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CARF-F-277

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March 2012

CARF is presently supported by Bank of Tokyo-Mitsubishi UFJ, Ltd., Citigroup, Dai-ichi Mutual Life Insurance Company, Meiji Yasuda Life Insurance Company, Nippon Life Insurance Company, Nomura Holdings, Inc. and Sumitomo Mitsui Banking Corporation (in alphabetical order). This financial support enables us to issue CARF Working Papers.

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APPLICATION OF THE KUSUOKA APPROXIMATION TO BARRIER OPTIONS

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Abstract. The authors focuses on numerical experiments of application of the Kusuoka approximation to pricing barrier options which is one of the problems with a boundary condition. The killing functions play a role of giving probability of hitting the boundary. The numerical experiments show that second-order approximation is achieved as done in pricing European style options ([3][4]).

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0. Introduction

The Kusuoka approximation is known as one of the higher-order weak approximation schemes for stochastic differential equations (SDEs). Weak approximation is calculation of $E\left[f\left(X(T,x)\right)\right]$ for a diffusion process $X(t,x)$. Since the only condition imposed to f in the Kusuoka approximation is Lipschitz continuity([2]), this scheme is directly applicable in dealing with European style options. The algorithms for this scheme were developed by Ninomiya–Victoir(NV) and Ninomiya–Ninomiya(NN) in [3] and [4] respectively to attain second-order approximation.

Our interest here is in applying this scheme to some other types of problems. In this paper, we consider pricing barrier options. A barrier option is one of the popular derivatives in finance. In this paper, we focus on pricing the barrier option whose price can be written as $E\left[g\left(T,x\right),\min_{0\le t\le T}X(t,x)>0\right]$ for an N-dimensional diffusion process such that

$$X(t,x)=x+\int_0^t V_0(X(s,x))\mathrm{d}s+\int_0^t V_1(X(s,x))\circ\mathrm{d}B(s), \tag{0.1}$$

where $\circ$ denotes the Stratonovich integral, $B(t)$ is a standard Brownian motion, and $V_0,V_1\in C_b^\infty(\mathbb{R}^N;\mathbb{R}^N)$. If we let f be a function such that $E\left[f\left(T,x\right)\right]=E\left[g\left(X(T,x)\right),\min_{0\le t\le T}X(t,x)>0\right]$, then f does not satisfy the Lipschitz condition imposed to f in the Kusuoka approximation.

Kusuoka suggested to separate the problem to two parts. One is approximation of $g\left(X(T,x)\right)$. The other is boundary condition. For approximation of $g\left(X(T,x)\right)$, the NV or the NN algorithm are directly applicable. For boundary condition, we apply a killing function which represents probability of hitting boundary $X(t,x)\le 0$

Date: March 20, 2012.

Key words and phrases. weak approximation, stochastic differential equations, numerical methods, mathematical finance, boundary conditions, barrier options.

for $t \in [0,T]$. Two types of killing functions are used in this research. One is called a "simple" killing function which simply takes 1 if $X(t,x) \leq 0$ and 0 otherwise. The other killing function, "standard" killing function, takes it into account that $X(t,x) > 0$ and $X(s,x) \leq 0$ for $s \in (t-1,t)$. The definitions of these functions are given in Section 2.

1. Problems and settings

Definitions and notations here follow [3].

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $B(t)$ a standard Brownian Motion. $C_b^\infty(\mathbb{R}^N;\mathbb{R}^N)$ denotes the set of $\mathbb{R}^N$-valued infinitely differentiable functions defined in $\mathbb{R}^N$ whose derivatives are all bounded.

Our interest is in approximation of $E\left[g(X(T,x)), \min_{0\leq t\leq T} X(t,x) > 0\right]$ where $g \in C_b^\infty(\mathbb{R}^N;\mathbb{R})$ and $X(t,x)$ is a solution to the Stratonovich stochastic integral equation

$$X(t,x) = x + \int_0^t V_0(X(s,x)) \circ \mathrm{d}s + \int_0^t V_1\left(X(s,x)\right) \circ \mathrm{d}B(s), \tag{1.1}$$

where $V_0, V_1 \in C_b^\infty(\mathbb{R}^N;\mathbb{R}^N)$. $V_i \in C_b^\infty\left(\mathbb{R}^N;\mathbb{R}^N\right)$, $i = 0,1$, is regarded as a vector field in the following way : for $f \in C_b^\infty(\mathbb{R}^N;\mathbb{R})$,

$$V_i f(x) = \sum_{j=1}^N V_i^j(x) \frac{\partial f}{\partial x_j}(x).$$

We let $T = 1$ in this paper.

1.1. Weak approximation schemes.

We here focus on the approximation of $g(1,x)$ in $E\left[g\left(X(1,x)\right), \min_{0\leq t\leq 1} X(t,x) > 0\right]$.

To avoid confusion, we suppose a general type of SDEs in this section:

$$X^j(t,x) = x_j + \sum_{i=0}^d \int_0^t V_i^j\left(X(s,x)\right) \circ dB^i(s) \tag{1.2}$$

for $j = 1,\ldots,N$, where $V_i \in C_b^\infty(\mathbb{R}^N;\mathbb{R}^N)$, $B^0(t) = t$ and $(B^1(t),\ldots,B^d(t))$ is a d-dimensional standard Brownian motion. Let L be a differential operator defined by

$$L = V_0 + \frac{1}{2}\sum_{k=1}^d V_i^2.$$

In this paper, we take three candidates of algorithms for approximation of $g\left(X(1,x)\right)$. One is the Euler–Maruyama scheme (EM) which is one of the most popular first-order weak approximation schemes. The other two are NV and NN algorithms for the Kusuoka approximation.

The EM scheme is widely adopted in finance, because of its simplicity and ease of implementation. The following random variables are to be constructed in this

scheme:

$$
\begin{aligned}
&X_0^{(\mathrm{EM}),n} = x,\\
&X_{(k+1)/n}^{(\mathrm{EM}),n} = X_{k/n}^{(\mathrm{EM}),n} + \frac{1}{n}\tilde{V}_0\left(X_{k/n}^{(\mathrm{EM}),n}\right) + \frac{1}{\sqrt{n}}\sum_{i=1}^{d} V_i\left(X_{k/n}^{(\mathrm{EM}),n}\right) Z_{k+1}^i
\end{aligned}
\tag{1.3}
$$

where $\tilde{V}_0$ denotes the conversion of V_0 to a drift in an Ito-form SDE. $Z_1, Z_2, \cdots, Z_n$ are n independent d-dimensional random variables distributed as $N(0,1)$.

Notation 1.1. $\exp(V)x$ *denotes the solution at time 1 of the ODE*

$$\frac{\mathrm{d}z_t}{\mathrm{d}t} = V(z_t), \quad z_0 = x.$$

The NV algorithm can be represented by the following family of random variables $\left\{X_{i/n}^{(\mathrm{NV}),n}\right\}_{i=0}^{n}$:

(1.4)

$$
\begin{aligned}
&X_0^{(\mathrm{NV}),n} := x\\
&X_{(k+1)/n}^{(\mathrm{NV}),n} :=\\
&\begin{cases}
\exp\left(\frac{V_0}{2n}\right)\exp\left(\frac{Z_k^1 V_1}{\sqrt{n}}\right)\exp\left(\frac{Z_k^2 V_2}{\sqrt{n}}\right)\cdots\exp\left(\frac{Z_k^d V_d}{\sqrt{n}}\right)\exp\left(\frac{V_0}{2n}\right) X_{k/n}^{(\mathrm{NV}),n}, & \text{if } \Lambda_k = +1\\
\exp\left(\frac{V_0}{2n}\right)\exp\left(\frac{Z_k^d V_d}{\sqrt{n}}\right)\exp\left(\frac{Z_k^{d-1} V_{d-1}}{\sqrt{n}}\right)\cdots\exp\left(\frac{Z_k^1 V_1}{\sqrt{n}}\right)\exp\left(\frac{V_0}{2n}\right) X_{k/n}^{(\mathrm{NV}),n}, & \text{if } \Lambda_k = -1,
\end{cases}
\end{aligned}
$$

where $(\Lambda_i, Z_i)_{i\in\{0,\ldots,n-1\}}$ are n-independent random variables such that Λ_i is a Bernoulli random variable independent of Z_i which is a standard d-dimensional normal random variable.

Remark 1.1. *In the NV algorithm, only when there does not exist the exact solution to an ODE*

$$\frac{\mathrm{d}z_t}{\mathrm{d}t} = Z_k^i V_i(z_t), \quad z_0 = y,$$

the Runge–Kutta method is applied to approximation of $\exp\left(Z_k^i V_i\right) y$*. Therefore, a great advantage of the NV algorithm is that there can be opportunities to avoid calculation for approximation of each* $\exp\left(Z_k^i V_i\right) y$*.*

The NN algorithm is defined by the following family of random variables $\left\{X_{i/n}^{(\mathrm{NN}),n}\right\}_{i=0}^{n}$:

$$
\begin{aligned}
&X_0^{(\mathrm{NN}),n} := x,\\
&X_{(k+1)/n}^{(\mathrm{NN}),n} := \exp\left(\frac{1}{2n}V_0 + \sum_{i=1}^{d}\frac{S_{1,k}^i}{\sqrt{n}}V_i\right)\exp\left(\frac{1}{2n}V_0 + \sum_{i=1}^{d}\frac{S_{2,k}^i}{\sqrt{n}}V_i\right) X_{k/n}^{(\mathrm{NN}),n}
\end{aligned}
\tag{1.5}
$$

where $\left(S_{j,k}^i\right)_{\substack{i\in\{1,\ldots,d\},j\in\{1,2\}\\ k\in\{0,\ldots,n-1\}}}$ are constructed by

$$
\begin{pmatrix} S_{1,k}^i \\ S_{2,k}^i \end{pmatrix} = \begin{pmatrix} 1/2 & 1/\sqrt{2} \\ 1/2 & -1/\sqrt{2} \end{pmatrix}\begin{pmatrix} \eta_{1,k}^i \\ \eta_{2,k}^i \end{pmatrix}, \quad \text{where } \eta_{j,k}^i \overset{i.i.d.}{\sim} N(0,1).
\tag{1.6}
$$

Remark 1.2. *In the NN algorithm, the ODE*

$$\frac{\mathrm{d}z_t}{\mathrm{d}t} = \frac{1}{2n}V_0(z_t) + \sum_{i=1}^{d}\frac{S_{2,k}^i}{\sqrt{n}}V_i(z_t), \quad z_0 = y \tag{1.7}$$

is unlikely to have an exact solution because of its complexity. A wider range including (1.7) *of application of the original Runge–Kutta method is indicated in* [3].

Both algorithms for the Kusuoka approximation attain second-order approximation as mentioned above.

2. Operator for approximation

Notation 2.1. *Let*

$$(P_t g)(x) = E\left[g(X(t,x)), \min_{0\le s\le t} X(s,x) > 0\right]. \tag{2.1}$$

For simplification of notation, we let $F : (0,T] \times \mathbb{R}^N \times \mathbb{R}^{D(d)} \to \mathbb{R}^N$ *be a function determined by each algorithm as*

$$\begin{aligned} X_0^{(\mathrm{alg}),n} &= x, \\ X_{t_k}^{(\mathrm{alg}),n} &= F\left(t_k, X_{t_{k-1}}^{(\mathrm{alg}),n}, S(t_k)\right), \end{aligned}$$

where $D(d)$ *is the number of random variables included in one-step calculation of each algorithm.*

$$D(d) = \begin{cases} d & \text{if } EM \\ d+1 & \text{if } NV \\ 2d & \text{if } NN \end{cases} \tag{2.2}$$

and $S(t_k)$ *denotes a set of* $D(d)$ *random numbers needed to obtain* $X_{t_k}^{(alg),n}$ *from* $X_{t_{k-1}}^{(alg,n)}$.

A "killing function " is constructed for representation of the probability density of B(t)'s hitting the boundary.

Definition 2.1. *Let* $k : (0,1] \times [0,\infty) \times \mathbb{R} \to [0,1]$ *be a measurable function.*

- *If*

$$k(s,x,y) = \begin{cases} 0, & \text{if } y > 0 \\ 1, & \text{if } y \le 0, \end{cases}$$

  then k *is called a simple killing function.*
- *If*

$$k(s,x,y) = \begin{cases} \exp -\frac{2xy}{s}, & \text{if } y > 0 \\ 1, & \text{if } y \le 0, \end{cases}$$

  then k *is called a standard killing function.*

Then we approximate $(P_t g)(x)$ by a sub-Markov operator $\left(Q_{(s)} g\right)(x)$ defined by

$$\left(Q_{(s)} g\right)(x) = E\left[g\left(F(s,x,S(s))\right)\left(1 - k\left(s,x,F(s,x,S(s))\right)\right)\right]. \tag{2.3}$$

3. Numerical experiments

We deal with the following two types of combinations of diffusion processes and functions for g :

(1) linear drift :

$$X(t,x) = x + \int_0^t (1 + X(s,x)) \circ \mathrm{d}s + \int_0^t \circ \mathrm{d}B(s).$$

For this process, we take g as a digital call option defined as

$$g(y) = \begin{cases} 1 & \text{if } K < y \\ 0 & \text{otherwise} \end{cases}$$

where K is a strike price.

(2) cosine drift :

$$X(t,x) = x + \int_0^t \left(\cos\left(X(s,x)\right) + \frac{3}{2} \right) \circ \mathrm{d}s + \int_0^t \circ \mathrm{d}B(s).$$

For this process, we take g as a European call option defined as

$$g(y) \max\{0, K - y\}$$

where K is a strike price.

Our interest is in relation between the number of partitions and the accuracy attained by each approximation scheme with the number partitions.

For approximation $X_1^{(ord\,p),n}$ of $X(1,x)$ with n partitions of $[0,1]$,

$$\left| E\left[f\left(X_1^{(ord\,p),n} \right) \right] - E\left[f\left(X(1,x) \right) \right] \right|$$

is called discretization error. If

$$E\left[f\left(X_1^{(ord\,p),n} \right) \right] = E\left[f\left(X(1,x) \right) \right] + O\left(n^{-p} \right),$$

then this approximation is said to be of order p.

It is shown that $p = 1$ for the EM scheme in [1] and [5] for C^4-function f. On the other hand, $p = 2$ for the NV and the NN algorithms if f is Lipschitz continuous ([2][3] [4]).

In partitioning $[0,T]$, we take the following procedure : Let $(0 = t_0^{(n)}, \ldots, t_n^{(n)} = T)$ be a partitioning of the interval $[0,T)$ defined by $t_k^{(n)} = k^\gamma T / n^\gamma$, where $n \in \mathbb{N}$ and $\gamma > 0$. Then $P_T g$ is approximated by

$$Q_{(s_n)} Q_{(s_{n-1})} \cdots Q_{(s_1)} g$$

where $s_k = t_k^{(n)} - t_{k-1}^{(n)}$. Hence, if $\gamma \neq 1.0$, then the length of each interval differs.

Initial value, strike price, etc. are determined as follows :

(1) linear drift
- initial value : 0.5
- strike price : 1.5
- the number of sample points : 100,000,000

(2) cosine drift
- initial value : 1.0
- strike price : 1.5
- killing function : standard killing

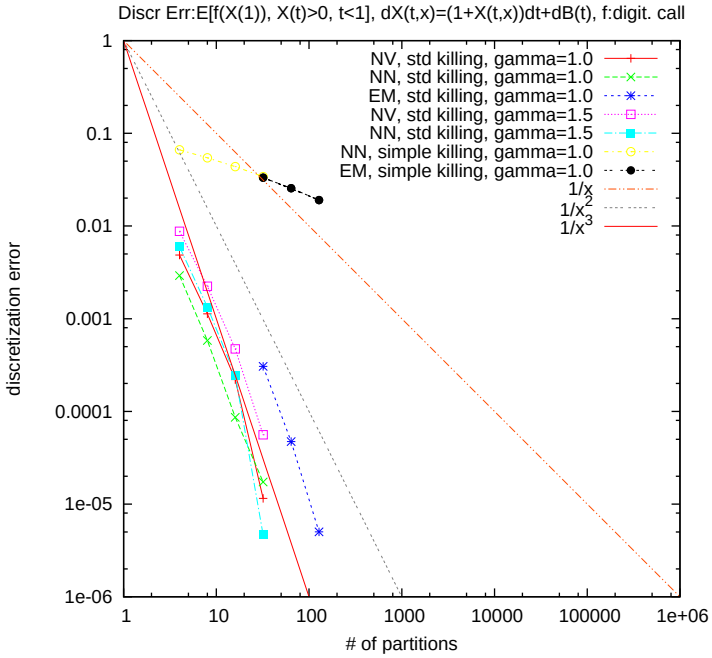

FIGURE 3.1. Discretization error(1)

- the number of sample points : 100,000,000

3.1. **Result.** Figure 3.1 and Figure 3.2 describes the discretization error of the experiments.

The true values for (1) linear drift and (2) cosine drift are replaced with the following values :

(1) linear drift

$$\text{true value} = 7.2806243601e-01$$

obtained by the NV algorithm with
$\gamma = 1.5$
256 partitions
$420,000,000$ sample points
standard killing

(2) cosine drift

$$\text{true value} = 1.0606646755e+00$$

obtained by the NV algorithm with
$\gamma = 1.0$
256 partitions
$300,000,000$ sample points
standard killing

We see some facts from these figures :

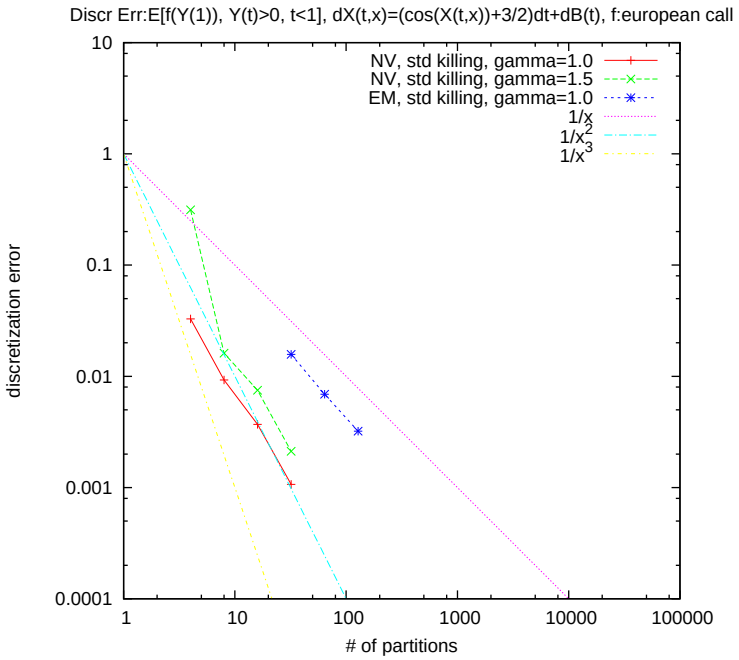

FIGURE 3.2. Discretization error(2)

- It is not clear how γ works, which means that the effect of the difference of lengths of interval is not remarkable as long as these experiments.
- For the experiment of (1) linear drift,
  - the Euler–Maruyama scheme performs much better than expected. More precisely, p, the order of approximation, is almost 3 which is even larger than 1.
  - both the NV and NN algorithms perform almost as good as the case applied to Lipschitz continuous cases, which means that the order of these algorithms is 2 as proved in [3] and [4].
  - we see the effect of choice of a killing function. The fact that the NN algorithm and the EM scheme both with a simple killing function gave the same slope indicates that the error occurring from hitting probability is so large that the difference of orders of approximation for these schemes is canceled.
- For the experiment of (2) cosine drift,
  - the slope of the EM scheme is -1 which is the exactly the same as proved.
  - it could be said that the NV algorithm performs a little worse than the original order. The slope of it looks slightly larger than -2.

4. Discussion

- All algorithms performed even better than expected in the model (1) as if it had been an experiment for the European style model. This result might come from the simpleness of the model. The error from approximation of

hitting boundary seems to be extremely small. It has not been cleared why the EM scheme worked so well, however.

- Since the choice of a killing function might seriously affect the order of approximation, better killing functions need be developed.
- From Figure 3.2, it is seemingly valuable to do more research for this type of models. From the result of the case of the NV algorithm, we perceive some other effect than the error from approximation of $g(X(t,x))$, though somehow the EM scheme still worked well in the sense that the error from the boundary condition is not observable.
- We have a larger class of possible models for (2.1) in the sense that various combinations of forms of g and V_0 can be considered. Therefore more numerical experiments should be conducted in order to compare the results dependent on combinations of models of V_0 and types of g.

  Multi-dimensional models ($N \geq 2$) also interest us from the practical point of view though only one-dimensional models are considered in these experiments.

Acknowledgments

This research was partly supported by the following grants : Center for Advanced Research in Finance at Graduate School of Economics, the University of Tokyo, the Grant-in-Aid for Young Scientists(B) of Japan Society for the Promotion of Science, 23730297, 2011, Ministry of Education, Science, Sports and Culture, Grant-Aid for Science Research (C) 22540115.

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