CARF Working Paper

CARF-F-257

# ON ADMISSIBLE STRATEGIES IN ROBUST UTILITY MAXIMIZATION

Keita Owari
The University of Tokyo

First version: October 2011
Current version: March 2012

CARF is presently supported by Bank of Tokyo-Mitsubishi UFJ, Ltd., Citigroup, Dai-ichi Mutual Life Insurance Company, Meiji Yasuda Life Insurance Company, Nippon Life Insurance Company, Nomura Holdings, Inc. and Sumitomo Mitsui Banking Corporation (in alphabetical order). This financial support enables us to issue CARF Working Papers.

# ON ADMISSIBLE STRATEGIES IN ROBUST UTILITY MAXIMIZATION

KEITA OWARI

*Graduate School of Economics, The University of Tokyo*
*7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan*

The existence of optimal strategy in robust utility maximization is addressed when the utility function is finite on the entire real line. A delicate problem in this case is to find a "good definition" of admissible strategies to admit an optimizer. Under certain assumptions, especially a kind of time-consistency property of the set $\mathcal{P}$ of probabilities which describes the model uncertainty, we show that an optimal strategy is obtained in the class of those whose wealths are supermartingales under all local martingale measures having a finite generalized entropy with one of $P \in \mathcal{P}$.

## 1. INTRODUCTION

This paper analyzes a *qualitative aspect* of the problem of robust utility maximization. Given a utility function $U$ and a set $\mathcal{P}$ of probabilities which describes the model uncertainty, the basic problem of this paper is to maximize the *robust utility functional*

$$X \mapsto \inf_{P \in \mathcal{P}} E_P[U(X)]$$

over all terminal wealths $x + \theta \cdot S_T = x + \int_0^T \theta dS$ of admissible strategies $\theta$, where $S$ is an underlying semimartingale. When $U$ is finite only on the positive half-line, the duality theory for this problem in the spirit of [15, 16] has been studied in both *quantitative* and *qualitative* aspects (e.g. [23], [22], [8]). In the case of utility taking finite values for all $x \in \mathbb{R}$, [18] shows the key duality, while [8] and [17] give a *partial result* on the existence of optimal strategy which we shall complete in this paper. See also [9] for more comprehensive reference and the background of the robust utility maximization problem.

A key subtlety intrinsic to the case of utility on $\mathbb{R}$ is *the "good definition" of admissible strategies* $\theta$, which will constitute the central theme of this paper. In this case, a universal and conceptually natural definition of admissibility is that $\theta \cdot S$ is uniformly bounded from below by some constant, which completely determines the quantitative nature of the problem. This class, however, typically fails to admit an optimizer. On the other hand, if $U$ is $-\infty$ on $\mathbb{R}_-$, the only natural (non-redundant) definition of admissibility is that the stochastic integral $\theta \cdot S$ is bounded from below by $-x$, and an optimal strategy is indeed obtained in this class under certain mild assumptions (see [23, 22]).

In the classical case (i.e., $\mathcal{P} = \{P\}$, say), the question of the good definition of admissibility is closely analyzed by [21] following the observation by [7] and [14] in the case of

---

*E-mail address*: keita.owari@gmail.com.

*Key words and phrases.* Robust utility maximization · Convex duality · Supermartingale · Martingale measures.
*Mathematics Subject Classification (2010).* 60G44 · 60H05 · 91G10 · 91G80
*JEL Classification.* C61 · G11
Forthcoming in *Mathematics and Financial Economics*. DOI: 10.1007/s11579-012-0068-3.

exponential utility. [21] shows that a "good definition" which yields us an optimal strategy is that $\theta \cdot S$ is a supermartingale under all local martingale measures $Q$ which has a "finite entropy" with the physical probability $P$. We denote the class of such $\theta$ by $\Theta_V(P)$ (see Section 2 for precise definitions including the meaning of "finite entropy"). Note that this class contains the usual admissible class, and the supermartingale property is consistent to the "No-Arbitrage philosophy". Thus $\Theta_V(P)$ is acceptably natural choice *when a single physical probability is specified*.

In the general robust case with $\mathcal{P}$ containing (infinitely) many elements, [8] (see also [17] for a slight generalization) provides a partial analogue of the above result which states that, under certain stronger assumptions, an optimal strategy is obtained in the class of $\theta$ with $\theta \cdot S$ being a supermartingale under all local martingale measures $Q$ having a finite entropy w.r.t. a certain element $\widehat{P} \in \mathcal{P}$ called a *least favorable measure*, i.e., in the class $\Theta_V(\widehat{P})$. Here a dissatisfaction comes of course from the dependence of admissibility on $\widehat{P}$. In *philosophy*, $\mathcal{P}$ is the set of *candidates* of real world models, and we do not know which one is true. Thus an "admissible strategy" should be *universally* admissible for all candidates $P \in \mathcal{P}$. Also, the least favorable probability $\widehat{P}$ is a part of solution to the dual problem of robust utility maximization, hence the class $\Theta_V(\widehat{P})$ is not *a priori* available.

In this view, a seemingly natural admissible class is $\bigcap_{P\in\mathcal{P}} \Theta_V(P)$ which is universal and contains all $\theta$ whose stochastic integrals are bounded below. Thus our central question in this paper is:

**Question 1.** Does the class $\bigcap_{P\in\mathcal{P}} \Theta_V(P)$ admit an optimal strategy?

The main result (Theorem 3.2) states that this is indeed the case if (in addition to standard assumptions) the set $\mathcal{P}$ of candidate models has a *time-consistency* property. We proceed as follows. The first step is to construct a so-called "optimal claim" for the abstract version of robust utility maximization, from which a candidate of optimal strategy $\hat{\theta}$ is derived through a predictable representation argument. This part is mostly standard excepting some technicality, but we give a slightly better description of optimal claim. Note that the additional time-consistency assumption is not required at this stage. The crucial step is to verify the supermartingale property of $\hat{\theta} \cdot S$ under all local martingale measures $Q$ which has a finite entropy with *some* $P \in \mathcal{P}$ *but its entropy with* $\widehat{P}$ *is infinite*. We shall do this by a (slight surprisingly) simple trick.

## 2. FORMULATION

We fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ as well as a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfying *the usual conditions*, where $T \in (0, \infty)$ is a fixed time horizon. Though many probabilities on $(\Omega, \mathcal{F})$ will appear in the sequel, the probability $\mathbb{P}$ plays the role of reference probability, i.e., every probabilistic notion is defined under $\mathbb{P}$ unless other probability is explicitly specified as $E_P[\cdot]$, $L^1(P)$ etc. In particular, the underlying asset prices $S$ is a $d$-dimensional $\mathbb{P}$-càdlàg semimartingale, and we assume:

(A1) $$S \text{ is } \mathbb{P}\text{-locally bounded.}$$

Let $\mathcal{P}$ be a set of probabilities $P \ll \mathbb{P}$, which we can (and do) embed into $L^1$ via the mapping $P \mapsto dP/d\mathbb{P}$. In this sense, we assume:

(A2) $$\mathcal{P} \text{ is convex and } \sigma(L^1, L^\infty)\text{-compact.}$$

We work with a utility function $U : \mathbb{R} \to \mathbb{R}$ which we assume

(A3) $U$ is differentiable, strictly concave on $\mathbb{R}$, and $U'(-\infty) = \infty$, $U'(\infty) = 0$, and satisfies the condition of *reasonable asymptotic elasticity*:

$$\liminf_{x\to-\infty}\frac{xU'(x)}{U(x)}>1 \text{ and } \limsup_{x\to\infty}\frac{xU'(x)}{U(x)}<1. \tag{A4}$$

The conjugate of utility function $U$ is denoted by $V$, i.e.,

$$V(y):=\sup_{x\in\mathbb{R}}(U(x)-xy),\quad y\in\mathbb{R}. \tag{2.1}$$

The assumptions (A3) and (A4) guarantee that $V$ is a "nice" convex function (see [10], [19] for details). Using this function, we introduce a generalized entropy:

$$V(\nu|P):=\begin{cases}E_P[V(d\nu/dP)] & \text{if } \nu\ll P,\\ +\infty & \text{otherwise}\end{cases} \tag{2.2}$$

for any positive finite measure $\nu\ll\mathbb{P}$ and $P\in\mathcal{P}$. When $U(x)=1-e^{-x}$ (exponential utility) and $Q$ is a probability with $Q\ll P$, we have $V(Q|P)=E_Q[\log(dQ/dP)]$, i.e., the relative entropy. Abusing the terminology, we still call the map $V(\cdot|\cdot)$ the *generalized entropy* associated to $V$. We define also the *robust generalized entropy* by

$$V(Q|\mathcal{P}):=\inf_{P\in\mathcal{P}}V(Q|P)<\infty. \tag{2.3}$$

Let $\mathcal{M}_{loc}$ be the set of all local martingale measures for $S$, i.e., probabilities $Q\ll\mathbb{P}$ under which $S$ is a local martingale. We then set

$$\mathcal{M}_V:=\{Q\in\mathcal{M}_{loc}:\ V(Q|\mathcal{P})<\infty\}. \tag{2.4}$$

Generically, for any set $\mathcal{Q}$ of probabilities $Q\ll\mathbb{P}$, we denote by $\mathcal{Q}^e$ the set of $Q\in\mathcal{Q}$ with $Q\sim\mathbb{P}$. We assume the existence of *equivalent* local martingale measure with finite entropy in the following sense:

$$\mathcal{M}_V^e:=\{Q\in\mathcal{M}_V:\ Q\sim\mathbb{P}\}\neq\emptyset. \tag{A5}$$

In particular, this implies the existence of $(Q,P)\in\mathcal{M}_V^e\times\mathcal{P}$ such that $Q\sim P\sim\mathbb{P}$ and $V(Q|P)<\infty$. See [17] for detail and other consequences of these assumptions.

Let $L(S)$ be the totality of all $(S,\mathbb{P})$-integrable $d$-dimensional predictable processes, $L_0(S):=\{\theta\in L(S):\theta_0=0\}$, and we denote by $\theta\cdot S$ the stochastic integral of $\theta\in L(S)$ w.r.t. $S$. See e.g., [12] or [13] for more information. When the utility function is finite on the entire real line, a conceptually natural choice of $\Theta$ is

$$\Theta_{bb}:=\{\theta\in L_0(S):\ \theta\cdot S_0=0,\ \theta\cdot S \text{ is bounded from below}\}. \tag{2.5}$$

Then the value function of the robust utility maximization problem is given by

$$u(x):=\sup_{\theta\in\Theta_{bb}}\inf_{P\in\mathcal{P}}E_P[U(x+\theta\cdot S_T)],\quad x\in\mathbb{R}. \tag{2.6}$$

When we seek an optimal strategy, however, the class $\Theta_{bb}$ is typically too small to admit an optimal strategy. We thus have to enlarge the admissible class. Our choice is the following.

$$\Theta_V:=\{\theta\in L_0(S):\ \theta\cdot S \text{ is a } Q\text{-supermartingale},\ \forall Q\in\mathcal{M}_V\}. \tag{2.7}$$

**Remark 2.1** (Another equivalent formulation)**.** We have defined the classes $\mathcal{M}_V$ and $\Theta_V$ through the *robust* generalized entropy $Q\mapsto V(Q|\mathcal{P})$. But the following equivalent formulation is sometimes useful for comparison. For each $P\in\mathcal{P}$, we set

$$\mathcal{M}_V(P):=\{Q\in\mathcal{M}_{loc}:\ V(Q|P)<\infty\}, \tag{2.8}$$

$$\Theta_V(P):=\{\theta\in L_0(S):\ \theta\cdot S \text{ is a } Q\text{-supermartingale } \forall Q\in\mathcal{M}_V(P)\}. \tag{2.9}$$

When a single $P \in \mathcal{P}$ is fixed as the physical probability, the class $\Theta_V(P)$ is shown to be an appropriate domain of utility maximization in [21]. Recalling (2.3), our choices $\mathcal{M}_V$ and $\Theta_V$ are rewritten respectively as

$$\mathcal{M}_V = \bigcup_{P \in \mathcal{P}} \mathcal{M}_V(P), \quad \Theta_V = \bigcap_{P \in \mathcal{P}} \Theta_V(P).$$

Thus our definition (2.7) is consistent to what we wrote in introduction.

Under the assumptions (A1) – (A5), a duality result (Theorem 2.3 of [18]) is applicable, which states in our case that for any $\Theta$ with $\Theta_{bb} \subset \Theta \subset \Theta_V$, we have

$$u(x) = \sup_{\theta \in \Theta} \inf_{P \in \mathcal{P}} E_P[U(x + \theta \cdot S_T)] = \inf_{\lambda > 0} \inf_{Q \in \mathcal{M}_V} (V(\lambda Q|\mathcal{P}) + \lambda x). \tag{2.10}$$

In particular, the value function is unchanged if we replace $\Theta_{bb}$ by the larger class $\Theta_V$. Under the same assumptions, the right hand side, the *dual problem* of the (2.6), admits a solution $(\hat{\lambda}, \widehat{Q}) \in (0, \infty) \times \mathcal{M}_V$, and the infimum $V(\hat{\lambda}\widehat{Q}|\mathcal{P}) = \inf_{P \in \mathcal{P}} V(\hat{\lambda}\widehat{Q}|P)$ is attained by a $\widehat{P} \in \mathcal{P}$ since $\mathcal{P}$ is weakly compact, and $V(\cdot|\cdot)$ is lower semicontinuous. Thus the right hand side of (2.10) is also written as $V(\hat{\lambda}\widehat{Q}|\widehat{P})$, and we call the triplet $(\hat{\lambda}, \widehat{Q}, \widehat{P})$ a dual optimizer.

A way of proving (2.10) and the existence of a solution $(\hat{\lambda}, \widehat{Q})$ is to closely analyze the robust utility functional $X \mapsto \inf_{P \in \mathcal{P}} E_P[U(X)]$ on $L^\infty$ characterizing $V(\cdot|\mathcal{P})$ as its conjugate. Then the duality and the existence of $(\hat{\lambda}, \widehat{Q})$ follow *simultaneously* from Fenchel's duality theorem. See [18] for detail. Alternatively, one can separate the dual problem into the minimization of $\lambda \mapsto \inf_{Q \in \mathcal{M}_V} V(\lambda Q|\mathcal{P}) + \lambda x$ and of $Q \mapsto V(\lambda Q|\mathcal{P})$ for each $\lambda$. For the latter problem, called the *robust $f$-projection*, [8] proves the existence by establishing a uniform integrability criterion in terms of $V(\cdot|\mathcal{P})$ in the spirit of the de la Vallée-Poussin theorem.

In contrast to the standard utility maximization, neither the uniqueness of $(\hat{\lambda}, \widehat{Q})$ (hence of the triplet $(\hat{\lambda}, \widehat{Q}, \widehat{P})$) nor the equivalence $\widehat{Q} \sim \mathbb{P}$ hold in the robust case, as the following trivial example illustrates:

**Example 2.2.** Suppose $\mathcal{M}^e_{loc} \neq \emptyset$, and that $\mathcal{M}_{loc}$ contains an element $Q_0$ which is not equivalent to $\mathbb{P}$. Then we take $\mathcal{P}$ so that $Q_0 \in \mathcal{P} \subset \mathcal{M}_{loc}$. In this case, $\hat{\lambda}$ is uniquely determined as the minimizer of $\lambda \mapsto V(\lambda) + \lambda x$. Then a triplet $(\hat{\lambda}, Q, P)$ is a dual optimizer if (and only if) $P = Q \in \mathcal{P} \subset \mathcal{M}_{loc}$. Indeed, by Jensen's inequality and the strict convexity of $V$, $V(\lambda Q|P) = E_P[V(\lambda dQ/dP)] \geq V(\lambda)$ whenever $Q \ll P$, and the "equality" holds if and only if $Q = P$. Hence $(\hat{\lambda}, \widehat{Q}, \widehat{P})$ is not unique, and $(\hat{\lambda}, Q_0, Q_0)$ is a solution with $Q_0 \not\sim \mathbb{P}$.

As for the equivalence, we still have $\widehat{Q} \sim \widehat{P}$ whenever $(\hat{\lambda}, \widehat{Q}, \widehat{P})$ is a dual optimizer (see [17], Theorem 2.7). Also, by an exhaustion argument, there exists a *maximal solution* $(\hat{\lambda}, \widehat{Q}, \widehat{P})$ in the sense that if $(\lambda, Q, P)$ is another dual optimizer, then $P \ll \widehat{P}$ (hence $Q \ll \widehat{Q}$) and $\lambda dQ/dP = \hat{\lambda} d\widehat{Q}/d\widehat{P}$, $P$-a.s., where the density $d\widehat{Q}/d\widehat{P}$ is defined $\mathbb{P}$-a.s. in the sense of *Lebesgue decomposition*. In particular, if $(\hat{\lambda}, \widehat{Q}, \widehat{P})$ and $(\tilde{\lambda}, \tilde{Q}, \tilde{P})$ are two maximal solution, then

$$\tilde{\lambda} d\tilde{Q}/d\tilde{P} = \hat{\lambda} d\widehat{Q}/d\widehat{P}, \ \mathbb{P}\text{-a.s.}, \tag{2.11}$$

See [17, Theorem 2.5 and Proposition 4.7]. This uniqueness is still useful in our purpose. Note finally that even such a maximal $\widehat{Q}$ may fail to be equivalent to the reference probability $\mathbb{P}$. See [23, Example 2.5] for a counter example. In the sequel, we fix such a maximal dual optimizer, and call $\widehat{P}$ a *least favorable measure*.

The duality (2.10) completely characterizes the *quantitative nature* of the problem (2.6). But our aim in this paper is to discuss the *qualitative nature*, especially the existence of optimal strategy in $\Theta_V$. To do this, assumptions (A1) – (A5) are not enough, and we assume additionally

$$\sup_{\theta\in\Theta_{bb}} E_P[U(\theta\cdot S_T)] < \infty, \quad \forall P\in\mathcal{P}^e. \tag{A6}$$

**Remark 2.3.** Several remarks on assumption (A6) are in order.

1. This assumption is automatically satisfied if $U(\infty) := \sup_x U(x) < \infty$ as exponential utility, and in this case, $U(X)^+ \in \bigcap_{P\in\mathcal{P}} L^1(P)$ for *any* random variable $X$. Therefore, the robust utility functional $X \mapsto \inf_{P\in\mathcal{P}} E_P[U(X)]$ is well-defined on $L^0$ as a $[-\infty,\infty)$-valued concave functional.
2. If $U(\infty) = \infty$, [2, Th. 1.1 and Remark 1.2] show under (A4) that (A6) is equivalent to:
$$\forall P\in\mathcal{P}^e,\ \exists Q\in\mathcal{M}_V \text{ such that } V(Q|P) < \infty. \tag{2.12}$$
This is further equivalent to saying that $v_P(y) < \infty$ for all $y > 0$ and $P\in\mathcal{P}^e$, where $v_P$ is the dual value function
$$v_P(y) := \inf_{Q\in\mathcal{M}_V} V(yQ|P), \quad y > 0.$$
3. We could state (A6) with the whole $\mathcal{P}$ rather than $\mathcal{P}^e$. But for our purpose, (A6) is enough. Recall that (A5) implies in particular $\mathcal{P}^e \neq \emptyset$. If $\bar{P}\in\mathcal{P}^e$, we have $(P+\bar{P})/2 \in \mathcal{P}^e$ for all $P\in\mathcal{P}$, and $\|X\|_{L^1((P+\bar{P})/2)} = (\|X\|_{L^1(P)} + \|X\|_{L^1(\bar{P})})/2 \geq \|X\|_{L^1(P)}/2$. Hence we have, for instance,
   (a) $\bigcap_{P\in\mathcal{P}} L^1(P) = \bigcap_{P\in\mathcal{P}^e} L^1(P)$;
   (b) if $(X^n)$ is bounded in $L^1(P)$ for all $P\in\mathcal{P}^e$, then the same is true for all $P\in\mathcal{P}$.
   In particular, (A6) (hence (2.12)) guarantees even in the case $U(\infty)=\infty$ that
$$X \in \bigcap_{Q\in\mathcal{M}_V} L^1(Q) \Rightarrow U(X)^+ \in \bigcap_{P\in\mathcal{P}} L^1(P). \tag{2.13}$$
In fact, if $V(Q|P) < \infty$ and $X\in L^1(Q)$, Young's inequality implies $U(X) \leq V(dQ/dP) + (dQ/dP)X \in L^1(P)$, and we can take such a $Q\in\mathcal{M}_V$ by (2.12) for all $P\in\mathcal{P}^e$.

**Remark 2.4** (Continuation of Remark 2.1)**.** We give a brief comparison of admissible classes considered in literature. In [17], the class $\Theta_V(\widehat{P})$ is used to discuss the existence of optimal strategy, while [8] considered (implicitly) a slightly smaller class:

$$\mathcal{M}_V^0(\widehat{Q},\widehat{P}) := \{Q\in\mathcal{M}_{loc} : V(\alpha Q + (1-\alpha)\widehat{Q}|\widehat{P}) < \infty,\ \exists\alpha\in(0,1)\}, \tag{2.14}$$

$$\Theta_V^0(\widehat{Q},\widehat{P}) := \left\{\theta\in L_0(S) : \begin{array}{c}\theta\cdot S \text{ is a } Q\text{-supermartingale,}\\ \forall Q\in\mathcal{M}_V^0(\widehat{Q},\widehat{P})\end{array}\right\}. \tag{2.15}$$

Note that $\Theta_V^0(\widehat{Q},\widehat{P}) \subset \Theta_V(\widehat{P})$ since $\mathcal{M}_V(\widehat{P}) \subset \mathcal{M}_V^0(\widehat{Q},\widehat{P})$, while if we set $\Theta_m(\widehat{Q}) := \{\theta\in L_0(S) : \theta\cdot S \text{ is a } \widehat{Q}\text{-martingale}\}$,

$$\Theta_V \cap \Theta_m(\widehat{Q}) \subset \Theta_V(\widehat{P}) \cap \Theta_m(\widehat{Q}) = \Theta_V^0(\widehat{Q},\widehat{P}) \cap \Theta_m(\widehat{Q}).$$

Thus $\Theta_V(\widehat{P})$ and $\Theta_V^0(\widehat{Q},\widehat{P})$ are essentially equivalent for the existence of optimal strategy (see Theorem 3.2). We just emphasize here that our class $\Theta_V$ depends neither on particular $P\in\mathcal{P}$ nor $Q\in\mathcal{M}_V$, while $\Theta_V(\widehat{P})$ and $\Theta_V^0(\widehat{Q},\widehat{P})$ do.

We conclude this section by recalling a stability property of a set of probability measures, called *m-stability*, which will be used in Theorem 3.2 below.

**Definition 2.5** ([5], Definition 1)**.** A set $\mathcal{Q}$ of probability measures is said to be *m-stable* (multiplicatively stable) if for any $Q \in \mathcal{Q}$, $Q' \in \mathcal{Q}^e$ with the density processes $Z_t = (dQ/d\mathbb{P})|_{\mathcal{F}_t}$ and $Z'_t = (dQ'/d\mathbb{P})|_{\mathcal{F}_t}$, as well as any stopping time $\tau \leq T$, a new probability $\bar{Q}$ defined by $d\bar{Q}/d\mathbb{P} := Z_\tau(Z'_T/Z'_\tau)$ is an element of $\mathcal{Q}$.

This property is equivalent to the *time-consistency* of the corresponding *dynamic coherent monetary utility function* $\phi_\tau(X) := \operatorname{ess\,inf}_{Q\in\mathcal{Q}} E_Q[X|\mathcal{F}_\tau]$: for any $X, Y \in L^\infty$ and stopping times $\sigma \leq \tau$, $\phi_\tau(X) \leq \phi_\tau(Y)$ implies $\phi_\sigma(X) \leq \phi_\sigma(Y)$. This is further equivalent (under (A3)) to the time-consistency of the dynamic robust utility functional $\mathcal{U}_\tau(X) := \operatorname{ess\,inf}_{Q\in\mathcal{Q}} E_Q[U(X)|\mathcal{F}_\tau]$. See [5, Theorem 12] for details and precise formulation. Note that the set $\mathcal{M}_{loc}$ of *all* local martingale measures is m-stable.

## 3. Main Results

We first state a result on a “weak solution” to the problem (2.6), which yields a candidate of optimal strategy. Let

$$\mathcal{X} := \left\{X \in L^0 : X \in \textstyle\bigcap_{Q\in\mathcal{M}_V} L^1(Q),\ \sup_{Q\in\mathcal{M}_V} E_Q[X] \leq 0\right\}. \tag{3.1}$$

Note that $\theta \cdot S_T \in \mathcal{X}$ if $\theta \in \Theta_V$, and $X \in \mathcal{X}$ implies $U(x+X)^+ \in \bigcap_{P\in\mathcal{P}} L^1(P)$ for any $x \in \mathbb{R}$, by (A6) and Remark 2.3. Thus the robust utility functional $X \mapsto \inf_{P\in\mathcal{P}} E_P[U(x+X)]$ is well-defined on $\mathcal{X}$.

**Theorem 3.1.** *Suppose (A1) – (A6), and let $x \in \mathbb{R}$ and $(\hat{\lambda}, \widehat{Q}, \widehat{P})$ be a maximal dual optimizer. Then there exists an $\widehat{X} \in \mathcal{X}$ such that $U(x+\widehat{X}) \in \bigcap_{P\in\mathcal{P}} L^1(P)$ and*

$$u(x) = \sup_{X\in\mathcal{X}} \inf_{P\in\mathcal{P}} E_P[U(x+X)] = \inf_{P\in\mathcal{P}} E_P[U(x+\widehat{X})], \tag{3.2}$$

*where the infimum is attained by $\widehat{P}$. Moreover, there exists an $(S, \widehat{Q})$-integrable predictable process $\hat{\theta}$ with $\hat{\theta}_0 = 0$ such that $\hat{\theta}\cdot S$ is a $\widehat{Q}$-martingale (not only local) and*

$$x + \widehat{X} = -V'(\hat{\lambda}\, d\widehat{Q}/d\widehat{P}) = x + \hat{\theta}\cdot S_T,\ \widehat{Q}\text{-a.s.} \tag{3.3}$$

*In particular, $\widehat{X}$ is $\widehat{Q}$-a.s. unique, and $\hat{\theta}$ is unique in the sense that $\hat{\theta}\cdot S$ is unique up to $\widehat{Q}$-indistinguishability.*

The proof is given in Section 4. The first equality in (3.2) states that the robust utility maximization over *terminal wealths* $x + \theta\cdot S_T$ is (quantitatively) equivalent to the *indirect utility maximization* :

$$u_{\mathcal{M}_V}(x) = \sup_{X\in\mathcal{X}} \inf_{P\in\mathcal{P}} E_P[U(x+X)],$$

while the random variable $\widehat{X}$ is the so-called *optimal contingent claim*. Such arguments are quite standard in (non-robust) utility maximization, and also in the robust case, [8, Theorem 3.11] shows a similar result: under (A1) – (A5), the assertions of Theorem 3.1 hold true *except that* the sets $\mathcal{M}_V$ (in the definition (3.1)) and $\mathcal{P}$ are replaced by $\mathcal{M}_V^0(\widehat{Q}, \widehat{P})$ and $\mathcal{P}^0(\widehat{Q}, \widehat{P})$ defined respectively by Remark 2.4 and

$$\mathcal{P}^0(\widehat{Q}, \widehat{P}) := \{P \in \mathcal{P} : V(\widehat{Q}|\alpha P + (1-\alpha)\widehat{P}) < \infty, \exists\alpha \in (0,1)\}.$$

Note that our finite utility assumption (A6) is automatic if $\mathcal{P}$ is replaced by $\mathcal{P}^0(\widehat{Q}, \widehat{P})$. Also, when $U(\infty) < \infty$, the set $\mathcal{P}^0(\widehat{Q}, \widehat{P})$ actually coincides with the whole set $\mathcal{P}$ ([8], Remark 3.10). However, $\mathcal{M}_V^0(\widehat{Q}, \widehat{P})$ still depends on $(\widehat{Q}, \widehat{P})$ which is the *solution* to the dual problem, hence not *a priori* available. On the other hand, our formulation is universal, which is a slight, but qualitatively crucial contribution.

Theorem 3.1 suggests that the "strategy" $\hat{\theta}$ is a candidate of optimal strategy. However, we still have to prove that this strategy is indeed *admissible*.

**Theorem 3.2.** *In addition to (A1) – (A6), we assume that $\widehat{Q} \sim \mathbb{P}$ and $\mathcal{P}$ is m-stable. Then $\hat{\theta}$ is $(S, \mathbb{P})$-integrable (hence $(S, P)$-integrable for all $P \in \mathcal{P}$), and $\hat{\theta} \cdot S$ is a supermartingale under all $Q \in \mathcal{M}_V$. In particular, $\hat{\theta}$ belongs to $\Theta_V$ and is an optimal strategy.*

The proof is given in Section 5. When $\mathcal{P} = \{\mathbb{P}\}$, the question of *uniform supermartingale property* of this type goes back to the "six-author paper" [7] which shows that the optimal wealth in *exponential utility* maximization is a *martingale* under all local martingale measures having a finite relative entropy with $\mathbb{P}$, under an additional assumption on reverse Hölder inequality which is later removed by [14]. Although this uniform martingale property is no longer true for other utility functions, [21] shows that the optimal wealth is a supermartingale under all $Q \in \mathcal{M}_V(\mathbb{P})$, for any utility functions on $\mathbb{R}$ with reasonable asymptotic elasticity. There are also some extensions to the case where the semimartingale $S$ is not locally bounded. See e.g. [3] and [4].

In the robust case, the $Q$-supermartingale property for all $Q \in \mathcal{M}_V(\widehat{P})$ (hence all $Q \in \mathcal{M}_V^0(\widehat{Q}, \widehat{P})$ since $\hat{\theta} \cdot S$ is a $\widehat{Q}$-martingale) is shown by [8] (see also [17] for a slight extension). We emphasize that the difference between $\mathcal{M}_V(\widehat{P})$ and $\mathcal{M}_V$ is essential here. Note that $\widehat{X}$ is also optimal for the utility maximization problem under the fixed measure $\widehat{P}$, and the same is true for $(\hat{\lambda}, \widehat{Q})$ in the dual side. Thus the result of [21] cited in the previous paragraph is still applicable (under the assumption $\widehat{Q} \sim \mathbb{P}$) for $Q$ with $V(Q|\widehat{P}) < \infty$, while we have to consider the case where $V(Q|P) < \infty$ for *some* $P \in \mathcal{P}$ *but possibly* $V(Q|\widehat{P}) = \infty$.

To grasp the situation, we try to describe the heuristics behind the argument in [21] (from our point of view), and our idea of extending it. In what follows in this section, we suppose all the assumptions of Theorem 3.2, especially $\widehat{Q} \sim \mathbb{P}$.

For a moment, we *suppose* that $\hat{\theta} \cdot S$ is a $Q$-supermartingale for some $Q \in \mathcal{M}_V$. Then the $\widehat{Q}$-martingale property and the representation (3.3) imply: for any stopping time $\tau \leq T$,

$$E_{\widehat{Q}}[V'(\hat{\lambda} d\widehat{Q}/d\widehat{P})|\mathcal{F}_\tau] \leq E_Q[V'(\hat{\lambda} d\widehat{Q}/d\widehat{P})|\mathcal{F}_\tau], \ Q\text{-a.s.} \tag{3.4}$$

On the other hand, Ansel-Stricker's lemma [1] shows that $\hat{\theta} \cdot S$ is a $Q$-supermartingale if and only if there exists a $Q$-*martingale lower bound*, i.e., a $Q$-martingale $M^Q$ such that $\hat{\theta} \cdot S \geq M^Q$, $Q$-a.s. In particular, if (3.4) holds true for any stopping time $\tau \leq T$, the process defined by $M^Q_\tau = -E_Q[V'(\hat{\lambda} d\widehat{Q}/d\widehat{P})|\mathcal{F}_\tau]$ provides a desired lower bound, hence (3.4) is a necessary and *sufficient* condition for $\hat{\theta} \cdot S$ to be a $Q$-supermartingale.

When $V(Q|\widehat{P}) < \infty$, the inequality (3.4) is obtained as the *variational inequality* which characterizes $\widehat{Q}$ as a minimizer of the functional $Q \mapsto V(\hat{\lambda} Q|\widehat{P})$ when $\tau = 0$, and a "Bellman-type" principle using the m-stability of the set of local martingale measures shows the case of general $\tau \leq T$.

If $\inf_{P \in \mathcal{P}} V(Q|P) < \infty$ but $V(Q|\widehat{P}) = \infty$, this argument is no longer applicable at least directly. Mathematically speaking, we loose some important estimates to guarantee the necessary convergences, or more intuitively, any element $Q$ with $V(Q|\widehat{P}) = \infty$ is in no way optimal at very early stage, and we can not draw further information from the optimality of $\widehat{Q}$ in the minimization of $Q \mapsto V(\hat{\lambda} Q|\widehat{P})$. However, we have used only a part of information of $\widehat{Q}$ so far, and it is natural to expect that a better information may improve the result. More specifically,

**Step 1** the optimality of $(\widehat{Q},\widehat{P})$ in the minimization of $(Q,P)\mapsto V(\hat{\lambda}Q|P)$ should yield a variational inequality similar to (3.4) but with an additional term involving $P$:

$$\text{“ } E_{\widehat{Q}}[V'(\hat{\lambda}d\widehat{Q}/d\widehat{P})|\mathcal{F}_\tau]+F_\tau(\widehat{P})\le E_Q[V'(\hat{\lambda}d\widehat{Q}/d\widehat{P})|\mathcal{F}_\tau]+F_\tau(P)\text{”.}$$

**Step 2** Though we may not take $P=\widehat{P}$ in general, it seems natural to expect that we may take $P$ "arbitrarily close to $\widehat{P}$" keeping $V(Q|P)<\infty$ with fixed $Q$.

**Step 3** If this is the case, we may expect (3.4) by an approximation argument:

$$\text{“ } F_\tau(P)\to F_\tau(\widehat{P})\text{”.}$$

The formal inequality in **Step 1** will be realized as Proposition 5.4 below, where the m-stability of $\mathcal{P}$ will play an important role. On the other hand, **Steps 2** and **3** will be justified in a certain sense by a simple trick which is a consequence of reasonable asymptotic elasticity (Lemma 5.5).

**Remark 3.3** (What happens when $\widehat{Q}\not\sim\mathbb{P}$?)**.** The equivalence $\widehat{Q}\sim\mathbb{P}$ is automatic if all elements of $\mathcal{P}$ are equivalent to $\mathbb{P}$. When the filtration $\mathbb{F}$ is *continuous* (i.e., every $(\mathbb{F},\mathbb{P})$-martingale is continuous, especially if it is generated by a Brownian motion), the latter condition is already implied by the m-stability of $\mathcal{P}$ and (A2) (see [5, Theorem 8]), thus it is not a further restriction in that case.

In general, however, the equivalence $\widehat{Q}\sim\mathbb{P}$ may fail (see [23, Example 2.5] for a counter example), thus it is worth asking what happens in that case. When $U$ is finite only on the positive half-line, the optimal claim $\widehat{X}$ (which does not require the assumption $\widehat{Q}\sim\mathbb{P}$) is super-hedged by some $(S,\mathbb{P})$-integrable process $\tilde{\theta}$ with $\tilde{\theta}\cdot S=\hat{\theta}\cdot S$, $\widehat{Q}$-a.s. By the monotonicity of robust utility functional, we see that $\tilde{\theta}$ is an optimal strategy without the additional assumption $\widehat{Q}\sim\mathbb{P}$ (see [23] and [22]). However, this argument essentially relies on the fact that $\widehat{X}$ is bounded below by $-x$ (since $U(x)=-\infty$ for $x<0$), and no longer works when the utility function is finite on the entire real line. Thus we can not drop the assumption $\widehat{Q}\sim\mathbb{P}$ (at now).

**Remark 3.4** (Random Endowment)**.** The results of this paper may also be stated with a *random endowment* $B$ as long as it is an $\mathcal{F}_T$-measurable random variable satisfying

$$\begin{aligned}&\forall P\in\mathcal{P},\ \exists\varepsilon_P>0 \text{ such that } U(-\varepsilon_P B^+)\in L^1(P),\\ &\exists\varepsilon>0 \text{ such that } \{U(-(1+\varepsilon)B^-)dP/d\mathbb{P}\}_{P\in\mathcal{P}} \text{ is uniformly integrable.}\end{aligned}\tag{3.5}$$

Then the robust utility maximization problem (2.6) reads as

$$u_B(x):=\sup_{\theta\in\Theta_{bb}}\inf_{P\in\mathcal{P}}E_P[U(x+\theta\cdot S_T+B)],\tag{3.6}$$

Assumption (3.5) implies that $B\in\bigcap_{Q\in\mathcal{M}_V}L^1(Q)$, and guarantees under (A1) – (A5) that a duality corresponding to (2.10) holds true [18, Theorem 2.3]:

$$\sup_{\theta\in\Theta_V}\inf_{P\in\mathcal{P}}E_P[U(x+\theta\cdot S_T+B)]=\inf_{\lambda>0}\inf_{Q\in\mathcal{M}_V}(V(\lambda Q|\mathcal{P})+\lambda x+\lambda E_Q[B]),\tag{3.7}$$

With the same assumptions, the dual problem admits a maximal solution with the unique density in the sense of (2.11). Then Theorems 3.1 and 3.2 remain true with similar proofs, and with obvious modifications, e.g., (3.3) is replaced by $x+\widehat{X}+B=-V'(\hat{\lambda}d\widehat{Q}/d\widehat{P})=x+\hat{\theta}\cdot S_T+B$, $\widehat{Q}$-a.s. We omit the details. See [18] for the treatment of random endowment and other implications of (3.5).

## 4. Optimal Claim

We first note that we have only to consider the case $x = 0$. Indeed, assumptions (A3) and (A4) on the utility function are invariant under the translation of utility function from $U$ to $U_x(\xi) := U(x + \xi)$, and all the results for $x \neq 0$ follow from those for $x = 0$ applied to the new utility function $U_x$. Thus we assume $x = 0$ in what follows.

The next technical lemma is a collection of several arguments in [4].

**Lemma 4.1** ([4]). *Let $(Q, P)$ be a pair of probabilities with $V(Q|P) < \infty$, and $(k^n)_n$ a sequence of random variables such that $E_P[U(k^n)]$ is bounded from below and $E_Q[k^n] \leq 0$ for all $n$. Then*

*(a) $(k^n)_n$ is bounded in $L^1(Q)$;*
*(b) $(U(k^n))_n$ is bounded in $L^1(P)$;*
*(c) If in addition $k^n$ converges a.s. to some $k \in L^0$, we have $k \in L^1(Q)$, $U(k) \in L^1(P)$ and that*

$$E_Q[k] \leq 0 \text{ and } \limsup_n E_P[U(k^n)] \leq E_P[U(k)]. \tag{4.1}$$

*Proof.* We just fill the gap from [4]. As we are assuming the reasonable asymptotic elasticity (A4), assertions (a) and (b) are contained in Proposition 6.3 of [4]. The assertion (c) also appears (implicitly) in the proof of their Theorem 4.10, which we briefly recall here.

Assume $k^n \to k$, $P$-a.s. Since $(k^n)$ (resp. $(U(k^n))$ is bounded in $L^1(Q)$ (resp. $L^1(P)$), Fatou's lemma applied to the sequence $(|k^n|)_n$ (resp. $(|U(k^n)|)_n$) shows that $k \in L^1(Q)$ (resp. $U(k) \in L^1(P)$). By Young's inequality, we have $U(k^n) - \lambda(dQ/dP)k^n \leq V(\lambda dQ/dP) \in L^1(P)$ for all $n \in \mathbb{N}$ and $\lambda > 0$, where the $P$-integrability of the right hand side for all $\lambda$ follows from the reasonable asymptotic elasticity. By this *integrable upper bound* as well as the assumption $E_Q[k^n] \leq 0$, (reverse) Fatou's lemma shows that

$$\begin{aligned}\limsup_n E_P[U(k^n)] &\leq \limsup_n E_P[U(k^n) - \lambda(dQ/dP)k^n] \\ &\leq E_P[U(k) - \lambda(dQ/dP)k] = E_P[U(k)] - \lambda E_Q[k], \ \forall \lambda > 0.\end{aligned}$$

Letting $\lambda \downarrow 0$, we have (4.1), while $E_Q[k] \leq 0$ follows by letting $\lambda \uparrow \infty$. □

*Proof of Theorem 3.1.* We choose a maximizing sequence $(\theta^n)_n \subset \Theta_{bb}$, that is

$$\inf_{P \in \mathcal{P}} E_P[U(\theta^n \cdot S_T)] \nearrow u(0). \tag{4.2}$$

This sequence does not have to converge, thus we appeal to a Komlós type argument. Let $(\bar{Q}, \bar{P}) \in \mathcal{M}_V \times \mathcal{P}$ be such that $\bar{Q} \sim \bar{P} \sim \mathbb{P}$ and $V(\bar{Q}|\bar{P}) < \infty$ which exists by (A5). Since $E_{\bar{P}}[U(\theta^n \cdot S_T)] \geq \inf_{P \in \mathcal{P}} E_P[U(\theta^1 \cdot S_T)]$ and $E_{\bar{Q}}[\theta^n \cdot S_T] \leq 0$ by construction, Lemma 4.1 (a) shows that $(\theta^n \cdot S_T)_n$ is bounded in $L^1(\bar{Q})$. Hence Komlós' theorem (see e.g. [6, Theorem 15.1.3]) yields another sequence $(\tilde{k}^n)_n$ such that

$$\begin{cases} \tilde{k}^n \in \operatorname{conv}(\theta^n \cdot S_T, \theta^{n+1} \cdot S_T, \cdots) \\ \tilde{k}^n \text{ converges } \bar{Q}\text{-a.s. (hence } \mathbb{P}\text{-a.s.) to some } \widehat{X} \in L^1(\bar{Q}). \end{cases}$$

By construction, each $\tilde{k}^n$ is again the terminal value of a stochastic integral $\tilde{\theta}^n \cdot S_T$ where $\tilde{\theta}^n$ is the convex combination of $(\theta^n, \theta^{n+1}, \cdots)$ with the same convex weights as $\tilde{k}^n$, hence $\tilde{\theta}^n \in \Theta_{bb}$ and $E_Q[\tilde{k}^n] \leq 0$ for each $n$ and $Q$, in particular.

Since the robust utility functional $X \mapsto \inf_{P \in \mathcal{P}} E_P[U(X)]$ is concave as a pointwise infimum of concave functionals, we have $\inf_{P \in \mathcal{P}} E_P[U(\tilde{k}^n)] \geq \inf_{P \in \mathcal{P}} E_P[U(\theta^n \cdot S_T)]$ for each $n$. Hence we still have $\lim_n \inf_{P \in \mathcal{P}} E_P[U(\tilde{k}^n)] = u(0)$, and the sequence $(E_P[U(\tilde{k}^n)])_n$ is bounded from below for all $P \in \mathcal{P}$.

If $Q \in \mathcal{M}_V$, there is a $P \in \mathcal{P}$ with $V(Q|P) < \infty$ by the definition of $\mathcal{M}_V$, hence another application of Lemma 4.1 to the sequence $(\tilde{k}^n)$ with the pair $(Q, P)$ shows that $\widehat{X} \in L^1(Q)$ and $E_Q[\widehat{X}] \leq 0$. Hence $\widehat{X} \in \mathcal{X}$.

We next show that $U(\widehat{X}) \in \bigcap_{P \in \mathcal{P}} L^1(P)$ and

$$\limsup_n E_P[U(\tilde{k}^n)] \leq E_P[U(\widehat{X})], \quad \forall P \in \mathcal{P}. \tag{4.3}$$

This is immediate from Fatou's lemma if $U$ is bounded from above. When $U(\infty) = \infty$ and $P \in \mathcal{P}^e$, we can take a $Q \in \mathcal{M}_V$ with $V(Q|P) < \infty$ by (2.12), hence Lemma 4.1 shows (4.3) and that $(U(\tilde{k}^n))_n$ is bounded in $L^1(P)$. Then Remark 2.3 shows that $U(\widehat{X}) \in L^1(P)$ and $(U(\tilde{k}^n))_n$ is still bounded in $L^1(P)$ for arbitrary $P \in \mathcal{P}$ which need not be equivalent to $\mathbb{P}$. To prove (4.3) in the case $P \not\sim \mathbb{P}$, we take $(\bar{Q}, \bar{P})$ as above, and set $P_\alpha := \alpha P + (1-\alpha)\bar{P}$ for $\alpha \in (0, 1)$. Since $P_\alpha \sim \mathbb{P}$, the claim is true for $P_\alpha$ for all $\alpha \in (0, 1)$, while we see that $\sup_n |E_{P_\alpha}[U(\tilde{k}^n)] - E_P[U(\tilde{k}^n)]| \leq 2(1-\alpha)\sup_n(\|U(\tilde{k}^n)\|_{L^1(P)} \vee \|U(\tilde{k}^n)\|_{L^1(\bar{P})}) \to 0$, as $\alpha \uparrow 1$. Thus we deduce

$$\begin{aligned}\limsup_n E_P[U(\tilde{k}^n)] &= \lim_{\alpha\uparrow 1}\limsup_n E_{\alpha P+(1-\alpha)\bar{P}}[U(\tilde{k}^n)]\\ &\leq \lim_{\alpha\uparrow 1} E_{\alpha P+(1-\alpha)\bar{P}}[U(\widehat{X})] = E_P[U(\widehat{X})].\end{aligned}$$

Hence (4.3) holds for all $P \in \mathcal{P}$.

We now prove (3.2). Note first that for all $\lambda > 0$, $X \in \mathcal{X}$, $Q \in \mathcal{M}_V$ and $P \in \mathcal{P}$,

$$E_P[U(X)] \leq V(\lambda Q|P) + \lambda E_Q[X] \leq V(\lambda Q|P).$$

In particular,

$$\inf_{P\in\mathcal{P}} E_P[U(X)] \leq \inf_{\lambda>0}\inf_{(Q,P)\in\mathcal{M}_V} V(\lambda Q|P) \overset{(2.10)}{=} u(0), \quad \forall X \in \mathcal{X},$$

On the other hand, (4.3) shows

$$u(0) = \lim_n \inf_{P\in\mathcal{P}} E_P[U(\tilde{k}^n)] \leq \inf_{P\in\mathcal{P}}\limsup_n E_P[U(\tilde{k}^n)] \leq \inf_{P\in\mathcal{P}} E_P[U(\widehat{X})].$$

This concludes the proof of (3.2).

We proceed to (3.3). Notice that

$$U(\widehat{X}) = V(\hat{\lambda} d\widehat{Q}/d\widehat{P}) + \hat{\lambda}(d\widehat{Q}/d\widehat{P})\widehat{X}, \ \widehat{P}\text{-a.s.} \tag{4.4}$$

Indeed, "$\leq$" is just a Young's inequality, while "$\geq$" follows from

$$\begin{aligned}u(0) = \inf_{P\in\mathcal{P}} E_P[U(\widehat{X})] \leq E_{\widehat{P}}[U(\widehat{X})] &\overset{\text{(i)}}{\leq} E_{\widehat{P}}\left[V\left(\hat{\lambda}\frac{d\widehat{Q}}{d\widehat{P}}\right) + \hat{\lambda}\frac{d\widehat{Q}}{d\widehat{P}}\widehat{X}\right]\\ &\overset{\text{(ii)}}{\leq} V(\hat{\lambda}\widehat{Q}|\widehat{P}) \overset{(2.10)}{=} u(0).\end{aligned}$$

Here (i) follows from the "$\leq$" part, and (ii) from $\widehat{X} \in \mathcal{X}$. In particular, $\widehat{P}$ attains the infimum in (3.2) and we obtain (4.4). But an elementary knowledge from convex analysis shows that this is possible only if

$$\widehat{X} = -V'(\hat{\lambda} d\widehat{Q}/d\widehat{P}), \ \widehat{P}\text{-a.s.}$$

This is the first equality in (3.3), and the $\widehat{Q}$-a.s. uniqueness of $\widehat{X}$ follows from that of $\hat{\lambda} d\widehat{Q}/d\widehat{P}$ (see (2.11)). On the other hand, the existence of $\hat{\theta} \in L(S, \widehat{Q})$ with $\theta_0 = 0$ and $\hat{\theta} \cdot S$ being a $\widehat{Q}$-martingale, which represents $-V'(\hat{\lambda} d\widehat{Q}/d\widehat{P})$ as (3.3), follows from Theorem 3.2 of [11] (see also [20, Theorem 2.2 (iv)]). Finally, $\widehat{Q}$-a.s. uniqueness of the process $\hat{\theta} \cdot S$ follows from the $\widehat{Q}$-a.s. uniqueness of the terminal value $\hat{\theta} \cdot S_T$ and the fact that $\hat{\theta} \cdot S$ is a $\widehat{Q}$-martingale. $\square$

## 5. Uniform Supermartingale Property of Optimal Wealth

We now proceed to the uniform supermartingale property of the optimal wealth, that is, we shall show that $\hat{\theta} \cdot S$ is a supermartingale under all local martingale measures $Q$ with finite entropy w.r.t. some $P \in \mathcal{P}$. As outlined in Section 3, this will follow if we can prove the dynamic variational inequality (3.4) for every $Q \in \mathcal{M}_V$. Therefore, the key of this section is the next proposition which should be compared with [8, Lemma 3.12]. Recall that we have only to consider the case $x = 0$. *In what follows, all the assumptions of Theorem 3.2 are in force, and we do not cite them in each statement.*

**Proposition 5.1.** *We have*

1. *for all $Q \in \mathcal{M}_V$, and for all stopping time $\tau \le T$,*

$$E_Q\left[V'\left(\hat{\lambda} d\widehat{Q}/d\widehat{P}\right) \,\Big|\, \mathcal{F}_\tau\right] \ge E_{\widehat{Q}}\left[V'\left(\hat{\lambda} d\widehat{Q}/d\widehat{P}\right) \,\Big|\, \mathcal{F}_\tau\right], \; Q\text{-a.s.} \tag{5.1}$$

2. *for all $P \in \mathcal{P}$, and for all stopping time $\tau \le T$,*

$$E_P[U(\hat{\theta} \cdot S_T)|\mathcal{F}_\tau] \ge E_{\widehat{P}}[U(\hat{\theta} \cdot S_T)|\mathcal{F}_\tau], \; P\text{-a.s.} \tag{5.2}$$

We introduce some notations. If $L$ is a strictly positive martingale, we denote $L_{\tau,T} := L_T/L_\tau$, for any stopping time $\tau \le T$. Recall that any probability $Q \ll \mathbb{P}$ is identified with a (uniformly integrable) martingale, namely the *density process* $Z^Q_\cdot = (dQ/d\mathbb{P})|_{\mathcal{F}_\cdot}$. In what follows, we denote by $\widehat{Z}$ (resp. $\widehat{D}$) the density process of $\widehat{Q}$ (resp. $\widehat{P}$). Also, when a pair $(Q, P) \in \mathcal{M}_{loc} \times \mathcal{P}$ is fixed, the density process of $Q$ (resp. $P$) is denoted by $Z$ (resp. $D$), and set:

$$Z^\alpha_{\tau,T} := \alpha Z_{\tau,T} + (1-\alpha)\widehat{Z}_{\tau,T}, \quad D^\alpha_{\tau,T} := \alpha D_{\tau,T} + (1-\alpha)\widehat{D}_{\tau,T}, \quad \alpha \in [0,1]. \tag{5.3}$$

We make a couple of simple reductions. The first one is just a notational reduction. In our purpose, we can assume without loss of generality that $\hat{\lambda} = 1$ *since we already know* $\hat{\lambda}$. Indeed, $(\hat{\lambda}\widehat{Q}, \widehat{P})$ minimizes $(\nu, P) \mapsto V(\nu|P)$ if and only if $(\widehat{Q}, \widehat{P})$ minimizes $(\nu, P) \mapsto V_{\hat{\lambda}}(\nu|P) := \frac{1}{\hat{\lambda}} V(\hat{\lambda}\nu|P)$. Next, we have only to prove (5.1) and (5.2) for all $Q \in \mathcal{M}^e_V$ and $P \in \mathcal{P}^e$, respectively. Indeed, if we could show (5.1) for all $Q' \in \mathcal{M}^e_V$ for instance, we have $\bar{Q} := (Q + \widehat{Q})/2 \in \mathcal{M}^e_V$ for any $Q \in \mathcal{M}_V$ on the one hand, and on the other hand, Bayes' formula implies

$$\begin{aligned} E_{\widehat{Q}}[\Phi|\mathcal{F}_\tau] &\le E_{\bar{Q}}[\Phi|\mathcal{F}_\tau] \\ &= \frac{Z_\tau}{Z_\tau + \widehat{Z}_\tau} E_Q[\Phi|\mathcal{F}_\tau] + \frac{\widehat{Z}_\tau}{Z_\tau + \widehat{Z}_\tau} E_{\widehat{Q}}[\Phi|\mathcal{F}_\tau] \text{ a.s. on } \{Z_\tau > 0\} \end{aligned}$$

where $\Phi = V'(d\widehat{Q}/d\widehat{P})$, hence (5.1). A similar argument applies also to (5.2).

The first step is to show a "Bellman-type" principle for a *time-consistent* optimization. Note that the set $\mathcal{M}_{loc}$ of *all* local martingale measures is m-stable, while $\mathcal{M}_V$ is not. The next simple lemma allows us to avoid this difficulty.

**Lemma 5.2.** *Let $(Q, P) \in \mathcal{M}^e_V \times \mathcal{P}^e$ with $V(Q|P) < \infty$, and $(Z, D)$ the corresponding density processes as well as $\alpha \in [0.1]$. Then for any stopping time $\tau \le T$, the random variable $\widehat{D}_\tau D^\alpha_{\tau,T} V\left(\frac{\widehat{Z}_\tau Z^\alpha_{\tau,T}}{\widehat{D}_\tau D^\alpha_{\tau,T}}\right)$ is $\mathcal{F}_\tau$-locally integrable i.e., there exists an increasing sequence $A_n \in \mathcal{F}_\tau$ such that*

$$\mathbb{P}(A_n) \nearrow 1 \quad \text{and} \quad 1_{A_n} \widehat{D}_\tau D^\alpha_{\tau,T} V\left(\frac{\widehat{Z}_\tau Z^\alpha_{\tau,T}}{\widehat{D}_\tau D^\alpha_{\tau,T}}\right) \in L^1, \; \forall n. \tag{5.4}$$

*Proof.* Since $\widehat{D}_\tau D^\alpha_{\tau,T} V\left(\frac{\widehat{Z}_\tau Z^\alpha_{\tau,T}}{\widehat{D}_\tau D^\alpha_{\tau,T}}\right) \leq \alpha \widehat{D}_\tau D_{\tau,T} V\left(\frac{\widehat{Z}_\tau Z_{\tau,T}}{\widehat{D}_\tau D_{\tau,T}}\right) + (1-\alpha)\widehat{D}_T V\left(\frac{\widehat{Z}_T}{\widehat{D}_T}\right)$ (see the proof of Lemma 5.5 below), and the second term is integrable, it suffices to prove the case $\alpha = 1$.

Recall from [10] that the condition (A4) of reasonable asymptotic elasticity is equivalent to: for any $a \geq 1$, there exists $C_a, C'_a > 0$ such that

$$V(\lambda y) \leq C_a V(y) + C'_a(y+1), \quad \forall \lambda \in [a^{-1}, a],\ \forall y > 0. \tag{5.5}$$

Since $V$ is bounded from below by $U(0)$, we can choose the constant $C'_a$ so that the right hand side is always positive. For the sequence $A_n$, we take

$$A_n := \{\widehat{Z}_\tau, Z_\tau, \widehat{D}_\tau, D_\tau \in (n^{-1}, n)\} \in \mathcal{F}_\tau, \quad \forall n.$$

Noting that $\varphi := \widehat{D}_\tau D_{\tau,T} V\left(\frac{\widehat{Z}_\tau Z_{\tau,T}}{\widehat{D}_\tau D_{\tau,T}}\right) = \frac{\widehat{D}_\tau}{D_\tau} D_T V\left(\frac{\widehat{Z}_\tau D_\tau}{\widehat{D}_\tau Z_\tau}\frac{Z_T}{D_T}\right)$, (5.5) implies that

$$\varphi \leq n^2 C_{n^4} D_T V(Z_T/D_T) + n^2 C'_{n^4}(Z_T + D_T) \text{ a.s. on } A_n.$$

Thus $1_{A_n}\varphi \in L^1$ for each $n$. Finally, $\mathbb{P}(A_n) \nearrow 1$ since $\widehat{Q} \sim \widehat{P} \sim Q \sim P \sim \mathbb{P}$ by assumption. □

**Lemma 5.3.** *For any $(Q, P) \simeq (Z, D) \in \mathcal{M}^e_V \times \mathcal{P}^e$ with $V(Q|P) < \infty$, $\alpha \in [0,1]$,*

$$E\left[\widehat{Z}_T V\left(\frac{\widehat{Z}_T}{\widehat{D}_T}\right) \,\Big|\, \mathcal{F}_\tau\right] \leq E\left[\widehat{D}_\tau D^\alpha_{\tau,T} V\left(\frac{\widehat{Z}_\tau Z^\alpha_{\tau,T}}{\widehat{D}_\tau D^\alpha_{\tau,T}}\right) \,\Big|\, \mathcal{F}_\tau\right] \text{ a.s.} \tag{5.6}$$

*Proof.* Note first that the conditional expectation of the right hand side is well-defined and a.s. finite by Lemma 5.2. Let $C'$ be the set on which the inequality (5.6) fails, which is $\mathcal{F}_\tau$-measurable. Then we suppose by way of contradiction that $\mathbb{P}(C') > 0$.

Take a sequence $(A_n) \subset \mathcal{F}_\tau$ as in Lemma 5.2 and a large $n$ so that $\mathbb{P}(C' \cap A_n) > 0$. Setting $C := C' \cap A_n$, we define a new pair $(\bar{Q}, \bar{P}) \simeq (\bar{Z}, \bar{D})$ by

$$\bar{Z}_T = 1_{C^c}\widehat{Z}_T + 1_C \widehat{Z}_\tau Z^\alpha_{\tau,T} \text{ and } \bar{D}_T = 1_{C^c}\widehat{D}_T + 1_C \widehat{D}_\tau D^\alpha_{\tau,T}.$$

First, $(\bar{Q}, \bar{P}) \in \mathcal{M}_{loc} \times \mathcal{P}$ by the m-stability of $\mathcal{M}_{loc}$ and $\mathcal{P}$. Also, since

$$\bar{D}_T V\left(\frac{\bar{Z}_T}{\bar{D}_T}\right) = 1_{C^c}\widehat{D}_T V\left(\frac{\widehat{Z}_T}{\widehat{D}_T}\right) + 1_C \widehat{D}_\tau D^\alpha_{\tau,T} V\left(\frac{\widehat{Z}_\tau Z^\alpha_{\tau,T}}{\widehat{D}_\tau D^\alpha_{\tau,T}}\right),$$

we have $V(\bar{Q}|\bar{P}) < \infty$ by the construction of $C$ and Lemma 5.2, hence $\bar{Q} \in \mathcal{M}_V$. Finally,

$$\begin{aligned} V(\bar{Q}|\bar{P}) &= E\left[\bar{D}_T V\left(\frac{\bar{Z}_T}{\bar{D}_T}\right)\right] \\ &= E\left[1_{C^c} E\left[\widehat{D}_T V\left(\frac{\widehat{Z}_T}{\widehat{D}_T}\right) \,\Big|\, \mathcal{F}_\tau\right] + 1_C E\left[\widehat{D}_\tau D^\alpha_{\tau,T} V\left(\frac{\widehat{Z}_\tau Z^\alpha_{\tau,T}}{\widehat{D}_\tau D^\alpha_{\tau,T}}\right) \,\Big|\, \mathcal{F}_\tau\right]\right] \\ &< V(\widehat{Q}|\widehat{P}). \end{aligned}$$

This contradict to the optimality of $(\widehat{Q}, \widehat{P})$. □

Now the formal inequality in **Step 1** at the end of Section 3 is realized as follows.

**Proposition 5.4.** *For any $(Q, P) \simeq (Z, D) \in \mathcal{M}^e_V \times \mathcal{P}^e$ with $V(Q|P) < \infty$,*

$$\begin{aligned} &\widehat{Z}_\tau \left\{E_Q\left[V'\left(d\widehat{Q}/d\widehat{P}\right) \,\Big|\, \mathcal{F}_\tau\right] - E_{\widehat{Q}}\left[V'\left(d\widehat{Q}/d\widehat{P}\right) \,\Big|\, \mathcal{F}_\tau\right]\right\} \\ &\quad + \widehat{D}_\tau \left\{E_P[U(\widehat{X})|\mathcal{F}_\tau] - E_{\widehat{P}}[U(\widehat{X})|\mathcal{F}_\tau]\right\} \geq 0, \text{ a.s.} \end{aligned} \tag{5.7}$$

*Proof.* Let $(Z, D)$, $\tau$, $\alpha$ be as above, and set

$$G_\tau(\alpha) := \widehat{D}_\tau D^\alpha_{\tau,T} V(\widehat{Z}_\tau Z^\alpha_{\tau,T} / \widehat{D}_\tau D^\alpha_{\tau,T}).$$

Then $\alpha \mapsto G_\tau(\alpha)$ is convex (a.s.) by (the proof of) Lemma 5.5 below, hence $(G_\tau(\alpha) - G(0))/\alpha$ decreases a.s. to the limit $\Xi_\tau(Q, P)$ as $\alpha \searrow 0$. Here $\Xi_\tau(Q, P)$ is explicitly computed as:

$$\Xi_\tau(Q, P) = \widehat{Z}_\tau V'\left(\frac{d\widehat{Q}}{d\widehat{P}}\right)(Z_{\tau,T} - \widehat{Z}_{\tau,T}) + \widehat{D}_\tau U(\widehat{X})(D_{\tau,T} - \widehat{D}_{\tau,T}),$$

using $\widehat{Z}_T/\widehat{D}_T = d\widehat{Q}/d\widehat{P}$ and $U(\widehat{X}) = V(d\widehat{Q}/d\widehat{P}) - (d\widehat{Q}/d\widehat{P})V'(d\widehat{Q}/d\widehat{P})$. Since $G_\tau(1)$ is $\mathcal{F}_\tau$-locally integrable and $E[(G_\tau(\alpha) - G_\tau(0))/\alpha|\mathcal{F}_\tau] \geq 0$ a.s. by Lemma 5.3, the (generalized) conditional monotone convergence theorem shows that $E[\Xi(Q, P)|\mathcal{F}_\tau] \geq 0$. Noting that $V'(d\widehat{Q}/d\widehat{P}) = -\widehat{X} \in L^1(Q)$ and $U(\widehat{X}) \in L^1(P)$ by Theorem 3.1, we deduce (5.7) from Bayes' formula. □

We proceed to **Step 2**. Fixing $Q \in \mathcal{M}_V$, we want to take $P$ "arbitrarily close" to $\widehat{P}$. The next simple lemma gives a precise form of this argument.

**Lemma 5.5.** *Let $(Q, P)$ and $(Q', P')$ be any two pairs of probability measures absolutely continuous w.r.t. $\mathbb{P}$. Then for any $\alpha, \gamma \in (0, 1)$, we have*

$$\begin{aligned} V(\alpha Q + &(1-\alpha)Q' | \gamma P + (1-\gamma)P') \\ &\leq \gamma V\left(\frac{\alpha}{\gamma} Q \,\Big|\, P\right) + (1-\gamma) V\left(\frac{1-\alpha}{1-\gamma} Q' \,\Big|\, P'\right). \end{aligned} \tag{5.8}$$

*In particular, $V(Q|P) < \infty$ and $V(Q'|P') < \infty$ imply $V(\alpha Q + (1-\alpha)Q'|\gamma P + (1-\gamma)Q') < \infty$ for any $\alpha, \gamma \in (0, 1)$.*

*Proof.* Note that for any positive numbers $x, x', y, y'$,

$$\frac{\alpha x + (1-\alpha)x'}{\gamma y + (1-\gamma)y'} = \frac{\gamma y}{\gamma y + (1-\gamma)y'} \frac{\alpha}{\gamma} \frac{x}{y} + \frac{(1-\gamma)y'}{\gamma y + (1-\gamma)y'} \frac{1-\alpha}{1-\gamma} \frac{x'}{y'}.$$

Thus the convexity of $V$ shows that

$$\begin{aligned} (\gamma y + (1-\gamma)y') V\left(\frac{\alpha x + (1-\alpha)x'}{\gamma y + (1-\gamma)y'}\right) \\ \leq \gamma y V\left(\frac{\alpha}{\gamma} \frac{x}{y}\right) + (1-\gamma)y' V\left(\frac{1-\alpha}{1-\gamma} \frac{x'}{y'}\right). \end{aligned}$$

Putting $dQ/d\mathbb{P}$ (resp. $dQ'/d\mathbb{P}$, $dP/d\mathbb{P}$, $dP'/d\mathbb{P}$) into $x$ (resp. $x'$, $y$, $y'$), and taking the $\mathbb{P}$-expectation, this implies (5.8). The second claim follows from the fact that $V(Q|P) < \infty \Rightarrow V(\lambda Q|P) < \infty$ for any $\lambda > 0$, as a consequence of reasonable asymptotic elasticity. □

*Proof of Proposition 5.1.* As noted after the statement of Proposition 5.1, we have only to consider the case $(Q, P) \in \mathcal{M}^e_V \times \mathcal{P}^e$ with $V(Q|P) < \infty$. Fixing such a pair $(Q, P)$, we put $Q_\alpha := \alpha Q + (1-\alpha)\widehat{Q}$ and $P_\gamma := \gamma P + (1-\gamma)\widehat{P}$ for any $\alpha, \gamma \in (0, 1)$. By Lemma 5.5, the auxiliary variational inequality (5.7) is valid for any $(Q_\alpha, P_\gamma)$ with arbitrary $\alpha, \gamma \in (0, 1)$. Noting that $E_{Q_\alpha}[\Phi|\mathcal{F}_\tau] - E_{\widehat{Q}}[\Phi|\mathcal{F}_\tau] = \frac{\alpha Z_\tau}{\alpha Z_\tau + (1-\alpha)\widehat{Z}_\tau}\{E_Q[\Phi|\mathcal{F}_\tau] - E_{\widehat{Q}}[\Phi|\mathcal{F}_\tau]\}$

etc, we have

$$\begin{aligned}
&\widehat{Z}_\tau \frac{\alpha Z_\tau}{\alpha Z_\tau + (1-\alpha)\widehat{Z}_\tau}\{E_Q[V'(d\widehat{Q}/d\widehat{P})|\mathcal{F}_\tau] - E_{\widehat{Q}}[V'(d\widehat{Q}/d\widehat{P})|\mathcal{F}_\tau]\} \\
&\quad + \widehat{D}_\tau \frac{\gamma D_\tau}{\gamma D_\tau + (1-\gamma)\widehat{D}_\tau}\{E_P[U(\widehat{X})|\mathcal{F}_\tau] - E_{\widehat{P}}[U(\widehat{X})|\mathcal{F}_\tau]\} \\
&\quad \geq 0, \text{a.s. } \forall \alpha, \gamma \in (0,1).
\end{aligned}$$

Since $\gamma D_\tau/(\gamma D_\tau + (1-\gamma)\widehat{D}_\tau) \overset{\gamma\downarrow 0}{\to} 0$ and $\alpha Z_\tau/(\alpha Z_\tau + (1-\alpha)\widehat{Z}_\tau) \overset{\alpha\downarrow 0}{\to} 0$, we deduce (5.1) and (5.2) by letting $\gamma \downarrow 0$ (resp. $\alpha \downarrow 0$) with $\alpha$ (resp. $\gamma$) being fixed, whenever $V(Q|P) < \infty$. Finally, any $Q \in \mathcal{M}_V^e$ (resp. $P \in \mathcal{P}^e$) admits a $P \in \mathcal{P}$ (resp. $Q \in \mathcal{M}_V$) with $V(Q|P) < \infty$ by definition (resp. by Remark 2.3). □

*Proof of Theorem 3.2.* Under the assumption $\widehat{Q} \sim \mathbb{P}$, the $(S, \mathbb{P})$-integrability of $\hat{\theta}$ is clear. We verify that $\hat{\theta} \cdot S$ is a supermartingale under each $Q \in \mathcal{M}_V$. Since $V'(d\widehat{Q}/d\widehat{P}) \in L^1(Q)$, the process defined by $M_\tau^Q = -E_Q[V'(d\widehat{Q}/d\widehat{P})|\mathcal{F}_\tau]$ is a $Q$-martingale. Then (3.3), (5.1) as well as the fact that $\hat{\theta} \cdot S$ is a $\widehat{Q}$-martingale show that

$$\hat{\theta} \cdot S_\tau = -E_{\widehat{Q}}[V'(d\widehat{Q}/d\widehat{P})|\mathcal{F}_\tau] \geq M_\tau^Q, \; Q\text{-a.s.}$$

for any stopping time $\tau \leq T$. A stochastic integral w.r.t. a $Q$-local martingale dominated below by a $Q$-(uniformly integrable) martingale is a $Q$-supermartingale by [24, Theorem 1], which is a variant of Ansel-Stricker's lemma [1, Proposition 3.3]. □

## Acknowledgements

Part of this research was carried out during the author's visit to ETH Zürich. He warmly thanks Prof. Martin Schweizer for a number of valuable suggestions and his hospitality. The author gratefully acknowledges the financial support from The Norinchukin Bank and the Global COE program "The research and training center for new development in mathematics".

## References

[1] Ansel, J.-P., and C. Stricker (1994): Couverture des actifs contingents et prix maximum. *Ann. Inst. H. Poincaré Probab. Statist.* **30**, 303–315.
[2] Bellini, F., and M. Frittelli (2002): On the existence of minimax martingale measures. *Math. Finance* **12**, 1–21.
[3] Biagini, S., and M. Frittelli (2007): The supermartingale property of the optimal wealth process for general semimartingales. *Finance Stoch.* **11**, 253–266.
[4] Biagini, S., and A. Černý (2011): Admissible strategies in semimartingale portfolio selection. *SIAM J. Control Optim.* **49**, 42–72.
[5] Delbaen, F. (2006): The structure of m-stable sets and in particular of the set of risk neutral measures. In: In memoriam Paul-André Meyer: Séminaire de Probabilités XXXIX, *Lecture Notes in Math.*, vol. 1874, pp. 215–258. Springer, Berlin.
[6] Delbaen, F., and W. Schachermayer (2006): The mathematics of arbitrage. Springer Finance. Springer-Verlag, Berlin.
[7] Delbaen, F., P. Grandits, T. Rheinländer, D. Samperi, M. Schweizer, and C. Stricker (2002): Exponential hedging and entropic penalties. *Math. Finance* **12**, 99–123.
[8] Föllmer, H., and A. Gundel (2006): Robust projections in the class of martingale measures. *Illinois J. Math.* **50**, 439–472.

[9] Föllmer, H., A. Schied, and S. Weber (2009): Robust preferences and robust portfolio choice. In: A. Bensoussan, Q. Zhang, P.G. Ciarlet (eds.) *Mathematical Modelling and Numerical Methods in Finance,* Handbook of Numerical Analysis*, vol. 15*, pp. 29–88. North-Holland.

[10] Frittelli, M., and E. Rosazza Gianin (2004): Equivalent formulations of reasonable asymptotic elasticity. Tech. Rep. 12, Dept. Matematica per le Decisioni, University of Florence.

[11] Goll, T., and L. Rüschendorf (2001): Minimax and minimal distance martingale measures and their relationship to portfolio optimization. *Finance Stoch.* **5**, 557–581.

[12] Jacod, J. (1980): Intégrales stochastiques par rapport à une semimartingale vectorielle et changements de filtration. In: Séminaire de Probabilités, XIV, *Lecture Notes in Math.*, vol. 784, pp. 161–172. Springer, Berlin.

[13] Jacod, J., and A. N. Shiryaev (2003): Limit Theorems for Stochastic Processes. *Grundlehren Math. Wiss.*, vol. 288. Springer-Verlag, 2nd edn.

[14] Kabanov, Y. M., and C. Stricker (2002): On the optimal portfolio for the exponential utility maximization: remarks to the six-author paper. *Math. Finance* **12**, 125–134.

[15] Kramkov, D., and W. Schachermayer (1999): The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Ann. Appl. Probab.* **9**, 904–950.

[16] Kramkov, D., and W. Schachermayer (2003): Necessary and sufficient conditions in the problem of optimal investment in incomplete markets. *Ann. Appl. Probab.* **13**, 1504–1516.

[17] Owari, K. (2011): Robust utility maximization with unbounded random endowment. *Adv. Math. Econ.* **14**, 147–181.

[18] Owari, K. (2011): Duality in robust utility maximization with unbounded claim via a robust extenstion of Rockafellar's theorem. Preprint, arXiv:1101.2968v1.

[19] Owari, K. (2011): A note on utility maximization with unbounded random endowment. *Asia-Pacific Financial Markets* **18**, 89–103.

[20] Schachermayer, W. (2001): Optimal investment in incomplete markets when wealth may become negative. *Ann. Appl. Probab.* **11**, 694–734.

[21] Schachermayer, W. (2003): A super-martingale property of the optimal portfolio process. *Finance Stoch.* **7**, 433–456.

[22] Schied, A. (2007): Optimal investments for risk- and ambiguity-averse preferences: a duality approach. *Finance Stoch.* **11**, 107–129.

[23] Schied, A., and C.-T. Wu (2005): Duality theory for optimal investments under model uncertainty. *Statist. Decisions* **23**, 199–217.

[24] Strasser, E. (2003): Necessary and sufficient conditions for the supermartingale property of a stochastic integral with respect to a local martingale. In: Séminaire de Probabilités XXXVII, *Lecture Notes in Math.*, vol. 1832, pp. 385–393. Springer, Berlin.