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An Asymptotic Expansion for Forward-Backward SDEs: A Malliavin Calculus Approach *

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September 2, 2013

Abstract

This paper proposes a new closed-form approximation scheme for the representation of the forward-backward stochastic differential equations (FBSDEs) of Ma and Zhang (2002). In particular, we obtain an error estimate for the scheme applying Malliavin calculus method of Kunitomo and Takahashi (2001, 2003), Kusuoka (2003), Takahashi and Yamada (2012) for the forward SDEs combined with the Picard iteration scheme for the BSDEs. We also show numerical examples for pricing options with counterparty risk under the local and stochastic volatility models, where the credit value adjustment (CVA) is taken into account.

Keywords: Forward-Backward Stochastic Differential Equations (FBSDEs), Asymptotic expansion, Malliavin calculus, CVA

1 Introduction

In this paper, we propose a new asymptotic expansion scheme with its error estimate for the forward-backward stochastic differential equations (FBSDEs). As an application, we derive recursive expansion formulas for the option price with CVA under the local and stochastic volatility models and show numerical examples.

Bismut (1973) introduced the backward stochastic differential equations (BSDEs) for the linear case, and Pardoux and Peng (1990) initiated the study for the non-linear BSDEs. Since then, in addition to its theoretical researches, substantial numbers of numerical schemes for the solutions to the BSDEs have been proposed. The one of the main reasons is that the BSDEs are closely related to various valuation problems in finance (e.g. pricing securities under asymmetric/imperfect collateralization, optimal portfolio and indifference pricing issues in incomplete and/or constrained markets). They also become particularly useful for modeling credit risks (e.g. Duffie and Huang (1996), Crépey (2012a,b), Fujii and Takahashi (2010, 2011)) as well as for the study of recursive utilities (e.g. Duffie and Epstein (1992), Nakamura et al. (2009)). Their financial applications are discussed in details for example, El Karoui et al. (1997), Ma and Yong (2000), a recent book edited by Carmona (2009), Crépey (2012a,b), and references therein.

As for numerical methods, Ma et al. (1994) showed the four-step scheme for the BSDEs and its numerical method has been proposed in Douglas et al. (1996). Bouchard and Touzi (2004) has developed a discrete-time approximation for Monte-Carlo simulation based on Malliavin calculus. Also, a least-square Monte-Carlo method for the BSDEs has been proposed by Gobet et al. (2005). Moreover, Bender and Denk (2007) has presented a Picard-type approximation, and showed its theoretical and numerical validity. Recently, Gobet and Labart (2010) and Briand and Labart (2012) have extended the Monte-Carlo scheme for the BSDEs using the Picard-type iteration.

Although a large number of finite difference methods and simulation-based methods were proposed for numerical approximations of the solutions to BSDEs, their closed form approximation methods have been rarely discussed. Fujii and Takahashi (2012a,b,c) are exceptions, where they presented a simple analytical approximation with perturbation or/and interacting particle scheme for non-linear fully coupled FBSDEs without error estimate. Especially, Fujii and Takahashi (2012b) derived an approximation formula for dynamic optimal portfolio in an incomplete market with stochastic volatility, and confirmed its validity through numerical experiment.

This paper presents a new closed-form approximation method for the forward-backward stochastic differential equations based on a Picard-type iteration and an asymptotic expansion in Malliavin calculus. Also, our method can be regarded as an extension of the representation theorem by Ma and Zhang (2002) and the approximation method in Takahashi and Yamada (2012). Roughly speaking, considering a perturbed forward SDE X^{ε}, $\varepsilon \in (0,1]$ and an associated backward SDE $(Y^{\varepsilon}, Z^{\varepsilon})$, we have the following recursive asymptotic expansion around some

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non-degenerate gaussian model $\bar{X}^0$: *i.e.*, for $k \geq 0$, $N \geq 1$

$$Y_t^{\varepsilon,t,x} \simeq u^{\varepsilon,k+1,N}(t,x) = E[g(\bar{X}_T^{0,t,x})] + E\left[\int_t^T f(s,\bar{X}_s^{0,t,x},Y_s^{\varepsilon,k,N,t,x},Z_s^{\varepsilon,k,N,t,x})ds\right]$$
$$+\sum_{i=1}^{N}\varepsilon^i E[g(\bar{X}_T^{0,t,x})\pi_{i,T}^{0,t}] + \sum_{i=1}^{N}\varepsilon^i E\left[\int_t^T f(s,\bar{X}_s^{0,t,x},Y_s^{\varepsilon,k,N,t,x},Z_s^{\varepsilon,k,N,t,x})\pi_{i,s}^{0,t}ds\right], \quad (1)$$

$$Z_t^{\varepsilon,t,x} \simeq (\nabla u^{\varepsilon,k+1,N}\sigma)(t,x) = \Bigg\{ E[g(\bar{X}_T^{0,t,x})N_{0,T}^{0,t}] + E\left[\int_t^T f(s,\bar{X}_s^{0,t,x},Y_s^{\varepsilon,k,N,t,x},Z_s^{\varepsilon,k,N,t,x})N_{0,s}^{0,t}ds\right]$$
$$+\sum_{i=1}^{N}\varepsilon^i E[g(\bar{X}_T^{0,t,x})N_{i,T}^{0,t}] + \sum_{i=1}^{N}\varepsilon^i E\left[\int_t^T f(s,\bar{X}_s^{0,t,x},Y_s^{\varepsilon,k,N,t,x},Z_s^{\varepsilon,k,N,t,x})N_{i,s}^{0,t}ds\right]\Bigg\}\varepsilon\sigma(t,x), \quad (2)$$

where $Y_s^{\varepsilon,k,N,t,x} = u^{\varepsilon,k,N}(s,\bar{X}_s^{0,t,x})$ and $Z_s^{\varepsilon,k,N,t,x} = (\nabla_x u^{\varepsilon,k,N}\sigma)(s,\bar{X}_s^{0,t,x})$. Here, the processes $\pi_{i,t}^0$ and $N_{i,t}^0$, $i = 1,\cdots,N$ are the *Malliavin weights* and in particular, $N_{0,t}^0$ corresponds to the weight appeared in the Ma-Zhang's representation theorem. Moreover, applying properties of so called *Kusuoka-Stroock functions* introduced by Kusuoka (2003), we obtain an error estimate of our scheme to show its mathematical validity.

The organization of this paper is as follows: The next section describes an idea for our method using a well-known example. Section 3 generalizes the idea and summarizes our algorithm in a general setting. After Section 4 provides the notations and basic results used in later sections, Section 5 presents our main result with its proof. Applying our scheme, Section 6 provides a simple numerical example for pricing options with counterparty risk under the local and stochastic volatility model. Section 7 concludes.

2 Motivated Example

In this section, we show an idea for our approximation method using the BSDE appearing in a well-known example of mathematical finance, so called "*hedging claims with higher interest rate for borrowing*" (Cvitanic and Karatzas (1993), El Karoui et al. (1997)).

Specifically, let us consider the following FBSDE examined by Gobet et al. (2005), Bender and Denk (2007) and Fujii and Takahashi (2012a):

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t, & (3)\\ S_0 &= s_0, & \\ dY_t &= rY_t dt - f(Y_t,Z_t)dt + Z_t dW_t, & (4)\\ Y_T &= g(S_T) = \max(S_T-K_1,0) - 2\max(S_T-K_2,0), & (5) \end{aligned}$$

where

$$f(y,z) = (R-r)\max\left(\frac{z}{\sigma}-y,0\right) - \left(\frac{\mu-r}{\sigma}\right)z. \quad (6)$$

When the borrowing rate R is higher than the lending rate r (i.e. $R > r$), the solution to the FBSDE above, $Y = \{Y_t : 0 \leq t \leq T\}$ represents the value process of a self-financing hedging strategy for a target payoff given by $g(S_T)$, and Z stands for the hedging strategy where Z_t/σ is the amount invested at time t in the risky asset whose price process is given by S.[1] In particular, we note that the specification of $g(S_T)$ as an option spread creates both lending and borrowing in the strategy. Here, r, R, μ and σ are assumed to be positive constants.

$Y = \{Y_t : 0 \leq t \leq T\}$ is represented as the following non-linear expectation:

$$Y_t = e^{-r(T-t)}E\left[g(S_T)|\mathcal{F}_t\right] + e^{-r(T-t)}E\left[\int_t^T f(Y_u,Z_u)du|\mathcal{F}_t\right],$$

where $\mathcal{F}_t$ is the filtration generated by W, *i.e.*, $\mathcal{F}_t = \sigma(W_s; s \leq t)$. Next, define u as

$$u(t,s) := Y_t^{t,s} = e^{-r(T-t)}E\left[g(S_T^{t,s})\right] + e^{-r(T-t)}E\left[\int_t^T f(Y_u^{t,s},Z_u^{t,s})du\right].$$

Then, using this u, $Z = \{Z_t : 0 \leq t \leq T\}$ is obtained as follows:

$$Z_t = \sigma S_t \frac{\partial}{\partial s}u(t,S_t).$$

[1] The problem is considered under the physical measure and $\left(\frac{\mu-r}{\sigma}\right)$ represents the market price of risk.

Moreover, applying a representation result by Ma and Zhang (2002), one has

$$Z_t = e^{-r(T-t)} \left\{ E[g(S_T^{t,s})N_T^{t,s}] + E[\int_t^T f(Y_u^{t,s}, Z_u^{t,s})N_u^{t,s} du] \right\},$$

where $N^{t,s} = \{N_u^{t,s} : 0 \le t \le s \le T\}$ is the Malliavin weight process given $S_t = s$:

$$N_u^{t,s} = \frac{1}{u-t} \int_t^u \sigma^{-1}(S_\tau^{t,s}) \frac{\partial}{\partial s} S_\tau^{t,s} dW_\tau.$$

Next, let us show an example of a closed form approximation for the BSDE using the Picard-type iteration. In the first place, define $u^0(t,s)$ as

$$u^0(t,s) \quad := \quad e^{-r(T-t)} E\left[g(S_T^{t,s})\right]. \tag{7}$$

Then, *the Malliavin representation for the Delta* under Black-Scholes model (3) is well-known, that is given by

$$\frac{\partial}{\partial s} u^0(t,s) = e^{-r(T-t)} E\left[g(S_T^{t,s}) \frac{1}{T-t} \int_t^T \frac{1}{s\sigma} dW_u\right]. \tag{8}$$

See Fournié et al (1999) for the details.

In this simple model, we are capable of its evaluation through one dimensional integrations. That is, given $\log S_t = x$, set the density of $\log S_T$ under (3) as $p(t,T,x,y)$:

$$p(t,T,x,y) = \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} \exp\left(-\frac{(y - x - \mu(T-t) + \frac{1}{2}\sigma^2(T-t))^2}{2\sigma^2(T-t)}\right). \tag{9}$$

Then, we have

$$u^0(t,s) \quad = \quad e^{-r(T-t)} \int_{\mathbf{R}} g(e^y) p(t,T,x,y) dy, \tag{10}$$

and

$$\frac{\partial}{\partial s} u^0(t,s) \quad = \quad e^{-r(T-t)} \int_{\mathbf{R}} g(e^y) w(t,x,y) p(t,T,x,y) dy,$$

where the finite dimensional Malliavin weight $w(t,x,y)$ is given by

$$w(t,x,y) = E\left[\frac{1}{T-t} \int_t^T \frac{1}{s\sigma} dW_u | X_{T-t} = y\right] = \frac{(y - x - \mu(T-t) + \frac{1}{2}\sigma^2(T-t))}{e^x \sigma^2(T-t)}. \tag{11}$$

Hence, we get the 0-th iteration $(Y^0, Z^0) = \{(Y_t^0, Z_t^0) : 0 \le t \le T\}$ as

$$\begin{aligned} Y_t^0 &= u^0(t, S_t), \\ Z_t^0 &= \sigma S_t \frac{\partial}{\partial s} u^0(t, S_t). \end{aligned} \tag{12}$$

Next, using the function $u^0(t,s)$, we define $u^1(t,s)$ as

$$u^1(t,e^x) \quad := \quad u^0(t,e^x) + e^{-r(T-t)} \int_t^T \int_{\mathbf{R}} f\left(u^0(v,e^y), \sigma e^y \frac{\partial}{\partial s} u^0(v,e^y)\right) p(t,v,x,y) dy dv, \tag{13}$$

where $x = \log s$. Then, applying the same weight w as (11), we are able to evaluate $\frac{\partial}{\partial s} u^1(t,s)$:

$$\begin{aligned} \frac{\partial}{\partial s} u^1(t,e^x) \quad &= \quad \frac{\partial}{\partial s} u^0(t,e^x) \\ &+ e^{-r(T-t)} \int_t^T \int_{\mathbf{R}} f\left(u^0(v,e^y), \sigma e^y \frac{\partial}{\partial s} u^0(v,e^y)\right) w(v,x,y) p(t,v,x,y) dy dv. \end{aligned} \tag{14}$$

Therefore, the first iteration is given by

$$\begin{aligned} Y_t^1 &= u^1(t, S_t), \\ Z_t^1 &= \sigma S_t \frac{\partial}{\partial s} u^1(t, S_t). \end{aligned} \tag{15}$$

Thus, for $k \ge 1$ let us recursively define $u^{k+1}(t,s) = u^{k+1}(t,e^x)$ (where $x = \log s$) as

$$u^{k+1}(t,e^x) \quad := \quad u^0(t,e^x) + e^{-r(T-t)} \int_t^T \int_{\mathbf{R}} f\left(u^k(v,e^y), \sigma e^y \frac{\partial}{\partial s} u^k(v,e^y)\right) p(t,v,x,y) dy dv, \tag{16}$$

which leads to the evaluation of $\frac{\partial}{\partial s}u^{k+1}(t,s)$ with the same weight w as (11):

$$\begin{aligned}\frac{\partial}{\partial s}u^{k+1}(t,e^x) &= \frac{\partial}{\partial s}u^0(t,e^x) \qquad (17)\\ &+e^{-r(T-t)}\int_t^T\int_{\mathbf{R}} f\left(u^k(v,e^y),\sigma e^y\frac{\partial}{\partial s}u^k(v,e^y)\right)w(v,x,y)p(t,v,x,y)dydv. \qquad (18)\end{aligned}$$

Hence, the $k+1$-iteration is obtained by

$$\begin{aligned}Y_t^{k+1} &= u^{k+1}(t,S_t), \qquad (19)\\ Z_t^{k+1} &= \sigma S_t\frac{\partial}{\partial s}u^{k+1}(t,S_t).\end{aligned}$$

Finally, applying the same parameters as in an example of Gobet et al. (2005) so that $S_0=100$, $\sigma=0.2$, $\mu=0.05$, $r=0.01$, $R=0.06$, $T=0.25$, $K_1=95$, $K_2=105$, let us show a numerical comparison of this iterated approximation scheme with their result.

- Benchmark value of Y_0 by Gobet et al. (2005): 2.95 with standard deviation 0.01, where they have tried various sets of basis functions in their regression-based Monte Carlo simulation to achieve this value.
- Our approximation values: 0-**th iteration** = 2.7864, **the first iteration** = 2.9671, and **the second iteration** = 2.9531.

It is observed that our approximation values become closer to the benchmark one as the more iterations are implemented. We also remark that a perturbed approximation method of Fujii and Takahashi (2012a)[2] has provided 2.7863, 2.968, and 2.953 for the 0-th, the first and the second order approximations, respectively, which are very close to our result. In the following sections, we extend our method in a more general setting.

3 Summary of Algorithm of Closed-form Approximation

In the example of section 2, we are able to make use of an explicit Gaussian density since the forward process is given by Black-Scholes model (3). However, when we consider a more complex forward process, the explicit density is no longer obtained in general. For the case of general forward processes on a probability space $(\Omega,\mathcal{F},P)$, let us introduce a perturbation parameter $\varepsilon\in(0,1]$ as

$$dX_t^\varepsilon=\mu(t,X_t^\varepsilon)dt+\varepsilon\sigma(t,X_t^\varepsilon)dW_t.$$

Then, for $\varepsilon>0$ we are able to derive a semi-closed form density applying an asymptotic expansion around some simple model $\bar{X}_T^{0,t,x}$ under a suitable condition, that is, for $N\in\mathbf{N}$,

$$p^\varepsilon(t,T,x,y) \simeq p^0(t,T,x,y)+\sum_{i=1}^N\varepsilon^iE[\pi_{i,T}^{0,t,x}|\bar{X}_T^{0,t,x}=y]p^0(t,T,x,y), \qquad (20)$$

with the density $p^0(t,T,x,y)$ of $\bar{X}_T^{0,t,x}$ and some Malliavin weights $\pi_{i,T}^{0,t,x}$, $i=1,\cdots,N$. For the following general BSDE;

$$Y_t^\varepsilon = g(X_T^\varepsilon)+\int_t^T f(s,X_s^\varepsilon,Y_s^\varepsilon,Z_s^\varepsilon)ds-\int_t^T Z_s^\varepsilon dW_s, \qquad (21)$$

we define the function u as

$$u^\varepsilon(t,x)=Y_t^{\varepsilon,t,x}=E[g(X_T^{\varepsilon,t,x})]+E\left[\int_t^T f(s,X_s^{\varepsilon,t,x},Y_s^{\varepsilon,t,x},Z_s^{\varepsilon,t,x})ds\right]. \qquad (22)$$

We approximate u^ε using a sequence $(u^{\varepsilon,k,N})_k$ in the following way.

1. $u^{\varepsilon,0,N}(t,x)$**: An approximation of the 0-th iteration**
   Here, the 0-th iteration is defined by
   $u^{\varepsilon,0}(t,x)=E[g(X_T^{\varepsilon,t,x})]+E\left[\int_t^T f(s,X_s^{\varepsilon,t,x},0,0)ds\right]$. Then,

$$\begin{aligned}Y_t^{\varepsilon,t,x} &= u^{\varepsilon,0}(t,x)\\ &\simeq u^{\varepsilon,0,N}(t,x)\\ &= E[g(\bar{X}_T^{0,t,x})]+\sum_{i=1}^N\varepsilon^iE[g(\bar{X}_T^{0,t,x})\pi_{i,T}^{0,t}]\end{aligned}$$

[2]See their paper for the details.

$$
\begin{aligned}
& +E\left[\int_t^T f(s,\bar{X}_s^{0,t,x},0,0)ds\right] + \sum_{i=1}^N \varepsilon^i E\left[\int_t^T f(s,\bar{X}_s^{0,t,x},0,0)\pi_{i,s}^{0,t}ds\right] \\
= & \int_{\mathbf{R}^d} g(y)p^0(t,T,x,y)dy + \sum_{i=1}^N \varepsilon^i \int_{\mathbf{R}^d} g(y)E[\pi_{i,T}^{0,t}|\bar{X}_T^{0,t,x}=y]p^0(t,T,x,y)dy \\
& + \int_t^T \int_{\mathbf{R}^d} f(s,y,0,0)p^0(t,s,x,y)dyds \\
& + \sum_{i=1}^N \varepsilon^i \int_t^T \int_{\mathbf{R}^d} f(s,y,0,0)E[\pi_{i,s}^{0,t}|\bar{X}_s^{0,t,x}=y]p^0(t,s,x,y)dyds.
\end{aligned}
$$

Note that the Malliavin weights $\pi_{i,s}^0$, $i=1,\cdots,N$ are same as in (20).

2. $u^{\varepsilon,1,N}(t,x)$**: An approximation of the first iteration**
Here, the first iteration is defined by
$u^{\varepsilon,1}(t,x) = E[g(X_T^{\varepsilon,t,x})] + E\left[\int_t^T f(s,X_s^{\varepsilon,t,x},u^{\varepsilon,0}(s,X_s^{\varepsilon,t,x}),(\nabla_x u^{\varepsilon,0}\sigma)(s,X_s^{\varepsilon,t,x}))ds\right]$.
Firstly, define

$$
\hat{u}^{\varepsilon,1}(t,x) = E[g(X_T^{\varepsilon,t,x})] + E\left[\int_t^T f(s,X_s^{\varepsilon,t,x},u^{\varepsilon,0,N}(s,X_s^{\varepsilon,t,x}),(\nabla_x u^{\varepsilon,0,N}\sigma)(s,X_s^{\varepsilon,t,x}))ds\right].
$$

$\hat{u}^{\varepsilon,1}(t,x)$ is an approximation of $u^{\varepsilon,1}(t,x)$:

$$
u^{\varepsilon,1}(t,x) \simeq \hat{u}^{\varepsilon,1}(t,x).
$$

We can not compute $\hat{u}^{\varepsilon,1}(t,x)$ explicitly because the density $p^\varepsilon(t,T,x,y)$ of $X_T^{\varepsilon,t,x}$ has no closed-form expression. Then, using the approximation of the density in (20) again, we expand $\hat{u}^{\varepsilon,1}(t,x)$ with respect to ε as follows:

$$
\begin{aligned}
\hat{u}^{\varepsilon,1}(t,x) = & E[g(X_T^{\varepsilon,t,x})] + E\left[\int_t^T f(s,X_s^{\varepsilon,t,x},u^{\varepsilon,0,N}(s,X_s^{\varepsilon,t,x}),(\nabla_x u^{\varepsilon,0,N}\sigma)(s,X_s^{\varepsilon,t,x}))ds\right] \qquad (23) \\
\simeq & u^{\varepsilon,1,N}(t,x) \\
= & E[g(\bar{X}_T^{0,t,x})] + \sum_{i=1}^N \varepsilon^i E[g(\bar{X}_T^{0,t,x})\pi_{i,T}^{0,t}] \\
& + E\left[\int_t^T f(s,\bar{X}_s^{0,t,x},u^{\varepsilon,0,N}(s,\bar{X}_s^{0,t,x}),(\nabla_x u^{\varepsilon,0,N}\sigma)(s,X_s^{0,t,x}))ds\right] \\
& + \sum_{i=1}^N \varepsilon^i E\left[\int_t^T f(s,\bar{X}_s^{0,t,x},u^{\varepsilon,0,N}(s,\bar{X}_s^{0,t,x}),(\nabla_x u^{\varepsilon,0,N}\sigma)(s,\bar{X}_s^{0,t,x}))\pi_{i,s}^{0,t}ds\right] \qquad (24) \\
= & \int_{\mathbf{R}^d} g(y)p^0(t,T,x,y)dy + \sum_{i=1}^N \varepsilon^i \int_{\mathbf{R}^d} g(y)E[\pi_{i,T}^{0,t}|\bar{X}_T^{0,t,x}=y]p^0(t,T,x,y)dy \\
& + \int_t^T \int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,0,N}(s,y),(\nabla_x u^{\varepsilon,0,N}\sigma)(s,y))p^0(t,s,x,y)dyds \\
& + \sum_{i=1}^N \varepsilon^i \int_t^T \int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,0,N}(s,y),(\nabla_x u^{\varepsilon,0,N}\sigma)(s,y))E[\pi_{i,s}^{0,t}|\bar{X}_s^{0,t,x}=y]p^0(t,s,x,y)dyds. \qquad (25)
\end{aligned}
$$

Since $Y_t^{\varepsilon,1,t,x} = u^{\varepsilon,1}(t,x)$, we get an approximation using (24)

$$
\begin{aligned}
Y_t^{\varepsilon,1,t,x} \simeq & u^{\varepsilon,1,N}(t,x) \\
= & E[g(\bar{X}_T^{0,t,x})] + \sum_{i=1}^N \varepsilon^i E[g(\bar{X}_T^{0,t,x})\pi_{i,T}^{0,t}] \\
& + E\left[\int_t^T f(s,\bar{X}_s^{0,t,x},Y_s^{\varepsilon,0,N,t,x},Z_s^{\varepsilon,0,N,t,x})ds\right] \\
& + \sum_{i=1}^N \varepsilon^i E\left[\int_t^T f(s,\bar{X}_s^{0,t,x},Y_s^{\varepsilon,0,N,t,x},Z_s^{\varepsilon,0,N,t,x})\pi_{i,s}^{0,t}ds\right]. \qquad (26)
\end{aligned}
$$

Here, $Y_s^{\varepsilon,0,N,t,x} = u^{\varepsilon,0,N}(s,\bar{X}_s^{0,t,x})$ and $Z_s^{\varepsilon,0,N,t,x} = (\nabla_x u^{\varepsilon,0,N}\sigma)(s,\bar{X}_s^{0,t,x})$.

3. **We iterate the procedure above.**
   Then, in general we obtain the following numerical approximation for $u^\varepsilon(t,x) = Y_t^{\varepsilon,t,x}$.
4. **Numerical approximation for** $u^\varepsilon(t,x) = Y_t^{\varepsilon,t,x}$

$$\begin{aligned}
Y_t^{\varepsilon,t,x} &= u^\varepsilon(t,x)\\
&\simeq u^{\varepsilon,k,N}(t,x)\\
&= E[g(\bar{X}_T^{0,t,x})] + \sum_{i=1}^N \varepsilon^i E[g(\bar{X}_T^{0,t,x})\pi_{i,T}^{0,t}]\\
&\quad +E\left[\int_t^T f(s,\bar{X}_s^{0,t,x}, Y_s^{\varepsilon,k-1,N,t,x}, Z_s^{\varepsilon,k-1,N,t,x})ds\right]\\
&\quad +\sum_{i=1}^N \varepsilon^i E\left[\int_t^T f(s,\bar{X}_s^{0,t,x}, Y_s^{\varepsilon,k-1,N,t,x}, Z_s^{\varepsilon,k-1,N,t,x})\pi_{i,s}^{0,t}ds\right]\\
&= \int_{\mathbf{R}^d} g(y)p^0(t,T,x,y)dy + \sum_{i=1}^N \varepsilon^i \int_{\mathbf{R}^d} g(y)E[\pi_{i,T}^{0,t}|\bar{X}_T^{0,t,x}=y]p^0(t,T,x,y)dy\\
&\quad +\int_t^T\int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k-1,N}(s,y),(\nabla_x u^{\varepsilon,k-1,N}\sigma)(s,y))p^0(t,s,x,y)dyds\\
&\quad +\sum_{i=1}^N \varepsilon^i \int_t^T\int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k-1,N}(s,y),(\nabla_x u^{\varepsilon,k-1,N}\sigma)(s,y))E[\pi_{i,s}^{0,t}|\bar{X}_s^{0,t,x}=y]p^0(t,s,x,y)dyds.
\end{aligned}$$

   Here, $Y_s^{\varepsilon,k-1,N,t,x} = u^{\varepsilon,k-1,N}(s,\bar{X}_s^{0,t,x})$ and $Z_s^{\varepsilon,k-1,N,t,x} = (\nabla_x u^{\varepsilon,k-1,N}\sigma)(s,\bar{X}_s^{0,t,x})$.

   We prove this conjecture rigorously using Malliavin calculus in Section 5.

4 Notations and Basic Results

Hereafter, we use the following notations.

- For $x\in\mathbf{R}^d$, $\nabla_x = (\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_d})$.
- $C(T,x)$ stands for a generic non-negative, non-decreacing and finite function of at most polynomial growth in x depending on $T>0$.
- $C_b^k(\mathbf{R}^d)$ is the space of the k-times continuously differential functions on $\mathbf{R}^d$ such that the partial derivatives are uniformly bounded.
- (Ω,H,P) is the Wiener space. H is the Cameron-Martin subspace.
- $\mathbf{D}^{k,p}$ is the space of the k-times Malliavin differentiable L^p-Wiener functionals for $k\in\mathbf{N}, p\in[1,\infty)$. We denote $\|\cdot\|_{\mathbf{D}^{k,p}}$ as the norm of $\mathbf{D}^{k,p}$.
- $\mathbf{D}^\infty$ is the space of the smooth Wiener functionals in the sense of Malliavin, that is, $\mathbf{D}^\infty = \cap_{k,p}\mathbf{D}^{k,p}$.
- $\mathbf{D}^{-\infty}$ is the space of the Watanabe distributions (the dual of $\mathbf{D}^\infty$).
- We say $F^\varepsilon = O(\varepsilon^n)$ *in* $\mathbf{D}^{k,p}$ as $\varepsilon\downarrow 0$ if $F^\varepsilon\in\mathbf{D}^{k,p}$ for all $\varepsilon\in(0,1]$ and

$$\limsup_{\varepsilon\downarrow 0}\|F^\varepsilon\|_{\mathbf{D}^{k,p}}/\varepsilon^n < \infty, \tag{27}$$

  where n is some real constant.

Let D be the *Malliavin derivative* operator (a densely defined, closed linear operator from $\mathbf{D}^{1,2}$ to $L^2(\Omega\times[0,T])$) and δ be its adjoint operator (so-called the *Skorohod integral*) $\delta : Dom(\delta)\to L^2(\Omega;\mathbf{R})$: for all $F\in\mathbf{D}^{1,2}$ and $u\in Dom(\delta)$,

$$E[F\delta(u)] = E\left[\int_0^T D_tFu_tdt\right], \tag{28}$$

where $Dom(\delta) = \left\{u\in L^2(\Omega\times[0,T]) : \left|E\left[\int_0^T D_tFu_tdt\right]\right| \le C\|F\|_{1,2}, \forall F\in\mathbf{D}^{1,2}\right\}$. It is well-known that the Skorohod integral has the following property. For the proof, see Nualart (2006), for instance.

Lemma 4.1 *Suppose that $F\in\mathbf{D}^{1,2}$. For any $u\in Dom(\delta)$ such that $Fu\in L^2([0,T]\times\Omega)$, one has $Fu\in Dom(\delta)$, and it holds that*

$$\delta(Fu) = F\int_0^T u_tdW_t - \int_0^T D_tFu_tdt. \tag{29}$$

In our algorithm summarized in section 3, we have to compute the asymptotic expansion $u^{k,N}$ recursively. From a numerical viewpoint, the stability of integration must be checked. In particular, the asymptotic behavior of our approximation is crucial when $t \uparrow T$. Hence, we introduce the Kusuoka-Stroock functions (Kusuoka (2003)) which help to clarify the order of a Wiener functional with respect to time t.

Definition 4.1 (Kusuoka-Stroock functions) *Given $r \in \mathbf{R}$ and $n \in \mathbf{N}$, we denote by $\mathcal{K}_r^T(n)$ the set of functions $G : (0, T] \times \mathbf{R}^d \to \mathbf{D}^{n,\infty}$ satisfying the followings:*

1. *$G(t, \cdot)$ is n-times continuously differentiable and $[\partial^\alpha G/\partial x^\alpha]$ is continuous in $(t, x) \in (0, T] \times \mathbf{R}^d$ a.s. for any multi-index α of the elements of $\{1, \cdots, d\}$ with length $|\alpha| \leq n$.*
2. *For all $k \leq n - |\alpha|$, $p \in [1, \infty)$,*

$$\sup_{t\in(0,T],x\in\mathbf{R}^d} t^{-r/2} \left\| \frac{\partial^\alpha G}{\partial x^\alpha}(t, x) \right\|_{\mathbf{D}^{k,p}} < \infty. \tag{30}$$

The above definition corresponds to Definition 2.1 of Crisan and Delarue (2012) of modified version of Kusuoka (2003). We write $\mathcal{K}_r^T$ for $\mathcal{K}_r^T(\infty)$.

Lemma 4.2 *[Properties of Kusuoka-Stroock functions] The followings hold.*

1. *Suppose $G \in \mathcal{K}_r^T(n)$ where $r \geq 0$. Then, for $i = 1, \cdots, d$,*

$$\int_0^\cdot G(s, x)dW_s^i \in \mathcal{K}_{r+1}^T(n) \text{ and } \int_0^\cdot G(s, x)ds \in \mathcal{K}_{r+2}^T(n). \tag{31}$$

2. *If $G_i \in \mathcal{K}_{r_i}^T(n_i)$, $i = 1, \cdots, N$, then*

$$\prod_i^N G_i \in \mathcal{K}_{r_1+\cdots+r_N}^T(\min_i n_i) \text{ and } \sum_{i=1}^N G_i \in \mathcal{K}_{\min_i r_i}^T(\min_i n_i). \tag{32}$$

Proof. See Lemma 5.1.2 of Nee (2010) for instance. □

Let $(X_t)_{t\in[0,T]}$ be the solution to the following stochastic differential equation:

$$\begin{aligned} dX_t^x &= V_0(X_t^x)dt + \sum_{i=1}^N V_i(X_t^x)dW_{i,t}, \\ X_0 &= x \in \mathbf{R}^d, \end{aligned} \tag{33}$$

where each V_i, $i = 0, 1, \cdots, N$ is bounded and belongs to $C_b^\infty(\mathbf{R}^d; \mathbf{R}^d)$. We assume that the UFG condition of Kusuoka (2003) holds. See p. 262 of Kusuoka (2003) for the definition of the UFG condition. Next, we summarize the Malliavin's integration by parts formula using Kusuoka-Stroock functions. For any multi-index $\alpha^{(k)} := (\alpha_1, \cdots, \alpha_k) \in \{1, \cdots, d\}^k$, $k \geq 1$, we denote by $\partial_{\alpha^{(k)}}$ the partial derivative $\frac{\partial^k}{\partial x_{\alpha_1} \cdots \partial x_{\alpha_k}}$.

Proposition 4.1 *Let $G : (0, T] \times \mathbf{R}^d \to \mathbf{D}^\infty = \mathbf{D}^{\infty,\infty}(\mathbf{R})$ be an element of $\mathcal{K}_r^T$ and let f be a function that belongs to the space $C_b^\infty(\mathbf{R}^d)$. Then for any multi-index $\alpha^{(k)} \in \{1, \cdots, d\}^k$, $k \geq 1$, there exists $H_{\alpha^{(k)}}(X_t^x, G(t, x)) \in \mathcal{K}_{r-|\alpha^{(k)}|}^T$ such that*

$$E\left[\partial_{\alpha^{(k)}} f(X_t^x)G(t, x)\right] = E\left[f(X_t^x)H_{\alpha^{(k)}}(X_t^x, G(t, x))\right], \tag{34}$$

with

$$\|H_{\alpha^{(k)}}(X_t^x, G(t, x))\|_{L^p} \leq C(T, x)t^{(r-|\alpha^{(k)}|)/2}, \tag{35}$$

where $H_{\alpha^{(k)}}(X_t^x, G(t, x))$ is recursively given by

$$H_{(i)}(X_t^x, G(t, x)) = \delta\left(\sum_{j=1}^N G\gamma_{ij}^{X_t^x} DX_t^{x,j}\right), \tag{36}$$

$$H_{\alpha^{(k)}}(X_t^x, G(t, x)) = H_{(\alpha_k)}(X_t^x, H_{\alpha^{(k-1)}}(X_t^x, G(t, x))), \tag{37}$$

and a positive constant $C(T, x)$ is depending on T and x. Here, $(\gamma_{ij}^{X_t^x})_{1\leq i,j\leq n}$ is the inverse matrix of the Malliavin covariance of X_t^x.

Proof. Apply Corollary 3.7 of Kusuoka-Stroock (1984) and Lemma 8-(3) of Kusuoka (2003) with Proposition 2.1.4 of Nualart (2006). □

5 Asymptotic Expansion for FBSDEs

5.1 Forward-Backward SDE

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space on which a d-dimensional Brownian motion $W = (W^1, \cdots, W^d)$ is defined. Let $(\mathcal{F}_t)_{t \geq 0}$ be the natural filtration generated by W, augmented by the P-null sets of $\mathcal{F}$. Consider the following d-dimensional forward stochastic differential equation $X_t = (X_t^1, \cdots, X_t^d)$;

$$dX_t^i = b^i(t, X_t)dt + \sum_{j=1}^{d} \sigma_j^i(t, X_t)dW_t^j, \quad i = 1, \cdots, d, \tag{38}$$

where $b : [0,T] \times \mathbf{R}^d \to \mathbf{R}^d$ and $\sigma : [0,T] \times \mathbf{R}^d \to \mathbf{R}^{d \times d}$.

Next, let us introduce a backward stochastic differential equation Y:

$$Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \tag{39}$$

where $g : \mathbf{R}^d \to \mathbf{R}$ and $f : [0,T] \times \mathbf{R}^d \times \mathbf{R} \times \mathbf{R}^d \to \mathbf{R}$.

We put some conditions below on the above forward-backward SDE.

Assumption 5.1

1. *The coefficients of forward process b, σ are bounded Borel functions and C_b^∞ in x.*
2. *There exist constants $a_i > 0$, $i = 1, 2$ such that for any vector ξ in $\mathbf{R}^d$ and any $(t,x) \in [0,T] \times \mathbf{R}^d$,*

$$a_1|\xi|^2 \leq \sum_{i,j=1}^{d} [\sigma\sigma^T]_{i,j}(t,x)\xi_i\xi_j \leq a_2|\xi|^2. \tag{40}$$

3. *The driver $f : [0,T] \times \mathbf{R}^d \times \mathbf{R} \times \mathbf{R}^{d\times d} \to \mathbf{R}$ is continuous in t and uniformly Lipschitz continuous in x, y, z with constant C_L, i.e. for all $t \in [0,T]$, (x_1, y_1, z_1), $(x_2, y_2, z_2) \in \mathbf{R}^d \times \mathbf{R} \times \mathbf{R}^{d\times d}$,*

$$|f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)| \leq C_L(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|). \tag{41}$$

   Also, we assume

$$|f(t,x,y,z)| \leq C_L(1 + |x| + |y| + |z|). \tag{42}$$

   for $(t,x,y,z) \in [0,T] \times \mathbf{R}^d \times \mathbf{R} \times \mathbf{R}^{d\times d}$.

4. *g is Lipschitz continuous function with constant C_L on $\mathbf{R}^d$ and $|g(x)| \leq C_L(1 + |x|)$ for $x \in \mathbf{R}^d$.*

5.2 Small Diffusion Expansion

In this subsection, we deal with a *small diffusion expansion* which corresponds to the framework in Kunitomo and Takahashi (2001, 2003) and derive a general approximation formula for FBSDEs. Consider the following d-dimensional perturbed forward stochastic differential equation $X_t^\varepsilon = (X_t^{1,\varepsilon}, \cdots, X_t^{d,\varepsilon})$:

$$dX_t^{i,\varepsilon} = b^i(t, X_t^\varepsilon)dt + \varepsilon \sum_{j=1}^{d} \sigma_j^i(t, X_t^\varepsilon)dW_t^j, \quad i = 1, \cdots, d, \tag{43}$$

where $b : [0,T] \times \mathbf{R}^d \to \mathbf{R}^d$, $\sigma : [0,T] \times \mathbf{R}^d \to \mathbf{R}^{d\times d}$ and $\varepsilon \in (0,1]$.

We introduce the associated BSDE as follows:

$$Y_t^\varepsilon = g(X_T^\varepsilon) + \int_t^T f(s, X_s^\varepsilon, Y_s^\varepsilon, Z_s^\varepsilon)ds - \int_t^T Z_s^\varepsilon dW_s, \tag{44}$$

where $g : \mathbf{R}^d \to \mathbf{R}$ and $f : [0,T] \times \mathbf{R}^d \times \mathbf{R} \times \mathbf{R}^{d\times d} \to \mathbf{R}$. We put Assumption 5.1. Remark that for $\varepsilon = 0$, the forward SDE X_t^0 degenerates, then BSDE Y_t^ε is well-defined for $\varepsilon \in (0,1]$.

Let $(Y^{\varepsilon,k})_k$ be a sequence of linear BSDEs :

$$\begin{aligned}
Y_t^{\varepsilon,0} &= g(X_T^\varepsilon) + \int_t^T f(s, X_s^\varepsilon, 0, 0)ds - \int_t^T Z_s^{\varepsilon,0} dW_s, \\
Y_t^{\varepsilon,1} &= g(X_T^\varepsilon) + \int_t^T f(s, X_s^\varepsilon, Y_s^{\varepsilon,0}, Z_s^{\varepsilon,0})ds - \int_t^T Z_s^{\varepsilon,1} dW_s, \\
Y_t^{\varepsilon,k+1} &= g(X_T^\varepsilon) + \int_t^T f(s, X_s^\varepsilon, Y_s^{\varepsilon,k}, Z_s^{\varepsilon,k})ds - \int_t^T Z_s^{\varepsilon,k+1} dW_s, \quad k \geq 0.
\end{aligned}$$

It is well-known that this sequence converges to non-linear BSDE Y^ε under a suitable norm:

$$Y^{\varepsilon,k} \to Y^\varepsilon, \quad as \quad k \to \infty.$$

For $\varepsilon \in (0,1]$, we define $u^\varepsilon : [0,T] \times \mathbf{R}^d \to \mathbf{R}$ as

$$u^\varepsilon(t,x) := Y_t^{\varepsilon,t,x} = E[g(X_T^{\varepsilon,t,x})] + E\left[\int_t^T f(s, X_s^{\varepsilon,t,x}, Y_s^{\varepsilon,t,x}, Z_s^{\varepsilon,t,x}) ds\right], \tag{45}$$

where $(X^{\varepsilon,t,x}, Y^{\varepsilon,t,x}, Z^{\varepsilon,t,x})$ denote the adapted solutions to the SDE's (38) and (44), restricted to $[t,T]$ with $X_t^{\varepsilon,t,x} = x$, *a.s.*. Under Assumption 5.1, the representation of Ma and Zhang (2002) holds, and for $\varepsilon \in (0,1]$ we define $\nabla_x u^\varepsilon \sigma$ on $[0,T] \times \mathbf{R}^d$ as

$$\begin{aligned}(\nabla_x u^\varepsilon \sigma)(t,x) &:= (\nabla_x u^\varepsilon(t,x))\varepsilon\sigma(t,x) \\ &= E[g(X_t^{\varepsilon,t,x}) N_T^{\varepsilon,t}]\varepsilon\sigma(t,x) + E\left[\int_t^T f(s, X_s^{\varepsilon,t,x}, Y_s^{\varepsilon,t,x}, Z_s^{\varepsilon,t,x}) N_s^{\varepsilon,t} ds\right]\varepsilon\sigma(t,x), \end{aligned} \tag{46}$$

where

$$N_u^{\varepsilon,t} = \frac{1}{\varepsilon(u-t)} \int_t^u \sigma^{-1}(\tau, X_\tau^{\varepsilon,t,x}) \nabla_x X_\tau^{\varepsilon,t,x} dW_\tau. \tag{47}$$

Also, under Assumption 5.1, remark that the solution to SDE $X_s^{\varepsilon,t,x}$ $(0 \le t < s \le T)$ has a smooth density $p^\varepsilon(t,s,x,y)$ and then we define a sequence $(u^{\varepsilon,k}, \nabla_x u^{\varepsilon,k}\sigma)_{k\ge 0}$.

$$\begin{aligned}
u^{\varepsilon,0}(t,x) &:= E[g(X_T^{\varepsilon,t,x})] + E\left[\int_t^T f(s, X_s^{\varepsilon,t,x}, 0, 0) ds\right] \\
&= \int_{\mathbf{R}^d} g(y) p^\varepsilon(t,T,x,y) dy + \int_t^T \int_{\mathbf{R}^d} f(s,y,0,0) p^\varepsilon(t,s,x,y) dy ds, \\
(\nabla_x u^{\varepsilon,0}\sigma)(t,x) &:= (\nabla_x u^{\varepsilon,0}(t,x))\varepsilon\sigma(t,x) \\
&= E[g(X_T^{\varepsilon,t,x}) N_T^{\varepsilon,t}]\varepsilon\sigma(t,x) + E\left[\int_t^T f(s, X_s^{\varepsilon,t,x}, 0, 0) N_s^{\varepsilon,t} ds\right]\varepsilon\sigma(t,x) \\
&= \int_{\mathbf{R}^d} g(y) E[N_T^{\varepsilon,t} | X_T^{\varepsilon,t,x} = y] p^\varepsilon(t,T,x,y) dy \varepsilon\sigma(t,x) \\
&\quad + \int_t^T \int_{\mathbf{R}^d} f(s,y,0,0) E[N_s^{\varepsilon,t} | X_s^{\varepsilon,t,x} = y] p^\varepsilon(t,s,x,y) dy ds \varepsilon\sigma(t,x), \\
u^{\varepsilon,k+1}(t,x) &:= E[g(X_T^{\varepsilon,t,x})] + E\left[\int_t^T f(s, X_s^{\varepsilon,t,x}, Y_s^{\varepsilon,k}, Z_s^{\varepsilon,k}) ds\right] \\
&= \int_{\mathbf{R}^d} g(y) p^\varepsilon(t,T,x,y) dy \\
&\quad + \int_t^T \int_{\mathbf{R}^d} f(s, y, u^{\varepsilon,k}(s,y), \nabla_x u^{\varepsilon,k}\sigma(s,y)) p^\varepsilon(t,s,x,y) dy ds, \quad k \in \mathbf{N}, \\
\nabla_x u^{\varepsilon,k+1}\sigma(t,x) &:= (\nabla_x u^{\varepsilon,k+1}(t,x))\varepsilon\sigma(t,x) \\
&= E[g(X_T^{\varepsilon,t,x}) N_T^{\varepsilon,t}]\varepsilon\sigma(t,x) + E\left[\int_t^T f(s, X_s^{\varepsilon,t,x}, Y_s^{\varepsilon,k}, Z_s^{\varepsilon,k}) N_s^{\varepsilon,t} ds\right]\varepsilon\sigma(t,x) \\
&= \int_{\mathbf{R}^d} g(y) E[N_T^{\varepsilon,t} | X_T^{\varepsilon,t,x} = y] p^\varepsilon(t,T,x,y) dy \varepsilon\sigma(t,x) \\
&\quad + \int_t^T \int_{\mathbf{R}^d} f(s, y, u^{\varepsilon,k}(s,y), (\nabla_x u^{\varepsilon,k}\sigma)(s,y)) E[N_s^{\varepsilon,t} | X_s^{\varepsilon,t,x} = y] p^\varepsilon(t,s,x,y) dy ds \varepsilon\sigma(t,x), \quad k \in \mathbf{N}.
\end{aligned}$$

5.2.1 Asymptotic Expansion Formula

We approximate X_t^ε by an *asymptotic expansion around the solution to ordinary differential equation* X_t^0

$$dX_t^0 = b(t, X_t^0) dt, \ X_0^0 = x. \tag{48}$$

Hereafter, let us denote $X_{i,T}^{t,x,\varepsilon}$ by $\frac{1}{i!}\frac{\partial^i}{\partial\varepsilon^i} X_T^{t,x,\varepsilon}$, $i \in \mathbf{N}$. In the first place, we provide a key result as the lemma below.

Lemma 5.1 *For* $s \in (t,T]$,

$$X_{i,s}^{t,x,\varepsilon} \in \mathcal{K}_i^T, \ i \in \mathbf{N}. \tag{49}$$

Let $X_{i,s}^{0,t,x}$ by $\frac{1}{i!}\frac{\partial^i}{\partial\varepsilon^i}X_s^{\varepsilon,t,x}|_{\varepsilon=0}$, $i\in\mathbf{N}$. For every $p\in(1,\infty)$, $k\in\mathbf{N}$ and $N\in\mathbf{N}$,

$$X_s^{\varepsilon,t,x}=X_t^0+\sum_{i=1}^{N}\varepsilon^i X_{i,s}^{0,t,x}+O(\varepsilon^{N+1})\ \ in\ \ \mathbf{D}^{k,p}\ \ as\ \varepsilon\downarrow 0. \tag{50}$$

Hereafter, we derive an asymptotic expansion of density of $X_T^{\varepsilon,t,x}$. Let

$$F_T^{\varepsilon,t,x}:=\frac{X_T^{\varepsilon,t,x}-X_T^{0,t,x}}{\varepsilon}. \tag{51}$$

Then,

$$F_T^{\varepsilon,t,x}=F_T^{0,t,x}+\sum_{i=1}^{N}\varepsilon^i F_{i,T}^{0,t,x}+O(\varepsilon^{N+1})\ \ in\ \mathbf{D}^\infty, \tag{52}$$

where $F_T^{0,t,x}=X_{1,T}^{t,x}$, $F_{i,T}^{0,t,x}=X_{i-1,T}^{0,t,x}$, $i\geq 1$.

Remark that although obviously $F_t^{\varepsilon,t,x}=0$, we use the notations $F_u^{\varepsilon,t,x}$, $X_{1,u}^{t,x}$, $X_{k,u}^{t,x}$, $k\geq 0$ meaning its dependence on x when $u>t$.

Let $\Sigma(t,T;x)=\{\Sigma_{i,j}(t,T;x)\}_{i,j}$ be the $d\times d$-matrix whose element is defined by

$$\Sigma_{i,j}(t,T;x)=\sum_{k=1}^{d}\int_t^T\hat{\sigma}_k^i(s,X_s^{0,t,x})\hat{\sigma}_k^j(s,X_s^{0,t,x})ds,\ \ 1\leq i,j\leq d, \tag{53}$$

where

$$\hat{\sigma}_k^i(s,X_s^{0,t,x})=(\nabla_x X_T^{0,t,x}(\nabla_x X_s^{0,t,x})^{-1}\sigma_k(s,X_s^{0,t,x}))^i. \tag{54}$$

Hereafter, we use abbreviated notations such as $F_T^{\varepsilon,t}$, $X_{1,T}^t$, $X_{k,T}^t$, $k\geq 0$ and $\Sigma_{i,j}(t,T)$, $1\leq i,j\leq d$ in stead of $F_t^{\varepsilon,t,x}$, $X_{1,T}^{t,x}$, $X_{k,T}^{t,x}$, $k\geq 0$ and $\Sigma_{i,j}(t,T;x)$, $1\leq i,j\leq d$ respectively. Under Assumption 5.1 we obtain the following expansions for $E[\varphi(X_T^{\varepsilon,t,x})]$ with φ of polynomial growth rate and $E[g(X_T^{\varepsilon,t,x})]$ with Lipschitz function g: they are useful for giving the properties of the expansion of Y^ε and proving our main result Theorem 5.1. We also characterize the Malliavin weights appearing in expansions as Kusuoka functions.

Proposition 5.1 *1. For a measurable function $\varphi:\mathbf{R}^d\to\mathbf{R}$ of at most polynomial growth, there exists non-negative, non-decreasing and finite function $C(T,N,x)$ of at most polynomial growth in x depending on T and N such that*

$$\left|E[\varphi(X_T^{\varepsilon,t,x})]-\left\{E[\varphi(\bar{X}_T^{0,t,x})]+\sum_{i=1}^{N}\varepsilon^i E[\varphi(\bar{X}_T^{0,t,x})\pi_{i,T}^{0,t,x}]\right\}\right|\leq\varepsilon^{N+1}C(T,N,x)(T-t)^{(N+1)/2}, \tag{55}$$

where $\bar{X}_T^{0,t,x}=X_T^{0,t,x}+\varepsilon X_{1,T}^{t,x}$ and $\pi_{i,T}^{0,t,x}=\sum_k^{(i)}H_{\alpha^{(k)}}(X_{1,T}^{0,t,x},\prod_{l=1}^{k}X_{\beta_l+1,T}^{0,t,x,\alpha_l})\in\mathcal{K}_i^T$, $i=1,\cdots,N$. Here,

$$\sum_k^{(i)}\equiv\sum_{k=1}^{i}\sum_{\beta_1+\cdots+\beta_k=i,\beta_j\geq 1}\sum_{\alpha^{(k)}\in\{1,\cdots,d\}^k}\frac{1}{k!}.$$

2. For a Lipschitz function $g:\mathbf{R}^d\to\mathbf{R}$ with constant C_g, there exists $C(T,N,x)$ depending on C_g, T, N and x such that

$$\left|E[g(X_T^{\varepsilon,t,x})]-\left\{E[g(\bar{X}_T^{0,t,x})]+\sum_{i=1}^{N}\varepsilon^i E[g(\bar{X}_T^{0,t,x})\pi_{i,T}^{0,t,x}]\right\}\right|\leq\varepsilon^{N+1}C(T,N,x)(T-t)^{(N+2)/2}, \tag{56}$$

where $\bar{X}_T^{0,t,x}$ and $\pi_{i,T}^{0,t,x}$, $i=1,\cdots,N$ are same in 1.

Proof.

1. Let $\delta_y(\cdot)$ be the delta function. Then, $\delta_y(F_T^{\varepsilon,t,x})\in\mathbf{D}^{-\infty}$ is expanded as follows:

$$\begin{aligned}\delta_y(F_T^{\varepsilon,t,x}) &= \delta_y(F_T^{0,t,x})+\sum_{i=1}^{N}\frac{\varepsilon^i}{i!}\frac{\partial^i}{\partial\varepsilon^i}\delta_y(F_T^{\varepsilon,t,x})|_{\varepsilon=0}\\ &\quad+\varepsilon^{N+1}\int_0^1\frac{(1-u)^N}{N!}\frac{\partial^{N+1}}{\partial\nu^{N+1}}\delta_y(F_T^{\nu,t,x})|_{\nu=0}du.\end{aligned} \tag{57}$$

Therefore, the density of $F_T^{\varepsilon,t,x}$ is calculated as follows:

$$
\begin{aligned}
& p^{F^\varepsilon}(t,T,0,y) = E[\delta_y(F_T^{\varepsilon,t,x})] && (58)\\
= \quad & E[\delta_y(F_T^{0,t,x})] + \sum_{i=1}^N \frac{\varepsilon^i}{i!} E\left[\frac{\partial^i}{\partial\varepsilon^i}\delta_y(F_T^{\varepsilon,t,x})|_{\varepsilon=0}\right] && (59)\\
& +\varepsilon^{N+1}\int_0^1 \frac{(1-u)^N}{N!} E\left[\frac{\partial^{N+1}}{\partial\nu^{N+1}}\delta_y(F_T^{\nu,t,x})|_{\nu=\varepsilon u}\right] du \\
= \quad & E[\delta_y(F_T^{0,t,x})] + \sum_{i=1}^N \varepsilon^i \sum_k^{(i)} E[\partial_\alpha^k \delta_y(F_T^{0,t,x})\prod_{l=1}^k F_{\beta_l,T}^{0,\alpha_l,t,x}] && (60)\\
& +\varepsilon^{N+1}\int_0^1 (1-u)^N (N+1) \sum_k^{(N+1)} E\left[\partial_\alpha^k\delta_y(F_T^{\varepsilon u,t,x})\prod_{l=1}^k F_{\beta_l,T}^{\varepsilon u,\alpha_l,t,x}\right] du \\
= \quad & E[\delta_y(F_T^{0,t,x})] + \sum_{i=1}^N \varepsilon^i E[\delta_y(F_T^{0,t,x})\pi_{i,T}^{0,t,x}] + \varepsilon^{N+1}\int_0^1 (1-u)^N E[\delta_y(F_T^{\varepsilon u,t,x})\tilde{\pi}_{N+1,T}^{\varepsilon u,t,x}]du. && (61)
\end{aligned}
$$

Here, we use the integration by parts

$$
\sum_k^{(i)} E[\partial_\alpha^k\delta_y(F_T^{0,t,x})\prod_{l=1}^k F_{\beta_l,T}^{0,\alpha_l,t,x}] = E[\delta_y(F_T^{0,t,x})\pi_{i,T}^{0,t,x}] \tag{62}
$$

with

$$
\pi_{i,T}^{0,t,x} = \sum_k^{(i)} H_\alpha(F_T^{0,t,x}, \prod_{l=1}^k F_{\beta_l,T}^{0,\alpha_l,t,x}) = \sum_k^{(i)} H_\alpha(X_{1,T}^{0,t,x}, \prod_{l=1}^k X_{\beta_l+1,T}^{0,\alpha_l,t,x}) \in \mathcal{K}_i^T, \tag{63}
$$

and

$$
(N+1)\sum_k^{(N+1)} E[\partial_\alpha^k\delta_y(F_T^{\varepsilon,t,x})\prod_{l=1}^k F_{\beta_l,T}^{\varepsilon,\alpha_l,t,x}] = E[\delta_y(F_T^{\varepsilon,t,x})\tilde{\pi}_{N+1,T}^{\varepsilon,t,x}] \tag{64}
$$

with $\tilde{\pi}_{N+1,T}^{\varepsilon,t,x} = (N+1)\sum_k^{(N+1)} H_\alpha(F_T^{\varepsilon,t,x}, \prod_{l=1}^k F_{\beta_l,T}^{\varepsilon,\alpha_l,t,x}) \in \mathcal{K}_{N+1}^T$. Remark that the following relation holds:

$$
\begin{aligned}
E\left[\delta_y(\bar{X}_T^{0,t,x})\right] &= E\left[\delta_{(y-X_T^0)/\varepsilon}(F_T^{0,t,x})\right]\frac{1}{\varepsilon^d},\\
E\left[\delta_y(X_T^{\varepsilon,t,x})\right] &= E\left[\delta_{(y-X_T^0)/\varepsilon}(F_T^{\varepsilon,t,x})\right]\frac{1}{\varepsilon^d}.
\end{aligned}
$$

We have

$$
\begin{aligned}
p^\varepsilon(t,T,x,y) \quad &= \quad \varepsilon^{-d}E[\delta_{(y-X_T^0)/\varepsilon}(X_{1,T}^{0,t,x})] + \sum_{i=1}^N \varepsilon^{i-d}E[\delta_{(y-X_T^0)/\varepsilon}(X_{1,T}^{0,t,x})\pi_{i,T}^{0,t,x}] \\
& \quad\quad +\varepsilon^{N+1}\int_0^1 (1-u)^N (u\varepsilon)^{-d} E[\delta_{(y-X_T^0)/u\varepsilon}(F_T^{u\varepsilon,t})\tilde{\pi}_{N+1,T}^{u\varepsilon,t,x}]du && (65)\\
&= \quad E[\delta_y(\bar{X}_T^0)] + \sum_{i=1}^N \varepsilon^i E[\delta_y(\bar{X}_T^0)\pi_{i,T}^{0,t,x}] \\
& \quad\quad +\varepsilon^{N+1}\int_0^1 (1-u)^N E[\delta_y(X_T^{u\varepsilon,t,x})\tilde{\pi}_{N+1,T}^{u\varepsilon,t,x}]du && (66)\\
&= \quad p^0(t,T,x,y) + \sum_{i=1}^N \varepsilon^i E[\pi_{i,T}^{0,t,x}|X_{1,T}^{0,t,x}=y]p^0(t,T,x,y) && (67)\\
& \quad\quad +\varepsilon^{N+1}\int_0^1 (1-u)^N E[\tilde{\pi}_{N+1,T}^{u\varepsilon,t,x}|X_T^{u\varepsilon,t,x}=y]p^\varepsilon(t,T,x,y)du. && (68)
\end{aligned}
$$

where $\bar{X}_T^{0,t,x} = X_T^{0,t,x} + \varepsilon X_{1,T}^t$ and

$$
p^0(t,T,x,y) = \frac{1}{(2\pi\varepsilon^2)^{d/2}\det(\Sigma(t,T))^{1/2}} e^{-\frac{(y-X_T^{0,t,x})\Sigma^{-1}(t,T)(y-X_T^{0,t,x})^T}{2\varepsilon^2}}. \tag{69}
$$

Therefore, we have

$$E[\varphi(X_T^{\varepsilon,t,x})] = \int_{\mathbf{R}^d}\varphi(y)p^\varepsilon(t,T,x,y)dy \tag{70}$$

$$= \int_{\mathbf{R}^d}\varphi(y)p^0(t,T,x,y)dy + \sum_{i=1}^N \varepsilon^i \int_{\mathbf{R}^d}\varphi(y)E[\pi_{i,T}^{0,t,x}|\bar{X}_T^{0,t,x}=y]p^0(t,T,x,y)dy \tag{71}$$

$$+\varepsilon^{N+1}\int_0^1(1-u)^N\int_{\mathbf{R}^d}\varphi(y)E[\tilde{\pi}_{N+1,T}^{\varepsilon u,t,x}|X_T^{\varepsilon u,t,x}=y]p^{\varepsilon u}(t,T,x,y)dydu \tag{72}$$

$$= E[\varphi(\bar{X}_T^{0,t,x})] + \sum_{i=1}^N \varepsilon^i E[\varphi(\bar{X}_T^{0,t,x})\pi_{i,T}^{0,t,x}] + \varepsilon^{N+1}\int_0^1 (1-u)^N E[\varphi(X_T^{\varepsilon,t,x})\tilde{\pi}_{N+1,T}^{u\varepsilon,t,x}]du. \tag{73}$$

The residual terms is estimated by the following inequality:

$$|E[\varphi(X_T^{\varepsilon,t,x})\tilde{\pi}_{N+1,T}^{\varepsilon,t,x}]| \le \|\varphi(X_T^{\varepsilon,t,x})\|_{L^q}\|\tilde{\pi}_{N+1,T}^{\varepsilon,t,x}\|_{L^p} \le C(T,N,x)(T-t)^{(N+1)/2}. \tag{74}$$

2. We have

$$\begin{aligned}\int_{\mathbf{R}^d} g(y)p^\varepsilon(t,T,x,y)dy &= \int_{\mathbf{R}^d} g(y)p^0(t,T,x,y)dy \\ &\quad + \sum_{i=1}^N \varepsilon^i \int_{\mathbf{R}^d} g(y)E[\pi_{i,T}^{0,t}|\bar{X}_T^{0,t,x}=y]p^0(t,T,x,y)dy \\ &\quad +\varepsilon^{N+1}\int_0^1(1-u)^N\int_{\mathbf{R}^d} g(y)E[\tilde{\pi}_{N+1,T}^{u\varepsilon,t}|X_T^{u\varepsilon,t,x}=y]p^{u\varepsilon}(t,T,x,y)dy,\end{aligned}$$

Let $(g_n)_{n\in\mathbf{N}}\subset C_b^\infty$ be a mollifier converging to g. For $i\in\mathbf{N}$, there exists $\zeta_{i,T}^{\varepsilon,t}\in\mathcal{K}_{i+1}^T$ such that

$$\left|E\left[g_n(X_T^{\varepsilon,t,x})\tilde{\pi}_{i,T}^{\varepsilon,t}\right]\right| = \left|E\left[\nabla_x g_n(X_T^{\varepsilon,t,x})\zeta_{i,T}^{\varepsilon,t}\right]\right| \le \|\nabla_x g_n\|_\infty \|\zeta_{i,T}^{\varepsilon,t}\|_{L^1}. \tag{75}$$

Then,

$$\left|E\left[g(X_T^{\varepsilon,t,x})\tilde{\pi}_{N+1,T}^{\varepsilon,t}\right]\right| \le C_g\|\zeta_{N+1,T}^{\varepsilon,t}\|_{L^1} \le C(T,N)(T-t)^{(N+2)/2}. \tag{76}$$

We also obtain expansions for $E[\varphi(X_T^{\varepsilon,t,x})N_T^{\varepsilon,t,x}]\varepsilon\sigma(t,x)$ with φ of polynomial growth rate and $E[g(X_T^{\varepsilon,t,x})N_T^{\varepsilon,t,x}]\varepsilon\sigma(t,x)$ with Lipschitz function g: they are useful for giving the properties of the expansion of Z^ε.

Proposition 5.2 *1. For a measurable function $\varphi:\mathbf{R}^d\to\mathbf{R}$ of at most polynomial growth, there exists non-negative, non-decreasing and finite function $C(T,N,x)$ of at most polynomial growth in x depending on T and N such that*

$$\begin{aligned}&\left|E[\varphi(X_T^{\varepsilon,t,x})N_{0,T}^{\varepsilon,t,x}]\varepsilon\sigma(t,x) - \left\{E[\varphi(\bar{X}_T^{0,t,x})N_{0,T}^{0,t,x}] + \sum_{i=1}^N \varepsilon^i E[\varphi(\bar{X}_T^{0,t,x})N_{i,T}^{0,t,x}]\right\}\varepsilon\sigma(t,x)\right| \\ &\le \varepsilon^{N+1}C(T,N,x)(T-t)^{N/2},\end{aligned} \tag{77}$$

where $\bar{X}_T^{0,t,x}=X_T^{0,t,x}+\varepsilon X_{1,T}^{t,x}$, $N_{0,T}^{0,t,x}=(N_{0,T}^{0,t,x,1}\cdots,N_{0,T}^{0,t,x,d})$ and $N_{i,T}^{0,t,x}=(N_{i,T}^{0,t,x,1},\cdots,N_{i,T}^{0,t,x,d})$, $i=1,\cdots,d$ are given by

$$N_{0,T}^{0,t,x,k} = \sum_{j=1}^d H_{(j)}(\bar{X}_T^{0,t,x},\partial_k\bar{X}_T^{0,t,x,j}) \in \mathcal{K}_{-1}^T,\quad 1\le k\le d, \tag{78}$$

and

$$N_{i,T}^{0,t,x,k} = \sum_{j=1}^d H_{(j)}(\bar{X}_T^{0,t,x},\partial_k\bar{X}_T^{0,t,x,j}\pi_{i,T}^{0,t,x}) + \partial_k\pi_{i,T}^{0,t,x} \in \mathcal{K}_{i-1}^T,\quad 1\le k\le d. \tag{79}$$

2. For a Lipschitz function $g:\mathbf{R}^d\to\mathbf{R}$ with constant C_g, there exists $C(T,N,x)$ depending on C_g, T, N and x such that

$$\begin{aligned}&\left|E[g(X_T^{\varepsilon,t,x})N_{0,T}^{\varepsilon,t}]\varepsilon\sigma(t,x) - \left\{E[g(\bar{X}_T^{0,t,x})N_{0,T}^{0,t}] + \sum_{i=1}^N \varepsilon^i E[g(\bar{X}_T^{0,t,x})N_{i,T}^{0,t}]\right\}\varepsilon\sigma(t,x)\right| \\ &\le \varepsilon^{N+1}C(T,N,x)(T-t)^{(N+1)/2},\end{aligned} \tag{80}$$

where $\bar{X}_T^{0,t,x}$, $N_{0,T}^{0,t,x}$ and $N_{i,T}^{0,t,x}$, $i=1,\cdots,d$ are same in 1.

Proof.

1. We differentiate the expansion of $E[\varphi(X_T^{\varepsilon,t,x})]$ with respect to initial x as follows:

$$\begin{aligned}\nabla_x E[\varphi(X_T^{\varepsilon,t,x})] &= \nabla_x E[\varphi(\bar{X}_T^{0,t,x})] + \sum_{i=1}^N \varepsilon^i \nabla_x E[\varphi(\bar{X}_T^{0,t,x})\pi_{i,T}^{0,t,x}] \\ &\quad + \varepsilon^{N+1} \int_0^1 (1-u)^N \nabla_x E[\varphi(X_T^{\varepsilon u,t,x})\tilde{\pi}_{N+1,T}^{u\varepsilon,t,x}] du. \end{aligned} \tag{81}$$

For a smooth sequence $(\varphi_n)_{n\in\mathbf{N}}$ converges to φ, we have

$$\nabla_x E[\varphi_n(X_T^{\varepsilon,t,x})] = E[\varphi_n(X_T^{\varepsilon,t,x})N_T^{\varepsilon,t,x}], \tag{82}$$

with $N_T^{\varepsilon,t,x} \in \mathcal{K}_{-1}^T$ and for $1 \le k \le d$,

$$\begin{aligned}\frac{\partial}{\partial x_k} E[\varphi_n(\bar{X}_T^{0,t,x})] &= \sum_{j=1}^d E[\partial_j \varphi_n(\bar{X}_T^{0,t,x})\partial_k \bar{X}_T^{0,t,x,j}] \\ &= E[\varphi_n(\bar{X}_T^{0,t,x})N_{0,T}^{t,x,k}], \end{aligned} \tag{83}$$

with

$$N_{0,T}^{t,x,k} = \sum_{j=1}^d H_{(j)}(\bar{X}_T^{0,t,x}, \partial_k \bar{X}_T^{0,t,x,j}) \in \mathcal{K}_{-1}^T. \tag{84}$$

Also, we have for $1 \le i \le N$, $1 \le k \le d$

$$\begin{aligned}&\frac{\partial}{\partial x_k} E[\varphi_n(\bar{X}_T^{0,t,x})\pi_{i,T}^{0,t,x}] \\ &= \sum_{j=1}^d \{E[\partial_j \varphi_n(\bar{X}_T^{0,t,x})\partial_k \bar{X}_T^{0,t,x,j}\pi_{i,T}^{0,t,x}] + E[\varphi_n(\bar{X}_T^{0,t,x})\partial_k \pi_{i,T}^{0,t,x}]\} \\ &= E[\varphi(\bar{X}_T^{0,t,x})N_{i,T}^{t,x,k}], \end{aligned} \tag{85}$$

with

$$N_{i,T}^{t,x,k} = \sum_{j=1}^d H_{(j)}(\bar{X}_T^{0,t,x}, \partial_k \bar{X}_T^{0,t,x,j}\pi_{i,T}^{0,t,x}) + \partial_k \pi_{i,T}^{0,t,x} \in \mathcal{K}_{i-1}^T, \tag{86}$$

and

$$\begin{aligned}&\frac{\partial}{\partial x_k} E[\varphi_n(X_T^{\varepsilon,t,x})\tilde{\pi}_{N+1,T}^{\varepsilon,t,x}] \\ &= \sum_{j=1}^d E[\partial_j \varphi_n(X_T^{\varepsilon,t,x})\partial_k X_T^{\varepsilon,t,x,j}\tilde{\pi}_{N+1,T}^{u\varepsilon,t,x}] + E[\varphi(X_T^{\varepsilon,t,x})\partial_k \tilde{\pi}_{N+1,T}^{\varepsilon,t,x}] \\ &= E[\varphi(X_T^{\varepsilon,t,x})\tilde{N}_{N+1,T}^{\varepsilon,t,x,k}]. \end{aligned} \tag{87}$$

with

$$\tilde{N}_{N+1,T}^{\varepsilon,t,x,k} = \sum_{j=1}^d H_{(j)}(X_T^{\varepsilon,t,x}, \partial_k X_T^{\varepsilon,t,x,j}\tilde{\pi}_{N+1,T}^{\varepsilon,t,x}) + \partial_k \tilde{\pi}_{N+1,T}^{\varepsilon,t,x}. \tag{88}$$

Here, we have for $1 \le k \le d$,

$$\partial_j X_T^{\varepsilon,t,x}\tilde{\pi}_{N+1,T}^{\varepsilon,t,x} \in \mathcal{K}_{N+1}^T,\ \ \partial_j \tilde{\pi}_{N+1,T}^{\varepsilon,t,x} \in \mathcal{K}_{N+1}^T, \tag{89}$$

and then $\tilde{N}_{N+1,T}^{\varepsilon,t,x} \in \mathcal{K}_N^T$. Therefore,

$$\begin{aligned}&E[\varphi(X_T^{\varepsilon,t,x})N_{0,T}^{\varepsilon,t}] - \left\{E[\varphi(\bar{X}_T^{0,t,x})N_{0,T}^{0,t}] + \sum_{i=1}^N \varepsilon^i E[\varphi(\bar{X}_T^{0,t,x})N_{i,T}^{0,t}]\right\} \\ &= \varepsilon^{N+1} \int_0^1 (1-u)^N E[\varphi(X_T^{\varepsilon u,t,x})\tilde{N}_{N+1,T}^{u\varepsilon,t,x}] du, \end{aligned} \tag{90}$$

and

$$|E[\varphi(X_T^{\varepsilon u,t,x})\tilde{N}_{N+1,T}^{u\varepsilon,t,x}]| \le C(T,N,x)(T-t)^{N/2}. \tag{91}$$

2. Let $(g_n)_{n\in\mathbf{N}}\subset C_b^\infty$ be a mollifier converging to g. For $i\in\mathbf{N}$, there exists $\zeta^{\varepsilon,t,x}_{N+1,T}\in\mathcal{K}^T_{(N+1)+1}$ such that

$$E\left[g_n(X_T^{\varepsilon,t,x})\tilde{N}^{\varepsilon,t}_{N+1,T}\right]=\frac{\partial}{\partial x_k}E\left[g_n(X_T^{\varepsilon,t,x})\tilde{\pi}^{\varepsilon,t}_{N+1,T}\right]=\frac{\partial}{\partial x_k}\sum_{j=1}^d E\left[\frac{\partial}{\partial x_j}g_n(X_T^{\varepsilon,t,x})\zeta^{\varepsilon,t,x,j}_{N+1,T}\right].$$

Then,

$$\begin{aligned}E\left[g_n(X_T^{\varepsilon,t,x})\tilde{N}^{\varepsilon,t}_{N+1,T}\right] &= \sum_{l,j=1}^d E\left[\frac{\partial^2}{\partial x_l\partial x_j}g_n(X_T^{\varepsilon,t,x})\frac{\partial}{\partial x_k}X_T^{\varepsilon,t,x,l}\zeta^{\varepsilon,t,x,j}_{N+1,T}\right]\\ &\quad+\sum_{j=1}^d E\left[\frac{\partial}{\partial x_j}g_n(X_T^{\varepsilon,t,x})\frac{\partial}{\partial x_k}\zeta^{\varepsilon,t,x,j}_{N+1,T}\right]\\ &= \sum_{j=1}^d E\left[\frac{\partial}{\partial x_j}g_n(X_T^{\varepsilon,t,x})\Phi^{\varepsilon,t,x,j}_{N+1,T}\right]\end{aligned}$$

with

$$\Phi^{\varepsilon,t,x,j}_{N+1,T}=\left\{\sum_{l=1}^d H_{(l)}(X_T^{\varepsilon,t,x,j},\frac{\partial}{\partial x_k}X_T^{\varepsilon,t,x,l}\zeta^{\varepsilon,t,x,j}_{N+1,T})+\frac{\partial}{\partial x_k}\zeta^{\varepsilon,t,x,j}_{N+1,T}\right\}\quad\in\mathcal{K}^T_{N+1}.$$

Therefore,

$$\left|E\left[g(X_T^{\varepsilon,t,x})N^{\varepsilon,t}_{N+1,T}\right]\right|\le C(T,N,x)(T-t)^{(N+1)/2}. \tag{92}$$

□

Using the weights $\pi^{0,t,x}_{i,s}$ and $N^{0,t,x}_{i,s}$, $i=0,1,\cdots,N$, in expansions in Proposition 5.1 and Proposition 5.2 , we define recursive approximation formulas for $(u^\varepsilon,\nabla_x u^\varepsilon\sigma)$. For $\varepsilon\in(0,1]$, define $u^{\varepsilon,k,N}$ and $\nabla_x u^{\varepsilon,k,N}\sigma$ on $[0,T]\times\mathbf{R}^d$, $k\ge 0, N\ge 1$ as follows. Let $u^{\varepsilon,0,N}$ be

$$\begin{aligned}u^{\varepsilon,0,N}(t,x)&:=\int_{\mathbf{R}^d}g(y)\left\{p^0(t,T,x,y)+\sum_{i=1}^N\varepsilon^i E[\pi^{0,t}_{i,T}|\bar{X}^{0,t,x}_T=y]p^0(t,T,x,y)\right\}dy\\ &+\int_t^T\int_{\mathbf{R}^d}f(s,y,0,0)\\ &\qquad\left\{p^0(t,s,x,y)+\sum_{i=1}^N\varepsilon^i E[\pi^{0,t}_{i,T}|\bar{X}^{0,t,x}_s=y]p^0(t,s,x,y)\right\}dyds,\end{aligned}$$

and let $\nabla_x u^{\varepsilon,0,N}\sigma$ be

$$\begin{aligned}(\nabla_x u^{\varepsilon,0,N}\sigma)(t,x)&:=(\nabla_x u^{\varepsilon,0,N}(t,x))\varepsilon\sigma(t,x)\\ &=\int_{\mathbf{R}^d}g(y)E[N^{0,t}_{0,T}|\bar{X}^{0,t,x}_T=y]p^0(t,T,x,y)dy\varepsilon\sigma(t,x)\\ &\quad+\sum_{i=1}^N\varepsilon^i\int_{\mathbf{R}^d}g(y)E[N^{0,t}_{i,T}|\bar{X}^{0,t,x}_T=y]p^0(t,T,x,y)dy\varepsilon\sigma(t,x)\\ &+\int_t^T\int_{\mathbf{R}^d}f(s,y,0,0)E[N^{0,t}_{0,s}|\bar{X}^{0,t,x}_s=y]p^0(t,s,x,y)dyds\varepsilon\sigma(t,x)\\ &\quad+\sum_{i=1}^N\varepsilon^i\int_t^T\int_{\mathbf{R}^d}f(s,y,0,0)E[N^{0,t}_{i,s}|\bar{X}^{0,t,x}_s=y]p^0(t,s,x,y)dyds\varepsilon\sigma(t,x).\end{aligned}$$

For $k\ge 0$, let $u^{\varepsilon,k+1,N}$ be

$$\begin{aligned}u^{\varepsilon,k+1,N}(t,x)&:=\int_{\mathbf{R}^d}g(y)\left\{p^0(t,T,x,y)+\sum_{i=1}^N\varepsilon^i E[\pi^{0,t}_{i,T}|\bar{X}^{0,t,x}_T=y]p^0(t,T,x,y)\right\}dy \qquad (93)\\ &+\int_t^T\int_{\mathbf{R}^d}f(s,y,u^{\varepsilon,k,N}(s,y),(\nabla_x u^{\varepsilon,k,N}\sigma)(s,y))\\ &\qquad\left\{p^0(t,s,x,y)+\sum_{i=1}^N\varepsilon^i E[\pi^{0,t}_{i,T}|\bar{X}^{0,t,x}_s=y]p^0(t,s,x,y)\right\}dyds,\end{aligned}$$

and let $\nabla_x u^{\varepsilon,k+1,N}\sigma$ be

$$
\begin{aligned}
(\nabla_x u^{\varepsilon,k+1,N}\sigma)(t,x) &:= (\nabla_x u^{\varepsilon,k+1,N}(t,x))\varepsilon\sigma(t,x) \qquad (94)\\
&= \int_{\mathbf{R}^d} g(y)E[N^{0,t}_{0,T}|\bar{X}^{0,t,x}_T = y]p^0(t,T,x,y)dy\varepsilon\sigma(t,x)\\
&\quad + \sum_{i=1}^N \varepsilon^i \int_{\mathbf{R}^d} g(y)E[N^{0,t}_{i,T}|\bar{X}^{0,t,x}_T = y]p^0(t,T,x,y)dy\varepsilon\sigma(t,x)\\
&\quad + \int_t^T \int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\nabla_x u^{\varepsilon,k,N}\sigma)(s,y))E[N^{0,t}_{0,s}|\bar{X}^{0,t,x}_s = y]p^0(t,s,x,y)dyds\varepsilon\sigma(t,x)\\
&\quad + \sum_{i=1}^N \varepsilon^i \int_t^T \int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\nabla_x u^{\varepsilon,k,N}\sigma)(s,y))E[N^{0,t}_{i,s}|\bar{X}^{0,t,x}_s = y]p^0(t,s,x,y)dyds\varepsilon\sigma(t,x).
\end{aligned}
$$

Then,

$$
\begin{aligned}
u^{\varepsilon,k+1,N}(t,x) &= E[g(\bar{X}^{0,t,x}_T)] + E\left[\int_t^T f(s,\bar{X}^{0,t,x}_s, Y^{\varepsilon,k,N,t,x}_s, Z^{\varepsilon,k,N,t,x}_s)ds\right]\\
&\quad + \sum_{i=1}^N \varepsilon^i E[g(\bar{X}^{0,t,x}_T)\pi^{0,t}_{i,T}] + \sum_{i=1}^N \varepsilon^i E\left[\int_t^T f(s,\bar{X}^{0,t,x}_s, Y^{\varepsilon,k,N,t,x}_s, Z^{\varepsilon,k,N,t,x}_s)\pi^{0,t}_{i,s}ds\right], \qquad (95)\\
(\nabla u^{\varepsilon,k+1,N}\sigma)(t,x) &= \Bigg\{E[g(\bar{X}^{0,t,x}_T)N^{0,t}_{0,T}] + E\left[\int_t^T f(s,\bar{X}^{0,t,x}_s, Y^{\varepsilon,k,N,t,x}_s, Z^{\varepsilon,k,N,t,x}_s)N^{0,t}_{0,s}ds\right]\\
&\quad + \sum_{i=1}^N \varepsilon^i E[g(\bar{X}^{0,t,x}_T)N^{0,t}_{i,T}] + \sum_{i=1}^N \varepsilon^i E\left[\int_t^T f(s,\bar{X}^{0,t,x}_s, Y^{\varepsilon,k,N,t,x}_s, Z^{\varepsilon,k,N,t,x}_s)N^{0,t}_{i,s}ds\right]\Bigg\}\varepsilon\sigma(t,x),
\end{aligned}
$$

(96)

where $Y^{\varepsilon,k,N,t,x}_s = u^{\varepsilon,k,N}(s,\bar{X}^{0,t,x}_s)$ and $Z^{\varepsilon,k,N,t,x}_s = (\nabla_x u^{\varepsilon,k,N}\sigma)(s,\bar{X}^{0,t,x}_s)$.

Here, the term

$$
\left\{E[g(\bar{X}^{0,t,x}_T)N^{0,t}_{0,T}] + E\left[\int_t^T f(s,\bar{X}^{0,t,x}_s, Y^{\varepsilon,k,N,t,x}_s, Z^{\varepsilon,k,N,t,x}_s)N^{0,t}_{0,s}ds\right]\right\}\varepsilon\sigma(t,x)
$$

is similar to the representation of $Z^{t,x}_t$ shown in Ma and Zhang (2002) (or Civitanic, Ma and Zhang (2003) when $f=0$). Hence, $(u^{\varepsilon,k,N}, \nabla_x u^{\varepsilon,k,N}\sigma)$ is regarded as a recursive expansion around the representation formula of Ma and Zhang (2002). Especially, by Lipschitz continuity of g, the following property holds for $(u^{\varepsilon,k,N}, \nabla_x u^{\varepsilon,k,N}\sigma)$.

Lemma 5.2 *For $k \geq 0$, $N \in \mathbf{N}$,*

$$
\begin{aligned}
|u^{\varepsilon,k,N}(t,x)| &\leq C(T,x), \qquad (97)\\
|\nabla_x u^{\varepsilon,k,N}\sigma(t,x)| &\leq C(T,x). \qquad (98)
\end{aligned}
$$

where $C(T,x)$ denotes a generic non-negative, non-decreasing and finite function of at most polynomial growth in x depending on T.

5.2.2 Error Estimate

For any $\beta,\mu > 0$, let $H_{\beta,\mu}$ be the space of functions $v : [0,T]\times\mathbf{R}^d \to \mathbf{R}^n$ such that

$$
\|v\|^2_{H_{\beta,\mu}} = \int_0^T \int_{\mathbf{R}^d} e^{\beta s}|v(s,x)|^2 e^{-\mu|x|}dxds < \infty.
$$

We also define the space $H_{\beta,\mu,X}$, For any $\beta,\mu > 0$ and any diffusion process X_s, $0 \leq s \leq T$ starting from x at time 0, let $H_{\beta,\mu,X}$ be the space of functions $v : [0,T]\times\mathbf{R}^d \to \mathbf{R}^n$ such that

$$
\|v\|^2_{H_{\beta,\mu,X}} = \int_0^T \int_{\mathbf{R}^d} e^{\beta s}E[|v(s,X_s)|^2]e^{-\mu|x|}dxds < \infty.
$$

Remark that the following norm equivalence result holds (see Gobet and Labart (2010) for more details). Suppose that b and σ are bounded measurable functions on $[0,T]\times\mathbf{R}^d$ and are Lipschitz continuous with respect to x, and σ satisfies the ellipticity condition. Then, there exist two constants $c_1, c_2 > 0$ such that $v \in L^2([0,T]\times\mathbf{R}^d, e^{\beta s}ds \times e^{-\mu|x|}dx)$

$$
c_1\|v\|^2_{H_{\beta,\mu}} \leq \|v\|^2_{H_{\beta,\mu,X}} \leq c_2\|v\|^2_{H_{\beta,\mu}}. \qquad (99)
$$

The next theorem is our main result, which evaluates a global approximation error of $(u^{\varepsilon,k,N}, \nabla_x u^{\varepsilon,k,N}\sigma)$ (in (93) and (94)) for $(u^\varepsilon, \nabla_x u^\varepsilon\sigma)$ (in (45) and (46)).

Theorem 5.1 *Suppose that Assumption 5.1 holds. Let C be $C = c_2/c_1$ and β be such that $2(1+T)CC_L^2 < \beta$ and set $\delta := \frac{2CC_L^2(T+1)}{\beta} < 1$. Then, for arbitrary $k \geq 0$ and $N \in \mathbf{N}$, there exists $C_0(T)$ depending on T and $C_1(T,N)$ depending on T and N such that*

$$\|u^\varepsilon - u^{\varepsilon,k,N}\|^2_{H_{\beta,\mu}} + \|(\nabla_x u^\varepsilon \sigma) - (\nabla_x u^{\varepsilon,k,N}\sigma)\|^2_{H_{\beta,\mu}} \leq \left\{ C_0(T)\cdot\delta^k + \varepsilon^{2(N+1)}C_1(T,N)\cdot\left(\frac{1-\delta^k}{1-\delta}\right)\right\}, \quad \varepsilon \in (0,1].$$

Proof.
Note that the following inequality holds:

$$\begin{aligned}
& \|u^\varepsilon - u^{\varepsilon,k,N}\|^2_{H_{\beta,\mu}} + \|\nabla_x u^\varepsilon\sigma - \nabla_x u^{\varepsilon,k,N}\sigma\|^2_{H_{\beta,\mu}} \\
\leq\ & 2(\|u^\varepsilon - u^{\varepsilon,k}\|^2_{H_{\beta,\mu}} + \|\nabla_x u^\varepsilon\sigma - \nabla_x u^{\varepsilon,k}\sigma\|^2_{H_{\beta,\mu}}) + 2(\|u^{\varepsilon,k} - u^{\varepsilon,k,N}\|^2_{H_{\beta,\mu}} + \|\nabla_x u^{\varepsilon,k}\sigma - \nabla_x u^{\varepsilon,k,N}\sigma\|^2_{H_{\beta,\mu}}).
\end{aligned}$$

First, we show the error $\|u^\varepsilon - u^{\varepsilon,k}\|^2_{H_{\beta,\mu}} + \|(\nabla_x u^\varepsilon\sigma) - (\nabla_x u^{\varepsilon,k}\sigma)\|^2_{H_{\beta,\mu}}$ by using the norm equivalence, (99) and the similar argument in the proof of Theorem 2.1 in El Karoui et al. (1997):

$$\|u^\varepsilon - u^{\varepsilon,k}\|^2_{H_{\beta,\mu}} + \|(\nabla_x u^\varepsilon\sigma) - (\nabla_x u^{\varepsilon,k}\sigma)\|^2_{H_{\beta,\mu}} \leq \frac{2CC_L^2(T+1)}{\beta}\{\|u^\varepsilon - u^{\varepsilon,k-1}\|^2_{H_{\beta,\mu}} + \|(\nabla_x u^\varepsilon\sigma) - (\nabla_x u^{\varepsilon,k-1}\sigma)\|^2_{H_{\beta,\mu}}\}.$$

Therefore,

$$\|u^\varepsilon - u^{\varepsilon,k}\|^2_{H_{\beta,\mu}} + \|(\nabla_x u^\varepsilon\sigma) - (\nabla_x u^{\varepsilon,k}\sigma)\|^2_{H_{\beta,\mu}} \leq C_0(T)\cdot\left(\frac{2CC_L^2(T+1)}{\beta}\right)^k,$$

where $C_0(T)$ such that $\|u^\varepsilon - u^{\varepsilon,0}\|^2_{H_{\beta,\mu}} + \|(\nabla_x u^\varepsilon\sigma) - (\nabla_x u^{\varepsilon,0}\sigma)\|^2_{H_{\beta,\mu}} \leq C_0(T)$.

Next, we estimate the error $\|u^{\varepsilon,k} - u^{\varepsilon,k,N}\|^2_{H_{\beta,\mu}} + \|\nabla_x u^{\varepsilon,k}\sigma - \nabla_x u^{\varepsilon,k,N}\sigma\|^2_{H_{\beta,\mu}}$.
The difference $u^{\varepsilon,k+1} - u^{\varepsilon,k+1,N}$ is represented as follows:

$$\begin{aligned}
& u^{\varepsilon,k+1}(t,x) - u^{\varepsilon,k+1,N}(t,x) \\
=\ & \int_{\mathbf{R}^d} g(y)p^\varepsilon(t,T,x,y)dy + \int_t^T\int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k}(s,y),(\nabla_x u^{\varepsilon,k}\sigma)(s,y))p^\varepsilon(t,s,x,y)dyds \\
& - \int_{\mathbf{R}^d} g(y)\left\{p^0(t,T,x,y) + \sum_{i=1}^N \varepsilon^i E[\pi^{0,t}_{i,T}|\bar{X}^{0,t,x}_T = y]p^0(t,T,x,y)\right\}dy \\
& - \int_t^T\int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\nabla_x u^{\varepsilon,k,N}\sigma)(s,y)) \\
& \qquad\qquad \left\{p^0(t,s,x,y) + \sum_{i=1}^N \varepsilon^i E[\pi^{0,t,x}_{i,T}|\bar{X}^{0,t,x}_s = y]p^0(t,s,x,y)\right\}dyds \\
=\ & \int_{\mathbf{R}^d} g(y)p^\varepsilon(t,T,x,y)dy - \int_{\mathbf{R}^d} g(y)\left\{p^0(t,T,x,y) + \sum_{i=1}^N \varepsilon^i E[\pi^{0,t}_{i,T}|\bar{X}^{0,t,x}_T = y]p^0(t,T,x,y)\right\}dy \\
& + \int_t^T\int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k}(s,y),(\nabla_x u^{\varepsilon,k}\sigma)(s,y))p^\varepsilon(t,s,x,y)dyds \\
& - \int_t^T\int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\nabla_x u^{\varepsilon,k,N}\sigma)(s,y))p^\varepsilon(t,s,x,y)dyds \\
& + \int_t^T\int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\nabla_x u^{\varepsilon,k,N}\sigma)(s,y))p^\varepsilon(t,s,x,y)dyds \\
& - \int_t^T\int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\nabla_x u^{\varepsilon,k,N}\sigma)(s,y)) \\
& \qquad\qquad \left\{p^0(t,s,x,y) + \sum_{i=1}^N \varepsilon^i E[\pi^{0,t}_{i,T}|\bar{X}^{0,t,x}_s = y]p^0(t,s,x,y)\right\}dyds.
\end{aligned}$$

Remark that after the second equality, we add the terms $\pm\int_t^T\int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\nabla_x u^{\varepsilon,k,N}\sigma)(s,y))p^\varepsilon(t,s,x,y)dyds$.
Let I_1, I_2 and I_3 be

$$I_1(t,x) := \int_{\mathbf{R}^d} g(y)p^\varepsilon(t,T,x,y)dy - \int_{\mathbf{R}^d} g(y)\left\{p^0(t,T,x,y) + \sum_{i=1}^N \varepsilon^i E[\pi^{0,t}_{i,T}|\bar{X}^{0,t,x}_T = y]p^0(t,T,x,y)\right\}dy,$$

$$I_2(t,x) := \int_t^T \int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k}(s,y),(\nabla_x u^{\varepsilon,k}\sigma)(s,y))p^{\varepsilon}(t,s,x,y)dyds - \int_t^T \int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\nabla_x u^{\varepsilon,k,N}\sigma)(s,y))p^{\varepsilon}(t,s,x,y)dyds,$$

$$I_3(t,x) := \int_t^T \int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\nabla_x u^{\varepsilon,k,N}\sigma)(s,y))p^{\varepsilon}(t,s,x,y)dyds - \int_t^T \int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\nabla_x u^{\varepsilon,k,N}\sigma)(s,y)) \left\{ p^0(t,s,x,y) + \sum_{i=1}^N \varepsilon^i E[\pi_{i,T}^{0,t} | \bar{X}_s^{0,t,x} = y] p^0(t,s,x,y) \right\} dyds.$$

The difference $(\nabla_x u^{\varepsilon,k+1}\sigma) - (\nabla_x u^{\varepsilon,k+1,N}\sigma)$ is represented as

$$\begin{aligned}
&(\nabla_x u^{\varepsilon,k+1}\sigma) - (\nabla_x u^{\varepsilon,k+1,N}\sigma) \\
=\ & \int_{\mathbf{R}^d} g(y) E[N_T^{\varepsilon,t} | X_T^{\varepsilon,t,x} = y] p^{\varepsilon}(t,T,x,y) dy \varepsilon\sigma(t,x) \\
&+ \int_t^T \int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k}(s,y),(\nabla_x u^{\varepsilon,k}\sigma)(s,y)) E[N_s^{\varepsilon,t} | X_s^{\varepsilon,t,x} = y] p^{\varepsilon}(t,s,x,y) dyds \varepsilon\sigma(t,x) \\
&- \int_{\mathbf{R}^d} g(y) E[N_{0,T}^{0,t} | \bar{X}_T^{0,t,x} = y] p^0(t,T,x,y) dy \varepsilon\sigma(t,x) \\
&- \sum_{i=1}^N \varepsilon^i \int_{\mathbf{R}^d} g(y) E[N_{i,T}^{0,t} | \bar{X}_T^{0,t,x} = y] p^0(t,T,x,y) dy \varepsilon\sigma(t,x) \\
&- \int_t^T \int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\nabla_x u^{\varepsilon,k,N}\sigma)(s,y)) E[N_{0,s}^{0,t} | \bar{X}_s^{0,t,x} = y] p^0(t,s,x,y) dyds \varepsilon\sigma(t,x) \\
&- \sum_{i=1}^N \varepsilon^i \int_t^T \int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\nabla_x u^{\varepsilon,k,N}\sigma)(s,y)) E[N_{i,T}^{0,t} | \bar{X}_T^{0,t,x} = y] p^0(t,T,x,y) dyds \varepsilon\sigma(t,x) \\
=\ & \int_{\mathbf{R}^d} g(y) E[N_T^{\varepsilon,t} | X_T^{\varepsilon,t,x} = y] p^{\varepsilon}(t,T,x,y) dy \varepsilon\sigma(t,x) \\
&- \int_{\mathbf{R}^d} g(y) E[N_{0,T}^{0,t} | \bar{X}_T^{0,t,x} = y] p^0(t,T,x,y) dy \varepsilon\sigma(t,x) \\
&- \sum_{i=1}^N \varepsilon^i \int_{\mathbf{R}^d} g(y) E[N_{i,T}^{0,t} | \bar{X}_T^{0,t,x} = y] p^0(t,T,x,y) dy \varepsilon\sigma(t,x) \\
&+ \int_t^T \int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k}(s,y),(\nabla_x u^{\varepsilon,k}\sigma)(s,y)) E[N_s^{\varepsilon,t} | X_s^{\varepsilon,t,x} = y] p^{\varepsilon}(t,s,x,y) dyds \varepsilon\sigma(t,x) \\
&- \int_t^T \int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\nabla_x u^{\varepsilon,k,N}\sigma)(s,y)) E[N_s^{\varepsilon,t} | X_s^{\varepsilon,t,x} = y] p^{\varepsilon}(t,s,x,y) dyds \varepsilon\sigma(t,x) \\
&+ \int_t^T \int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\nabla_x u^{\varepsilon,k,N}\sigma)(s,y)) E[N_s^{\varepsilon,t} | X_s^{\varepsilon,t,x} = y] p^{\varepsilon}(t,s,x,y) dyds \varepsilon\sigma(t,x) \\
&- \int_t^T \int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\nabla_x u^{\varepsilon,k,N}\sigma)(s,y)) E[N_{0,s}^{0,t} | \bar{X}_s^{0,t,x} = y] p^0(t,s,x,y) dyds \varepsilon\sigma(t,x) \\
&- \sum_{i=1}^N \varepsilon^i \int_t^T \int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\nabla_x u^{\varepsilon,k,N}\sigma)(s,y)) E[N_{i,s}^{0,t} | \bar{X}_s^{0,t,x} = y] p^0(t,s,x,y) dyds \varepsilon\sigma(t,x).
\end{aligned}$$

Let

$$J_1(t,x) := \int_{\mathbf{R}^d} g(y) E[N_T^{\varepsilon,t} | X_T^{\varepsilon,t,x} = y] p^{\varepsilon}(t,T,x,y) dy \varepsilon\sigma(t,x) - \int_{\mathbf{R}^d} g(y) E[N_{0,T}^{0,t} | \bar{X}_T^{0,t,x} = y] p^0(t,T,x,y) dy \varepsilon\sigma(t,x) - \sum_{i=1}^N \varepsilon^i \int_{\mathbf{R}^d} g(y) E[N_{i,T}^{0,t} | \bar{X}_T^{0,t,x} = y] p^0(t,T,x,y) dy \varepsilon\sigma(t,x),$$

$$
\begin{aligned}
J_2(t,x) \quad := \quad & \int_t^T \int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k}(s,y),(\nabla_x u^{\varepsilon,k}\sigma)(s,y)) E[N_s^{\varepsilon,t}|X_s^{\varepsilon,t,x}=y] p^{\varepsilon}(t,s,x,y) dyds\varepsilon\sigma(t,x) \\
& -\int_t^T \int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\nabla_x u^{\varepsilon,k,N}\sigma)(s,y)) E[N_s^{\varepsilon,t}|X_s^{\varepsilon,t,x}=y] p^{\varepsilon}(t,s,x,y) dyds\varepsilon\sigma(t,x) \\
J_3(t,x) \quad := \quad & \int_t^T \int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\nabla_x u^{\varepsilon,k,N}\sigma)(s,y)) E[N_s^{\varepsilon,t}|X_s^{\varepsilon,t,x}=y] p^{\varepsilon}(t,s,x,y) dyds\varepsilon\sigma(t,x) \\
& -\int_t^T \int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\nabla_x u^{\varepsilon,k,N}\sigma)(s,y)) E[N_{0,s}^{0,t}|\bar{X}_s^{0,t,x}=y] p^{0}(t,s,x,y) dyds\varepsilon\sigma(t,x) \\
& -\sum_{i=1}^N \varepsilon^i \int_t^T \int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\nabla_x u^{\varepsilon,k,N}\sigma)(s,y)) E[N_{i,s}^{0,t}|\bar{X}_s^{0,t,x}=y] p^{0}(t,s,x,y) dyds\varepsilon\sigma(t,x)
\end{aligned}
$$

Then,

$$
\|u^{\varepsilon,k+1}-u^{\varepsilon,k+1,N}\|^2_{H_{\beta,\mu}} \leq 3\|I_1\|^2_{H_{\beta,\mu}} + 3\|I_2\|^2_{H_{\beta,\mu}} + 3\|I_3\|^2_{H_{\beta,\mu}},
$$

$$
\|(\nabla_x u^{\varepsilon,k+1}\sigma)-(\nabla_x u^{\varepsilon,k+1,N}\sigma)\|^2_{H_{\beta,\mu}} \leq 3\|J_1\|^2_{H_{\beta,\mu}} + 3\|J_2\|^2_{H_{\beta,\mu}} + 3\|J_3\|^2_{H_{\beta,\mu}}.
$$

By Proposition 5.1 and Proposition 5.2, we have the following estimates

$$
\begin{aligned}
|I_1(t,x)| \quad = \quad & \left| \int_{\mathbf{R}^d} g(y) \left\{ p^{\varepsilon}(t,T,x,y) - p^0(t,T,x,y) - \sum_{i=1}^N \varepsilon^i E[\pi_{i,T}^{0,t}|\bar{X}_T^{0,t,x}=y] p^0(t,T,x,y) \right\} dy \right| \\
\leq \quad & c(T,x,N)\varepsilon^{N+1}(T-t)^{(N+2)/2},
\end{aligned} \tag{100}
$$

$$
\begin{aligned}
|J_1(t,x)| \quad = \quad & \left| \int_{\mathbf{R}^d} g(y) \left\{ \nabla_x p^{\varepsilon}(t,T,x,y) \right.\right. \\
& \left.\left. -E[N_{0,T}^{0,t,x}|\bar{X}_T^{0,t,x}=y] p^0(t,T,x,y) - \sum_{i=1}^N \varepsilon^i E[N_{i,T}^{0,t}|\bar{X}_T^{0,t,x}=y] p^0(t,T,x,y) \right\} dy \varepsilon\sigma(t,x) \right| \\
\leq \quad & r(T,x,N)\varepsilon^{N+1}(T-t)^{(N+1)/2},
\end{aligned} \tag{101}
$$

and

$$
\begin{aligned}
|I_3(t,x)| \quad = \quad & \left| \int_t^T \int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\nabla_x u^{\varepsilon,k,N}\sigma)(s,y)) \right. \\
& \left. \left\{ p^{\varepsilon}(t,s,x,y) - p^0(t,s,x,y) - \sum_{i=1}^N \varepsilon^i E[\pi_{i,s}^{0,t}|\bar{X}_s^{0,t,x}=y] p^0(t,s,x,y) \right\} dyds \right| \\
\leq \quad & C(T,x,N)\varepsilon^{N+1} \int_t^T (s-t)^{(N+1)/2} ds \\
= \quad & C(T,x,N)\varepsilon^{N+1}(T-t)^{(N+3)/2},
\end{aligned} \tag{102}
$$

$$
\begin{aligned}
|J_3(t,x)| \quad = \quad & \left| \int_t^T \int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\nabla_x u^{\varepsilon,k,N}\sigma)(s,y)) \left\{ \nabla_x p^{\varepsilon}(t,s,x,y) \right.\right. \\
& \left.\left. -E[N_{0,s}^{0,t}|\bar{X}_s^{0,t,x}=y] p^0(t,s,x,y) - \sum_{i=1}^N \varepsilon^i E[N_{i,s}^{0,t}|\bar{X}_s^{0,t,x}=y] p^0(t,s,x,y) \right\} dyds\sigma(t,x) \right| \\
\leq \quad & R(T,x,N)\varepsilon^{N+1} \int_t^T (s-t)^{N/2} ds \\
= \quad & R(T,x,N)\varepsilon^{N+1}(T-t)^{(N+2)/2}.
\end{aligned} \tag{103}
$$

Here, $c(T,x,N)$, $C(T,x,N)$, $r(T,x,N)$ and $R(T,x,N)$ are some non-negative, non-decreasing and finite functions of at most polynomial growth in x depending on T and N.

Therefore, we obtain

$$
\begin{aligned}
\|I_1\|^2_{H_{\beta,\mu}} \leq \varepsilon^{2(N+1)} K_1(T,N), \quad \|I_3\|^2_{H_{\beta,\mu}} \leq \varepsilon^{2(N+1)} K_3(T,N), \\
\|J_1\|^2_{H_{\beta,\mu}} \leq \varepsilon^{2(N+1)} L_1(T,N), \quad \|J_3\|^2_{H_{\beta,\mu}} \leq \varepsilon^{2(N+1)} L_3(T,N),
\end{aligned}
$$

for some $K_1(T,N)$, $K_3(T,N)$, $L_1(T,N)$ and $L_3(T,N)$ depending on T and N.

In order to estimate $\|I_2\|^2_{\beta,\mu}$ and $\|J_2\|^2_{\beta,\mu}$, we define

$$\hat{u}^{\varepsilon,k+1}(t,x) = E[g(X_T^{\varepsilon,t,x})] + E\left[\int_t^T f(s, X_s^{\varepsilon,t,x}, u^{\varepsilon,k,N}(s, X_s^{\varepsilon,t,x}), (\nabla_x u^{\varepsilon,k,N}\sigma)(s, X_s^{\varepsilon,t,x}))ds\right]. \tag{104}$$

Since f is Lipschitz with constant C_L, again using the norm equivalence result, (99) and the similar argument in the proof of Theorem 2.1 in El Karoui et al. (1997) we obtain

$$\begin{aligned}
&\|I_2\|^2_{H_{\beta,\mu}} \leq c_1^{-1}\|u^{\varepsilon,k+1} - \hat{u}^{\varepsilon,k+1}\|^2_{\beta,\mu,X^\varepsilon} = c_1^{-1}\int_{\mathbf{R}^d}\int_0^T e^{\beta s}E[|u^{\varepsilon,k+1}(s,X_s^\varepsilon) - \hat{u}^{\varepsilon,k+1}(s,X_s^\varepsilon)|^2]dse^{-\mu|x|}dx \\
\leq\ & c_1^{-1}\frac{T}{\beta}\int_{\mathbf{R}^d} E\left[\int_0^T e^{\beta s}|f(s,X_s^\varepsilon,u^{\varepsilon,k}(s,X_s^\varepsilon),\nabla_x u^{\varepsilon,k}\sigma(s,X_s^\varepsilon)) - f(s,X_s^\varepsilon,u^{\varepsilon,k,N}(s,X_s^\varepsilon),(\nabla_x u^{\varepsilon,k,N}\sigma)(s,X_s^\varepsilon))|^2 ds\right]e^{-\mu|x|}dx \\
\leq\ & \frac{2c_1^{-1}C_L^2T}{\beta}\int_{\mathbf{R}^d} E\left[\int_0^T e^{\beta s}\{|u^{\varepsilon,k}(s,X_s^\varepsilon) - u^{\varepsilon,k,N}(s,X_s^\varepsilon)|^2 + |(\nabla_x u^{\varepsilon,k}\sigma)(s,X_s^\varepsilon) - (\nabla_x u^{\varepsilon,k,N}\sigma)(s,X_s^\varepsilon)|^2\}ds\right]e^{-\mu|x|}dx \\
\leq\ & \frac{2CC_L^2T}{\beta}\{\|u^{\varepsilon,k} - u^{\varepsilon,k,N}\|^2_{H_{\beta,\mu}} + \|(\nabla_x u^{\varepsilon,k}\sigma) - (\nabla_x u^{\varepsilon,k,N}\sigma)\|^2_{H_{\beta,\mu}}\},
\end{aligned}$$

$$\begin{aligned}
&\|J_2\|^2_{H_{\beta,\mu}} \leq c_1^{-1}\|(\nabla_x u^{\varepsilon,k+1}\sigma) - (\nabla_x \hat{u}^{\varepsilon,k+1}\sigma)\|^2_{\beta,\mu,X^\varepsilon} \\
=\ & c_1^{-1}\int_{\mathbf{R}^n}\int_0^T e^{\beta s}E[|(\nabla_x u^{\varepsilon,k+1}\sigma)(s,X_s^\varepsilon) - (\nabla_x \hat{u}^{\varepsilon,k+1}\sigma)(s,X_s^\varepsilon)|^2]dse^{-\mu|x|}dx \\
\leq\ & c_1^{-1}\frac{1}{\beta}\int_{\mathbf{R}^d} E\left[\int_0^T e^{\beta s}|f(s,X_s^\varepsilon,u^{\varepsilon,k}(s,X_s^\varepsilon),(\nabla_x u^{\varepsilon,k}\sigma)(s,X_s^\varepsilon)) - f(s,X_s^\varepsilon,u^{\varepsilon,k,N}(s,X_s^\varepsilon),(\nabla_x u^{\varepsilon,k,N}\sigma)(s,X_s^\varepsilon))|^2 ds\right]e^{-\mu|x|}dx \\
\leq\ & \frac{2c_1^{-1}C_L^2}{\beta}\int_{\mathbf{R}^d} E\left[\int_0^T e^{\beta s}\{|u^{\varepsilon,k}(s,X_s^\varepsilon) - u^{\varepsilon,k,N}(s,X_s^\varepsilon)|^2 + |(\nabla_x u^{\varepsilon,k}\sigma)(s,X_s^\varepsilon) - (\nabla_x u^{\varepsilon,k,N}\sigma)(s,X_s^\varepsilon)|^2\}ds\right]e^{-\mu|x|}dx \\
\leq\ & \frac{2CC_L^2}{\beta}\{\|u^{\varepsilon,k} - u^{\varepsilon,k,N}\|^2_{H_{\beta,\mu}} + \|(\nabla_x u^{\varepsilon,k}\sigma) - (\nabla_x u^{\varepsilon,k,N}\sigma)\|^2_{H_{\beta,\mu}}\}.
\end{aligned}$$

Then, we have the following estimate for $\|u^{\varepsilon,k+1} - u^{\varepsilon,k+1,N}\|^2_{H_{\beta,\mu}}$ and $\|(\nabla_x u^{\varepsilon,k+1}\sigma) - (\nabla_x u^{\varepsilon,k+1,N}\sigma)\|^2_{H_{\beta,\mu}}$,

$$\begin{aligned}
&\|u^{\varepsilon,k+1} - u^{\varepsilon,k+1,N}\|^2_{H_{\beta,\mu}} \\
\leq\ & \varepsilon^{2(N+1)}K(T,N) + \frac{2CC_L^2T}{\beta}\{\|u^{\varepsilon,k} - u^{\varepsilon,k,N}\|^2_{H_{\beta,\mu}} + \|(\nabla_x u^{\varepsilon,k}\sigma) - (\nabla_x u^{\varepsilon,k,N}\sigma)\|^2_{H_{\beta,\mu}}\},
\end{aligned} \tag{105}$$

$$\begin{aligned}
&\|(\nabla_x u^{\varepsilon,k+1}\sigma) - (\nabla_x u^{\varepsilon,k+1,N}\sigma)\|^2_{H_{\beta,\mu}} \\
\leq\ & \varepsilon^{2(N+1)}L(T,N) + \frac{2CC_L^2}{\beta}\{\|u^{\varepsilon,k} - u^{\varepsilon,k,N}\|^2_{H_{\beta,\mu}} + \|(\nabla_x u^{\varepsilon,k}\sigma) - (\nabla_x u^{\varepsilon,k,N}\sigma)\|^2_{H_{\beta,\mu}}\},
\end{aligned} \tag{106}$$

where $K(T,N) = 2\max\{K_1(T,N), K_3(T,N)\}$ and $L(T,N) = 2\max\{L_1(T,N), L_3(T,N)\}$. Therefore, by (105) and (106), we obtain

$$\begin{aligned}
&\|u^{\varepsilon,k+1} - u^{\varepsilon,k+1,N}\|^2_{H_{\beta,\mu}} + \|(\nabla_x u^{\varepsilon,k+1}\sigma) - (\nabla_x u^{\varepsilon,k+1,N}\sigma)\|^2_{H_{\beta,\mu}} \\
\leq\ & \varepsilon^{2(N+1)}\gamma(T,N) \\
&+ \frac{2CC_L^2(T+1)}{\beta}\{\|u^{\varepsilon,k} - u^{\varepsilon,k,N}\|^2_{H_{\beta,\mu}} + \|(\nabla_x u^{\varepsilon,k}\sigma) - (\nabla_x u^{\varepsilon,k,N}\sigma)\|^2_{H_{\beta,\mu}}\},
\end{aligned} \tag{107}$$

where $\gamma(T,N) = 2\max\{K(T,N), L(T,N)\}$.

Remark that the differences $u^{\varepsilon,0} - u^{\varepsilon,0,N}$ and $\nabla_x u^{\varepsilon,0}\sigma - \nabla_x u^{\varepsilon,0,N}\sigma$ are given as follows:

$$\begin{aligned}
&u^{\varepsilon,0}(t,x) - u^{\varepsilon,0,N}(t,x) \\
=\ & \int_{\mathbf{R}^d} g(y)p^\varepsilon(t,T,x,y)dy \\
&- \int_{\mathbf{R}} g(y)\left\{p^0(t,T,x,y) + \sum_{i=1}^N \varepsilon^i E[\pi_{i,T}^{0,t}|\bar{X}_T^{0,t,x} = y]p^0(t,T,x,y)\right\}dy \\
&+ \int_t^T\int_{\mathbf{R}^d} f(s,y,0,0)p^\varepsilon(t,s,x,y)dyds \\
&- \int_t^T\int_{\mathbf{R}^d} f(s,y,0,0)\left\{p^0(t,s,x,y) + \sum_{i=1}^N \varepsilon^i E[\pi_{i,T}^{0,t}|\bar{X}_s^{0,t,x} = y]p^0(t,s,x,y)\right\}dyds
\end{aligned}$$

and

$$\begin{aligned}
&(\nabla_x u^{\varepsilon,0}\sigma)(t,x) - (\nabla_x u^{\varepsilon,0,N}\sigma)(t,x) \\
= & \int_{\mathbf{R}^d} g(y)E[N_T^{\varepsilon,t}|X_T^{\varepsilon,t} = y]p^\varepsilon(t,T,x,y)dy\varepsilon\sigma(t,x) \\
&\quad - \int_{\mathbf{R}^d} g(y)E[N_{0,T}^{0,t}|\bar{X}_T^{0,t,x} = y]p^0(t,T,x,y)dy\varepsilon\sigma(t,x) \\
&\quad\quad - \sum_{i=1}^N \varepsilon^i \int_{\mathbf{R}^d} g(y)E[N_{i,T}^{0,t}|\bar{X}_T^{0,t,x} = y]p^0(t,T,x,y)dy\varepsilon\sigma(t,x) \\
&+ \int_t^T \int_{\mathbf{R}^d} f(s,y,0,0)E[N_s^{\varepsilon,t}|X_s^{\varepsilon,t,x} = y]p^\varepsilon(t,s,x,y)dy\varepsilon\sigma(t,x) \\
&\quad\quad - \int_t^T \int_{\mathbf{R}^d} f(s,y,0,0)E[N_{0,s}^{0,t}|\bar{X}_s^{0,t,x} = y]p^0(t,s,x,y)dy\varepsilon\sigma(t,x) \\
&\quad\quad\quad - \sum_{i=1}^N \varepsilon^i \int_t^T \int_{\mathbf{R}^d} f(s,y,0,0)E[N_{i,s}^{0,t}|\bar{X}_s^{0,t,x} = y]p^0(t,s,x,y)dyds\varepsilon\sigma(t,x).
\end{aligned}$$

Then, the term $\|u^{\varepsilon,0} - u^{\varepsilon,0,N}\|^2_{H_{\beta,\mu}} + \|(\nabla_x u^{\varepsilon,0}\sigma) - (\nabla_x u^{\varepsilon,0,N}\sigma)\|^2_{H_{\beta,\mu}}$ is estimated by the asymptotic error, that is,

$$\|u^{\varepsilon,0} - u^{\varepsilon,0,N}\|^2_{H_{\beta,\mu}} + \|(\nabla_x u^{\varepsilon,0}\sigma) - (\nabla_x u^{\varepsilon,0,N}\sigma)\|^2_{H_{\beta,\mu}} \le \varepsilon^{2(N+1)}K_0(T,N),$$

for some $K_0(T,N)$.

Therefore, we obtain

$$\begin{aligned}
&\|u^{\varepsilon,k+1} - u^{\varepsilon,k+1,N}\|^2_{H_{\beta,\mu}} + \|(\nabla_x u^{\varepsilon,k+1}\sigma) - (\nabla_x u^{\varepsilon,k+1,N}\sigma)\|^2_{H_{\beta,\mu}} \\
\le\ & \varepsilon^{2(N+1)}C_1(T,N) + \frac{2CC_L^2(T+1)}{\beta}\{\|u^{\varepsilon,k} - u^{\varepsilon,k,N}\|^2_{H_{\beta,\mu}} + \|(\nabla_x u^{\varepsilon,k}\sigma) - (\nabla_x u^{\varepsilon,k,N}\sigma)\|^2_{H_{\beta,\mu}}\} \\
\le\ & \varepsilon^{2(N+1)}C_1(T,N) \\
& + \frac{2CC_L^2(T+1)}{\beta}\Bigg\{\varepsilon^{2(N+1)}C_1(T,N) + \\
&\qquad \frac{2CC_L^2(T+1)}{\beta}\{\|u^{\varepsilon,k-1} - u^{\varepsilon,k-1,N}\|^2_{H_{\beta,\mu}} + \|(\nabla_x u^{\varepsilon,k-1}\sigma) - (\nabla_x u^{\varepsilon,k-1,N}\sigma)\|^2_{H_{\beta,\mu}}\}\Bigg\} \\
& \cdots \\
\le\ & \varepsilon^{2(N+1)}C_1(T,N)\left\{\left(\frac{2CC_L^2(T+1)}{\beta}\right)^{k+1} + \left(\frac{2CC_L^2(T+1)}{\beta}\right)^{k} + \cdots + \left(\frac{2CC_L^2(T+1)}{\beta}\right) + 1\right\} \\
=\ & \varepsilon^{2(N+1)}C_1(T,N) \cdot \left(\frac{1 - \left(\frac{2CC_L^2(T+1)}{\beta}\right)^{k+1}}{1 - \left(\frac{2CC_L^2(T+1)}{\beta}\right)}\right),
\end{aligned} \tag{108}$$

where $C_1(T,N) = \max\{\gamma(T,N), K_0(T,N)\}$.

Finally, Choose β such that $2CC_L^2(T+1) < \beta$ and set $\delta = \frac{2CC_L^2(T+1)}{\beta} < 1$, by (100) and (108) we obtain the global error

$$\|u^\varepsilon - u^{\varepsilon,k,N}\|^2_{H_{\beta,\mu}} + \|(\nabla_x u^\varepsilon\sigma) - (\nabla_x u^{\varepsilon,k,N}\sigma)\|^2_{H_{\beta,\mu}} \le \left\{C_0(T)\cdot\delta^k + \varepsilon^{2(N+1)}C_1(T,N)\cdot\left(\frac{1-\delta^k}{1-\delta}\right)\right\}.$$

□

Remark 5.1 *Consider the following small diffusion setting under a weaker condition:*

$$X_t^\varepsilon = x + \int_0^t b(X_s^\varepsilon)ds + \varepsilon\sum_{j=1}^d \int_0^t \sigma_j(X_s^\varepsilon)dW_s^j \tag{109}$$

with smooth coefficients and Hörmander's condition.

Using Malliavin calculus, Ben Arous and Léandre (1991) showed the Varadhan estimate for the density $p^\varepsilon(x,y)$ of $X_1^{\varepsilon,x}$

$$\lim_{\varepsilon\downarrow 0} 2\varepsilon^2 \log p^\varepsilon(x,y) = -d_B^2(x,y), \tag{110}$$

where $d_B^2(x,y)$ *is the Bismutian distance is given by*

$$d_B^2(x,y) = \inf_{\Phi(h)_1 = y, \det \gamma_{\Psi(h)_1} > 0} \|h\|_H^2. \tag{111}$$

Here, $\Phi(h)_t$ *is a skeleton of the process* X_t^ε

$$\Phi(h)_t = x + \int_0^t b(\Phi(h)_s)ds + \varepsilon \sum_{j=1}^d \int_0^t \sigma_j(\Phi(h)_s) h_s^j ds \tag{112}$$

and $\gamma(\Psi(h)_t)$ *is the deterministic Malliavin covariance*

$$\langle D\Phi(h)_t, D\Phi(h)_t \rangle_H. \tag{113}$$

See Chapter 4 in Barlow and Nualart (1995) and Léandre (2006) for more details. Using the above large deviation (110), we conjecture that an approximation for FBSDEs similar to Theorem 5.1 could be constructed, which seems a interesting and a challenging task.

6 Applications: Pricing Options with Counterparty Risk under the Local and Stochastic Volatility Models

This section applies our approximation algorithm to option pricing with counterparty risk in a simple FBSDE setting. Here, we omit a discussion on modeling and pricing issues under default risk, and concentrate on the concrete description of our approximation scheme with investigation of its validity by using a simple example.[3] Particularly, we use the local and stochastic volatility models for the underlying (forward) price process S under the risk-neutral measure. Let Y be the solution to the following non-linear BSDE:

$$Y_t = g(S_T) - (1-R)\beta \int_t^T (Y_s)^+ ds - \int_t^T Z_s dW_s^1. \tag{114}$$

Here, Y represents the value process with a target payoff $g(S_T)$ taking the *risky (substitution) closing out CVA* into account; $R \geq 0$ and $\beta > 0$ denote a constant recovery rate and a constant default intensity, respectively. Also, the risk-free interest rate and the dividend rate of the underlying asset are assumed to be zero for simplicity. Next, let $(Y^k, Z^k)_{k \geq 0}$ be a sequence of the following linear BSDEs:

$$\begin{aligned} Y_t^0 &= g(S_T) - \int_t^T Z_s^0 dW_s^1. \\ Y_t^1 &= g(S_T) - (1-R)\beta \int_t^T (Y_s^0)^+ ds - \int_t^T Z_s^1 dW_s^1. \\ Y_t^{k+1} &= g(S_T) - (1-R)\beta \int_t^T (Y_s^k)^+ ds - \int_t^T Z_s^{k+1} dW_s^1, \quad k \geq 1, \end{aligned} \tag{115}$$

which is an approximation sequence of the value process Y.

Remark 6.1 *Under the setting above, suppose we consider plain-vanilla options, that is* $g(S_T) = (S_T - K)^+$ *or* $(K - S_T)^+$*. Then, given constant values of* R *and* β *as well as* $Y^k > 0$ *for usual setup of parameters in practice, due to the martingale property of the (risk-free) option value* Y^0 *under the risk-neutral measure, we are able to express* $u^k(t,s) := Y_t^{k,t,s}$ *for each* $k = 0, 1, 2, \cdots$ *as follows:*

$$u^k(t,s) = u^0(t,s) \left[1 + \sum_{i=1}^k \frac{q^i}{i!} \right], \tag{116}$$

where $q = (-1)(1-R)\beta(T-t)$*. Hence, for this simplest case we can easily obtain the benchmark values* $u^k(t,s)$ *through evaluation of* $u^0(t,s)$ *by numerical computation such as the Monte Carlo simulation, against which the validity of our approximation scheme is examined. However, note that it is much more tough task to get the benchmark values under the situation with stochastic intensity and recovery, while our scheme is applicable under the setting without substantial effort.*

[3] See Fujii and Takahashi (2010, 2011) for the detail of modeling and pricing issues under default risk, for instance.

6.1 Local Volatility Model

We consider the local volatility model

$$dS_t = r_t S_t dt + \sigma(t, S_t) dW_t, \quad S_0 > 0, \tag{117}$$

where W is an one dimensional Brownian motion and $\sigma(t,x)$ is the local volatility function. For simplicity, we assume $r_t \equiv 0$. In our framework, we assume the following perturbed model

$$dS_t^\varepsilon = \varepsilon\sigma(t, S_t^\varepsilon) dW_t^1, \quad S_0^\varepsilon = S_0. \tag{118}$$

Define

$$u^\varepsilon(t,s) := Y_t^{\varepsilon,t,s} = E\left[g(S_T^{\varepsilon,t,s})\right] - E\left[\int_t^T (1-R)\beta(Y_\tau^{\varepsilon,t,s})^+ d\tau\right]. \tag{119}$$

Then, $(\partial_x u^\varepsilon \sigma)(t,s) := Z_t^{\varepsilon,t,s}$ is given by

$$(\partial_x u^\varepsilon \sigma)(t,x) = Z_t^{\varepsilon,t,x} = E\left[g(S_T^{\varepsilon,t,s}) N_T^t\right]\sigma(t,x) - E\left[\int_t^T (1-R)\beta(Y_\tau^{\varepsilon,t,s})^+ N_\tau^{\varepsilon,t} d\tau\right]\sigma(t,x), \tag{120}$$

where N is the Malliavin weight for the delta for the local volatility model

$$N_\tau^{\varepsilon,t} = \frac{1}{\tau - t}\int_t^\tau \sigma^{-1}(v, S_v^{\varepsilon,t,s})\partial_s S_v^{\varepsilon,t,s} dW_v. \tag{121}$$

$S_T^{\varepsilon,t,s}$ is expanded as follows:

$$S_T^{\varepsilon,t,s} = S_T^{0,t,s} + \varepsilon S_{1,T}^{t,s} + \varepsilon^2 S_{2,T}^{t,s} + O(\varepsilon^3). \tag{122}$$

In this case, $S_T^{0,t,x}$, $S_{1,T}^{t,s}$ and $S_{2,T}^{t,s}$ are given by

$$S_T^{0,t,s} = s, \tag{123}$$

$$S_{1,T}^{t,s} = \int_t^T \sigma(u, S_u^{0,t,s}) dW_u = \int_t^T \sigma(u,s) dW_u \in \mathcal{K}_1^T, \tag{124}$$

$$S_{2,T}^{t,s} = \int_t^T \partial_x\sigma(u, S_u^{0,t,s})\int_t^u \sigma(v, S_v^{0,t,s}) dW_v dW_u = \int_t^T \partial_x\sigma(u,s)\int_t^u \sigma(v,s) dW_v dW_u \in \mathcal{K}_2^T. \tag{125}$$

Then, the density $p^{LV,\varepsilon}(t,T,s,S)$ of $S_T^{\varepsilon,t,s}$ can be expanded as follows

$$\begin{aligned}
p^{LV,\varepsilon}(t,T,s,S) &\simeq p_{approx}^{LV,\varepsilon}(t,T,s,S) \\
&= \frac{1}{\varepsilon}\left\{E\left[\delta_{(S-s)/\varepsilon}(S_{1,T}^{t,s})\right] + \varepsilon E\left[\delta_{(S-s)/\varepsilon}(S_{1,T}^{t,s})\pi_{t,T}^{LV}\right]\right\} \\
&= \frac{1}{\sqrt{2\pi\varepsilon^2\int_t^T \sigma^2(u,s)du}}\exp\left\{\frac{-(S-s)^2}{2\varepsilon^2\int_t^T \sigma^2(u,s)du}\right\}\left\{1 + \varepsilon E[\pi_{t,T}^{LV} | S_{1,T}^{t,s} = (S-s)/\varepsilon]\right\},
\end{aligned} \tag{126}$$

where $\pi_{t,T}^{LV}$ is the Malliavin weight for the local model in the small diffusion expansion

$$\pi_{t,T}^{LV} = \frac{1}{T-t}\left\{S_{2,T}^{t,s}\int_t^T \sigma^{-1}(\tau,s) dW_\tau - \int_t^T D_\tau S_{2,T}^{t,s}\sigma^{-1}(\tau,s) dW_\tau\right\}, \tag{127}$$

with the Malliavin derivative D for the Brownian motion W.

Then,

$$\begin{aligned}
u^0(t,s) &:= Y_t^{0,t,s}, \\
u^1(t,s) &:= Y_t^{1,t,s}, \\
u^{k+1}(t,s) &:= Y_t^{k+1,t,s}, \ k \geq 1,
\end{aligned}$$

are approximated by

$$\begin{aligned}
u^0(t,s) &\simeq u_{approx}^0(t,s) \\
&= \int_{\mathbf{R}} g(S) p_{approx}^{LV,\varepsilon}(t,T,s,S) dS. \\
u^1(t,s) &\simeq u_{approx}^1(t,s)
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbf{R}} g(S)p^{LV,\varepsilon}_{approx}(t,T,s,S)dS - (1-R)\beta\int_t^T\int_{\mathbf{R}}(u^0_{approx}(\tau,S))^+ p^{LV,\varepsilon}_{approx}(t,\tau,s,S)dSd\tau. \\
u^{k+1}(t,s) &\simeq u^{k+1}_{approx}(t,s) \\
&= \int_{\mathbf{R}} g(S)p^{LV,\varepsilon}_{approx}(t,T,s,S)dS - (1-R)\beta\int_t^T\int_{\mathbf{R}}(u^k_{approx}(\tau,S))^+ p^{LV,\varepsilon}_{approx}(t,\tau,s,S)dSd\tau, \quad k\geq 1.
\end{aligned} \tag{128}
$$

For example, we take $\varepsilon\sigma(t,x) = \varepsilon\sigma x^{\alpha}$ (CEV volatility). For the case of CEV model, $S^{\varepsilon,t,x}_T$ is expanded as follows:

$$
S^{\varepsilon,t,s}_T = S^{0,t,s}_T + \varepsilon S^{t,s}_{1,T} + \varepsilon^2 S^{t,s}_{2,T} + O(\varepsilon^3). \tag{129}
$$

where $S^{0,t,s}_T$, $S^{t,s}_{1,T}$ and $S^{t,s}_{2,T}$ are given by

$$
S^{0,t,s}_T = s, \tag{130}
$$

$$
S^{t,s}_{1,T} = \int_t^T \sigma s^{\alpha}dW_u \in \mathcal{K}^T_1, \tag{131}
$$

$$
S^{t,s}_{2,T} = \int_t^T \alpha\sigma s^{\alpha-1}\int_t^u \sigma s^{\alpha}dW_v dW_u \in \mathcal{K}^T_2. \tag{132}
$$

Then, $\pi^{LV}_{t,T}$ is given as

$$
\begin{aligned}
\pi^{LV}_{t,T} &= \frac{1}{\sigma^2 s^{2\alpha}(T-t)}\alpha\sigma^2 s^{\alpha-1}s^{\alpha}\left\{\int_t^T\int_t^u dW_v dW_u\int_t^T\sigma s^{\alpha}dW_u - \int_t^T D_{\tau}\int_t^T\int_t^u dW_v dW_u \sigma s^{\alpha}dW_{\tau}\right\} \\
&= \frac{1}{\sigma^2 s^{2\alpha}(T-t)}\alpha\sigma^2 s^{2\alpha-1}\left\{\int_t^T\int_t^u dW_v dW_u\int_t^T\sigma s^{\beta}dW_u - \int_t^T\left(\int_t^{\tau}dW_v + \int_{\tau}^T dW_u\right)\sigma s^{\beta}d\tau\right\}.
\end{aligned} \tag{133}
$$

Therefore, the conditional expectation of $\pi^{LV}_{t,T}$ given $S^{t,s}_{1,T} = y$ is computed as follows:

$$
E[\pi^{LV}_{t,T}|S^{t,s}_{1,T} = y] = E[\pi^{LV}_{t,T}|\int_t^T\sigma s^{\alpha}dW_u = y] \tag{134}
$$

$$
= \frac{1}{\sigma^2 s^{2\alpha}(T-t)}\alpha\sigma^2 s^{2\alpha-1}\left\{E[\int_t^T\int_t^u dW_v dW_u\int_t^T\sigma s^{\alpha}dW_u|\int_t^T\sigma s^{\alpha}dW_u = y]\right. \tag{135}
$$

$$
\left. -E[\int_t^T\left(\int_t^{\tau}dW_v + \int_{\tau}^T dW_u\right)\sigma s^{\alpha}d\tau|\int_t^T\sigma s^{\alpha}dW_u = y]\right\} \tag{136}
$$

$$
= \alpha\sigma^4 s^{4\alpha-1}\int_t^T\int_t^u dvdu\left(\frac{y^3}{(\sigma^2 s^{2\beta}(T-t))^3} - \frac{3y}{(\sigma^2 s^{2\alpha}(T-t))^2}\right) \tag{137}
$$

$$
= \frac{1}{2}\alpha\sigma^4 s^{4\alpha-1}(T-t)^2\left(\frac{y^3}{(\sigma^2 s^{2\alpha}(T-t))^3} - \frac{3y}{(\sigma^2 s^{2\alpha}(T-t))^2}\right). \tag{138}
$$

Therefore, the approximated density of the CEV model is given as

$$
\begin{aligned}
&p^{LV,\varepsilon}_{approx}(t,T,s,S) \\
= &\frac{1}{\sqrt{2\pi\varepsilon^2\sigma^2 s^{2\alpha}(T-t)}}\exp\left\{\frac{-(S-s)^2}{2\varepsilon^2\sigma^2 s^{2\alpha}(T-t)}\right\}\left\{1+\varepsilon\frac{1}{2}\alpha\sigma^4 s^{4\alpha-1}(T-t)^2\left(\frac{((S-s)/\varepsilon)^3}{(\sigma^2 s^{2\alpha}(T-t))^3} - \frac{3(S-s)/\varepsilon}{(\sigma^2 s^{2\alpha}(T-t))^2}\right)\right\}.
\end{aligned} \tag{139}
$$

We show numerical examples of our approximation scheme (128) for the option price $u(t,x)$ under the CEV model with the call payoff function $g(x) = (x-K)^+$. In this case, using (139) with $\varepsilon = 1.0$, we easily obtain $u^0_{approx}(t,s)$ in (128) as follows:

$$
u^0_{approx}(t,s) = yN\left(\frac{y}{\sqrt{\Sigma(t,T)}}\right) + \left(\Sigma(t,T) - \frac{\zeta(t,T)}{\Sigma(t,T)}y\right)n[y:0,\Sigma(t,T)], \tag{140}
$$

where $N(x)$ and $n[x:\mu,\Sigma]$ denote the standard normal distribution function, and the normal density function with the mean μ and the variance Σ, respectively. Also, y, $\Sigma_{t,T}$ and $\zeta_{t,T}$ are defined in the following:

$$
\begin{aligned}
y &= s-K, \\
\Sigma(t,T) &= \sigma^2 s^{2\alpha}(T-t), \\
\zeta(t,T) &= \alpha\sigma^4 s^{4\alpha-1}\frac{(T-t)^2}{2}.
\end{aligned} \tag{141}
$$

- The parameters of the model are specified as follows:

  $t = 0.0,\ T = 2.0,\ r = 0.0,\ S_0 = 10,000,\ \sigma_{BS} = 0.1,\ \alpha = 0.5,\ \varepsilon = 1.0,\ \beta = 0.06$ (intensity), $R = 0.0$ (recovery rate).

Here, σ_{BS} denotes the instantaneous volatility of the log-normal (or the Black-Scholes) process, and we set the CEV volatility σ_{CEV} as $\sigma_{CEV} = \sigma_{BS} S_0^{1-\alpha}$ below.

The result is given in Table 1–3: **AE** $u^0_{approx}(= u^0_{approx}(0, S_0))$ is evaluated by the equation (140), and **AE** $u^k_{approx}(= u^k_{approx}(0, S_0))$, $(k = 1, 2)$ are evaluated based on the corresponding equations in (128) by numerical integration with the equations (139) and (140). **Exact value** $u(0, S_0)$ is approximated as in (116) by the equation (142) below with $k = 5$, which gives the sufficiently convergent value for this case. Also, **Benchmark** $u^k = u^k(0, S_0)$, $k = 1, 2$ are computed by the following equation (142) with $k = 1, 2$, respectively:

$$u^0(0, S_0)\left[1 + \sum_{i=1}^{k} \frac{q^i}{i!}\right], \tag{142}$$

where $q = (-1)(1 - R)\beta T$, and the value of $u^0(0, S_0)$ is obtained based on Monte Carlo simulation for the CEV process. In each simulation, the numbers of the trials and the time steps are 1,000,000 with the antithetic variable method and 750, respectively. Also, in Table 1–3 the relative errors denoted by **AE Error** u and **AE Error** u^k of our asymptotic expansion are computed by $(u^k_{approx}(0, S_0) - u(0, S_0))/u(0, S_0)$ and $(u^k_{approx}(0, S_0) - u^k(0, S_0))/u^k(0, S_0)$, respectively. It is observed that $u^k_{approx}(= u^k_{approx}(0, S_0))$, $k = 1, 2$ become closer towards $u(0, S_0)$.

Although this example uses only the ε^1-order expansion of the density, we already know from the existing work (e.g. Takahashi et al. (2012)) that higher order expansions produce much more better approximation for the risk-free option price u^0, which is expected to provide more precise approximations for the solution to the BSDE as well.

6.2 Stochastic Volatility Model

Let (S, v) be the Heston's stochastic volatility model

$$\begin{aligned} dS_t &= r_t S_t dt + \sqrt{v_t} S_t dW^1_t, \quad S_0 > 0 \\ dv_t &= \kappa(\theta - v_t)dt + \nu\sqrt{v_t}(\rho dW^1_t + \sqrt{1-\rho^2} dW^2_t), \quad v_0 > 0 \end{aligned} \tag{143}$$

where $W = (W^1, W^2)$ a two dimensional Brownian motion. For simplicity, we also assume $r_t \equiv 0$. Let $X_t := \log S_t$ and then by Itô formula we have the logarithm underlying price process:

$$\begin{aligned} dX_t &= -\frac{1}{2} v_t dt + \sqrt{v_t} dW^1_t, \quad x_0 = \log S_0, \\ dv_t &= \kappa(\theta - v_t)dt + \nu\sqrt{v_t}(\rho dW^1_t + \sqrt{1-\rho^2} dW^2_t), \quad v_0 > 0. \end{aligned} \tag{144}$$

We put a perturbation parameter, ε in the following way:

$$\begin{aligned} dX^\varepsilon_t &= -\frac{\varepsilon}{2} v^\varepsilon_t dt + \varepsilon\sqrt{v^\varepsilon_t} dW^1_t, \quad x_0 = \log S_0, \\ dv^\varepsilon_t &= \kappa(\theta - v^\varepsilon_t)dt + \varepsilon\nu\sqrt{v^\varepsilon_t}(\rho dW^1_t + \sqrt{1-\rho^2} dW^2_t), \quad v_0 > 0. \end{aligned} \tag{145}$$

Although the setting of the above FBSDE ((145) and (114)) does not rigorously satisfy the conditions in Section 5, our algorithm is still applicable to this model. We slightly modify the small diffusion expansion discussed in Section 5 and apply the expansion of Takahashi and Yamada (2012). We expand X^ε_t as follows:

$$X^\varepsilon_t = X^0_t + \varepsilon^2 X_{1t} + O(\varepsilon^3), \tag{146}$$

where

$$\begin{aligned} X^0_t &= x + \int_0^t \varepsilon\sqrt{\tilde{v}_s} dW^1_s - \frac{1}{2}\int_0^t \varepsilon\tilde{v}_s ds, \\ X_{1t} &= \nu\int_0^t \frac{1}{2\sqrt{\tilde{v}_s}} v_{1s} dW^1_s - \frac{1}{2}\nu\int_t^T v_{1s} ds, \\ \tilde{v}_t &= v_0 + \int_0^t \kappa(\theta - \tilde{v}_s) ds, \\ v_{1t} &= \int_0^t e^{-\kappa(t-s)}\sqrt{\tilde{v}_s}(\rho dW^1_s + \sqrt{1-\rho^2} dW^2_s). \end{aligned}$$

Note that $X^0_t - x \in \mathcal{K}^T_1$ and $X_{1t} \in \mathcal{K}^T_2$. When $\varepsilon = 1$, by Takahashi and Yamada (2012),

$$p^\varepsilon(t, T, x, y) \simeq E[\delta_y(X^{0,t,x}_T)] + \nu E[\delta_y(X^{0,t,x}_T)\pi^{SV}_{t,T}], \tag{147}$$

where $\pi^{SV}_{t,T}$ is the Malliavin weight for the Heston's stochastic volatility model in the expansion

$$\pi^{SV}_{t,T} = \frac{1}{\Sigma(t,T)}\left\{X^t_{1T}\int_t^T D_{s,1}X^{0,t,x}_T dW^1_s - \int_t^T D_{s,1}X^t_{1T}D_{s,1}X^{0,t,x}_T ds\right\}, \tag{148}$$

with

$$\Sigma(t,T) = \theta(T-t) + (v_0-\theta)e^{\kappa t}(1-e^{\kappa(T-t)})/\kappa. \tag{149}$$

Here, D_1 is the Malliavin derivative for the Brownian motion W^1. Therefore, we have the following density approximation

$$p^\varepsilon(t,T,x,y) \simeq p^{SV,\varepsilon}_{approx}(t,T,x,y) = \frac{1}{\sqrt{2\pi\Sigma(t,T)}}e^{-\frac{(y-x-\frac{1}{2}\Sigma(t,T))^2}{2\Sigma(t,T)}}\left\{1+\nu\zeta^t_T(x,y)\right\}. \tag{150}$$

Here,

$$\begin{aligned} &\zeta^t_T(x,y) = E[\pi^{SV}_{t,T}|X^{t,x,0}_T = y] \\ = \ & C(t,T)\left(\frac{(y-x-\frac{1}{2}\Sigma(t,T))^3}{\Sigma(t,T)^3} - \frac{3(y-x-\frac{1}{2}\Sigma(t,T))}{\Sigma(t,T)^2} - \frac{(y-x-\frac{1}{2}\Sigma(t,T))^2}{\Sigma(t,T)^2} + \frac{1}{\Sigma(t,T)}\right), \end{aligned} \tag{151}$$

with

$$C(t,T) = \frac{\rho}{2\kappa^2}e^{-\kappa(T-t)}\left\{\theta-(v_0-\theta)-(v_0-\theta)\kappa(T-t)+e^{\kappa(T-t)}(v_0-\theta+\theta(-1+\kappa(T-t)))\right\}. \tag{152}$$

Applying the approximate density (150) derived above, we are able to take an approximation sequence $(u^k_{approx})_k$ as follows:

$$\begin{aligned} u^0_{approx}(t,e^x) &= \int_{\mathbf{R}} g(e^y)p^{SV,\varepsilon}_{approx}(t,T,x,y)dy. \\ u^{k+1}_{approx}(t,e^x) &= \int_{\mathbf{R}} g(e^y)p^{SV,\varepsilon}_{approx}(t,T,x,y)dy \\ &\quad -(1-R)\beta\int_t^T\int_{\mathbf{R}}(u^k_{approx}(\tau,e^y))^+p^{SV,\varepsilon}_{approx}(t,\tau,x,y)dyd\tau, \quad k\geq 0, \end{aligned} \tag{153}$$

where $x=\log S$.

Finally, in Table 4–6 let us provide numerical examples of our approximation for the option price $u(t,x)$ in the stochastic volatility model with the call payoff function $g(x)=(x-K)^+$.

- The parameters of the model are specified as follows:

  $t=0.0,\ T=3.0,\ r=0.0,\ S_0=10,000,\ v_0=0.25,\ \kappa=1.0,\ \theta=0.25,\ \varepsilon=1,\ \nu=0.1, \rho=-0.25,$
  $\beta=0.06$ (intensity), $R=0.0$ (recovery rate).

  We remark that the computations of **Exact value** $u(0,S_0)$, **Benchmark** u^k, **AE Error** u and **AE Error** u^k are same as in Table 1–3, except that the benchmark value of $u^0(0,S_0)$ is calculated by the Fourier transform method for the Heston model (144). Also, the values of **AE** $u^k_{approx}(=u^k_{approx}(0,S_0))$ are computed in the following:

$$\begin{aligned} u^0_{approx}(0,S_0) &= \int_{\mathbf{R}} g(e^y)p^{SV,\varepsilon}_{approx}(0,T,x,y)dy \\ &= C_{BS}(x,\Sigma(0,T)) + \nu C(0,T)S_0 n(d_1(0,T,x):0,1)(-d_2(0,T,x))/\Sigma(0,T), \\ u^{k+1}_{approx}(0,S_0) &= C_{BS}(x,\Sigma(0,T)) + \nu C(0,T)S_0 n(d_1(0,T,x):0,1)(-d_2(0,T,x))/\Sigma(0,T) \\ &\quad -(1-R)\beta\int_t^T\int_{\mathbf{R}}(u^k_{approx}(\tau,e^y))^+p^{SV,\varepsilon}_{approx}(t,\tau,x,y)dyd\tau, \quad k\geq 0, \end{aligned} \tag{154}$$

  where $x=\log S_0$ and

$$C_{BS}(x,\Sigma(t,T)) = e^xN(d_1(t,T,x)) - KN(d_2(t,T,x)), \tag{155}$$

  with

$$\begin{aligned} N(x) &= \int_{-\infty}^x \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy, \\ d_1(t,T,x) &= \frac{\log(e^x/K)}{\sqrt{\Sigma(t,T)}} + \frac{1}{2}\sqrt{\Sigma(t,T)}, \\ d_2(t,T,x) &= \frac{\log(e^x/K)}{\sqrt{\Sigma(t,T)}} - \frac{1}{2}\sqrt{\Sigma(t,T)}. \end{aligned}$$

  As in the CEV case, it is observed that $u^k_{approx}(=u^k_{approx}(0,S_0))$, $k=1,2$ become closer towards $u(0,S_0)$.

Table 1: European call option price with CVA under the CEV model $\alpha = 0.5$ (In-the-money case : $K = 7500$, **Exact value** $u(0, S_0) = 2230.24$)

Iteration k	**Benchmark** u^k	**AE** u^k_{approx}	**AE Error** u	**AE Error** u^k
0th	2514.59	2514.49	12.75%	0.00%
1st	2212.84	2212.81	-0.78%	0.00%
2nd	2230.41	2231.11	0.04%	0.01%

Table 2: European call option price with CVA under the CEV model $\alpha = 0.5$ (At-the-money case : $K = 10000$, **Exact value** $u(0, S_0) = 499.45$)

Iteration k	**Benchmark** u^k	**AE** u^k_{approx}	**AE Error** u	**AE Error** u^k
0th	563.13	564.19	12.96%	0.19%
1st	495.55	496.51	-0.59%	0.19%
2nd	499.61	500.61	0.23%	0.20%

Table 3: European call option price with CVA under the CEV model $\alpha = 0.5$ (Out-of-the-money case : $K = 12500$, **Exact value** $u(0, S_0) = 26.01$)

Iteration k	**Benchmark** u^k	**AE** u^k_{approx}	**AE Error** u	**AE Error** u^k
0th	29.33	29.28	12.55%	-0.18%
1st	25.81	25.76	-0.97%	-0.20%
2nd	26.02	25.97	-0.16%	-0.20%

Table 4: European call option price with CVA under the Heston volatility model (In-the-money case : $K = 7500$, **Exact value** $u(0, S_0) = 3612.84$)

Iteration k	**Benchmark** u^k	**AE** u^k_{approx}	**AE Error** u	**AE Error** u^k
0th	4325.36	4327.84	19.79%	0.06%
1st	3546.80	3549.44	-1.75%	0.07%
2nd	3616.87	3620.13	0.20%	0.09%

Table 5: European call option price with CVA under the the Heston volatility model (At-the-money case : $K = 10000$, **Exact value** $u(0, S_0) = 2784.16$)

Iteration k	**Benchmark** u^k	**AE** u^k_{approx}	**AE Error** u	**AE Error** u^k
0th	3333.25	3336.51	19.84%	0.10%
1st	2733.27	2736.06	-1.73%	0.10%
2nd	2787.26	2790.55	0.23%	0.12%

Table 6: European call option price with CVA under the the Heston volatility model (Out-of-the-money case : $K = 12500$, **Exact value** $u(0, S_0) = 2178.74$)

Iteration k	**Benchmark** u^k	**AE** u^k_{approx}	**AE Error** u	**AE Error** u^k
0th	2608.42	2611.74	19.87%	0.13%
1st	2138.90	2141.33	-1.72%	0.13%
2nd	2181.16	2183.98	0.24%	0.13%

7 Conclusion

This paper has developed a new general approximation method for forward-backward stochastic differential equations (FBSDEs). In particular, we have proposed a closed-form approximation based on an asymptotic expansion for forward SDEs combined with Picard-type iteration scheme for BSDEs. Based on Malliavin calculus, especially applying so called *Kusuoka function* (Kusuoka (2003)), we have justified our method with its error estimate for the approximation.

From a practical viewpoint, examination of our scheme under more complex examples is an important and interesting problem. Moreover, a challenging task is to develop mathematical validity of approximations with perturbation for fully coupled FBSDEs. Those topics as well as our approximation method under weaker mathematical condition will be discussed in our future researches.

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A Proof of Lemma 5.1

We prove the assertion by induction. First,

$$\frac{\partial}{\partial\varepsilon}X_t^\varepsilon = \sum_{i=1}^d \int_0^t \nabla X_t^\varepsilon(\nabla X_s^\varepsilon)^{-1}\sigma_i(s,X_s^\varepsilon)dW_s^i \tag{156}$$

$$+\varepsilon\sum_{i=1}^d \int_0^t \nabla X_t^\varepsilon(\nabla X_s^\varepsilon)^{-1}\partial V_i(s,X_s^\varepsilon)V_i(s,X_s^\varepsilon)ds. \tag{157}$$

Since $\nabla X_t^\varepsilon, (\nabla X_t^\varepsilon)^{-1} \in \mathcal{K}_0^T$ and σ_i, $i=1,\cdots,n$ are bounded, $\frac{\partial}{\partial\varepsilon}X_t^\varepsilon \in \mathcal{K}_1^T$.

For $k \geq 2$, $\frac{1}{k!}\frac{\partial^k}{\partial\varepsilon^k}X_t^\varepsilon = \left(\frac{1}{k!}\frac{\partial^k}{\partial\varepsilon^k}X_t^{\varepsilon,1}, \cdots, \frac{1}{k!}\frac{\partial^k}{\partial\varepsilon^k}X_t^{\varepsilon,d}\right)$ is recursively determined by the following:

$$\frac{1}{k!}\frac{\partial^k}{\partial\varepsilon^k}X_t^{\varepsilon,j} = \sum_{\mathbf{l}_\beta,\mathbf{d}_\beta}^{(k)} \int_0^t \left(\prod_{j=1}^\beta \frac{1}{l_j!}\frac{\partial^{l_j}}{\partial\varepsilon^{l_j}}X_s^{\varepsilon,d_j}\right)\partial_{d_\beta}^\beta V_0^j(s,X_s^\varepsilon)ds \tag{158}$$

$$+\sum_{\mathbf{l}_\beta,\mathbf{d}_\beta}^{(k-1)}\int_0^t\left(\prod_{j=1}^{\beta}\frac{1}{l_j!}\frac{\partial^{l_j}}{\partial\varepsilon^{l_j}}X_s^{\varepsilon,d_j}\right)\sum_{i=1}^{d}\partial_{\mathbf{d}_\beta}^{\beta}\sigma_i^j(s,X_s^\varepsilon)dW_s^i \tag{159}$$

$$+\varepsilon\sum_{\mathbf{l}_\beta,\mathbf{d}_\beta}^{(k)}\int_0^t\left(\prod_{j=1}^{k}\frac{1}{l_j!}\frac{\partial^{l_j}}{\partial\varepsilon^{l_j}}X_s^{\varepsilon,d_j}\right)\sum_{i=1}^{d}\partial_{\mathbf{d}_k}^{k}\sigma_i^j(s,X_s^\varepsilon)dW_s^i \tag{160}$$

where $\partial_{d_\beta}^\beta=\frac{\partial^\beta}{\partial x_{d_1}\cdots\partial x_{d_\beta}}$,

$$\sum_{\mathbf{l}_\beta,\mathbf{d}_\beta}^{(l)}:=\sum_{\beta=1}^{l}\sum_{\mathbf{l}_\beta\in L_{l,\beta}}\sum_{\mathbf{d}_\beta\in\{1,\cdots,d\}^\beta}\frac{1}{\beta!}, \tag{161}$$

and

$$L_{l,\beta}:=\left\{\mathbf{l}_\beta=(l_1,\cdots,l_\beta);\ \sum_{j=1}^{\beta}l_j=l;\ (l,l_j,\beta\in\mathbf{N})\right\}. \tag{162}$$

The above SDEs is linear and the order of the Kusuoka function $\frac{1}{i!}\frac{\partial^i}{\partial\varepsilon^i}X_t^\varepsilon$ is determined inductively by the term

$$\sum_{\mathbf{l}_\beta,\mathbf{d}_\beta}^{(i-1)}\frac{1}{\beta!}\int_0^t\nabla X_t^\varepsilon\,(\nabla X_s^\varepsilon)^{-1}\left(\prod_{j=1}^{\beta}\frac{1}{l_j!}\frac{\partial^{l_j}}{\partial\varepsilon^{l_j}}X_s^{\varepsilon,d_j}\right)\sum_{i=1}^{d}\partial_{\mathbf{d}_\beta}^{\beta}\sigma_i(s,X_s^\varepsilon)dW_s^i\ \in\mathcal{K}_i^T. \tag{163}$$

Then, $\frac{1}{i!}\frac{\partial^i}{\partial\varepsilon^i}X_t^\varepsilon\in\mathcal{K}_i^T$. □

B Proof of Lemma 5.2

$u^{\varepsilon,0,N}$ and $\nabla_x u^{\varepsilon,0,N}\sigma$ are represented as

$$\begin{aligned}
u^{\varepsilon,0,N}(t,x) &= E[g(\bar{X}_T^{0,t,x})\vartheta_T]+E\left[\int_t^T f(s,\bar{X}_s^{0,t,x},0,0)\vartheta_s ds\right],\\
\nabla_x u^{\varepsilon,0,N}\sigma(t,x) &= \left\{E\left[g(\bar{X}_T^{0,t,x})\gamma_T\right]+E\left[\int_t^T f(s,\bar{X}_s^{0,t,x},0,0)\gamma_s ds\right]\right\}\varepsilon\sigma(t,x),
\end{aligned}$$

where $\vartheta_s=1+\sum_{i=1}^N\varepsilon^i\pi_{i,s}^{0,t}$ and $\gamma_s=\sum_{i=0}^N\varepsilon^iN_{i,s}^{0,t}$. Remark that $\vartheta_s\in\mathcal{K}^T_{\min\{0,1,\cdots,N\}}=\mathcal{K}_0^T$ and $\gamma_s\in\mathcal{K}^T_{\min\{-1,0,\cdots,N-1\}}=\mathcal{K}_{-1}^T$. Since g is Lipschitz continuous and of linear growth in x, we obtain

$$\left|E[g(\bar{X}_T^{0,t,x})\vartheta_T]\right|\le\|g(\bar{X}_T^{0,t,x})\|_{L^p}\|\vartheta_T\|_{L^q}\le C(T,x), \tag{164}$$

$$\left|E[g(\bar{X}_T^{0,t,x})\gamma_T]\varepsilon\sigma(t,x)\right|\le\varepsilon C(T,x). \tag{165}$$

Also, as f is of linear growth in x, we have

$$\left|E[\int_t^T f(s,\bar{X}_s^{0,t,x},0,0)\vartheta_s ds]\right|\le\int_t^T C(T,x)ds, \tag{166}$$

$$\left|E[\int_t^T f(s,\bar{X}_s^{0,t,x},0,0)\gamma_s ds]\varepsilon\sigma(t,x)\right|\le\int_t^T C(T,x)\frac{1}{\sqrt{s-t}}ds, \tag{167}$$

where $C(T,x)$ denotes a non-negative, non-decreasing and finite function of at most polynomial growth in x depending on T. Here, we use 4 and 5 of Proposition 5.2. Then, we obtain estimates for $u^{\varepsilon,0,N}$ and $\nabla_x u^{\varepsilon,0,N}\sigma$:

$$|u^{\varepsilon,0,N}(t,x)|\ \le\ C(T,x), \tag{168}$$

$$|\nabla_x u^{\varepsilon,0,N}\sigma(t,x)|\ \le\ C(T,x). \tag{169}$$

Note that for $k\ge 1$,

$$\begin{aligned}
u^{\varepsilon,k,N}(t,x) &= E[g(\bar{X}_T^{0,t,x})\vartheta_T]+E\left[\int_t^T f(s,X_s^{0,t,x},u^{\varepsilon,k-1,N}(s,\bar{X}_s^{0,t,x}),\nabla_x u^{\varepsilon,k-1,N}\sigma(s,\bar{X}_s^{0,t,x}))\vartheta_s ds\right],\\
\nabla_x u^{\varepsilon,k,N}\sigma(t,x) &= E[g(\bar{X}_T^{0,t,x})\gamma_T]\varepsilon\sigma(t,x)\\
&\quad+E\left[\int_t^T f(s,\bar{X}_s^{0,t,x},u^{\varepsilon,k-1,N}(s,\bar{X}_s^{0,t,x}),\nabla_x u^{\varepsilon,k-1,N}\sigma(s,\bar{X}_s^{0,t,x}))\gamma_s ds\right]\varepsilon\sigma(t,x).
\end{aligned}$$

Hence, by recursive applications of 4. and 5. in Proposition 5.2, we have

$$\left|E\left[\int_t^T f(s,\bar{X}_s^{0,t,x},u^{\varepsilon,k-1,N}(s,\bar{X}_s^{0,t,x}),\nabla_x u^{\varepsilon,k-1,N}\sigma(s,\bar{X}_s^{0,t,x}))\vartheta_s ds\right]\right| \leq \int_t^T C(T,x)ds, \tag{170}$$

$$\left|E\left[\int_t^T f(s,\bar{X}_s^{0,t,x},u^{\varepsilon,k-1,N}(s,\bar{X}_s^{0,t,x}),\nabla_x u^{\varepsilon,k-1,N}\sigma(s,\bar{X}_s^{0,t,x}))\gamma_s ds\right]\right| \leq \int_t^T C(T,x)\frac{1}{\sqrt{s-t}}ds. \tag{171}$$

Then, we obtain (97) and (98). □

Remark B.1 *Since f is a Lipschitz function, we are able to estimate (167) and (171) more precisely by using the mollifier argument. However, above is enough for our purpose here.*