrt{n}(v(x_0(X_{n:n_S}), X_{n:n_S}) - v(x_0, x_i(x_0)) \xrightarrow{d} \\ &N\left(0, \frac{\alpha(1-\alpha)}{f^2_{X_i|\theta}(x_i(x_0))} (\frac{\partial v(x_0, x_i(x_0))}{\partial x_0} \frac{\partial x_0(x_i(x_0))}{\partial x_i} + \frac{\partial v(x_0, x_i(x_0))}{\partial x_i})^2\right). \end{aligned} \tag{304}$$

We now show the second assertion. We proceed as follows. First we consider the sufficient conditions for the distribution of bids $H_n^*(b_i)$ under $P_n(\beta_n^*)$ such that $\sqrt{n}(P_n(\beta_n^*) - P_n(x_0)) \xrightarrow{p} 0$. Then we show that indeed $H_n^*(b_i)$ satisfies the condition.

272. Conditions for $\sqrt{n}(P_n(\beta_n^*) - P_n(x_0)) \xrightarrow{p} 0$.

We note that $\sqrt{n}(P_n(x_0) - P_n(x_0)) \to_p 0$ is equivalent to, for each a,

$$\begin{aligned} &\Pr(\sqrt{n}(P_n(\beta_n^*) - v(x_0, x_i(x_0)) \leq a) \\ &- \Pr(\sqrt{n}(P_n(x_0) - v(x_0, x_i(x_0)) \leq a) \to 0 \text{ as } n \to \infty. \end{aligned} \tag{305}$$

It is equivalent to

$$\begin{aligned} &\eta_n(a) \\ \equiv\ &\Pr(P_n(\beta_n^*) \leq v(x_0, x_i(x_0)) + \frac{a}{\sqrt{n}}) - \\ &\Pr(P_n(x_0) \leq v(x_0, x_i(x_0)) + \frac{a}{\sqrt{n}}) \\ \to\ &0 \text{ as } n \to \infty. \end{aligned} \tag{306}$$

We recall that, from David and Nagaraja (2004, equation 2.1.6)

$$\begin{aligned}
&\Pr(X_{i:n} \leq x) \\
&= \sum_{j=1}^{n} \binom{n}{j} F(x)^j (1-F(x))^{n-j} \\
&= \frac{1}{\text{Beta}(i, n-i+1)} \int_0^{F(x)} t^{i-1}(1-t)^{n-i} dt.
\end{aligned} \tag{307}$$

In order to derive the condition, assume, first, hypothetically, that each player uses an equilibrium strategy of a buyer $\beta^*_{n,b}(x_i)$. Under this assumption, each player's bid is iid from the distribution $H_{n,b}(b_i)$. It follows from (306) and (307) that

$$\eta_n(a) = \frac{1}{\text{Beta}(n-n_S, n_S+1)} \int_{H_n(v(x_0,x_i(x_0))+\frac{a}{\sqrt{n}})}^{H_{n,b}(v(x_0,x_i(x_0))+\frac{a}{\sqrt{n}})} t^{n-n_S-1}(1-t)^{n_S} dt \tag{308}$$

where $H^*_n(b_i)$ is the distribution function of bids when every player bids according to the strategy $v(x_0(x_i),x_i)$.

As a next step, we now evaluate $\text{Beta}(n-n_S,n_S+1)$. We let

$$\alpha = \frac{n-n_S}{n} \tag{309}$$

Then, from (308) and (309) that

$$\text{Beta}(n-n_S, n_S+1) = \text{Beta}(\alpha n, n-\alpha n+1) \tag{310}$$

We recall a standard result of

$$\text{Beta}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{311}$$

where $\Gamma(x)$ is a Gamma function. It follows from (311) that

$$\frac{1}{\text{Beta}(\alpha n, n-\alpha n+1)} = \frac{\Gamma(n+1)}{\Gamma(\alpha n)\Gamma(n-\alpha n+1)} \tag{312}$$

We now recall Stirling's formula for Gamma function:

$$\Gamma(x) = \sqrt{\frac{2\pi}{x}}\left(\frac{x}{e}\right)^x\left(1+O\left(\frac{1}{x}\right)\right) \approx \sqrt{\frac{2\pi}{x}}\left(\frac{x}{e}\right)^x. \tag{313}$$

By substituting (313) into (312), we get

$$\begin{aligned}
&\frac{1}{\text{Beta}(\alpha n, n-\alpha n+1)} \\
&\approx \frac{\sqrt{\frac{2\pi}{n+1}}\left(\frac{n+1}{e}\right)^{n+1}}{\sqrt{\frac{2\pi}{\alpha n}}\left(\frac{\alpha n}{e}\right)^{\alpha n}\sqrt{\frac{2\pi}{n-\alpha n+1}}\left(\frac{n-\alpha n+1}{e}\right)^{n-\alpha n+1}} \\
&= \frac{(n+1)^{n+1/2}}{\sqrt{2\pi}(\alpha n)^{\alpha n-1/2}(n-\alpha n+1)^{n-\alpha n+1/2}} \\
&= \frac{(n+1)^n(n+1)^{1/2}}{\sqrt{2\pi}n^n\alpha^{\alpha n-1/2}(1-\alpha+\frac{1}{n})^{n-\alpha n+1/2}}
\end{aligned} \tag{314}$$

We now evaluate

$$\int_{H_n(v(x_0,x_i(x_0))+\frac{a}{\sqrt{n}})}^{H_n^*(v(x_0,x_i(x_0))+\frac{a}{\sqrt{n}})} t^{n-n_S-1}(1-t)^{n_S}dt.$$

In order to get the maximum of the integrand, we now maximize

$$g(t) = t^{\alpha n-1}(1-t)^{n-\alpha n} \tag{315}$$

Then, by taking the logarithm,

$$\log g(t) = (\alpha n - 1)\log t + (n - \alpha n)\log(1-t) \tag{316}$$

The first order condition is

$$\frac{t}{\alpha n - 1} = \frac{1-t}{n - \alpha n} \tag{317}$$

Rewriting,

$$(n-\alpha n)t = (\alpha n - 1)(1-t) \tag{318}$$

Then,

$$t(n+1) = \alpha n - 1 \tag{319}$$

It follows from (319) that

$$t = \frac{\alpha n - 1}{n+1} \approx \alpha - \frac{1}{n} \tag{320}$$

and

$$1 - t \approx 1 - \alpha + \frac{1}{n} \tag{321}$$

Then, by substituting into (315),

$$g(t) \leq (\alpha - \frac{1}{n})^{\alpha n-1}(1 - \alpha + \frac{1}{n})^{n-\alpha n} \tag{322}$$

Thus we now have, by combining (314) and (322),

$$\begin{aligned}
\eta_n(a) &\leq \frac{(n+1)^n(n+1)^{1/2}}{\sqrt{2\pi}n^n\alpha^{\alpha n-1/2}(1-\alpha+\frac{1}{n})^{n-\alpha n+1/2}} && (323)\\
&\cdot(\alpha-\frac{1}{n})^{\alpha n-1}(1-\alpha+\frac{1}{n})^{n-\alpha n}\\
&\cdot[H_{n,b}(v(x_0,x_i(x_0))+\frac{a}{\sqrt{n}})-H_n(v(x_0,x_i(x_0))+\frac{a}{\sqrt{n}})]\\
&\approx \frac{1}{\sqrt{2\pi}}(1+\frac{1}{n})^{|N_n|}(n+1)^{1/2}\\
&\cdot\frac{(\alpha-\frac{1}{n})^{\alpha n-1}(1-\alpha+\frac{1}{n})^{n-\alpha n}}{\alpha^{\alpha n-1/2}(1-\alpha+\frac{1}{n})^{n-\alpha n+1/2}} && (324)\\
&\cdot[H_{n,b}(v(x_0,x_i(x_0))+\frac{a}{\sqrt{n}})-H_n(v(x_0,x_i(x_0))+\frac{a}{\sqrt{n}})]\\
&\approx \frac{1}{\sqrt{2\pi}}(1+\frac{1}{n})^{n}(n+1)^{1/2}\frac{1}{\sqrt{\alpha(1-\alpha)}}\\
&[H_{n,b}(v(x_0,x_i(x_0))+\frac{a}{\sqrt{n}})-H_n(v(x_0,x_i(x_0))+\frac{a}{\sqrt{n}})]\\
&\simeq \frac{1}{\sqrt{2\pi}}e\frac{1}{\sqrt{\alpha(1-\alpha)}} && (325)\\
&(n+1)^{1/2}[H_{n,b}(v(x_0,x_i(x_0))+\frac{a}{\sqrt{n}})-H_n(v(x_0,x_i(x_0))+\frac{a}{\sqrt{n}})]. && (326)
\end{aligned}$$

It follows that

$$\sqrt{n}\left(H^*_{n,b}(v(x_0,x_i(x_0))+\frac{a}{\sqrt{n}})-H_n(v(x_0,x_i(x_0))+\frac{a}{\sqrt{n}})\right)\to 0. \quad (327)$$

is suffice for

$$\sqrt{n}(P_n(\beta^*_{n,b})-P_n(x_0))\xrightarrow{p} 0$$

Similarly, we have

$$\sqrt{n}\left(H^*_{n,s}(v(x_0,x_i(x_0))+\frac{a}{\sqrt{n}})-H_n(v(x_0,x_i(x_0))+\frac{a}{\sqrt{n}})\right)\to 0 \quad (328)$$

is suffice for

$$\sqrt{n}(P_n(\beta^*_{n,s})-P_n(x_0))\xrightarrow{p} 0$$

It follows that, when (327) and (328) are satisfied,

$$\sqrt{n}(\max(H_{n,b}(v(x_0,x_i(x_0))+\frac{a}{\sqrt{n}}),H_{n,s}(v(x_0,x_i(x_0))+\frac{a}{\sqrt{n}})-P_n(x_0))\to 0 \quad (329)$$

and

$$\sqrt{n}(\min(H_{n,b}(v(x_0,x_i(x_0))+\frac{a}{\sqrt{n}}),H_{n,s}(v(x_0,x_i(x_0))+\frac{a}{\sqrt{n}})-P_n(x_0))\to 0 \quad (330)$$

Then, since H_n is sandwiched between $\max(H_{n,b}^{BNE}, H_{n,s}^{BNE})$ and $\min(H_{n,b}^{BNE}, H_{n,s}^{BNE})$, it follows that

$$\sqrt{n}(P_n(\beta_n^*) - P_n(x_0)) \to 0. \tag{331}$$

It follows that (327) and (328) are sufficient conditions.

We have now derived that both buyers and sellers bids converge at the rate faster than $O(\frac{1}{\sqrt{n}})$ are sufficient. We now derive the rate of convergence of buyers and sellers bids.

273. Rate of convergence. We first note that it is suffice to show that, ,for each signal x_i, the distance between $v(x_0(x_i),x_i)$ and the bids $\beta_{n,s}^*$ and $\beta_{n,b}^*$ vanishes at the rate faster than $O(\frac{1}{\sqrt{n}})$. Because, if it is so, the distribution of bids at b_i according to each of $\beta_{n,s}^*$ and $\beta_{n,b}^*$,which is bounded above by $H_n(b_i + O(\frac{1}{\sqrt{n}}))$, will converge to $H_n(b_i)$ at the rate of $O(\frac{1}{\sqrt{n}})$.

To show the rate of convergence of buyers' and sellers' bids, we first derive the first order condition for buyers and sellers.

274. First order conditions. We now derive the first order condition of a buyer following Rustichini, Satterthwaite, and Williams (1994). Suppose a buyer i increases a bid from b_i to $b_i + \Delta b_i$[6]. This change in bids can change the outcome of the auction for buyer i in three cases:

- If a bid b_i is between n_S-1st highest bid and n_Sth highest bid out of $n_B - 2$ buyers and n_S sellers, and if there is a buy bid between b_i and $b_i + \Delta b_i$, then a bid of b_i will not win, but increasing the bid to $b_i + \Delta$ will win the good by surpassing the bid by the buyer.
- If a bid b_i is between n_S-1st highest bid and n_Sth highest bid out of $n_B - 1$ buyers and $n_S - 1$ sellers, and if there is a sell bid between b_i and $b_i + \Delta b_i$, then a bid of b_i will not win, but increasing the bid to $b_i + \Delta$ will win the good by surpassing the bid by the seller.
- If a bid b_i is between the $n_S - 1$ st highest bid and the n_Sth highest bid out of $n_B - 1$ buyers and n_S sellers, then a bid of b_i will win and increasing the bid to $b_i + \Delta$ will increase the payment by $k\Delta$.

We now define the following notations.

Let

$$\begin{aligned} A &= \text{the event that } b_i \text{ is between } |N_{n,S}|\text{-1 st highest bid and } |N_{n,S}|\text{th highest bid out of} \\ &\quad |N_{n,B}|\text{-2 buyers and} n_S \text{sellers, and there is a buy bid between } b_i \text{ and } b_i + \Delta b_i, \\ B &= \text{the event that } b_i \text{ is between } |N_{n,S}|\text{-1 st highest bid and } |N_{n,S}|\text{th highest bid out of} \\ &\quad |N_{n,B}|\text{-1 buyers and} n_S\text{-1 sellers, and there is a sell bid between } b_i \text{ and } b_i + \Delta b_i, \end{aligned}$$

[6]Here Δ represents a miniscule amount of the increase in bid in the model with a continuous set of bids.

and

$$C = b_i \text{is between the} n_S - 1\text{st highest bid and the} n_S\text{th highest bid out of } n_B - 1 \text{buyers and } n_S \text{sellers}$$

Let us consider the first case. The payoff change from this case is

$$\left(E_n[v(x_0, x_i)|x_i, A] - b_i\right) \cdot \Pr_n (A|x_i) \tag{332}$$

The payoff change from the second case is

$$\left(E_n[v(x_0, x_i)|x_i, B] - b_i\right) \cdot \Pr_n (B|x_i) \tag{333}$$

The payoff change from the third case is

$$k\Delta \cdot \Pr_n (C|x_i) \tag{334}$$

Therefore, the first order condition is, from (332), (333), and (334)

$$(E_n[v(x_0, x_i)|x_i, A] - b_i) \cdot \Pr_n (A|x_i) + (E_n[v(x_0, x_i)|x_i, B] - b_i) \cdot \Pr_n (B|x_i) \\ -k\Delta \cdot \Pr_n (C|x_i) = 0 \tag{335}$$

A preliminary result. As a step for deriving the rate of convergence of bidding strategies, we now evaluate the first order condition (335).

It follows from (335) that

$$\left(E_n[v(x_0, x_i)|x_i, A] - b_i\right) \cdot \Pr_n (A|x_i) - k\Delta \cdot \Pr_n (C|x_i) \leq 0. \tag{336}$$

Therefore, it follows from (336) that

$$((E_n[v(x_0, x_i)|x_i, A) - b_{i,n}) \leq k\Delta \frac{\Pr_n (C|x_i)}{\Pr_n (A|x_i)} \tag{337}$$

We now evaluate the right hand side of (337). We note that,

$$\frac{\Pr_n (C|x_i)}{\Pr_n (A|x_i)} \tag{338}$$

$$= \frac{\int \Pr_n (C|x_0)\, f_{\theta|X_i}(x_0|x_i) dx_0}{\int \Pr_n (A|x_0)\, f_{\theta|X_i}(x_0|x_i) dx_0}$$

by conditioning and by the property that x_i is redundant after conditioning on x_0

$$= \frac{\int \Pr_n (C|x_0)\, f_{\theta|X_i}(x_0|x_i) dx_0}{\int \Pr_n (A|x_0) \Pr_n(\text{there is a buy bid between } b_i \text{ and } b_i + \Delta b_i) f_{\theta|X_i}(x_0|x_i) dx_0} \tag{339}$$

by conditional independence. (340)

In order to evaluate (338), we recall, from Lemma 5 in Milgrom (1979), that if there is $C > 0$ such that $\frac{a_i}{b_i} \leq C$ for every i, then

$$\frac{\sum_i a_i}{\sum_i b_i} \leq C. \tag{341}$$

Then it follows that, if we have, or each x_0,

$$\frac{\Pr_n (C|x_0)}{\Pr_n (A|x_0)} \to O(1) \tag{342}$$

and

$$\Pr_n(\text{there is a buy bid between } b_i \text{ and } b_i + \Delta b_i) \to O(n) \tag{343}$$

Then we have

$$\frac{\Pr_n (C|x_i)}{\Pr_n (A|x_i)} = O(\frac{1}{n}) \tag{344}$$

Thus now we need to show (342) and (343). We first show (342).

275. Show (342). We note, given x_0, each player's signal is independently and identically distributed with the distribution $F_{X_i|\theta}(x_i|x_0)$. Given this independence structure, we can

- $\Pr_n (C|x_0)$ corresponds to the probability $M_{n_S:n_B}$ in Rustichini, Satterthwaite, and Williams (1994), page 1060 where the distribution of types in Rustichini, Satterthwaite, and Williams (1994) is now $F_{X_i|\theta}(x_i|x_0)$.
- $\Pr_n (A|x_0)$ corresponds to the probability $L_{n_S:n_B}$ in Rustichini, Satterthwaite, and Williams (1994), page 1060.

It follows from Rustichini, Satterthwaite, and Williams (1994), equation (3.12), that

$$\begin{aligned} & \frac{\Pr_n (C|x_0)}{\Pr_n (A|x_0)} \\ \leq\ & 2F(x_i'|x_0)(1 + \frac{n_B}{n_S}) \\ \leq\ & O(1) \end{aligned} \tag{345}$$

Intuition is as follows: Both numerator and denominator calculate the probability that b_i will fall in the interval between adjacent order statistics of bids. In the numerator, there are $n_B + n_S - 1$ bids and in the denominator there are $n_B + n_S - 2$. This implies, on one hand, there are more bids in the event in the numerator to fall in the interval. But on the other hand, the length of the interval of the adjacent order statistics will decrease at the rate of $O(1/n)$. As a result of these two balances, the ratio does not grow and decline even if n changes.

276. Show (343). We note, from Rustichini, Satterthwaite, and Williams (1994), (3.1) that

$$\begin{aligned} & \Pr_n(\text{there is a buy bid between } b_i \text{ and } b_i + \Delta b_i \text{ out of } n_B - 1 \text{ buyers}) \quad (346) \\ = \; & (n_B - 1) h^*_{n,b}(b_i) \\ = \; & O(n) \end{aligned}$$

An intuition is as follows. We are now considering a possibility that there will be a bid hitting a fixed length of an interval. This probability increases at the order $O(n)$ since any one of $n_B - 1$ buyers can hit the interval.

It now follows from (337) and (344) that

$$E_n[v(x_0, x_i)|x_i, A\ ] - b_{i,n} \to 0 \text{ at the rate } O(\frac{1}{n}) \quad (347)$$

We can now derive the rate of convergence of bidding strategies building on the preliminary result

277. Characterization of the rate of convergence of bidding strategies. In the previous calculation, we obtained a bound on

$$E_n[v(x_0, x_i)|x_i, A] \ - b_{i,n} \to 0$$

by omitting terms for $E_n[v(x_0, x_i)|x_i, B] \ - b_i$

We can now, by omitting the first term instead, following a similar procedure used to derive (347), that

$$E_n[v(x_0, x_i)|x_i, B] \ - b_{i,n} \to 0 \text{ at the rate } O(\frac{1}{n}) \quad (348)$$

It now follows from (347) and (348) that

$$E_n[v(x_0, x_i)|x_i, A] \ \to \ E_n[v(x_0, x_i)|x_i, B] \ \text{ at the rate } O(\frac{1}{n}) \quad (349)$$

We note that these two conditioning events are only different by switching one buyer and one seller. Since (349) holds for every x_i, it follows that the distributions of bids by a buyer and a seller converge at the rate $O(\frac{1}{n})$. Since the distribution of signals is symmetric between a buyer and a seller, it has to be that the bidding strategies converge. That is,

$$\beta^*_{n,b}(x_i) \to \beta^*_{n,s}(x_i) \text{ at rate } O(\frac{1}{n}) \quad (350)$$

This in turn implies that, since the buyer's and seller's strategies are getting symmetric,

$$\begin{aligned} b_i \ & \to \ E_n[v(x_0, x_i)|x_i, A] \quad (351) \\ & \to \ E_n[v(x_0, x_i)|x_i, B] \end{aligned}$$

at the rate $O(\frac{1}{n})$. Since this holds for every n, it implies that asymptotically the equilibrium bidding strategies are symmetric across n. Thus, for each x_0, by Glivenko-Cantelli theorem, $E_n[v(x_0,x_i)|x_i,A]$ converges to $v(x_0(x_i),x_i)$ at the rate $O(\frac{1}{n})$ as $n \to \infty$.

Thus (327) and (328) follow.