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Auctioneer's Discretion in Combinatorial Auctions*

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Abstract

We argue that in order to achieve the VCG outcomes in combinatorial auctions, the auctioneer does not need to make a full contractual agreement on the protocol with participants. We can leave the detail of its design to the auctioneer's discretion. The auctioneer can even make it contingent on unverifiable information. We consider general dynamical protocols termed price-demand procedures, and introduce representative valuation functions, which are, with connectedness and revealed preferences, calculated easily from the occurred history. It is sufficient to examine whether the efficient allocations with and without any single buyer associated with the representative valuation functions were revealed.

Keywords: Combinatorial Auctions, Price-Demand Procedures, VCG Mechanisms, Connectedness, Representative Valuation Functions, Detail-Freeness

JEL Classification Numbers: D44, D47, D61, D82, D86

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1. Introduction

This paper investigates combinatorial auction problems wherein a single seller (auctioneer) sells multiple indivisible items to multiple buyers who have private and quasi-linear valuations. We investigate general open-bid dynamical auction protocols a la clock auctions in the continuous time horizon, termed *price-demand procedures*; the auctioneer continues to ask a price vector to each buyer and require this buyer to announce a demand correspondence. This paper examines whether an arbitrary given price-demand procedure can successfully gather sufficient information regarding the buyers' valuations for achieving the VCG outcomes, i.e., there exists an efficient and strategy-proof direct mechanism that is consistent with this procedure and is ex-post individually rational in a binding manner.[1].

This paper assumes that the considered price-demand procedure satisfies *connectedness* in the sense that the auctioneer does not make his (or her) asked price vector jump to a price vector that he has never asked before. With connectedness, we can automatically identify the difference in valuation between any pair of packages (subsets of items) that a buyer has revealed as elements of his (or her) demand correspondences before. According to this connectedness, we demonstrate the following tractable calculation method for elucidating whether there exists a VCG mechanism that is consistent with this price-demand procedure, and for deriving such a VCG mechanism in a concrete manner. Based on the history regarding price vectors and demand correspondences, we define the *representative valuation function* for each buyer by assigning any revealed package with the minimal relative valuations in a consistent manner with the history. At any time, irrespective of what has occurred in the history, the auctioneer can easily identify whether he has succeeded in gathering sufficient information for implementing the VCG outcome using this calculated profile of representative valuation functions alone. What the auctioneer has to do for the collection of sufficient information for the achievement of the VCG outcomes is to examine just whether the occurred history reveals the efficient allocations with and without any single buyer associated with the profile of representative valuation

[1] See Vickrey (1961), Clarke (1971), and Groves (1973). See also Rothkopf, Teisberg, and Kahn (1990), Milgrom (2004), Ausubel and Milgrom (2006), and Parkes (2006).

functions.

This result will play an important role in simplifying the contractual relationship between the auctioneer (seller) and the participants (buyers) from the viewpoint of contractual incompleteness and detail-freeness in mechanism design; we can leave the fine detail of procedure design to the auctioneer's discretion[2]. The following four requirements on the auctioneers fully describe what the auctioneer has to agree with the participants on; firstly, the auctioneer is required to continue to ask a price vector to each participant according to a price-demand procedure the selection of which the participants leave to the auctioneer's discretion. The auctioneer is not required to inform them of the fine detail of his selected procedure. Secondly, the auctioneer is required to make his asked price vectors consistent with connectedness. Thirdly, the auctioneer is required to continue to ask price vectors until the occurred history fully reveals the efficient allocations with and without any single buyer associated with the corresponding profile of representative valuation functions. Fourthly, the auctioneer is required to determine the VCG outcome associated with the profile of representative valuation functions.

It is possible for the participants to recognize whether these requirements to be satisfied even without knowing the detail of the selected procedure. Provided the auctioneer is penalized if he does not follow these requirements, the contractual agreement on these requirements sufficiently incentivizes the auctioneer to select a price-demand procedure that a VCG mechanism is consistent with, achieving the VCG outcomes. The success of incentivizing the auctioneer in this manner implies that the auctioneer, who prefers making the stopping time as quick as possible, can even make his procedure selection contingent on his unverifiable information concerning the participants' preferences.

The standard practice of VCG mechanisms, wherein buyers directly announce the entire valuations for the enormous number of packages, has flaws in terms of complexity and privacy preservation; it might be too complicated for any buyer to

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[2] Related works in the literature of incomplete contract are Maskin and Tirole (1999) and Tirole (1999), for instance. The detail freeness of auction protocol design compared with abstract mechanism design such as 'Wilson doctrine' was emphasized in the literature of auction design. See Krishna (2010), Matsushima (2005, 2008), for instance.

assess and report the entire valuations simultaneously. Moreover, any buyer might be concerned about preserving his privacy, because he is afraid that any information that is confidential but irrelevant to the decisions could be leaked to the public. In this case, it would be important to search for the possibility of replacing the standard practice with any (signal-contingent) price-demand procedure in order to make information gathering compatible with the addressing of practical issues; the sequential revelation of demand correspondences would be much easier to answer than the revelation of the entire valuations at once.[3]

Several works such as Kelso and Crawford (1982), Gul and Stacchetti (2000), Bikhchandani and Ostroy (2002), Ausubel (2006), Ausubel, Cramton, and Milgrom (2006), Parkes and Ungar (2002), Lahaie and Parkes (2004), Lahaie, Constantin, and Parkes (2005), and Mishra and Parkes (2007) have examined various concepts for open-bid dynamical auction protocols. Parkes and Ungar (2002), Lahaie, Constantin, and Parkes (2005), Lahaie and Parkes (2004), and Mishra and Parkes (2007) introduced an involved notion termed *universal competitive equilibrium*, and showed that the auctioneer can gather sufficient information for achieving the VCG outcomes if and only if the gathered information identifies a universal competitive equilibrium. The present paper provides a method for elucidating which universal competitive equilibrium to be identified; the profile of representative valuation functions could generally be a universal competitive equilibrium. Moreover, the profile of representative valuation functions along with the sets of all revealed packages could be the sufficient statistics for privacy preservation, i.e., for the extent to which the information regarding the buyers' valuations could be leaked.

In order to prevent the buyers from behaving strategically and promote their meaningful biddings, activity rules were proposed by several authors and incorporated into real situations. For example, see Milgrom (2004) and Ausubel, Cramton, and Milgrom (2006). Accordingly, the present paper assumes the *revealed preference* rule in the sense that any buyer is required to make his demand correspondences consistent with a valuation function.

It might be inevitable in general combinatorial auctions that ideally designed

[3] Note that the replacement of direct revelation to price-demand procedure does not imply the reduction of communication cost. See Segal (2006) and Nisan and Segal (2006).

protocols are too complicated to be understandable to the participants. On the other hand, it is not a very difficult problem for the participants to identify whether the protocol that the auctioneer designed in a discretional manner to successfully implement the VCG outcomes or not. This finding could calm the complexity issues from the contractual viewpoints.

The remainder of this paper is organized as follows: Section 2 models the combinatorial auction problem. Section 3 introduces the concepts of price-demand procedure. Section 4 introduces the concept of connectedness. Section 5 introduces the concept of representative valuation functions. Section 6 discusses the auctioneer's discretion. Section 7 concludes.

2. Model

Let us investigate a combinatorial auction problem wherein $l \geq 1$ multiple items are traded altogether. Let $L \equiv \{1,...,l\}$. The set of all buyers is denoted by $N \equiv \{1,...,n\}$, where $n \geq 2$. An allocation is defined as $a \equiv (a_1,...,a_n)$, where $a_i \subset L$ implies the package of items that is assigned to buyer i, and $a_i \cap a_j = \phi$ for $j \neq i$. Let A denote the set of all allocations. *An allocation without buyer* $i \in N$ is defined as $a^i \equiv (a_j)_{j \in N \setminus \{i\}}$, where $a_j \subset L$ and $a_j \cap a_h = \phi$ for $h \neq j$. Let A^i denote the set of all allocations without buyer $i \in N$.

We assume quasi-linearity. A *valuation function for buyer* $i \in N$ is defined as $u_i : 2^L \to R_+ \cup \{0\}$, where $u_i(\phi) = 0$, and

(1) $$u_i(a_i) > u_i(\tilde{a}_i) \text{ if } \tilde{a}_i \neq a_i \text{ and } \tilde{a}_i \subset a_i .$$

Let U_i denote the set of all valuation functions for buyer i. Let $U \equiv \prod_{i \in N} U_i$ and $U_{-i} \equiv \prod_{j \in N \setminus \{i\}} U_j$. An allocation $a \in A$ is said to be *efficient for* $u \in U$ if

$$\sum_{i \in N} u_i(a_i) \geq \sum_{i \in N} u_i(\tilde{a}_i) \text{ for all } \tilde{a} \in A .$$

Let $A^*(u) \subset A$ denote the set of all efficient allocations for $u \in U$. An allocation $a^i \in A^i$ without buyer i is said to be *efficient for* $u_{-i} = (u_j)_{j \in N \setminus \{i\}} \in U_{-i}$ if

$$\sum_{j\in N\setminus\{i\}} u_j(a_i) \geq \sum_{j\in N\setminus\{i\}} u_j(\tilde{a}_i) \quad \text{for all} \quad \tilde{a}^i \in A^i .$$

Let $A^{i*}(u_{-i}) \subset A^i$ denote the set of all efficient allocations without buyer i for $u_{-i} \in U_{-i}$.

A *direct mechanism*, hereinafter a *mechanism*, is defined as $G = (f, x)$, where $f : U \to A$ denotes an *allocation rule*, and $x : U \to R^n$ denotes a *payment rule*. Let us denote $f(u) = (f_i(u))_{i\in N} \in A$, $x = (x_i)_{i\in N}$, $x_i : U \to R$, and $x(u) = (x_i(u))_{i\in N} \in R^n$. A mechanism G is said to be *efficient* if

$$f(u) \in A^*(u) \quad \text{for all} \quad u \in U .$$

A mechanism G is said to be *VCG* if it is efficient and

$$x_i(u) = \max_{a^i \in A^i} \sum_{j\in N\setminus\{i\}} u_j(a_j) - \sum_{j\in N\setminus\{i\}} u_j(f_j(u)) \quad \text{for all} \quad i \in N \quad \text{and all} \quad u \in U .$$

Note that a VCG mechanism G is efficient and *strategy-proof* in the sense that for every $i \in N$, every $u \in U$, and every $\tilde{u}_i \in U_i$,

$$u_i(f(u)) - x_i(u) \geq u_i(f(\tilde{u}_i, u_{-i})) - x_i(\tilde{u}_i, u_{-i}) .$$

3. Price-Demand Procedures

A *price vector for buyer* $i \in N$ is denoted by $p_i = (p_i(a_i))_{a_i \subset L} \in R^{2^L}$, where $p_i(\phi) = 0$, and

(2) $$p_i(a_i) > p_i(\tilde{a}_i) \quad \text{if} \quad a_i \neq \tilde{a}_i \quad \text{and} \quad \tilde{a}_i \subset a_i .$$

Let P_i denote the set of all price vectors for buyer i. Let $p \equiv (p_i)_{i\in N} \in \prod_{i\in N} P_i$ denote a profile of price vectors.

Let us consider an open-bid dynamical auction protocol a la clock auction in the continuous time horizon $[0,\infty)$, termed price-demand procedure, wherein at any time $t \in [0,\infty)$, the auctioneer asks a price vector $p_i = p_i(t) \in P_i$ to each buyer $i \in N$ and requires this buyer to announce a demand correspondence $m_i = m_i(t) \subset 2^L$. A combination of price vector and demand correspondence $(p_i, m_i) \in P_i \times 2^L$ is said to be *consistent with* $u_i \in U_i$ for buyer i if m_i is the set of all best responses to p_i, i.e.,

$$m_i = \arg\max_{a_i \in 2^L} \{u_i(a_i) - p_i(a_i)\}.$$

A *history for each buyer* $i \in N$ *up to time* $t \in (0,\infty)$ is denoted by $h_i^t : [0,t) \to P_i \times 2^L$, where we denote $h_i^t(\tau) = (p_i(\tau), m_i(\tau))$. It is said to be *consistent with* $u_i \in U_i$ if $h_i^t(\tau)$ is consistent with u_i for all $\tau \in [0,t)$. Let h_i^0 denote the *null* history. Let $H_i^t(u_i)$ denote the set of all histories for buyer i up to time t that is consistent with u_i. Let $H_i^t \equiv \bigcup_{u_i \in U_i} H_i^t(u_i)$, $H^t \equiv \prod_{i \in N} H_i^t$, $H \equiv \bigcup_{t \in [0,\infty)} H^t$, $h^t = (h_i^t)_{i \in N} \in H^t$, and $H_i^0 \equiv \{h_i^0\}$. For every $h_i^t \in H_i^t$, let us define the set of all valuation functions for buyer i that h_i^t is consistent with by

$$U_i(h_i^t) \equiv \{u_i \in U_i \mid h_i^t \in H_i^t(u_i)\}.$$

For every $h_i^t \in H_i^t$, let us define the set of all packages for buyer i that he announces as his demand in the history h_i^t by

$$A_i(h_i^t) \equiv \{a_i \in A_i \mid a_i \in m_i(\tau) \text{ for some } \tau \in [0,t)\}.$$

Let $A(h^t) \equiv \prod_{i \in N} A_i(h_i^t)$ and $A^i(h^t) \equiv \prod_{j \in N \setminus \{i\}} A_j(h_j^t)$.

A *price-demand procedure* is defined by (γ, T), where $\gamma = (\gamma_i)_{i \in N}$ denotes a *pricing rule* and $T : U \to (0,\infty)$ denotes a *stopping rule*. At any time $t \in [0,\infty)$, where $h^t \in H^t$ has occurred, the auctioneer asks the price vector $\gamma_i(h^t) \in P_i$ to each buyer $i \in N$. Let $h^t = h^t(u,\gamma)$ denote the history up to time t that occurs when the buyers continue to announce their demand correspondences in a consistent manner with u:

$$h^t(u,\gamma) \in H^t(u), \text{ and } p_i(\tau) = \gamma_i(h^\tau) \text{ for all } i \in N \text{ and all } \tau \in [0,t),$$

where we denote $h^t(u,\gamma) = (h_i^t(u,\gamma))_{i \in N}$, and

$$h_i^t(u,\gamma)(\tau) = h_i^t(\tau) = (p_i(\tau), m_i(\tau)) \text{ for all } i \in N \text{ and all } \tau \in [0,t).$$

We assume the *revealed preference* rule in that any buyer is required to make his demand correspondences in a consistent manner with a valuation function. When the auctioneer follows the pricing rule γ and the buyers continue to make their demand correspondence in a consistent manner with $u \in U$, the auctioneer stops asking price vectors at the time given by $t = T(u) \in (0, \infty)$. In this case, the stopping rule T should be contingent only on the history; for every $\{u, u'\} \subset U$, if $h^{T(u)}(u, \gamma)$ is consistent with u', i.e., if $u' \in U(h^{T(u)}(u, \gamma))$, then it must hold that

$$T(u') = T(u), \text{ and } h^{T(u)}(u, \gamma) = h^{T(u')}(u', \gamma).$$

Let us define the set of all histories that can occur according to the revealed preference rule by

$$H(\gamma, T) \equiv \{h^t \in H \mid h^t = h^{T(u)}(u, \gamma) \textit{ for some } u \in U\}.$$

A mechanism $G = (f, x)$ is said to be consistent with a *price-demand procedur*e (γ, T) if for every $h^t \in H(\gamma, T)$ and every $\{u, u'\} \subset U(h^t)$,

$$(f(u), x(u)) = (f(u'), x(u')).$$

The following lemma shows that whenever a mechanism is efficient and consistent with a price-demand procedure, the allocation induced by this mechanism must be revealed.

Lemma 1: *If a mechanism* G *is efficient and consistent with a price-demand procedure* (γ, T)*, then*

$$f(u) \in A(h^{T(u)}(u, \gamma)) \quad \textit{for all} \quad u \in U.$$

Proof: For every $\varepsilon > 0$ and every $u \in U$, let us define $u_{i,\varepsilon} \in U_i$ as

$$u_{i,\varepsilon}(a_i) = u_i(a_i) + \varepsilon \quad \text{for all} \quad a_i \subset 2^L \setminus \{\phi\}.$$

Assume that there exists $u \in U$ and $i \in N$ such that $f_i(u) \notin A_i(h_i^{T(u)}(u, \gamma))$. Suppose that $f_i(u) = \phi \notin A_i(h_i^{T(u)}(u, \gamma))$. Then, $h_i^{T(u)}(u, \gamma)$ must be consistent with $u_{i,\varepsilon}$ for all $\varepsilon > 0$, i.e.,

$$u_{i,\varepsilon} \in U_i(h_i^{T(u)}(\gamma, T)) \quad \text{for all} \quad \varepsilon > 0,$$

which along with consistency and efficiency implies that

$$f(u_{i,\varepsilon}, u_{-i}) = f(u) \in A^*(u_{i,\varepsilon}, u_{-i}) \quad \text{for all} \quad \varepsilon > 0.$$

This is a contradiction, because any efficient allocation $a \in A^*(u_{i,\varepsilon}, u_{-i})$ for $(u_{i,\varepsilon}, u_{-i})$ never satisfies $a_i \neq \phi$, provided ε is sufficiently large.

Suppose that $f_i(u) \neq \phi$. Then, we can select $a_i \subset 2^L$ such that $a_i \neq f_i(u)$, $a_i \subset f_i(u)$, and that for every $a'_i \subset 2^L$ satisfying that $a'_i \notin \{a_i, f_i(u)\}$ and $a'_i \subset f_i(u)$,

$$u_i(f_i(u)) - u_i(a_i) \leq u_i(f_i(u)) - u_i(a'_i).$$

From (1), we can select $u'_i \neq u_i$ that $h_i^{T(u)}(\gamma, T)$ is consistent with, i.e.,

$$u'_i \in U_i(h_i^{T(u)}(\gamma, T)).$$

We can also select $j \in N \setminus \{i\}$ in a manner that $u'_i(f_i(u)) - u'_i(a_i)$ is close enough to zero, satisfying that

(3) $$u'_i(f_i(u)) - u'_i(a_i) < u_j(f_j(u) \cup f_i(u) \setminus a_i) - u_j(f_j(u)).$$

Let us specify $\hat{a} \in A$ by $\hat{a}_i = a_i$, $\hat{a}_j = f_j(u) \cup f_i(u) \setminus a_i$, and

$$\hat{a}_h = f_h(u) \text{ for all } h \in N \setminus \{i, j\}.$$

From (3),

$$u'_i(f_i(u)) + \sum_{h \in N \setminus \{i\}} u_h(f_h(u)) < u'_i(\hat{a}_i) + \sum_{h \in N \setminus \{i\}} u_h(\hat{a}_h),$$

implying that $f(u)$ is not efficient for (u'_i, u_{-i}). However, since $h_i^{T(u)}(u, \lambda)$ is consistent with u'_i, it must hold that $f(u'_i, u_{-i}) = f(u)$. This contradicts efficiency.

Q.E.D.

4. Connectedness

A history h_i^t for buyer i up to time t is said to be *connected* if for every $\tau \in (0, t)$, either $p_i(\tau) = \lim_{\tau' \uparrow \tau} p_i(\tau')$ or

$$p_i(\tau) = p_i(\tau') \text{ for some } \tau' \in (0, \tau).$$

The connectedness implies that the auctioneer never makes his asked price vector jump to any price vector that he has never asked before. The following lemma shows that with connectedness, the auctioneer can calculate the difference in valuation for any buyer between any pair of packages whenever this buyer revealed them.

Lemma 2: *For every connected history $h_i^t \in H_i^t$ and every $\{a_i, a'_i\} \subset A_i(h_i^t)$, there* *uniquely exists* $x_i(a_i, a_i', h_i^t) \in R$ *such that*

$$x_i(a_i, a_i', h_i^t) = u_i(a_i) - u_i(a_i') \text{ for all } u_i \in U_i(h_i^t).$$

Proof: Since h_i^t is connected, there exists a finite sequence $(\tau^{(l)}, a_i^{(l)})_{l=1}^k$ such that $k \geq 2$, $a_i^{(1)} = a_i'$, $a_i^{(k)} = a_i$,

$$\tau^{(l)} \in [0,t) \text{ and } a_i^{(l)} \in m_i(\tau^{(l)}) \text{ for all } l \in \{1,...,k\},$$

and

$$a_i^{(l-1)} \in m_i(\tau^{(l)}) \text{ for all } l \in \{2,...,k\}.$$

For every $u_i \in U_i(h_i^t)$ and every $l \in \{2,...,k\}$, since $\{a_i^{(l)}, a_i^{(l-1)}\} \subset m_i(\tau^{(l)})$,

$$u_i(a_i^{(l)}) - p_i(\tau^{(l)})(a_i^{(l)}) = u_i(a_i^{(l-1)}) - p_i(\tau^{(l)})(a_i^{(l-1)}).$$

Hence,

$$u_i(a_i) - u_i(a_i') = \sum_{l=2}^{k}\{u_i(a_i^{(l)}) - u_i(a_i^{(l-1)})\} = \sum_{l=2}^{k}\{p_i(\tau^{(l)})(a_i^{(l)}) - p_i(\tau^{(l)})(a_i^{(l-1)})\}. \tag{4}$$

Let us specify $x_i(a_i, a_i', h_i^t) \in R$ as

$$x_i(a_i, a_i', h_i^t) = \sum_{l=2}^{k}\{p_i(\tau^{(l)})(a_i^{(l)}) - p_i(\tau^{(l)})(a_i^{(l-1)})\}.$$

Since this specification does not depend on the selection of $u_i \in U_i(h_i^t)$, it follows from (4) that for every $u_i \in U_i(h_i^t)$ and every $\{a_i, a_i'\} \subset A_i(h_i^t)$,

$$u_i(a_i) - u_i(a_i') = x_i(a_i, a_i', h_i^t).$$

Q.E.D.

A price-demand procedure (γ, T) is said to be *connected* if for every $t \in (0, \infty)$, every $u \in U$, and every $i \in N$, $h_i^t(u, \gamma) \in H_i^t$ is connected. The following proposition shows a necessary and sufficient condition for the existence of a VCG mechanism that is consistent with a connected price-demand procedure; it is necessary and sufficient that, associated with any profile of valuation functions, the efficient allocations with and without any single buyer are revealed.

Proposition 3: *There exists a VCG mechanism* G *that is consistent with a connected price-demand procedure* (γ,T) *if and only if for every* $h^t \in H(\gamma,T)$ *, there exist* $a^*(h^t)\in A(h^t)$ *, and* $a^{i*}(h_{-i}^t)\in A^i(h_{-i}^t)$ *for each* $i\in N$ *, such that for every* $u \in U(h^t)$ *,*

(5) $$a^*(h^t)\in A^*(u),$$

and

(6) $$a^{i*}(h_{-i}^t)\in A^{i*}(u_{-i}) \text{ for all } i\in N.$$

Proof: We prove the "if" part as follows. Suppose that for every $h^t \in H(\gamma,T)$, there exist $a^*(h^t)\in A(h^t)$, and $a^{i*}(h_{-i}^t)\in A^i(h_{-i}^t)$ for each $i\in N$, that satisfy (5) and (6) for all $u\in U(h^t)$. Then, we can specify $f:U\to A$ in a manner that for every $h^t\in H(\gamma,T)$ and every $u\in U(h^t)$,

$$f(u)=a^*(h^t).$$

We can also specify $x_i:U\to R$ for each $i\in N$ in a manner that for every $h^t\in H(\gamma,T)$ and every $u\in U(h^t)$,

$$x_i(u)=\sum_{j\in N\setminus\{i\}} x_j(a_j^{i*}(h_{-i}^t), f_j(u), h^t).$$

From Lemma 2 and (6), it follows that

$$x_i(u)=\max_{a^i\in A^i}\sum_{j\in N\setminus\{i\}} u_j(a_j)-\sum_{j\in N\setminus\{i\}} u_j(f_j(u)),$$

which along with (5) implies that the specified mechanism $G=(f,x)$ is VCG.

We prove the "only if" part as follows. Assume that $G=(f,x)$ is VCG and consistent with (γ,T). Note from (1) and (2) that for every $i\in N$, every $h_i^t\in H_i^t$, and every $\{a_i,\tilde{a}_i\}\not\subset A_i(h_i^t)$, there exists $\{u_i,\tilde{u}_i\}\subset U_i(h_i^t)$ such that

$$u_i(a_i)-u_i(\tilde{a}_i)\neq \tilde{u}_i(a_i)-\tilde{u}_i(\tilde{a}_i).$$

Hence, for every $u\in U$, if either $f(u)\notin A(h^{T(u)}(u,\lambda))$ or

$$A^j(h^{T(u)}(u,\lambda))\bigcap A^{j*}(u_{-j})=\phi \text{ for some } j\in N,$$

then there exist $j\in N$ and $\tilde{u}_j\in U_j$ such that

$$(\tilde{u}_j,u_{-j})\in U(h^{T(u)}(u,\lambda)),$$

and for every $i \in N \setminus \{j\}$,

$$x_i(\tilde{u}_j, u_{-j}) = \max_{a^i \in A^i}\{\tilde{u}_j(a_j) + \sum_{h \in N\setminus\{i,j\}} u_h(a_h)\} - \{\tilde{u}_j(f_j(u)) + \sum_{h \in N\setminus\{i,j\}} u_h(f_h(u))\}$$

$$\neq \max_{a^i \in A^i} \sum_{h \in N\setminus\{i\}} u_h(a_h) - \sum_{h \in N\setminus\{i\}} u_h(f_h(u)) = x_i(u).$$

This contradicts the supposition that G is consistent with (γ, T). Hence, we have proved that for every $u \in U$,

$$f(u) \in A(h^{T(u)}(u,\lambda)) \text{, and}$$

$$A^j(h^{T(u)}(u,\lambda)) \cap A^{j*}(u_{-j}) \neq \phi \quad \text{for all } j \in N.$$

Suppose that there exist $\{u, \tilde{u}\} \subset U$, $j \in N$, and $a^j \in A^j$ such that

$$\tilde{u} \in U(h^{T(u)}(u,\lambda)),\ a^j \in A^j(h_{-j}^{T(u)}(u,\lambda)),\ a^j \in A^{j*}(u_{-j}) \text{, and}$$

$$a^j \notin A^{j*}(\tilde{u}_{-j}).$$

Without loss of generality, we can select $\tilde{u}$ satisfying that

$$x_j(\tilde{u}) = \max_{\tilde{a}^j \in A^j} \sum_{i \in N\setminus\{j\}} \tilde{u}_i(\tilde{a}_i) - \sum_{i \in N\setminus\{j\}} \tilde{u}_i(f_i(u)) > \sum_{i \in N\setminus\{j\}} \tilde{u}_i(a_i) - \sum_{i \in N\setminus\{j\}} \tilde{u}_i(f_i(u)).$$

Since

$$f(u) \in A(h^{T(u)}(u,\lambda)) \quad \text{and} \quad a^j \in A^j(h_{-j}^{T(u)}(u,\lambda)),$$

it follows that

$$\sum_{i \in N\setminus\{j\}} \tilde{u}_i(a_i) - \sum_{i \in N\setminus\{j\}} \tilde{u}_i(f_i(u)) = \sum_{i \in N\setminus\{j\}} u_i(a_i) - \sum_{i \in N\setminus\{j\}} u_i(f_i(u)) = x_i(u),$$

which implies that $x_i(\tilde{u}) \neq x_i(u)$. This contradicts the supposition that G is consistent with (γ, T). Hence, we have proved that for every $u \in U$ and every $j \in N$,

$$A^j(h_{-j}^{T(u)}(u,\lambda)) \cap (\bigcap_{\tilde{u}_{-j} \in U_{-j}(h_{-j}^{T(u)}(u,\lambda))} A^{j*}(\tilde{u}_{-j})) \neq \phi,$$

which implies that there exists $a^{i*}(h_{-i}^t) \in A^i(h_{-i}^t)$ that satisfies (6). Moreover, Lemma 1 implies that there exists $a^*(h^t) \in A(h^t)$ for each $i \in N$ satisfying (5).

From the above observations, we have proved the "only if" part.

Q.E.D.

5. Representative Valuation Functions

For every $i \in N$, every $t \in (0,\infty)$, and every connected history $h_i^t \in H_i^t$, let us define the *representative valuation function* $u_i^{[h_i^t]} \in U_i$ as follows. Assume $u_i^{[h_i^t]}(\phi) = 0$, and fix an arbitrary package for buyer i that belongs to $A_i(h_i^t)$, denoted by $\tilde{a}_i \in A_i(h_i^t)$. For every $a_i \in A_i(h_i^t) \setminus \{\tilde{a}_i\}$, let us specify

$$u_i^{[h_i^t]}(a_i) \equiv u_i^{[h_i^t]}(\tilde{a}_i) - x_i(\tilde{a}_i, a_i, h_i^t),$$

and for every $a_i \notin A_i(h_i^t)$,

$$u_i^{[h_i^t]}(a_i) \equiv \inf_{\tau \in [0,t), a_i' \in m_i(\tau)} \{u_i^{[h_i^t]}(a_i') - p_i(\tau)(a_i') + p_i(\tau)(a_i)\}.$$

The latter part of the specifications implies that the representative valuation function assigns the maximal absolute value to any unrevealed non-null package in the consistent manner with the history. It is clear that the representative valuation function $u_i^{[h_i^t]}$ exists uniquely. Let $u^{[h^t]} = (u_i^{[h_i^t]})_{i \in N}$.

The following proposition shows that the representative valuation function $u_i^{[h_i^t]}$ assigns any *revealed* package $a_i \in A_i(h_i^t)$ with the *minimal* possible valuation in relative terms. It also shows that $U_i(h_i^t)$ can be uniquely identified from $u_i^{[h_i^t]}$ and $A_i(h_i^t)$.

Proposition 4: *For every $t \in [0,\infty)$, every connected history $h_i^t \in H_i^t$, and every $u_i \in U_i$, it holds that $u_i \in U_i(h_i^t)$ if and only if for every $a_i \in A_i(h_i^t)$,*

$$u_i(a_i) - u_i(a_i') = u_i^{[h_i^t]}(a_i) - u_i^{[h_i^t]}(a_i') \quad \textit{for all} \quad a_i' \in A_i(h_i^t), \textit{ and}$$

$$u_i(a_i) - u_i(a_i') > u_i^{[h_i^t]}(a_i) - u_i^{[h_i^t]}(a_i') \quad \textit{for all} \quad a_i' \notin A_i(h_i^t).$$

In this case,

$$u_i(a_i) \geq u_i^{[h_i^t]}(a_i), \textit{ and}$$

$$u_i(a_i) = u_i^{[h_i^t]}(a_i) \quad \textit{if and only if} \quad \phi \in A_i(h_i^t).$$

Proof: The proof of the "if" part is straightforward from the definition of $u_i^{[h_i^t]}$. From the definition of $u_i^{[h_i^t]}$ and $a_i \in A_i(h_i^t)$, if $u_i \in U_i(h_i^t)$, then for every $a_i' \in A_i$,

$$u_i(a_i) - u_i(a_i') \geq u_i^{[h_i^t]}(a_i) - u_i^{[h_i^t]}(a_i'),$$

and $a_i' \in A_i(h_i^t)$ if and only if

$$u_i(a_i) - u_i(a_i') = u_i^{[h_i^t]}(a_i) - u_i^{[h_i^t]}(a_i') = x_i(a_i, a_i', h_i^t),$$

where we have used the assumption of revealed preference rule and Lemma 2.

By letting $a_i' = \phi$, from $u_i(\phi) = u_i^{[h_i^t]}(\phi) = 0$ and $u_i(a_i) \geq u_i^{[h_i^t]}(a_i)$, it follows that

$$u_i(a_i) = u_i^{[h_i^t]}(a_i) \text{ if and only if } \phi \in A_i(h_i^t).$$

Q.E.D.

The representative valuation function $u_i^{[h_i^t]}$ along with the set of revealed packages $A_i(h_i^t)$ could be regarded as the sufficient statistics concerning the extent to which the information about buyer $i's$ valuation function u_i was leaked in the history h_i^t.

The following theorem shows that the necessary and sufficient condition in Proposition 3 can be replaced with another condition implying that, associated with the profile of representative valuation functions, the efficient allocations with and without any single buyer are revealed. This condition could be much simpler than that in Proposition 3, because all we have to do for evaluating the sufficiency is to *examine just the representative valuation functions.*

Theorem 5: *There exists a VCG mechanism* G *that is consistent with a connected price-demand procedure* (γ, T) *if and only if for every* $h^t \in H(\gamma, T)$,

(7) $$A(h^t) \cap A^*(u^{[h^t]}) \neq \phi,$$

and

(8) $$A^i(h_{-i}^t) \cap A^{i*}(u_{-i}^{[h_{-i}^t]}) \neq \phi \quad \textit{for all } i \in N.$$

Proof: From Proposition 4 and the specification of $u^{[h^t]}$, it follows that for every $i \in N$, every $a_i \in A_i(h_i^t)$, and every $\tilde{a}_i \in A_i$,

$$u_i^{[h_i^t]}(a_i) - u_i^{[h_i^t]}(\tilde{a}_i) \leq u_i(a_i) - u_i(\tilde{a}_i) \quad \text{for all} \quad u_i \in U_i(h_i^t).$$

Hence, for every $a \in A(h^t)$,

$$a \in A^*(u) \quad \text{for all} \quad u \in U(h^t) \quad \text{if} \quad a \in A^*(u^{[h^t]}).$$

From the specification of $u^{[h^t]}$, it follows that for every $a \in A(h^t)$,

$$a \in A^*(u^{[h^t]}) \quad \text{if} \quad a \in A^*(u) \quad \text{for all} \quad u \in U(h^t).$$

Hence, we have proved that for every $a \in A(h^t)$,

$$a \in A^*(u^{[h^t]}) \quad \text{if and only if} \quad a \in A^*(u) \quad \text{for all} \quad u \in U(h^t).$$

This implies that (7) is equivalent to (5). In the same manner, for every $j \in N$ and every $a^j \in A^j(h_{-j}^t)$,

$$a^j \in A^{j*}(u_{-j}^{[h^t]}) \quad \text{if and only if} \quad a^j \in A^{j*}(u_{-j}) \quad \text{for all} \quad u_{-j} \in U_{-j}(h_{-j}^t).$$

This implies that (8) is equivalent to (6).

Q.E.D.

Since the profile of representative valuation functions $u^{[h^t]}$ minimizes the differences in valuation between the efficient allocations and other allocations, the requirements of efficiency for $u^{[h^t]}$ would be the severest among all relevant profiles of valuation functions $u \in U(h^t)$; it is sufficient to just examine $u^{[h^t]}$.

We should recall the implication of Proposition 4 that $U_i(h_i^t)$ can be uniquely identified from $u_i^{[h_i^t]}$ and $A_i(h_i^t)$. Hence, the extent to which the information regarding the buyers' valuations is leaked to the public can be fully expressed by $u_i^{[h_i^t]}$ and $A_i(h_i^t)$.

A profile of price vectors $p \equiv (p_i)_{i \in N} \in \prod_{i \in N} P_i$ is said to be a *competitive equilibrium for* $u \in U$ if there exists an allocation $a^{CE}(u) \in A$ that maximizes the payoffs for the seller and the buyers, i.e.,

$$\sum_{i \in N} p_i(a_i^{CE}) \geq \sum_{i \in N} p_i(a_i) \quad \text{for all} \quad a \in A,$$

and for every $i \in N$ and $a_i \in A_i$,

$$u_i(a_i^{CE}) - p_i(a_i^{CE}) \geq u_i(a_i) - p_i(a_i) .$$

A profile of price vectors $p \in \prod_{i \in N} P_i$ is said to be a *competitive equilibrium without buyer* i *for* $u_{-i} \in U_{-i}$ if there exists an allocation without buyer i, $a^{i,CE}(u) \in A^i$, that maximizes the payoffs for the sellers and the buyers except for buyer i satisfying that

$$\sum_{j \in N \setminus \{i\}} p_j(a_j^{i,CE}) \geq \sum_{j \in N \setminus \{i\}} p_j(a_j) \quad \text{for all} \quad a^i \in A^i ,$$

and for every $j \in N \setminus \{i\}$ and $a_j \in A_j$,

$$u_j(a_j^{i,CE}) - p_j(a_j^{i,CE}) \geq u_j(a_j) - p_j(a_j) .$$

According to the previous works such as Parkes and Ungar (2002) and Mishra and Parkes (2004), a profile of price vectors $p \in \prod_{i \in N} P_i$ is said to be a *universal competitive equilibrium for* $u \in U$ if it is a competitive equilibrium for u, and for every $i \in N$, it is a competitive equilibrium without buyer i for u_{-i}. Note that whenever p is a universal competitive equilibrium for u, then the allocations $a^{CE}(u)$ and $a_{-i}^{i,CE}(u)$ could satisfy efficiency in that $a^{CE}(u) \in A^*(u)$, and $a^{i,CE}(u) \in A^{j*}(u_{-i})$ for all $i \in N$.

Because of (1) and (2), we can express the representative valuation function by a $|A_i|$-dimensional vector as $u_i^{[h_i^t]} = (u_i^{[h_i^t]}(a_i))_{a_i \in A_i}$, which could be regarded as a price vector for buyer i, i.e., $u_i^{[h_i^t]} \in P_i$.

Proposition 6: *For every* $h^t \in H^t$, *if properties (7) and (8) are satisfied, then the profile of representative valuation functions* $u_i^{[h_i^t]}$ *is a universal competitive equilibrium for all* $u \in U(h_i^t)$.

Proof: From (7) and (8), we can select $a^*(h^t) \in A(h^t)$, and $a^{i*}(h_{-i}^t) \in A^i(h_{-i}^t)$ for each $i \in N$, such that $a^*(h^t) \in A(h^t) \cap A^*(u^{[h^t]})$, and

$$a^{i*}(h_{-i}^t) \in A^i(h_{-i}^t) \cap A^{i*}(u_{-i}^{[h_{-i}^t]}) \quad \text{for all} \quad i \in N .$$

Hence,

$$\sum_{i\in N} u_i^{[h_i^t]}(a_i^*(h_i^t)) \geq \sum_{i\in N} u_i^{[h_i^t]}(a_i) \quad \text{for all} \quad a \in A,$$

and for every $i \in N$,

$$\sum_{j\in N\setminus\{i\}} u_j^{[h_j^t]}(a_i^{j*}(h_i^t)) \geq \sum_{j\in N\setminus\{i\}} u_j^{[h_j^t]}(a_j) \quad \text{for all} \quad a^i \in A^i.$$

From Proposition 4, $a^*(h^t) \in A(h^t)$, and $a^{i*}(h_{-i}^t) \in A^i(h_{-i}^t)$, it follows that for every $u \in U(h^t)$ and every $i \in N$,

$$u_i(a_i^*(h_i^t)) - u_i^{[h_i^t]}(a_i^*(h_i^t)) \geq u_i(a_i) - u_i^{[h_i^t]}(a_i) \quad \text{for all} \quad a_i \in A_i,$$

and for every $j \in N \setminus \{i\}$,

$$u_j(a_j^{i*}(h_j^t)) - u_j^{[h_j^t]}(a_j^{i*}(h_j^t)) \geq u_j(a_j) - u_j^{[h_j^t]}(a_j) \quad \text{for all} \quad a_j \in A_j.$$

These observations imply that $u^{[h^t]} \in P$ is a universal competitive equilibrium.

Q.E.D.

Proposition 6 is related to several works such as Parkes and Ungar (2002), Lahaie, Constantin, and Parkes (2005), and Lahaie and Parkes (2004), which showed that there exists a VCG mechanism that is consistent with a price-demand procedure if and only if the occurred history always reveals a universal competitive equilibrium. Proposition 6 provides a method for elucidating which universal competitive equilibrium to be identified, by showing that the profile of representative valuation functions could be a universal competitive equilibrium.

6. Auctioneer's Discretion

Let us consider the situation in which the auctioneer (seller) makes a pre-play contractual agreement with the participants (buyers) in the following manner.

(i) The auctioneer is required to continue to ask a price vector to each participant according to a price-demand procedure the selection of which the participants leave to the auctioneer's discretion. The auctioneer is not required to inform them of the detail of his selected procedure.

(ii) The auctioneer is required to make his asked price vectors consistent with connectedness.

(iii) The auctioneer is required to continue to ask price vectors until the occurred history h^t fully reveals the efficient allocations with and without any single buyer associated with the corresponding profile of representative valuation functions, i.e.,

$$A(h^t)\bigcap A^*(u^{[h^t]})\neq\phi\text{, and }\; A^i(h^t_{-i})\bigcap A^{i*}(u_{-i}^{[h^t_{-i}]})\neq\phi\;\text{ for all }\; i\in N\text{ .}$$

(iv) The auctioneer is required to determine the VCG outcome associated with the profile of representative valuation functions, i.e., determine $(a,s)\in A\times R^n$ such that

$$a\in A(h^t)\bigcap A^*(u^{[h^t]})\text{, and}$$

$$s_i=\max_{\tilde{a}^i\in A^i}\sum_{j\in N\setminus\{i\}}u_j^{[h^t]}(\tilde{a}^i_j)-\sum_{j\in N\setminus\{i\}}u_j^{[h^t]}(a^i_j)\;\text{ for all }\; i\in N\text{ .}$$

It is possible for the participants to recognize whether these requirements to be satisfied even without knowing the detail of the selected procedure. Provided the auctioneer is penalized if he does not follow these requirements, the contractual agreement on these requirements sufficiently incentivizes the auctioneer to select a price-demand procedure that a VCG mechanism is consistent with, achieving the VCG outcomes. He does not need to make a pre-play agreement with the participants in terms of the fine detail of the procedure.

The success of incentivizing the auctioneer in this manner implies that the auctioneer, who prefers making the stopping time as quick as possible, can even make his procedure selection contingent on unverifiable information concerning the participants' preferences. Let Ω denote the set of possible *signals*. The auctioneer observes signal $\omega=\xi(u)\in\Omega$ that is dependent on the profile of valuation functions u. By observing this signal, the auctioneer recognizes that the profile of valuation functions is included in the set $\xi^{-1}(\omega)\subset U$. The auctioneer attempts to select a connected price-demand procedure $\lambda(\omega)=(\gamma,T)$, with which a VCG mechanism is consistent, that makes the stopping times as quick as possible.[4][5]

[4] The participants do not need to know Ω or $\xi:U\to\Omega$.

[5] We require $\lambda(\omega)$ to induce the VCG outcome even if the profile of valuation functions is not included in $\xi^{-1}(\omega)$. This is necessary for preventing buyers from cheating.

For example, consider the situation wherein the auctioneer possesses unverifiable but complete information regarding the profile of valuation functions:

$$\Omega = U \text{, and } \xi(u) = u \text{ for all } u \in U .$$

In this case, the auctioneer will select a connected price-demand procedure $\lambda(\omega) = (\gamma, T)$ that is consistent with a VCG mechanism, according to which, at the time close to the initial time 0, the auctioneer will ask a universal competitive equilibrium corresponding to $u = \xi^{-1}(\omega)$. Hence, the stopping time $T(u)$ could be selected as close to zero as possible; the auctioneer can immediately verify the efficient allocations with and without any single buyer. Moreover, by selecting the smallest one amongst all universal competitive equilibria, the auctioneer can even preserve the participants' privacy as much as possible.

7. Conclusion

We investigated the combinatorial auction problem. With connectedness, we demonstrated a tractable calculation method for elucidating whether there exists a VCG mechanism that is consistent with an arbitrary given price-demand procedure and for explicitly deriving such a VCG mechanism. The concept of representative valuation functions played the central role in this method, which was easily calculated on the basis of the history as the history-consistent minimal relative valuations. All the auctioneer had to do for these elucidations was to examine the profile of representative valuation functions alone; it was necessary and sufficient that the efficient allocations with and without any single buyer associated with the profile of representative valuation functions were revealed. The profile of representative valuation functions could be a universal competitive equilibrium in this case. Our characterization result could play the important role in simplifying the contractual relationship between the auctioneer and the participants; we could leave the detail of procedure design to the auctioneer's discretion and even permit the auctioneer to make the procedure design contingent on any unverifiable information.

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