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Optimal Multiunit Exchange Design

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Abstract

We investigate multiunit exchange where a central planner and participants both bring commodities to sell and the central planner plays the role of platform provider. The central planner has restrictions on allocations. We characterize the optimal mechanism concerning his (her) revenue under incentive compatibility and individual rationality in the ex-post term. We introduce modified virtual valuation and show that the optimization problem can be replaced with the maximization of modified virtual valuations. We apply our results to important problems of single-unit demands and position exchanges. We demonstrate a clock auction design that implements the optimal position allocation through dominant strategies.

Keywords: Revenue-Maximization, Multiunit Exchanges, Outside Options, Modified Virtual Valuation, Single-Unit Demands, Position Exchanges.

JEL Classification Numbers: D44, D61, D82

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1. Introduction

This paper investigates the optimal design of trading mechanisms with incomplete information under the standard assumptions in auction theory such as quasi-linearity, private values, independent type distributions, risk-neutrality, and no externality. We also assume single-dimensional type spaces. Multiple homogeneous commodities are allocated among players (participants). The central planner design**s** the trading mechanism in order to maximize his expected revenue in a consistent manner with incentive compatibility and individual rationality in the ex-post term.

We demonstrate an approach to extend the basic concepts explored by Myerson (1981) such as virtual valuations in the single-unit auction problem to the more general multiunit allocation problem[3]. The present paper has the following two substantial points of extension:

(i) Each participant may possess multiple units as his initial endowment, which he has the option to sell instead of purchasing additional units. Moreover, he has the outside option not to participate in the allocation problem and instead consume his initial endowment by himself.

(ii) An allocation that the central planner can select may be restricted to a particular subset of profiles of commodity bundles.

The point (i) implies that the central planner attempts to enhance his revenue not only by selling his initial endowment but also by (directly or indirectly) exploiting brokerage fee for assisting the exchanges across players as an intermediary. Based on this point of extension, this paper investigates the optimal mechanism design not only for the multiunit auction framework but also for the more general framework termed *multiunit exchange*, which combines auction with *bargaining* such as Myerson and Satterthwaite (1983). This paper regards the central planner as the intermediary who monopolistically provides a platform that enables transaction among players.

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[3] There exist previous works concerning optimal multi-unit auction design such as Maskin and Riley (1989), Palfrey (1983), Branco (1996), and Monteiro (2002). Ulku (2009) and Edelman and Schwarz (2010) investigated optimal auction design in some classes with heterogeneous commodities.

The outside opportunity value that each player can obtain by consuming his endowment by himself could be dependent on his type. This type-dependence makes it substantial to incentivize each player to participate in the allocation problem, because he (she) might require the central planner to pay the excessive *bargaining rent* induced by his outside option whose value is unknown to the central planner.

The incorporation of auction with bargaining has been studied by Myerson and Satterthwaite (1983), Cramton, Gibbons, and Klemperer (1987), Segal and Whinston (2010), and Matsushima (2011) in terms of the achievement of efficiency without the central planner's having the deficit. Instead of the achievement of efficiency, the present paper focuses on revenue-maximization.

Based on these observations, we make a modification of the key concept termed *virtual valuation* in the optimal auction design literature; we replace the virtual valuation for each player with the valuation reduced not only by his informational rent but also by his bargaining rent, which is termed the *modified virtual valuation (MVV)*. The single-unit term of MVV, i.e., *modified unit virtual valuation (MUVV*), implies marginal revenue in terms of type if this player purchases additional units, whereas it implies marginal cost in terms of type if he sells his endowment.

With this concept replacement, we can show a characterization result according to the similar method to Myerson (1981) in that the optimization problem for the central planner's revenue can be replaced with the problem of maximizing the expected sum of the participants' MVVs in terms of allocation rule. This characterization result holds irrespective of whether there is any restriction on the central planner's allocation selection as implied by the point (ii).

We demonstrate a manner of designing the optimal mechanism. We show a tractable *monotonicity* condition concerning MUVV, which could be sufficient for the *regularity* property on the trading environment that makes it much easier to construct the optimal mechanism. With this regularity, the optimization problem can be replaced with the more tractable problem of maximizing the sum of the participants' MVVs in terms of allocation at every state of the world.

By utilizing these results, we can investigate two important cases named *single-unit demands* and *position exchanges*. The single-unit demand case assumes that each participant is either a seller of a single unit or buyer for a single unit. We show that in the associated optimal mechanism, the central planner offers the *bid* price to each seller, and the *ask* price to each buyer, and each player decides to either reject or accept this offer. The bid or ask price for each player does not depend on his type revelation, whereas it is *increasing* in the other players' type revelations. Hence, the bid-ask spread increases as the difference between a seller and a buyer in valuation, i.e., the surplus induced by their trade, decreases. This implies that the central planner attempts to obtain greater revenue when the surplus is smaller, without being afraid of a risk that the trade fails to take place. This finding is in contrast with the case of intermediary that can be thought about in which the brokerage fee could be proportional to the surplus. We also investigate the case of large double auctions.

By taking the point (ii) into account, we can model problems of allocating *heterogeneous* commodities such as *position auctions* explored by Edelman, Ostrovsky, and Schwarz (2007), Varian (2007), and Athey and Ellison (2011) within our framework, where different bundles of homogeneous commodity are regarded as heterogeneous commodities (positions). Edelman and Schwarz (2010) demonstrated the optimal position auction design on the assumption of symmetric type distributions across players. This paper extends their results to the more general case and shows the optimal mechanism design for *position exchange with asymmetry*.

Edelman and Schwarz (2010) introduced an ascending clock auction format that explicitly determines the allocation and payment vector through a dynamical price-adjustment process, which was termed the generalized English auction (GEA) with reserve price. They showed that it implements the optimal position allocations through Nash equilibria. The present paper introduces a new design of ascending clock auction format termed the *generalized Japanese auction* (GJA) as a substantial extension of the GEA with reserve price, which implements the optimal position allocations through (mostly) *dominant* strategies.

According to the GJA, the auctioneer offers and ascends the unit price (pay per click for sponsored search) to each player in the continuous time horizon; each player decides the time to quit the GJA. The later a player makes the time to quit relatively to the other player, the better position he can obtain. In this case, he pays the unit price that the auctioneer has offered to him at the last time that some other player has quitted before him. In the GJA, there are substantial departures from the GEA in that the auctioneer can offer *different* unit prices across the participants, and that the auctioneer can make the price-adjustment dependent on history of play. Because of these departures, it could be a (mostly) dominant strategy for any player to remain active until a particular type-dependent time of quitting, which is equivalent to the value of his MUVV, irrespective of the other player's quitting time selections.

The organization of this paper is as follows. Section 2 shows the basic model for multiunit exchange. Section 3 defines the optimization problem for the central planner's revenue, defines the concept of modified virtual valuation (MVV), and shows a characterization result for optimal mechanism design. Section 4 introduces the regularity condition, and provides the monotonicity condition on modified unit virtual valuations (MUVVs) that is sufficient for this regularity. This section also shows a tractable specification of the optimal mechanism for the cases without restrictions on allocations. Section 5 investigates the single-unit demand case. Section 6 investigates position exchanges. Section 7 investigates position auctions and demonstrates the generalized Japanese auction. Section 8 concludes.

2. The Model

We investigate the following allocation problem named *multiunit exchange* as a generalized concept of multiunit auction, in which each participant can not only purchase but also sell commodities. Let $N = \{1,...,n\}$ denote the set of *players* (participants), where $n \geq 2$. Each player $i \in N$ has a *type* $\omega_i \in [0,1]$ in a single-dimensional type space that is randomly and independently determined according to a probability density function $p_i(\omega_i) \geq 0$,[4] where $\int_{\omega_i=0}^{1} p(\omega_i)d\omega_i = 1$. Let us denote by $P_i(\omega_i) \equiv \int_{s_i=0}^{\omega_i} p(s_i)ds_i$ the associated cumulative distribution.

There exist e units of homogenous commodity in totality, where e is a fixed positive integer. Each player $i \in N$ possesses e_i units as his initial endowment, where e_i is a fixed non-negative integer. We assume $\sum_{i \in N} e_i \leq e$; there exists a *central planner* who possesses the remaining $e_0 \equiv e - \sum_{i \in N} e_i$ units as his initial endowment.

An *allocation* is defined as a vector of non-negative integers $a = (a_i)_{i=1}^n$, where we assume $\sum_{i \in N} a_i \leq e$, according to which, each player $i \in N$ is assigned a_i units, whereas the remaining $e - \sum_{i \in N} a_i$ units are assigned to the central planner. Let us denote by A the set of all allocations. Each player $i's$ payoff function has a *quasi-linear* and *risk-neutral* form with private values, i.e., $v_i(a_i, \omega_i) + s_i$, where $s_i \in R$ denotes the monetary transfer to him from the central planner, and $v_i : \{0,...,e\} \times [0,1] \to R$ denotes his *valuation function*. We assume that it is differentiable in $\omega_i \in [0,1]$. We assume that

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[4] In order to apply the basic concept of Myerson (1981), we need to assume single-dimensional type spaces. Most previous works concerning the optimal multi-unit auction and position action such as Branco (1996), Monteiro (2002), Athey and Elison (2009), and Edelman and Schwarz (2011) commonly assumed this single-dimensionality. Ulku (2009), who investigated more general multi-object auctions, also assumed it.

(1) $v_i(0,\omega_i)=0$ for all $\omega_i \in [0,1]$, and $v_i(a_i,0)=0$ for all $a_i \in \{0,...,e\}$,

and that the central planner has zero valuation at all times. We define the *unit valuation* for player $i \in N$ associated with $(\omega_i, a_i) \in [0,1]\times\{1,...,e\}$ by

$$w_i(a_i,\omega_i) \equiv v_i(a_i,\omega_i) - v_i(a_i - 1,\omega_i),$$

implying player $i's$ valuation for the (a_i)-th unit consumption.

Assumption 1: For every $i \in N$ and every $\omega_i \in [0,1]$,

(2) $w_i(a_i,\omega_i) \geq 0$ for all $a_i \in \{1,...,e\}$,

and

(3) $\dfrac{\partial w_i(a_i,\omega_i)}{\partial \omega_i} \geq 0$ for all $a_i \in \{1,...,e\}$.

The inequalities (2) imply free disposal in that his valuation for any unit consumption is non-negative. From the inequalities (2), it follows that

(4) $v_i(a_i,\omega_i)$ is non-decreasing in $a_i \in \{0,...,e\}$.

The inequalities (3) imply that the higher each player's type is, the higher his unit valuation is. From the inequalities (3), it follows that

(5) $\dfrac{\partial v_i(a_i,\omega_i)}{\partial \omega_i}$ is non-decreasing in $a_i \in \{0,...,e\}$.

From the inequalities (3), it follows that v_i satisfies *increasing difference* in that for every $(a_i, a_i') \in \{0,...,e\}^2$ and every $(\omega_i,\omega_i') \in [0,1]^2$,

(6) $v_i(a_i',\omega_i) - v_i(a_i,\omega_i) \leq v_i(a_i',\omega_i') - v_i(a_i,\omega_i')$ if $a_i < a_i'$ and $\omega_i < \omega_i'$.

From the inequalities (3), it follows that

(7) $\dfrac{\partial v_i(a_i,\omega_i)}{\partial \omega_i} \geq 0$ for all $a_i \in \{1,...,e\}$.

A *direct revelation mechanism*, shortly a *mechanism*, is defined as (f,x), where $f=(f_i)_{i\in N}:[0,1]^n \to A$ is an *allocation rule*, $x=(x_i)_{i\in N}:[0,1]^n \to R^n$ is a *payment rule*, $f_i:[0,1]^n \to \{0,...,e\}$, and $x_i:[0,1]^n \to R$. We denote by F the set of all allocation rules. We denote by X the set of all payment rules. We require ex post incentive compatibility for a mechanism (f,x) as follows.

Ex Post Incentive compatibility (EPIC): For every $i \in N$ and every $\omega \in [0,1]^n$,

$$v_i(f_i(\omega),\omega_i) + x_i(\omega) \geq v_i(f_i(\omega_i',\omega_{-i}),\omega_i) + x_i(\omega_i',\omega_{-i}) \text{ for all } \omega_i' \in [0,1].$$

Let us denote by $\tilde{F} \subset F$ the set of all allocation rules f satisfying that for every $i \in N$, f_i is *non-decreasing* in $\omega_i \in [0,1]$.

Lemma 1: *With Assumption 1, a mechanism* (f,x) *satisfies EPIC if and only if*

$$f \in \tilde{F},$$

and for every $(i,\omega) \in N \times [0,1]^n$,

(8) $$x_i(\omega) = \int_{s_i=0}^{\omega_i} \frac{\partial v_i(f_i(s_i,\omega_{-i}),s_i)}{\partial s_i} ds_i - v_i(f_i(\omega),\omega_i) + D_i(\omega_{-i}),$$

where $D_i : [0,1]^{n-1} \to R$ *is an arbitrary function.*

Proof: It is clear from the envelope theorem in the auction theory literature (See Milgrom (2004) and Krishna (2010), for instance) that the equalities (8) are necessary for (f,x) to satisfy EPIC. Moreover, if (f,x) satisfies EPIC, then, for every $(\omega,\omega_i') \in [0,1]^{n+1}$,

$$v_i(f_i(\omega),\omega_i) - v_i(f_i(\omega_i',\omega_{-i}),\omega_i) \geq x_i(\omega_i',\omega_{-i}) - x_i(\omega)$$
$$\geq v_i(f_i(\omega),\omega_i') - v_i(f_i(\omega_i',\omega_{-i}),\omega_i'),$$

which along with (6) implies $f \in \tilde{F}$.

Suppose that (f,x) satisfies $f \in \tilde{F}$ and the inequalities (8). Then, from (5) and (7), it follows that for every $i \in N$, and every $(\omega_i,\omega_i') \in [0,1]^2$, if $\omega_i > \omega_i'$, then

$$v_i(f_i(\omega),\omega_i) + x_i(\omega) - \{v_i(f_i(\omega_i',\omega_{-i}),\omega_i') + x_i(\omega_i',\omega_{-i})\}$$
$$= \int_{s_i=\omega_i'}^{\omega_i} \frac{\partial v_i(f_i(s_i,\omega_{-i}),s_i)}{\partial s_i} ds_i \leq v_i(f_i(\omega),\omega_i) - v_i(f_i(\omega),\omega_i'),$$

which implies

$$v_i(f_i(\omega),\omega_i') + x_i(\omega) \le v_i(f_i(\omega_i',\omega_{-i}),\omega_i') + x_i(\omega_i',\omega_{-i}) .$$

In the same manner,

$$v_i(f_i(\omega),\omega_i) + x_i(\omega) - \{v_i(f_i(\omega_i',\omega_{-i}),\omega_i') + x_i(\omega_i',\omega_{-i})\}$$

$$= \int_{s_i=\omega_i'}^{\omega_i} \frac{\partial v_i(f_i(s_i,\omega_{-i}),s_i)}{\partial s_i} ds_i \ge v_i(f_i(\omega_i',\omega_{-i}),\omega_i) - v_i(f_i(\omega_i',\omega_{-i}),\omega_i') ,$$

which implies

$$v_i(f_i(\omega),\omega_i) + x_i(\omega) \ge v_i(f_i(\omega_i',\omega_{-i}),\omega_i') + x_i(\omega_i',\omega_{-i}) .$$

Hence, (f,x) satisfies EPIC.

Q.E.D.

Each player has the *outside option* not to participate in the allocation problem and consume his initial endowment by himself. Each player can exercise this option at any time. Hence, we require for a mechanism ex post individual rationality in that any player never wants to exercise this option in the ex post term, because the mechanism guarantees him at least the same value as his type-dependent outside opportunity value $v_i(e_i,\omega_i)$.

Ex Post Individual Rationality (EPIR): For every $i \in N$ and every $\omega \in [0,1]^n$,

$$v_i(f_i(\omega),\omega_i) + x_i(\omega) \ge v_i(e_i,\omega_i) .$$

3. Revenue-Maximization

The central planner attempts to maximize his expected revenue under the constraints of EPIC and EPIR. We denote by a non-empty subset $\hat{A} \subset A$ the set of *feasible* allocations, which is exogenously given[5]. An allocation rule $f \in F$ is said to be *feasible* if

$$f(\omega) \in \hat{A} \text{ for all } \omega \in [0,1]^n .$$

Let us denote by a non-empty subset $\hat{F} \subset F$ the set of all feasible allocation rules. We assume that the central planner is restricted to select any allocation rule f from this subset $\hat{F}$. We define the optimization problem concerning the central planner's expected revenue as

$$(9) \qquad \max_{(f,x)\in \hat{F}\times X} E[-\sum_{i\in N} x_i(\omega)] \text{ subject to EPIC and EPIR.}$$

According to Myerson (1981), we define the *virtual valuation* for player $i \in N$ associated with $(a_i, \omega_i) \in \{0,...,e\} \times [0,1]$ as

$$u_i(a_i,\omega_i) \equiv v_i(a_i,\omega_i) - \frac{1-P_i(\omega_i)}{p_i(\omega_i)} \frac{\partial v_i(a_i,\omega_i)}{\partial \omega_i} .$$

The auction theory literature has typically assumed that $e_i = 0$ for all $i \in N$; with this assumption, the central planner can extract each player $i's$ valuation $v_i(a_i,\omega_i)$ minus his *informational rent* given by

$$\frac{1-P_i(\omega_i)}{p_i(\omega_i)} \frac{\partial v_i(a_i,\omega_i)}{\partial \omega_i} ,$$

which implies to the virtual valuation $u_i(a_i,\omega_i)$ defined above.

In order to investigate the more general case of multiunit exchange such that $e_i > 0$ for some $i \in N$, we further define the *modified virtual valuation* (MVV) for player $i \in N$ associated with $(a_i, \omega_i) \in \{0,...,e\} \times [0,1]$ as

$$u_i^*(a_i,\omega_i) = u_i(a_i,\omega_i) \qquad \text{if } a_i \geq e_i ,$$

and

[5] Palfrey (1983) investigated the case that the revenue-maximizing central planner endogenously determines available commodity bundles as a commitment. See also Armstrong (2000).

$$u_i^*(a_i,\omega_i) = u_i(a_i,\omega_i) - \frac{1}{p_i(\omega_i)}\{\frac{\partial v_i(e_i,\omega_i)}{\partial \omega_i} - \frac{\partial v_i(a_i,\omega_i)}{\partial \omega_i}\} \text{ if } a_i < e_i .$$

The value of

$$\frac{1}{p_i(\omega_i)}\{\frac{\partial v_i(e_i,\omega_i)}{\partial \omega_i} - \frac{\partial v_i(a_i,\omega_i)}{\partial \omega_i}\}$$

represents *player $i's$ bargaining rent induced by his outside option*. When the central planner assigns to player i lesser than his initial endowment, in order to prevent him from exercising his outside option, the central planner has to make up for the loss that this player takes by paying back to him his bargaining rent.

Theorem 2: *With Assumption 1, a mechanism (f,x) is the solution to the optimization problem (9) if and only if*

$$f \in \hat{F} \cap \tilde{F} ,$$

(10) $$E[\sum_{i\in N} u_i^*(f_i(\omega),\omega_i)] \geq E[\sum_{i\in N} u_i^*(g_i(\omega),\omega_i)] \quad \textit{for all} \quad g \in \hat{F} \cap \tilde{F} ,$$

and x is given by the inequalities (8), where for every $i \in N$ and every $\omega_{-i} \in [0,1]^{n-1}$,

(11) $$D_i(\omega_{-i}) = \max_{\omega_i}\{v_i(e_i,\omega_i) - \int_{s_i=0}^{\omega_i} \frac{\partial v_i(f_i(s_i,\omega_{-i}),s_i)}{\partial s_i} ds_i\} ,$$

that is,

(12) $$x_i(\omega) = \int_{s_i=0}^{\omega_i} \frac{\partial v_i(f_i(s_i,\omega_{-i}),s_i)}{\partial s_i} ds_i - v_i(f_i(\omega),\omega_i)$$

$$+\max_{\omega_i}\{v_i(e_i,\omega_i) - \int_{s_i=0}^{\omega_i} \frac{\partial v_i(f_i(s_i,\omega_{-i}),s_i)}{\partial s_i} ds_i\} .$$

Proof: It is clear from Lemma 1 that if (f,x) is the solution to the optimization problem (9), it must satisfy $f \in \hat{F} \cap \tilde{F}$, the inequalities (8), and the equalities (11).

Suppose that $f \in \hat{F} \cap \tilde{F}$. For each $i \in N$ and each $\omega_{-i} \in [0,1]^{n-1}$, we define $\omega_i(\omega_{-i}) \in [0,1]$ by

$$\omega_i(\omega_{-i}) = 1 \qquad \text{if} \quad f_i(\omega) < e_i \quad \text{for all} \quad \omega_i \in [0,1] ,$$

$$\omega_i(\omega_{-i}) = 0 \qquad \text{if} \quad f_i(\omega) > e_i \quad \text{for all} \quad \omega_i \in [0,1],$$

and

$$f_i(\omega_i(\omega_{-i}), \omega_{-i}) = e_i \quad \text{and} \quad f_i(\omega) < e_i \quad \text{for all} \quad \omega_i \in [0, \omega_i(\omega_{-i})) \quad \text{otherwise.}$$

Hence, player i is assigned less than his initial endowment if and only if $\omega_i < \omega_i(\omega_{-i})$.

From (1), note that $\omega_i = \omega_i(\omega_{-i})$ maximizes the value of

$$v_i(e_i, \omega_i) - \int_{s_i=0}^{\omega_i} \frac{\partial v_i(f_i(s_i, \omega_{-i}), s_i)}{\partial s_i} ds_i \ .$$

Hence, from (11),

$$D_i(\omega_{-i}) = \int_{s_i=0}^{\omega_i(\omega_{-i})} \{\frac{\partial v_i(e_i, s_i)}{\partial s_i} - \frac{\partial v_i(f_i(s_i, \omega_{-i}), s_i)}{\partial s_i}\} ds_i \ ,$$

and therefore,

$$x_i(\omega) = \int_{s_i=0}^{\omega_i} \frac{\partial v_i(f_i(s_i, \omega_{-i}), s_i)}{\partial s_i} ds_i - v_i(f_i(\omega), \omega_i)$$

$$+ \int_{s_i=0}^{\omega_i(\omega_{-i})} \{\frac{\partial v_i(e_i, s_i)}{\partial s_i} - \frac{\partial v_i(f_i(s_i, \omega_{-i}), s_i)}{\partial s_i}\} ds_i \ .$$

Let us specify $z_i : \{0,...,e\} \times [0,1] \to R$ by

$$z_i(a_i, \omega_i) = 0 \qquad \text{if} \quad a_i \geq e_i,$$

and

$$z_i(a_i, \omega_i) = \frac{\partial v_i(e_i, \omega_i)}{\partial \omega_i} - \frac{\partial v_i(a_i, \omega_i)}{\partial \omega_i} \qquad \text{if} \quad a_i < e_i,$$

implying player $i's$ bargaining rent, i.e.,

$$u_i^*(a_i, \omega_i) = u_i(a_i, \omega_i) - \frac{z_i(a_i, \omega_i)}{p_i(\omega_i)} .$$

Hence,

$$E[x_i(\omega) \mid \omega_{-i}] = \int_{\omega_i=0}^{1} \{ \int_{s_i=0}^{\omega_i} \frac{\partial v_i(f_i(s_i, \omega_{-i}), s_i)}{\partial s_i} ds_i - v_i(f_i(\omega), \omega_i)\} p_i(\omega_i) d\omega_i$$

$$+ \int_{\omega_i=0}^{1} z_i(f_i(\omega), \omega_i) d\omega_i$$

$$= \int_{\omega_i=0}^{1} [\frac{\partial v_i(f_i(\omega),\omega_i)}{\partial \omega_i}\{\frac{1-P_i(\omega_i)}{p_i(\omega_i)}\} - v_i(f_i(\omega),\omega_i) + \frac{z_i(f_i(\omega),\omega_i)}{p_i(\omega_i)}] p_i(\omega_i) d\omega_i$$

$$= \int_{\omega_i=0}^{1} \{-u_i(f_i(\omega),\omega_i) + \frac{z_i(f_i(\omega),\omega_i)}{p_i(\omega_i)}\} p_i(\omega_i) d\omega_i$$

$$= -\int_{\omega_i=0}^{1} u_i^*(f_i(\omega),\omega_i) p_i(\omega_i) d\omega_i .$$

From these observations, we have proven that

$$E[-\sum_{i\in N} x_i(\omega)] = E[\sum_{i\in N} u_i^*(f(\omega),\omega_i)] .$$

Clearly, the inequalities (10) imply the solution to the optimization problem (9).

Q.E.D.

From the proof of Theorem 2, it must be noted that the expected revenue for the central planner induced by the solution (f, x) to the optimization problem (9) is equivalent to the expected value of the sum of the players' MVVs, i.e.,

$$E[-\sum_{i\in N} x_i(\omega)] = E[\sum_{i\in N} u_i^*(f(\omega),\omega_i)] .$$

4. Regularity

The optimization problem (9) is said to be *regular* if there exists a feasible and non-decreasing allocation rule $f \in \hat{F} \cap \tilde{F}$ such that

(13) $$\sum_{i \in N} u_i^*(f_i(\omega), \omega_i) \geq \sum_{i \in N} u_i^*(a_i, \omega_i) \text{ for all } a \in \hat{A} \text{ and all } \omega \in [0,1]^n .$$

With regularity, we can replace the inequalities (10) with the inequalities (13), which can dramatically simplify the optimization problem. From Theorem 2, it is clear that with regularity, a mechanism (f, x) is the solution to the optimization problem (9) if and only if $f \in \hat{F} \cap \tilde{F}$ and it satisfies the equalities (12) and the inequalities (13). This section demonstrates a sufficient condition for this regularity.

We define the *unit virtual valuation* for player $i \in N$ associated with each $(\omega_i, a_i) \in [0,1] \times \{1, ..., e\}$ as

$$y_i(a_i, \omega_i) \equiv u_i(a_i, \omega_i) - u_i(a_i - 1, \omega_i) .$$

We further define the *modified unit virtual valuation* (MUVV) for player $i \in N$ associated with each $(\omega_i, a_i) \in [0,1] \times \{1, ..., e\}$ as

$$y_i^*(a_i, \omega_i) \equiv u_i^*(a_i, \omega_i) - u_i^*(a_i - 1, \omega_i) .$$

Note that

$$y_i^*(a_i, \omega_i) = w_i(a_i, \omega_i) - \frac{1 - P_i(\omega_i)}{p_i(\omega_i)} \frac{\partial w_i(a_i, \omega_i)}{\partial \omega_i} \qquad \text{if } a_i > e_i ,$$

and

$$y_i^*(a_i, \omega_i) = w_i(a_i, \omega_i) + \frac{P_i(\omega_i)}{p_i(\omega_i)} \frac{\partial w_i(a_i, \omega_i)}{\partial \omega_i} \qquad \text{if } a_i \leq e_i .$$

Note that the above defined MUVV implies *marginal revenue* in terms of type, i.e., $\dfrac{\frac{\partial}{\partial \omega_i}\{(1 - P_i(\omega_i)) w_i(\omega_i)\}}{p_i(\omega_i)}$, if player i purchases additional units ($a_i > e_i$), whereas it implies *marginal cost* $\dfrac{\frac{\partial}{\partial \omega_i}\{P_i(\omega_i) w_i(\omega_i)\}}{p_i(\omega_i)}$ if he sells his endowment ($a_i \leq e_i$). We assume the following mild properties of *monotonicity* for MUVV.

Assumption 2: For every $i \in N$,

(14) $\quad y_i^*(a_i, \omega_i)$ is non-increasing in $a_i \in \{0, ..., e-1\}$ for all $\omega_i \in [0,1]$,

(15) $\quad y_i^*(a_i, \omega_i)$ is non-decreasing in $\omega_i \in [0,1]$ for all $a_i \in \{0, ..., e\}$.

The following theorem shows that the monotonicity implied by Assumption 2 are sufficient for the regularity.

Theorem 3: *With Assumptions 1 and 2, the optimization problem (9) is regular.*

Proof: Suppose that there exists no allocation rule that is included in $\hat{F} \cap \tilde{F}$ and satisfies the inequalities (13). Then, there exists $f \in \hat{F}$ that satisfies the inequalities (13) but is not included in $\tilde{F}$. Without loss of generality, we can assume that there exist $i \in N$, $\omega \in [0,1]^n$, and $\omega_i' > \omega_i$, such that

$$f_i(\omega) > f_i(\omega_i', \omega_{-i}),$$

$$\sum_{j \in N} u_j^*(f_j(\omega), \omega_j) \geq \sum_{j \in N} u_j^*(f_j(\omega_j', \omega_{-j}), \omega_j),$$

and

$$u_i^*(f_i(\omega), \omega_i') + \sum_{j \in N \setminus \{i\}} u_j^*(f_j(\omega), \omega_j)$$

$$\leq u_i^*(f_i(\omega_i', \omega_{-i}), \omega_i') + \sum_{j \in N \setminus \{i\}} u_j^*(f_j(\omega_j', \omega_{-j}), \omega_j).$$

In this case, without loss of generality, we can also assume that one of the last two inequalities strictly holds. Hence,

$$u_i^*(f_i(\omega), \omega_i) - u_i^*(f_i(\omega_i', \omega_{-i}), \omega_i) > u_i^*(f_i(\omega), \omega_i') - u_i^*(f_i(\omega_i', \omega_{-i}), \omega_i'),$$

that is,

$$\sum_{a_i = f_i(\omega_i', \omega_{-i})}^{f_i(\omega)} y_i^*(a_i, \omega_i) > \sum_{a_i = f_i(\omega_i', \omega_{-i})}^{f_i(\omega)} y_i^*(a_i, \omega_i').$$

This implies $\omega_i > \omega_i'$, because of the property (15). This is a contradiction.

Q.E.D.

5. Single-Unit Demands

Suppose that the central planner can select any allocation in the entire set A without restriction, i.e.,

$$A = \hat{A}, \text{ that is, } \hat{F} = F .$$

Then, we can specify the solution to the optimization problem (9) as the mechanism denoted by (f^*, x^*) in the manner that *the commodities are assigned to players who have high MUVV with precedence*. In this case, f^* is well characterized according to the method that for every $(i, j) \in N^2$ and every $\omega \in [0,1]^n$,

$$y_i^*(f_i^*(\omega)+1, \omega_i) \leq y_j^*(f_j^*(\omega), \omega_j) \quad \text{if} \quad f_j^*(\omega) > 0 \quad \text{and} \quad f_i^*(\omega) < e,$$

and

$$y_i^*(f_i^*(\omega)+1, \omega_i) \leq 0 \quad \text{for all} \quad i \in N \quad \text{if} \quad \sum_{h \in N} f_h^*(\omega) < e \quad \text{and} \quad f_i^*(\omega) < e .$$

From (15) in Assumption 2, we can select such a f^* from the set $\tilde{F}$. From (14) in Assumption 2, f^* satisfies the inequalities (13). Moreover, we can specify the payment rule $x = x^*$ according to the inequalities (12) for $f = f^*$. Hence, the proof of the following theorem is straightforward from Theorems 2 and 3.

Theorem 4: *With Assumptions 1 and 2, and with the assumption of* $\hat{A} = A$, *the specified mechanism* (f^*, x^*) *is the solution to the optimization problem (9).*

This section investigates a special case of multiunit exchange with no restriction on allocations, i.e., $\hat{A} = A$, which is termed *single-unit demands*; each player prefers at most one unit, where

$$v_i(a_i, \omega_i) = \omega_i \quad \text{for all} \quad (i, a_i, \omega_i) \in N \times \{1, ..., e\} \times [0,1] .$$

For convenience, we assume that

$$e < n, \text{ and } e_i \in \{0,1\} \quad \text{for all} \quad i \in N .$$

In this single-unit demand case, we can easily calculate

$$w_i(1, \omega_i) = \omega_i \quad \text{and} \quad w_i(a_i, \omega_i) = 0 \quad \text{for all} \quad a_i > 1 ,$$

$$u_i^*(a_i,\omega_i)=\omega_i-\frac{1-P_i(\omega_i)}{p_i(\omega_i)} \qquad \text{if } a_i\geq 1,$$

$$u_i^*(0,\omega_i)=-\frac{1}{p_i(\omega_i)} \qquad \text{if } e_i=1,$$

and

$$u_i^*(0,\omega_i)=0 \qquad \text{if } e_i=0.$$

Hence, each player $i's$ MUVV is given by

$$y_i^*(a_i,\omega_i)=0 \quad \text{for all } a_i\geq 2,$$

$$y_i^*(1,\omega_i)=\omega_i-\frac{1-P_i(\omega_i)}{p_i(\omega_i)} \qquad \text{if } e_i=0,$$

and

$$y_i^*(1,\omega_i)=\omega_i+\frac{P_i(\omega_i)}{p_i(\omega_i)} \qquad \text{if } e_i=1.$$

Note that Assumption 1 and the inequalities (14) automatically hold. We assume the inequalities (15); both Assumptions 1 and 2 hold.

Fix $i\in N$ and $\omega_{-i}\in[0,1]^{n-1}$ arbitrarily. Let us define $j(\omega_{-i})\in N\setminus\{i\}$ as the player who has the (e)-th highest MUVV among all players except for player i, where

$$y_i^*(1,\omega_h)\geq y_i^*(1,\omega_{j(\omega_{-i})}) \quad \text{for at least } e \text{ players } h \text{ in } N\setminus\{i\},$$

and

$$y_i^*(1,\omega_h)\leq y_i^*(1,\omega_{j(\omega_{-i})}) \quad \text{for at least } n-e \text{ players } h \text{ in } N\setminus\{i\}.$$

We define the *pivotal type* for player i, $\underline{\omega}_i(\omega_{-i})\in[0,1]$, by

$$y_i^*(1,\underline{\omega}_i(\omega_{-i}))=y_{j(\omega_{-i})}^*(1,\omega_{j(\omega_{-i})}) \qquad \text{if } y_{j(\omega_{-i})}^*(1,\omega_{j(\omega_{-i})})\geq 0,$$

and

$$y_i^*(1,\underline{\omega}_i(\omega_{-i}))=0 \qquad \text{if } y_{j(\omega_{-i})}^*(1,\omega_{j(\omega_{-i})})<0.$$

Note that player i has the (e)-th highest or even higher MUVV if and only if

$$\omega_i\geq\underline{\omega}_i(\omega_{-i}).$$

The following theorem shows that according to the optimal mechanism (f^*,x^*) specified above, each player i who has the null initial endowment will purchase a single unit of commodity if and (almost) only if his type is greater than the pivotal type

$\underline{\omega}_i(\omega_{-i})$, where he pays the same monetary amount as this pivotal type valuation $\underline{\omega}_i(\omega_{-i})$. On the other hand, each player i who has the non-null initial endowment will sell his initial endowment if and (almost) only if his type is lesser than the pivotal type $\underline{\omega}_i(\omega_{-i})$, where he receives the same monetary amount as the pivotal type valuation $\underline{\omega}_i(\omega_{-i})$.

Theorem 5: *In the single-unit demands case, the solution* (f^*, x^*) *to the optimization problem (9) is characterized as follows: for every* $i \in N$ *and every* $\omega \in [0,1]^n$,

$$f_i^*(\omega) = 1 \qquad \text{if } \omega_i > \underline{\omega}_i(\omega_{-i}),$$

$$f_i^*(\omega) = 0 \qquad \text{if } \omega_i < \underline{\omega}_i(\omega_{-i}).$$

$$x_i^*(\omega) = 0 \qquad \text{if } e_i = 1 \text{ and } \omega_i > \underline{\omega}_i(\omega_{-i}),$$

$$x_i^*(\omega) = \underline{\omega}_i(\omega_{-i}) \qquad \text{if } e_i = 1 \text{ and } \omega_i < \underline{\omega}_i(\omega_{-i}),$$

$$x_i^*(\omega) = -\underline{\omega}_i(\omega_{-i}) \qquad \text{if } e_i = 0 \text{ and } \omega_i > \underline{\omega}_i(\omega_{-i}),$$

and

$$x_i^*(\omega) = 0 \qquad \text{if } e_i = 0 \text{ and } \omega_i < \underline{\omega}_i(\omega_{-i}).$$

Proof: Based on the arguments of this section, all we have to do for this proof is just to show the part of the characterization of $x_i^*(\omega)$. From (11) and the specifications of the model in this section, it follows that for $f = f^*$,

$$D_i(\omega_{-i}) = \underline{\omega}_i(\omega_{-i}) \qquad \text{if } e_i = 1,$$

and

$$D_i(\omega_{-i}) = 0 \qquad \text{if } e_i = 0.$$

Moreover, from the specifications of this section, it follows that

$$\int_{s_i=0}^{\omega_i} \frac{\partial v_i(f_i^*(s_i, \omega_{-i}), s_i)}{\partial s_i} ds_i - v_i(f_i^*(\omega), \omega_i) = 0 \qquad \text{if } \omega_i < \underline{\omega}_i(\omega_{-i}),$$

and

$$\int_{s_i=0}^{\omega_i} \frac{\partial v_i(f_i^*(s_i,\omega_{-i}),s_i)}{\partial s_i} ds_i - v_i(f_i^*(\omega),\omega_i) = -\underline{\omega}_i(\omega_{-i}) \qquad \text{if} \quad \omega_i > \underline{\omega}_i(\omega_{-i}) .$$

Hence, from (12), we have shown the specifications of $x_i^*(\omega)$.

Q.E.D.

We can interpret the central planner as the intermediary in double auctions in the following manner; the central planner bids the price equal to the pivotal type, i.e., $\underline{\omega}_i(\omega_{-i})$, to any player i who has the non-null initial endowment. This player sells a single unit to the central planner if and (almost) only if his valuation ω_i is less than $\underline{\omega}_i(\omega_{-i})$. The central planner asks the price equal to the pivotal type, i.e., $\underline{\omega}_i(\omega_{-i})$, to any player i who has the null initial endowment. This player purchases a single unit from the central planner if and (almost) only if his valuation ω_i is greater than $\underline{\omega}_i(\omega_{-i})$. From the specification of the pivotal types $\underline{\omega}_i(\omega_{-i})$, the case of excess demand for the central planner never takes place.

Note that the bid or ask price $\underline{\omega}_i(\omega_{-i})$ for each player i is independent of his type revelation ω_i. From the monotonicity (15) of MUVV $y_i^*(a_i,\omega_i)$ in terms of ω_i, it follows that the bid or ask price $\underline{\omega}_i(\omega_{-i})$ is *non-decreasing* in $\omega_{-i} \in [0,1]^{n-1}$. Hence, the bid-ask spread that the central planner offers to any pair of a seller and a buyer may decrease as the difference between their valuations, i.e., the surplus induced by their trade, shrinks. Needless to say, the trade may fail to take place when this difference is too small, because the bid-ask spread is set too large. Hence, the central planner tends to aim at high revenue without being afraid of a risk that the trade fails to take place when the surplus induced by the trade is not very large. These points are in contrast with the case of intermediary that can be thought about in which the brokerage fee could be proportional to the surplus induced by the trade.

In order to clarify the above points further, let us investigate an example of bilateral trades a la Myerson and Satterthwaite (1983); we assume that $n = 2$, and that player 1 is the only seller, i.e.,

$$e = e_1 = 1 \text{ and } e_2 = 0 .$$

In this example, note that

$$\underline{\omega}_1(\omega_2) = 0 \qquad \text{if} \quad \omega_2 - \frac{1-P_2(\omega_2)}{p_2(\omega_2)} \le 0,$$

$$\underline{\omega}_1(\omega_2) + \frac{1-P_1(\underline{\omega}_1(\omega_2))}{p_1(\underline{\omega}_1(\omega_2))} = \omega_2 - \frac{1-P_2(\omega_2)}{p_2(\omega_2)} \qquad \text{if} \quad \omega_2 - \frac{1-P_2(\omega_2)}{p_2(\omega_2)} > 0,$$

$$\underline{\omega}_2(\omega_1) = 1 \qquad \text{if} \quad \omega_1 + \frac{1-P_1(\omega_1)}{p_1(\omega_1)} \ge 1,$$

and

$$\omega_1 + \frac{1-P_1(\omega_1)}{p_1(\omega_1)} = \underline{\omega}_2(\omega_1) - \frac{1-P_2(\underline{\omega}_2(\omega_1))}{p_2(\underline{\omega}_2(\omega_1))} \qquad \text{if} \quad \omega_1 + \frac{1-P_1(\omega_1)}{p_1(\omega_1)} < 1.$$

Note also that

$$\omega_1 > \underline{\omega}_1(\omega_2) \quad \text{if and only if} \quad \omega_2 < \underline{\omega}_2(\omega_1).$$

If $\omega_1 > \underline{\omega}_1(\omega_2)$, then the trade never takes place, i.e.,

$$f_1^*(\omega) = 1, \quad f_2^*(\omega) = 0, \text{ and } \quad x_i^*(\omega) = x_i^*(\omega) = 0.$$

If $\omega_1 < \underline{\omega}_1(\omega_2)$, then the trade does take place, i.e.,

$$f_1^*(\omega) = 0, \quad f_2^*(\omega) = 1, \quad x_1^*(\omega) = \underline{\omega}_1(\omega_2), \text{ and } \quad x_2^*(\omega) = -\underline{\omega}_2(\omega_1).$$

Hence, the central planner's revenue is given by

$$-x_1^*(\omega) - x_2^*(\omega) = 0 \qquad \text{if} \quad \omega_1 > \underline{\omega}_1(\omega_2),$$

and

$$-x_1^*(\omega) - x_2^*(\omega) = -\underline{\omega}_1(\omega_2) + \underline{\omega}_2(\omega_1) \quad \text{if} \quad \omega_1 < \underline{\omega}_1(\omega_2).$$

Since $\underline{\omega}_i(\omega_{-i})$ is non-decreasing in ω_{-i}, the central planner's revenue induced by the trade is non-increasing in the buyer's type ω_2 and is non-decreasing in the seller's type ω_1.

Let us further assume uniform distributions;

$$p_i(\omega_i) = 1 \quad \text{for all} \quad i \in \{1,2\} \quad \text{and all} \quad \omega_i \in [0,1].$$

Hence,

$$\underline{\omega}_1(\omega_2) = \max[\omega_2 - 1/2, 0] \quad \text{and} \quad \underline{\omega}_2(\omega_1) = \min[\omega_1 + 1/2, 1],$$

where

$$\omega_1 + \omega_2 = \tfrac{1}{2} \text{ if and only if } \underline{\omega}_1(\omega_2) = \omega_1 \text{ and } \underline{\omega}_2(\omega_1) = \omega_2 .$$

Hence, the central planner's revenue is given by

$$-x_1^*(\omega) - x_2^*(\omega) = 0 \qquad \text{if } \omega_2 - \omega_1 < \tfrac{1}{2},$$

and

$$-x_1^*(\omega) - x_2^*(\omega) = 1 - (\omega_2 - \omega_1) \qquad \text{if } \omega_2 - \omega_1 \geq \tfrac{1}{2}.$$

The central planner's revenue is maximal if the surplus $\omega_2 - \omega_1$ is equal to $\tfrac{1}{2}$. It declines as the surplus decreases. It is equal to zero, i.e., minimal if the surplus is maximal, i.e., $\omega_2 - \omega_1 = 1$. See Figure 1.

[Figure 1]

Remark (Large Double Auction): Let us consider the large double auction in which there are rn sellers and $(1-r)n$ buyers and n is assumed to be sufficiently large; $0 < r < 1$ is assumed. For convenience, we assume that the central planner brings no commodities to sell. Each seller's type and each buyer's type are determined according to the same distributions denoted by P_s and P_b, respectively. In such large double auctions, according to the optimal mechanism (f^*, x^*), the central planner almost certainly makes his ask and bid prices close to particular prices ω_s^* and ω_b^*, which are specified as the solution to the constrained maximization:

$$(\omega_s^*, \omega_b^*) \equiv \underset{(\omega_s, \omega_b) \in [0,1]^2}{\arg\max} [\omega_s (1-r)\{1 - P_b(\omega_s)\} - \omega_b r P_s(\omega_b)]$$

subject to

$$(1-r)\{1 - P_b(\omega_s)\} = r P_s(\omega_b).$$

The interpretation of this central planner's behavioral mode in the limit of double auctions is as follows. The central planner purchases commodities for the bid price ω_b from a large group of sellers whose supply function is given by $S(\omega_b) = r P_s(\omega_b)$, and then sells these commodities for the ask price ω_s to a large group of buyers whose demand function is given by $D(\omega_s) = (1-r)\{1-P_b(\omega_s)\}$. Hence, the central planner's optimization problem is given by the following profit maximization:

$$\max_{(\omega_s,\omega_b)\in[0,1]^2} \{\omega_s D_b(\omega_s) - \omega_b S_s(\omega_b)\}$$

subject to the equivalence of demand and supply, i.e.,

$$D(\omega_s) = S(\omega_b)\ .$$

6. Position Exchanges

This section investigates the problem of allocating *heterogeneous* items termed the *position exchange*; we can model it within our framework. There exist $m+1$ positions as the heterogeneous items to be traded. Each player $i \in N$ possesses a single position $l_i \in \{1,...,m+1\}$ as his initial endowment. Position $m+1$ implies the null position for which any player has zero valuation. We assume that the players' initial endowments are different with each other whenever some of them is a non-null position, i.e., for every $i \in N$ and every $j \in N \setminus \{i\}$,

$$l_i \neq l_j \qquad \text{if } l_i \neq m+1.$$ [6]

A special case of position exchange is so-called *the position auction*, where the central planner possesses the entire non-null positions, i.e.,

$$l_i = m+1 \text{ for all } i \in N.$$

The sponsored search auction is an example[7].

Each player i has the valuation for each position $l \in \{1,...,m+1\}$ given by a linear form $\alpha(l)\omega_i$, where $\alpha(l)$ is a fixed non-negative integer, implying, for instance, the click number in position l in the context of sponsored search. Note that $\alpha(l)$ is common across players, and that ω_i implies player $i's$ valuation per click. We assume that the lower the position is, the more valuable it is for any player, that is,

$$\alpha(1) > \alpha(2) > \cdots > \alpha(m) > \alpha(m+1) = 0.$$

We can model this position exchange as multiunit exchange with restriction on allocations by regarding each position as a bundle of homogeneous commodities (regarding one click as one unit), i.e.,

$$e = \sum_{l=1}^{m} \alpha(l), \text{ and } e_i = \alpha(l_i) \text{ for all } i \in N.$$

The set of feasible allocations $\hat{A}$ is specified as the proper subset of A such that $a \in \hat{A}$ if and only if for every $i \in N$,

[6] There might be multiple players who have the null positions as their initial endowments.

[7] See Edelman, Ostrovsky, and Schwarz (2007), Varian (2007), Edelman and Schwarz (2010), and Athey and Ellison (2011), for instance.

$$a_i \in \{\alpha(1), ..., \alpha(m), \alpha(m+1)\},$$

and for every $j \in N \setminus \{i\}$,

$$a_i \neq a_j \qquad \text{if } a_j > 0.$$

Note that the profile of their initial endowments is feasible, i.e., $(\alpha(l_i))_{i \in N} \in \hat{A}$. Each player $i's$ valuation function v_i is given by

$$v_i(a_i, \omega_i) = a_i \omega_i \text{ for all } (a_i, \omega_i) \in \{0, ..., e\} \times [0,1].$$

We can calculate unit valuations, MVVs, and MUVVs as

$$w_i(a_i, \omega_i) = \omega_i \text{ for all } (a_i, \omega_i) \in \{1, ..., e\} \times [0,1],$$

$$u_i^*(a_i, \omega_i) = a_i\{\omega_i - \frac{1 - P_i(\omega_i)}{p_i(\omega_i)}\} \qquad \text{if } a_i \geq e_i,$$

$$u_i^*(a_i, \omega_i) = a_i\{\omega_i + \frac{P_i(\omega_i)}{p_i(\omega_i)}\} - \frac{e_i}{p_i(\omega_i)} \qquad \text{if } a_i < e_i,$$

$$y_i^*(a_i, \omega_i) = \omega_i - \frac{1 - P_i(\omega_i)}{p_i(\omega_i)} \qquad \text{if } a_i \geq e_i,$$

and

$$y_i^*(a_i, \omega_i) = \omega_i + \frac{P_i(\omega_i)}{p_i(\omega_i)} \qquad \text{if } a_i < e_i.$$

Assumption 1 and the property (14) automatically hold. We assume that the property of (15) in Assumption 2 holds in a strict sense, i.e., $y_i^*(a_i, \omega_i)$ is increasing in ω_i.

Let us denote by $(f, x) = (\hat{f}, \hat{x})$ the solution to the optimization problem (9) for the position exchange modeled as the multiunit exchange with restriction on allocations. It is clear that for every $i \in N$, there exists $\hat{\omega}_i : \{1, ..., m\} \times [0,1]^{n-1} \to [0,1]$ such that $\hat{\omega}_i(l, \omega_{-i})$ is decreasing in $l \in \{1, ..., m\}$, and for every $\omega \in [0,1]^n$,

$$\hat{f}_i(\omega) = \alpha(1) \qquad \text{if } \omega_i > \hat{\omega}_i(1, \omega_{-i}),$$

$$\hat{f}_i(\omega) = 0 \qquad \text{if } \omega_i < \hat{\omega}_i(m, \omega_{-i}),$$

and for every $l \in \{2, ..., m\}$,

$$\hat{f}_i(\omega) = \alpha(l) \qquad \text{if } \hat{\omega}_i(l-1, \omega_{-i}) > \omega_i > \hat{\omega}_i(l, \omega_{-i}).$$

We call $\hat{\omega}_i(l,\omega_{-i})$ the $(l)-th$ *pivotal* type for player i; he obtains position l or any better position if and (almost) only if his type is greater than $\hat{\omega}_i(l,\omega_{-i})$.

Theorem 5: *For every* $l \in \{2,...,m\}$, *suppose that* $\hat{f}_i(\omega) = \alpha(l)$. *Then,*

$$\hat{x}_i(\omega) = 0 \qquad \textit{if} \quad \hat{f}_i(\omega) = e_i, \textit{ i.e.,} \quad l = l_i,$$

$$\hat{x}_i(\omega) = -\sum_{k=l}^{l_i-1}\{\alpha(k)-\alpha(k+1)\}\hat{\omega}_i(k,\omega_{-i}) \quad \textit{if} \quad f_i(\omega) > e_i, \textit{ i.e.,} \quad l < l_i,$$

and

$$\hat{x}_i(\omega) = \sum_{k=l_i}^{l-1}\{\alpha(k)-\alpha(k+1)\}\hat{\omega}_i(k,\omega_{-i}) \quad \textit{if} \quad f_i(\omega) < e_i, \textit{ i.e.,} \quad l > l_i.$$

Proof: From the equalities (12) and $\hat{f}_i(\omega) = \alpha(l)$, it follows that

$$\hat{x}_i(\omega) = -\sum_{k=l}^{m}\{\alpha(k)-\alpha(k+1)\}\hat{\omega}_i(k,\omega_{-i}) + \sum_{k=l_i}^{m}\{\alpha(k)-\alpha(k+1)\}\hat{\omega}_i(k,\omega_{-i}),$$

which implies the equalities of this theorem.

Q.E.D.

According to the optimal mechanism $(\hat{f},\hat{x})$, if each player i purchases a better position $l < l_i$ than his initial endowment, then he *pays* the pivotal type valuation $\hat{\omega}_i(k,\omega_{-i})$ for any position k between his initial endowment and his purchased position ($l \le k < l_i$) multiplied by the increase in click number $\alpha(k)-\alpha(k+1)$. If each player i purchases a worse position $l > l_i$ than his initial endowment, then he *earns* the pivotal type valuation $\hat{\omega}_i(k,\omega_{-i})$ for any position k between his initial endowment and his purchased position ($l_i \le k < l$) multiplied by the increase in click number $\alpha(k)-\alpha(k+1)$.

The optimal design for sponsored search auction with symmetry, addressed by Edelman and Schwarz (2010), termed the *generalized second price auction with reserve price* is closely related to our design $(\hat{f},\hat{x})$. Theorem 5 generalizes their result by taking into account the *asymmetry* across players in terms of their type distributions, and also by extending the auction framework to the *exchange* framework.

7. Position Auctions: Generalized Japanese Auction

We further investigate *the position auction* as a special case of position exchange, where

$$e_i = 0\text{, i.e., } l_i = m+1 \text{ for all } i \in N .$$

In this case, for every $\omega \in [0,1]^n$, every $i \in N$, every $j \in N \setminus \{i\}$, and $l \in \{1,...,m\}$,

$$\omega_i - \frac{1-P_i(\omega_i)}{p_i(\omega_i)} \geq \omega_j - \frac{1-P_j(\omega_j)}{p_j(\omega_j)} \qquad \text{if } \hat{f}_i(\omega) > \hat{f}_j(\omega) .$$

The $(l)-th$ pivotal type $\hat{\omega}_i(l,\omega_{-i})$ is regarded as the type ω_i such that $\omega_i - \frac{1-P_i(\omega_i)}{p_i(\omega_i)}$ is equivalent to the $l-th$ largest $\omega_j - \frac{1-P_j(\omega_j)}{p_j(\omega_j)}$ among the other players $j \neq i$. We define $\underline{\omega}_i \in (0,1)$ by

$$\underline{\omega}_i - \frac{1-P_i(\underline{\omega}_i)}{p_i(\underline{\omega}_i)} = 0 ,$$

which implies the *reserve price* for player i. Note that

$$\hat{f}_i(\omega) = 0 \qquad \text{if } \omega_i < \underline{\omega}_i .$$

From Theorem 5, it follows that in the position auction, the optimal payment rule $\hat{x}$ is rewritten in a more tractable manner; for every $(i,\omega) \in N \times [0,1]^n$,

$$\hat{x}_i(\omega) = 0 \qquad \text{if } \hat{f}_i(\omega) = 0 ,$$

and for every $l \in \{1,...,m\}$,

$$\hat{x}_i(\omega) = -\sum_{k=l}^{m} \{\alpha(k) - \alpha(k+1)\} \hat{\omega}_i(k,\omega_{-i}) \quad \text{if } \hat{f}_i(\omega) = l .$$

The optimal mechanism $(\hat{f},\hat{x})$ in this case implies a generalized concept of the generalized second price auction with reserve price addressed by Edelman and Schwarz (2010); the asymmetry in terms of type distribution is newly taken into account in the present paper.

It is important to note that in (not the position exchange but) the position auction, the optimal allocation and payment $(\hat{f}_i(\omega), \hat{x}_i(\omega))$ for each player i do not depend on the types of the other players who are assigned better positions than him; for every $i \in N$ and every $(\omega, \omega') \in [0,1]^{2n}$, whenever for every $j \in N \setminus \{i\}$,

$$\omega_j = \omega'_j \qquad \text{if} \quad \hat{f}_i(\omega) > \hat{f}_j(\omega),$$

and

$$\omega_j \leq \omega'_j \qquad \text{if} \quad \hat{f}_i(\omega) < \hat{f}_j(\omega),$$

then it holds that

$$(\hat{f}_i(\omega), R_i(\omega)) = (\hat{f}_i(\omega'), R_i(\omega')).$$

This *irrelevance* property can make the optimal mechanism $(\hat{f}, \hat{x})$ easy to implement in practice: we introduce below an ascending clock auction format termed the *generalized Japanese auction (GJA),* which implements the optimal allocations and payment vectors induced by $(\hat{f}, \hat{x})$ through (mostly) dominant strategies, and can bring the process of determining the allocation and payment vector to light.[8]

For every $i \in N$ and every $\omega_i \in \Omega_i$, let us define

$$t_i(\omega_i) \equiv \omega_i - \frac{1 - P_i(\omega_i)}{p_i(\omega_i)}.$$

Note that

$$t_i(\omega_i) > 0 \quad \text{if and only if} \quad \omega_i > \underline{\omega}_i$$

For every $t \in [0,1]$, let us denote

$$\tilde{\omega}_i(t) = t_i^{-1}(t).$$

Note that the time t is equivalent to type $\tilde{\omega}_i(t)'s$ MUVV, i.e.,

$$\tilde{\omega}_i(t) - \frac{1 - P_i(\tilde{\omega}_i(t))}{p_i(\tilde{\omega}_i(t))} = t \quad \text{for all} \quad (i,t) \in N \times [0,1].$$

For every $(t, l, r) \in [0,1] \times \{1, ..., n\} \times R$, we define $\tilde{r}_i(t, l, r) \in [0,1]$ by

$$\tilde{r}_i(t, l, r) \equiv \frac{\{\alpha(l) - \alpha(l+1)\}\tilde{\omega}_i(t) + \alpha(l+1)r}{\alpha(l)} \qquad \text{if} \quad l \leq m,$$

and

$$\tilde{r}_i(t, l, r) = \tilde{\omega}_i(t) \qquad \text{if} \quad l \geq m+1.$$

[8] It might be a difficult problem to implement the optimal position exchange through a clock auction format, because the irrelevance property no longer holds.

Based on these definitions, we define the GJA as the following ascending clock auction format in the continuous time horizon $[0,1]$. At the initial time $t=0$, the auctioneer offers the unit price, equal to the reserve price, $r_i(0)=\underline{\omega}_i$ for each player i. The auctioneer continues to ascend the unit price $r_i(t)$. It is important to note that the auctioneer is permitted to offer *different* unit prices across players and to make the price-adjustment *history-dependent*. Any active player decides on whether to 'quit the GJA' or 'keep active'. Once he decides to quit, he never comes back to the GJA.

Fix an arbitrary time $t\in(0,1]$. We denote by $r_i(\tau)\in(0,1)$ the unit price that the auctioneer has offered each player $i\in N$ at any previous time $\tau\in[0,t)$. We denote by $\tilde{N}\subset N$ the set of all players who are active at the time t. We denote by $\tilde{\tau}\in[0,t)$ the last time at which there exists a player who has quitted the GJA; we let $\tilde{\tau}=0$ whenever $\tilde{N}=N$. Based on these notations, we specify the unit price that the auctioneer offers each active player i at time t as

$$r_i(t)=\tilde{r}_i(t,\left|\tilde{N}\right|-1,r_i(\tilde{\tau}))=\frac{\{\alpha(\left|\tilde{N}\right|-1)-\alpha(\left|\tilde{N}\right|)\}\tilde{\omega}_i(t)+\alpha(\left|\tilde{N}\right|)r}{\alpha(\left|\tilde{N}\right|-1)}.$$

If player i decides to quit the GJA at time t, he obtains position $\min[m+1,\left|\tilde{N}\right|]$ for the unit price $r_i(\tilde{\tau})$, which the auctioneer has offered him at the previous time $\tilde{\tau}$.

What each player has to do in the GJA is just to select the time at which he intends to quit. Clearly, any player i whose type ω_i is lesser than $\underline{\omega}_i$ prefers not to participate in the GJA, while any player i whose type ω_i is greater than $\underline{\omega}_i$ prefers to participate. Suppose that the time at which each player i with type $\omega_i\in[\underline{\omega}_i,1]$ intends to quit is given by $t_i\in[0,1]$. Let us denote

$$\tilde{\omega}_j=\tilde{\omega}_j(t_j)\ \text{ for all }\ j\in N\text{, and }\ l=\hat{f}_i(\tilde{\omega}).$$

Then, the resulting allocation and payment for each player i is given by

$$\hat{f}_i(\tilde{\omega})\ \text{ and }\ \hat{x}_i(\tilde{\omega})=-\sum_{k=l}^{m}\{\alpha(k)-\alpha(k+1)\}\hat{\omega}_i(k,\tilde{\omega}_{-i}).$$

This, along with ex post incentive compatibility of $(\hat{f},\hat{x})$, implies that it is a best response for each player i with type $\omega_i\in[\underline{\omega}_i,1]$ to make the time to quit the GJA equivalent to $t_i = t_i(\omega_i)$, i.e., the time that is equivalent to his MUVV value, irrespective of the other players' time selections. Hence, we have shown that the GJA can achieve the same allocation and payment vector as those induced by $(\hat{f}, \hat{x})$ as the outcome of the mostly dominant strategy profiles.

Edelman and Schwarz (2010) showed that with the symmetry assumption, the optimal sponsored search auction termed the generalized second price auction with reserve price can be implemented by the *generalized English auction* (GEA) with reserve price through Nash equilibria, where the auctioneer was restricted to offer the same unit price across players and prohibited from making the price-adjustment history-dependent. In the GJA, the auctioneer has the more controllability of price adjustment in term of history-dependence and heterogeneity across players than the GEA with reserve price. Because of this high price controllability, the best strategy for each player in the GJA could be much simpler than in the GEA; each player $i's$ best time $t_i = \tilde{\omega}_i^{-i}(\omega_i)$ to quit equals his MUVV in the history-independent manner.

In order to make the manner of determining the allocation and payments in the GJA clearer, let us consider an arbitrary type profile $\omega \in [0,1]^n$, where players are ordered according to the size of MUVV, and every player's MUVV is positive, i.e.,

$$\omega_1 - \frac{1-P_1(\omega_1)}{p_1(\omega_1)} > \omega_2 - \frac{1-P_2(\omega_2)}{p_2(\omega_2)} > \cdots > \omega_n - \frac{1-P_n(\omega_n)}{p_n(\omega_n)} > 0 .$$

According to the (mostly) dominant strategy, each player $i \in N$ intends to quit at the time $t_i(\omega_i) = \omega_i - \dfrac{1-P_i(\omega_i)}{p_i(\omega_i)}$. Since

$$t_1(\omega_1) > t_2(\omega_2) > \cdots > t_n(\omega_n) > 0 ,$$

each player $i \in \{1,...,m\}$ receives the non-null position i for the unit price that the auctioneer have offered him when player $i+1$ has quitted, i.e., $r_i(t_{i+1}(\omega_{i+1}))$; any player $i \in \{m+1,...,n\}$ receives the null position $m+1$.

Until the time of $t_{m+1}(\omega_{m+1})$, the auctioneer continues to offer any active player i the unit price given by

$$r_i(t) = \tilde{\omega}_i(t) .$$

At any time $t \in (t_{m+1}, t_m]$, the auctioneer offers any active player i the unit price given by

$$r_i(t) = \frac{\{\alpha(m-1) - \alpha(m)\}\tilde{\omega}_i(t) + \alpha(m) r_i(t_{m+1})}{\alpha(m-1)}$$

$$= \frac{\{\alpha(m-1) - \alpha(m)\}\tilde{\omega}_i(t) + \alpha(m)\tilde{\omega}_i(t_{m+1})}{\alpha(m-1)}.$$

Recursively, for every $l \in \{2,...,m\}$, at any time $t \in (t_{l+1}, t_l]$, the auctioneer offers any active player i the unit price given by

$$r_i(t) = \frac{\{\alpha(l-1) - \alpha(l)\}\tilde{\omega}_i(t) + \alpha(l) r_i(t_{l+1})}{\alpha(l-1)}.$$

Hence, the unit price that each player $i \in \{1,...,m\}$ pays for the purchase of position i is equivalent to

$$r_i(t_{i+1}) = \frac{\sum_{k=i}^{m} \{\alpha(k) - \alpha(k+1)\}\tilde{\omega}_i(t_{k+1})}{\alpha(i)}.$$

Since

$$\tilde{\omega}_i(k, \omega_{-i}) = \tilde{\omega}_i(t_{k+1}) \quad \text{for all} \quad k \in \{i,...,m\},$$

it holds that $\hat{x}_i(\omega) = -\alpha(i) r_i(t_{i+1})$.

8. Conclusion

This paper investigated a class of allocation problems termed multiunit exchanges with/without restriction on allocations, where not only the central planner but also the participants brought commodities to sell. This class included important situations such as platforms in double auctions and heterogeneous position allocations as special cases. By introducing the concept termed modified virtual valuation (MVV), which was defined as the valuation minus the bargaining rent as well as the informational rent, the unit term of which (MUVV) implied a hybrid of marginal revenue and marginal cost, we demonstrated a tractable characterization result of optimal mechanism design with ex post incentive compatibility and ex post individual rationality. With mild monotonicity assumptions, this optimization problem could be replaced with the maximization of the sum of expected values of MVVs. By utilizing these results, we investigated the platform-provider's revenue maximization in the single-unit demand case and in position exchanges. We further showed a new design of ascending clock auction format that implements the optimal position auction through mostly dominant strategies.

The arguments in this paper depended on the standard assumptions such as quasi-linearity, private values, independent type distributions, risk-neutrality, and no externality. It might be important as future researches to eliminate or weaken these assumptions. For instance, Figueroa and Skreta (2011) argued that the presence of externalities makes their outside opportunity values type-dependent in a different manner. The present paper did not take such externalities into account; the type-dependence of the outside opportunity value for each player was caused by the presence of his non-null initial endowment.

This paper assumed one-dimensional type spaces. It might be substantial to investigate multi-dimensional type spaces if we attempts to extend this paper to general dynamical multi-object trading; it is inevitable that we are confronted with such multi-dimensionality issues if players gradually receive private information in dynamical trading procedures. See Bergemann and Said (2010), for instance.

It was implicit to assume that the participants could not either access other competing platforms or directly trade with each other. It might be interesting to investigate strategic behavior of competing platforms as an extension of this paper, where the central planners might be able to provide differentiated contents with each other. In this respect, the literature of multi-sided markets could be relevant. See Rochet and Tirole (2003), for instance.

This paper required incentive compatibility and individual rationality not in the interim term but in the ex-post term. Mookherjee and Riechelstein (1992) showed that any incentive compatible mechanism with interim individual rationality could be implemented by an ex-post incentive compatible mechanism with interim individual rationality. However, it is not necessarily possible for any incentive compatible mechanism with interim individual rationality to be implemented by an ex post incentive compatible mechanism with *ex-post* individual rationality. The effect of replacing the ex-post term with the interim term on the optimization problem remains unsolved.

References

Armstrong, M. (2000): “Optimal Multi-Object Auctions,” *Review of Economic Studies* 67, 181-455.

Athey, S. and G. Ellison (2011): “Position Auctions with Consumer Search,” *Quarterly Journal of Economics* 126, 1213-1270.

Bergemann, D. and M. Said (2010): “Dynamic Auctions: Survey,” mimeo.

Branco, F. (1996): “Multiple Unit Auctions of an Indivisible Good,” *Economic Theory* 8, 77–101.

Cramton, P., R. Gibbons, and P. Klemperer (1987): “Dissolving a Partnership Efficiently,” *Econometrica* 55, 615–632.

Edelman, B., M. Ostrovsky, and M. Schwarz (2007): “Internet Advertising and the Generalized Second Price Auction,” *American Economic Review* 97, 242-259.

Edelman, B. and M. Schwarz (2010): “Optimal Auction Design and Equilibrium Selection in Sponsored Search Auctions,” *American Economic Review* 100, 597-602.

Figueroa, N. and V. Skreta (2009): “The Role of Optimal Threats in Auction Design,” *Journal of Economic Theory* 14, 884–897

Krishna, V. (2010): *Auction Theory*, Academic Press.

Maskin, E. and J. Riley (1989): “Optimal Multi-Unit Auctions,” in *The Economics of Missing Markets, Information, and Games* (F. Hahn, Ed.), 1989, Oxford University Press.

Matsushima, H. (2010): “Efficient Combinatorial Exchanges,” Discussion Paper CIRJE-F-826, University of Tokyo.

Milgrom, P. (2004): *Putting Auction Theory to Work*, Cambridge University Press, Cambridge.

Monteiro, P. (2002): “Optimal Auctions in a General Model of Identical Goods,” *Journal of Mathematical Economics* 37, 71-79.

Mookherjee, D. and S. Reichelstein (1992): “Dominant Strategy Implementation of Bayesian Incentive Compatible Allocation Rules,” *Journal of Economic Theory* 56, 378-399.

Myerson, R. (1981): “Optimal Auction Design,” *Mathematics of Operations Research* 6, 58-73.

Myerson, R. and M. Satterthwaite (1983): “Efficient Mechanisms for Bilateral Trading,” *Journal of Economic Theory* 29, 265-281.

Palfrey, T. (1983): “Bundling Decisions by a Multiproduct Monopolist with Incomplete Information,” *Econometrica* 51, 463-484.

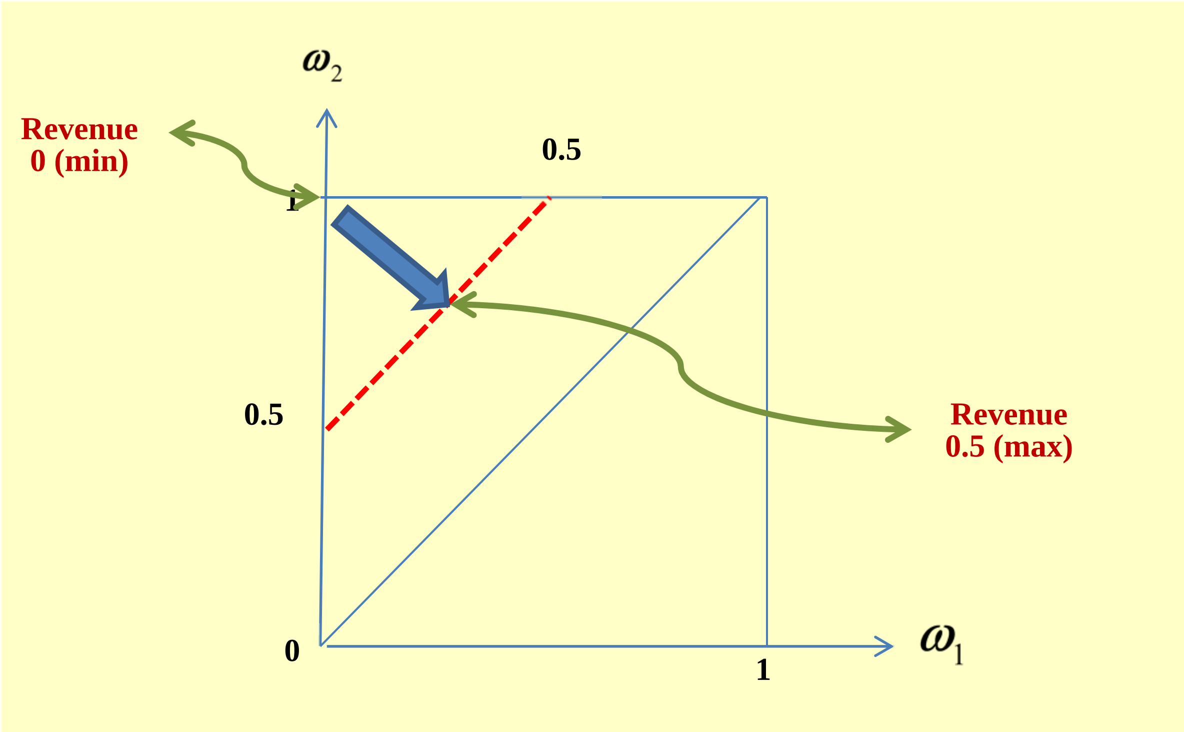
Rochet, J.-C. and J. Tirole (2003): “Platform Competition in Two-Sided Markets,” *Journal of the European Economic Association* 1, 990-1029.

Riley, J. and W. Samuelson (1981): “Optimal Auctions,” *American Economic Review* 71, 381-392.

Segal, I. and M. Whinston (2010): “Property Right,” forthcoming in *the Handbook of Organizational Economics* (Gibbons, R. and J. Roberts, eds.), Princeton University Press.

Ulku, L. (2009): “Optimal Combinatorial Mechanism Design,” ITAM Working Paper 0903.

Varian, H. (2007): “Position Auctions,” *International Journal of Industrial Organization* 25, 1163-1178.



[Figure 1]