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Role of Leverage in Bubbles and Crashes*

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Abstract

This paper investigates the possibility that an unproductive company with limited debt capacity raises huge funds through share issuances by utilizing a small sign of enthusiasm. We generalize the timing game of Matsushima (2012) by permitting arbitrageurs to use high leverage for purchasing the shares. Thanks to this leverage, any arbitrageur has strong incentive to ride the bubble by continuing to purchase them, instead of timing the market quickly. We show that the harmful bubble persists for a long time as the unique Nash equilibrium. Importantly, this result holds even if the underlying positive feedback traders are not very enthusiastic.

Keywords: Harmful Bubbles, Leverage, Share Issuance, Timing Games, Behavioral Arbitrageurs.

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1. Introduction

This paper investigates the possibility that an unproductive company with limited debt capacity raises huge funds through *share issuances* by utilizing a small sign of enthusiasm. We demonstrate a theoretical model of the stock market describing the phenomenon of *bubbles and crashes* as a generalization of the timing game explored by Matsushima (2012). In the model of this paper, we will permit arbitrageurs to use high *leverage* for purchasing the company's newly issued shares. Thanks to this leverage, any arbitrageur has strong incentive to ride the bubble by continuing to purchase them instead of timing the market quickly; the company can conspire with these arbitrageurs in terms of their absorption of a part of its share issuance, making its fund raising compatible with the long persistence of the bubble.

We show that the bubble persists for a long time as the *unique* Nash equilibrium; importantly, *this result holds even if the underlying positive feedback traders are not very enthusiastic*. In this case, the bubble should be regarded as being *socially harmful*; the unproductive company wastes the raised funds on its personal usage, while the arbitrageurs are willing to invest all their collateralized borrowings in this company's inefficient enterprise.

In the same manner as Matsushima (2012), multiple arbitrageurs as the players of this game compete with each other to time the market at the earliest within a continuous time interval. Compared with the other previous models that identified conditions under which the bubble takes place[1], the basic model of Matsushima (2012) has substantial advantages in that Nash equilibria can be analyzed in a constructive manner according to the tractable standard method of preemption games, and that the arbitrageurs' perception that the bubble will inevitably crash despite its long persistence can be explicitly taken into account.

It is implicit in the timing game to assume that there are many *positive feedback traders* who are slave of euphoria; they incorrectly perceive that each share of this

[1] See Harrison and Kreps (1978), Blanchard and Watson (1982), Tirole (1985), Allen and Gordon (1993), Allen, Morris, and Postlewaite (1993), and Scheinkman and Xiong (2003), for example. Abreu and Brunnermeier (2003) is closely related to Matsushima (2012), which firstly but in a different manner applied the preemption game analysis with the closed time interval to bubbles and crashes.

company will sell at the price that is greater than its fundamental value, and keep reinforcing their misperception during the bubble. These positive feedback traders, however, could be sensitive to a trend of the professional arbitrageurs in such manners that once the proportion of shares that these arbitrageurs possess all together declines below some critical level, the resultant selling pressure forces the positive feedback traders out of their euphoria, bursting the bubble immediately.

It might be the case that the arbitrageurs examine the possibility of profiting by riding the bubble for a while and selling up later on. In this case, however, they have to remain in competition with each other over who becomes the winner by selling up before others. This competitive pressure will quickly undo the mispricing caused by the positive feedback traders' euphoria, as long as the increase in winner's gain by keeping the bubble persistent is not much expected.

In order to overcome the above-mentioned difficulty of describing the persistence of bubble in theory, Matsushima (2012) assumed in his model that each arbitrageur is expected by the other arbitrageurs to be not rational but *behavioral* with a non-negligible probability in that he is committed not to time the market of his own accord. Matsushima then showed that any rational arbitrageur is willing to ride the bubble for a long time as the unique Nash equilibrium if the underlying positive feedback traders are so enthusiastic that the growth rate of bubble price is sufficiently high. In the model of Matsushima, however, whenever the positive feedback traders are not very enthusiastic, then it is inevitable that any rational arbitrageur is willing to time the market quickest, i.e., the bubble never persists.

In contrast to Matsushima (2012), this paper will show that the bubble can persist for a long time even if the positive feedback traders are not very enthusiastic; as the main departures from Matsushima, the present paper's model will permit the arbitrageurs to use leverage and permit the company to issue new shares. Thanks to this leverage, any arbitrageur can dramatically increase the winner's gain by keeping the bubble persistent longer as follows. By using leverage for the purchase of the company's newly issued shares, he can make his earned capital gain much greater than the capital gain that he can earn without leverage. By adding this extra capital gain to his personal capital and expanding his collateralized borrowing in the accelerative manner, he can continue purchasing more shares, dramatically enhancing his winner's gain at the later times.

It is implicit in this paper's model to assume that, besides the positive feedback traders, there are also many other traders who do not know the company's fundamental value and are naive enough to misperceive that the market value correctly reflects the true fundamental value at all times and not to mind the risk that the share price slumps. By making non-recourse short-term debt contracts backed by his shareholdings with these traders, each arbitrageur could have strong incentive to continue purchasing the shares.

Without this arbitrageur's leverage, the company might hesitate to issue new shares, because the selling pressure caused by this issuance will dampen the positive feedback traders' enthusiasm. With it, on the other hand, the company is safe from the failure in catering to their enthusiasm as long as it continues issuing shares just at moderate speed.

By introducing the arbitrageurs' usage of leverage and the company's share issuance into the model, we can show that with only weakly regulated leverage, the socially harmful bubble can persist for a long time as the unique Nash equilibrium; importantly, this result holds even if the positive feedback traders are not very enthusiastic and each arbitrageur is expected to be rational almost certainly. Hence, we can conclude that *any opportunity of trifling enthusiasm could cause serious social harm*; with the help of the arbitrageurs' leverage, the company can continue raising funds for its personal benefit without bursting the bubble.

Matsushima (2012) particularly assumed that the bubble price grows in an exponential manner. In contrast, the present paper will investigate more general price formations, which could be compatible with the case that the positive feedback traders are not very enthusiastic at all times.

There are relevant empirical researches such as Stein (1996), Baker, Stein, and Wurgler (2003), and Polk and Sapienza (2008) concerning about whether deviation from the fundamental value affects the company's investment policy. Baker, Stein, and Wurgler (2003) pointed out that the investment of a company that has limited debt capacity and heavily dependent on external equity is quite sensitive to its stock price fluctuations. There are other relevant empirical researches such as Adrian and Shin (2010), pointing out that financial intermediaries are sensitive to the fluctuations in asset prices. There are also works such as Kiyotaki and Moore (1997) and Farhi and Tirole

(2009), pointing out that the bubble encourages the economic performance because it makes capital more valuable as collateral; in contrast, the present paper claims that in order to keep the bubble persistent, the arbitrageurs may have to use their collateralized borrowings for the purchase of the unproductive company's shares.

The organization of this paper is as follows. Sections 2 and 3 generalize the model of Matsushima (2012) by introducing leverage and share issuance and by generalizing price formations. Section 4 shows the basic results for characterizing all symmetric Nash equilibria. Section 5 shows as the main contribution of this paper that the bubble persists for a long time as the unique Nash equilibrium if the regulation on the arbitrageurs' leverage is sufficiently weak. Section 6 examines about the social cost caused by the company's wasting the raised funds. Section 7 concludes.

2. Leverage and Share Issuance

We model the market for a company's stock as a timing game (or a preemption game) with a continuous time horizon $[0,1]$ that is a generalization of the model explored by Matsushima (2012). We assume that the market interest rate equals zero, that no dividends are paid, that no short selling is permitted. We also assume that the company has no profitable business opportunity. Hence, its fundamental value is set equal to zero, and this company has limited debt capacity.

This paper generalizes the model of Matsushima (2012) from the following three viewpoints:

(i) The company is permitted to issue new shares.

(ii) The arbitrageurs are permitted to utilize leverage for purchasing the company's newly issued shares.

(iii) More general price formations than Matsushima (2012) are considered.

The total number of shares that the company issues up to any time $t \in [0,1]$ is denoted by $S(t) > 0$, where $S(t)$ is assumed to be non-decreasing in t; during any short time interval $[t, t+\Delta]$, the company newly issues $S'(t)\Delta$ number of shares. There exist $n \geq 2$ arbitrageurs each of whom decides the time to sell up his (or her) shares. Provided the bubble does not crash at or before time $t \in [0,1]$, the number of shares that each arbitrageur $i \in N$ owns at this time is denoted by $S_i(t)$, where $S_i(t)$ is assumed to be non-decreasing in t; during any short time interval $[t, t+\Delta]$, he purchases $S_i'(t)\Delta$ number of the company's newly issued shares.

The bubble persists as long as at least $n - \tilde{n} + 1$ arbitrageurs continues to hold a claim to $c \times 100\,\%$ of the company's assets, where $\tilde{n}$ is a positive integer, $\tilde{n} < n$, and $0 < c < 1/n$. During the bubble, the share price grows according to a continuous and increasing price formation function $P : [0,1] \to (0,\infty)$, where $P(t)$ implies the share price at time $t \in [0,1]$ provided the bubble persists.

Once any arbitrageur sells up his shares, this selling pressure triggers the other arbitrageurs to sell up immediately, bursting the bubble; the share price declines to zero immediately. Even if no arbitrageur sells up, the bubble automatically crashes just after the terminal time 1.

According to Matsushima (2012), it is implicit in this paper to assume that there are many *positive feedback traders* who are slaves to euphoria; at any time $t \in [0,1]$ during the bubble, they incorrectly perceive that each share will sell for the price $P(t)$ in the future, and they reinforce their misperception about the share price according to the price formation function P. The moment $\tilde{n}$ or more arbitrageurs sell up, the resultant selling pressure forces them out of their euphoria.

Thanks to the bubble, it is possible for the company to raise funds by issuing new shares. Any arbitrageur is permitted to use leverage for purchasing this company's newly issued shares. In order to maintain the persistence of the bubble, the company has to keep the number of shares that each arbitrageur owns not less than $c \times 100\,\%$ of this company's assets. In this case, it is appropriate to assume that

(1) $$cS(t) = \min_{i \in \{1,\dots,n\}} S_i(t) \quad \text{for all} \quad t \in [0,1],$$

because the company prefers raising funds as much as possible.

Each arbitrageur should be considered to purchase the newly issued shares by short-term debt finance such as repurchase agreement with traders other than the positive feedback traders, where the *leverage ratio* $L \geq 1$ is exogenously given, and we will explain about these traders later. Let $S_i(t)$ denote the number of shares that arbitrageur i holds at the time $t \in [0,1]$ during the bubble. Because his leverage ratio is set equal to L, he has the debt obligation of monetary amount $\frac{L-1}{L} S_i(t)P(t)$ with zero interest rate to the debt holders in this case. Hence, his personal capital, denoted by $W_i(t)$, is given by the market value of his shareholdings minus his debt obligation:

(2) $$W_i(t) = S_i(t)P(t) - \frac{L-1}{L} S_i(t)P(t) = \frac{S_i(t)P(t)}{L}.$$

It is implicit in this paper's model to assume that besides the positive feedback traders, there are many other traders who do not know the company's fundamental value and are naive enough to misperceive that the market value always reflects the fundamental value correctly; they do not mind that there is the risk that the company's stock price slumps. Hence, by making debt contracts with these traders, any arbitrageur can let these debt holders bear $\frac{L-1}{L}\times 100\%$ of the loss caused by the crash of the bubble.

Provided the bubble does not crash at or before time $t+\Delta$, arbitrageur i can earn the capital gain

$$S_i(t)\{P(t+\Delta)-P(t)\}$$

from time t to time $t+\Delta$. In this case, his personal capital increases by this amount:

$$W_i(t+\Delta)=W_i(t)+S_i(t)\{P(t+\Delta)-P(t)\}.$$

By letting Δ close to zero, it follows that

(3) $$W_i'(t)=S_i(t)P'(t).$$

Moreover, from (2), it follows that

$$W_i'(t)=\frac{S_i'(t)P(t)+S_i(t)P'(t)}{L},$$

which, along with (3), implies that for every $t\in[0,1]$,

$$S_i(t)P'(t)=\frac{S_i'(t)P(t)+S_i(t)P'(t)}{L},$$

that is,

(4) $$\frac{S_i'(t)}{S_i(t)}=(L-1)\frac{P'(t)}{P(t)}.$$

Hence, by arbitrageur $i's$ utilizing leverage, the growth rate of the number of shares that arbitrageur i owns, $\frac{S_i'(t)}{S_i(t)}$, could be the same as the growth rate of the share price, $\frac{P'(t)}{P(t)}$, multiplied by the leverage ratio minus unity, $L-1$. From (4), it follows that the price formation function P, the leverage ratio L, and the initial shareholding $S_i(0)$ uniquely determine arbitrageur $i's$ shareholding $S_i(t)$;

$$S_i(t)=S_i(0)(\frac{P(t)}{P(0)})^{L-1} \quad \text{for all} \quad t\in[0,1],$$

which, along with (2), implies that

(5) $$S_i(t)=\frac{LW_i(0)}{P(0)}(\frac{P(t)}{P(0)})^{L-1}.$$

Throughout this paper, we assume that any arbitrageur possesses the same amount of personal capital at the initial time 0:

$$W_i(0) = W_1(0) \text{, i.e., } S_i(0) = S_1(0) \text{, for all } i \in \{1, ..., n\},$$

which, along with the constraint on the company's share issuance given by (1), and along with (5), implies that

$$cS(t) = S_1(t) \text{ for all } t \in [0,1].$$

Hence, the specification of (L, P, c) uniquely determines $S(t)$. From (4), for every $t \in [0,1]$,

$$\frac{S'(t)}{S(t)} = (L-1)\frac{P'(t)}{P(t)},$$

which determines the total number of shares at each time t;

(6) $$S(t) = S(0)(\frac{P(t)}{P(0)})^{L-1} = \frac{LW_1(0)}{cP(0)}(\frac{P(t)}{P(0)})^{L-1}.$$

The model of Matsushima (2012) could be regarded as a special case; it was assumed that the company never issued new shares, i.e., $S(t)$ and $S_i(t)$ were constant across times, that the arbitrageurs never used leverage, i.e., $L = 1$, and that particular price formation functions were considered, according to which, the share price exponentially grew at a constant growth rate $\rho > 0$, i.e.,

$$P(t) = e^{\rho t} \text{ for all } t \in [0,1].$$

3. Timing Games with Behavioral Types

We model the timing game with behavioral types as a generalization of Matsushima (2012) as follows. Each arbitrageur $i \in N$ as a player selects the time to sell up during the time interval $[0,1]$. A *strategy* for each arbitrageur $i \in N$ is defined as a cumulative distribution $q_i : [0,1] \to [0,1]$ that is non-decreasing, right continuous, and satisfies $q_i(1) = 1$. Following any strategy q_i, arbitrageur i plans to sell up at or before any time $a_i \in [0,1]$ with the probability $q_i(a_i) \in [0,1]$. Let Q_i denote the set of strategies for arbitrageur i. We will consider $q_i = a_i$ to be a *pure* strategy, where $q_i(\tau) = 0$ for all $\tau \in [0, a_i)$, and $q_i(\tau) = 1$ for all $\tau \in [a_i, \tau_0]$.

Let us fix any arbitrary positive real number $\varepsilon \in (0,1)$. Each arbitrageur is expected by the other arbitrageurs to be rational with a probability $1 - \varepsilon > 0$, while he is expected to be *behavioral* with the remaining probability $\varepsilon > 0$. If an arbitrageur is behavioral, he continues to purchase and never sells up before the other arbitrageurs sell up; i.e., he is committed not to burst the bubble of his own accord.

At any time $t \in [0,1]$, if he can sell up before the bubble crashes, then he obtains the monetary amount $S_i(t)P(t)$ and repays the debt obligation $\frac{L-1}{L} S_i(t)P(t)$; his earned payoff could be the same as his personal capital at the crash time:

$$S_i(t)P(t) - \frac{L-1}{L} S_i(t)P(t) = \frac{S_i(t)P(t)}{L} = W_i(t).$$

We assume non-recourse that any arbitrageur is exempted from repayment beyond the market value of his shareholding; whenever this market value declines below his debt obligation, he is seized only his shareholding without repaying the total obligation. If he fails to sell up before the bubble crashes, the market value of shares declines to zero. However, because of this non-recourse assumption, he is exempted from repayment by being seized his shareholdings to the lenders, whose market value has already declined to zero. Hence, his corresponding payoff should be equal to zero in this case.

It must be noted that the traders with whom the arbitrageurs make debt contracts are so naive that they are unconcerned about the risk that the share price declines. This becomes the driving force that the arbitrageurs make advantageous contracts with them.

Suppose that each arbitrageur $i \in N$ plans to sell up at time $a_i \in [0,1]$. Let us arbitrarily set any nonempty subset of arbitrageurs $H \subset N$. Suppose that any arbitrageur $i \in H$ is rational, while any arbitrageur $i \in N \setminus H$ is behavioral. We denote by $\tau = \min\{a_j \mid j \in H\}$ the earliest time at which the rational arbitrageurs plan to sell up. Let $l = \left|\{j \in H \mid a_j = \tau\}\right|$ denote the number of rational arbitrageurs who plans to sell up at this earliest time τ. If $l > \tilde{n}$, then, with the probability $\frac{\tilde{n}}{l}$, any rational arbitrageur $i \in H$ who plans to sell up at time τ can sell up before the crash of the bubble, and therefore, can earn $W_i(t) = \frac{S_i(t)P(t)}{L}$. With regard to the remaining probability $1 - \frac{\tilde{n}}{l}$, he fails to sell up before the crash and earns zero. If $l \le \tilde{n}$, then he can certainly sell up before the crash. In this case, $\tilde{n} - l$ further arbitrageurs can sell up before the crash; even any arbitrageur who either is behavioral or plans to sell up after time τ has the opportunity to sell up before the crash with regard to the positive probability $\frac{\tilde{n} - l}{n - l}$. Based on the above explanation[2], let us define the expected earning of any rational arbitrageur $i \in H$ by

$$v_i(H, a) = \min[1, \frac{\tilde{n}}{l}] W_i(\tau) \qquad \text{if } a_i = \tau,$$

and

$$v_i(H, a) = \max[\frac{\tilde{n} - l}{n - l}, 0] W_i(\tau) \qquad \text{if } a_i > \tau,$$

where $a = (a_j)_{j \in N}$ and $W_i(\tau) = \frac{S_i(\tau)P(\tau)}{L}$.

Let $Q = Q_1 \times \cdots \times Q_n$. Let $q = (q_1, ..., q_n) \in Q$ denote a strategy profile. The payoff function $u_i : Q \to R$ for each arbitrageur $i \in N$ is defined; for every strategy profile

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[2] The explanation basically follows Matsushima (2012).

$q \in Q$, we specify $u_i(q)$ as the expected value of $v_i(H,a)$ in terms of (a,H), that is,

$$u_i(q) \equiv E[\sum_{H \subset N: i \in H} v_i(H,a)\varepsilon^{n-|H|}(1-\varepsilon)^{|H|-1} \mid q],$$

where $E[\cdot \mid q]$ is the expectation operator in terms of $a \equiv (a_j)_{j \in N}$. A strategy profile $q \in Q$ is said to be *symmetric* if

$$q_i = q_1 \text{ for all } i \in N.$$

A strategy profile $q \in Q$ is said to be a *Nash equilibrium* if

$$u_i(q) \geq u_i(q_i', q_{-i}) \text{ for all } i \in N \text{ and all } q_i' \in Q_i.$$

4. Basic Results

For each strategy profile $q \in Q$, we can denote the probability that the bubble has crashed at or before any time $t \in [0,1]$ by

$$D(t;q) \equiv 1 - \prod_{i \in N} \{1 - (1-\varepsilon) q_i(t)\}.$$

Let us define the *hazard rate* at which the bubble crashes at any time $t \in [0,1]$ as

$$\theta(t;q) \equiv \frac{\frac{\partial}{\partial t} D(t;q)}{1 - D(t;q)}.$$

Moreover, for each arbitrageur $i \in N$, we can denote the probability that the bubble has crashed at or before any time t, provided arbitrageur i never bursts the bubble of his own accord, by

$$D_i(t;q_{-i}) = 1 - \prod_{j \in N \setminus \{i\}} \{1 - (1-\varepsilon) q_j(t)\}.$$

Let us fix an arbitrary strategy profile $q \in Q$, and consider any time $t \in [0,1]$ at which $q_i(t)$ is continuous and increasing for every $i \in \{1,...,n\}$. Because of the continuity of q at time t, it is a negligible case that two or more arbitrageurs time the market in the neighborhood of time t. This implies that the difference in expected payoff between the case that arbitrageur i firstly times the market and the case that he does not do it should be equivalent to

$$W_i(t) - \frac{\tilde{n}-1}{n-1} W_i(t) = \frac{n-\tilde{n}}{n-1} W_i(t).$$

Hence, if q is a Nash equilibrium, the following first order condition must hold:

$$\frac{n-\tilde{n}}{n-1} W_i(t) \frac{\partial}{\partial t} D_i(t;q_{-i}) = W_i'(t)\{1 - D_{-i}(t;q_{-i})\}. \tag{7}$$

The left hand side of (7) implies the marginal loss induced by the decrease in winning probability, while its right hand side implies that the marginal gain induced by the increase in personal capital. The equality (7), along with the equalities (2) and (3), implies that the first order condition at time t is equivalent to the following equality:

(8) $$\frac{\frac{\partial}{\partial t} D_i(t;q_{-i})}{1-D_i(t;q_{-i})} = L\frac{n-1}{n-\tilde{n}}\frac{P'(t)}{P(t)}.$$

Let us further suppose that q is symmetric. Then, from the equality (8), we can derive the following equalities, both of which express the same first order condition in different manners:

(9) $$\theta(t;q) = L\frac{n}{n-\tilde{n}}\frac{P'(t)}{P(t)},$$

(10) $$\frac{(1-\varepsilon)q_1'(t)}{1-(1-\varepsilon)q_1(t)} = \frac{L}{n-\tilde{n}}\frac{P'(t)}{P(t)}.$$

According to the expression of the first order condition given by (10), depending on $\varepsilon \in (0,1)$ and $L \geq 1$, we can uniquely define a critical time $\tilde{\tau} = \tilde{\tau}(\varepsilon, L) \in [0,1]$ and a symmetric and continuous strategy profile $\tilde{q} = \tilde{q}(\varepsilon, L) \in Q$ satisfying that

(11) $$\frac{(1-\varepsilon)\tilde{q}_1'(t)}{1-(1-\varepsilon)\tilde{q}_1(t)} = \frac{L}{n-\tilde{n}}\frac{P'(t)}{P(t)} \quad \text{for all } t \in [\tilde{\tau}, 1],$$

and

$$\tilde{q}_1(t) = 0 \quad \text{for all } t \in [0, \tilde{\tau}).$$

From the continuity of $\tilde{q}$ at time $\tilde{\tau}$, it must hold that $\tilde{\tau} = \tilde{\tau}(\varepsilon, L) < 1$. Note that $\tilde{\tau}(\varepsilon, L)$ is increasing in L, because the hazard rate given by (9), $\theta(t;\tilde{q}(\varepsilon, L))$, is increasing in L. Note also that $\lim_{L\to\infty} \tilde{\tau}(\varepsilon, L) = 1$, i.e., the bubble certainly persists up to the time near the terminal time 1, provided the leverage ratio is sufficiently large.

From (11), it follows that

(12) $$\tilde{q}_1(t) = \frac{1-\{1-(1-\varepsilon)\tilde{q}_1(\tilde{\tau})\}(\frac{P(\tilde{\tau})}{P(t)})^{\frac{L}{n-\tilde{n}}}}{1-\varepsilon},$$

which, along with $\tilde{q}_1(1) = 1$, and with the fact that $\tilde{q}_1(\tilde{\tau}) = 0$ if $\tilde{\tau} > 0$, implies that

(13) $$\varepsilon = (\frac{P(\tilde{\tau})}{P(1)})^{\frac{L}{n-\tilde{n}}} \qquad \text{if } \tilde{\tau} > 0,$$

and

$$\varepsilon = \{1-(1-\varepsilon)\tilde{q}_1(0)\}(\frac{P(0)}{P(1)})^{\frac{L}{n-\tilde{n}}} \qquad \text{if } \tilde{\tau}=0 .$$

Based on the specifications of $\tilde{\tau}$ and $\tilde{q}$, for every $\hat{\tau}\in[\tilde{\tau},1]$, we can specify another continuous and symmetric strategy profile, $q^{\hat{\tau}}=(q_i^{\hat{\tau}})_{i\in N}\in Q$, as follows:

$$q_1^{\hat{\tau}}(t)=\tilde{q}_1(t) \quad \text{for all } t\in[\hat{\tau},1],$$

and

$$q_1^{\hat{\tau}}(t)=\tilde{q}_1(\hat{\tau}) \quad \text{for all } t\in[0,\hat{\tau}).$$

According to $q^{\hat{\tau}}$, any rational arbitrageur plans to sell up at the initial time 0 with the probability $\tilde{q}_1(\hat{\tau})$; with the remaining probability $1-\tilde{q}_1(\hat{\tau})$, he plans to ride the bubble up to the time $\hat{\tau}$, and then follow $\tilde{q}$ afterwards. Note that $\tilde{q}=q^{\tilde{\tau}}$.

The following theorem characterizes all symmetric Nash equilibria.

Theorem 1: *A symmetric strategy profile $q\in Q$ is a Nash equilibrium if and only if there exists $\hat{\tau}\in[\tilde{\tau},1]$ such that*

$$q=q^{\hat{\tau}},$$

and

(14) $$u_1(0,q_{-1})=u_1(\hat{\tau},q_{-1}) \quad \textit{whenever } \hat{\tau}<1 \textit{ and } q_1(0)>0,$$

and

(15) $$u_1(0,q_{-1})\geq u_1(\hat{\tau},q_{-1}) \quad \textit{whenever } \hat{\tau}=1.$$

Proof: See The Appendix.

The necessity part of the proof of Theorem 1 follows the proof of Proposition 1 in Matsushima (2012); according to the standard method of the preemption game analysis, it follows that q is continuous and increasing, and the first order condition holds in $t\in[\hat{\tau},1]$, implying this necessity part. The proof of the sufficiency part is self-explanatory.

In the same manner as Matsushima (2012), the strategy profiles $q^{\hat{\tau}}$ can be categorized into three types: bubble-crash, quick-crash, and hybrid. The strategy profiles $q^{\hat{\tau}}$ is said to be *bubble-crash* if the bubble never bursts at the initial time 0, i.e., $q_1^{\hat{\tau}}(0) = 0$. It is said to be *quick-crash* if the bubble certainly bursts at the initial time 0 whenever there is a rational arbitrageur, i.e., $q_1^{\hat{\tau}}(0) = 1$. It is said to be *hybrid* if the bubble bursts at the initial time 0 with a positive probability but not with certain, i.e., $0 < q_1^{\hat{\tau}}(0) < 1$. Among them, this paper will focus just on bubble-crash Nash equilibria; from the specifications of $\tilde{q}$ and $q^{\hat{\tau}}$, it is clear that $q^{\hat{\tau}}$ is bubble-crash if and only if $\hat{\tau} = \tilde{\tau}$, i.e.,

$$q^{\hat{\tau}} = \tilde{q}, \text{ and } \tilde{q}_1(0) = 0.$$

The following theorem shows a necessary and sufficient condition for bubble-crash Nash equilibrium.

Theorem 2: *The strategy profile* $\tilde{q}$ *is a Nash equilibrium if and only if*

(16) $$\varepsilon \geq (\frac{P(0)}{P(1)})^{\frac{L}{n-\tilde{n}}}.$$

Proof: From Theorem 1, $\tilde{\tau} < 1$, and $\tilde{q} = q^{\tilde{\tau}}$, it follows that $\tilde{q}$ is a Nash equilibrium if and only if

$$\text{either } \tilde{q}_1(0) = 0 \text{ or } u_1(0, \tilde{q}_{-1}) = u_1(\tilde{\tau}, \tilde{q}_{-1}).$$

The inequality (16), along with the increasing property of $P(t)$, implies (13), implying that $\tilde{q}_1(0) = 0$, i.e., $\tilde{q}$ is a Nash equilibrium.

Suppose that the equality (16) does not hold. Then, it must hold that $\tilde{\tau} = 0$ and $\tilde{q}_1(0) > 0$. This, however, contradicts the Nash equilibrium property that any time choice in $[0,1]$ is a best response: any arbitrageur prefers time 0 to any time slightly later than time 0, because he can increase his winning probability without any substantial decrease in winner's gain.

Q.E.D.

The proof of Theorem 2 showed that (16) implies that $\tilde{q}_1(0) = 0$, and therefore, from (12) and (13), it follows that

$$\tilde{q}_1(t) = \frac{1-\varepsilon(\frac{P(1)}{P(t)})^{\frac{L}{n-\tilde{n}}}}{1-\varepsilon} \quad \text{for all} \quad \tau \in [\tilde{\tau},1],$$

and

$$\tilde{q}_1(t) = 0 \quad \text{for all} \quad \tau \in [0,\tilde{\tau}],$$

where $\tilde{\tau}$ was defined as

$$\varepsilon = (\frac{P(\tilde{\tau})}{P(1)})^{\frac{L}{n-\tilde{n}}}.$$

Note also that the probability that the bubble has crashed at or before any time $t \in [\tilde{\tau},1]$ is given by

$$D(t;\tilde{q}) \equiv 1-\varepsilon^n(\frac{P(1)}{P(t)})^{\frac{nL}{n-\tilde{n}}}.$$

The following theorem shows the necessary and sufficient condition under which the bubble-crash strategy profile $\tilde{q}$ is the *unique* Nash equilibrium even if we take all asymmetric Nash equilibria into account, the proof of which follows the proof of Theorem 3 in Matsushima (2012).

Theorem 3: *The bubble-crash strategy profile* $\tilde{q}$ *is the unique Nash equilibrium if*

(17) $$\varepsilon > (\frac{P(0)}{P(1)})^{\frac{L}{n-1}}.$$

Proof: See The Appendix.

The sufficient condition (17) could be regarded as being almost necessary for uniqueness; it is just the strict inequality version of (16) that is the necessary and sufficient condition for $\tilde{q}$ to be a Nash equilibrium. Whenever the positive feedback traders are not very enthusiastic, i.e., $\frac{P(0)}{P(1)}$ is close to unity, and each arbitrageur is expected to be rational almost certainly, i.e., ε is close to zero, then, without leverage, i.e., $L=1$, the inequality (16) never holds; the bubble-crash strategy profile $\tilde{q}$ never becomes a Nash equilibrium. In the similar manner to Section 5 in Matsushima (2012), we can show that the quick-crash strategy profile is the only Nash equilibrium without leverage, while, with leverage, by selecting L sufficiently large, the bubble-crash strategy profile $\tilde{q}$ could become the unique Nash equilibrium.

5. Long-Persistent Bubble

This section shows that by letting the leverage ratio L sufficiently large, we can make the critical time $\tilde{\tau}$ as close to unity as possible. Hence, the bubble can persist for a long time even if the positive feedback traders are not very enthusiastic and the possibility of each arbitrageur being behavioral is negligible.

For every $t \in [0,1)$ and every $\varepsilon \in (0,1]$, we define a particular leverage ratio $L(t,\varepsilon) \geq 1$ by (13), i.e.,

$$\varepsilon = (\frac{P(t)}{P(1)})^{\frac{L(t,\varepsilon)}{n-\tilde{n}}} \text{. i.e., } L(t,\varepsilon) = \frac{(n-\tilde{n})\log\varepsilon}{\log P(t) - \log P(1)} \quad \text{if} \quad \varepsilon \in (0, (\frac{P(t)}{p(1)})^{\frac{1}{n-\tilde{n}}}),$$

and

$$L(t,\varepsilon) = 1 \qquad \text{if} \quad \varepsilon \in [(\frac{P(t)}{p(1)})^{\frac{1}{n-\tilde{n}}}, 1].$$

Note that $L(t,\varepsilon)$ is increasing in t, and

$$\lim_{t \to 1} L(t,\varepsilon) = \infty .$$

The following theorem shows that, provided the regulation on leverage is sufficiently weak, the bubble can persist for a long time, even if the positive feedback traders are not very enthusiastic and the possibility of each arbitrageur being behavioral is negligible, i.e., irrespective of the specification of (P,ε).

Theorem 4: *For every $t \in [0,1)$ and every $\varepsilon \in (0,1]$, if we let $L = L(t,\varepsilon)$, then it holds that $\tilde{\tau} = t$ and $\tilde{q}$ is a Nash equilibrium.*

Proof: Theorem 4 holds straightforwardly from Theorem 2 and the specification of $L(t,\varepsilon)$.

Q.E.D.

Theorem 5: *For every $t \in [0,1)$ and every $\varepsilon \in (0,1]$, if*

(18) $$P(t)^{n-1} > P(0)^{n-\tilde{n}} P(1)^{\tilde{n}-1},$$

and we let $L = L(t,\varepsilon)$, then it holds that $\tilde{\tau} = t$ and $\tilde{q}$ is the unique Nash equilibrium.

Proof: From the inequality (18),

$$\left(\frac{P(t)}{P(1)}\right)^{\frac{L(t,\varepsilon)}{n-\tilde{n}}} > \left(\frac{P(0)}{P(1)}\right)^{\frac{L(t,\varepsilon)}{n-1}},$$

which, along with $\varepsilon = \left(\frac{P(t)}{P(1)}\right)^{\frac{L(t,\varepsilon)}{n-\tilde{n}}}$, implies the inequality (17). Hence, Theorem 3 implies that $\tilde{q}$ is the unique Nash equilibrium.

Q.E.D.

Clearly, any time t that is sufficiently close to the terminal time 1 satisfies (18). Hence, from Theorem 5, we can conclude that even if the positive feedback traders are not very enthusiastic and the possibility of each arbitrageur being behavioral is negligible, the bubble persists for a long time as the *unique* Nash equilibrium, where the regulation on leverage was assumed to be sufficiently weak.

6. Social Cost

In any short time interval $[\tau, \tau+\Delta]$ during the bubble, the company issues $S'(\tau)\Delta$ number of shares for the price $P(t)$. Since the company has no profitability, it just wastes the raised fund $p(\tau)S'(\tau)\Delta$. Hence, the total money raised by share issuance from the initial time 0 to time t at which the bubble bursts, which is defined as

$$C(t) \equiv \int_{\tau=0}^{t} P(\tau)S'(\tau)d\tau ,$$

could be regarded as the *social cost* induced by the persistence of bubble. From (6), we can calculate this social cost as

$$C(t) = P(0)S(0)\int_{\tau=0}^{t}(L-1)\frac{P(\tau)^{L-1}}{P(0)^{L}}d\tau = P(0)S(0)\frac{L-1}{L}\{(\frac{P(t)}{P(0)})^{L-1} - 1\} .$$

Note that $C(t)$ does not depend on the detail of the process of price increase from the initial time 0 to the crash time t (depends just on $P(0)$ and $P(1)$) and that it is increasing in L and $\frac{P(t)}{P(0)}$. Since the market value of the company's stock at time t is given by

$$P(t)S(t) = P(0)S(0)(\frac{P(t)}{P(0)})^{L} ,$$

we can calculate the ratio between the social cost and market value of the company:

$$\frac{C(t)}{P(t)S(t)} = \frac{L-1}{L}\cdot\frac{P(0)}{P(t)}\{1-(\frac{P(0)}{P(t)})^{L-1}\} ,$$

which is increasing in L but decreasing in $\frac{P(t)}{P(0)}$. In particular, if the positive feedback traders are not enthusiastic and the regulation on leverage is weak, i.e., if $\frac{P(t)}{P(0)} > 1$ is close to unity and L is sufficiently large, then the ratio $\frac{C(t)}{P(t)S(t)}$ could be close to unity; the most part of the market value should be nothing but the social cost.

7. Conclusion

As generalizing the timing game explored by Matsushima (2012), which provided a foundation for describing the phenomenon of bubbles and crashes in the stock market, this paper demonstrated a theoretical model in which an unproductive company with limited debt capacity can continue issuing new shares for financing its personal benefit, while the arbitrageurs finance their purchases of this company’s newly issued shares through short-term debt contracts with naive traders. We showed that with a sufficiently weak regulation on leverage for these arbitrageurs, the bubble can persist for a long time as the unique Nash equilibrium; importantly, this result held even if the underlying positive feedback traders are not very enthusiastic and the arbitrageurs are expected to be rational almost certainly. We should be concerned about the bubble adversely affecting the society, because the company may pour the raised fund into unproductive enterprise. Hence, it would be important as future researches to innovate any conceptual device for prudential regulation that can prevent such harmful bubbles from persisting without disturbing the growth of any potential company that has truly high productivity.

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The Appendix

Proof of Theorem 1: We set any symmetric Nash equilibrium $q \in Q$ arbitrarily. It is clear that the inequality (15) is necessary and sufficient for the Nash equilibrium property if $q = q^1$. We assume that $q \neq q^1$, i.e., $q_1(0) < 1$.

We show that $q_1(\tau)$ is continuous in $[0,1]$. Suppose that $q_1(\tau)$ is not continuous in $[0,1]$; there exists $\tau' > 0$ such that $\lim_{\tau \uparrow \tau'} q_1(\tau) < q_1(\tau')$. Since

$$\min[1, \frac{\tilde{n}}{l+1}] > \max[0, \frac{\tilde{n}-l}{n-l}] \text{ for all } l \in \{0,...,n-1\},$$

it follows from the symmetry of q that by selecting any time that is slightly earlier than time τ', any arbitrageur can drastically increase his wining probability. This implies that no arbitrageur selects time τ', which is a contradiction.

Let us specify

$$\hat{\tau} = \max\{\tau \in (0,1] : q_1(\tau) = q_1(0)\}.$$

We show that $q_1(\tau)$ is increasing in $[\hat{\tau},1]$. Suppose that $q_1(\tau)$ is not increasing in $[\hat{\tau},1]$. From the continuity of q_1 and the specification of $\hat{\tau}$, we can select $\tau', \tau'' \in [\hat{\tau},1]$ such that $\tau' < \tau''$, $q_1(\tau') = q_1(\tau'')$, and the time choice τ' is a best response. Since no arbitrageur selects any time τ in (τ', τ''), it follows from the continuity of q that by selecting time τ'' instead of τ', any arbitrageur can increase the winner's gain from $W_i(\tau')$ to $W_i(\tau'')$ without decreasing his wining probability. This is a contradiction.

Any time choice $\tau \in [\hat{\tau},1]$ must be a best response, because $q_1(\tau)$ is increasing in $[\hat{\tau},1]$. This implies that the first-order condition for all $\tau \in [\hat{\tau},1]$, i.e., $q = q^{\hat{\tau}}$. Given $\hat{\tau} < 1$, it is clear from the fact that the winner's gain $W_i(t)$ is increasing that $q^{\hat{\tau}}$ is a Nash equilibrium if and only if

$$u_1(0, q_{-1}) = u_1(\hat{\tau}, q_{-1}) \text{ whenever } q_1(0) > 0.$$

This implies that (14) is necessary and sufficient.

Q.E.D.

Proof of Theorem 3: We will show that $\tilde{q}$ is the unique symmetric Nash equilibrium. Note from (6) that for every $q'_{-1} \in Q_{-1}$,

$$u_1(0, q'_{-1}) \leq W_1(0) = \frac{P(0)S_i(0)}{L},$$

and

$$u_1(1, q'_{-1}) \geq \varepsilon^{n-1} W_1(1) = \varepsilon^{n-1} \frac{P(0)S_i(0)}{L} \left(\frac{P(1)}{P(0)}\right)^L.$$

These inequalities, along with (17), imply that the time choice 0 is dominated by the time choice 1, i.e.,

$$u_1(1, q'_{-1}) > u_1(0, q'_{-1}) \quad \text{for all} \quad q'_{-1} \in Q_{-1}.$$

Hence, any symmetric Nash equilibrium $q \in Q$ must satisfy $q_1(0) = 0$, which along with Theorem 1 implies $q = \tilde{q}$. Since (17) implies (16), it follows from Theorem 2 that $\tilde{q}$ is the unique symmetric Nash equilibrium.

We will show that $\tilde{q}$ is the unique Nash equilibrium, even if all asymmetric Nash equilibria are taken into account. We set any Nash equilibrium $q \in Q$ arbitrarily.

First, we show that $q_i(\tau)$ must be continuous in $[0,1]$ for all $i \in N$. Suppose that $q_i(\tau)$ is not continuous in $[0,1]$. Then, there exists $\tau' > 0$ such that $\lim_{\tau \uparrow \tau'} q_i(\tau) < q_i(\tau')$ for some $i \in N$. Since $\min[1, \frac{\tilde{n}}{l+1}] - \max[0, \frac{\tilde{n}-l}{n-l}] > 0$ for all $l \in \{0, ..., n-1\}$, it follows that any other arbitrageur can drastically increase his winning probability by selecting any time that is slightly earlier than time τ'. Hence, any other arbitrageur never selects any time that is either the same as, or slightly later than, the time τ'. Hence, arbitrageur i can increase the winner's gain by postponing timing the market further without decreasing his winning probability. This is a contradiction.

Second, we show that $D(\tau; q)$ must be increasing in $[\tau^1, 1]$, where we denote

$$\tau^1 = \max\{\tau \in (0,1] : q_i(\tau) = q_i(0) \textit{ for all } i \in N\}.$$

Suppose that $D(\tau; q)$ is not increasing in $[\tau^1, 1]$. Hence, from the continuity of q, we can select $\tau', \tau'' \in (\tau^1, 1]$ such that $\tau' < \tau''$, $D(\tau'; q) = D(\tau''; q)$, and the time choice τ'

is a best response for some arbitrageur. Since no arbitrageur selects any time τ in (τ', τ''), it follows from the continuity of q that by selecting time τ'' instead of τ', any arbitrageur can increase the winner's gain from $W_i(\tau')$ to $W_i(\tau'')$ without decreasing his winning probability. This is a contradiction.

Third, we show that q must be symmetric. Suppose that q is asymmetric. Since (17) implies that the time choice 0 is a dominated strategy, it follows $\tau^1 > 0$, and

$$q_i(\tau) = 0 \quad \text{for all} \quad i \in N \quad \text{and all} \quad \tau \in [0, \tau^1].$$

Since q is continuous and $D(\tau; q)$ is increasing in $[\tau^1, 1]$, from the supposition $\tau^1 > 0$ and (12), it follows that there exist $\tau' > 0$, $\tau'' > \tau'$, and $i \in N$ such that

$$q_1(t) = q_j(t) \quad \text{for all} \quad j \in N \quad \text{and all} \quad t \in [0, \tau'],$$

$$\text{(A-1)} \qquad \frac{\frac{\partial D_i(\tau; q)}{\partial t}}{1 - D_i(\tau; q)} > \min_{h \neq i} \frac{\frac{\partial D_h(\tau; q)}{\partial t}}{1 - D_h(\tau; q)} \quad \text{for all} \quad t \in (\tau', \tau''),$$

and

$$\text{(A-2)} \qquad \frac{\frac{\partial D_i(\tau''; q)}{\partial t}}{1 - D_i(\tau''; q)} = \min_{h \neq i} \frac{\frac{\partial D_h(\tau''; q)}{\partial t}}{1 - D_h(\tau''; q)} > 0,$$

where the last inequality was derived from the increasing property of $D(\tau; q)$ in $[\tau^1, 1]$. Since $D(\tau; q)$ is increasing in $[\tau^1, 1]$, any time choice t in (τ', τ'') must be a best response for any arbitrageur $j \in N$ satisfying that

$$\frac{\frac{\partial D_j(t; q)}{\partial t}}{1 - D_j(t; q)} = \min_{h \neq i} \frac{\frac{\partial D_h(t; q)}{\partial t}}{1 - D_h(t; q)}.$$

Since this equality implies $\frac{\partial q_j(t)}{\partial t} > 0$, it follows from the continuity of q that the first-order condition holds for arbitrageur j; for every $t \in (\tau', \tau'')$,

$$\frac{(1-\varepsilon) q_j{}'(t)}{1 - (1-\varepsilon) q_j(t)} = L \frac{n-1}{n - \tilde{n}} \frac{P'(t)}{P(t)}.$$

Hence, from (A-1),

$$\frac{(1-\varepsilon)q_i{}'(t)}{1-(1-\varepsilon)q_i(t)} > L\frac{n-1}{n-\tilde{n}}\frac{P'(t)}{P(t)},$$

which implies that the first-order condition does not hold for arbitrageur i for every $t \in (\tau', \tau'')$, where the inequality $\frac{\partial u_i(\tau, q_{-i})}{\partial \tau} < 0$ holds in this case. This inequality implies that arbitrageur i prefers time τ' rather than any time in $(\tau', \tau'' + \varepsilon)$, and therefore,

$$\frac{\partial D_i(\tau; q)}{\partial t} = 0 \quad \text{for all} \quad \tau \in (\tau', \tau'' + \eta),$$

where η was positive but close to zero. This is a contradiction, because the inequality of (A-2) implied $\frac{\partial D_i(\tau''; q)}{\partial t} > 0$. Hence, we have proved that any Nash equilibrium q must be symmetric.

Q.E.D.