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Optimal Multiunit Exchange Design with Single-Dimensionality

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Optimal Exchange Mechanism Design with Single-Dimensionality[1]

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Abstract

We investigate allocation problems that generalize auction and bargaining, namely multiunit exchanges, where both a central planner and participants bring homogeneous commodities to sell altogether, and there exist restrictions on feasible allocations. We characterize the optimal mechanism in terms of revenue-maximization under dominant strategy incentive compatibility and ex-post individual rationality. We introduce modified virtual valuation, and show that, irrespective of the restrictions, the optimization can be replaced with the maximization of the sum of modified virtual valuations. We apply this result to an important class of allocation problems for heterogeneous items with single-item demands based on Mussa and Rosen (1978).

Keywords: Revenue-Maximization, Auction and Bargaining, Single-Dimensional Types, Outside Options, Modified Virtual Valuation, Heterogeneous Items.

JEL Classification Numbers: D44, D61, D82.

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1. Introduction

This paper investigates the problem of optimal design of trading mechanisms with asymmetric information, which generalizes auction and bargaining, namely *multiunit exchange*, under the standard assumptions such as quasi-linearity, private values, independent type distributions, risk-neutrality, and no externality. We assume single-dimensional type spaces. Multiple homogeneous commodities are allocated among players (participants). The central planner design**s** the trading mechanism in order to maximize his expected revenue in a consistent manner with the incentive constraints of dominant strategy incentive compatibility and ex post individual rationality.

We demonstrate an approach to extend the basic concepts explored by Myerson (1981) and Myerson and Satterthwaite (1983) such as virtual valuations from a limited class of single-unit allocation problems to a more general class of multiunit allocation problems with restrictions on the set of feasible allocations[3]. The present paper has the following two substantial points of generalization from auction and bargaining:

(i) Both the central planner and the participants bring commodities to sell altogether at the same time. Each participant possesses multiple units as his initial endowment, which he has the option to sell instead of purchasing additional units. Moreover, he has the outside option not to participate in the allocation problem and instead consume his initial endowment by himself.

(ii) Feasible allocations that the central planner can select are restricted to a particular subset of commodity bundle profiles.

The point (i) implies that the central planner earns his revenue not only by selling his endowment but also by exploiting brokerage fee for assisting the exchanges across players. Based on this point of generalization, this paper investigates the optimal mechanism design not only for the auction framework but also for the more general framework termed multiunit exchange, which combines auction with bargaining a la

[3] There exist previous works concerning optimal multi-object auction in terms of the central planner's revenue such as Maskin and Riley (1989), Palfrey (1983), Branco (1996), Monteiro (2002), Ulku (2009), and Edelman and Schwarz (2010).

Myerson and Satterthwaite (1983).[4] Hence, this paper regards the central planner as the hybrid of a seller and an intermediary who monopolistically provides a platform that enables transaction among players.

The outside opportunity value that each player can obtain by consuming his initial endowment by himself could be *dependent* on his type. This type-dependence makes it non-trivial to incentivize each player to participate in the allocation problem at all times, because he (she) might require the central planner to pay the excessive bargaining rent induced by his outside option whose value is unknown to the central planner. Hence, we make a modification of the key concept termed *virtual valuation* in the optimal auction design literature; we replace the virtual valuation for each player with the valuation reduced not only by his informational rent but also by his bargaining rent, which is termed the *modified virtual valuation (MVV)*. In this case, the single-unit term of MVV, i.e., *modified unit virtual valuation (MUVV*), implies *marginal revenue* if this player purchases additional units, whereas it implies *marginal cost* if he sells his endowment.

With this concept replacement, we can show a characterization result according to the similar method to Myerson (1981) and Myerson and Satterthwaite (1983); the optimization problem in terms of the central planner's revenue can be replaced with the problem of maximizing the expected sum of the participants' MVVs in terms of non-decreasing allocation rule. Importantly, this characterization result holds irrespective of what kind of restrictions are imposed on the set of feasible allocations, as implied by the point (ii).

We then demonstrate a tractable manner of constructing the optimal exchange mechanism; we show a mild condition termed *monotonicity* concerning MUVV, which could be sufficient for the *regularity* property on the trading environment that makes it much easier to construct the optimal mechanism. With this regularity, the optimization problem can be replaced with the much simpler problem of maximizing the sum of the participants' MVVs in terms of allocation in the ex post term.

By utilizing these results, we can investigate an important class concerning *exchanges* of multiple heterogeneous items associated with the multiplicative structure explored by Mussa and Rosen (1978); these heterogeneous items can be treated like

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[4] For other related works to bargaining with asymmetric information, see Chatterjee and Samuelson (1983) and Segal and Whinston (2011).

different bundles of a homogeneous commodity. We assume *single-item demands* in that each player demands at most a single item. We show that the allocation problem for heterogeneous items with single-item demands can be replaced with a particular multiunit exchange problem with restrictions on feasible allocations; we demonstrate a tractable manner of constructing the optimal trading mechanism, which could be regarded as generalizing Theorem 4 in Myerson and Satterthwaite (1983) and the arguments on optimal sponsored search auctions by Edelman and Schwarz (2010).

The companion paper by Matsushima (2012) firstly investigated multi-object exchange mechanisms as unifying auction and bargaining, which focused on the achievement of efficiency instead of revenue-maximization. Milgrom (2007) and Cramton (2011) are also related to the present paper.

The organization of this paper is as follows. Section 2 shows the basic model. Section 3 defines the optimization problem for the central planner's revenue, defines the concept of modified virtual valuation (MVV), and shows a characterization result for optimal mechanism design. Section 4 introduces the regularity condition, and provides the monotonicity condition on modified unit virtual valuations (MUVVs) that is sufficient for this regularity. Section 5 investigates the allocation problems for multiple heterogeneous items with single-item demands. Section 6 concludes.

2. The Model

We investigate an allocation problem termed *multiunit exchange*, in which both a central planner and participants bring homogeneous commodities to sell altogether, and there exist some restrictions on the set of feasible allocations. Let $N = \{1,...,n\}$ denote the set of all *players* (participants), where $n \geq 2$. Each player $i \in N$ has a *type* $\omega_i \in [0,1]$ in a single-dimensional type space that is randomly and independently determined according to a probability density function $p_i(\omega_i) \geq 0$, [5] where $\int_{\omega_i=0}^{1} p(\omega_i)d\omega_i = 1$. Let us denote by $P_i(\omega_i) \equiv \int_{s_i=0}^{\omega_i} p(s_i)ds_i$ the associated cumulative distribution. There exist e units of the homogenous commodity to be traded, where e is a fixed positive integer. Each player $i \in N$ possesses e_i units of this commodity as his initial endowment, where e_i is a non-negative integer. We assume that $\sum_{i \in N} e_i \leq e$; the central planner possesses the remaining $e_0 \equiv e - \sum_{i \in N} e_i$ units. An *allocation* is defined as a vector of non-negative integers $a = (a_i)_{i=1}^{n}$, where $\sum_{i \in N} a_i \leq e$; each player $i \in N$ is assigned a_i units, whereas the central planner is assigned the remaining $e - \sum_{i \in N} a_i$ units. Let us denote by A the set of all allocations. Each player $i's$ payoff function has a *quasi-linear* and *risk-neutral* form with *private values,* i.e., $v_i(a_i, \omega_i) + s_i$, where $s_i \in R$ denotes the monetary transfer to him from the central planner, and $v_i : \{0,...,e\} \times [0,1] \to R$ denotes his *valuation function.* [6] We assume that it is differentiable in $\omega_i \in [0,1]$. We assume that

(1) $$v_i(0, \omega_i) = 0 \text{ for all } \omega_i \in [0,1], \text{ and } v_i(a_i, 0) = 0 \text{ for all } a_i \in \{0,...,e\}.$$

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[5] Most previous works concerning optimal multi-object auction design such as Branco (1996), Monteiro (2002), Athey and Elison (2011), Edelman and Schwarz (2011), and Ulku (2009) commonly assumed single-dimensional type spaces.

[6] The central planner has zero valuation for any commodity bundle

We define the *unit valuation* for player $i \in N$ associated with $(\omega_i, a_i) \in [0,1] \times \{1,...,e\}$ by

$$w_i(a_i, \omega_i) \equiv v_i(a_i, \omega_i) - v_i(a_i - 1, \omega_i),$$

implying player $i's$ valuation for the (a_i)-th unit consumption.

Assumption 1: For every $i \in N$, every $\omega_i \in [0,1]$, and every $a_i \in \{1,...,e\}$,

(2) $$w_i(a_i, \omega_i) \geq 0,$$

and

(3) $$\frac{\partial w_i(a_i, \omega_i)}{\partial \omega_i} \geq 0.$$

The inequalities (2) imply *free disposal* in that his valuation for any unit consumption is non-negative. From (2), it follows that

(4) $v_i(a_i, \omega_i)$ is non-decreasing in $a_i \in \{0,...,e\}$.

The inequalities (3) imply that the higher each player's type is, the higher his unit valuation is. From (3), it follows that

(5) $\dfrac{\partial v_i(a_i, \omega_i)}{\partial \omega_i}$ is non-decreasing in $a_i \in \{0,...,e\}$.

From (3), it follows that v_i satisfies *increasing difference* in that for every $(a_i, a_i') \in \{0,...,e\}^2$ and every $(\omega_i, \omega_i') \in [0,1]^2$,

(6) $v_i(a_i', \omega_i) - v_i(a_i, \omega_i) \leq v_i(a_i', \omega_i') - v_i(a_i, \omega_i')$ if $a_i < a_i'$ and $\omega_i < \omega_i'$.

From (3), it follows that

(7) $\dfrac{\partial v_i(a_i, \omega_i)}{\partial \omega_i} \geq 0$ for all $a_i \in \{1,...,e\}$.

A *direct revelation mechanism*, shortly a *mechanism*, is defined as (f, x), where $f = (f_i)_{i \in N} : [0,1]^n \to A$ is an *allocation rule*, $x = (x_i)_{i \in N} : [0,1]^n \to R^n$ is a *payment rule*, $f_i : [0,1]^n \to \{0,...,e\}$, and $x_i : [0,1]^n \to R$. We denote by F the set of all allocation rules. We denote by X the set of all payment rules. We require *dominant strategy incentive compatibility* as follows.

Dominant Strategy Incentive compatibility (DIC): For every $i \in N$ and every $\omega \in [0,1]^n$,

$$v_i(f_i(\omega), \omega_i) + x_i(\omega) \geq v_i(f_i(\omega_i', \omega_{-i}), \omega_i) + x_i(\omega_i', \omega_{-i}) \quad \text{for all} \quad \omega_i' \in [0,1].$$

Let us denote by $\tilde{F} \subset F$ the set of all allocation rules f satisfying that for every $i \in N$, f_i is *non-decreasing* in $\omega_i \in [0,1]$.

Lemma 1: *With Assumption 1, a mechanism* (f, x) *satisfies DIC if and only if*

$$f \in \tilde{F},$$

and for every $(i, \omega) \in N \times [0,1]^n$,

(8) $$x_i(\omega) = \int_{s_i=0}^{\omega_i} \frac{\partial v_i(f_i(s_i, \omega_{-i}), s_i)}{\partial s_i} ds_i - v_i(f_i(\omega), \omega_i) + D_i(\omega_{-i}),$$

where $D_i : [0,1]^{n-1} \to R$ *is an arbitrary function.*

Proof: It is clear from the envelope theorem in the auction theory literature (See Milgrom (2004) and Krishna (2010), for instance) that the equalities (8) are necessary for (f, x) to satisfy DIC. Moreover, if (f, x) satisfies DIC, then, for every $(\omega, \omega_i') \in [0,1]^{n+1}$,

$$v_i(f_i(\omega), \omega_i) - v_i(f_i(\omega_i', \omega_{-i}), \omega_i) \geq x_i(\omega_i', \omega_{-i}) - x_i(\omega)$$
$$\geq v_i(f_i(\omega), \omega_i') - v_i(f_i(\omega_i', \omega_{-i}), \omega_i'),$$

which along with (6) implies $f \in \tilde{F}$.

Suppose that (f, x) satisfies $f \in \tilde{F}$ and (8). Then, from (5) and (7), it follows that for every $i \in N$, and every $(\omega_i, \omega_i') \in [0,1]^2$, if $\omega_i > \omega_i'$, then

$$v_i(f_i(\omega), \omega_i) + x_i(\omega) - \{v_i(f_i(\omega_i', \omega_{-i}), \omega_i') + x_i(\omega_i', \omega_{-i})\}$$
$$= \int_{s_i=\omega_i'}^{\omega_i} \frac{\partial v_i(f_i(s_i, \omega_{-i}), s_i)}{\partial s_i} ds_i \leq v_i(f_i(\omega), \omega_i) - v_i(f_i(\omega), \omega_i'),$$

which implies that

$$v_i(f_i(\omega),\omega_i')+x_i(\omega)\le v_i(f_i(\omega_i',\omega_{-i}),\omega_i')+x_i(\omega_i',\omega_{-i})\,.$$

In the same manner, it follows that

$$v_i(f_i(\omega),\omega_i)+x_i(\omega)-\{v_i(f_i(\omega_i',\omega_{-i}),\omega_i')+x_i(\omega_i',\omega_{-i})\}$$

$$=\int_{s_i=\omega_i'}^{\omega_i}\frac{\partial v_i(f_i(s_i,\omega_{-i}),s_i)}{\partial s_i}ds_i\ge v_i(f_i(\omega_i',\omega_{-i}),\omega_i)-v_i(f_i(\omega_i',\omega_{-i}),\omega_i')\,,$$

which implies that

$$v_i(f_i(\omega),\omega_i)+x_i(\omega)\ge v_i(f_i(\omega_i',\omega_{-i}),\omega_i')+x_i(\omega_i',\omega_{-i})\,.$$

Hence, we have proven that (f,x) satisfies DIC.

Q.E.D.

Each player has the *outside option* not to participate in the allocation problem and instead consume his initial endowment by himself. Each player can exercise this option at any time in the ex post stage. Hence, we require for a mechanism *ex post individual rationality* (EPIR) in that any player never wants to exercise this option in the ex post stage; the mechanism guarantees him at least the same value as his type-dependent outside opportunity value that is given by $v_i(e_i,\omega_i)$.

Ex Post Individual Rationality (EPIR): For every $i\in N$ and every $\omega\in[0,1]^n$,

$$v_i(f_i(\omega),\omega_i)+x_i(\omega)\ge v_i(e_i,\omega_i)\,.$$

Let us fix an arbitrary subset $\hat{A}\subset A$ as the set of *feasible* allocations; we assume that the profile of the players' initial endowments is feasible, i.e., $(e_i)_{i\in N}\in\hat{A}$. An allocation rule $f\in F$ is said to be *feasible* if

$$f(\omega)\in\hat{A}\ \text{ for all }\ \omega\in[0,1]^n.$$

Let us denote by a non-empty subset $\hat{F}\subset F$ the set of all feasible allocation rules. We assume that the central planner is restricted to select any allocation rule f from this subset $\hat{F}$.

3. Revenue-Maximization

The central planner attempts to maximize his expected revenue by selecting any mechanism under the constraints of DIC and EPIR. We define the optimization problem concerning the central planner's expected revenue as

(9) $$\max_{(f,x)\in\hat{F}\times X} E[-\sum_{i\in N} x_i(\omega)] \text{ subject to DIC and EPIR.}$$

Let us define the *virtual valuation* for player $i \in N$ associated with $(a_i, \omega_i) \in \{0,...,e\}\times[0,1]$ as

$$u_i(a_i,\omega_i) \equiv v_i(a_i,\omega_i) - \frac{1-P_i(\omega_i)}{p_i(\omega_i)} \frac{\partial v_i(a_i,\omega_i)}{\partial \omega_i}.$$

We further define the *modified virtual valuation* (MVV) for player $i \in N$ associated with $(a_i, \omega_i) \in \{0,...,e\}\times[0,1]$ as

$$u_i^*(a_i,\omega_i) = u_i(a_i,\omega_i) \qquad \text{if } a_i \geq e_i,$$

and

$$u_i^*(a_i,\omega_i) = u_i(a_i,\omega_i) - \frac{1}{p_i(\omega_i)}\{\frac{\partial v_i(e_i,\omega_i)}{\partial \omega_i} - \frac{\partial v_i(a_i,\omega_i)}{\partial \omega_i}\} \text{ if } a_i < e_i.$$

Note that the value of

$$\frac{1}{p_i(\omega_i)}\{\frac{\partial v_i(e_i,\omega_i)}{\partial \omega_i} - \frac{\partial v_i(a_i,\omega_i)}{\partial \omega_i}\}$$

implies player $i's$ *bargaining rent* induced by his outside option. When the central planner assigns to player i lesser than his initial endowment, in order to prevent him from exercising his outside option, the central planner has to make up for the loss that this player takes by paying back to him this bargaining rent.

Theorem 2: *With Assumption 1, a mechanism* (f, x) *is the solution to the optimization problem (9) if and only if*

$$f \in \hat{F} \cap \tilde{F},$$

(10) $$E[\sum_{i\in N} u_i^*(f_i(\omega),\omega_i)] \geq E[\sum_{i\in N} u_i^*(g_i(\omega),\omega_i)] \quad \textit{for all} \quad g \in \hat{F}\cap\tilde{F},$$

and x *is given by (8), where for every* $i \in N$ *and every* $\omega_{-i} \in [0,1]^{n-1}$,

(11) $$D_i(\omega_{-i}) = \max_{\omega_i}\{v_i(e_i,\omega_i) - \int_{s_i=0}^{\omega_i} \frac{\partial v_i(f_i(s_i,\omega_{-i}),s_i)}{\partial s_i} ds_i\},$$

that is,

(12) $$x_i(\omega) = \int_{s_i=0}^{\omega_i} \frac{\partial v_i(f_i(s_i,\omega_{-i}),s_i)}{\partial s_i} ds_i - v_i(f_i(\omega),\omega_i)$$

$$+\max_{\omega_i}\{v_i(e_i,\omega_i) - \int_{s_i=0}^{\omega_i} \frac{\partial v_i(f_i(s_i,\omega_{-i}),s_i)}{\partial s_i} ds_i\}.$$

Proof: It is clear from Lemma 1 that if (f,x) is the solution to the optimization problem (9), it must satisfy $f \in \hat{F} \cap \tilde{F}$, (8), and (11).

Suppose that $f \in \hat{F} \cap \tilde{F}$. For each $i \in N$ and each $\omega_{-i} \in [0,1]^{n-1}$, let us define $\omega_i(\omega_{-i}) \in [0,1]$ by

$$\omega_i(\omega_{-i}) = 1 \quad \text{if } f_i(\omega) < e_i \text{ for all } \omega_i \in [0,1],$$

$$\omega_i(\omega_{-i}) = 0 \quad \text{if } f_i(\omega) > e_i \text{ for all } \omega_i \in [0,1],$$

and

$$f_i(\omega_i(\omega_{-i}),\omega_{-i}) = e_i \text{ and } f_i(\omega) < e_i \text{ for all } \omega_i \in [0,\omega_i(\omega_{-i})) \text{ otherwise.}$$

Player i is assigned less than his initial endowment if and only if $\omega_i < \omega_i(\omega_{-i})$. From (1), it follows that $\omega_i = \omega_i(\omega_{-i})$ maximizes the value of

$$v_i(e_i,\omega_i) - \int_{s_i=0}^{\omega_i} \frac{\partial v_i(f_i(s_i,\omega_{-i}),s_i)}{\partial s_i} ds_i.$$

Hence, from (11),

$$D_i(\omega_{-i}) = \int_{s_i=0}^{\omega_i(\omega_{-i})} \{\frac{\partial v_i(e_i,s_i)}{\partial s_i} - \frac{\partial v_i(f_i(s_i,\omega_{-i}),s_i)}{\partial s_i}\} ds_i,$$

and therefore,

$$x_i(\omega) = \int_{s_i=0}^{\omega_i} \frac{\partial v_i(f_i(s_i,\omega_{-i}),s_i)}{\partial s_i} ds_i - v_i(f_i(\omega),\omega_i)$$

$$+\int_{s_i=0}^{\omega_i(\omega_{-i})} \{\frac{\partial v_i(e_i,s_i)}{\partial s_i} - \frac{\partial v_i(f_i(s_i,\omega_{-i}),s_i)}{\partial s_i}\} ds_i.$$

Let us specify $z_i : \{0,...,e\} \times [0,1] \to R$ by

$$z_i(a_i, \omega_i) = 0 \qquad \text{if } a_i \geq e_i,$$

and

$$z_i(a_i, \omega_i) = \frac{\partial v_i(e_i, \omega_i)}{\partial \omega_i} - \frac{\partial v_i(a_i, \omega_i)}{\partial \omega_i} \qquad \text{if } a_i < e_i,$$

implying player $i's$ bargaining rent, i.e.,

$$u_i^*(a_i, \omega_i) = u_i(a_i, \omega_i) - \frac{z_i(a_i, \omega_i)}{p_i(\omega_i)}.$$

Hence,

$$E[x_i(\omega) \mid \omega_{-i}] = \int_{\omega_i=0}^{1} \{ \int_{s_i=0}^{\omega_i} \frac{\partial v_i(f_i(s_i, \omega_{-i}), s_i)}{\partial s_i} ds_i - v_i(f_i(\omega), \omega_i)\} p_i(\omega_i) d\omega_i$$

$$+ \int_{\omega_i=0}^{1} z_i(f_i(\omega), \omega_i) d\omega_i$$

$$= \int_{\omega_i=0}^{1} [\frac{\partial v_i(f_i(\omega), \omega_i)}{\partial \omega_i} \{\frac{1-P_i(\omega_i)}{p_i(\omega_i)}\} - v_i(f_i(\omega), \omega_i) + \frac{z_i(f_i(\omega), \omega_i)}{p_i(\omega_i)}] p_i(\omega_i) d\omega_i$$

$$= \int_{\omega_i=0}^{1} \{-u_i(f_i(\omega), \omega_i) + \frac{z_i(f_i(\omega), \omega_i)}{p_i(\omega_i)}\} p_i(\omega_i) d\omega_i$$

$$= -\int_{\omega_i=0}^{1} u_i^*(f_i(\omega), \omega_i) p_i(\omega_i) d\omega_i .$$

From these observations, we have proven that

$$E[-\sum_{i \in N} x_i(\omega)] = E[\sum_{i \in N} u_i^*(f(\omega), \omega_i)].$$

Clearly, the inequalities (10) imply the solution to the optimization problem (9).

Q.E.D.

From the proof of Theorem 2, it must be noted that the expected revenue induced by the solution (f, x) to the optimization problem (9) is equivalent to the expected value of the sum of the players' MVVs, i.e.,

$$E[-\sum_{i \in N} x_i(\omega)] = E[\sum_{i \in N} u_i^*(f(\omega), \omega_i)].$$

It is substantial in this paper to note that Theorem 2 generally holds irrespective of what kind of restrictions are imposed on the set of feasible allocations $\hat{A}$. This generality plays a crucial role in investigating an important class of allocation problems with multiple heterogeneous items, the detail of which will be explained in Section 5.

4. Regularity

The optimization problem (9) is said to be *regular* if there exists a feasible and non-decreasing allocation rule $f \in \hat{F} \cap \tilde{F}$ such that

(13) $$\sum_{i \in N} u_i^*(f_i(\omega), \omega_i) \geq \sum_{i \in N} u_i^*(a_i, \omega_i) \quad \text{for all} \quad a \in \hat{A} \quad \text{and all} \quad \omega \in [0,1]^n .$$

With regularity, we can replace (10) with (13), which can simplify the optimization problem; from Theorem 2, it is clear that a mechanism (f, x) is the solution to the optimization problem (9) if and only if $f \in \hat{F} \cap \tilde{F}$ and it satisfies (12) and (13). Based on this observation, this section will demonstrate a sufficient condition for this regularity.

We define the *unit virtual valuation* for player $i \in N$ associated with each $(\omega_i, a_i) \in [0,1] \times \{1, ..., e\}$ as

$$y_i(a_i, \omega_i) \equiv u_i(a_i, \omega_i) - u_i(a_i - 1, \omega_i) .$$

We further define the *modified unit virtual valuation* (MUVV) for player $i \in N$ associated with each $(\omega_i, a_i) \in [0,1] \times \{1, ..., e\}$ as

$$y_i^*(a_i, \omega_i) \equiv u_i^*(a_i, \omega_i) - u_i^*(a_i - 1, \omega_i) .$$

Note that

$$y_i^*(a_i, \omega_i) = w_i(a_i, \omega_i) - \frac{1 - P_i(\omega_i)}{p_i(\omega_i)} \frac{\partial w_i(a_i, \omega_i)}{\partial \omega_i} \qquad \text{if} \quad a_i > e_i ,$$

and

$$y_i^*(a_i, \omega_i) = w_i(a_i, \omega_i) + \frac{P_i(\omega_i)}{p_i(\omega_i)} \frac{\partial w_i(a_i, \omega_i)}{\partial \omega_i} \qquad \text{if} \quad a_i \leq e_i .$$

The above defined MUVV implies *marginal revenue*, $\dfrac{\frac{\partial}{\partial \omega_i}\{(1 - P_i(\omega_i) w_i(\omega_i)\}}{p_i(\omega_i)}$, if player i purchases additional units ($a_i > e_i$), whereas it implies *marginal cost*

$\dfrac{\frac{\partial}{\partial \omega_i}\{P_i(\omega_i)w_i(\omega_i)\}}{p_i(\omega_i)}$ if he sells his endowment ($a_i \le e_i$). We assume the following mild properties of *monotonicity* for MUVV.[7]

Assumption 2: For every $i \in N$,

(14) $y_i^*(a_i,\omega_i)$ is non-increasing in $a_i \in \{0,...,e-1\}$ for all $\omega_i \in [0,1]$,

(15) $y_i^*(a_i,\omega_i)$ is non-decreasing in $\omega_i \in [0,1]$ for all $a_i \in \{0,...,e\}$.

Theorem 3: *With Assumptions 1 and 2, the optimization problem (9) is regular.*

Proof: Suppose that there exists no allocation rule that is included in $\hat{F} \cap \tilde{F}$ and satisfies (13). Then, there exists $f \in \hat{F}$ that satisfies (13) but is not included in $\tilde{F}$. Without loss of generality, we can assume that there exist $i \in N$, $\omega \in [0,1]^n$, and $\omega_i' > \omega_i$ such that

$$f_i(\omega) > f_i(\omega_i', \omega_{-i}),$$

$$\sum_{j\in N} u_j^*(f_j(\omega),\omega_j) \ge \sum_{j\in N} u_j^*(f_j(\omega_j',\omega_{-j}),\omega_j),$$

and

$$u_i^*(f_i(\omega),\omega_i') + \sum_{j\in N\setminus\{i\}} u_j^*(f_j(\omega),\omega_j)$$
$$\le u_i^*(f_i(\omega_i',\omega_{-i}),\omega_i') + \sum_{j\in N\setminus\{i\}} u_j^*(f_j(\omega_j',\omega_{-j}),\omega_j).$$

We can also assume without loss of generality that one of the last two inequalities strictly holds. Hence, it follows that

$$u_i^*(f_i(\omega),\omega_i) - u_i^*(f_i(\omega_i',\omega_{-i}),\omega_i) > u_i^*(f_i(\omega),\omega_i') - u_i^*(f_i(\omega_i',\omega_{-i}),\omega_i'),$$

that is,

$$\sum_{a_i = f_i(\omega_i',\omega_{-i})}^{f_i(\omega)} y_i^*(a_i,\omega_i) > \sum_{a_i = f_i(\omega_i',\omega_{-i})}^{f_i(\omega)} y_i^*(a_i,\omega_i').$$

[7] The requirement in Assumption 2 is independent of the specification of $\hat{A}$.

This implies $\omega_i > \omega_i'$, because of (15). This is a contradiction.

Q.E.D.

5. Heterogeneous Items

This section investigates an important class of allocation problems concerning multiple (not homogeneous but) *heterogeneous* items that is associated with the multiplicative structures explored by Mussa and Rosen (1978).[8] We will demonstrate a tractable manner for modeling this class as a special case of the analysis in the previous sections.

Let us denote by $L \equiv \{1,...,m\}$ a finite and nonempty set of heterogeneous items, where $m \geq n$. Each player $i \in N$ possesses a single item $l_i \in L$ as his initial endowment, where

$$l_i \neq l_j \text{ for all } i \in N \text{ and all } j \in N \setminus \{i\}.$$

According to Mussa and Rosen, let us denote by $q(l) \in R_+ \bigcup \{0\}$ the *quality* of each item $l \in L$; each player i with type ω_i has valuation for each item l that is expressed by a linear form of

$$q(l)\omega_i .$$

For convenience, we assume that

$$q(1) \geq q(2) \geq \cdots \geq q(m).$$

The central planner attempts to maximize his expected revenue on the assumption of *single-item demands* in that each player is assigned just a single item.

We can model the above problem as a special case of our allocation problems in the following manner. For convenience of arguments, we assume that there exist a positive real number $\alpha > 0$ and a non-negative integer $e(l)$ for each $l \in L$ such that

$$q(l) = \alpha e(l) \text{ for all } l \in N .$$[9]

We can treat each item $l \in L$ just like the bundle of $q(l)$ units of some homogeneous commodity. Let us specify

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[8] There are related works to Mussa and Rosen in auction theory such as Edelman, Ostrovsky, and Schwarz (2007), Varian (2007), Gershkov and Moldovanu (2009), Edelman and Schwarz (2010), and Athey and Ellison (2011).

[9] However, we can eliminate this assumption without any substantial change, because any profile of qualities $(q(l))_{l \in L}$ can be approximately described by a well-selected $(\alpha, (e(l))_{l \in L})$.

$$e = \sum_{l=1}^{m} e(l)\text{, and } \ e_i = e(l_i) \ \text{ for all } \ i \in N .$$

The set of feasible allocations $\hat{A}$ is specified as the subset of A such that $a \in \hat{A}$ if and only if there exists a function $\mu : N \to L$ satisfying that

$$\mu(i) \neq \mu(j) \ \text{ for all } \ i \in N \ \text{ and all } \ j \in N \setminus \{i\},$$

and

$$a_i = e(\mu(i)) \ \text{ for all } \ i \in N .$$[10]

Each player $i's$ valuation function v_i is specified as

$$v_i(a_i, \omega_i) = \alpha a_i \omega_i \ \text{ for all } \ (a_i, \omega_i) \in \{0,...,e\} \times [0,1] .$$

From the above specifications, it follows that

$$w_i(a_i, \omega_i) = \alpha \omega_i \ \text{ for all } \ (a_i, \omega_i) \in \{1,...,e\} \times [0,1] ,$$

$$u_i^*(a_i, \omega_i) = \alpha a_i \{\omega_i - \frac{1 - P_i(\omega_i)}{p_i(\omega_i)}\} \qquad \text{if } \ a_i \geq e_i ,$$

$$u_i^*(a_i, \omega_i) = \alpha [a_i \{\omega_i + \frac{P_i(\omega_i)}{p_i(\omega_i)}\} - \frac{e_i}{p_i(\omega_i)}] \qquad \text{if } \ a_i < e_i ,$$

$$y_i^*(a_i, \omega_i) = \alpha \{\omega_i - \frac{1 - P_i(\omega_i)}{p_i(\omega_i)}\} \qquad \text{if } \ a_i \geq e_i ,$$

and

$$y_i^*(a_i, \omega_i) = \alpha \{\omega_i + \frac{P_i(\omega_i)}{p_i(\omega_i)}\} \qquad \text{if } \ a_i < e_i .$$

Assumption 1 and (14) automatically hold. We assume that the property (15) holds in the strict sense, i.e., $y_i^*(a_i, \omega_i)$ is increasing in ω_i .

Based on the replacement of the allocation problem for multiple heterogeneous items with the allocation problem for multiple homogeneous commodities with restrictions on the set of feasible allocations in the above manner, the solution to the central planner’s revenue-maximization in the former problem could be well described by the solution $(f, x) = (\hat{f}, \hat{x})$ to the optimization problem (9) in the latter problem. Since $y_i^*(a_i, \omega_i)$ is increasing in ω_i, we can assume without loss of generality that for

[10] Note that the profile of their initial endowments is feasible.

every $i \in N$, there exists $\hat{\omega}_i : L \times [0,1]^{n-1} \to [0,1]$ such that $\hat{\omega}_i(l, \omega_{-i})$ is decreasing in $l \in L$, and for every $\omega \in [0,1]^n$,

$$\hat{f}_i(\omega) = e(1) \qquad \text{if} \quad \omega_i > \hat{\omega}_i(1, \omega_{-i}),$$

$$\hat{f}_i(\omega) = 0 \qquad \text{if} \quad \omega_i < \hat{\omega}_i(m, \omega_{-i}),$$

and for every $l \in \{2,...,m\}$,

$$\hat{f}_i(\omega) = e(l) \qquad \text{if} \quad \hat{\omega}_i(l-1, \omega_{-i}) > \omega_i > \hat{\omega}_i(l, \omega_{-i}).$$

Let us call $\hat{\omega}_i(l, \omega_{-i})$ the $(l)-th$ *pivotal* type for player i; he obtains item l or any better item if and (almost) only if his type is greater than $\hat{\omega}_i(l, \omega_{-i})$.

Theorem 4: *For every* $i \in N$, *every* $\omega \in [0,1]^n$, *and every* $l \in \{2,...,m\}$, *let us suppose that* $\hat{f}_i(\omega) = q(l)$. *Then, it follows that*

$$\hat{x}_i(\omega) = 0 \qquad \textit{if} \quad \hat{f}_i(\omega) = e_i, \textit{ i.e., } l = l_i,$$

$$\hat{x}_i(\omega) = -\sum_{k=l}^{l_i-1} \{q(k) - q(k+1)\} \hat{\omega}_i(k, \omega_{-i}) \qquad \textit{if} \quad \hat{f}_i(\omega) > e_i, \textit{ i.e., } l < l_i,$$

and

$$\hat{x}_i(\omega) = \sum_{k=l_i}^{l-1} \{q(k) - q(k+1)\} \hat{\omega}_i(k, \omega_{-i}) \qquad \textit{if} \quad \hat{f}_i(\omega) < e_i, \textit{ i.e., } l > l_i.$$

Proof: From (12) and $\hat{f}_i(\omega) = q(l)$, it follows that

$$\hat{x}_i(\omega) = -\alpha \sum_{k=l}^{m} \{e(k) - e(k+1)\} \hat{\omega}_i(k, \omega_{-i}) + \alpha \sum_{k=l_i}^{m} \{e(k) - e(k+1)\} \hat{\omega}_i(k, \omega_{-i}),$$

which along with $q(k) = \alpha e(k)$ implies this theorem.

Q.E.D.

If each player i purchases a better item $l < l_i$ than his initial endowment, then he *pays* the pivotal type $\hat{\omega}_i(k, \omega_{-i})$ for any item k between his initial endowment and his purchased item ($l \le k < l_i$) multiplied by the increase in quality $q(k) - q(k+1)$. If each player i purchases a worse item $l > l_i$ than his initial endowment, then he *earns* the pivotal type $\hat{\omega}_i(k,\omega_{-i})$ for any item k between his initial endowment and his purchased item ($l_i \leq k < l$) multiplied by the increase in quality $q(k) - q(k+1)$.

Let us investigate a more special case in which there exists a positive integer $\tilde{m} \in \{1,...,m\}$ such that

$$\tilde{m} < n,$$

$$q(l) = 0 \quad \text{for all} \quad l \in \{\tilde{m}+1,...,m\},$$

and

$$q(l) > 0 \quad \text{for all} \quad l \in \{1,...,\tilde{m}\}.$$

Any item that is included in $\{1,...,\tilde{m}\}$ implies a non-trivial item, while any item that is included in $\{\tilde{m}+1,...,m\}$ implies a null item. For each $i \in N$ and each $\omega_{-i} \in [0,1]^{n-1}$, let us define $j(\omega_{-i}) \in N \setminus \{i\}$ as the player who has the $(\tilde{m})$-th highest MUVV among all players except for player i;

$$y_i^*(1,\omega_h) \geq y_i^*(1,\omega_{j(\omega_{-i})}) \quad \text{for at least} \quad \tilde{m} \quad \text{players} \quad h \quad \text{in} \quad N \setminus \{i\},$$

and

$$y_i^*(1,\omega_h) \leq y_i^*(1,\omega_{j(\omega_{-i})}) \quad \text{for at least} \quad n-\tilde{m} \quad \text{players} \quad h \quad \text{in} \quad N \setminus \{i\}.$$

Let us define $\tilde{\omega}_i(\omega_{-i}) \in [0,1]$ by

$$y_i^*(1,\tilde{\omega}_i(\omega_{-i})) = y_{j(\omega_{-i})}^*(1,\omega_{j(\omega_{-i})}) \qquad \text{if} \quad y_{j(\omega_{-i})}^*(1,\omega_{j(\omega_{-i})}) \geq 0,$$

and

$$y_i^*(1,\tilde{\omega}_i(\omega_{-i})) = 0 \qquad \text{if} \quad y_{j(\omega_{-i})}^*(1,\omega_{j(\omega_{-i})}) < 0.$$

Note that player i has the $(\tilde{m})$-th highest or even higher MUVV if and only if

$$\omega_i \geq \tilde{\omega}_i(\omega_{-i}).$$

Hence, it follows from Theorem 4 that

$$\hat{\omega}_i(\tilde{m},\omega_{-i}) = \tilde{\omega}_i(\omega_{-i}) \quad \text{for all} \quad i \in N,$$

implying that any player i is assigned a non-trivial item if and (almost) only if his type is greater than $\tilde{\omega}_i(\omega_{-i})$.

It must be noted that if any non-trivial item has the same quality, i.e.,

$$q(l) = q(1) \quad \text{for all} \quad l \in \{1,...,\tilde{m}\},$$

then it follows that

$$\hat{x}_i(\omega) = 0 \qquad \text{if } \hat{f}_i(\omega) = e_i,$$

$$\hat{x}_i(\omega) = q(1)\tilde{\omega}_i(\omega_{-i}) \qquad \text{if } \hat{f}_i(\omega) > e_i,$$

and

$$\hat{x}_i(\omega) = -q(1)\tilde{\omega}_i(\omega_{-i}) \qquad \text{if } \hat{f}_i(\omega) < e_i.$$

This case can be regarded this case as a generalization of Theorem 4 in Myerson and Satterthwaite (1983). The interpretation is as follows; the central planner bids the price $q(1)\tilde{\omega}_i(\omega_{-i})$ to any player i who has a non-trivial item. This player sells his item to the central planner if and (almost) only if his type ω_i is less than $\tilde{\omega}_i(\omega_{-i})$. On the other hand, the central planner asks the price $q(1)\tilde{\omega}_i(\omega_{-i})$ to any player i who has only a null item. This player purchases a non-trivial item from the central planner if and (almost) only if his type ω_i is greater than $\tilde{\omega}_i(\omega_{-i})$.

Let us further specify the allocation problem for multiple heterogeneous items as the problem of *auction* with single-item demands, where we assume that all players have only null items, i.e.,

$$e_i = 0 \text{ for all } i \in N.$$

It is clear in this case that for every $\omega \in [0,1]^n$, every $i \in N$, every $j \in N \setminus \{i\}$, and $l \in \{1,...,m\}$,

$$\omega_i - \frac{1-P_i(\omega_i)}{p_i(\omega_i)} \geq \omega_j - \frac{1-P_j(\omega_j)}{p_j(\omega_j)} \qquad \text{if } \hat{f}_i(\omega) > \hat{f}_j(\omega).$$

Hence, the $(l)-th$ pivotal type $\hat{\omega}_i(l,\omega_{-i})$ is regarded as the type ω_i such that $\omega_i - \frac{1-P_i(\omega_i)}{p_i(\omega_i)}$ is equivalent to the $(l)-th$ largest $\omega_j - \frac{1-P_j(\omega_j)}{p_j(\omega_j)}$ among the other players $j \neq i$. Let us define $\underline{\omega}_i \in (0,1)$ by

$$\underline{\omega}_i - \frac{1-P_i(\underline{\omega}_i)}{p_i(\underline{\omega}_i)} = 0,$$

implying the *reserve price* for player i. Clearly,

$$\hat{f}_i(\omega) = 0 \qquad \text{if } \omega_i < \underline{\omega}_i.$$

Hence, it follows that for every $(i,\omega) \in N \times [0,1]^n$,

$$\hat{x}_i(\omega) = 0 \qquad \text{if} \quad \hat{f}_i(\omega) = 0,$$

and for every $l \in \{1,\ldots,m\}$,

$$\hat{x}_i(\omega) = -\sum_{k=l}^{m}\{q(k) - q(k+1)\}\hat{\omega}_i(k,\omega_{-i}) \qquad \text{if} \quad \hat{f}_i(\omega) = l.$$

The optimal mechanism $(\hat{f},\hat{x})$ in this case can be regarded as a generalization of the optimal sponsored search auction addressed by Edelman and Schwarz (2010); the asymmetry in terms of type distribution is explicitly taken into account in the present paper.

6. Conclusion

This paper investigated the allocation problems termed multiunit exchanges with restrictions on the set of feasible allocations, where both the central planner and the participants brought homogeneous commodities to sell altogether. We introduced the concept termed modified virtual valuation (MVV), which was defined as the valuation minus the bargaining rent as well as the informational rent, the unit term of which (MUVV) implied a hybrid of marginal revenue and marginal cost. We made standard assumptions such as quasi-linearity, private values, independent type distributions, risk-neutrality, and no externality. We then demonstrated a tractable characterization of the optimal mechanism design in terms of the central planner’s expected revenue under the constraints of dominant strategy incentive compatibility and ex post individual rationality. With a mild monotonicity assumption, irrespective of the restrictions imposed on the set of feasible allocations, this optimization problem could be replaced with the maximization of the sum of MVVs in the ex post term. Hence, we could successfully apply these results to an important class of allocation problems for multiple heterogeneous items with single-item demands that was associated with the multiplicative structure explored by Mussa and Rosen (1978).

This paper required incentive compatibility and individual rationality in the ex-post term. Gershkov et al (2012) showed that any Bayesian incentive compatible mechanism with interim individual rationality could be implemented by a dominant strategy incentive compatible mechanism with interim individual rationality. However, it is not necessarily possible for any Bayesian incentive compatible mechanism with interim individual rationality to be implemented by a dominant strategy incentive compatible mechanism with ex-post individual rationality. Hence, the effect of replacing the ex-post term with the interim term on the optimization problem remains unsolved.

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