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Given a convex risk measure on L^∞ having the Lebesgue property, we construct a solid space of random variables on which the original risk measure is extended preserving the Lebesgue property (on the entire space). This space is an order-continuous Banach lattice, and is maximum among all solid spaces admitting such a regular extension. We then characterize the space in terms of uniform integrability of certain families. As a byproduct, we present a generalization of Jouini-Schachermayer-Touzi's theorem on the weak-compactness characterization of Lebesgue property, which is valid for any solid vector spaces of random variables, and does not require any topological property of the space.

1. INTRODUCTION

In financial mathematics, a *convex risk measure* is a monotone (decreasing) convex function ρ on a vector space of random variables containing the constants, verifying the property that $\rho(X + a) = \rho(X) - a$ whenever a is a constant (cash-invariance). This notion was introduced by [6, 18, 20] to replace the widely used *Value at Risk* which is still the industry standard, but has an essential drawback as a measure of risk that diversification may *increase* the risk in terms of the Value at Risk. Since then, convex risk measures on L^∞ (i.e. for bounded risks) have been extensively studied, establishing a number of their fine properties as well as examples [see e.g. 14, 19]. However, L^∞ is clearly too small to capture the actual risks, and a key current direction is the study of risk measures *beyond bounded risks*. Several authors considered those on particular bigger spaces, e.g., L^p spaces [1, 23], Orlicz spaces and their Morse subspaces [10, 27, 4, 5], abstract locally convex Fréchet lattices [8], and L^0 [25] to mention a few.

Another direction towards unbounded risks, which we shall explore, is to *extend* convex risk measures originally defined on L^∞ to some big space. In this line, [11] considered (essentially) an extension of risk measure on L^∞ to a *possibly improper* monotone convex function on L^0, based on "approximation by bounded variables", then provided necessary and sufficient conditions for the resulting function to be proper (hence a risk measure). It is, however, not a *unique* extension, and no regularity of resulting risk measure is considered there. More recently, [16] investigated (in our language) an extension of risk measure from L^∞ to L^1 *preserving the Fatou property*. The method of [16] is mathematically the *topological* (L^1-) *closure* based on the Fenchel-Moreau dual representation, and it is

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proved that if the original risk measure is *law-invariant* (which already implies the Fatou property on L^∞), the L^1-closure gives a unique lower semi-continuous extension to L^1.

The Fatou property (σ-order lower semi-continuity) is necessary and sufficient for the risk measure to have a dual representation by *σ-additive* probabilities, which is the *minimal* requirement for practical use. In many applications, however, this is not enough, and one often needs the stronger *Lebesgue property* (σ-order continuity) corresponding to the dominated convergence in measure theory. This property presents a number of pleasant features, e.g., it is stable under inf-convolution, is connected to a useful weak compactness property and provides the *σ-additive* subgradients at everywhere. See [7, 24, 32, 15] for the practical implications of these properties.

The aim of this paper is to investigate extensions of convex risk measures *preserving the Lebesgue property*. In contrast to the "Fatou extension" by [16], even law-invariant risk measures can not typically retain the Lebesgue property to L^1 (see Example 2.6). Then a natural question is to ask, given a convex risk measure with the Lebesgue property on L^∞, how far it can be extended preserving the Lebesgue property, or if there is a "maximal" space which accommodate a Lebesgue-preserving extension, and what such a space is (if exists). Note that the Lebesgue property here refers to that on the entire space (not in restriction to L^∞), and such extension does make sense.

Our heuristic behind this study is as follows. Among structural properties of the spaces, the crucial one in the analysis of risk measures is not the topology, but the order structure. Also, as long as we consider spaces of random variables on a fixed probability space, there is a universal order of *almost sure inequality* in terms of which key properties of risk measures are described. Especially, the Fatou and Lebesgue properties are regularities w.r.t. this order (see (Fa($\mathscr{X}$)) and (Le($\mathscr{X}$)) for precise definition), thus in a certain sense, these properties are compatible between different spaces, in contrast to the topological regularities (see Remark 2.4). In particular, the "maximal extension preserving the Lebesgue property" makes sense.

Our analysis is based on a simple *uniform-integrability-like* property of risk measures implied by the Lebesgue property (Lemma 3.1). Given a risk measure on L^∞, this suggest us to introduce (formally) a solid vector space of random variables *beyond which the risk measure can not have a Lebesgue extension*. We then verify that the space thus constructed is (well-defined and) an *order-continuous Banach lattice* under a natural gauge norm. Exploiting this and an extended Namioka-Klee theorem by [8], we show further that this space admits a unique Lebesgue extension of the original risk measure, and the space is maximum among all solid spaces of random variables admitting such an extension (Theorem 3.5). We also characterize this maximum space as a subspace of more natural Orlicz-type space in terms of certain uniform integrability property (Theorem 3.9).

The maximality and uniqueness of our Lebesgue extension implies that if ρ is a convex risk measure on a solid space $\mathscr{X}$ having the Lebesgue property, then ρ must agree on $\mathscr{X}$ with the unique Lebesgue extension of $\rho|_{L^\infty}$. This allows us to investigate the Lebesgue property on *arbitrary solid space*, in the spirit of *Jouini-Schachermayer-Touzi's (JST) theorem* [22] which asserts that for a risk measure on L^∞, the Lebesgue property, the weak compactness of lower level sets of the minimal penalty function and the attainability of the supremum in the robust representation are all equivalent. See also [27] for a recent generalization of this result to a certain class of Orlicz spaces. We shall provide a generalization of JST theorem (Theorem 3.13) on arbitrary solid space where the weak compactness of the level sets is replaced by the *uniform integrability* of the algebraic products of level sets and arbitrary single element of $\mathscr{X}$ which agrees with the one by [22] when $\mathscr{X} = L^\infty$.

As an immediate consequence, we show that on any solid space, the Lebesgue property is sufficient for the everywhere subdifferentiability with σ-additive subgradients. Finally, some typical examples are examined in Section 6.

2. Preliminaries

We fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ throughout the paper. All random variables are defined on $(\Omega, \mathcal{F})$, and any probabilistic notation without reference to the probability is understood with respect to $\mathbb{P}$. By convention, we identify without further notice random variables which are equal *almost surely* (a.s.). $L^0 := L^0(\Omega, \mathcal{F}, \mathbb{P})$ denotes the space of (equivalence classes given a.s. equality) of *a.s. finite* ($\mathbb{P}(|X| < \infty) = 1$) random variables. The space L^0 is an *order complete Riesz space* (vector lattice) endowed with the partial order of *almost sure inequality*, i.e., "$X \leq Y$ a.s.". A vector subspace $\mathscr{X} \subset L^0$ is called *solid* (in L^0) if $X \in \mathscr{X}$ and $|Y| \leq |X|$ (a.s.) imply $Y \in \mathscr{X}$. Any solid subspace $\mathscr{X}$ is a *lattice* on its own right with respect to the same a.s. order, thus in particular, if $X \in \mathscr{X}$, then $X^+ := X \vee 0$, $X^- := -(X \wedge 0)$, $|X| := X^+ + X^-$ are elements of $\mathscr{X}$, and the positive cone $\mathscr{X}_+ := \{X \in \mathscr{X} : X \geq 0\}$ is well-defined.

The expectation of a random variable $X \in L^0$ with respect to $\mathbb{P}$ is denoted by $\mathbb{E}[X] := \int_\Omega X(\omega)\mathbb{P}(d\omega)$ whenever well-defined, and for each $p \in (0, \infty]$, $L^p := L^p(\Omega, \mathcal{F}, \mathbb{P})$ denotes the standard Lebesgue space, i.e., $X \in L^p$ iff $\mathbb{E}[|X|^p] < \infty$ (resp. X is essentially bounded) when $p < \infty$ (resp. $p = \infty$). For all $p \in [0, \infty]$, L^p is a solid subspace of L^0, and for $1 \leq p < \infty$ (resp. $p = \infty$), it is a Banach space endowed with the norm $\|X\|_p := \mathbb{E}[|X|^p]^{1/p}$ (resp. $\|X\|_\infty := \operatorname{ess\,sup}|X|$).

By $\mathcal{P}$, we denote the set of all probability measures Q absolutely continuous with respect to $\mathbb{P}$ ($Q \ll \mathbb{P}$), and we use the notation $Q \sim P$ to mean $Q \ll P$ and $P \ll Q$ (equivalent). We identify a probability $Q \in \mathcal{P}$ with its *Radon-Nikodým density* $dQ/d\mathbb{P}$ with respect to $\mathbb{P}$. Then $\mathcal{P}$ becomes a bounded convex closed subset of L^1. For each $Q \in \mathcal{P}$, the Q-expectation as well as the Lebesgue spaces under Q and their norms are denoted respectively by $E_Q[X]$, $L^p(Q) := L^p(\Omega, \mathcal{F}, Q)$ and $\|X\|_{L^p(Q)}$, where we explicitly indicate the dependence on Q unless $Q = \mathbb{P}$.

Finally, we make a couple of remarks. First, all spaces appearing in the sequel are subspaces of L^0 (i.e., consists of random variables on $(\Omega, \mathcal{F})$), thus we omit mentioning the "master space" L^0 in lattice related notation. For instance, "a solid space" refers to a "solid vector subspace of L^0". Second, a solid space $\mathscr{X}$ contains L^∞ as soon as it contains the constants or more simply, a single non-zero constant, say 1. For any other unexplained notation and detail of Riesz space terminology, we refer the reader to [2, Ch. 8, 9].

2.1. Convex Risk Functions on L^∞

Throughout the paper, we will consider a *convex risk function* that we always assume:

Assumption 2.1 (and Definition). $\rho^0 : L^\infty \to \mathbb{R}$ is a normalized sensitive (relevant) *convex risk function* on L^∞, that is

- (A1) monotone: $X, Y \in L^\infty$, $X \leq Y$ a.s. $\Rightarrow \rho^0(X) \leq \rho^0(Y)$;
- (A2) convex: $\rho^0(\alpha X + (1-\alpha)Y) \leq \alpha\rho^0(X) + (1-\alpha)\rho^0(Y)$, $\forall \alpha \in [0, 1]$, $X, Y \in L^\infty$;
- (A3) cash-invariant: $\rho^0(X + c) = \rho^0(X) + c$ for all $X \in L^\infty$, $c \in \mathbb{R}$;
- (A4) normalized: $\rho^0(0) = 0$;
- (A5) sensitive (relevant): $\rho^0(\varepsilon \mathbb{1}_A) > 0$ if $\varepsilon > 0$ and $\mathbb{P}(A) > 0$.

Remark 2.2 (on terminology)**.** If ρ^0 satisfies (A1-3) above, then $X \mapsto \rho^0(-X)$ (resp. $X \mapsto -\rho^0(-X)$) is a *convex risk measure* (resp. *concave monetary utility function*). Though the latter two notions seem more common in literature, we prefer *convex* and *increasing* functions, and one can freely move between three notions with obvious change(s) of sign. The normalizing assumption (A4) has no loss in generality, since we can always replace ρ^0 by $\rho^0 - \rho^0(0)$.

(A1,3) imply that ρ^0 is $\|\cdot\|_\infty$-Lipschitz continuous, hence weakly lower semi-continuous on L^∞. This implies the Fenchel-Moreau dual representation in terms of *finitely additive probabilities*. This type of regularity, however, is not interesting enough in practice, and we usually require at least the *Fatou property* (on L^∞):

$$\sup_n \|X_n\| < \infty,\ X_n \to X \text{ a.s. } \Rightarrow\ \rho^0(X) \le \liminf_n \rho^0(X_n). \tag{2.1}$$

This property is understood in two ways: the weak* ($\sigma(L^\infty, L^1)$-)lower semi-continuity (Krein-Šmulian theorem), and as the (σ-) order lower semi-continuity w.r.t. the *a.s. order*. The latter view is essential for our analysis.

We define the (minimal) penalty function of ρ^0 as its Fenchel-Legendre transform

$$\gamma(Z) := \sup_{X \in L^\infty} (\mathbb{E}[XZ] - \rho^0(X)), \quad \forall Z \in L^1. \tag{2.2}$$

Then $\gamma \ge 0$ by (A4), γ is convex and $\sigma(L^1, L^\infty)$-lower semi-continuous, while (A1,3) mean in terms of γ that $\gamma(Z)$ is finite only if Z is the density of some $Q \in \mathcal{P}$, i.e.,

$$\gamma(Z) < \infty\ \Rightarrow\ Z \ge 0 \text{ a.s. and } \mathbb{E}[Z] = 1, \tag{2.3}$$

We thus regard γ as a function on $\mathcal{P}$, and write $\gamma(Q) = \gamma(dQ/d\mathbb{P})$. We set also

$$\mathcal{Q}_\gamma := \{Q \in \mathcal{P} :\ \gamma(Q) < \infty\}. \tag{2.4}$$

In terms of γ, the Fatou property (2.1) is equivalent to the *robust representation*

$$\rho^0(X) = \sup_{Q \in \mathcal{P}} (E_Q[X] - \gamma(Q)) = \sup_{Q \in \mathcal{Q}_\gamma} (E_Q[X] - \gamma(Q)), \quad X \in L^\infty. \tag{2.5}$$

Given (A1-3) and (2.1), ρ^0 satisfies (A4) iff $\inf_{Q \in \mathcal{Q}_\gamma} \gamma(Q) = 0$, and (A5) iff

$$\exists Q^0 \sim \mathbb{P} \text{ such that } \gamma(Q^0) < \infty. \tag{2.6}$$

The function ρ^0 is said to satisfy the *Lebesgue property* (on L^∞) If

$$\sup_n \|X_n\|_\infty < \infty,\ X_n \to X \text{ a.s. } \Rightarrow\ \rho^0(X) = \lim_n \rho^0(X_n). \tag{2.7}$$

Clearly, (2.7) implies (2.1). It is known as the Jouini-Schachermayer-Touzi theorem (cited below as Theorem 3.11) that given (2.1), the Lebesgue property (2.7) is equivalent to any of the following two conditions: (1) $\{Q \in \mathcal{P} : \gamma(Q) \le c\}$ is weakly compact in L^1 for all $c > 0$, (2) the supremum in (2.5) is attained for every $X \in L^\infty$.

2.2. Convex Risk Functions on Solid Spaces

The notion of convex risk functions can make sense on ordered vector spaces containing the "constants". Here we restrict our attention to *solid vector spaces* of random variables.

Definition 2.3 (Convex Risk Functions)**.** Let $\mathscr{X} \subset L^0$ be a solid space containing the constants. Then a function $\rho : \mathscr{X} \to (-\infty, \infty]$ is called a convex risk function on $\mathscr{X}$ if it is monotone ($X \le Y$ a.s. $\Rightarrow \rho(X) \le \rho(Y)$), proper ($\rho \not\equiv +\infty$), convex and cash-invariant:

$$\rho(X + c) = \rho(X) + c,\ \forall X \in \mathscr{X},\ \forall c \in \mathbb{R}.$$

A convex risk function ρ on $\mathscr{X}$ is said to be normalized (resp. sensitive) if $\rho(0) = 0$ (resp. $\rho(\varepsilon \mathbb{1}_A) > 0$ whenever $\varepsilon > 0$ and $\mathbb{P}(A) > 0$).

A difference between Assumption (and Definition) 2.1 and Definition 2.3 (other than the obvious change of space) is that we generally allow for convex risk functions on $\mathscr{X}$ to take the value $+\infty$, while on L^∞, (A1) and (A3) already implies ρ^0 is finite valued.

The Fatou and Lebesgue properties also make senses on solid space $\mathscr{X}$ in the forms:

$$\text{(Fa}(\mathscr{X})) \quad \exists Y \in \mathscr{X},\ |X_n| \le |Y| \text{ a.s. } \forall n \text{ and } X_n \to X \text{ a.s. } \Rightarrow \rho(X) \le \liminf_n \rho(X_n),$$

$$\text{(Le}(\mathscr{X})) \quad \exists Y \in \mathscr{X},\ |X_n| \le |Y| \text{ a.s. } \forall n \text{ and } X_n \to X \text{ a.s. } \Rightarrow \rho(X) = \lim_n \rho(X_n).$$

Remark 2.4. A couple of remarks are in order.

(1) When $\mathscr{X} = L^\infty$, the common assumption of (Fa($\mathscr{X}$)) and (Le($\mathscr{X}$)) is equivalent to $\sup_n \|X_n\|_\infty < \infty$, hence these properties agree with (2.1) and (2.7), respectively.
(2) The Fatou and Lebesgue properties are compatible: if $\mathscr{X}$ and $\mathscr{Y}$ are solid spaces with $\mathscr{X} \subset \mathscr{Y}$, then (Fa($\mathscr{Y}$)) (resp. (Le($\mathscr{Y}$))) implies (Fa($\mathscr{X}$)) (resp. (Le($\mathscr{X}$))).

Many authors studied convex risk functions on particular spaces that we briefly review here in a highly selective manner. We divide the literature into three categories.

1. The first class of spaces are L^p ($p < \infty$) or slightly more generally the Morse subspaces M^Ψ of the Orlicz spaces L^Ψ (also known as *Orlicz hearts*) [23, 10, 9, 1]. As the duals of these spaces have no "singular parts", the treatment of risk functions is rather easier (than L^∞), and the "norm" regularities already give sufficiently nice description of risk functions. In this case, the Lebesgue (resp. Fatou) property coincides with the norm continuity (resp. lower semi-continuity).
2. The second class is Orlicz spaces L^Ψ or more generally *locally convex Fréchet lattices*. As L^∞, the duals of these spaces generally have "singular part", thus the topological lower semi-continuity gives only a dual representation by possibly "singular measures". To obtain the σ-additive representation and further nice description, we need the regularities in terms of some weaker topology. In case of L^∞, this is exactly those for the weak* topology, which are characterized as the regularities in terms of order, namely Fatou and Lebesgue properties. [8] generalized this observation to locally convex Fréchet lattices, making clear the connection of topological and order regularities and the role of order structure of the spaces. See also [4] for related direction.
3. Another conceptually natural choice of space is the space L^0 of all random variables. As is well known, however, the dual of L^0 is degenerate ($\{0\}$ if atomless), thus the standard duality in convex analysis à la Fenchel-Moreau no longer works (directly). [25] (see below) and [11] studied this case, appealing extensively to the *monotonicity*.

2.3. Question and Related Directions

As noted in Remark 2.4, the Fatou and Lebesgue properties are stable under restriction of spaces. In particular, if $\mathscr{X}$ is solid subspace of L^0 and ρ is a convex risk function on $\mathscr{X}$ in the sense of Definition 2.3 with (Le($\mathscr{X}$)) (resp. (Fa($\mathscr{X}$))), then ρ (more precisely its restriction to L^∞) is also a convex risk function on L^∞ in the sense of Assumption 2.1 having (2.7) (resp. (2.1)). We are interested in the converse direction.

Definition 2.5 (Lebesgue Extension)**.** Let ρ be a convex risk function on a solid space $\mathscr{X} \subset L^0$ and $\mathscr{Y} \subset L^0$ be another solid space with $\mathscr{X} \subset \mathscr{Y}$. We say that ρ has a *Lebesgue extension* to $\mathscr{Y}$ if there is a convex risk function ρ' on $\mathscr{Y}$ such that $\rho'|_{\mathscr{X}} = \rho$ and ρ' has the

Lebesgue property on $\mathscr{Y}$ (hence on $\mathscr{X}$ too). If this is the case, we call $(\rho', \mathscr{Y})$ a Lebesgue extension of $(\rho, \mathscr{X})$ (or simply, of ρ).

Question 1. Suppose Assumption 2.1 and (2.7). Then does there exists a maximum solid space of random variables which admits a Lebesgue extension of ρ^0?

Here we briefly review some related directions.

Fatou Extension of Law-Invariant Risk Functions. A natural related question is the extension preserving the Fatou property (instead of Lebesgue). Given a convex risk function ρ^0 on L^∞, [16] considered its L^1*-closure* $\bar{\rho}^1(X) := \sup_{Z \in L^\infty}(\mathbb{E}[XZ] - \gamma(Z))$ on L^1. This function is proper and (weakly) lower semi-continuous on L^1 (hence Fatou) as soon as $\mathcal{Q}_\gamma \cap L^\infty \neq \emptyset$. On the other hand, it is not clear that $\bar{\rho}^1$ is an extension of ρ^0, i.e. $\bar{\rho}^1|_{L^\infty} = \rho^0$. [16] proved that this is the case if ρ^0 is *law-invariant*, and then $\bar{\rho}^1$ is a *unique* lower semi-continuous extension of ρ^0 to L^1. In particular, every *law-invariant* convex risk function has a "Fatou" extension to L^1. In contrast, the Lebesgue property may not be preserved to L^1 (even if law-invariant) as the next example illustrates.

Example 2.6 (Entropic Risk Function)**.** Let

$$\rho_{\mathrm{ent}}(X) := \log \mathbb{E}[\exp(X)], \quad X \in L^0. \tag{2.8}$$

Clearly, this is a convex risk function with the Fatou property on the whole L^0. Indeed, when $\mathscr{X} = L^0$, $X_n \to X$ already implies $\sup_n |X_n| =: Y \in L^0$, hence (Fa($\mathscr{X}$)) on L^0 follows from Fatou's lemma applied to $(\exp(X_n))_n$. Let

$$L^{\exp} := \{X \in L^0 : \mathbb{E}[\exp(\lambda|X|)] < \infty, \exists \lambda > 0\} \quad \text{(Orlicz space)},$$
$$M^{\exp} := \{X \in L^0 : \mathbb{E}[\exp(\lambda|X|)] < \infty, \forall \lambda > 0\} \quad \text{(Morse subspace)}.$$

Then $M^{\exp} \subset L^{\exp} \subset L^1$ and the inclusions are strict if the probability space is atomless (e.g. exponential random variables). The function ρ_{ent} satisfies the Lebesgue property on $M^{\exp}$. Indeed, if $(X_n) \subset M^{\exp}$, $|X_n| \leq Y \in M^{\exp}$ and $X_n \to X$ a.s., we can apply the dominated convergence theorem to the sequence $(\exp(X_n))_n$ dominated by $\exp(Y) \in L^1$. On the other hand, the Lebesgue property fails to hold on $L^{\exp}$. To see this, pick an $X \in L^{\exp} \setminus M^{\exp}$ and a positive constant $\lambda > 0$ with $\mathbb{E}[\exp(\lambda|X|)] = \infty$. Note that

$$e^{\lambda|X|} = e^{\lambda|X|}\mathbb{1}_{\{|X|>n\}} + e^{\lambda|X|}\mathbb{1}_{\{|X|\leq n\}} \leq \exp(\lambda|X|\mathbb{1}_{\{|X|>n\}}) + e^{\lambda n}.$$

Thus if we take $X_n = \lambda|X|\mathbb{1}_{\{|X|>n\}}$, then $|X_n| \leq \lambda|X| \in L^{\exp}$, $X_n \to 0$ a.s., but $\rho_{\mathrm{ent}}(X_n) \equiv +\infty \neq 0 = \rho_{\mathrm{ent}}(0)$. In summary, $(\rho_{\mathrm{ent}}|_{M^{\exp}}, M^{\exp})$ is a regular extension of $(\rho_{\mathrm{ent}}, L^\infty)$, while $(\rho_{\mathrm{ent}}|_{L^{\exp}}, L^{\exp})$ is not.

Approximation by Bounded Variables. One may have an intuition that L^∞ well-works as a *skeleton*, i.e., even if a risk function is defined on a big space, its structure is more or less determined by its values on L^∞. The *maximum Lebesgue extension* gives a precise limit of this reasoning. Suppose that ρ^0 has a Lebesgue extension ρ to a solid space $\mathscr{X} \subset L^0$. Then the Lebesgue property implies for any $X \in \mathscr{X}$

$$\rho(X) = \lim_{m\to\infty} \lim_{n\to\infty} \rho^0((X \vee (-m)) \wedge n) =: \rho_{\mathrm{ext}}(X), \tag{2.9}$$

and the two limits are interchangeable. In particular, the Lebesgue extension (to $\mathscr{X}$) is unique (if exists) and it inherits the basic structure of the original ρ^0. The equality (2.9), the interchangeability of limits as well as the uniqueness may not be true for Fatou extension.

For general ρ^0 (not necessarily Lebesgue), ρ_{ext} itself is well-defined on the whole L^0 as a *possibly improper* monotone convex cash-invariant function, and $\rho_{\mathrm{ext}}|_{L^\infty} = \rho^0$. [11]

investigated this extension (in the context of càdlàg processes) providing necessary and sufficient conditions for ρ_{ext} to be proper (hence a convex risk function).

Finite-Valued Extension and the Lebesgue Property on L^∞. We note that our standing assumption (2.7) is reasonable, and is related to a *finite-valued extension* of ρ^0. In fact,

Theorem 2.7 ([12], Theorem 3)**.** *If $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless and if ρ^0 has a finite-valued extension to a solid space $\mathscr{X} \supsetneq L^\infty$ which is* rearrangement-invariant *($X \in \mathscr{X}$ and* $\text{law}(Y) = \text{law}(X) \Rightarrow Y \in \mathscr{X}$*), then ρ^0 has the Lebesgue property (on L^∞).*

The assumption of being atomless is harmless in practice, and all L^p spaces are solid and rearrangement-invariant. In particular, most of risk functions of interest have the Lebesgue property on L^∞.

Inf-convolution. Another feature of the Lebesgue property that the Fatou property does not have is the stability under *infimal convolution*. Given a risk function ρ and an arbitrary convex function g on a space $\mathscr{X}$, their infimal convolution is defined by $\rho \Box g(X) = \inf_{Y \in \mathscr{X}}(\rho(X-Y) + g(Y))$ which is again a convex function, and it is proper iff $\rho\Box g(0) > -\infty$. Then if *either ρ or g* has the Lebesgue property, then so does $\rho\Box g$ (hence *a fortiori* Fatou too) as long as proper, while the Fatou properties of *both of ρ and g* do not imply that of $\rho\Box g$, thus not enough even for the σ-additive robust representation of the convolution. In the financial context, this type of convolution appears in (1) risk/asset allocation problems where g is another convex risk function, and in (2) hedging and indifference pricing based on a convex risk functions (measure or monetary utility) where g is the indicator function (in the sense of convex analysis) of a convex set. In case (1), *if both risk functions are law-invariant*, the Lebesgue property on L^∞ of one of risk functions provide the robust representation of the convolution on L^1 thanks to Fatou-extendability to L^1 above. In case (2), however, we note that the convex set determining g as its indicator function is typically the convex cone (hence unbounded) of attainable claims which is not law-invariant. Thus the argument of [16] no longer works and we can not impose the Lebesgue property on g.

3. MAIN RESULTS

3.1. ELEMENTARY OBSERVATIONS

Lemma 3.1. *Suppose that $\mathscr{X}$ is a solid subspace of L^0 and ρ is a (normalized) convex risk function on $\mathscr{X}$. If ρ has the Lebesgue property (Le($\mathscr{X}$)), then for any $X \in \mathscr{X}$,*

$$\lim_{N\to\infty} \rho\left(\alpha|X|\mathbb{1}_{\{|X|>N\}}\right) = 0, \quad \forall \alpha > 0. \tag{3.1}$$

Proof. For any $\alpha > 0$, $Y_N^\alpha := \alpha|X|\mathbb{1}_{\{|X|>N\}} \to 0$, a.s. $(N \uparrow \infty)$, $|Y_N^\alpha| \leq |\alpha X|$ with $\alpha X \in \mathscr{X}$, thus $Y_N^\alpha \in \mathscr{X}$ (since solid) and $\lim_N \rho(Y_N^\alpha) = 0$ by the Lebesgue property. □

Remark 3.2. Some comments on the condition (3.1) are in order.

(1) (3.1) implies $\rho(\alpha|X|) < \infty$ for every $\alpha > 0$. Indeed, by the convexity and monotonicity, $\rho(\alpha|X|) \leq \frac{1}{2}(\rho(2\alpha|X|\mathbb{1}_{\{|X|>N\}}) + \rho(2\alpha|X|\mathbb{1}_{\{|X|\leq N\}}))$. The first term is eventually finite by (3.1) while the second term is bounded by $2\alpha N$ by the cash-invariance and the monotonicity. In particular, if ρ has the Lebesgue property on $\mathscr{X}$, then it must be finite valued since $\rho(X) \leq \rho(|X|)$ again by the monotonicity.

(2) If X and Y satisfy the condition (3.1), then so does $X + Y$. To see this, observe that $(|X|+|Y|)\mathbb{1}_{\{|X+Y|>N\}} \leq 2|X|\mathbb{1}_{\{|X|>N/2\}} + 2|Y|\mathbb{1}_{\{|Y|>N/2\}}$, then apply the convexity.

(3) Finally, if Y satisfies (3.1), then so does any X with $|X| \leq |Y|$ by the monotonicity.

Lemma 3.1 suggests us that the Lebesgue extension is impossible beyond

$$\text{“}\{X \in L^0 : \lim_N \rho^0(\alpha|X|\mathbb{1}_{\{|X|>N\}}) = 0,\ \forall \alpha > 0\}\text{”}.$$

Though this ρ^0 has *not* been defined outside L^∞, thus this "space" is still formal, observe that to give a rigorous meaning to this space, ρ^0 needs only to be defined for *positive* random variables, and (the second expression of) (2.5) makes sense on L^0_+. We thus *define*

$$\hat{\rho}(X) := \sup_{Q\in\mathcal{Q}_\gamma} (E_Q[X] - \gamma(Q)),\ \forall X \in \mathcal{D}_\gamma, \tag{3.2}$$

$$\mathcal{D}_\gamma := \{X \in L^0 : X^- \in \textstyle\bigcap_{Q\in\mathcal{Q}_\gamma} L^1(Q)\}. \tag{3.3}$$

$\mathcal{D}_\gamma$ is not *linear*, but is a convex cone with $L^\infty \cup L^0_+ \subset \mathcal{D}_\gamma$, and is *upward solid*: $X \geq Y \in \mathcal{D}_\gamma \Rightarrow X \in \mathcal{D}_\gamma$. The next lemma says that $\hat{\rho}$ is a "convex risk function on $\mathcal{D}_\gamma$":

Lemma 3.3. *$\hat{\rho} : \mathcal{D}_\gamma \to (-\infty, +\infty]$ is a proper monotone convex function with $\hat{\rho}(X + c) = \hat{\rho}(X) + c$ for all $X \in \mathcal{D}_\gamma$ and $c \in \mathbb{R}$, and $\hat{\rho}|_{L^\infty} = \rho^0$.*

Proof. For each $Q \in \mathcal{Q}_\gamma$, $X \mapsto E_Q[X] - \gamma(Q)$ on $\mathcal{D}_\gamma$ is $\mathbb{R} \cup \{+\infty\}$-valued, monotone, and convex, hence so is the point-wise supremum $\hat{\rho}$. It is clear from the definition that $\rho = \rho^0$ on L^∞, and in particular $\hat{\rho}(0) = 0$ (hence proper too). The cash-invariance is also immediate by a direct computation. □

3.2. The Maximum Solid Space Admitting Lebesgue Extension

Now the following space is well-defined:

$$M_u^{\hat{\rho}} := \left\{ X \in L^0 : \lim_{N\to\infty} \hat{\rho}\left(\alpha|X|1_{\{|X|>N\}}\right) = 0,\ \forall \alpha > 0 \right\}. \tag{3.4}$$

By Remark 3.2, $M_u^{\hat{\rho}}$ is a solid vector space. We introduce the *gauge*:

$$\|X\|_{\hat{\rho}} := \inf\{\lambda > 0 : \hat{\rho}(|X|/\lambda) \leq 1\} \quad (\inf\emptyset := \infty). \tag{3.5}$$

Lemma 3.4. *$\|\cdot\|_{\hat{\rho}}$ is a $[0,\infty]$-valued seminorm on L^0. Moreover, for all $X \in L^0$,*

$$|X| \leq |Y| \text{ a.s.} \ \Rightarrow\ \|X\|_{\hat{\rho}} \leq \|Y\|_{\hat{\rho}}; \tag{3.6}$$

$$\|X\|_{\hat{\rho}} < \infty \ \Leftrightarrow\ \hat{\rho}(\alpha|X|) < \infty,\ \exists \alpha > 0; \tag{3.7}$$

$$\|X\|_{\hat{\rho}} = 0 \ \Leftrightarrow\ X = 0 \text{ a.s.} \tag{3.8}$$

Proof. It is standard that the gauge of the convex set $\{X : \hat{\rho}(|X|) \leq 1\}$ is an $[0,\infty]$-valued seminorm. (3.6) is clear from the monotonicity of $\hat{\rho}$. As for (3.7), "$\Rightarrow$" is clear from (3.5), while if $\hat{\rho}(\alpha|X|) < \infty$ for an $\alpha > 0$, $\hat{\rho}(\varepsilon\alpha|X|) = \hat{\rho}(\varepsilon\alpha|X| + (1-\varepsilon)0) \leq \varepsilon\hat{\rho}(\alpha|X|)$ for all $\varepsilon \in (0,1)$ by convexity. To see "$\Rightarrow$" in (3.8), suppose $X \neq 0$, then there is a set A with $|X| > \varepsilon > 0$ on A and $\mathbb{P}(A) > 0$. By (A5), $\hat{\rho}|_{L^\infty} = \rho^0$, monotonicity and convexity,

$$\hat{\rho}(|X|/\lambda) \geq \hat{\rho}((\varepsilon/\lambda)\mathbb{1}_A) \geq \rho^0(\varepsilon\mathbb{1}_A)/\lambda \to \infty \quad (1 > \lambda \downarrow 0).$$

Thus $\|X\|_{\hat{\rho}} > 0$. □

A norm on a Riesz space satisfying (3.6) is called a *lattice norm*. A Riesz space equipped with such a norm is called a *Banach lattice* if it is complete (w.r.t. the norm). By Lemma 3.4 and Remark 3.2 (1), we see that $\|\cdot\|_{\hat{\rho}}$ is a lattice norm on $M_u^{\hat{\rho}}$.

Now our first result is the following which will be proved in Section 4:

Theorem 3.5. *Suppose Assumption 2.1 and (2.7). Then*

(1) $M_u^{\hat\rho}$ *is solid in* L^0*, and* $(M_u^{\hat\rho}, \|\cdot\|_{\hat\rho})$ *is a* σ*-order-continuous Banach lattice, i.e.,* $M_u^{\hat\rho}$ *is complete w.r.t. the lattice norm* $\|\cdot\|_{\hat\rho}$ *and* $\|\cdot\|_{\hat\rho}$ *is* σ*-order continuous:*

$$\exists Y \in M_u^{\hat\rho},\ |X_n| \le |Y| \text{ a.s. } (\forall n), \text{ and } |X_n| \to 0 \text{ a.s.} \Rightarrow \lim_n \|X_n\|_{\hat\rho} = 0. \tag{3.9}$$

(2) $(\hat\rho, M_u^{\hat\rho})$ *is a Lebesgue extension of* ρ^0*, i.e.,* $\hat\rho : M_u^{\hat\rho} \to \mathbb{R}$ *is well-defined as a convex risk function satisfying (Le($\mathscr{X}$)) on* $M_u^{\hat\rho}$ *with* $\hat\rho|_{L^\infty} = \rho^0$*.*

(3) $M_u^{\hat\rho}$ *is the maximum solid space admitting a Lebesgue extension of* ρ^0*, i.e., if* $\mathscr{X} \subset L^0$ *is a solid space and* $(\rho', \mathscr{X})$ *is a Lebesgue extension of* ρ^0*, then* $\mathscr{X} \subset M_u^{\hat\rho}$ *and* $\hat\rho|_{\mathscr{X}} = \rho'$*. In particular,* $\hat\rho$ *is the unique Lebesgue extension of* ρ^0 *to* $M_u^{\hat\rho}$*.*

Remark 3.6. The property (3.9) remains true if *sequences* (X_n) are replaced by *nets* $(X_\lambda)_\lambda$, thus $(M_u^{\hat\rho}, \|\cdot\|_{\hat\rho})$ is actually *order-continuous* (not only "σ"). Indeed, L^0 is order-complete, hence so is its ideal $M_u^{\hat\rho}$ (see [3, Theorem 7.73], [2, Lemma 8.14]), and the order-continuity of the norm of an order-complete Banach lattice is equivalent to the generally weaker σ-order-continuity (see [2, Theorem 9.22]).

We already know that $M_u^{\hat\rho}$ is solid, and is a normed Riesz space with the lattice norm $\|\cdot\|_{\hat\rho}$, while it is *heuristically* clear that ρ^0 has no Lebesgue extension beyond $M_u^{\hat\rho}$. It thus remains essentially that $(M_u^{\hat\rho}, \|\cdot\|_{\hat\rho})$ is complete, and $\|\cdot\|_{\hat\rho}$ is *order-continuous*. In fact, if $\|\cdot\|_{\hat\rho}$ is order-continuous, every norm-continuous function is automatically order-continuous (hence Lebesgue), while if $M_u^{\hat\rho}$ is complete (hence is a Banach lattice), every *finite-valued* monotone convex function is norm-continuous by the extended Namioka-Klee theorem due to [8].

Obviously, our idea for the definition of $M_u^{\hat\rho}$ stems from the theory of (Musielak-)Orlicz spaces. Let us define the *Orlicz space* and its *Morse subspace* associated to $\hat\rho$:

$$L^{\hat\rho} := \{X \in L^0 : \hat\rho(\alpha|X|) < \infty, \exists\alpha > 0\} \overset{(3.7)}{=} \left\{X \in L^0 : \|X\|_{\hat\rho} < \infty\right\}, \tag{3.10}$$

$$M^{\hat\rho} := \left\{X \in L^0 : \hat\rho(\alpha|X|) < \infty, \forall\alpha > 0\right\}. \tag{3.11}$$

In analogy to the standard Orlicz spaces both $L^{\hat\rho}$ and $M^{\hat\rho}$ are Banach lattices (cf. [25, 31]), and $\hat\rho$ is well-defined ($\mathbb{R} \cup \{+\infty\}$-valued) on $L^{\hat\rho}$ with the Fatou property, and it is finite-valued on $M^{\hat\rho}$ (Proposition 4.5 below). In general, $M_u^{\hat\rho} \subset M^{\hat\rho} \subset L^{\hat\rho}$ by Remark 3.2 (2) and (3.7), and the inclusions can generally be strict as the following two examples illustrate.

Example 3.7 (Entropic Risk Function revisited)**.** Put $\rho^0 = \rho_{\mathrm{ent}}$ (defined by (2.8)). Then $\gamma(Q) = \sup_{X \in L^\infty}(E_Q[X] - \rho_{\mathrm{ent}}(X)) = E_Q[\log(dQ/d\mathbb{P})] =: \mathcal{H}(Q|\mathbb{P})$, the relative entropy (thus entropic risk function, see [19, Lemma 3.29]). Therefore

$$\hat\rho_{\mathrm{ent}}(X) = \sup_{Q \ll \mathbb{P}, \mathcal{H}(Q|\mathbb{P}) < \infty} (E_Q[X] - \mathcal{H}(Q|\mathbb{P})),$$

and the identity $\hat\rho_{\mathrm{ent}}(X) = \log \mathbb{E}[\exp(X)]$ remains true for all $X \in L^0_+$. In particular, $M^{\hat\rho_{\mathrm{ent}}} = M^{\exp} \subsetneq L^{\exp} = L^{\hat\rho_{\mathrm{ent}}}$ if $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless. Further, we see that $M_u^{\hat\rho_{\mathrm{ent}}} = M^{\hat\rho_{\mathrm{ent}}} (= M^{\exp})$. Indeed, if $X \in M^{\exp}$, $\mathbb{E}[\exp(\lambda|X|\mathbb{1}_{\{|X|>N\}})] = \mathbb{E}[\exp(\lambda|X|)\mathbb{1}_{\{|X|>N\}}] + \mathbb{P}(|X| \le N) \to 1$ by the dominated convergence for every $\lambda > 0$.

The next example shows that the inclusion $M_u^{\hat\rho} \subset M^{\hat\rho}$ may be strict.

Example 3.8. Let $(\Omega, \mathcal{F}) = (\mathbb{N}, 2^{\mathbb{N}})$ and define a sequence of probabilities on $(\mathbb{N}, 2^{\mathbb{N}})$:

$$Q_1(\{1\}) = 1,\ Q_n(\{1\}) = 1 - 1/n,\ Q_n(\{n\}) = 1/n, \quad n \in \mathbb{N},$$

and set $\mathbb{P}(\{n\}) = 2^{-n}$ ($\forall n \in \mathbb{N}$), $\mathcal{Q} := \overline{\mathrm{conv}}(Q_n : n \in \mathbb{N})$ and $\gamma(Q) = 0$ (resp. $= +\infty$) if $Q \in \mathcal{Q}$ (resp. $Q \notin \mathcal{Q}$). The corresponding risk function ρ^0 is coherent. We see easily that $\mathcal{Q}$ is weakly compact (hence ρ^0 has the Lebesgue property on $L^\infty \simeq l^\infty$) and

$$\hat{\rho}(X) = \sup_{Q \in \mathcal{Q}} E_Q[X] = \sup_n E_{Q_n}[X], \quad X \in L^0_+,$$

Now if we take $X(k) = k$, then

$$E_{Q_n}[X] = (1 - 1/n) + n \cdot (1/n) = 2 - 1/n \;\Rightarrow\; \hat{\rho}(X) = \sup_n E_{Q_n}[X] = 2,$$

thus $X \in M^{\hat{\rho}}$ by coherence. On the other hand, for any $N \in \mathbb{N}$,

$$E_{Q_n}[X \mathbb{1}_{\{X>N\}}] = \mathbb{1}_{\{n>N\}} \;\Rightarrow\; \rho(X \mathbb{1}_{\{X>N\}}) = \sup_n E_{Q_n}[X \mathbb{1}_{\{X>N\}}] \equiv 1.$$

Hence $X \notin M^{\hat{\rho}}_u$.

We now state our second result which explains the reason for the subscript "u". The proof will be given in Section 5.1.

Theorem 3.9. *For $X \in M^{\hat{\rho}}$, the following are equivalent:*

(1) $X \in M^{\hat{\rho}}_u$;
(2) for any $c > 0$, the family $\{X dQ/d\mathbb{P} : \gamma(Q) \le c\}$ is uniformly integrable;
(3) for every $Y \in L^\infty$, the supremum $\sup_{Q \in \mathcal{Q}_\gamma}(E_Q[|X|Y] - \gamma(Q))$ is attained.

Remark 3.10. In the coherent case, the condition (2) simplifies to the uniform integrability of families $\{X dQ/d\mathbb{P}\}_{Q \in \mathcal{Q}}$ for a suitable set $\mathcal{Q}$, which repeatedly appeared in the study of duality theory for robust utility indifference prices by the author [28, 29]. There the difference between the "integrability" ($\sup_{Q\in\mathcal{Q}} E_Q[|X|] < \infty$) and the "uniform integrability" in the sense above plays a crucial role for the validity of the key duality formula. In analogy, the condition (2) is crucial when we generalize the duality theory to the case of penalized robust utility, which was an original motivation of the current study.

3.3. JOUINI-SCHACHERMAYER-TOUZI'S THEOREM AND SUBDIFFERENTIABILITY

Here we discuss a generalization of the following Jouini-Schachermayer-Touzi (JST) theorem, obtained first by [22] under an additional assumption of separability of L^1, and the latter condition was removed by [13] using a homogenization technique.

Theorem 3.11 ([22, Theorem 5.2], [13, Theorem 2])**.** *For a convex risk function $\rho^0 : L^\infty \to \mathbb{R}$ with the Fatou property (2.1), the following are equivalent:*

(1) ρ^0 has the Lebesgue property (2.7);
(2) $\{dQ/d\mathbb{P} : \gamma(Q) \le c\}$ is weakly compact for each $c > 0$;
(3) for each $X \in L^\infty$, the supremum $\sup_{Q\in\mathcal{Q}_\gamma}(E_Q[X] - \gamma(Q))$ is attained.

Several comments are in order.

Item (3) is also stated in terms of subdifferentiability of ρ^0. In fact, noting that

$$\rho^0(X) = E_Q[X] - \gamma(Q) \;\Leftrightarrow\; E_Q[X] - \rho^0(X) = \gamma(Q) = \sup_{Y \in L^\infty}(E_Q[Y] - \rho^0(Y)), \tag{3.12}$$

the maximizer Q is a σ*-additive* subgradient of ρ^0 at X. Thus (1) $\Leftrightarrow$ (3) tells us that the Lebesgue property is a necessary and sufficient condition for everywhere subdifferentiability in this sense. Here the σ-additivity is essential since *finitely additive* subgradient always exists as long as ρ^0 is Fatou in the L^∞ case (by Banach- Alaoglu theorem). [8] then shows that the latter type of subdifferentiability is still true for *finite-valued* convex risk functions on any locally convex Fréchet lattice (including all Banach lattices), raising a natural question: given a finite-valued convex risk function having a dual representation by σ-additive probabilities, is the supremum in the representation always attained? The answer is generally no of course, but [8] proved that this is the case if the space is a locally convex Fréchet lattice and the risk function have the Lebesgue property, namely the implication (1) $\Rightarrow$ (3) is still true for "good spaces".

(2) $\Leftrightarrow$ (3) is viewed as an analogue of James' theorem: a bounded closed convex subset B of a Banach space E is weakly compact if and only if every continuous linear functional on E attains the maximum on B. This theorem can also be stated in a *perturbed form*: the set B is weakly compact iff $\langle x, y\rangle - \delta_B(x)$ attains the maximum over the whole E for every $y \in E^*$, while $\{x \in E : \delta_B(x) \leq c\} = B$ for all $c > 0$. In this view, the equivalence (2) $\Leftrightarrow$ (3) is a version of perturbed James' theorem with the indicator δ_B replaced by a convex perturbation function ρ^0. In this line, [26] recently obtained a general form of perturbed James' theorem for *coercive* perturbation functions (see Theorem 5.2 below).

Recently, [27] obtained a complete generalization of JST theorem for an Orlicz space L^Ψ whose order continuous dual coincides with M^{Ψ^*}. The precise statement is:

Theorem 3.12 ([27], Theorem 1)**.** *Let Ψ be a Young function with finite conjugate Ψ^*, and $\rho : L^\Psi \to \mathbb{R}$ a finite-valued convex risk function which is lower semi-continuous w.r.t. $\sigma(L^\Psi, M^{\Psi^*})$ (weak* topology). Then the following are equivalent:*

(1) ρ has the Lebesgue property (Le($\mathscr{X}$)) on L^Ψ;
(2) for each $c \geq 0$, $\{Z \in M^{\Psi^*} : \rho^*(Z) \leq c\}$ is $\sigma(M^{\Psi^*}, L^\Psi)$-compact;
(3) for each $X \in L^\Psi$, the supremum $\sup_{Z\in M^{\Psi^*}}(\mathbb{E}[XZ] - \rho^*(Z))$ is attained.

The $\sigma(L^\Psi, M^{\Psi^*})$-lower semi-continuity is equivalent to the representation $\rho(X) = \sup_{Z\in M^{\Psi^*}}(\mathbb{E}[XZ] - \rho^*(Z))$ with $Z \in M^{\Psi^*}$ only, which is generally stronger than the Fatou property on L^Ψ ($\Leftrightarrow$ the representation with all $Z \in L^{\Psi^*}$). The two conditions are in fact equivalent if the conjugate Ψ^* satisfies the so-called Δ_2-condition (since then $L^{\Psi^*} = M^{\Psi^*}$). On the other hand, if Ψ satisfies the Δ_2-condition, then $L^\Psi = M^\Psi$ and this case is thoroughly studied by [10, 9].

Using Theorems 3.5 and 3.9 as well as simple comparisons, we can give yet another extension of Theorem 3.11. For the space $\mathscr{X}$ we require only that it is a solid space containing the constants, which is the case for most of common spaces including all Orlicz spaces/Morse subspaces and L^p with $p \in [0, \infty]$, so our setting is quite general. Besides its generality, our characterization is also universal (and elementary) in the sense that it does not involve any topological structure of the particular space.

Theorem 3.13 (Generalization of JST-Theorem [22])**.** *Let $\mathscr{X} \subset L^0$ be a solid space containing the constants and $\rho : \mathscr{X} \to \mathbb{R}$ be a finite-valued convex risk function with $\rho|_{L^\infty}$ satisfying (A1-5) and the Fatou property (2.1). Then the following are equivalent:*

(1) ρ has the Lebesgue property (Le($\mathscr{X}$)) on $\mathscr{X}$;
(2) for all $X \in \mathscr{X}$ and $c \geq 0$, $\{XdQ/d\mathbb{P} : \gamma_\infty(Q) \leq c\}$ is uniformly integrable where $\gamma_\infty(Q) = \sup_{X\in L^\infty}(E_Q[X] - \rho(X))$;
(3) for all $X \in \mathscr{X}$, the supremum $\sup_{Q\in\mathcal{Q}_{\gamma_\infty}}(E_Q[X] - \gamma_\infty(Q))$ is (well-defined and) attained.
(4) it holds that

$$\rho(X) = \max_{Q\in\mathcal{Q}_{\gamma_\infty}}(E_Q[X] - \gamma_\infty(Q)), \quad \forall X \in \mathscr{X}, \tag{3.13}$$

Proof will be given in Section 5.2.

Remark 3.14. Some remarks on (3) and (4) are in order.

(1) We have not *a priori* assumed that $\rho(X) = \sup_{Q\in\mathcal{Q}_{\gamma_\infty}}(E_Q[X]-\gamma_\infty(Q))$ on the whole $\mathscr{X}$ (recall that γ_∞ is the conjugate of $\rho|_{L^\infty}$), thus (3) $\Rightarrow$ (4) is not completely trivial.
(2) The finiteness of ρ on $\mathscr{X}$ and the Fatou property on L^∞ already imply that $\mathscr{X}$ is contained in $\bigcap_{Q\in\mathcal{Q}_{\gamma_\infty}} L^1(Q)$. See Lemma 5.4, or observe more directly that $E_Q[|X|] = \sup_n E_Q[|X|\wedge n] \leq \sup_n \rho(|X|\wedge n) + \gamma_\infty(Q) \leq \rho(|X|) + \gamma_\infty(Q)$. Thus the supremum in (3) is well-defined without any further assumption.
(3) When $\mathscr{X} = L^\infty$, Theorem 3.13 recovers JST theorem since then (2) is equivalent to the weak compactness of all lower level sets of the minimal penalty function.

Here we emphasize that our extension is not only general, but also elementary and universal. In fact, our statements are completely free of topological structure of the particular space $\mathscr{X}$, as everything is done by the minimal penalty function γ_∞ of $\rho|_{L^\infty}$. In particular, we do not need *a priori* to mind whether $\mathscr{X}$ is locally convex, what the *order-continuous dual* $\mathscr{X}_n^\sim$ is (if $\mathscr{X}$ is a Fréchet lattice), and how good is the topology $\sigma(\mathscr{X}, \mathscr{X}_n^\sim)$ etc. If ρ has the Lebesgue property, then ρ is represented by γ_∞ and $\mathcal{Q}_{\gamma_\infty}$ on the whole $\mathscr{X}$ as (3.13) and $\mathscr{X}$ *a fortiori* has a good topology as a subspace of $M_u^{\hat{\rho}}$. However, the *a priori* assumed topology of $\mathscr{X}$ does not matter as long as we hope the Lebesgue property.

Nevertheless, it is better to mention what is deduced additionally if we are given some topological information of $\mathscr{X}$. When $\mathscr{X}$ is a locally convex Fréchet lattice as in [8], the finiteness of ρ and the Fatou property on L^∞ imply additionally that $\mathcal{Q}_{\gamma_\infty} \subset \mathscr{X}_n^\sim$ since then $\mathscr{X} \subset \bigcap_{Q\in\mathcal{Q}_{\gamma_\infty}} L^1(Q)$ (Remark 3.14 (2)), while (3.13) implies $\gamma_\infty = \rho^*$. Thus (4) can be stated as: $\sup_{Z\in\mathscr{X}_n^\sim}(\mathbb{E}[XZ]-\rho^*(Z))$ is attained for all $X \in \mathscr{X}$ as obtained in [8, Lemma 7]. If more specifically $\mathscr{X}$ is an Orlicz space L^Ψ, then Item (2) in Theorem 3.13 is actually equivalent to

$$\{Z \in L^{\Psi^*} : \rho^*(Z) \leq c\} \text{ is } \sigma(L^{\Psi^*}, L^\Psi)\text{-compact for all } c > 0. \tag{3.14}$$

That (2) implies (3.14) is an easy consequence of a characterization of weakly compact sets in L^Ψ (see [30, p.144, Corollary 2]). The converse implication follows from the observation that (3.14) implies with the help of a minimax theorem that $\rho = \widehat{\rho|_{L^\infty}}$ on L^Ψ, hence *a fortiori* $\rho^* = \gamma_\infty$.

Finally, we make a comment on the subdifferentiability. Let $\mathscr{X}$ be a solid space and $\rho : \mathscr{X} \to \mathbb{R}$ a finite-valued convex risk function as in Theorem 3.13. From the implication (1) $\Rightarrow$ (3), if ρ has the Lebesgue property on $\mathscr{X}$, there exists some Q such that

$$E_Q[X] - \rho(X) = \gamma_\infty(Q) \geq E_Q[Y] - \rho(Y), \quad \forall Y \in L^\infty.$$

Since $\mathscr{X} \subset \bigcap_{Q\in\mathcal{Q}_{\gamma_\infty}} L^1(Q)$ by Remark 3.14 (2), another application of the Lebesgue property shows $E_Q[Y] - \rho(Y) = \lim_n (E_Q[Y\mathbb{1}_{\{|Y|\leq n\}}] - \rho(Y\mathbb{1}_{\{|Y|\leq n\}}))$ for all $Y \in \mathscr{X}$, hence the above inequality is valid for all $Y \in \mathscr{X}$. We thus obtain:

Corollary 3.15. *Let $\mathscr{X}$ be a solid space and ρ a finite-valued convex risk function on $\mathscr{X}$. If ρ has the Lebesgue property, then it admits a σ-additive subgradient everywhere in the sense that for every $X \in \mathscr{X}$, there exists a $Q \in \mathcal{P}$ such that $\mathscr{X} \subset L^1(Q)$ and*

$$E_Q[X] - \rho(X) \geq E_Q[Y] - \rho(Y), \quad \forall Y \in \mathscr{X}.$$

4. Proof of Theorem 3.5

Lemma 4.1. *$\hat{\rho} : \mathcal{D}_\gamma \to [0, \infty]$ is continuous from below:*

$$X_n \nearrow X \text{ a.s.}, \ X_1 \in \mathcal{D}(\gamma) \ \Rightarrow \ \hat{\rho}(X) = \lim_n \hat{\rho}(X_n). \tag{4.1}$$

In particular,

(4.2) $X_n \to X \in L^0$ *a.s. and* $\exists Y \in \mathcal{D}(\gamma)$ *s.t.* $X_n \geq Y$, $\forall n \Rightarrow \hat{\rho}(X) \leq \liminf_n \hat{\rho}(X_n)$;

(4.3) $\|X\|_{\hat{\rho}} \neq 0 \Rightarrow \hat{\rho}(|X|/\|X\|_{\hat{\rho}}) \leq 1$;

(4.4) $\forall \lambda > 0$, $\|X\|_{\hat{\rho}} \leq \lambda \Leftrightarrow \hat{\rho}(|X|/\lambda) \leq 1$

(4.5) $L^0 \ni X_n \to X \in L^0 \Rightarrow \|X\|_{\hat{\rho}} \leq \liminf_n \|X_n\|_{\hat{\rho}}$.

Proof. Note first that $X_n \geq X_1 \in \mathcal{D}(\gamma)$ implies that $X_n, X \in \mathcal{D}(\gamma)$, thus $\hat{\rho}(X_n)$ are well-defined, and (4.1) is equivalent to $\hat{\rho}(X) = \sup_n \hat{\rho}(X_n)$ by monotonicity. Since $X_1^- \in \bigcap_{Q \in \mathcal{Q}_\gamma} L^1(Q)$, we can use the monotone convergence (under each $Q \in \mathcal{Q}_\gamma$) to deduce

$$\begin{aligned}\hat{\rho}(X) &= \sup_{Q \in \mathcal{Q}_\gamma} (E_Q[X] - \gamma(Q)) = \sup_{Q \in \mathcal{Q}_\gamma} \sup_n (E_Q[X_n] - \gamma(Q)) \\ &= \sup_n \sup_{Q \in \mathcal{Q}_\gamma} (E_Q[X_n] - \gamma(Q)) = \sup_n \hat{\rho}(X_n),\end{aligned}$$

hence (4.1). To deduce (4.2) from (4.1), we set $Y_n := \inf_{k \geq n} X_k$. Then $\mathcal{D}_\gamma \ni Y \leq Y_n \nearrow X$, hence $\hat{\rho}(X) = \sup_n \hat{\rho}(Y_n) \leq \sup_n \inf_{k \geq n} \hat{\rho}(X_k)$ by (4.1) and the monotonicity.

If $\|X\|_{\hat{\rho}} \neq 0$, (4.1) shows that $\hat{\rho}(|X|/\|X\|_{\hat{\rho}}) = \lim_n \hat{\rho}(|X|/(\|X\|_{\hat{\rho}} + 1/n)) \leq 1$ where the last inequality follows from the definition of $\|\cdot\|_{\hat{\rho}}$. The implication "$\Leftarrow$" in (4.4) is clear from the definition, while (4.3) shows that $0 < \|X\|_{\hat{\rho}} \leq \lambda$ implies $\hat{\rho}(|X|/\lambda) \leq \hat{\rho}(|X|/\|X\|_{\hat{\rho}}) \leq 1$, and if $\|X\|_{\hat{\rho}} = 0$, $\hat{\rho}(|X|/\lambda) \leq 1$ for *any* $\lambda > 0$.

Finally, if $X_n \to X \in L^0$ a.s., the non-negative sequence $(|X_n|/\lambda)$ satisfies the assumption of (4.2) for any $\lambda > 0$. In particular, for any $0 < \varepsilon < \|X\|_{\hat{\rho}}$,

$$1 < \hat{\rho}\left(\frac{|X|}{\|X\|_{\hat{\rho}} - \varepsilon}\right) \leq \liminf_n \hat{\rho}\left(\frac{|X_n|}{\|X\|_{\hat{\rho}} - \varepsilon}\right).$$

Thus for any such ε, there exists an n_ε such that for any $k \geq n_\varepsilon$, $\hat{\rho}(|X_k|/(\|X\|_{\hat{\rho}} - \varepsilon)) > 1 \Leftrightarrow \|X_k\|_{\hat{\rho}} > \|X\|_{\hat{\rho}} - \varepsilon$ by (4.4), hence $\liminf_n \|X_n\|_{\hat{\rho}} \geq \inf_{k \geq n_\varepsilon} \|X_k\|_{\hat{\rho}} \geq \|X\|_{\hat{\rho}} - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have (4.5). □

Lemma 4.2. *For any* $X \in L^0$*, we have*

(4.6) $$\|X\|_Q \leq (1 + \gamma(Q))\|X\|_{\hat{\rho}}, \quad \textit{for any } Q \in \mathcal{Q}_\gamma.$$

Proof. This is trivial if $\|X\|_{\hat{\rho}} = \infty$. If $\|X\|_{\hat{\rho}} < \infty$, (3.2) and (3.5) imply that for any $\varepsilon > 0$, $E_Q[|X|/(\|X\|_{\hat{\rho}} + \varepsilon)] - \gamma(Q) \leq \hat{\rho}(|X|/(\|X\|_{\hat{\rho}} + \varepsilon)) \leq 1$, hence

$$\|X\|_{L^1(Q)} \leq (1 + \gamma(Q))(\|X\|_{\hat{\rho}} + \varepsilon), \ \forall \varepsilon > 0.$$

Letting $\varepsilon \downarrow 0$, we obtain (4.6). □

Lemma 4.3. $\|\cdot\|_{\hat{\rho}}$ *is* σ*-order-continuous on* L^∞ *i.e.,*

(4.7) $$\sup_n \|X_n\|_\infty < \infty \textit{ and } X_n \to 0 \textit{ a.s.} \Rightarrow \|X_n\|_{\hat{\rho}} \to 0.$$

Proof. Since $\hat{\rho}|_{L^\infty} = \rho^0$ and ρ^0 has the Lebesgue property on L^∞ by assumption, we have $\lim_n \hat{\rho}(X_n/\lambda) = 0$ for any $\lambda > 0$. Thus (4.7) follows from the definition of $\|\cdot\|_{\hat{\rho}}$. □

Lemma 4.4. *For any* $X \in L^0$*, the following conditions are equivalent:*

(1) $X \in M_u^{\hat{\rho}}$;
(2) $\lim_{N \to \infty} \|X 1_{\{|X| > N\}}\|_{\hat{\rho}} = 0$;
(3) for any decreasing sequence $(A_n) \subset \mathcal{F}$ *with* $\mathbb{P}(A_n) \downarrow 0$, $\|X 1_{A_n}\|_{\hat{\rho}} \downarrow 0$;
(4) for any decreasing sequence $(A_n) \subset \mathcal{F}$ *with* $\mathbb{P}(A_n) \downarrow 0$ *and* $\alpha > 0$, $\hat{\rho}(\alpha |X| 1_{A_n}) \downarrow 0$;

Proof. Observe that for any decreasing sequence $Y_n \searrow 0$ a.s.,

$$\hat{\rho}(\alpha Y_n) \searrow 0,\ \forall \alpha > 0 \ \Leftrightarrow\ \forall \alpha > 0,\ \exists n_\alpha \text{ s.t. } \hat{\rho}(\alpha Y_{n_\alpha}) \leq 1.$$

Indeed, "$\Rightarrow$" is clear, while the latter condition implies that for any $\varepsilon \in (0, 1)$,

$$(1/\varepsilon)\hat{\rho}(\alpha Y_{n_{\alpha/\varepsilon}}) \leq \hat{\rho}((\alpha/\varepsilon) Y_{n_{\alpha/\varepsilon}}) \leq 1, \text{ hence } \hat{\rho}(\alpha Y_{n_{\alpha/\varepsilon}}) \leq \varepsilon.$$

This shows (1) $\Leftrightarrow$ (2), and (3) $\Leftrightarrow$ (4). The implication (3) $\Rightarrow$ (2) is obvious.

Suppose (2) and let (A_n) be a decreasing sequence with $\mathbb{P}(A_n) \downarrow 0$. Then observe that

$$\|X 1_{A_n}\|_{\hat{\rho}} \leq \|X 1_{A_n \cap \{|X|>N\}}\|_{\hat{\rho}} + \|X 1_{A_n \cap \{|X|\leq N\}}\|_{\hat{\rho}} \leq \|X 1_{\{|X|>N\}}\|_{\hat{\rho}} + N\|1_{A_n}\|_{\hat{\rho}}.$$

The first term in the right hand side tends to 0 by (2), hence we can take a large N so that $\|X 1_{\{|X|\geq N\}}\|_{\hat{\rho}} \leq \varepsilon/2$, while $\|1_{A_n}\|_\infty \leq 1$ and $1_{A_n} \searrow 0$ a.s. by assumption. Then Lemma 4.3 shows that $\|1_{A_n}\|_{\hat{\rho}} \searrow 0$, thus it is less than $\varepsilon/2N$ for large n, hence (3). □

Proposition 4.5. *Both $M^{\hat{\rho}}$ and $L^{\hat{\rho}}$ are Banach lattices, $\hat{\rho}$ is well-defined as a convex risk function on $L^{\hat{\rho}}$ satisfying the Fatou property (Fa($\mathscr{X}$)), and it is finite-valued on $M^{\hat{\rho}}$.*

Proof. By definition, both $M^{\hat{\rho}}$ and $L^{\hat{\rho}}$ are solid vector spaces, $M^{\hat{\rho}} \subset L^{\hat{\rho}}$ and $\|\cdot\|_{\hat{\rho}}$ is a lattice norm on $L^{\hat{\rho}}$ (hence on $M^{\hat{\rho}}$) by Lemma 3.4 and the subsequent comments. Also, by (4.6), $L^{\hat{\rho}} \subset \mathcal{D}_\gamma \cap (-\mathcal{D}_\gamma)$, hence $\hat{\rho}$ is a well-defined convex risk function on $L^{\hat{\rho}}$ (hence on $M^{\hat{\rho}}$) satisfying (Fa($\mathscr{X}$)) by Lemma 3.3 and (4.2), while it is finite on $M^{\hat{\rho}}$ since $\hat{\rho}(X) \leq \hat{\rho}(|X|) < \infty$.

It thus remains only that $L^{\hat{\rho}}$ is complete for the norm $\|\cdot\|_{\hat{\rho}}$, and $M^{\hat{\rho}}$ is closed in $L^{\hat{\rho}}$. So let $(X_n)_n$ be a Cauchy sequence in $L^{\hat{\rho}}$ for the norm $\|\cdot\|_{\hat{\rho}}$, and take $Q^0 \sim \mathbb{P}$ with $\gamma(Q^0) < \infty$ by (2.6). Then by (4.6), (X_n) is still Cauchy in $L^1(Q^0)$, thus admits a limit X in $L^1(Q^0)$. Then we may pick a subsequence $(X_{n_k})_k$ such that $X_{n_k} \to X$ a.s. and

$$\|X_l - X_m\|_{\hat{\rho}} \leq 2^{-k} \text{ whenever } l, m \geq n_k.$$

We deduce from (4.5) that $\|X_l - X\|_{\hat{\rho}} \leq \liminf_{m\to\infty} \|X_l - X_m\|_{\hat{\rho}} \leq 2^{-k}$ for $l \geq n_k$, hence $\|X_n - X\|_{\hat{\rho}} \to 0$. Thus $L^{\hat{\rho}}$ is complete. If each X_n is in $M^{\hat{\rho}}$, we have further that

$$\hat{\rho}(\alpha|X|) \leq \hat{\rho}(\alpha|X - X_n| + \alpha|X_n|) \leq \frac{1}{2}\hat{\rho}(2\alpha|X - X_n|) + \frac{1}{2}\hat{\rho}(2\alpha|X_n|),$$

by the monotonicity and the convexity. The second term is always finite since $X_n \in M^{\hat{\rho}}$, while if we take e.g., n_α so that $\|X - X_{n_\alpha}\|_{\hat{\rho}} \leq 1/2\alpha$, (4.4) implies that the first term is not greater than $1/2$. Consequently, $X \in M^{\hat{\rho}}$, thus $M^{\hat{\rho}}$ is closed. □

Proof of Theorem 3.5 (1). We already know that $M^{\hat{\rho}}_u$ is a solid vector subspace of a Banach lattice $M^{\hat{\rho}}$. Thus to complete (1), we need only to show that $M^{\hat{\rho}}_u$ is closed in $M^{\hat{\rho}}$, and the norm $\|\cdot\|_{\hat{\rho}}$ is order-continuous.

Pick a sequence $(X_n) \subset M^{\hat{\rho}}_u$ and $X \in M^{\hat{\rho}}$ such that $\|X - X_n\|_{\hat{\rho}} \to 0$. Then note that

$$\|X 1_{\{|X|>N\}}\|_{\hat{\rho}} \leq \|X_n 1_{\{|X|>N\}}\|_{\hat{\rho}} + \|X - X_n\|_{\hat{\rho}}, \quad \forall n, N.$$

The second term tends to zero as $n \to \infty$ regardless to N by assumption, while for each n, the first term tends to zero as $N \to \infty$ by Lemma 4.4 (3) since $X_n \in M^{\hat{\rho}}_u$. A diagonal argument then prove that the limit X is in $M^{\hat{\rho}}_u$, hence $M^{\hat{\rho}}_u$ is closed in $M^{\hat{\rho}}$.

For (3.9), we may assume w.o.l.g that $X = 0$. So let $X_n \in M^{\hat{\rho}}_u$, $X_n \to 0$ a.s., and $|X_n| \leq |Y|$ for some $Y \in M^{\hat{\rho}}_u$, and observe from (3.6) that

$$\|X_n\|_{\hat{\rho}} \leq \|X_n \mathbb{1}_{\{|Y|>N\}}\|_{\hat{\rho}} + \|X_n \mathbb{1}_{\{|Y|\leq N\}}\|_{\hat{\rho}} \leq \|Y \mathbb{1}_{\{|Y|>N\}}\|_{\hat{\rho}} + \|X_n \mathbb{1}_{\{|X_n|\leq N\}}\|_{\hat{\rho}}.$$

Since $Y \in M_u^{\hat\rho}$, $\|Y\mathbb{1}_{\{|Y|>N\}}\|_{\hat\rho} \overset{N}{\to} 0$ regardless to n by (1) $\Rightarrow$ (2) of Lemma 4.4. For the second term, note that $\sup_n \|X_n\mathbb{1}_{\{|X_n|\le N\}}\|_\infty \le N$ and $X_n\mathbb{1}_{\{|X|\le N\}} \to 0$ a.s. for any N. By (4.7) (order continuity of $\|\cdot\|_{\hat\rho}$ on L^∞), we deduce that $\lim_n \|X_n\mathbb{1}_{\{|X_n\le N\}}\|_{\hat\rho} = 0$ for each N. Then a diagonal argument concludes the proof of (3.9). □

Proof of Theorem 3.5 (2). By the *extended Namioka-Klee theorem* [8, Th.1], any *finite-valued* monotone convex function on a Banach lattice is norm-continuous. On the other hand, we know from Proposition 4.5 that $\hat\rho$ is a finite-valued convex risk function on the Banach lattice $M_u^{\hat\rho} \subset M^{\hat\rho}$, and $\|\cdot\|_{\hat\rho}$ is σ-order continuous on $M_u^{\hat\rho}$. Combining these two facts, we have that $\hat\rho$ has the Lebesgue property on $M_u^{\hat\rho}$. □

Proof of Theorem 3.5 (3). Let ρ' be a Lebesgue extension of ρ^0 to a *solid* space $\mathscr{X}$. We first show that $\hat\rho = \rho'$ on $\mathscr{X} \cap M_u^{\hat\rho}$. Indeed, if $X \in \mathscr{X} \cap M_u^{\hat\rho}$, the sequence $Y_n := X\mathbb{1}_{\{|X|\le n\}}$ satisfies $Y_n \in \mathscr{X} \cap M_u^{\hat\rho} \cap L^\infty$ (thus $\hat\rho(Y_n) = \rho'(Y_n)$) since solid, $|Y_n| \le |X|$ and $Y_n \to X$ a.s. by definition. Hence the Lebesgue property of ρ (resp. ρ') on $M_u^{\hat\rho}$ (resp. on $\mathscr{X}$) shows

$$\hat\rho(X) = \lim_n \hat\rho(Y_n) = \lim_n \rho'(Y_n) = \rho'(X),$$

Next, we show $\mathscr{X} \subset M_u^{\hat\rho}$, hence $(\hat\rho, M_u^{\hat\rho})$ is the maximum Lebesgue extension. We have to show that if $X \in \mathscr{X}$, then $\lim_N \hat\rho(\alpha|X|\mathbb{1}_{\{|X|>N\}}) = 0$ for all $\alpha > 0$ (this makes sense because $\hat\rho$ is well-defined on L^0_+ by (3.2)). In fact, since $\mathscr{X}$ (and $M_u^{\hat\rho}$) is solid,

$$\lim_N \hat\rho(\alpha|X|\mathbb{1}_{\{|X|>N\}}) \overset{(4.1)}{=} \lim_N \lim_k \hat\rho(\alpha|X|\mathbb{1}_{\{N<|X|\le k\}}) \overset{\text{(i)}}{=} \lim_N \lim_k \rho'(\alpha|X|\mathbb{1}_{\{N<|X|\le k\}})$$
$$\overset{\text{(ii)}}{=} \lim_N \rho'(\alpha|X|\mathbb{1}_{\{|X|>N\}}) \overset{\text{(ii)}}{=} 0.$$

Here (i) follows from $\hat\rho = \rho^0 = \rho'$ on L^∞, and (ii) from the Lebesgue property of ρ' on $\mathscr{X}$ since $\alpha|X|\mathbb{1}_{\{N<|X|\le k\}} \uparrow \alpha|X|\mathbb{1}_{\{|X|>N\}}$ $(k\uparrow\infty)$ and $\alpha|X|\mathbb{1}_{\{|X|>N\}} \downarrow 0$ a.s. $(N\uparrow\infty)$. □

5. Proof of Theorem 3.9 and Theorem 3.13

5.1. Proof of Theorem 3.9

We proceed as (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (1) and (3), then (3) $\Rightarrow$ (2).

Proof of "(1) $\Rightarrow$ (2)". Let $X \in M_u^{\hat\rho}$. By the inequality (4.6), we have for any $A \in \mathcal{F}$ that

$$\sup_{\gamma(Q)\le c} E_Q[|X|1_A] \le \sup_{\gamma(Q)\le c} E_Q[|X|1_{A\cap\{|X|>N\}}] + \sup_{\gamma(Q)\le c} E_Q[|X|1_{A\cap\{|X|\le N\}}]$$
$$\le (1+c)\|X1_{\{|X|>N\}}\|_{\hat\rho} + N \sup_{\gamma(Q)\le c} Q(A).$$

Since $X \in M_u^{\hat\rho}$, Lemma 4.4 implies that for any $\varepsilon > 0$ there exists an $N_\varepsilon > 0$ such that the first term in the right hand side is less than $\varepsilon/2$. On the other hand, since ρ^0 is assumed to satisfy the Lebesgue property on L^∞, hence $\{dQ/d\mathbb{P} : \gamma(Q) \le c\}$ is uniformly integrable by Theorem 3.11 (JST), we can choose a δ_ε so that $\mathbb{P}(A) \le \delta_\varepsilon$ implies $\sup_{\gamma(Q)\le\alpha} Q(A) \le \varepsilon/2N_\varepsilon$. Now the result follows from the standard characterization of uniform integrability. □

The proof of (2) $\Rightarrow$ (1) and (3) is a bit more involved.

Lemma 5.1. *Suppose $X \in M^{\hat\rho}$ and $\{|X|dQ/d\mathbb{P} : \gamma(Q) \le c\}$ is uniformly integrable for all $c \ge 0$. Then for any $\beta \in \mathbb{R}$ and $Y \in L^\infty$, the map $Q \mapsto E_Q[|X|Y] - \gamma(Q)$ is level compact, i.e., $A_\beta := \{Q \in \mathcal{Q}_\gamma : E_Q[|X|Y] - \gamma(Q) \ge \beta\}$ is $\sigma(L^1, L^\infty)$-compact.*

Proof. Since A_β is convex as $Q \mapsto E_Q[|X|Y] - \gamma(Q)$ is concave, the claim is equivalent to saying that A_β is (norm)-closed and uniformly integrable due to the Dunford-Pettis theorem. We begin with the uniform integrability. Let us fix a $Q_0 \in \mathcal{Q}_\gamma$ and estimate

$$\begin{aligned} E_Q[|X|Y] &\le E\left[2|X|\|Y\|_\infty \frac{1}{2}\frac{dQ}{d\mathbb{P}}\right] \le E\left[2|X|\|Y\|_\infty \frac{1}{2}\left(\frac{dQ}{d\mathbb{P}} + \frac{dQ_0}{d\mathbb{P}}\right)\right] \\ &\le \hat{\rho}\left(2\|Y\|_\infty |X|\right) + \gamma\left((Q+Q_0)/2\right) \\ &\le \hat{\rho}\left(2\|Y\|_\infty |X|\right) + \frac{1}{2}(\gamma(Q) + \gamma(Q_0)), \quad \forall Q \in \mathcal{Q}_\gamma. \end{aligned}$$

Noting that $\hat{\rho}(2\|Y\|_\infty|X|) < \infty$ since $X \in M^{\hat{\rho}}$, we see that $E_Q[|X|Y] - \gamma(Q) \le \hat{\rho}(2\|Y\|_\infty|X|) + (\gamma(Q_0) - \gamma(Q))/2$ for all $Q \in \mathcal{Q}_\gamma$, hence

$$Q \in A_\beta \;\Rightarrow\; \gamma(Q) \le 2(\hat{\rho}(\|Y\|_\infty|X|) - \beta) + \gamma(Q_0) =: \delta_\alpha.$$

Since $\gamma(Q)$ is level-compact by the JST theorem (Theorem 3.11) and the Lebesgue property of $\hat{\rho}|_{L^\infty} = \rho^0$ on L^∞, A_β is uniformly integrable.

Next, let $(Q_n)_n \subset A_\beta$ be a sequence converging in L^1 to Q, and we prove $Q \in A_\beta$. Passing to a subsequence (still denoted by (Q_n)), we may suppose that $dQ_n/d\mathbb{P} \to dQ/d\mathbb{P}$ a.s., and since $\gamma(Q_n) \le \delta_\alpha$ for all n by what we have shown above, the family $\{|X|Y dQ_n/d\mathbb{P}\}_n$ is uniformly integrable as well by the assumption (since $Y \in L^\infty$). Therefore, $E_Q[|X|Y] = \lim_n E_{Q_n}[|X|Y]$ by the dominated convergence, while $\gamma(Q) \le \liminf_n \gamma(Q_n)$ by the lower semi-continuity. Summing up, we have

$$E_Q[|X|Y] - \gamma(Q) \ge \limsup_n (E_{Q_n}[|X|Y] - \gamma(Q_n)) \ge \beta.$$

Thus $Q \in A_\beta$, obtaining that A_β is closed. □

Proof of Theorem 3.9: (2) ⇒ (3). Given (2), $X \in M^{\hat{\rho}}$ and $Y \in L^\infty$, Lemma 5.1 shows that the concave function $Q \mapsto E_Q[|X|Y] - \gamma(Q)$ is weakly upper semi-continuous on $\mathcal{Q}_\gamma$, and all the upper level sets are weakly compact, hence attains its maximum on $\mathcal{Q}_\gamma$. □

Proof of Theorem 3.9: (2) ⇒ (1). Given (2), Lemma 5.1 allows us to apply a minimax theorem (Theorem A.1) to obtain: for any convex set $\mathcal{C} \subset L^\infty$,

$$\inf_{Y \in \mathcal{C}} \sup_{Q \in \mathcal{Q}_\gamma} \big(E_Q[|X|Y] - \gamma(Q)\big) = \sup_{Q \in \mathcal{Q}_\gamma} \inf_{Y \in \mathcal{C}} \big(E_Q[|X|Y] - \gamma(Q)\big) \tag{5.1}$$

Indeed, the function $g(Q,Y) := \gamma(Q) - E_Q[|X|Y]$ satisfies the assumption of Theorem A.1. We apply this minimax equality to the sets $\alpha\mathcal{C}_1 = \alpha\mathrm{conv}(\mathbb{1}_{\{|X|>N\}};\, N \in \mathbb{N})$. Observe that if $N_1 < \cdots < N_n$, $\beta_k \ge 0$ and $\beta_1 + \cdots + \beta_n = 1$,

$$\mathbb{1}_{\{|X|>N_n\}} \le \beta_1 \mathbb{1}_{\{|X|>N_1\}} + \cdots \beta_n \mathbb{1}_{\{|X|>N_n\}} \le \mathbb{1}_{\{|X|>N_1\}},$$

hence $\lim_N \hat{\rho}(\alpha|X|\mathbb{1}_{\{|X|>N\}}) = \inf_{Y \in \alpha\mathcal{C}_1} \hat{\rho}(|X|Y)$, while for each $Q \in \mathcal{Q}_\gamma$ ($\Rightarrow X \in L^1(Q)$), $\inf_{Y \in \alpha\mathcal{C}_1} E_Q[|X|Y] = \inf_N E_Q[\alpha|X|\mathbb{1}_{\{|X|>N\}}] = 0$. Thus (5.1) implies that

$$\begin{aligned} \lim_N \hat{\rho}(\alpha|X|\mathbb{1}_{\{|X|>N\}}) &= \inf_{Y \in \alpha\mathcal{C}_1} \sup_{Q \in \mathcal{Q}_\gamma} (E_Q[|X|Y] - \gamma(Q)) \\ &= \sup_{Q \in \mathcal{Q}_\gamma} \inf_{Y \in \alpha\mathcal{C}_1} (E_Q[|X|Y] - \gamma(Q)) = -\inf_{Q \in \mathcal{Q}_\gamma} \gamma(Q) = 0. \end{aligned}$$

This is exactly (1). □

For the proof of (3) ⇒ (2), we need a version of *perturbed James' theorem* due to [26].

Theorem 5.2 ([26], Theorem 2). *Let E be a real Banach space and $f : E \to \mathbb{R}\cup\{+\infty\}$ be a function which is coercive, i.e.,*

$$\lim_{\|x\|\to\infty} \frac{f(x)}{\|x\|} = +\infty. \tag{5.2}$$

Then if for every $x^* \in E^*$, the supremum

$$\sup_{x\in E} (x^*(x) - f(x)) \tag{5.3}$$

is attained, the level set $\{x \in E : f(x) \le c\}$ is relatively weakly compact for each $c \in \mathbb{R}$.

To apply this theorem, we make a "change of variable". For any $X \in M^{\hat\rho}$ with $X \ge 1$ a.s., we define

$$\tilde\gamma_X(Z) := \gamma(Z/X) = \sup_{Y\in L^\infty} (\mathbb{E}[(Z/X)Y] - \hat\rho(Y)), \quad \forall Z \in L^1. \tag{5.4}$$

In view of (2.3), we see that

$$\tilde\gamma_X(Z) < \infty \Leftrightarrow \exists Q \in \mathcal{Q}_\gamma \text{ s.t. } Z = XdQ/d\mathbb{P}. \tag{5.5}$$

(Recall that $M^{\hat\rho} \subset \bigcap_{Q\in\mathcal{Q}_\gamma} L^1(Q)$ by (4.6)). Moreover,

Lemma 5.3. *If $X \in M^{\hat\rho}$ with $X \ge 1$ a.s., then $\lim_{\|Z\|_1\to\infty} \tilde\gamma_X(Z)/\|Z\|_1 = +\infty$ where $\|\cdot\|_1$ is the L^1-norm, i.e., $\tilde\gamma_X$ is coercive function on L^1.*

Proof. By (5.4), for every $Z \in \mathrm{dom}(\tilde\gamma_X) \subset L^1_+$,

$$\begin{aligned}\tilde\gamma_X(Z) &\ge \sup_{n\in\mathbb{N}, Y\in L^\infty_+} \left(\mathbb{E}[(Z/X)YX\mathbb{1}_{\{X\le n\}}] - \hat\rho(YX\mathbb{1}_{\{X\le n\}})\right)\\ &= \sup_{n\in\mathbb{N}, Y\in L^\infty_+} \left(\mathbb{E}[ZY\mathbb{1}_{\{X\le n\}}] - \hat\rho(YX\mathbb{1}_{\{X\le n\}})\right).\end{aligned}$$

(Remember that we are assuming $X \ge 1$ a.s.) On the other hand, for every $Y \in L^\infty_+$,

$$\mathbb{E}[ZY] = \lim_n \mathbb{E}[ZY\mathbb{1}_{\{X\le n\}}] \quad \text{and} \quad \hat\rho(YX) = \lim_n \hat\rho(YX\mathbb{1}_{\{X\le n\}})$$

by the dominated convergence theorem and (4.1) respectively. Consequently,

$$\tilde\gamma_X(Z) \ge \mathbb{E}[ZY] - \hat\rho(YX), \quad \forall Z \in \mathrm{dom}(\tilde\gamma_X) \subset L^1, \ \forall Y \in L^\infty_+. \tag{5.6}$$

In particular, taking $Y = n+1$ (constant), and noting $\|Z\|_1 = \mathbb{E}[Z]$ if $Z \ge 0$ a.s.,

$$\frac{\tilde\gamma_X(Z)}{\|Z\|_1} \ge n + 1 - \frac{\hat\rho((n+1)X)}{\|Z\|_1}$$

Making use of the assumption $X \in M^{\hat\rho}$ ($\Leftrightarrow \hat\rho(\alpha|X|) < \infty$ for any $\alpha > 0$), we deduce that $\tilde\gamma_X(Z)/\|Z\|_1 \ge n$ for any $\|Z\|_1 > \hat\rho((n+1)X)$, which concludes the proof. □

Proof of Theorem 3.9: (3) ⇒ (2). Since $|X| \le |X|\vee 1 \le |X|+1$ and $\{Q \in \mathcal{Q}_\gamma : \gamma(Q) \le c\}$ is uniformly integrable for any $c \ge 0$, it suffices to deduce (2) for $X \in M^{\hat\rho}$ with $X \ge 1$ a.s. We apply Theorem 5.2 to $\tilde\gamma_X$ on the Banach space L^1.

By (3), for any $Y \in L^\infty$, there exists a $Q_{X,Y} \in \mathcal{Q}_\gamma$ such that $E_{Q_{X,Y}}[XY] - \gamma(Q_{X,Y}) = \sup_{Q\in\mathcal{Q}_\gamma}(E_Q[XY] - \gamma(Q))$. Letting $Z_{X,Y} = XdQ_{X,Y}/d\mathbb{P}$, (5.4) and (5.5) imply

$$\begin{aligned}\mathbb{E}[Z_{X,Y}Y] - \tilde\gamma_X(Z_{X,Y}) = E_{Q_{X,Y}}[XY] - \gamma(Q_{X,Y}) &= \sup_{Q\in\mathcal{Q}_\gamma} (E_Q[XY] - \gamma(Q))\\ &= \sup_{Z\in L^1} (\mathbb{E}[ZY] - \tilde\gamma_X(Z)),\end{aligned}$$

for all $Y \in L^\infty$, i.e., the supremum $\sup_{Z\in L^1}(\mathbb{E}[ZY] - \tilde\gamma_X(Z))$ is attained for all $Y \in L^\infty$. Then by Lemma 5.3, we can apply Theorem 5.2 to deduce that $\{Z \in L^1 : \tilde\gamma_X(Z) \le c\}$ is relatively weakly compact in L^1 ($\Leftrightarrow$ uniformly integrable by Dunford-Pettis) for all $c \geq 0$. On the other hand, $\{Z \in L^1 : \tilde{\gamma}_X(Z) \leq c\} = \{XdQ/d\mathbb{P} : \gamma(Q) \leq c\}$ by (5.5) □

5.2. Proof of Theorem 3.13

We begin with a couple of comparison results. Let ρ^0 satisfy Assumption 2.1 and (2.7).

Lemma 5.4. *Let $\mathscr{X}$ be a solid space and $\rho : \mathscr{X} \to \mathbb{R}$ a finite valued convex risk function. If $\rho|_{L^\infty} = \rho^0$, then $\mathscr{X} \subset M^{\hat{\rho}}$.*

Proof. Since $\rho|_{L^\infty} = \rho^0$ and ρ is finite on the solid space $\mathscr{X}$, $\rho(\alpha|X|) < \infty$, hence

$$\hat{\rho}(\alpha|X|) \stackrel{(4.1)}{=} \sup_n \hat{\rho}(\alpha|X| \wedge n) = \sup_n \rho(\alpha|X| \wedge n) \leq \rho(\alpha|X|) < \infty$$

for all $\alpha > 0$ and $X \in \mathscr{X}$. Thus $\mathscr{X} \subset M^{\hat{\rho}}$. □

Lemma 5.5. *Let $\mathscr{X}$ be solid and $\rho : \mathscr{X} \to \mathbb{R}$ be a finite valued convex risk function. If $\mathscr{X} \subset M^{\hat{\rho}}_u$ and $\rho|_{L^\infty} = \rho^0$, then ρ has the Lebesgue property (Le($\mathscr{X}$)) on $\mathscr{X}$, thus a fortiori $\rho = \hat{\rho}$ on $\mathscr{X}$.*

Proof. The claim amounts to showing that for any sequence $(X_n)_n \subset \mathscr{X}$,

$$\exists Y \in \mathscr{X},\ |X_n| \leq |Y| \text{ and } X_n \to X \text{ a.s. } \Rightarrow \rho(X) = \lim_n \rho(X_n). \tag{5.7}$$

We fix such a sequence (X_n) with the limit X as well as Y verifying the condition. Put

$$B_{|Y|} := \{Z \in \mathscr{X} : |Z| \wedge n|Y| \uparrow |Z|\}.$$

$B_{|Y|}$ is the *principal band* in $\mathscr{X}$ generated by $Y \in \mathscr{X} \subset M^{\hat{\rho}}_u$ (see [2, p.324]). Observe that $B_{|Y|} \subset \mathscr{X} \subset M^{\hat{\rho}}_u$ (of course) and $B_{|Y|}$ is also a band in the Banach lattice $M^{\hat{\rho}}_u$. Thus [2, Theorem 8.43] shows that $B_{|Y|}$ is norm-closed in $M^{\hat{\rho}}_u$, hence $(B_{|Y|}, \|\cdot\|_{\hat{\rho}})$ is again a Banach lattice. By the extended Namioka-Klee theorem, the finite valued monotone convex function $\rho|_{B_{|Y|}}$ on the Banach lattice $(B_{|Y|}, \|\cdot\|_{\hat{\rho}})$ is $\|\cdot\|_{\hat{\rho}}$-continuous, while $\|\cdot\|_{\hat{\rho}}$ is (σ-)order continuous on $M^{\hat{\rho}}_u$ by Theorem 3.5 (1), hence on the subspace $B_{|Y|}$. The original sequence (X_n), its limit X as well as Y are all contained in $B_{|Y|}$, thus we deduce $\rho(X) = \lim_n \rho(X_n)$. Since $(X_n) \subset \mathscr{X}$ is arbitrary, we see that ρ has the Lebesgue property (Le($\mathscr{X}$)) on $\mathscr{X}$. Then Theorem 3.5 (3) shows that ρ must agree with $\hat{\rho}$ on $\mathscr{X}$. □

Proof of Theorem 3.13. Put $\rho^0 := \rho|_{L^\infty}$. By the Fatou property on L^∞, ρ^0 has the robust representation (2.5) on L^∞ with $\gamma = \gamma_\infty$ and $\mathcal{Q}_\gamma = \mathcal{Q}_{\gamma_\infty}$, that is $\rho^0 = \hat{\rho}|_{L^\infty}$ with this choice of γ. Moreover, any of (1) - (3) implies that ρ^0 has the Lebesgue property (2.7). This is trivial in the case of (1), and each of (2) and (3) restricted to L^∞ shows ρ^0 has the Lebesgue property by the original JST theorem. Thus $\mathscr{X} \subset M^{\hat{\rho}}$ by Lemma 5.4 since ρ is finite, and we can use Theorems 3.5 and 3.9 for the choice $\rho^0 = \rho|_{L^\infty}$.

We proceed as (1) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (1), and (2) $\Leftrightarrow$ (3).

(1) $\Rightarrow$ (4) $\Rightarrow$ (3). If (1) holds, Theorem 3.5 (3) shows that $\mathscr{X} \subset M^{\hat{\rho}}_u$ and $\rho = \hat{\rho}|_{\mathscr{X}}$ while "(1) $\Rightarrow$ (3)" of Theorem 3.9 with $Y = \mathrm{sign}(X)$ implies that $\sup_{Q \in \mathcal{Q}_{\gamma_\infty}}(E_Q[X] - \gamma_\infty(Q))$ is attained for all $X \in \mathscr{X} \subset M^{\hat{\rho}}_u$. Hence (4) hold, and (4) $\Rightarrow$ (3) is obvious.

(2) $\Leftrightarrow$ (3) $\Rightarrow$ (1). The solidness of $\mathscr{X}$ implies that $XY = |X|(\mathrm{sign}(X)Y) \in \mathscr{X}$ for any $X \in \mathscr{X}$ and $Y \in L^\infty$. Thus given $\mathscr{X} \subset M^{\hat{\rho}}$, (2) $\Leftrightarrow$ $\mathscr{X} \subset M^{\hat{\rho}}_u$ $\Leftrightarrow$ (3) by Theorem 3.9. On the other hand, given $\mathscr{X} \subset M^{\hat{\rho}}_u$, Lemma 5.5 shows that ρ has the Lebesgue property, hence (1). □

6. Examples

Here we examine some typical risk functions deriving the explicit forms of the space $M_u^{\hat\rho}$. We begin with a simple remark. Though we defined $\hat\rho$ using the robust representation of ρ^0 on L^∞, it may be more convenient to use other more explicit formula ρ^0 if available. By Lemma 4.1, we know that $\hat\rho$ is continuous from below on $\mathcal{D}_\gamma$ hence on L^0_+. In particular,

$$\hat\rho(|X|) = \lim_n \rho^0(|X| \wedge n), \quad \forall X \in L^0. \tag{6.1}$$

Note that this formula may not be true for $X \in L^0 \setminus L^0_+$, but we need only to consider positive random variables, or equivalently all $|X|$ with $X \in L^0$ to derive the spaces $M_u^{\hat\rho}$ and $M^{\hat\rho}$.

6.1. Utility Based Shortfall Risk

Let $l : \mathbb{R} \to \mathbb{R}$ be a strictly increasing convex function with $l(0) > \inf_x l(x)$ (thus not identically constant). We define the associated *shortfall risk function* by

$$\rho_l(X) := \inf\{x \in \mathbb{R} : \mathbb{E}[l(X-x)] \le l(0)\}, \quad \forall X \in L^\infty. \tag{6.2}$$

Then ρ_l satisfies (A1-5) as well as (2.7) and its penalty function is given by

$$\gamma_l(Q) = \inf_{\lambda>0} \frac{1}{\lambda}\left(l(0) + \mathbb{E}\left[l^*\left(\lambda\frac{dQ}{d\mathbb{P}}\right)\right]\right). \tag{6.3}$$

(See [19, Ch.4]). Also, (6.1) implies that

$$\hat\rho_l(|X|) = \sup_n \inf\{x : \mathbb{E}[l(|X|\wedge n - x)] \le l(0)\} \le \inf\{x : \mathbb{E}[l(|X| - x)] \le l(0)\},$$

while if $\hat\rho(|X|) < \infty$, we have $\mathbb{E}[l(|X| - \hat\rho(|X|))] \le \lim_n \mathbb{E}[l(|X|\wedge n - \hat\rho(|X|))] \le \limsup_n \mathbb{E}[l(|X|\wedge n - \rho^0(|X|\wedge n))] \le l(0)$ by monotone convergence and $\rho^0(|X|\wedge n) \le \hat\rho(|X|)$, thus

$$\hat\rho_l(|X|) = \inf\{x : \mathbb{E}[l(|X| - x)] \le l(0)\}, X \in L^0. \tag{6.4}$$

In this case, two spaces $M_u^{\hat\rho_l}$ and $M^{\hat\rho_l}$ coincide and equal to the Morse subspace associated to the Young function $\Psi(x) := l(|x|) - l(0)$, i.e.,

Proposition 6.1. *$M_u^{\hat\rho_l} = M^{\hat\rho} = M^\Psi$ where $\Psi(|X|) := l(|x|) - l(0)$.*

Proof. 1. $M^\Psi \subset M_u^{\hat\rho_l}$. It suffices to show that for any $c > 0$, the family $\{XdQ/d\mathbb{P} : \gamma_l(Q) \le c\}$ is uniformly integrable. So let us fix $c > 0$ and $X \in M^\Psi$. Observe that if $\gamma_l(Q) \le c$, then there exists a $\lambda_Q > 0$ such that $\frac{1}{\lambda_Q}\left(l(0) + \mathbb{E}\left[l^*(\lambda_Q dQ/d\mathbb{P})\right]\right) \le c+1$, and such λ_Q is bounded below by a constant depending only on c and l. Indeed,

$$c+1 \ge \frac{1}{\lambda_Q}\left(l(0) + \mathbb{E}\left[l^*\left(\lambda_Q\frac{dQ}{d\mathbb{P}}\right)\right]\right) \ge \frac{l(0) + l^*(\lambda_Q)}{\lambda_Q} \tag{6.5}$$

by Jensen's inequality, and taking $x_0 < 0$ so that $l(x_0) < l(0)$ (such exists since $l(0) > \inf_x l(x)$), $(l(0) + l^*(\lambda_Q))/\lambda_Q = \sup_x(x + (l(0) - l(x))/\lambda_Q) \ge x_0 + (l(0) - l(x_0))/\lambda_Q$, hence (6.5) implies $\lambda \ge \frac{l(0)-l(x_0)}{c+1-x_0} =: \Lambda(c)$. On the other hand, noting that $l(\alpha|X|\mathbb{1}_A) = \Psi(|X|)\mathbb{1}_A + l(0)$ for any $A \in \mathcal{F}$ and $\alpha > 0$, Young's inequality shows

$$E_Q[\alpha\lambda_Q|X|\mathbb{1}_A] \le \mathbb{E}[\Psi(\alpha|X|)\mathbb{1}_A] + \left(l(0) + \mathbb{E}\left[l^*\left(\lambda_Q\frac{dQ}{d\mathbb{P}}\right)\right]\right),$$

from which we have

$$E_Q[|X|\mathbb{1}_A] \le \frac{1}{\alpha\lambda_Q}\mathbb{E}[\Psi(\alpha|X|)\mathbb{1}_A] + \frac{c+1}{\alpha} \le \frac{1}{\alpha}\frac{\mathbb{E}[\Psi(\alpha|X|)\mathbb{1}_A]}{\Lambda(c)} + \frac{c+1}{\alpha}$$

for any Q with $\gamma_l(Q) \le c$. Since $X \in M^\Psi$, the desired uniform integrability follows from a diagonal argument.

2. $M^{\hat\rho_l} \subset M^\Psi$, hence three spaces agree. This follows from (6.4). Indeed, we have the implications: $\hat\rho_l(\alpha|X|) < \infty \Rightarrow \exists x \in \mathbb{R}$ s.t. $l(\alpha|X| - x) \in L^1 \Rightarrow l(\alpha|X|/2) \le \frac{1}{2}l(\alpha|X| - x) + \frac{1}{2}l(x) \in L^1$. We deduce that $M^{\hat\rho} \subset M^\Psi$. □

Remark 6.2. In definition (6.2), we chose $l(0)$ for the acceptance level so that $\rho_l(0) = 0$ (and sensitive). If ρ_l is defined with other acceptance level δ instead of $l(0)$, we can normalize it by adding the constant $a^l(\delta) := \sup\{x : l(x) \le \delta\}$ or equivalently replacing the function l by $x \mapsto l(x + a^l(\delta))$. Also, the case $l(0) = \inf_x l(x)$ corresponds to the worst case risk function $\rho^{\text{worst}}(X) = \operatorname{ess\,sup} X$.

6.2. Robust Shortfall Risk

Let l be as above and fix a set $\mathcal{P}_0$ of probabilities $P \ll \mathbb{P}$ such that

$$\mathcal{P}_0 \text{ is convex and weakly compact in } L^1. \tag{6.6}$$

Then we consider a *robust shortfall risk function*

$$\rho_{l,\mathcal{P}_0}(X) := \inf\{x \in \mathbb{R} : \sup_{P \in \mathcal{P}} E_P[l(X - x)] \le l(0)\}, \quad X \in L^\infty. \tag{6.7}$$

The function $\rho_{l,\mathcal{P}_0}$ on L^∞ is a convex risk function with the minimal penalty function

$$\gamma_{l,\mathcal{P}_0}(Q) := \inf_{\lambda > 0} \frac{1}{\lambda}\left(l(0) + \inf_{P \in \mathcal{P}_0} E_P\left[l^*\left(\lambda \frac{dQ}{dP} \right) \right] \right) \tag{6.8}$$

with the convention $l^*(\infty) := \infty$ and $\frac{dQ}{dP} := \frac{dQ/d\mathbb{P}}{dP/d\mathbb{P}} \mathbb{1}_{\{dP/d\mathbb{P} > 0\}} + \infty \cdot \mathbb{1}_{\{dQ/d\mathbb{P} > 0, dP/d\mathbb{P} = 0\}}$ (see [19, Corollary 4.119]). Under (6.6), we have further that $\rho_{l,\mathcal{P}_0}$ has the Lebesgue property on L^∞. This follows from a robust version of de la Vallée-Poussin theorem due to [17] (this result is stated there for sets of *probability measures*, but their proof does not use the latter fact, and the exactly same proof applies to sets of *positive finite measures*). Also, slightly modifying the argument for (6.4), we have

$$\hat\rho_{l,\mathcal{P}_0}(|X|) := \inf\{x \in \mathbb{R} : \sup_{P \in \mathcal{P}_0} E_P[l(|X| - x)] \le l(0)\}, \quad X \in L^0. \tag{6.9}$$

We introduce a couple of robust analogues of M^Ψ:

$$M^\Psi(\mathcal{P}) := \{X \in L^0 : \sup_{P \in \mathcal{P}} E_P[\Psi(\lambda|X|)] < \infty, \ \forall \lambda > 0\}$$
$$M_u^\Psi(\mathcal{P}) := \{X \in L^0 : \lim_{N \to \infty} \sup_{P \in \mathcal{P}} E_P[\Psi(\lambda|X|)\mathbb{1}_{\{|X| > N\}}] = 0, \ \forall \lambda > 0\}.$$

When $\mathcal{P}_0 = \{\mathbb{P}\}$, the two spaces coincide with M^Ψ. Now we have:

Proposition 6.3. *Assume (6.6). Then*

$$M_u^\Psi(\mathcal{P}_0) = M_u^{\hat\rho_{l,\mathcal{P}_0}} \subset M^{\hat\rho_{l,\mathcal{P}_0}} \subset M^\Psi(\mathcal{P}_0).$$

Proof. 1. $M_u^\Psi(\mathcal{P}_0) \subset M_u^{\hat\rho_{l,\mathcal{P}_0}}$. It suffices that if $X \in M_u^\Psi(\mathcal{P}_0)$ and $c > 0$, then $\{X dQ/d\mathbb{P} : \gamma_{l,\mathcal{P}_0}(Q) \le c\}$ is uniformly integrable. With a similar reasoning and notation as Proposition 6.1, we see that $\gamma_{l,\mathcal{P}_0}(Q) \le c$ implies the existence of $\lambda_Q \ge \Lambda(c) =$

$(l(0) - l(x_0))/(c + 1 - x_0)$ and $P_Q \in \mathcal{P}_0$ such that $\frac{1}{\lambda_Q}\left(l(0) + E_{P_Q}\left[l^*\left(\lambda_Q \frac{dQ}{dP_Q}\right)\right]\right) \leq c + 1$. By Young's inequality and $l(\alpha|X|\mathbb{1}_A) = \Psi(\alpha|X|)\mathbb{1}_A + l(0)$, we see that

$$\begin{aligned} E_Q[\lambda_Q \alpha |X| \mathbb{1}_A] &\leq E_{P_Q}[\Psi(\alpha|X|)\mathbb{1}_A] + \left(l(0) + E_{P_Q}\left[l^*\left(\lambda_Q \frac{dQ}{dP_Q}\right)\right]\right) \\ &\leq \sup_{P \in \mathcal{P}_0} E_P[\Psi(\alpha|X|)\mathbb{1}_A] + \lambda_Q(c+1) \end{aligned}$$

for all Q with $\gamma_{l,\mathcal{P}_0}(Q) \leq c$, $A \in \mathcal{F}$ and $\alpha > 0$. Hence

$$\sup\{E_Q[|X|\mathbb{1}_A] : \gamma_{l,\mathcal{P}_0}(Q) \leq c\} \leq \frac{1}{\alpha \Lambda(c)} \sup_{P \in \mathcal{P}_0} E[\Psi(\alpha|X|)\mathbb{1}_A] + \frac{c+1}{\alpha},$$

from which the uniform integrability follows by a diagonal technique.

2. $M_u^{\Psi}(\mathcal{P}_0) \supset M_u^{\hat{\rho}_{l,\mathcal{P}_0}}$. Let $X \in M_u^{\hat{\rho}_{l,\mathcal{P}_0}}$ and $\alpha > 0$. By the definition of $M_u^{\hat{\rho}_{l,\mathcal{P}_0}}$, there is a sequence $(N_n)_n \subset \mathbb{N}$ such that $\hat{\rho}_{l,\mathcal{P}_0}(n\alpha|X|\mathbb{1}_{\{|X|>N_n\}}) < 2^{-n}$. Then by (6.7),

$$\sup_{P \in \mathcal{P}_0} E_P[l(n\alpha|X|\mathbb{1}_{\{|X|>N_n\}} - 2^{-n})] \leq l(0).$$

Noting that $\Psi(\alpha|X|\mathbb{1}_{A_n}) = l(\alpha|X|\mathbb{1}_{A_n}) - l(0) \leq n^{-1} l(n\alpha|X|\mathbb{1}_{A_n} - 2^{-n}) + \frac{n-1}{n} l(\frac{2^{-n}}{n-1}) - l(0)$ with $A_n := \{|X| > N_n\}$ by the convexity, we have

$$\begin{aligned} \sup_{P \in \mathcal{P}_0} E_P[\Psi(\alpha|X|)\mathbb{1}_{A_n}] &\leq \frac{1}{n} \sup_{P \in \mathcal{P}_0} E_P[l(n\alpha|X|\mathbb{1}_{A_n} - 2^{-n})] + \frac{n-1}{n} l\left(\frac{2^{-n}}{n-1}\right) - l(0) \\ &\leq \frac{l(0)}{n} + \frac{n-1}{n} l\left(\frac{2^{-n}}{n-1}\right) - l(0) \to 0 + l(0) - l(0) = 0. \end{aligned}$$

Since $\alpha > 0$ is arbitrary, we have $X \in M_u^{\Psi}(\mathcal{P}_0)$.

3. $M^{\hat{\rho}_{l,\mathcal{P}_0}} \subset M^{\Psi}(\mathcal{P}_0)$. If $X \in M^{\hat{\rho}_{l,\mathcal{P}_0}}$, we have for every $\alpha > 0$,

$$\begin{aligned} \sup_{P \in \mathcal{P}_0} E_P[\Psi(\alpha|X|)] &= \sup_{P \in \mathcal{P}_0} E_P[l(\alpha|X|)] - l(0) \\ &\leq \frac{1}{2} \sup_{P \in \mathcal{P}_0} E_P[l(2\alpha|X| - x)] + \frac{1}{2} l(x) - l(0) < \infty. \end{aligned}$$

for $x > \hat{\rho}_{l,\mathcal{P}_0}(\alpha|X|)$ by (6.9). Thus $M^{\hat{\rho}_{l,\mathcal{P}_0}} \subset M^{\Psi}(\mathcal{P}_0)$. □

Example 6.4 (Robust Entropic Risk Functions). Let $l(x) = e^x$. Then $\rho_{l,\mathcal{P}_0}$ is the entropic one, and the associated Young function is $\Psi_e(x) := e^{|x|} - 1$. In this case, we have $M_u^{\Psi_e}(\mathcal{P}_0) = M^{\Psi_e}(\mathcal{P}_0)$, thus $M_u^{\hat{\rho}_{l,\mathcal{P}_0}} = M^{\hat{\rho}_{l,\mathcal{P}_0}}$. Indeed, by Hölder's inequality,

$$\begin{aligned} \sup_{P \in \mathcal{P}_0} E_P[e^{\alpha|X|}\mathbb{1}_{\{|X|>N\}}] &\leq \sup_{P \in \mathcal{P}_0} \left(E_P[e^{2\alpha|X|}]^{1/2} P(|X| > N)^{1/2}\right) \\ &\leq \sup_{P \in \mathcal{P}_0} E_P[e^{2\alpha|X|}]^{1/2} \sup_{P \in \mathcal{P}} P(|X| > N)^{1/2}. \end{aligned}$$

This and the uniform integrability of $\mathcal{P}_0$ show that $\lim_N \sup_{P \in \mathcal{P}_0} E_P[e^{\alpha|X|}\mathbb{1}_{\{|X|>N\}}] = 0$ for every $\alpha > 0$ as soon as $X \in M^{\Psi_e}(\mathcal{P}_0)$, hence $M_u^{\Psi_e}(\mathcal{P}_0) = M^{\Psi_e}(\mathcal{P}_0)$.

6.3. Law-Invariant Case

Recall that a convex risk function ρ^0 on L^∞ is called *law-invariant* if $\rho^0(X) = \rho^0(Y)$ whenever X and Y have the same distribution. Any law-invariant convex risk function on L^∞ has the following representation (Kusuoka's representation):

$$\rho^0(X) = \sup_{\mu \in \mathcal{M}_1((0,1])} \left(\int_{(0,1]} v_\lambda(X)\mu(d\lambda) - \beta(\mu)\right) \tag{6.10}$$

where $v_\lambda(X) := \frac{1}{\lambda}\int_0^\lambda q_X(1-t)dt$, the *average value at risk at level* λ (up to change of sign) with $q_X(t) := \inf\{x : \mathbb{P}(X \leq x) > t\}$, $\mathcal{M}_1((0,1])$ is the set of all Borel probability measures on $(0,1]$ and β is a lower semi-continuous penalty function. Then ρ^0 has the Lebesgue property on L^∞ if and only if all the level sets $\{\mu : \beta(\mu) \leq c\}$ are relatively weak* compact in $\mathcal{M}_1((0,1])$ or equivalently tight ([14, Ch. 5] or [22]). In particular, for any relatively weak* compact convex set $\mathcal{M} \subset \mathcal{M}_1((0,1])$,

$$\rho_{\mathcal{M}}(X) := \sup_{\mu\in\mathcal{M}} \int_{(0,1]} v_\lambda(X)\mu(d\lambda)$$

is a law-invariant coherent risk measure on L^∞ satisfying the Lebesgue property.

Example 6.5 (AV@R)**.** For every $\lambda \in (0,1]$, v_λ admits the representation:

$$v_\lambda(X) = \sup\{E_Q[X] : Q \in \mathcal{P},\ dQ/d\mathbb{P} \leq 1/\lambda\}, \tag{6.11}$$

for all $X \in L^\infty$, and since $\hat{v}_\lambda(|X|) = \sup_n v_\lambda(|X| \wedge n)$,

$$\|X\|_{L^1} \leq \hat{v}_\lambda(|X|) = \|X\|_{\hat{v}_\lambda} \leq \frac{1}{\lambda}\|X\|_{L^1}, \quad X \geq 0,$$

hence we have $M_u^{\hat{v}_\lambda} = M^{\hat{v}_\lambda} = L^1$ for every $\lambda \in (0,1]$, and the representation (6.11) extends to L^1. In particular, $\hat{v}_\lambda$ has the Lebesgue property on L^1.

Example 6.6 (Concave Distortions)**.** Let $\mu \in \mathcal{M}_1((0,1])$ and define

$$\rho_\mu(X) := \int_{(0,1]} v_t(X)\mu(dt).$$

This type of risk functions are called concave distortion, and it is known that if the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless, every law-invariant *comonotonic* risk function is written in this form (see [19, Theorem 4.93]). For ρ_μ, two spaces $M_u^{\hat{\rho}_\mu}$ and $M^{\hat{\rho}_\mu}$ coincide. Indeed, $\hat{\rho}_\mu(|X|) < \infty$ implies that $\hat{v}_\cdot(|X|) \in L^1((0,1],\mu)$, hence $\hat{v}_t(|X|) < \infty$ for μ-a.e. $t \in (0,1]$. Since $\hat{v}_t(|X|\mathbb{1}_{\{|X|>N\}}) \downarrow 0$ as $N \to \infty$ and $\hat{v}_\cdot(|X|\mathbb{1}_{\{|X|>N\}}) \leq v_\cdot(|X|) \in L^1((0,1],\mu)$ for μ-a.e. $t \in (0,1]$, the dominated convergence theorem shows that

$$\lim_N \int_{(0,1]} \hat{v}_t(|X|\mathbb{1}_{\{|X|>N\}})\mu(dt) = \int_{(0,1]} \lim_N \hat{v}_t(|X|\mathbb{1}_{\{|X|>N\}})\mu(dt) = 0.$$

Repeating the same argument for $\alpha|X|$ ($\alpha > 0$) instead of X, we have $M_u^{\hat{\rho}_\mu} = M^{\hat{\rho}_\mu}$.

Recall that any *finite-valued* convex risk function on a solid and rearrangement-invariant space strictly bigger than L^∞ has the Lebesgue property *restricted to* L^∞ (Theorem 2.7). Then it is natural to ask how about the Lebesgue property on the *entire space*. In our context, $M_u^{\hat{\rho}}$ and $M^{\hat{\rho}}$ are solid by very definitions, and rearrangement-invariant too if the original ρ^0 is law-invariant, while $M^{\hat{\rho}}$ is the maximal solid space on which $\hat{\rho}$ is finite-valued. Thus it is worthwhile to ask if $M_u^{\hat{\rho}} = M^{\hat{\rho}}$ when ρ^0 is law-invariant. The answer is generally no.

Example 6.7 (Law-invariant risk function with $M_u^{\hat{\rho}} \subsetneq M^{\hat{\rho}}$)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be atomless and for each n, we define an element of $\mathcal{M}_1((0,1])$ by

$$\mu_n(dt) := \left(1 - \frac{1}{n}\right)\frac{e}{e-1}\mathbb{1}_{(e^{-1},1]}(t)dt + \frac{1}{n}\frac{e^n}{e-1}\mathbb{1}_{(e^{-n},e^{-n+1}]}(t)dt. \tag{6.12}$$

Then the family $(\mu_n)_n$ and hence $\overline{\mathrm{conv}}(\mu_n; n \in \mathbb{N})$ is uniformly integrable in $L^1((0,1],dt)$ ($\Leftrightarrow$ weakly compact in $\mathcal{M}_1((0,1])$). Hence the law-invariant coherent risk function

$$\rho^0(X) := \sup_n \int_{(0,1]} v_t(X)\mu_n(dt) \quad \left(\Rightarrow \hat{\rho}(|X|) = \sup_n \int_{(0,1]} \hat{v}_\lambda(|X|)\mu_n(d\lambda)\right)$$

has the Lebesgue property on L^∞. In this case, $M_u^{\hat{\rho}} \subsetneq M^{\hat{\rho}}$. Indeed, let X be an exponential random variable with parameter 1, i.e., $F_X(x) = 1 - e^{-x} \Leftrightarrow q_X(t) = -\log(1-t)$. Then

$$\hat{v}_\lambda(X) = \frac{1}{\lambda}\int_0^\lambda (-\log t)dt = 1 - \log\lambda.$$

For each n, $\int_{(0,1]} \hat{v}_t(X)\mu_n(dt) = 4 - \frac{e}{e-1} - \frac{1}{n}$, so $\hat{\rho}(X) = \sup_n \int_{(0,1]} \hat{v}_t(X)\mu_n(dt) = 4 - \frac{e}{e-1} < \infty$. This shows that $X \in M^{\hat{\rho}}$. We next compute $\lim_N \rho(X\mathbb{1}_{\{X>N\}})$. Since $q_{X\mathbb{1}_{\{X>N\}}}(t) = q_X\mathbb{1}_{\{q_X(t)>N\}}$ and $q_X(1-t) > N \Leftrightarrow t < 1 - F_X(N) = e^{-N}$,

$$\begin{aligned}\hat{v}_\lambda(X\mathbb{1}_{\{X>N\}}) &= \frac{1}{\lambda}\int_0^\lambda q_X(1-t)\mathbb{1}_{\{q_X(1-t)>N\}}dt \\ &= \{\lambda\wedge e^{-N} - (\lambda\wedge e^{-N})\log(\lambda\wedge e^{-N}))\}/\lambda.\end{aligned}$$

Thus for $n > N+1$,

$$\begin{aligned}&\int_{(0,1]} \hat{v}_t(X\mathbb{1}_{\{X>N\}})\mu_n(dt) \\ &= \left(1-\frac{1}{n}\right)\frac{e}{e-1}\left(e^{-N} - e^{-N}\log e^{-N}\right) + \frac{1}{n}\left(2 + n - \frac{e}{e-1}\right) \\ &= 1 + \frac{e}{e-1}\left(e^{-N} - e^{-N}\log e^{-N}\right) + \frac{1}{n}\left\{2 - \frac{e}{e-1}\left(1 + e^{-N} - e^{-N}\log e^{-N}\right)\right\}\end{aligned}$$

Hence $\hat{\rho}(X\mathbb{1}_{\{X>N\}}) = \sup_n \int_{(0,1]} \hat{v}_t(X\mathbb{1}_{\{X>N\}})\mu_n(dt) = 1 + \frac{e}{e-1}\left(e^{-N} - e^{-N}\log e^{-N}\right)$. Consequently, $\lim_{N\to\infty}\rho(X\mathbb{1}_{\{X>N\}}) \geq 1 + \lim_N \frac{e}{e-1}\left(e^{-N} - e^{-N}\log e^{-N}\right) = 1$. Thus $X \notin M_u^{\hat{\rho}}$.

Appendix A. A Minimax Theorem

We have used the following version of minimax theorem which should be a known result, and is actually an immediate consequence of [21, Theorems 1 and 2]. We could not, however, find a reference, thus we give here a simple proof.

Theorem A.1. *Let C be a convex subset of a topological vector space, and D an arbitrary convex set. Suppose we are given a function $f : C \times D \to \mathbb{R}$ such that*

(1) for any $y \in D$, $x \mapsto f(x,y)$ is convex and level-compact, i.e., $\mathrm{lev}_{\leq\alpha} f(\cdot, y) := \{x \in C : f(x,y) \leq \alpha\}$ is compact for each $\alpha \in \mathbb{R}$;

(2) for any $x \in C$, $y \mapsto f(x,y)$ is concave on D.

Then we have

$$\inf_{x\in C}\sup_{y\in D} f(x,y) = \sup_{y\in D}\inf_{x\in C} f(x,y). \tag{A.1}$$

Proof. Note first that "$\geq$" is always true whatever C, D and f are. Thus there is nothing to prove if $\alpha := \sup_{y\in D}\inf_{x\in C} f(x,y) = \infty$, hence we assume $\alpha < \infty$.

For any $y \in D$ and $\beta \in \mathbb{R}$, we set $A_y^\beta := \{x \in C : f(x,y) \leq \beta\}$. Then [21, Theorem 1] implies that the family $\{A_y^{\alpha+\varepsilon}\}_{y\in D}$ has the finite intersection property for every $\varepsilon > 0$. Noting that each $A_y^{\alpha+\varepsilon}$ is compact by assumption made on f, we have $\bigcap_{y\in D} A_y^{\alpha+\varepsilon} \neq \emptyset$ (indeed, fixing arbitrary $y_0 \in D$, we have $A_{y_0}^{\alpha+\varepsilon}$ is compact, $A_y^{\alpha+\varepsilon} \cap A_{y_0}^{\alpha+\varepsilon}$ is its non-empty closed subset for each $y \in D$, and $\bigcap_{y\in D} A_y^{\alpha+\varepsilon} = \bigcap_{y\in D}(A_y^{\alpha+\varepsilon} \cap A_{y_0}^{\alpha+\varepsilon}) \neq \emptyset$). But this is a necessary and sufficient condition for the equality (A.1) by [21, Theorem 2]. □

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