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Choice of Collateral Currency *

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Abstract

Collateral has been used for a long time in the cash market and we have also experienced significant increase of its use as an important credit risk mitigation tool in the derivatives market for this decade. Despite its long history in the financial market, its importance for funding has been recognized relatively recently following the explosion of basis spreads in the crisis. This paper has demonstrated the impact of collateralization on derivatives pricing through its funding effects based on the actual data of swap markets. It has also shown the importance of the "choice" of collateral currency. In particular, when a contract allows multiple currencies as eligible collateral as well as its free replacement, the paper has found that the embedded "cheapest-to-deliver" option can be quite valuable and significantly change the fair value of a trade. The implications of these findings for risk management have been also discussed.

Keywords : swap, collateral, derivatives, Libor, currency, OIS, EONIA, Fed-Fund, CCS, basis, risk management, HJM, FX option, CSA, CVA, term structure

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1 Introduction

Collateralization in the OTC (over-the-counter) market has continued to grow at a rapid pace over the past decade. According to ISDA (International Swaps and Derivatives Association), about 70% of the trade volumes for all the OTC trades were collateralized at the end of 2009, which was merely 30% in 2003 [4]. A stringent collateral management will also be a crucial issue for successful installation of central clearing houses.

The role of collateralization is mainly twofold: 1) reduction of the counterparty credit risk, and 2) change of funding costs of trades. The first one has been well recognized and studied extensively. Although it is not as obvious as the first one, the second effect is also important. Recently, the latter effect has gained strong attention among practitioners, since they have experienced significant difference between Libors and funding costs of collateralized trades. The work of Johannes & Sundaresan (2007) [5] was the first focusing on the cost of collateralization, which studied the effect on swap prices based on empirical analysis. As a more recent work, Piterbarg (2010) [6] discussed the general option pricing using the similar formula to take the funding cost of collateral into account.

The impacts of collateralization are the most significant in interest rate and long dated FX markets, where they affect various types of basis spread and also FX forward. In previous two works Fujii, Shimada & Takahasi (2009)[1, 2], we have extended the formula used in [5, 6] to the situation where the payment and collateral currencies are different, which is crucial to handle multi-currency products. Based on the result, we have presented systematic procedures of curve construction in the presence of collateral and multiple currencies, and also their no-arbitrage dynamics in an HJM (Heath-Jarrow-Morton) framework.

In this article, we have constructed the collateralized swap curves consistently with the actual market data, and demonstrated the importance of collateralization in pricing of derivatives [1]. It is well-known among market participants that the existence of large basis spreads in cross currency swap (CCS) market is reflecting differences in funding costs among various currencies. Hence, it is a natural question to ask what is the impact on derivative pricing from different choice of collateral currency. In fact, by making use of the information in CCS markets, we have found that the choice of the collateral currency has non-negligible impact on derivative prices. This finding gives rise to another interesting twist. When the relevant CSA (credit support annex, which specifies all the details of collateral agreement) allows multiple choices of collateral currency and free replacement among them, a payer of the collateral has the "cheapest-to-deliver" (CTD) option. We have demonstrated the embedded option can significantly change the effective discounting factor and hence the fair value of the trade, especially when the CCS market is volatile.

2 Pricing under the collateralization

This section reviews [1], our results on pricing derivatives under the collateralization. Let us make the following simplifying assumptions about the collateral contract.

1. *Full collateralization (zero threshold) by cash.*
2. *The collateral is adjusted continuously with zero minimum transfer amount.*

[1] All the market data used in this article were taken from the Bloomberg.

Actually, daily margin call is now quite popular in the market, which makes the above assumptions a reasonable proxy for the reality. Since the assumptions allow us to neglect the loss given default of the counterparty, we can treat each trade/payment separately without worrying about the non-linearity arising from the netting effects and the asymmetric handling of exposure.

We consider a derivative whose payoff at time T is given by $h^{(i)}(T)$ in terms of currency "i". We suppose that currency "j" is used as the collateral for the contract. Note that instantaneous return (or cost when it is negative) by holding the cash collateral at time t is given by

$$y^{(j)}(t) = r^{(j)}(t) - c^{(j)}(t) \ , \tag{2.1}$$

where $r^{(j)}$ and $c^{(j)}$ denote the risk-free interest rate and the collateral rate of the currency j, respectively. A common practice in the market is to set $c^{(j)}$ as the overnight (ON) rate of currency j. Distinction between the theoretical risk-free rate and the market ON rate is required for unified treatment of different collaterals and also for calibration to a cross currency basis, which will become clearer in later discussions. If we denote the present value of the derivative at time t by $h^{(i)}(t)$ (in terms of currency i), collateral amount posted from the counterparty is given by $\left(h^{(i)}(t)/f_x^{(i,j)}(t)\right)$, where $f_x^{(i,j)}(t)$ is the foreign exchange rate at time t representing the price of the unit amount of currency j in terms of currency i. These considerations lead to the following calculation for the collateralized derivative price,

$$h^{(i)}(t) = E_t^{Q^i}\left[e^{-\int_t^T r^{(i)}(s)ds} h^{(i)}(T)\right] + f_x^{(i,j)}(t) E_t^{Q^j}\left[\int_t^T e^{-\int_t^s r^{(j)}(u)du} y^{(j)}(s) \left(\frac{h^{(i)}(s)}{f_x^{(i,j)}(s)}\right) ds\right] \ ,$$

where $E_t^{Q^i}[\cdot]$ is the time t conditional expectation under the risk-neutral measure of currency i, where the money-market account of currency i is used as the numeraire. By aligning the measure in the above formula, it is easy to see that

$$X(t) := e^{-\int_0^t r^{(i)}(s)ds} h^{(i)}(t) + \int_0^t e^{-\int_0^s r^{(i)}(u)du} y^{(j)}(s) h^{(i)}(s) ds \tag{2.2}$$

is a Q^i-martingale under appropriate integrability conditions. This tells us that the process of the option price can be written as

$$dh^{(i)}(t) = \left(r^{(i)}(t) - y^{(j)}(t)\right) h^{(i)}(t)dt + dM(t) \tag{2.3}$$

with some Q^i-martingale M.

As a result, we have the following theorem:

Theorem 1 [2] *Suppose that $h^{(i)}(T)$ is a derivative's payoff at time T in terms of currency "i" and that currency "j" is used as the collateral for the contract. Then, the value of the derivative at time t, $h^{(i)}(t)$ is given by*

$$\begin{aligned} h^{(i)}(t) &= E_t^{Q^i}\left[e^{-\int_t^T r^{(i)}(s)ds}\left(e^{\int_t^T y^{(j)}(s)ds}\right) h^{(i)}(T)\right] & (2.4) \\ &= D^{(i)}(t,T) E_t^{T^i}\left[e^{-\int_t^T y^{(i,j)}(s)ds} h^{(i)}(T)\right] \ , & (2.5) \end{aligned}$$

[2] Although we are dealing with continuous processes here, we obtain the same result as long as there is no simultaneous jump of underlying assets when the counterparty defaults.

where
$$y^{(i,j)}(s) = y^{(i)}(s) - y^{(j)}(s) \tag{2.6}$$
with $y^{(i)}(s) = r^{(i)}(s) - c^{(i)}(s)$ *and* $y^{(j)}(s) = r^{(j)}(s) - c^{(j)}(s)$*. Here, we have defined the collateralized zero-coupon bond of currency* i *as*
$$D^{(i)}(t,T) = E_t^{Q^i}\left[e^{-\int_t^T c^{(i)}(s)ds}\right] \ . \tag{2.7}$$
We have also defined the "collateralized forward measure" T^i *of currency* i*, for which* $E_t^{T^i}[\cdot]$ *denotes the time* t *conditional expectation where* $D^{(i)}(t,T)$ *is used as its numeraire* [3].

As a corollary of the theorem, we have
$$h(t) = E_t^Q\left[e^{-\int_t^T c(s)ds}h(T)\right] = D(t,T)E_t^T[h(T)] \tag{2.8}$$
when the payment and collateral currencies are the same. This is consistent with the result of Piterbarg (2010) [6]. In addition, by setting $h(T)=1$, it is easily seen by (2.5) that $E_t^{T^i}\left[e^{-\int_t^T y^{(i,j)}(s)ds}\right]$ is the ratio of two discount bonds, i.e. a relative value of the discount bond collateralized in a different currency j in terms of the one collateralized in its payment currency i.

3 Curve Construction in Single Currency

In this section, we will construct the relevant yield curves in a single currency market. For the details of the procedures, see [1, 3]. Here, we briefly summarize the set of formulas needed to strip the relevant discounting factors and forward Libors;
$$\begin{aligned}
&\mathrm{OIS}_N \sum_{n=1}^{N} \Delta_n D(0,T_n) = D(0,T_0) - D(0,T_N) \ , \\
&\mathrm{IRS}_M \sum_{m=1}^{M} \Delta_m D(0,T_m) = \sum_{m=1}^{M} \delta_m D(0,T_m) E^{T_m}[L(T_{m-1},T_m;\tau)] \ , \\
&\sum_{n=1}^{N} \delta_n D(0,T_n)\left(E^{T_n}[L(T_{n-1},T_n;\tau_S)] + TS_N\right) = \sum_{m=1}^{M} \delta_m D(0,T_m) E^{T_m}[L(T_{m-1},T_m;\tau_L)] \ .
\end{aligned}$$
These are the consistency conditions to give the market quotes of various swaps [4]. We have denoted the market observed OIS (Overnight Index Swap) rate, IRS (Interest Rate Swap) rate and TS (Tenor Swap) spread respectively as OIS_N, IRS_M and TS_N, where the subscripts represent the lengths of swaps. $\{T_n\}_{n\geq 0}$ are the reset/payment times of each instrument. We distinguish day-count fraction of fixed and floating legs by Δ and δ, which are not necessarily the same among different instruments. $L(T_{m-1},T_m;\tau)$ is the Libor with tenor τ whose reset and payment times are T_{m-1} and T_m, respectively. In the third formula, we have distinguished the two different tenors by τ_S and τ_L ($>\tau_S$). If $\tau_S = 3\text{m}$ and $\tau_L = 6\text{m}$, for example, then $N = 2M$ to match the length of two legs.

[3] Notice the difference from the usual forward measure where the numeraire is not collateralized.

[4] If payments are compounded in TS, the formula becomes slightly more complicated. However, the effect from compounding is negligibly small and does not cause any meaningful change to the result.

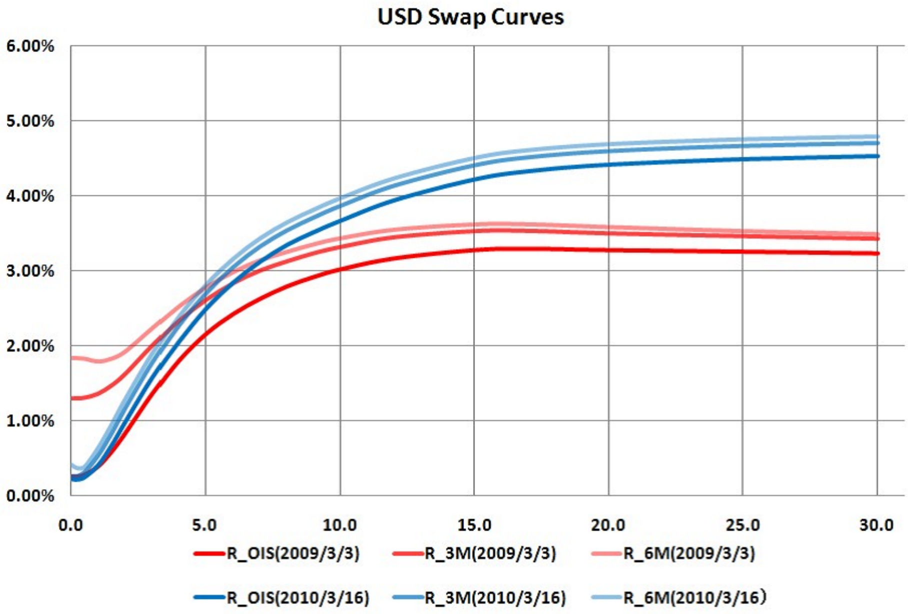

Figure 1: USD zero rate curves of Fed-Fund rate, $3m$ and $6m$ Libors.

In Fig. 1, we have given examples of calibrated yield curves for USD market on 2009/3/3 and 2010/3/16, where $R_{\rm OIS}$, R_{3m} and R_{6m} denote the zero rates for OIS (Fed-Fund rate), 3m and 6m forward Libor, respectively. $R_{\rm OIS}(\cdot)$ is defined as $R_{\rm OIS}(T) = -\ln(D(0,T))/T$. For the forward Libor, the zero-rate curve $R_\tau(\cdot)$ is determined recursively through the relation

$$E^{T_m}[L(T_{m-1},T_m;\tau)] = \frac{1}{\delta_m}\left(\frac{e^{-R_\tau(T_{m-1})T_{m-1}}}{e^{-R_\tau(T_m)T_m}} - 1\right) . \tag{3.1}$$

In the actual calculation of $D(0,\cdot)$, we have used the Fed-Fund vs 3m-Libor basis swap, where the two parties exchange 3m Libor and the compounded Fed-Fund rate with spread, which seems more liquid and a larger number of quotes available than the usual OIS. In Fig. 2, one can see the historical behavior of the spread between 1yr IRS and OIS for USD, JPY and EUR, where the underlying floating rates of IRS are 3m-Libor for USD and EUR and 6m-Libor for JPY.

Remarks: In the above calculations, we have assumed that the conditions given in the previous section are satisfied, and also that all the instruments are collateralized by the cash of domestic currency or its payment currency. Cautious readers may worry about the possibility that the market quotes contain significant contributions from market participants who use a foreign currency as collateral. However, the induced changes in IRS/TS quotes are very small and impossible to distinguish from the bid/offer spreads in normal circumstances, because the correction appears both in the fixed and floating legs which keeps the market quotes almost unchanged [5].

[5] As for cross currency swaps, the change can be a few bps, which can be comparable to the market bid/offer spreads.

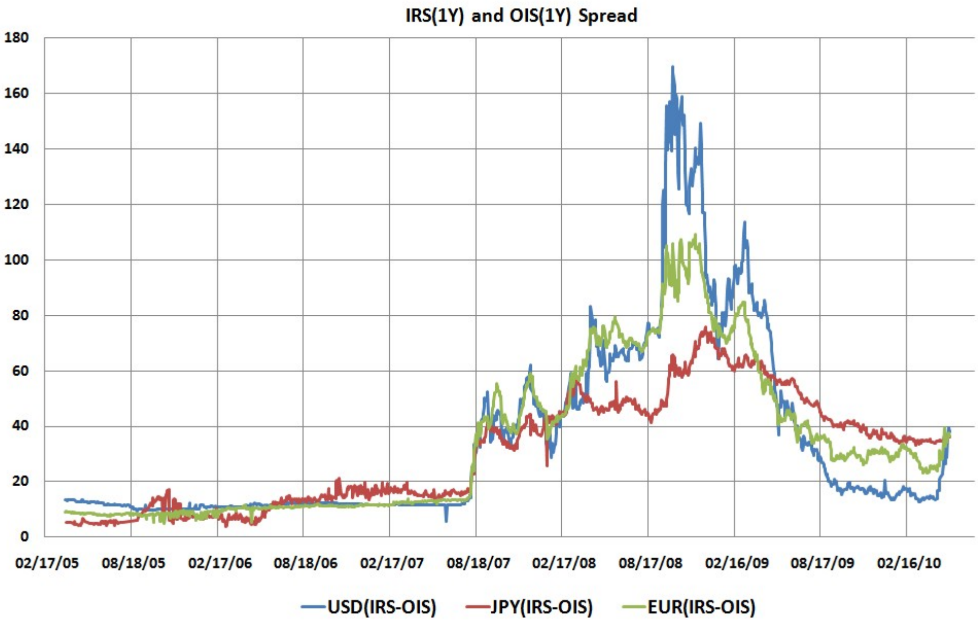

Figure 2: Difference between 1yr IRS and OIS. Underlying floating rates are 3m-Libor for USD and EUR, and 6m-Libor for JPY.

4 Curve Construction in Multiple Currencies

4.1 Calibration Procedures

In this section, we will discuss how to make the term structure consistent with CCS (cross currency swap) market. The current market is dominated by USD crosses where 3m USD Libor flat is exchanged with 3m Libor of a different currency with additional basis spread. The most popular type of CCS is called MtMCCS in which the notional of USD leg is reset at the start of every calculation period of Libor while the notional of the other leg is kept constant throughout the contract period [6].

We consider a MtMCCS of (i, j) currency pair, where the leg of currency i (intended to be USD) needs notional refreshments. We assume that the collateral is posted in currency i, which seems common in the market.

The value of j-leg of a T_0-start T_N-maturing MtMCCS is calculated as

$$\begin{aligned} PV_j &= -D^{(j)}(0,T_0)E^{T_0^j}\left[e^{-\int_0^{T_0} y^{(j,i)}(s)ds}\right] + D^{(j)}(0,T_N)E^{T_n^j}\left[e^{-\int_0^{T_N} y^{(j,i)}(s)ds}\right] \\ &+ \sum_{n=1}^{N} \delta_n^{(j)} D^{(j)}(0,T_n)E^{T_n^j}\left[e^{-\int_0^{T_n} y^{(j,i)}(s)ds}\left(L^{(j)}(T_{n-1},T_n;\tau)+B_N\right)\right], \end{aligned} \tag{4.1}$$

where the basis spread B_N is available as a market quote. In [2], we have assumed that all of the $\{y^{(k)}(\cdot)\}$ and hence $\{y^{(i,j)}(\cdot)\}$ are deterministic functions of time to make the curve construction simpler. Here, we slightly relax the assumption allowing randomness

[6] As for the details of MtMCCS and a different type of CCS, see [2, 3].

of $\{y^{(i,j)}(\cdot)\}$. As long as we assume that $\{y^{(i,j)}(\cdot)\}$ is independent from the dynamics of Libors and collateral rates, the procedures of bootstrapping given in [2] can be applied in the same way [7]. Under this assumption, we obtain

$$\begin{aligned} PV_j &= -D^{(j)}(0,T_0)e^{-\int_0^{T_0} y^{(j,i)}(0,s)ds} + D^{(j)}(0,T_N)e^{-\int_0^{T_N} y^{(j,i)}(0,s)ds} \\ &+ \sum_{n=1}^{N} \delta_n^{(j)} D^{(j)}(0,T_n)e^{-\int_0^{T_n} y^{(j,i)}(0,s)ds} \left(E^{T_n^j}[L^{(j)}(T_{n-1},T_n;\tau)] + B_N \right) . \end{aligned} \tag{4.2}$$

Here, we have defined, $y^{(j,i)}(t,s)$, the forward rate of $y^{(j,i)}(s)$ at time t as [8]

$$e^{-\int_t^T y^{(j,i)}(t,s)ds} = E_t^{Q^j}\left[e^{-\int_t^T y^{(j,i)}(s)ds} \right] . \tag{4.3}$$

Note that non-zero correlations among $\{y^{(i,k)}\}_{i,k}$ themselves do not pose any difficulty on curve construction.

On the other hand, the present value of i-leg in terms of currency j is given by

$$\begin{aligned} PV_i &= -\sum_{n=1}^{N} E^{Q^i}\left[e^{-\int_0^{T_{n-1}} c^{(i)}(s)ds} f_x^{(i,j)}(T_{n-1}) \right] / f_x^{(i,j)}(0) \\ &\quad + \sum_{n=1}^{N} E^{Q^i}\left[e^{-\int_0^{T_n} c^{(i)}(s)ds} f_x^{(i,j)}(T_{n-1}) \left(1 + \delta_n^{(i)} L^{(i)}(T_{n-1},T_n;\tau) \right) \right] / f_x^{(i,j)}(0) \\ &= \sum_{n=1}^{N} \delta_n^{(i)} D^{(i)}(0,T_n) E^{T_n^i}\left[\frac{f_x^{(i,j)}(T_{n-1})}{f_x^{(i,j)}(0)} B^{(i)}(T_{n-1},T_n;\tau) \right] , \end{aligned} \tag{4.4}$$

where

$$B^{(i)}(t,T_k;\tau) = E_t^{T_k^i}\left[L^{(i)}(T_{k-1},T_k;\tau) \right] - \frac{1}{\delta_k^{(i)}} \left(\frac{D^{(i)}(t,T_{k-1})}{D^{(i)}(t,T_k)} - 1 \right) , \tag{4.5}$$

which represents a Libor-OIS spread. Since we found no persistent correlation between FX and Libor-OIS spread in historical data, we have treated them as independent variables. Even if a non-zero correlation exists in a certain period, the expected correction seems not numerically important relative to the typical size of bid/offer spreads for MtMCCS (about a few bps at the time of writing). Since 3-month timing adjustment of FX is safely negligible, an approximate value of i-leg is obtained as

$$PV_i \simeq \sum_{n=1}^{N} \delta_n^{(i)} D^{(i)}(0,T_n) \frac{D^{(j)}(0,T_{n-1})}{D^{(i)}(0,T_{n-1})} e^{-\int_0^{T_{n-1}} y^{(j,i)}(0,s)ds} B^{(i)}(0,T_n;\tau), \tag{4.6}$$

where we have used the following result of the forward FX collateralized with currency i:

$$f_x^{(i,j)}(t,T) = f_x^{(i,j)}(t) \frac{D^{(j)}(t,T)}{D^{(i)}(t,T)} e^{-\int_t^T y^{(j,i)}(t,s)ds} . \tag{4.7}$$

[7] In practice, it would not be a problem even if there is a non-zero correlation as long as it does not meaningfully change the model implied quotes compared to the market bid/offer spreads.

[8] Since we are assuming the independence from the collateral rate, the measure change within the same currency gives no difference.

Using Eqs. (4.2) and (4.6), the term structure of $\{y^{(j,i)}(0,\cdot)\}$ can be extracted from the equality $PV_i = PV_j$, a consistency condition for the observed market spread.

Under the above approximation, (i,j)-MtMCCS par spread is expressed as

$$\begin{aligned} B_N \quad = \quad & \left[\left\{ \sum_{n=1}^{N} \delta_n^{(i)} D_{T_n}^{(i)} \left(\frac{D_{T_{n-1}}^{(j)}}{D_{T_{n-1}}^{(i)}} \right) e^{-\int_0^{T_{n-1}} y^{(j,i)}(0,s)ds} B_{T_n}^{(i)} - \sum_{n=1}^{N} \delta_n^{(j)} D_{T_n}^{(j)} e^{-\int_0^{T_n} y^{(j,i)}(0,s)ds} B_{T_n}^{(j)} \right\} \right. \\ & \left. - \sum_{n=1}^{N} D_{T_{n-1}}^{(j)} e^{-\int_0^{T_{n-1}} y^{(j,i)}(0,s)ds} \left(e^{-\int_{T_{n-1}}^{T_n} y^{(j,i)}(0,s)ds} - 1 \right) \right] \Big/ \sum_{n=1}^{N} \delta_n^{(j)} D_{T_n}^{(j)} e^{-\int_0^{T_n} y^{(j,i)}(0,s)ds} \ , \quad (4.8) \end{aligned}$$

where we have shortened the notations as $D^{(k)}(0,T) = D_T^{(k)}$ and $B^{(k)}(0,T;\tau) = B_T^{(k)}$.

4.2 Historical Behavior

Now, let us check the historical behavior of $R_{y(\mathrm{EUR,USD})}$ and $R_{y(\mathrm{JPY,USD})}$ given in Fig. 3 to 5 [9]. Here, the spread R_y is defined as

$$R_{y(j,i)}(T) = -\frac{\ln\left(E^{Q^j}\left[e^{-\int_0^T y^{(j,i)}(s)ds} \right] \right)}{T} = \frac{1}{T}\int_0^T y^{(j,i)}(0,s)ds \ . \qquad (4.9)$$

In Fig. 3, we have shown historical behaviors of basis spreads of 5y MtMCCS, corresponding $R_{y(\mathrm{X,USD})}(5y)$, and difference of $R_{3m}(5y) - R_{\mathrm{OIS}}(5y)$ between the two currency pairs denoted by ΔLibor-OIS$(5y;\mathrm{USD},\mathrm{X})$ [10]. Here, "X" stands for either EUR or JPY. As expected from Eq. (4.8), $R_{y(\mathrm{X,USD})}(5y) + \Delta$Libor-OIS$(5y;\mathrm{USD},\mathrm{X})$ well agrees with the 5y MtMCCS spread with typical error smaller than a few bps. From the figure, we observe that a significant portion of the movement of CCS spreads stems from the change of $y^{(i,j)}$, rather than the difference of Libor-OIS spread between two currencies. In fact, the level (difference)-correlation between R_y and CCS spread is quite high, which is about 93% (69%) for EUR and about 70% (48%) for JPY for the historical series used in the figure. On the other hand, the same quantities between ΔLibor-OIS and CCS spread are given by -56% (3%) for EUR and 9% (4%) for JPY.

The 3m-roll historical volatilities of $y^{(EUR,USD)}$ instantaneous forwards, which are annualized in absolute terms, are given in Fig. 7. In a calm market, they tend to be 50 bps or so, but they were more than a percentage point just after the market crisis, which is reflecting a significant widening of the CCS basis spread to seek USD cash in illiquid market. Except the CCS basis spread, y does not seem to have persistent correlations with other market variables such as OIS, IRS and FX forwards.

5 Implications for Derivatives Pricing and Summary

We now consider implications of collateralization for derivatives pricing. It is straightforward to see when payment and collateral currencies are the same. As in Eq. (2.8),

[9] Due to the lack of OIS data for JPY market, we have only a limited data for (JPY,USD) pair. We have used Cubic Monotone Spline for calibration although the figures are given in linear plots for ease.

[10] It can be interpreted as the difference of Libor-OIS spread between USD and X.

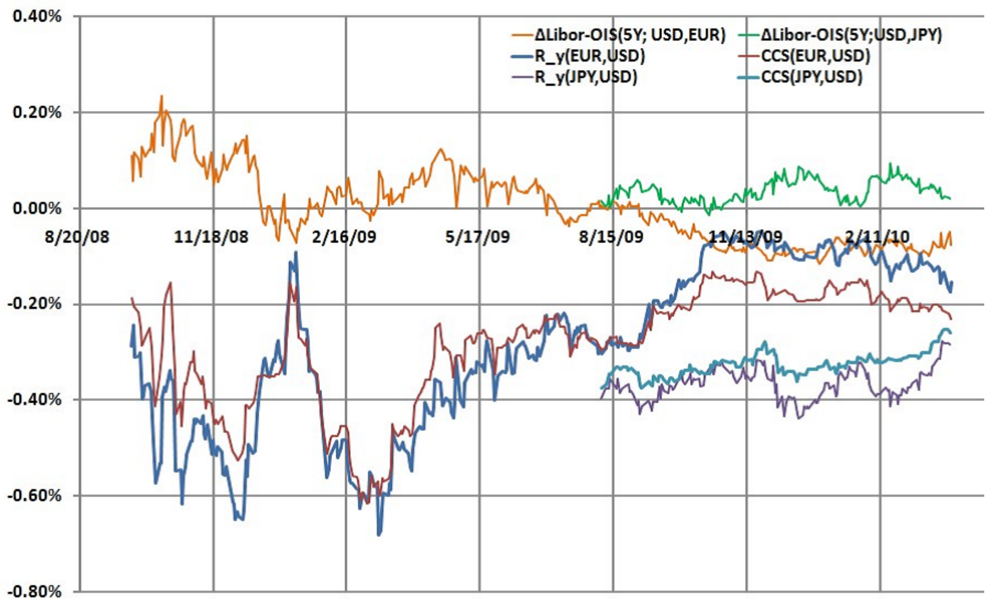

Figure 3: $R_{y(EUR,USD)}(5y)$, $R_{y(JPY,SD)}(5y)$, difference of Libor-OIS spreads, and corresponding quotes of 5y-MtMCCS.

discounting rate is now determined by collateral (or ON) rate rather than Libors. Hence, in the presence of the current level of Libor-OIS spread of $(10 \sim 20)$bps, the conventional Libor discounting method results in significant underestimation of the value of future payments, which can even be a few percentage points for long maturities. Considering the mechanism of collateralization, financial firms need to hedge the move of OIS in addition to Libors. In particular, the risk of floating-rate payments needs to be checked carefully, since the overnight rate can move in the opposite direction to Libor as was observed in this financial crisis. In Fig. 8, the present values of Libor floating legs with final principal $(= 1)$ payment

$$PV = \sum_{n=1}^{N} \delta_n D(0, T_n) E^{T_n} \left[L(T_{n-1}, T_n; \tau) \right] + D(0, T_N) \tag{5.1}$$

are given for various maturities. If traditional Libor discounting is used, the stream of Libor payments has the constant present value "1", which is obviously wrong from our results. This point is very important in risk-management point of view, since financial firms may overlook the quite significant interest-rate risk exposure when they use traditional interest rate models in their system.

If a trade with payment currency j is collateralized by foreign currency i, an additional modification to the discounting factor appears (See theorem1 with $h(T) = 1$.) [11]:

$$e^{-\int_t^T y^{(j,i)}(t,s)ds} = E_t^{Q^j} \left[e^{-\int_t^T y^{(j,i)}(s)ds} \right] . \tag{5.2}$$

From Figs. 5 and 6, one can see that posting USD as collateral tends to be expensive from the view point of collateral payers, which is particularly the case when everyone seeking

[11] Here, we are assuming independence of y from reference assets.

USD cash in illiquid market. For example, from Fig. 6, one can see that the value of JPY payment in 10 years time is more expensive by around 3% when it is collateralized by USD instead of JPY. The effects should be more profound for emerging currencies where the implied CCS basis spread can easily be 100 bps or more.

We now discuss the embedded CTD option in a collateral agreement. In some cases, financial firms make contracts with CSA allowing several currencies as eligible collateral. Suppose that the payer of collateral has a right to replace a collateral currency whenever he wants. In this case, the collateral payer should choose the cheapest collateral currency to post, which leads to the modification of the discounting factor of currency j as

$$E_t^{Q^j}\left[e^{-\int_t^T \max_{i\in C}\left\{y^{(j,i)}(s)\right\}ds}\right], \tag{5.3}$$

where C is the set of eligible currencies. Note that, by the definition of collateral payers, they want to make $(-PV)$ (>0) as small as possible. Although there is a tendency toward a CSA allowing only one collateral currency to reduce the operational burden, it does not seem uncommon to accept the domestic currency and USD as eligible collateral, for example. In this case, the above factor turns out to be

$$E_t^{Q^j}\left[e^{-\int_t^T \max\left\{y^{(j,\mathrm{USD})}(s),0\right\}ds}\right] . \tag{5.4}$$

In Fig. 9, we have plotted the modification factor given in Eq. (5.4), for $j=$ EUR as of 2010/3/16. We have used Hull-White model for the dynamics of $y^{(EUR,USD)}(\cdot)$ with a mean reversion parameter 1.5% per annum and the set of volatilities, $\sigma=0,25,50$ and 75 bps [12], respectively. As can be seen from the historical volatilities given in Fig. 7, σ can be much higher under volatile environment. The curve labeled by USD (EUR) denotes the modification of the discount factor when only USD (EUR) is eligible collateral for the ease of comparison. One can easily see that there is significant impact when the collateral currency chosen optimally. For example, from Fig. 9, one can see if the parties choose the collateral currency from EUR and USD optimally, it increases the effective discounting rate by roughly 50 bps annually even when the annualized volatility of spread $y^{(EUR,USD)}$ is 50 bps. We have qualitatively the same results for (JPY,USD) pair, although they are omitted due to the space limitation [13]. Although we expect that there are various obstacles to implement the optimal strategy in practice, the development of common electronic platform for collateral management as well as brisk startups of central clearing houses will make the optimal collateral strategy be an important issue in coming years.

Finally, let us emphasize a potential danger to use the traditional Libor-discounting model, which still seems quite common among financial firms. First of all, it can overlook large delta exposure to Libor-OIS and MtMCCS (or closely related "y") spreads. Note that, even if a desk is only dealing with single currency products, it inevitably has exposure to CCS spreads through modifications of discounting factors if it accepts foreign currencies as collateral. Furthermore, if the firm adopts a CSA allowing free replacement of collateral currency, there may exist non-negligible exposure on CCS volatility (with large negative gamma) through the embedded CTD options. Although we have cut the details of HJM framework under collateralization due to the limited space, the full list of relevant SDEs

[12] These are annualized volatilities in absolute terms.

[13] The analyzed data will be available upon request.

could be provided upon request. We emphasize that every building block of the framework is market observable, i.e. collateral rate $c^{(i)}$, Libor-OIS sprad $B^{(i)}$, $y^{(i,j)}$ spread, and $f_x^{(i,j)}$ for each currency and pairs, where the unobserved risk-free rate is embedded in c and y. See [2] for related discussions.

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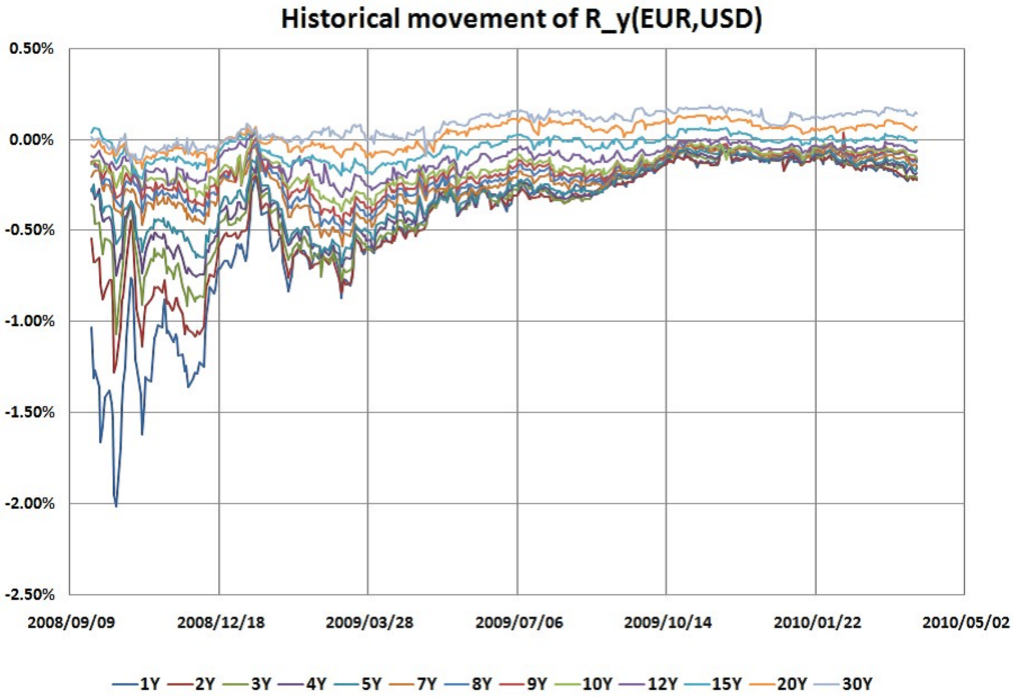

Figure 4: Historical movement of calibrated $R_{y(EUR,USD)}$.

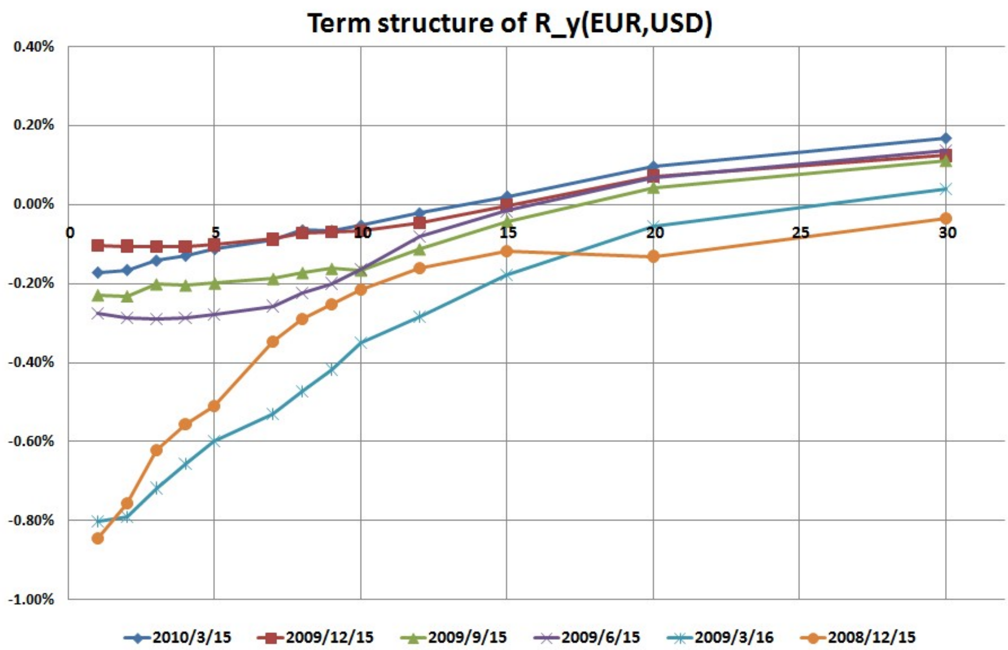

Figure 5: Examples of $R_{y(EUR,USD)}$ term structure.

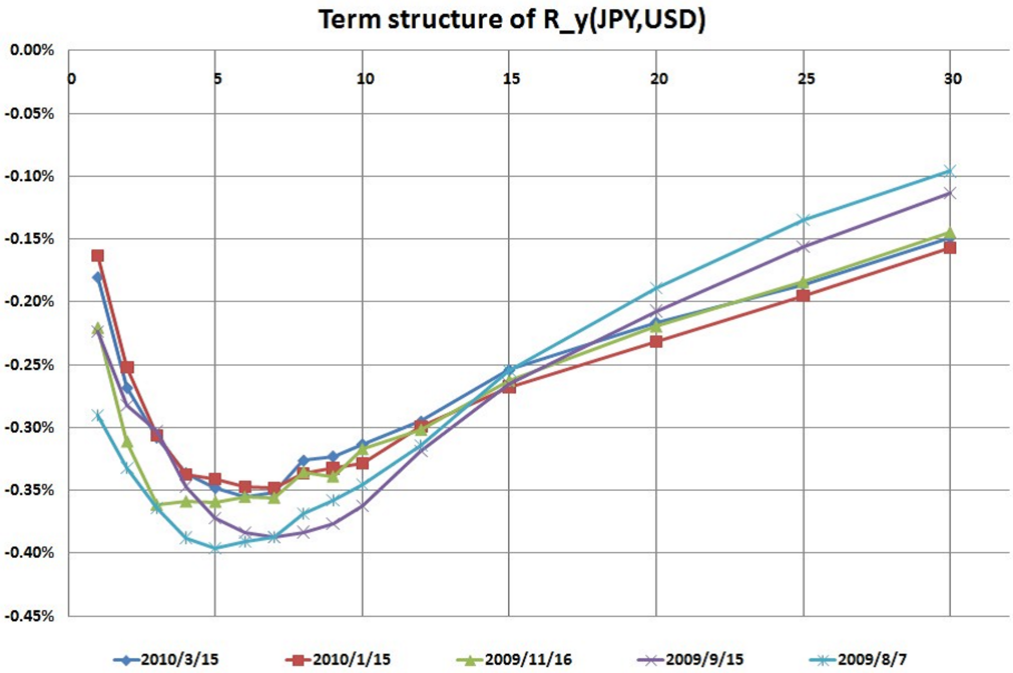

Figure 6: Examples of $R_{y(JPY,USD)}$ term structure.

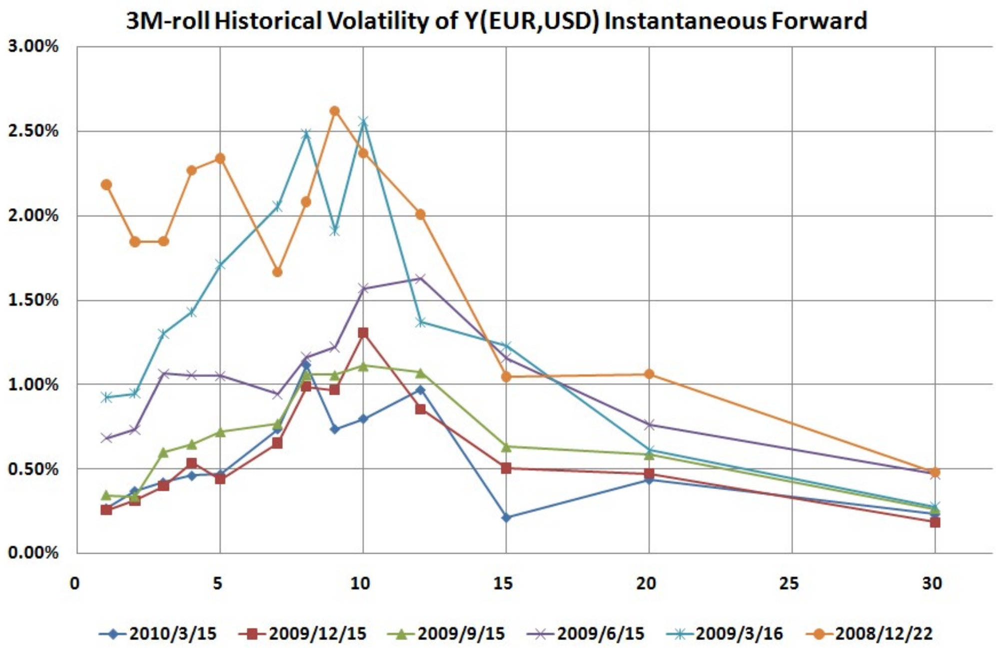

Figure 7: 3M-Roll historical volatility of $y^{(EUR,USD)}$ instantaneous forward. Annualized in absolute terms.

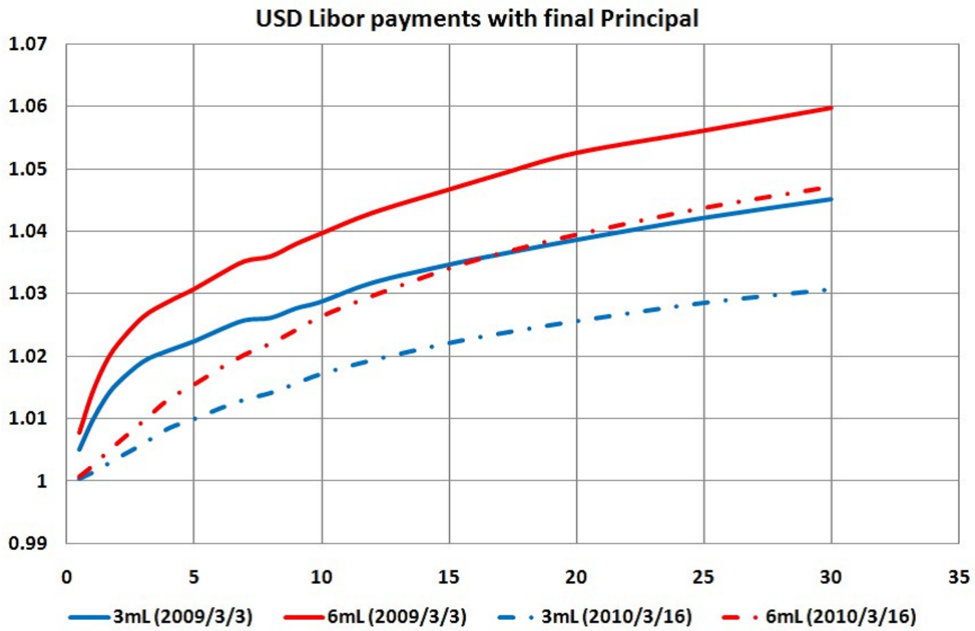

Figure 8: Present value of USD Libor stream with final principal (= 1) payment.

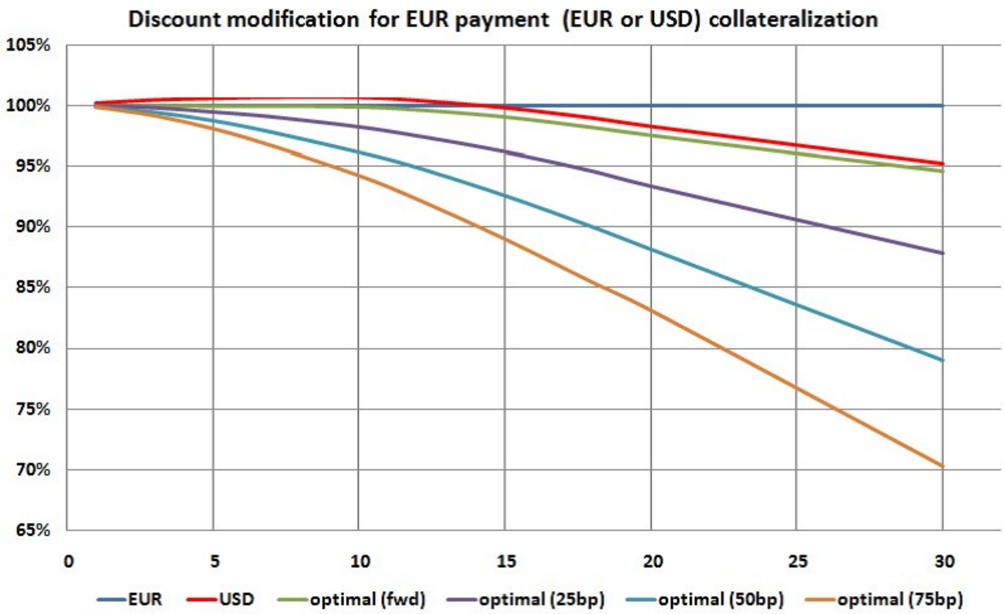

Figure 9: Modification of EUR discounting factors based on HW model for $y^{(EUR,USD)}$ as of 2010/3/16. The mean-reversion parameter is 1.5%, and the volatility is given at each label.