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Yasushi Asako
Waseda University
Kozo Ueda
Bank of Japan

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The Boy Who Cried Bubble: Public Warnings Against Riding Bubbles*

Yasushi Asako† and Kozo Ueda‡

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Abstract

Governments seemed unsuccessful in their attempts to stop bubbles through the use of warnings. This paper examines the effects of public warnings using a simple model of riding bubbles. We show that public warnings against a bubble can stop it if investors believe that the government issues such warnings only after bubbles start. Moreover, the bubble may crash before the warning. If there is the possibility that the government issues a warning even though bubble does not occur, then warnings cannot stop the bubble immediately. Our model suggests that, for public warnings, it is not type-II errors but rather type-I errors that are important in preventing bubbles. Public warnings are effective when they provide information to less-informed investors.

Keywords: riding bubbles, crashes, public warnings, asymmetric information

JEL Classification Numbers: C72, D82, D84, E58, G12, G18

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†Waseda University (yasushi.asako@aoni.waseda.jp)

‡Bank of Japan (kouzou.ueda@boj.or.jp)

There really was a wolf here! The flock has scattered!

I cried out, "Wolf!" Why didn't you come?

The Boy Who Cried Wolf in *Aesop's Fables*

1 Introduction

History is rife with examples of bubbles and crashes (see Kindleberger and Aliber [2005] and the recent financial crisis that started in the summer of 2007). The recent financial crisis, in particular, reminded policymakers that preventing bubbles is paramount to maintaining financial and economic stability. However, we have limited knowledge of how bubbles arise and how they can be prevented.

Using the model of riding bubbles, this paper considers the role of public policy in dealing with bubbles and specifically whether public warnings can prevent bubbles. In recent years, some studies have indicated that asymmetric information creates bubbles (see below for details). If only a fraction of investors know that a current stock price is over-priced, they may have an incentive to ride the bubble: they hold their stock for a period and then sell it for a higher price before other investors sell. If instead all the investors are equally aware that the current price is over-priced, then they will probably lose by riding the bubble. In this respect, public information is considered important to reducing the degree of asymmetric information and thus eliminating the bubble.

However, government authorities did not seem able to successfully stop bubbles by means of warnings. Kindleberger and Aliber (2005, p.16) state

> One question is whether manias can be halted by official warning—moral suasion or jawboning. The evidence suggests that they cannot, or at least that many crises followed warnings that were intended to head them off.

For example, in February 1929, Paul Warburg, Chairman and one of the founders of the Federal Reserve Board (Fed), warned that the U.S. stock prices were too high and the current situation was similar to the crisis in 1907. Despite the warning, stock prices continued to increase. In December 1996, Alan Greenspan, Chairman of the Fed, warned that the U.S. stock market was "irrationally exuberant," but the stock prices continued to increase (Kindleberger and Aliber [2005, p.80]). Japan encountered a similar challenge during the bubble economy in the late 1980s. Okina et al. (2001, p.422) indicate that the Bank of Japan "had already voiced concern over the massive increase in money supply and the rapid rise in asset prices in the summer of 1986." In fact, Yasushi Mieno, Deputy Governor of the Bank of Japan, described the situation as "dry wood" (which can easily catch fire, implying the risk of high inflation). However, according to Okina et al. (2001, p.430), "it could not succeed in persuading the public."

Why were those central banks unable to stop bubbles by means of public warnings? Why did the markets ignore these warnings even though the theory implies the importance of public information? This paper examines the effects of public warnings using a simple model of riding bubbles.

Our model is based on the model of riding bubbles by Abreu and Brunnermeier (2002, 2003). A bubble is interpreted as a situation in which an asset price is higher than its fundamentals. At some point, an investor receives a bubble signal and recognizes correctly that the current price is overpriced. However, the investor does not know either how many investors are aware of this price disparity or when the bubble began. The bubble crashes when a certain fraction of rational investors sell their stock. Rational investors thus face a trade-off: if they wait to sell their stock, the price may increase, but the probability of crashing the bubble also increases. As a result, they may ride the bubble for a certain time.[1] We simplify the model of Abreu and Brunnermeier (2002, 2003) so that it can be extended for this purpose. In our simplified model, two discrete types of rational investors, more informed and less informed, exist in terms of private information, instead of continuously distributed rational investors.[2]

We introduce public warnings to the model as a public signal. To the best of our knowledge, no attempt has been made to theoretically examine the role of public warnings on riding bubbles. In the model, public warnings are given exogenously; we do not analyze any

[1]This theory is supported by several empirical studies and experimental studies, such as Temin and Voth (2004) and Brunnermeier and Morgan (2010).

[2]This simplification not only yields the same riding bubble equilibrium as that in Abreu and Brunnermeier (2002, 2003) but also allows the model to be extended.

strategic choice of a government authority to issue warnings. By considering the following types of public warnings, we show that not type-II error but type-I error is important in preventing bubbles. The first two types of warnings are issued after a bubble starts. One is a deterministic warning, which is issued at a definite period after the bubble starts. The other type is a probabilistic warning. The date of a warning is distributed over several periods after the bubble occurs. In both cases, warnings involve the risk of delay. Warnings may not be issued although the bubble has started (type-II error). Nevertheless, in this case, warnings are effective: the bubble crashes at the date of the warning. Moreover, interestingly and importantly, the bubble may crash before the warning. Because investors know that the bubble crashes at the date of the warning, they may want to sell earlier. Finally, we consider a mixed case in which the government issues two types of warnings, noisy and deterministic. A noisy warning does not depend on when the bubble starts, but a deterministic warning does depend on the bubble onset. The warning is possibly issued after the bubble starts, but also possibly issued even though the asset price is supported by the fundamentals. In such a case, a warning helps shorten the bubble duration but cannot stop the bubble immediately.

Whether public warnings help investors deduce their types is the key to these results. In the cases of deterministic and probabilistic warnings, some investors are able to deduce that other investors were previously aware of the bubble. These less-informed investors recognize that they cannot sell their stock at a high price if they maintain their bubble-riding strategy; therefore, they sell their stock earlier. The bubble then crashes at the date of the warning. In the mixed case, the warning gives investors imperfect but partially helpful information about their types. Because of the warning, investors who became aware of the bubble later than did other investors revise their beliefs on their types, becoming more pessimistic about the opportunity to sell their stock at a high price. As a result, they shorten the period of riding the bubble.

1.1 Policy Implications and Previous Studies

Our paper has two policy implications. First, for the public warning to be effective, the type-I error is extraordinarily important: governments should not warn unless a bubble exists. With regards to this rationale, the literature offers two perspectives. Okina et al. (2001), Borio and Lowe (2002), and White (2006) propose implementing preemptive policy against bubbles. They argue that governments should not be too late to contain the accumulation of imbalances caused by a bubble, weighing the type-II error. Mishkin (2007), however, argues against such preemptive policy. Emphasizing the type-I error (i.e., errors in detecting bubbles), he instead proposes a mop-up policy after a bubble collapses. Regarding the importance of the type-I error, our paper may appear more similar to the latter perspective, but it in fact differs in its implications. The question at hand is not whether preemptive policy is preferable, but whether such policy is effective at preventing bubbles. For the public warning to be effective, a government should aim to minimize the type-I error. As long as the type-I error remains large, preemptive warnings have no consequence, so those conflicting views have no value.

Second, announcements targeted at less-informed investors are important in preventing bubbles or shortening the duration of bubbles. We find that once less-informed investors deduce their type, such investors need to change their bubble-riding strategy because they know that they cannot sell at a high price before more-informed investors do, as long as they maintain their strategy. As a result, the duration of the bubble-riding shortens.

Previous studies have implemented various frameworks to explain bubbles. Classically, bubbles are explained by rational-bubble models within a rational-expectations framework (Samuelson [1958] and Tirole [1985]). These models are used to analyze the macro-implications of bubbles. Bubbles and crashes are given exogenously, investors have symmetric information, and coordination expectation is exogenously assumed. Therefore, those studies do not focus on the strategies of individuals. Recently, some models have shown that investors hold a bubble asset because they believe that they can sell it for a higher price in the future. These models focus on the microeconomic aspect of bubbles, assuming asymmetric information.[3] Public warnings thus play an important role in mitigating asymmetric information

[3] It is well known that the asymmetric information held by investors cannot explain bubbles. The key is the no-trade theorem (see Brunnermeier [2001]): that investors do not hold a bubble asset when they have common knowledge on a true model because they can deduce the content of the asymmetric information (Allen et al. [1993] and Morris et al. [1995]). Therefore, some studies explain bubbles by introducing noise traders (DeLong et al. [1990]), heterogeneous belief (Harrison and Kreps [1978], Scheinkman and Xiong [2003]), or principal-agent problems between fund managers and investors (Allen and Gordon [1993], Allen

and affecting the occurrence of bubbles.

Section 2 presents the simple model of riding bubbles, and Section 3 analyzes the effects of public warnings and discusses implications. Section 4 concludes the paper.

2 Simple Model of Riding Bubbles

2.1 Setup

This section presents the model of riding bubbles, which is based on and simplified from the model of Abreu and Brunnermeier (2003), to analyze the effects of public warnings.

Prior to $t = 0$, the stock price index coincides with the fundamental value, growing at the risk-free rate r. We assume that $r = 0$ without a loss of generality. From $t = 0$ onwards, the stock price grows at a rate of $g > 0$, that is, $\exp(gt)$. Up to some random time t_0, the higher price is justified by the fundamental value, but is not after t_0. The bubble starts at t_0. The fundamental value grows at the rate of zero from t_0, so the price justified by the fundamental is $\exp(gt_o)$, and a bubble component is given by $\exp(gt) - \exp(gt_o)$ where $t > t_o$.

There exists a continuum of rational investors of size one. They receive a private signal, thus becoming aware of the bubble. Two types of rational investors exist. The fraction $\alpha \in (0, 1)$ of them are more-informed investors (type-H), while $1 - \alpha$ of them are less-informed investors (type-L). Type-H receives the private signal at t_0, while type-L receives it at $t_0 + \eta$, where $\eta > 0$. We denote their types as $i = H, L$. All type-i investors receive a private signal at t_i, where $t_H = t_0$ and $t_L = t_0 + \eta$.

Rational investors decide when to sell their stock. Assume that they cannot buy their stock back and that they never sell before they receive the signal because the price increases to at least $\exp(gt_0)$. Ex ante, no investor knows his type. When α of the rational investors sell their stock, the bubble crashes (endogenous crash).[4] Therefore, if all type-H investors

and Gale [2000]).

[4]This assumption can be justified by the existence of irrational investors. See Abreu and Brunnermeier (2002, 2003). To be precise, in Abreu and Brunnermeier (2003), when the fraction κ of investors sell their stock, the bubble crashes endogenously. In our model, the investors aware of the bubble before κ of the investors become aware are categorized as type-H, and the others are categorized as type-L. We use α

sell their stock, the bubble crashes.

If less than α of rational investors sell their stock before $t_o + \bar{\tau}$, the bubble crashes automatically at $t_o + \bar{\tau}$ (exogenous crash). This setup is described in Figure 1. We assume that $\bar{\tau} > \eta$ and that $\bar{\tau}$ and t_0 are multiples of η. We also assume that $\alpha \leq 1/2$. The fraction of type-L is higher than the fraction of type-H. In other words, if all type-L investors ($1-\alpha$ of investors) sell their stock, the bubble crashes.[5]

[Figure 1 Here]

Note that α has two meanings: (1) the fraction of type-H and (2) the fraction of investors such that the bubble would crash endogenously were these investors to sell their stock. Even if these two fractions differ, our results mentioned below do not change on the three conditions: (1) the fraction of type-H is α, (2) the bubble crashes endogenously when $\alpha'(< \alpha)$ of investors sell their stock, and (3) when less than $1 - \alpha$ of investors sell their stock at the same time before others sell, all of them can sell at a high price.

2.2 Equilibrium

In what follows, we concentrate on a pure strategy and symmetric Nash equilibrium.

First, there exists an equilibrium in which type-H sells their stock any period between t_0 and $t_0 + \eta$. Suppose that rational investors sell their stock at $t_i + \tau$, where $0 \leq \tau < \eta$. In this case, when an investor receives the signal, he deduces that he is type-H because the bubble crashes before $t_0 + \eta$ and the type-L investors never receive the signal. Therefore, the ex post probability of being type-H after the signal is received is 1. Even if one type-H investor deviates and sells their stock after $t_0 + \tau$, the bubble crashes at $t_0 + \tau$ because the investor is atomic and only one investor's deviation does not affect the market. The expected payoff thus decreases for the investor. As an opposite case, suppose that one type-H investor

instead of κ.

[5] We also assume that when less than $1 - \alpha$ of investors simultaneously sell their stock before others sell, these investors can sell at a high price. On the other hand, if more than $1-\alpha$ of investors simultaneously sell their stock, they cannot sell at a high price, receiving only $\exp(gt_o)$. The main implications do not change even if we assume that some fraction of investors can sell at a high price when too many investors sell at the same time t. In this case, the expected payoff lies between $\exp(gt_o)$ and $\exp(gt)$.

deviates and sells their stock before $t_0 + \tau$. After his sale, the price continues to increase until $t_0 + \tau$, so the payoff also decreases for the investor. Therefore, regardless of the value of α, this is equilibrium. This equilibrium represents the case in which no one rides the bubble, in that the bubble crashes before less-informed investors receive the private signal.

There is another more important equilibrium in which investors ride the bubble. Both type-H and type-L receive a signal, and they are uncertain about their types. Suppose that investors sell their stock at $t_i + \bar{\tau}$. In this case, the expected payoff is $\alpha \exp(g(t_i + \bar{\tau})) + (1 - \alpha) \exp(g(t_i - \eta))$. The first term corresponds to the payoff for type-H. If an investor is type-H, he can sell the asset at the price $\exp(g(t_i + \bar{\tau}))$. Its probability is given by α. The second term corresponds to the payoff for type-L. If he is type-L, the bubble crashes before he sells, so the price is $\exp(g(t_i - \eta))$. Its probability is given by $1 - \alpha$. Suppose an investor deviates and sells his stock at any period between $t_i - \eta + \bar{\tau}$ and $t_i + \bar{\tau}$. If he is type-L, regardless of this deviation, the expected payoff remains $\exp(g(t_i - \eta))$ because the bubble crashes at $t_i - \eta + \bar{\tau}$. If he is type-H, the expected utility decreases. Thus, they never deviate in this way. If they deviate and sell their stock after $t_i + \bar{\tau}$, the bubble crashes before they sell regardless of their types. Thus, they never deviate in this way, either. The only possibility to deviate is to sell their stock just before $t_i - \eta + \bar{\tau}$. The expected payoff from this deviation becomes $\exp(g(t_i - \eta + \bar{\tau}))$ because they can sell their stock at this price with certainty. Comparing the payoffs suggests that investors sell their stock at $t_i + \bar{\tau}$ if

$$\begin{aligned} & \alpha \exp(g(t_i + \bar{\tau})) + (1 - \alpha) \exp(g(t_i - \eta)) \\ \geq \; & \exp(g(t_i - \eta + \bar{\tau})), \end{aligned}$$

$$\alpha \geq \frac{\exp(-g\eta)\left[\exp(g\bar{\tau}) - 1\right]}{\exp(g\bar{\tau}) - \exp(-g\eta)}. \tag{1}$$

If this equation holds, investors do not have an incentive to deviate by selling earlier than $\exp(g(t_i + \bar{\tau}))$. On the other hand, if this equation does not hold, an investor has an incentive to deviate by selling at $t_i - \eta + \bar{\tau}$. Define τ^*, which satisfies

$$\alpha = \frac{\exp(-g\eta)\left[\exp(g\tau^*) - 1\right]}{\exp(g\tau^*) - \exp(-g\eta)}. \tag{2}$$

Investors do not deviate by selling earlier when they sell their stock at or before $t_i + \tau^*$. If (1) does not hold, then τ^* is below $\bar{\tau}$ because the right-hand side of (2) increases with τ^*.

If α is small enough to be below $\frac{1-\exp(-g\eta)}{\exp(g\eta)-\exp(-g\eta)}$, then τ^* is below η. Denote α^* such that $\tau^* < \eta$ if $\alpha < \alpha^*$, that is,

$$\alpha^* = \frac{1 - \exp(-g\eta)}{\exp(g\eta) - \exp(-g\eta)}. \tag{3}$$

Definition 1 *If (1) holds, then $\tau^* = \bar{\tau}$. If not, define τ^* such that $\alpha = \dfrac{\exp(-g\eta)\,[\exp(g\tau^*) - 1]}{\exp(g\tau^*) - \exp(-g\eta)}$.*

Lemma 1 *Assume $\alpha > \alpha^*$. Investors have an incentive to deviate and sell at $t_i - \eta + \tau$ if $\tau > \tau^*$, and they do not if $\tau < \tau^*$. Investors are indifferent to the strategy to sell their stock at $t_i + \tau^*$ and the strategy to sell their stock at $t_i - \eta + \tau^*$.*

The period τ^* is the upper bound of the periods during which investors wait to sell their stock. For all positive τ lower than τ^*, there is an equilibrium in which investors wait to sell their stock until $t_i + \tau$. As the fraction of type-H (α) increases, τ^* increases. That is, because they believe more that they are more informed ex ante, investors tend to wait longer. The bubble may continue for longer periods. As the interval of signals between type-H and L lengthens (higher η), investors tend to wait longer, and the bubble may last longer. Thus, we obtain the following proposition.

Proposition 1 *There are two Nash equilibria in this game.*

1. *Regardless of the value of α, the bubble crashes at any period between t_0 and $t_0 + \eta$.*

2. *If $\alpha \geq \alpha^*$, the bubble crashes at $t_o + \tau$, where $\eta \leq \tau \leq \tau^*$.*

In the following sections, we concentrate on the second case. To this end, we introduce the following assumption.[6]

Assumption 1 $\alpha \geq \alpha^*$.

[6] Even though we assume $\alpha \leq 1/2$, this assumption is not crucial for $\alpha \geq \dfrac{1 - \exp(-g\eta)}{\exp(g\eta) - \exp(-g\eta)}$. The right-hand side is at most $1/2$ when $g\eta$ is zero, decreasing to zero as $g\eta$ increases.

2.3 Uniqueness and Coalition-proof Nash equilibrium

Before considering the role of public warnings, it is worth comparing our model with that of Abreu and Brunnermeier (2003). As we stated in the Introduction, we simplify the model of Abreu and Brunnermeier (2002, 2003) so that we can incorporate public warnings. Without public warnings, the model of Abreu and Brunnermeier (2003) yields a unique Nash equilibrium, while our simplified model tends toward multiple Nash equilibria. In Abreu and Brunnermeier (2003), investors become aware of the bubble sequentially and continuously. If one investor waits slightly longer, the probability of a crash increases continuously. Thus, there is a unique period until which an investor waits to sell. On the other hand, our model depicts only two types of investors, and each investor is atomic. If all type-H investors choose a certain period to sell their stock, the bubble crashes in the period with certainty, even if one investor waits slightly longer. If all type-H investors wait slightly longer, then the probability of crash does not increase because there is a lag before type-L investors receive the signal. Thus, there exist continuous equilibria with respect to the time to sell. Despite the difference, the critical implication holds as it does in Abreu and Brunnermeier (2003). Because investors are aware of the bubble, but cannot know when the bubble starts, a trade-off arises: if investors wait longer, the bubble may crash before they sell, but they may be able to sell their stock at a higher price.

To obtain the same result as Abreu and Brunnermeier (2003), we assume that the bubble continues until the upper bound of the duration of the bubble, $t_0+\tau^*$. According to the above proposition, it is possible that the bubble ends before the upper bound. However, to simplify and to concentrate on the most interesting case, we introduce the following assumption.

Assumption 2 *The bubble continues until the upper bound, which is the longest period that investors do not have an incentive to deviate by selling earlier.*

Assumption 2 can be justified if we employ, with some minor changes of our settings, the (pure-strategy and symmetric) coalition-proof Nash equilibrium introduced by Bernheim, Peleg, and Whinston (1987).[7] In other words, the unique Nash equilibrium in Abreu and

[7] A coalition-proof equilibrium allows players to communicate prior to the game, reaching an agreement

Brunnermeier (2003) corresponds to the (pure-strategy and symmetric) coalition-proof Nash equilibrium in our simpler model.

Corollary 1 *Suppose Assumption 1. The pure-strategy and symmetric coalition-proof Nash equilibrium in our model is that all investors sell at* $t_i + \tau^*$ *and that the bubble crashes at* $t_o + \tau^*$. *This equilibrium is unique.*

The proof of this proposition is provided in Appendix A.1. In the following parts, we simply assume Assumption 2 and show the upper bound of the bubble duration.[8]

3 Public Warning

In this section we introduce a public warning to the above model. Let t_W denote the period during which the warning is issued. We analyze four types of warning: (1) deterministic warning, (2) probabilistic warning, (3) noisy warning, and (4) a mixture case of (1) and (3).[9]

to coordinate their actions in a mutually beneficial way. A coalition-proof Nash equilibrium requires that the agreement is not subject to an improving deviation, which is self-enforcing by any coalition of players. A deviation is self-enforcing if there is no further self-enforcing and improving deviation available to a proper subcoalition of players. If we employ the strong Nash equilibrium introduced by Aumann (1959), which does not require self-enforcing, then there is no equilibrium because when all investors sell the stock at $t_i + \tau^*$, all investors' payoffs can be improved by selling later than $t_i + \tau^*$. A strong Nash equilibrium is a coalition-proof Nash equilibrium, so there is no such strong Nash equilibrium.

[8] Although a coalition-proof Nash equilibrium can be applied to all analyses in the following parts, we only use Nash-equilibrium concepts and Assumption 2. Note that while the model of Abreu and Brunnermeier (2003) yields the unique bubble-riding equilibrium, our model embeds broader cases including when bubble riding does not occur (Proposition 1 (1)). That bubbles do not appear all the time in the real world is likely one advantage of our model. For that reason, even if we can derive the unique equilibrium by borrowing the idea of a coalition-proof Nash equilibrium, we do not want restrict ourselves to this case only. In addition, it is well known that a coalition-proof Nash equilibrium is a strict concept because it may not exist depending on game settings.

[9] Abreu and Brunnermeier (2003) also analyze the effect of uninformative events. The public warning differs from uninformative events in three ways. First, uninformative events do not depend on the beginning date of the bubble, t_0. Conversely, deterministic and probabilistic warnings in our model depend on the date of warning. Second, Abreu and Brunnermeier (2003) assume that events are only observed by investors who become aware of the bubble. Considering the nature of public warnings, we do not make such an assumption.

As we discussed in the previous section, we examine the upper bound until which the bubble continues and assume that the bubble continues until this upper bound by assuming Assumption 2. At any period before the upper bound, the bubble could crash endogenously, constituting another Nash equilibrium.

3.1 Deterministic Warning

Assume that the warning is issued at a certain period after the bubble starts (i.e., $t_W = t_0 + \tau_W$) and that all players know the value of $\tau_W > 0$.

For the first case, suppose that the warning is issued both earlier than the upper bound of the bubble period and later than the period in which type-L receives the private signal (i.e., $\eta < \tau_W < \tau^*$). In this case, all investors deduce their types before $t_0 + \tau^*$. When investors receive the warning at $t_i + \tau_W - \eta$, they deduce that they are type-L. They recognize that they cannot sell their stock before the bubble crashes if they wait until $t_i + \tau^*$. Thus, the strategy of type-L investors should be to sell their stock just before type-H investors sell their stock. On the other hand, type-H investors also deduce their type because they receive the warning at $t_i + \tau_W$, and their strategy should be to sell their stock just before type-L investors sell their stock. Therefore, if they receive the warning, all investors sell their stock immediately.

Interestingly, this type of deterministic warning is never issued at equilibrium. Investors do not wait until or later than $t_i + \tau_W$ because the bubble crashes with the warning, which is issued before or at the same time as the stock sale. If so, they cannot sell at a high price.

Suppose that they sell at $t_i + \tau$, where $\tau < \tau_W$. Because $\tau_W < \tau^*$, from Lemma 1, the expected utility decreases if an investor deviates from selling at $t_i + \tau$. Thus, any value $\tau < \tau_W$ can be an equilibrium in which they sell at $t_i + \tau$. As a result, type-H investors sell their stock at $t_0 + \tau$ before the warning is issued at $t_0 + \tau_W$, and the bubble crashes.[10]

For the second case, suppose that the warning period is earlier than the period at which type-L receives the private signal (i.e., $\tau_W < \eta$). If investors receive the warning before they get the private signal, they recognize that they are type-L. If investors get the private signal

[10] In this subsection, we do not suppose Assumption 2. If Assumption 2 holds, investors sell slightly before $t_i + \tau_W$.

before or at the same time as the warning, they recognize that they are type-H. All investors have an incentive to sell their stock immediately when they get the warning. However, this warning never be issued in equilibrium, either. Type-H can deduce their type before type-L can deduce, so investors who get the private signal before the warning sell their stock before the warning period. The bubble crashes before $t_0 + \tau_W$.

For the last case, suppose that the warning period is too late (i.e., $\tau^* < \tau_C$). The bubble crashes endogenously before the warning. In this case, the warning never affects the investors selling decisions.

Proposition 2 *Suppose Assumption 1. Suppose also that a public warning is deterministic and issued earlier than the endogenous crashing period without warning, $t_0 + \tau^*$. Then, the bubble crashes before the warning. The warning is never issued in equilibrium.*

3.2 Probabilistic Warning

As a more realistic case, we consider a probabilistic warning. The date of the warning $t_W = t_0 + \tau_W$ distributes over several periods after the bubble starts.

3.2.1 Two Possible Timings of Warning

Suppose that the warning period t_W is $t_0+2\eta$ with probability p and is $t_0+3\eta$ with probability $1-p$. Assume that $3\eta < \tau^*$. This timing is shown in Figure 2. Note that if $\tau^* \leq 2\eta$, both warnings have no meaning because the warning is issued after the bubble crashes endogenously. We discuss the case of $2\eta < \tau^* \leq 3\eta$ in Appendix A.2.

[Figure 2 Here]

We begin by examining what happens when a public warning is issued. First, suppose that investors do not receive the warning until $t_i+2\eta$. This assumption implies that they are type-H because type-L receives the warning by $t_i + 2\eta$ surely.[11] Such type-H investors will sell their stock just before type-L (who will receive the warning at $t_L + 2\eta$) sells their stock.

[11] Type-H receives the warning at either $t_H + 2\eta$ or $t_H + 3\eta$. Type-L receives the warning at either $t_L + \eta$ or $t_L + 2\eta$.

Second, suppose that investors receive the warning at $t_i + \eta$. This assumption implies that they are type-L. Such type-L investors will sell their stock just before type-H (who receives the warning at $t_H + 2\eta$) sells their stock. Consequently, when investors receive the warning at $t_i + 2\eta$, they recognize that their type is known to the other type of investors. They have no hope of selling their stock at a higher price regardless of their types if they sell their stock after the warning is issued. Therefore, investors who receive the warning at $t_i + 2\eta$ sell their stock immediately, and the bubble crashes. Anticipating this event, investors who receive the warning at $t_i + \eta$ also sell their stock immediately, implying that if the warning is issued, the bubble crashes immediately.

As is the case with the deterministic warning, the bubble may crash before the probabilistic warning is issued in equilibrium. We examine whether the situation in which investors sell their stock just before $t_i + 3\eta$ can be an equilibrium. Suppose that investors do not receive the warning and that η periods have passed after they receive the private signal. That is, investors are at $t_i + \eta$. In this case, investors have two choices: do not change the strategy (sell just before $t_i + 3\eta$) or deviate and sell before $t_i + 2\eta$. If the investor sells just before $t_i + 2\eta$, he succeeds in selling his stock before the bubble crashes. The expected payoff from this deviation is $\exp(g(t_i + 2\eta))$. If he does not change the strategy, his expected payoff is

$$\Pr(H|t_i+\eta)(1-p)\exp(g(t_i+3\eta))+\Pr(H|t_i+\eta)p\exp(g(t_i))+(1-\Pr(H|t_i+\eta))\exp(g(t_i-\eta)). \tag{4}$$

$\Pr(H|t_i+\eta)$ is the probability that this investor is type-H when he does not receive a warning at $t_i + \eta$. At $t_i + \eta$, there are two possible cases. First, this investor is type-H. Second, this investor is type-L, and the warning was not issued at $t_0 + 2\eta = t_L + \eta$. The joint probability that investors do not receive a warning at $t_i + \eta$ is $\alpha + (1 - \alpha)p$. Therefore, we obtain

$$\Pr(H|t_i + \eta) = \frac{\alpha}{\alpha + (1 - \alpha)p}.$$

If this investor is type-H, the probability that the warning is issued at $t_i + 2\eta$ is p. The first term of (4) suggests that, with probability $\Pr(H|t_i + \eta)(1 - p)$, the investor is type-H and the public warning will not be issued at $t_i + 2\eta$. Such an investor can succeed in selling his stock just before $t_i + 3\eta$ and receive $\exp(g(t_i + 3\eta))$. The second term suggests that, with probability $\Pr(H|t_i + \eta)p$, the investor is type-H and the public warning is issued at $t_i + 2\eta$. The bubble crashes immediately after the warning and the stock price falls to $\exp(g(t_i))$ because $t_0 = t_H$. The investor receives $\exp(g(t_i))$. The third term suggests that, with probability $1 - \Pr(H|t_i + \eta)$, the investor is type-L. Such an investor cannot sell his stock with the high price, so he gets only $\exp(g(t_i - \eta))$ because $t_0 = t_L - \eta$.

His expected payoff from selling just before $t_i + 2\eta$ is equal to or lower than that from selling just before $t_i + 3\eta$ if

$$\begin{aligned} & \exp(g(t_i + 2\eta)) \\ \leq \; & \frac{\alpha}{\alpha + (1-\alpha)p}(1-p)\exp(g(t_i + 3\eta)) + \frac{\alpha}{\alpha + (1-\alpha)p} p \exp(g(t_i)) + \frac{(1-\alpha)p}{\alpha + (1-\alpha)p}\exp(g(t_i - \eta)). \end{aligned}$$

This inequality is transformed to

$$\alpha \geq \frac{p\exp(-g\eta)\left[\exp(3g\eta) - 1\right]}{[1 - \exp(-g\eta)][p + (1-p)\exp(3g\eta)]}. \tag{5}$$

If (5) is satisfied, the equilibrium strategy of investors is to sell their stock just before $t_i + 3\eta$. If (5) is not satisfied, the equilibrium strategy of investors is to sell their stock just before $t_i + 2\eta$, which is the first period at which a probabilistic warning is issued.

Three cases are considered. First, if (5) does not hold, investors never wait until $t_i + 3\eta$ and try to sell just before $t_i + 2\eta$. The bubble crashes just before $t_0 + 2\eta$, which equals $t_H + 2\eta$ for type-H and $t_L + \eta$ for type-L. A warning is not issued. Second, if (5) holds and a warning is not issued at $t_i + 2\eta$, then investors wait until just before $t_i + 3\eta$. If the warning is not issued at $t_i + 2\eta$, then type-H investors do not receive any warning until $t_H + 2\eta$, so they deduce that their type is type-H and that $t_0 = t_i$ and thus sell just before $t_H + 3\eta = t_0 + 3\eta$. The bubble thus crashes just before $t_0 + 3\eta$, and a warning is not issued. For type-L, the bubble crashes at $t_L + 2\eta$ before they sell. Third, if (5) holds and a warning is issued at $t_0 + 2\eta$, then the bubble crashes instantaneously. The following proposition summarizes the above.

Proposition 3 *Suppose Assumptions 1 and 2. Suppose also that a public warning is issued at period $t_0 + 2\eta$ with probability p, and at $t_0 + 3\eta$ with probability $1 - p$. Assume that $3\eta < \tau^*$. Then, if (5) does not hold, a warning is never issued in equilibrium, and the bubble crashes just before $t_0 + 2\eta$. If (5) holds, and a warning is not issued at $t_0 + 2\eta$, a warning is never* *issued in equilibrium, and the bubble crashes just before $t_0 + 3\eta$. If (5) holds and a warning is issued at $t_0 + 2\eta$, the bubble crashes immediately after the warning.*

As the probability of early public warnings (p) increases, the condition for holding the stock longer tightens. Compared to the right-hand side of (1), the right-hand side of (5) is smaller if $p = 0$. Therefore, provided $p = 0$, (5) is more likely to be satisfied, loosening the condition for waiting until $t_i + 3\eta$. As p increases, the right-hand side of (5) increases because $\exp(3g\eta) - 1 > 0$. If p is sufficiently high, (5) does not hold, eliminating the incentive to wait until $t_i + 3\eta$. Investors sell their stock just before $t_i + 2\eta$, and the bubble crashes before the warning.

3.2.2 n Possible Timings of Warning

Suppose that warning periods, $t_0 + \tau_W$, are distributed on $\{t_0 + 2\eta, t_0 + 3\eta, ...t_0 + n\eta\}$, where $n > 3$, and each probability is given by $1/(n-1)$. Assume that $n\eta < \tau^*$. We discuss the case of $2\eta < \tau^* \leq n\eta$ in Appendix A.2. The details of the following discussion are provided in Appendix A.3, and the timing is shown in Figure 3.

[Figure 3 Here]

We again begin by examining what happens when a public warning is issued. If investors receive a warning at $t_i + \eta$, then they are type-L. Such type-L investors will sell their stock just before a type-H investor, who receives the warning at $t_i + 2\eta$. That is, with such a probabilistic warning, even type-H may not be able to sell earlier. On the other hand, if investors do not receive a warning until $t_i + (n-1)\eta$, then they are type-H because type-L receives a warning by $t_i + (n-1)\eta$ surely. Such type-H investors will sell their stock just before type-L investors receive the warning at $t_i + (n-1)\eta$. As a result, for the same reason as in the case with two possible timings, there exist investors who have no chance to sell their stock at a higher price, regardless of their types, if they keep their stock after the warning. Therefore, such investors sell their stock immediately, and the bubble crashes. Then, investors who receive the warning at the different periods also sell their stock immediately, so when there is the warning, the bubble crashes immediately.

As in the previous cases, the bubble may crash before the warning is issued in equilibrium. For example, let us consider the situation, before $t_i + (n-1)\eta$, in which they have an incentive to wait until $t_i + n\eta$ even though it is possible that the warning is issued at or before $t_i + (n-1)\eta$. As in the case of two possible timings, we find that investors sell their stock just before $t_i + n\eta$ if

$$\alpha \geq \frac{\exp(-g\eta)\,[\exp(ng\eta) - 1]}{[1 + \exp(ng\eta)]\,[1 - \exp(-g\eta)]}. \tag{6}$$

If (6) is satisfied, investors wait and sell just before $t_i + n\eta$ when a warning is not issued until $t_i + (n-1)\eta$. If (6) does not hold, investors have an incentive to sell just before $t_i + (n-1)\eta$ when a warning is not issued until $t_i + (n-2)\eta$.

If (6) does not hold, we need to examine a more general case, that is, whether an equilibrium can correspond to the situation in which investors wait and sell their stock just before $t_i + (n-m+2)\eta$ when a warning is not issued until $t_i + (n-m)\eta$, where $m \geq 2$. In the above, we examine the case of $m = 2$. The condition that investors do not sell just before $t_i + (n-m+1)\eta$ and wait to sell longer is

$$\alpha \geq \frac{(m-1)\,[\exp(g(n-m+1)\eta) - \exp(-g\eta)]}{(m-1)[\exp(g(n-m+2)\eta) - \exp(-g\eta)] - [\exp(g(n-m+1)\eta) - 1]}. \tag{7}$$

If (7) does not hold, investors have an incentive to sell just before $t_i + (n-m+1)\eta$ when a warning is not issued until $t_i + (n-m)\eta$.

The right-hand side of (7) decreases with m, as Lemma 2 in Appendix A.3 shows. Thus, when (7) is satisfied with $m = m'$, (7) is also satisfied with any $m \geq m'$. In other words, if investors have an incentive to wait and sell just before $t_i + (n-m'+2)\eta$ at $t_i + (n-m')\eta$, then investors also have an incentive to wait and sell just before (at least) $t_i + (n-m+2)\eta$ at $t_i + (n-m)\eta$, where $t_i + (n-m)\eta < t_i + (n-m')\eta$.

On the other hand, when (7) is not satisfied with $m = m'$, (7) is also not satisfied with any $m \leq m'$. Therefore, with $m = n-1$ (i.e., at $t_i + \eta$), if (7) is not satisfied, then (7) is also not satisfied for any m because $m \leq n-1$. If it is satisfied, investors sell just before $t_i + 2\eta$, which is the first period at which a possible warning is issued. Formally, (7) is not satisfied with $m = n-1$ if

$$\alpha < \alpha^{**} \equiv \frac{(n-2)\,[\exp(2g\eta) - \exp(-g\eta)]}{(n-2)[\exp(3g\eta) - \exp(-g\eta)] - [\exp(2g\eta) - 1]}. \tag{8}$$

Note that α^{**} is higher than α^* denoted by (3), so such an α that satisfies (8) exists under Assumption 1.[12] If (8) is not satisfied, there exists a value of $m \in [2, n-1)$ that does not satisfy (7); however, it is satisfied by $m-1$. The following definition encompasses such an m.

Definition 2 *Denote m^* such that*

1. *If (8) is satisfied, then $m^* = n - 1$.*

2. *If (8) is not satisfied, then m^* is an integer that satisfies*
$$\alpha \geq \frac{(m^*-2)\left[\exp(g(n-m^*)\eta) - \exp(-g\eta)\right]}{(m^*-2)[\exp(g(n-m^*+1)\eta) - \exp(-g\eta)] - [\exp(g(n-m^*)\eta) - 1]} \text{ and}$$
$$\alpha < \frac{(m^*-1)\left[\exp(g(n-m^*+1)\eta) - \exp(-g\eta)\right]}{(m^*-1)[\exp(g(n-m^*+2)\eta) - \exp(-g\eta)] - [\exp(g(n-m^*+1)\eta) - 1]}.$$

This definition implies that at $t_i + (n - m^* - 1)\eta$, investors chose to ride a bubble until $t_i + (n - m^* + 1)\eta$ without an incentive to deviate to sell at $t_i + (n - m^*)\eta$. However, at $t_i + (n - m^*)\eta$, investors have an incentive to deviate and sell at $t_i + (n - m^* + 1)\eta$. That is, the equilibrium strategy of investors is to sell their stock just before $t_i + (n - m^* + 1)\eta$. As a result, the following proposition is obtained.

Proposition 4 *Suppose Assumption 1 and 2. Suppose also that warning periods are distributed on $\{t_0 + 2\eta, t_0 + 3\eta, ...t_0 + n\eta\}$, each with probability $1/(n-1)$, where $n > 3$. Assume that $n\eta < \tau^*$. Then, if there is a warning, the bubble crashes when the warning is issued. If (6) does not hold, and there is no warning before $t_0 + (n - m^*)\eta$, then the bubble crashes just before $t_0 + (n - m^* + 1)\eta$. If (6) holds, and there is no warning before $t_0 + (n-1)\eta$, the bubble crashes just before $t_0 + n\eta$.*

The proof of this proposition is provided in Appendix A.3. As α increases, m^* decreases. Thus, if the probability of being type-H increases, investors have an incentive to wait longer.

[12] After some calculations, the right hand side of (8) is less than α^* iff $\exp(2g\eta) - \exp(g\eta) + \exp(-g\eta) > \frac{n-3}{n-1}$. The right-hand side of this condition is less than one. On the other hand, the left-hand side is greater than one because it equals one when $g\eta = 0$, and it increases with $g\eta$ when $g\eta > 0$.

3.2.3 Section Summary

In all cases, the bubble crashes immediately if a warning is issued. Moreover, in many cases, the bubble crashes before a warning is issued because it is possible that type-L deduces his type. Without any warning, type-L never deduces his type, although he is not able to sell at a high price. Type-L becomes a victim of the crash, but investors have an incentive to ride a bubble because it is possible that they are type-H and can sell at a high price. Such a situation changes if there is a warning. Type-L may deduce his own type and try to sell at a high price. As a result, no one has an incentive to hold a stock any longer when a warning is issued. These results suggest that public announcements targeted at less-informed investors are important in preventing bubbles, as discussed in the Introduction.[13]

Additionally, if the probability of being type-H decreases, and/or the probability of early warnings increases, the duration for which investors wait to sell shortens because they are less likely to sell their stock with a high price.

However, as we discussed in the Introduction, we did not encounter many cases in which public warnings stopped bubbles. The next two subsections consider the reason for this historical observation.

3.3 Noisy Warning

Let us consider a case in which warnings are only noisy. Warning periods are distributed on $\{0, \eta, 2\eta, , 3\eta, ...n\eta\}$ each with probability $1/(n+1)$. The timing of warnings does not depend on t_0. It is possible that the warning is too early for a bubble to start.

It is not possible that investors deduce their type when they are type-L, as in the previous case. Because the timing of warnings does not depend on t_0, they cannot deduce their type from the warning. The warning provides no information as to whether the current price is

[13] Even in the setting of Abreu and Brunnermeier (2003), if public warnings are deterministic, less-informed investors, who are aware of the bubble after more than a proportion κ of investors are aware of the bubble, can know that they received the private signal later than did other investors. Thus, they try to sell their stock earlier, and the bubble crashes at the same time as the public warning. In that respect, our simplification of Abreu and Brunnermeier (2003) is not the reason we obtain these results.

overpriced. As a result, the situation is exactly the same as the case without any warning.[14]

Corollary 2 *Suppose Assumptions 1 and 2. Suppose also that the warning's periods are distributed on* $\{0, \eta, 2\eta, , 3\eta, ...n\eta\}$*, each with probability* $1/(n+1)$*. The bubble crashes at* $t_0 + \tau^*$*. Even if there is a warning, the bubble does not crash at that time.*

3.4 Deterministic and Noisy Warning

Finally, suppose a combination of two types of warnings, deterministic and noisy. A warning is deterministic with probability p, and assume that such a warning is issued at t_0. With the remaining probability $1-p$, a warning is noisy, and such a warning's periods are distributed on $\{0, \eta, 2\eta, , 3\eta, ...n\eta\}$, each with probability $(1-p)/(n+1)$. Investors cannot observe the type of warning. The timing is shown in Figure 4. We assume that $n\eta \leq t_0$.[15] This case is considered more realistic than the previous cases because it encompasses a case in which a warning is right or wrong: the warning may be issued even though the asset price is not over-priced.

[Figure 4 Here]

If investors receive a private signal before the warning or 2η periods later than the warnings, then their posterior belief of being type-H does not change because such a warning

[14] It is possible that the warning may be too early, but its timing depends on the beginning period of the bubble, t_0. That is, the warning's periods are distributed on $\{t_0 - \underline{n}\eta, ..., t_0, ...t_0 + \overline{n}\eta\}$, each with probability $1/(\overline{n} + \underline{n} + 1)$. If investors receive the warning at $t_i - (\underline{n}+1)\eta$, these investors can know that their type is type-L when they receive the private signal. Such investors will sell their stock just before type-H, who receives the warning at $t_i - \underline{n}\eta$, sells their stock. If investors do not receive a warning until $t_i - (\overline{n} - 1)\eta$, these investors can know that their type is type-H. Thus, the situation is almost the same as that presented in the previous subsections. No one waits to sell his stock. The only difference is that even though there is a warning before the bubble begins, nothing happens until investors start to receive the private signal. Because the bubble has yet to start when the warning is issued and the asset price continues to rise (at least until t_0), investors have no incentive to sell their stock just when the early warning is issued. The bubble will crash just before $t_0 + \eta$. That is, if the warning is issued before the beginning date, the best response of investors is to sell their stock just before $t_i + \eta$. In this case, type-L never receives the private signal.

[15] If $n\eta < t_0$, all investors can know whether the warning is noisy or deterministic. Thus, the result is the same as the previous sections.

cannot be deterministic; if investors realize that it is noisy warning, it does not affect investors' posterior beliefs and decisions.

There are two important instances, which lead to the modification of investors' posterior beliefs. First, investors are aware of the bubble at the same time as the public warning. We call such an investor a type-0 investor. Second, investors are aware of the bubble one period later than the public warning. We call such an investor a type-1 investor. These two types are shown in Figure 5.

[Figure 5 Here]

Suppose that type-0 investors receive a private signal at $t_i \in \{\eta, 2\eta, , 3\eta, ..., (n-1)\eta\}$ at the same time as a warning. For simplicity, we exclude the beginning period $t_i = 0$ and the end period $t_i = n\eta$. From the prior belief, with probability p, this warning is deterministic. If so, the investors are both type-0 and type-H investors with probability α. On the other hand, the warning is noisy with the prior probability $(1-p)/(n+1)$. If the warning is noisy, the investors are type-H with probability α and type-L with probability $1-\alpha$. Thus, the probability that investors receive the warning at the same time as a private signal is $\Pr(0) = p\alpha + \frac{1-p}{n+1}$, and the joint probability that investors receive the warning at the same time as a private signal and are type-H is $\Pr(0 \cap H) = p\alpha + \frac{1-p}{n+1}\alpha$. Denote by $\gamma(H|0)$ the posterior belief to be type-H when investors are type-0, given by

$$\gamma(H|0) = \frac{\Pr(0 \cap H)}{\Pr(0)} = \frac{p\alpha + \frac{1-p}{n+1}\alpha}{p\alpha + \frac{1-p}{n+1}} = \frac{\alpha(1+np)}{(1+n)\alpha p + 1 - p}.$$

It is higher than α. If an investor receives both the private signal and the public warning simultaneously, this investor is more likely to be type-H because it is possible that this warning is deterministic and accurate. Thus, such an investor believes that he is more likely to be type-H.

We turn to type-1 investors who receive a private signal at $t_i \in \{\eta, 2\eta, , 3\eta, ..., (n-1)\eta\}$, η periods after a warning. From the prior belief, with probability p, this warning is deterministic. If so, the investors are both type-1 and type-L with probability $1-\alpha$. The warning is noisy with the prior probability $(1-p)/(n+1)$. If the warning is noisy, the investors are type-H with probability α and type-L with probability $1-\alpha$. Thus, the probability that investors receive the warning η periods before the private signal is $\Pr(1) = p(1-\alpha) + \frac{1-p}{n+1}$, and the joint probability that investors receive the warning at the same time as a private signal and are type-H is $\Pr(1 \cap H) = \frac{1-p}{n+1}\alpha$. As a result, the posterior belief of being type-H, $\gamma(H|1)$, is given by

$$\gamma(H|1) = \frac{\Pr(1 \cap H)}{\Pr(1)} = \frac{\frac{1-p}{n+1}\alpha}{p(1-\alpha) + \frac{1-p}{n+1}}.$$

It is lower than α because the probability of being type-H is zero in the case of deterministic warning.

In this situation, type-0 investors have an incentive to ride the bubble for longer than τ^*. Type-0 investors cannot sell at a high price if they are type-L, but because investors are more likely to be type-H due to $\gamma(H|0) > \alpha$, their incentive to wait longer increases.

At the same time, type-0 investors must take into account when type-1 investors sell their stock. Because of $\gamma(H|1) < \alpha$, type-1 investors have an incentive to wait for a shorter duration than the case without the public warning. Denoting the maximum timing of type-1 investors' sale by $t_i + \tau^{**}$, we obtain $\tau^{**} < \tau^*$. Considering this, type-0 investors shortens the duration of riding bubbles. For simplicity, assume that their stock can be sold only at the multiple period of η or that all the investors, including type-0 and type-1, choose the same strategy; then we may concentrate on the case in which all the investors sell their stock at $t_i + \tau^{**}$. The optimal τ^{**} satisfies

$$\begin{aligned} & \gamma(H|0)\exp(g(t_i + \tau^{**})) + (1-\gamma(H|0))\exp(g(t_i - \eta)) \\ \geq \ & \exp(g(t_i - \eta + \tau^{**})), \end{aligned}$$

for type-0 investors,

$$\begin{aligned} & \gamma(H|1)\exp(g(t_i + \tau^{**})) + (1-\gamma(H|1))\exp(g(t_i - \eta)) \\ \geq \ & \exp(g(t_i - \eta + \tau^{**})), \end{aligned}$$

for type-1 investors, and

$$\begin{aligned} & \alpha\exp(g(t_i + \tau^{**})) + (1-\alpha)\exp(g(t_i - \eta)) \\ \geq \ & \exp(g(t_i - \eta + \tau^{**})), \end{aligned}$$

for the other investors. Because of $\tau^{**} < \tau^*$, $\gamma(H|0) > \alpha$, and $\gamma(H|1) < \alpha$, only the second equation binds. Therefore, τ^{**} is given by

$$\gamma(H|1) = \frac{\frac{1-p}{n+1}\alpha}{p(1-\alpha) + \frac{1-p}{n+1}} = \frac{\exp(-g\eta)\left[\exp(g\tau^{**}) - 1\right]}{\exp(g\tau^{**}) - \exp(-g\eta)}.$$

Because of $\gamma(H|1) < \alpha$, it is easy to verify that $\tau^{**} < \tau^*$. The duration of the bubble-riding shortens compared with the case without public warnings. Unlike the deterministic warning case, in this case the bubble does not crash at the time of the warning. The following proposition states when the bubble crashes:

Proposition 5 *Suppose Assumption 1 and 2. Suppose also that there are two types of the warning: deterministic (same as Proposition 2) and noisy (same as Proposition 2). The bubble crashes at $t_0 + t^{**}$, which is shorter than $t_0 + t^*$.*

When p converges to zero, both $\gamma(H|0)$ and $\gamma(H|1)$ converge to α. The optimal τ^{**} increases to τ^*, and the outcome is the same as with the noisy warning. When p converges to one, $\gamma(H|0)$ converges to one and $\gamma(H|1)$ converges to zero. The optimal τ^{**} decreases to zero, and the outcome is the same as with the deterministic warning.

Note that even if the deterministic warning is not issued strictly at t_0, but within a certain range of periods after t_0, the above results hardly change. A public warning helps shorten the bubble periods but cannot stop it immediately.

3.5 Discussion

Using results obtained so far, we discuss some implications. Our results highlight the importance of announcements targeted at less-informed investors in preventing bubbles or shortening the duration of bubbles. Note that, unless a warning is issued, type-H investors can sell at a high price. Thus, even though type-H investors deduce their type, that type has no meaning in equilibrium. On the other hand, if type-L investors know their type, such investors need to change their strategy because they know that they cannot sell at a high price when they maintain their strategy. Type-L investors try to sell their stock earlier than type-H investors. As a result, the bubble crashes immediately when type-L investors deduce their type. Therefore, public information and, in the long-term, economic and financial education would be helpful in preventing bubbles by providing the information that the less-informed investors lack.

Type-L investors can or cannot deduce their type in the following cases. First, unless a warning is noisy, it allows type-L investors to deduce their type. Thus, in this case, the bubble crashes immediately when the warning is issued. Moreover, knowing that the bubble crashes when there is a warning, investors are prepared for the risk associated with riding the bubble. In some occasions, they sell their stock before warnings. Second, if the warning is purely noisy, investors ignore it because type-L investors cannot deduce their type. Third, in an intermediate case between the above first and second cases, a warning helps type-L investors revise their beliefs about their type. A warning cannot stop the bubble immediately, but it can shorten the duration of the bubble.

An important point is whether investors believe the warning. Even if the government authority issues a warning after a bubble starts, if investors regard it as noisy information, then the bubble cannot be stopped by the warning. In that respect, it is critical to reduce the type-I error: governments need to reduce the probability of spurious warnings. In other words, governments must not be like the boy who cried wolf.

Regarding macroeconomic policy that addresses bubbles, two opposing views seem to exist. Some cast doubt on the active preemptive role of macroeconomic policy in bubble prevention. For example, Mishkin (2007) argues that bubbles are difficult to detect and that central banks could cope with bubble bursts by reacting quickly after the collapse of asset prices. Others call for active preemptive public interventions. For example, Okina et al. (2001), Borio and Lowe (2002), and White (2006) emphasize the risk that less aggressive macroeconomic policy would result in disruptive booms and busts in real economic activity. They argue that identifying financial imbalances is not impossible. With this motivation, Borio and Lowe (2002) search for the indicators of financial imbalances, such as credit growth and asset price increases, from the perspective of noise to signal ratios. Before the crisis, the first view appeared to have dominated the second, but the detrimental effects of the recent financial crisis have created many proponents for the second.

Our model provides an answer from a different perspective, that is, whether public warnings are really effective at preventing bubbles. The first view is entirely correct if the type-I error is extremely high: bubbles are impossible or highly difficult to detect. In this case, preemptive communication policy is not needed. Even more strongly than the first view, our model suggests that, if their premise is correct, then public warnings do no harm or good because they are useless information for investors. However, our model does not deny the second view. Public warnings are useful in preventing bubbles if the type-I error is not too high. It is noteworthy, however, that the high risks associated with the type-II error, that is, late policy responses to financial imbalances, do not necessarily support early policy responses. What this paper shows is that, for the public warning, the type-I error is more important than the type-II error. Without resolving the type-I error, preemptive communication policy proves ineffective in preventing bubbles, even if a government is lucky enough to detect a bubble. In that respect, good bubble indicators need to be constructed on the basis of the type-I error. Borio and Lowe (2002) are one of several promising and important attempts, although they weigh the type-I and II errors equally by looking at noise to signal ratios.

4 Conclusion

This paper has examined the effects of public warnings against bubbles based on the model of riding bubbles and showed the reasons that several warnings had been ignored by investors. Without the type-I error, the bubble crashes when there is a public warning, and the bubble can crash before the announcement. However, the type-I error diminishes the effectiveness of the public warning. When the authority warns even if it is not the bubble, the bubble duration shortens, but the bubble does not crash immediately after the warning.

We touch upon two limitations of this paper. One is the lack of consideration concerning the strategies of the government authority. Warnings are given exogenously. In reality, the government authority gathers information and communicates whether or not the asset is overpriced based on its decision. Another limitation is a consideration concerning irrational investors. In the model, they are treated implicitly as an economic entity who herd and ride bubbles. Incorporating such features is important to our future research.

A Appendix

A.1 Proof of Proposition 1

Instead of assuming continuous investors of size one, suppose that there are $N \geq 5$ investors, and αN of them are type-H, where αN is a positive integer. Suppose also that (1) if αN investors sell their stock at the same time before others sell, the bubble crashes endogenously, and all of them can sell at a high price; and (2) if $\alpha N - 1$ investors sell their stock before others, the bubble also crashes endogenously. In other words, when all investors choose symmetric strategies, even though one investor deviates by selling later, the period at which the bubble crashes does not change. In this proof, we consider only a pure-strategy and symmetric equilibrium.

Suppose that all investors sell their stock at $t_i + \tau'$, where $\tau' < \tau^*$, that is, before the upper bound. Then, if all investors make a coalition and deviate by selling at $t_i + \tau^*$, all investors' payoffs can be improved because they can sell with a higher price if they are type-H, and the payoff does not change if they are type-L. Moreover, no one (and no subcoalition) has an incentive to deviate from this renewed deviation because no one has an incentive to sell earlier, as Lemma 1 shows. Thus, this deviation is self-enforcing, and selling at $t_i + \tau'$, where $\tau' < \tau^*$, is not a coalition-proof Nash equilibrium. Suppose that all investors sell their stock at $t_i + \tau^*$. From this state, suppose that all investors make a coalition and deviate by selling later than $t_i + \tau^*$. An investor has an incentive to deviate from this deviation by selling earlier, as Lemma 1 shows, so this deviation is not self-enforcing. A coalition-proof Nash equilibrium is a Nash equilibrium, so selling at $t_i + \tau^*$ is the unique (pure-strategy and proof) coalition-proof Nash equilibrium.

A.2 When the warning is too late

Suppose that warning periods, $t_0 + t_W$, are distributed on $\{t_0 + 2\eta, t_0 + 3\eta, ...t_0 + n\eta\}$, each with probability $1/(n-1)$, where $n \geq 3$. Also, suppose that $n\eta > \tau^*$. That is, it is possible that the warning is too late. This timing is shown in Figure 6. Denote $\overline{m}\eta = \tau^*$.

[Figure 6 Here]

If investors receive a warning at $t_i + \eta$, such investors deduce that they are type-L and will sell their stock just before type-H (who receives the warning at $t_i + 2\eta$) sells their stock. Assume that all investors decide to sell their stock at $t_i + \tau^*$. If investors do not receive any warning at $t_i + \tau^* - \eta$, then they are type-H because if they were type-L, the bubble would crash by $t_i + \tau^* - \eta$. Such investors will sell their stock at $t_i + \tau^*$, and type-L investors (who will receive the warning after $t_i + \tau^* - \eta$) cannot sell their stock at a high price. Thus, when investors receive the warning at $t_i + \tau^* - \eta$, they have no hope of selling their stock at a higher price regardless of their type if they still keep their stock. Therefore, such investors sell their stock immediately, and the bubble crashes.

Therefore, the situation becomes similar to that in Section 3.2.2 because the bubble crashes at the same time as a warning. Thus, for the same reason as in Section 3.2.2, there exists an m^* such that investors choose to ride on a bubble until $t_i + (n - m^* + 1)\eta$. That is, this is the case of a probabilistic warning in which $t_0 + t_W$ are distributed on $\{t_0 + 2\eta, t_0 + 3\eta, ...t_0 + \tau^*\}$ with equal probability.

A.3 Proof of Proposition 4

First, we examine what happens if there is a warning. We want to show that if a warning is issued, all investors sell their stock immediately and the bubble crashes. Suppose that all investors do not sell their stock even though a warning is issued.

A Warning is Issued: Who can deduce that his type is L?

If investors receive the warning at $t_i + \eta$, then they are type-L. Such investors will sell their stock just before type-H (who receives the warning at $t_i + 2\eta$) sells their stock. Thus, when investors receive the warning at $t_i + 2\eta$, they cannot sell at a higher price with probability α, that is, when they are type-H; and if investors who receive the warning at $t_i + 2\eta$ are type-H, and they plan to sell their stock later than $t_i + 2\eta$, they need to give up selling with a high price. However, there is an opportunity to sell with a high price if they are type-L, and it happens with probability $1 - \alpha$. That is, if investors who receive the warning at $t_i + 2\eta$ sell their stock earlier than type-H investors who receive the warning at $t_i + 3\eta$, such investors can sell their stock with a high price when such type-H plans to sell their stock later than $t_i + 3\eta$. For the same reason, investors who receive the warning at $t_i + m\eta$ sell their stock just before type-H investors (who receives the warning at $t_i + (m+1)\eta$) sell their stock, where $1 \leq m \leq n-2$.

A Warning is Issued: Who can deduce that his type is H?

On the other hand, if investors do not receive the warning until $t_i + (n-1)\eta$, they can deduce that they are type-H because type-L receives the warning until $t_L + (n-1)\eta$ surely. Such investors who find themselves type-H know that type-L investors receive the warning, if any, at $t_L + (n-1)\eta$ from type-L's perspective. They try to sell their stock just before this type-L.

The Bubble Crashes with a Warning

It is suggested that when investors receive the warning at $t_i + (n-1)\eta$, they have no hope of selling their stock at a higher price regardless of their types. That is, if they are type-L, the type-H investors who do not receive the warning at $t_i + (n-1)\eta$ will sell their stock before they sell their stock. If they are type-H, the type-L investors who receive the warning $t_L + (n-2)\eta$ will sell their stock before they sell.

Therefore, investors who receive the warning at $t_i + (n-1)\eta$ sell their stock at the same time as the warning regardless of their types. Knowing this, investors who receive the warning at $t_i+(n-2)\eta$ also sell their stock immediately regardless of their types because their strategy is to sell their stock before type-H investors who receive the signal at $t_i + (n-1)\eta$. For the same reason, investors who receive the warning at $t_i+m\eta$ sell their stock immediately for $m \in \{2, 3, ..., n-3\}$.

How Long do They Wait? The Last Two Periods

As in the cases of deterministic and two-period probabilistic warnings, the bubble may crash before a warning is issued in equilibrium. As we argued in the main text, investors sell just before $t_i + 2\eta$, if (8) is satisfied. So, we concentrate on the case in which (8) is not satisfied. First, we examine whether an equilibrium can occur when investors wait and sell their stock just before $t_i + n\eta$ if a warning is not issued until $t_i + (n-2)\eta$. Suppose that $(n-2)\eta$ periods have passed after investors receive the private signal and a warning is not yet been issued. That is, investors are at $t_i + (n-2)\eta$. Their choice is either (a) to maintain their strategy and sell just before $t_i + n\eta$ or (b) to deviate and sell their stock just before $t_i + (n-1)\eta$. If investors choose (b), they succeed in selling their stock before the bubble crashes, so the expected payoff from (b) is $\exp(g(t_i + (n-1)\eta))$. If they do not change the strategy, their expected payoff is

$$\Pr(H|t_i+(n-2)\eta)\frac{1}{2}\exp(g(t_i+n\eta))+\Pr(H|t_i+(n-2)\eta)\frac{1}{2}\exp(gt_i)+(1-\Pr(H|t_i+(n-2)\eta))\exp(g(t_i-\eta)). \tag{9}$$

$\Pr(H|t_i + (n-2)\eta)$ is the probability that this investor is type-H when a warning is not issued yet at $t_i + (n-2)\eta$. At $t_i + (n-2)\eta$, there are two possible cases. First, this investor is type-H and spends $(n-2)\eta$ periods after a private signal without a public warning. Second, this investor is type-L, and a warning is not issued until $t_L + (n-2)\eta = t_0 + (n-1)\eta$. The joint probability that a warning is not issued until $t_i + (n-2)\eta$ is given by

$$\alpha\frac{2}{n-1} + (1-\alpha)\frac{1}{n-1}. \tag{10}$$

This expression holds true because if they are type-H with the probability of α, then t_0 equals t_i, and a warning is issued either at $t_i + (n-1)\eta$ or $t_i + n\eta$ with the probability of $\frac{2}{n-1}$. If they are type-L with the probability of $1-\alpha$, then t_0 equals $t_i - \eta$, and a warning is issued at $t_i + (n-1)\eta$ with the probability of $\frac{1}{n-1}$. Therefore,

$$\Pr(H|t_i + (n-2)\eta) = \frac{\alpha\frac{2}{n-1}}{\alpha\frac{2}{n-1} + (1-\alpha)\frac{1}{n-1}} = \frac{2\alpha}{1+\alpha}$$

If this investor is type-H, the probability that the warning is issued at $t_H + (n-1)\eta = t_0 + (n-1)\eta$ is 1/2 because of the uniform distribution of possible warning periods. If this investor is type-L, this investor cannot sell their stock with a high price with certainty. Under the condition that a warning is not issued until $t_i + (n-2)\eta$, three cases are considered, each of which corresponds to each term in (9). First, investors are type-H and a warning is issued at $t_i + n\eta$. Its probability is $\frac{2\alpha}{1+\alpha}$ times 1/2. In this case, investors who do not receive a warning until $t_i + (n-1)\eta$ recognize that they are type-H. They also know that the warning is issued at $t_i + n\eta$, and the bubble crashes immediately if a warning is issued. Investors sell their stock with the price of $\exp(g(t_i + n\eta))$ just before $t_i + n\eta$. Second, investors are type-H and a warning is issued at $t_i + (n-1)\eta$. Its probability is also $\dfrac{2\alpha}{1+\alpha}$ times $1/2$. The bubble crashes at $t_i + (n-1)\eta$. The stock price falls to $\exp(gt_i)$. Third, investors are type-L. A warning is issued at $t_i + (n-1)\eta$ surely. Its probability is $1 - \dfrac{2\alpha}{1+\alpha} = \dfrac{1-\alpha}{1+\alpha}$. The bubble crashes at $t_i + (n-1)\eta$. The stock price falls to $\exp(g(t_i - \eta))$.

His expected payoff from choosing to sell just before $t_i + n\eta$ is equal to or higher than that from selling just before $t_i + (n-1)\eta$ iff

$$\begin{aligned} & \frac{\alpha}{1+\alpha}\exp(g(t_i + n\eta)) + \frac{\alpha}{1+\alpha}\exp(gt_i) + \frac{1-\alpha}{1+\alpha}\exp(g(t_i - \eta)) \\ \geq \; & \exp(g(t_i + (n-1)\eta)). \end{aligned} \tag{11}$$

This is simplified to the same equation as (6):

$$\alpha \geq \frac{\exp(-g\eta)\left[\exp(ng\eta) - 1\right]}{\left[1 + \exp(ng\eta)\right]\left[1 - \exp(-g\eta)\right]}. \tag{6}$$

If this condition holds, investors do not have an incentive to deviate to sell just before $t_i + (n-1)\eta$ when a warning is not issued until $t_i + (n-2)\eta$.

How Long do They Wait? The Last m Periods

If (6) does not hold, investors have an incentive to deviate, selling their stock just before $t_i + (n-1)\eta$. In this case, we need to check whether investors wait and sell their stock just before $t_i + (n-1)\eta$ if a warning is not issued until $t_i + (n-3)\eta$. More generally, we examine whether equilibrium can entail the situation in which investors wait and sell their stock just before $t_i + (n-m+2)\eta$ if a warning is not issued until $t_i + (n-m)\eta$. In the previous part, we examined the case of $m = 2$, and we examine the case of $m \geq 2$ in the following parts.

Suppose that investors are at $t_i + (n-m)\eta$ and a warning has yet to be issued. We examine whether investors have an incentive to wait and sell their stock just before $t_i + (n-m+2)\eta$. Their choice is either (a) to maintain the strategy and sell just before $t_i + (n-m+2)\eta$ or (b) to sell their stock just before $t_i + (n-m+1)\eta$. If investors choose (b), they succeed in selling their stock before the bubble crashes, so the expected payoff from this deviation is

$\exp(g(t_i + (n-m+1)\eta))$. If they do not change the strategy, their expected payoff is

$$\Pr(H|t_i + (n-m)\eta)\frac{m-1}{m}\exp(g(t_i + (n-m+2)\eta))$$
$$+\Pr(H|t_i + (n-m)\eta)\frac{1}{m}\exp(gt_i) + (1 - \Pr(H|t_i + (n-m)\eta))\exp(g(t_i - \eta)). \quad (12)$$

$\Pr(H|t_i + (n-m)\eta)$ is the probability that this investor is type-H when a warning is not issued yet at $t_i+(n-m)\eta$. At $t_i+(n-m)\eta$, there are two possible cases. First, this investor is type-H and spends $(n-m)\eta$ periods after a private signal without a public warning. Second, this investor is type-L, and a warning is not issued until $t_L + (n-m)\eta = t_0 + (n-m+1)\eta$. The joint probability that a warning is not issued until $t_i + (n-m)\eta$ is given by

$$\alpha\frac{m}{n-1} + (1-\alpha)\frac{m-1}{n-1}.$$

This expression holds true because if they are type-H with the probability of α, then t_0 equals t_i, and a warning is issued after $t_i + (n-m+1)\eta$ with the probability of $\frac{m}{n-1}$. If they are type-L with the probability of $1-\alpha$, then a warning is issued after $t_i+(n-m+1)\eta$ with the probability of $\frac{m-1}{n-1}$. Therefore,

$$\Pr(H|t_i + (n-m)\eta) = \frac{\alpha\frac{m}{n-1}}{\alpha\frac{m}{n-1} + (1-\alpha)\frac{m-1}{n-1}} = \frac{m\alpha}{m-1+\alpha}$$

Under the condition that a warning is not issued until $t_i+(n-m)\eta$, three cases are considered, each of which corresponds to each term in (12). First, investors are type-H and a warning is not issued at $t_i+(n-m+1)\eta$. Its probability is $\frac{m\alpha}{m-1+\alpha}$ times $\frac{m-1}{m}$. In this case, investors who do not receive a warning until $t_i + (n-m+1)\eta$ will sell just before $t_i + (n-m+2)\eta$ with the price of $\exp(g(t_i + (n-m+2)\eta))$. Second, investors are type-H and a warning is issued at $t_i + (n-m+1)\eta$. Its probability is $\frac{m\alpha}{m-1+\alpha}$ times $\frac{1}{m}$. The bubble crashes at $t_i + (n-m+1)\eta$. The stock price falls to $\exp(gt_i)$. Third, investors are type-L with the probability of $1 - \frac{m\alpha}{m-1+\alpha} = \frac{(m-1)(1-\alpha)}{m-1+\alpha}$. They cannot sell their stock with a high price, receiving $\exp(g(t_i - \eta))$.

Their expected payoff from choosing to sell just before $t_i + (n-m+2)\eta$ is equal to or higher than that from selling just before $t_i + (n-m+1)\eta$ iff

$$\frac{\alpha(m-1)}{m-1+\alpha}\exp(g(t_i + (n-m+2)\eta)) + \frac{\alpha}{m-1+\alpha}\exp(gt_i) + \frac{(m-1)(1-\alpha)}{m-1+\alpha}\exp(g(t_i-\eta))$$
$$\geq \exp(g(t_i + (n-m+1)\eta)).$$

This expression is simplified to the same equation as (7):

$$\alpha \geq \frac{(m-1)\left[\exp(g(n-m+1)\eta) - \exp(-g\eta)\right]}{(m-1)[\exp(g(n-m+2)\eta) - \exp(-g\eta)] - [\exp(g(n-m+1)\eta) - 1]}. \tag{7}$$

The important property of the above equation is that the right-hand side decreases with m.

Lemma 2 *The right-hand side of (7) decreases as the remaining periods m increases.*

PROOF: Differentiate the right-hand side of (7) with respect to m. From some calculations, it is negative if and only if the following equation is negative.

$$\begin{aligned}
&-[\exp(g(n-m+2)\eta) - 1][\exp(g(n-m+1)\eta) - 1] \\
&-(1 + m(m-1)g\eta)[\exp(g(n-m+2)\eta) - \exp(g(n-m+1)\eta)]
\end{aligned}$$

Because $g(n-m+1)\eta > 0$, $\exp(g(n-m+2)\eta) > \exp(g(n-m+1)\eta) > 1$. Thus, the above equation is strictly negative. $\square$

From Lemma 2, we can define an m^* that satisfies Definition 2. If $m > m^*$, investors have an incentive to wait after $t_i + (n-m+1)\eta$. That is, investors will wait until or later than $t_i + (n-m+2)\eta$. On the other hand, if $m < m^*$, they do not have an incentive to wait, selling just before $t_i + (n-m+1)\eta$. $\square$

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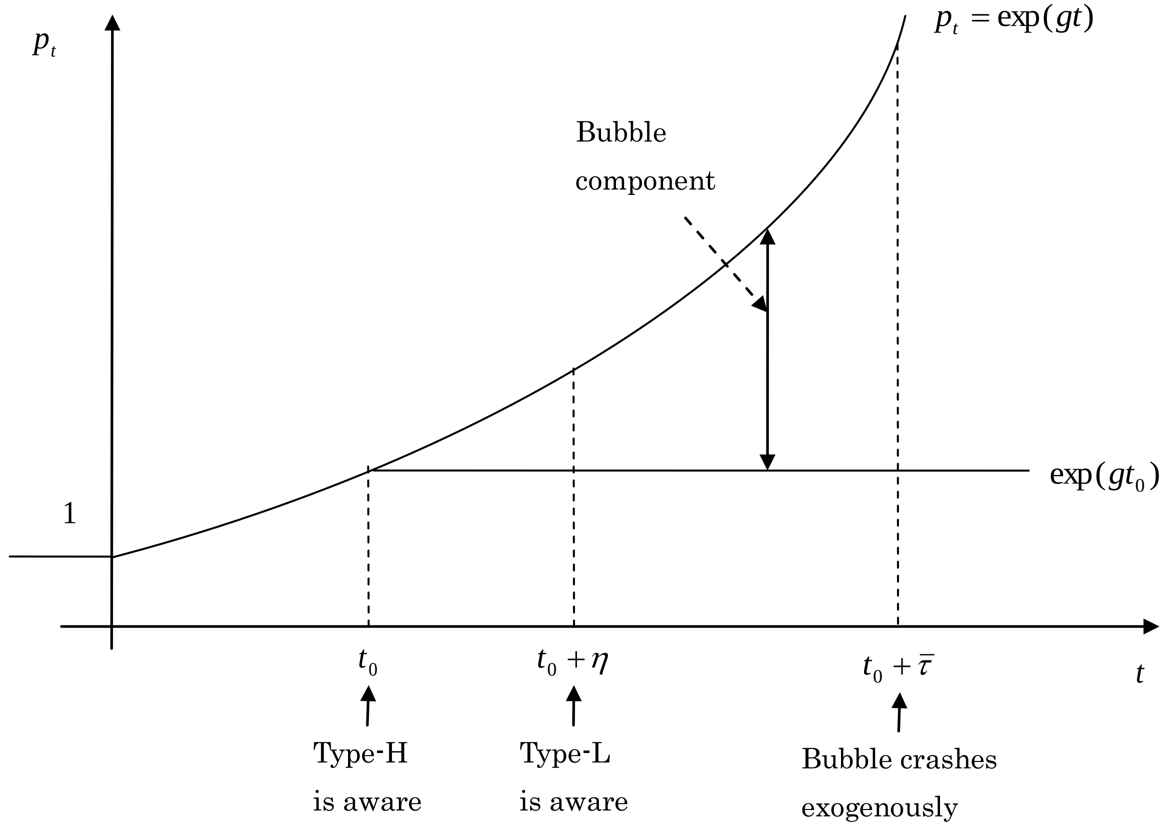

Figure 1: A Simple Model of Riding Bubbles

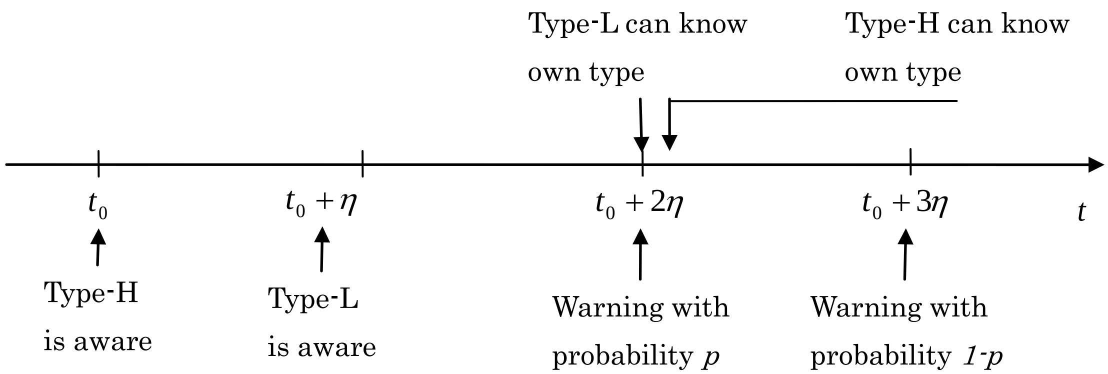

Figure 2: Two Possible Timings of Warning

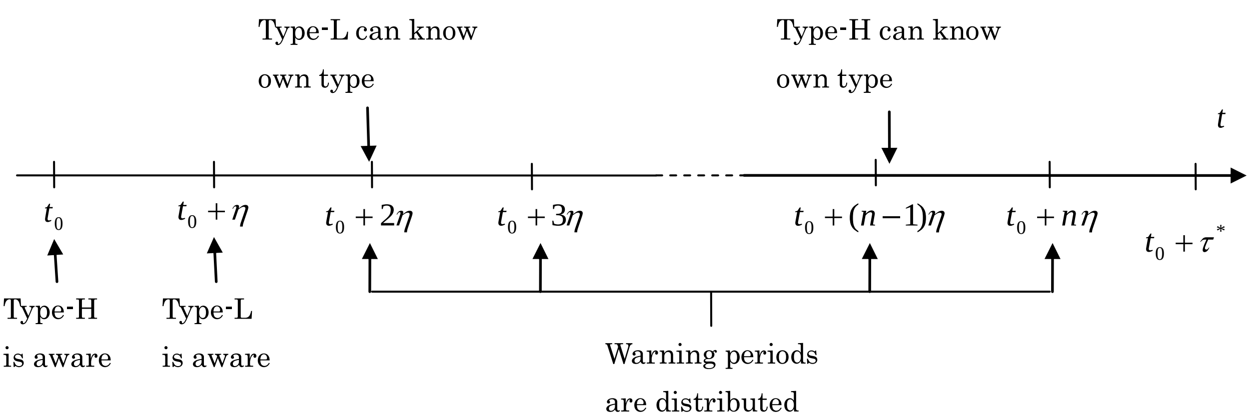

Figure 3: *n* Possible Timings of Warning

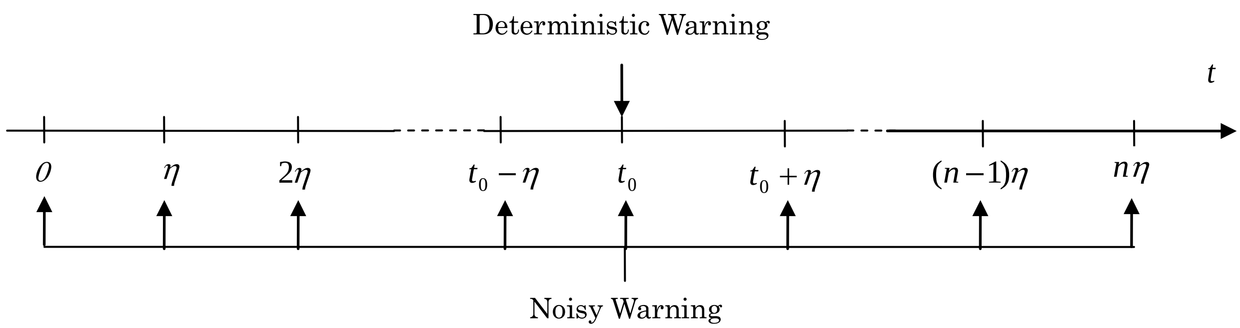

Figure 4: Noisy and Deterministic Warning

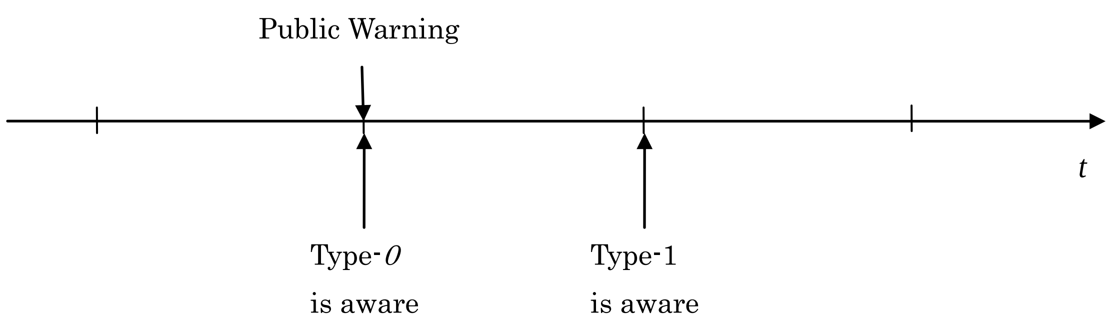

Figure 5: Type-0 and Type-1 Investors

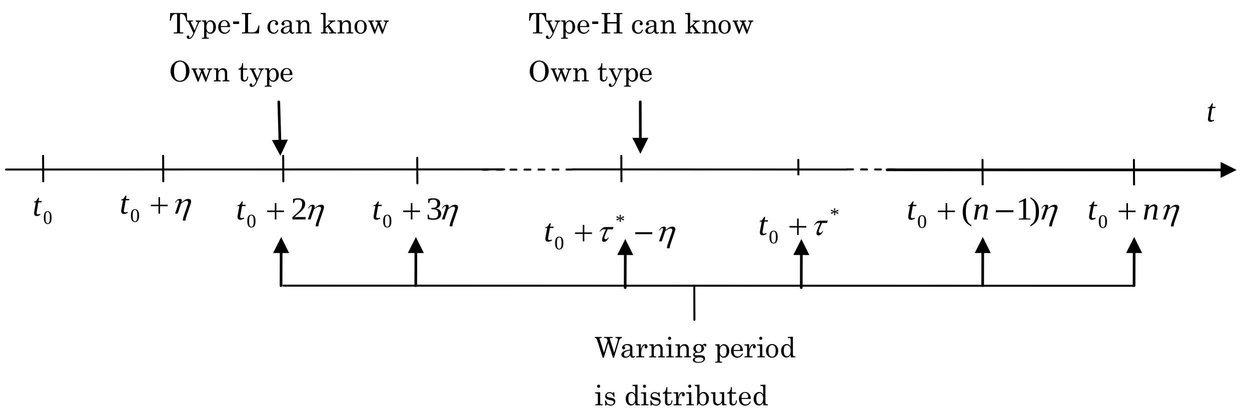

Figure 6: When the Warning is too Late