CARF-F-300

On the Optimal Super- and Sub-Hedging Strategies

Yukihiro Tsuzuki
The University of Tokyo

First Version: November 2012
Current Version: August 2013

On Optimal Super-Hedging and Sub-Hedging Strategies*

Yukihiro Tsuzuki†‡

August 25, 2013

Abstract

This paper proposes optimal super-hedging and sub-hedging strategies for a derivative on two underlying assets without any specification of the underlying processes. Moreover, the strategies are free from any model of the dependency between the underlying asset prices. We derive the optimal pricing bounds by finding a joint distribution under which the derivative price is equal to the hedging portfolio's value; the portfolio consists of liquid derivatives on each of the underlying assets. As examples, we obtain new super-hedging and sub-hedging strategies for several exotic options such as quanto options, exchange options, basket options, forward starting options, and knock-out options.

†Graduate School of Economics, University of Tokyo, 7-3-1, Hongo, Bunkyo-ku, Tokyo, 113-0033, Japan, E-mail: yukihirotsuzuki@gmail.com

‡The author is grateful to Professor Akihiko Takahashi (Graduate School of Economics, University of Tokyo) for his guidance.

1 Introduction

This paper proposes optimal super-hedging and sub-hedging strategies for a derivative on two underlying assets without any specifications of the underlying processes.

The standard approach to pricing and hedging derivatives is to postulate a particular model for the behavior of the underlying asset prices. Model-parameters are determined by calibration to market prices of liquid derivatives or by estimation from historical data, and hedging is carried out based on the model with only liquid derivatives in the market. For the case of multi-asset derivatives, the dependency structure among the assets is usually estimated and cannot be hedged because there does not exist any derivatives containing information on the dependency. The model with estimated parameters does not necessarily describe the actual behavior of the market, which leads to lack of robustness of the hedging strategy. This is problematic especially in financial turmoils such as the crisis in 2007.

In order to overcome the problem of the standard approach, many researchers have been investigating model-independent super-hedging and sub-hedging strategies for single- or multi-asset derivatives, which is one of the most challenging fields in mathematical finance: currency cross-rate options or spread options by [12] and [20]; basket options or asian option by [1], [2], [9], [16], [18], and [21]; barrier options by [7] and [25]; forward starting options by [17]; lookback options by [14]. They obtain the optimal super-hedging and sub-hedging portfolio for the multi-asset derivatives under an assumption that they have no information on the joint distribution of the assets. The assumption of no information on the joint distribution is reasonable because there is no derivatives including such information in most markets. On the other hand, there exist certain researches (e.g. [26] and [3]) that make use of information embedded in some markets. For single-asset derivatives, it is assumed that marginal distributions of the underlying price at each time are known. Moreover, they add an assumption that the underlying asset price itself is a martingale, which reduces the problem to finding the solution to a Skorohod embedding problem.

We are also in line with the previous works for multi-asset derivatives: our strategies are free from any dependency between two underlying asset prices. The hedging strategy is carried out with a static portfolio which consists of liquid derivatives on each underlying asset. Here, a static portfolio means a portfolio which does not require any transaction after the inception of the contract. A model-independent static hedging strategy with liquid derivatives is effective because it is easy to construct and maintain and never fails to hedge the derivative even in the financial turmoil periods when many models and hedges collapse. We derive the optimal pricing bounds through finding a joint distribution under which the derivative price is equal to the hedging portfolio's value as in the previous works.

On the other hand, we differ from the previous works studying some specific derivatives in that we deal with more general derivatives including existing works such as quanto options, exchange options, basket options, forward starting options and knock-out options. super-hedging and sub-hedging strategies for these apparently different options are derived based on a common well-known inequality, namely Young's inequality. We prove the optimality based on copulas theory which is introduced to mathematical finance by [11]. Under the joint distribution in the optimal case, random variables appearing in a payoff function are co-monotonic or counter-monotonic (see e.g. [13] for co-monotonicity). In particular, [9] derives optimal super-hedging strategies for basket call options using theory of comonotonicity and [21] for basket call and put options. Our approach is more robust than [7], [25] and [17], which assume a price process of the underlying asset to be a martingale and require a transaction after the inception of the contract. Obviously, as their assumption is violated in the real markets with nonzero interest rates, the result cannot be directly applied in practice. Moreover, it is not necessarily possible to trade during the turmoil periods, which may cause substantial hedging errors. In contrast, we neither impose this assumption nor require any transaction after the inception.

The rest of the paper is as follows. The next sections describes the setup and the problem considered in this paper. Section three proposes our new hedging strategies. In the fourth section, our result is applied to some derivatives including quanto options, exchange options, basket options, forward starting options and knock-out options.

2 Setup

We make some assumptions on the market environment. Suppose that two risky assets are traded in the market. Let S_t^X and S_t^Y be the time-t prices of the assets respectively for $t \in [0, T^*]$, where T^* is some arbitrarily determined time horizon. The risk-free interest rate and the dividend yields of the assets are assumed to be zero for simplicity. It is assumed that there exists a risk-neutral probability measure $\mathbb{Q}$, under which the instantaneous expected rate of return on every asset is equal to zero in our settings.

Let X and Y be random variables which are dependent on each asset price S^X and S^Y respectively. Then, a derivative considered in this paper is a product with maturity T whose payoff is expressed for some function $K : \mathbb{R} \times \mathbb{R} \longrightarrow [0, +\infty)$ which is integrable on $\mathbb{R} \times \mathbb{R}$ by

$$\Phi(X, Y) := \int_\alpha^X \int_\beta^Y K(x, y) dy dx, \tag{2.1}$$

where α and β are some real numbers which are less than the essential infimum of X and Y respectively. Note that the assumption that the function K takes a non-negative value is essential. Concrete examples are shown in Section 4.

Remark 1. *A payoff with both long and short is not represented in this form since $K(x, y)$ takes negative values in this case.*

Let us suppose that the marginal distribution functions of the random variable X and Y are known.

Definition 1. *The marginal distribution functions F and G of the random variable X and Y respectively are defined for $x, y \in \mathbb{R}$ by*

$$F(x) := \mathbb{Q}(X \leq x), \tag{2.2}$$

$$G(y) := \mathbb{Q}(Y \leq y). \tag{2.3}$$

Next, let us introduce some notation to express the joint distribution of the two random variables X and Y with a copula function as in Appendix A.

Definition 2. *A joint distribution function H^C of the random variables X and Y with a copula function C is for $x, y \in \mathbb{R}$ defined by*

$$\begin{aligned} H^C(x, y) &:= C(F(x), G(y)) \\ &= \mathbb{Q}^C(X \leq x, Y \leq y). \end{aligned} \tag{2.4}$$

Especially, if the copula function is Maximum copula M,

$$\begin{aligned} H^M(x, y) &:= M(F(x), G(y)) \\ &= \mathbb{Q}^M(X \leq x, Y \leq y), \end{aligned} \tag{2.5}$$

where $M(x, y) := \min(x, y)$. In addition, $\mathbb{E}^C$ is defined as the expectation operator under $\mathbb{Q}^C$ with a copula function C.

Remark 2. *The marginal distributions of X and Y under any risk-neutral probability measure $\mathbb{Q}^C$ are independent of choice of the copula function C. We may omit C in $\mathbb{Q}^C$ and $\mathbb{E}^C$ if it is concerned with only the marginal distributions.*

The problem in this paper is stated as follows.

Problem 1. *Suppose that the marginal distribution functions of the random variable X and Y are known, but the joint distribution function is not known. Then, what is the cheapest super-hedging strategy on a derivative with maturity T whose payoff is $\Phi(X, Y)$?*

In order to prove that the super-hedging strategy is the cheapest, one way is to compare the cost of a super-hedging portfolio with the price of the derivative under some measure $\mathbb{Q}^C$ with a copula function C. The upper bound on the price of the derivative with payoff $\Phi(X,Y)$ is

$$\sup_C \mathbb{E}^C(\Phi(X,Y)), \tag{2.6}$$

where C is an arbitrary copula function. Since the cost of any super-hedging portfolio is larger than $\mathbb{E}^C(\Phi(X,Y))$ with any copula function C, if we find a particular measure $\mathbb{Q}^C$ and a super-hedging portfolio whose cost is $\mathbb{E}^C(\Phi(X,Y))$, the strategy is the cheapest one.

3 Super-hedging and Sub-hedging Strategy

In this section, we first introduce the super-hedging strategy. Then, we derive the sub-hedging strategy using the super-hedging strategy (See [10] for the case where a pay-off function is not twice differentiable). The following lemma is an extended version of Young's inequality (See Theorem 2.3 in [23]).

Lemma 1. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a non-decreasing function. Then for every Lebesgue locally integrable function $K : \mathbb{R}\times\mathbb{R} \longrightarrow [0,+\infty)$ and real numbers X, Y and α, we have*

$$\begin{aligned}\int_\alpha^X \int_{f(\alpha)}^Y K(x,y)dydx \quad &\le \quad \int_\alpha^X \left(\int_{f(\alpha)}^{f(x)} K(x,y)dy\right)dx \\ &+ \quad \int_{f(\alpha)}^Y \left(\int_\alpha^{f_{sup}^{-1}(y)} K(x,y)dx\right)dy, \end{aligned}\tag{3.1}$$

where f_{sup}^{-1} is the right-continuous inverse function of f:

$$f_{sup}^{-1}(y) := \inf\{x \in \mathbb{R} \mid y < f(x)\}. \tag{3.2}$$

If in addition K is strictly positive almost everywhere, then the equality occurs if and only if $y \in [f(x-), f(x+)]$.

Applying Lemma 1 to our problem, the value of $\Phi(X,Y)$ is dominated by the payoff of the following portfolio, if $\beta = f(\alpha)$:

- a derivative with the payoff $\int_\alpha^X \left(\int_{f(\alpha)}^{f(x)} K(x,y)dy\right)dx$
- a derivative with the payoff $\int_{f(\alpha)}^Y \left(\int_\alpha^{f_{sup}^{-1}(y)} K(x,y)dx\right)dy$.

Although these payoffs seem to be complicated, they can be replicated with liquid derivatives. For example, when X and Y are dependent only on S_T^X and S_T^Y respectively, they can be replicated with plain-vanilla options on each asset with maturity T as in [6] and [8] (See [3] for the case where a payoff function is not twice differentiable).

The question is how to choose the function f in order for the portfolio to be cheap. The answer to the question is that the function f should be f^* in Definition 3, if we assume that Assumption 1 holds.

Assumption 1. *F is continuous.*

Definition 3. *Let $f^* : [\alpha_*, +\infty) \longrightarrow \mathbb{R}$ be a non-decreasing function defined by*

$$f^*(x) := \inf\{y \in \mathbb{R} \mid F(x) < G(y)\}, \tag{3.3}$$

where α_* is the essential infimum of X.

The following lemma shows that the function f^* can be viewed as a transform of X to some random variable which has the same distribution of Y.

Lemma 2. *Suppose that Assumption 1 holds. Then, the random variable $f^*(X)$ has the same distribution as the random variable Y:*

$$\mathbb{Q}(f^*(X) \leq y) = G(y). \tag{3.4}$$

Proof. See Appendix B. □

The cheapest super-hedging strategy is obtained by taking a particular probability space such that $Y = f^*(X)$, which means that the random variables X and Y are most "dependent".

Theorem 1. *Suppose that Assumption 1 holds and that f^* is a function defined by Definition 3 which is extended such that $\beta = f^*(\alpha)$ if needed. Then,*

$$\begin{aligned} \Phi(X, f^*(X)) &= \int_\alpha^X \left(\int_{f^*(\alpha)}^{f^*(x)} K(x,y)dy \right) dx \\ &+ \int_{f^*(\alpha)}^{f^*(X)} \left(\int_\alpha^{(f^*)^{-1}_{sup}(y)} K(x,y)dx \right) dy \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \mathbb{E}^M(\Phi(X,Y)) &= \mathbb{E}\left(\int_\alpha^X \left(\int_{f^*(\alpha)}^{f^*(x)} K(x,y)dy \right) dx \right) \\ &+ \mathbb{E}\left(\int_{f^*(\alpha)}^{Y} \left(\int_\alpha^{(f^*)^{-1}_{sup}(y)} K(x,y)dx \right) dy \right), \end{aligned} \tag{3.6}$$

where $\mathbb{E}^M$ is the expectation operator of the measure with Maximum copula M defined by Eq.(2.5).

Proof. See Appendix C. □

Remark 3. *Although we have defined α and β as some real numbers which are less than the essential infimum of X and Y respectively, Theorem 1 is also valid for any real numbers α and β such that $\beta = f^*(\alpha)$.*

Remark 4. *Theorem 1 is also valid for functions (e.g. delta function) that can be approximated by series of locally integrable functions.*

Remark 5. *Let $g^*(y) := \inf\{x \in \mathbb{R} \mid G(y) < F(x)\}$. Then, $g^*(Y)$ has the same distribution as X, if G is continuous. In case where both F and G are continuous, we have another hedging strategy where X and f^* are replaced with Y and g^*. However, the hedging strategy dose not depend on the choice, since $f^*(X) = (g^*)^{-1}_{sup}(X)$ holds almost surely.*

Corollary 1. *Suppose that Assumption 1 holds, that $X > 0$ and $Y = 1_A$ for some measurable set A and $\mathbb{Q}(A) > 0$. Then,*

$$-(x_* - X)_+ + x_* 1_A \leq X 1_A \leq (X - x^*)_+ + x^* 1_A, \tag{3.7}$$

where x^* and x_* are respectively defined by

$$x^* := \inf\{x \in \mathbb{R} \mid \mathbb{Q}(A^c) \leq F(x)\} \tag{3.8}$$

$$x_* := \inf\{x \in \mathbb{R} \mid \mathbb{Q}(A) \leq F(x)\}. \tag{3.9}$$

Proof. See Appendix D. □

Let α_* and β_* be the essential infimum of X and $-Y$ respectively. Then, we have

$$\begin{aligned}\Phi(X,Y) &= -\int_{\alpha}^{X}\int_{-\beta}^{-Y} K(x,-y)dydx &(3.10)\\ &= -\left(\int_{\alpha_*}^{X}\int_{\beta_*}^{-Y} + \int_{\alpha_*}^{X}\int_{-\beta}^{\beta_*} + \int_{\alpha}^{\alpha_*}\int_{-\beta}^{-Y} K(x,-y)dydx\right). &(3.11)\end{aligned}$$

We obtain the sub-hedging strategy by applying Theorem 1 to Eq.(3.10) or the first integral of Eq.(3.11), if all of the tree integrals of Eq.(3.11) are finite.

Remark 6. *Note that the sub-hedging strategy is not necessarily determined uniquely, while the lower pricing bound is unique.*

4 Examples

In this section, we assume that S_T^X and S_T^Y are positive and their distribution functions are continuous and strictly increasing for simplicity. Moreover, for random variables X and Y and $K : \mathbb{R}\times\mathbb{R} \longrightarrow [0,+\infty)$, let $\Phi_X(x)$ and $\Phi_Y(y)$ be

$$\begin{aligned}\Phi_X(x) &:= \int_{\alpha}^{x}\left(\int_{f^*(\alpha)}^{f^*(x)} K(x,y)dy\right)dx\\ \Phi_Y(y) &:= \int_{f^*(\alpha)}^{y}\left(\int_{\alpha}^{(f^*)^{-1}_{sup}(y)} K(x,y)dx\right)dy, &(4.1)\end{aligned}$$

where f^* is defined by Definition 3 for X and Y. Hereafter, we omit sup in f^{-1}_{sup} for simplicity.

4.1 Quanto Options

Quanto options are dependent on a price of a foreign asset at maturity and a pre-fixed foreign exchange rate. Let S_T^X be the time-T exchange rate and S_T^Y be the time-T foreign asset price. A quanto call option is a contract which pays the holder a total of

$$(S_T^Y - \kappa)_+ \tag{4.2}$$

in the domestic currency, where κ is a positive number. See [4] for more details of the Black formula and [5] for an application of copulas.

Consider the payoff (4.2) denominated in the foreign currency:

$$\frac{1}{S_T^X}(S_T^Y - \kappa)_+. \tag{4.3}$$

Then we can directly apply our result to the payoff (4.3) with $X = \frac{1}{S_T^X}$, $Y = (S_T^Y - \kappa)_+$ and $K = 1$ and obtain the super-hedging portfolio:

$$XY \le \int_0^X f^*(x)dx + \int_0^Y (f^*)^{-1}(y)dy. \tag{4.4}$$

This means that the quanto option is super-hedged by

- an option on the exchange rate whose payoff is $\int_0^{\frac{1}{S_T^X}} f^*(x)dx$

- an option on the foreign asset whose payoff is $\int_0^{(S_T^Y-\kappa)_+}(f^*)^{-1}(y)dy$,

where both of the payoffs are denominated in the foreign currency.

For sub-hedging, we have for any $\alpha > 0$,

$$(X-\alpha)(-Y-\tilde{f}(\alpha)) \le \int_\alpha^X (\tilde{f}(x)-\tilde{f}(\alpha))dx + \int_{\tilde{f}(\alpha)}^{-Y}(\tilde{f}^{-1}(y)-\alpha)dy. \tag{4.5}$$

where $\tilde{f}$ is defined by Definition 3 for X and $-Y$. We obtain the sub-hedging portfolio:

- an option on the exchange rate whose payoff is $-\int_0^{\frac{1}{S_T^X}}(\tilde{f}(x)-\tilde{f}(\alpha))dx - \tilde{f}(\alpha)\frac{1}{S_T^X}$
- an option on the foreign asset whose payoff is $-\int_0^{-(S_T^Y-\kappa)_+}(\tilde{f}^{-1}(y)-\alpha)dy - \alpha((S_T^Y-\kappa)_+ - \tilde{f}(\alpha))$,

where both of the payoffs are denominated in the foreign currency.

Remark 7. *[27] investigates pricing bounds on quanto options with several numerical examples.*

4.2 Exchange Options

Exchange options are options to exchange one risky asset for another (see [22]). The options are equivalent to many financial arrangements such as spread options and cross-currency option. [12] investigates the pricing bounds for a cross-currency option and [20] does for spread options. We obtain the same result as theirs.

Let us consider an exchange option whose payoff is:

$$(S_T^X - S_T^Y - \kappa)_+, \tag{4.6}$$

where κ is a positive number.

Let $X = S_T^X$, $Y = -S_T^Y$ and $K(x,y) = \delta(x+y-\kappa)$ for deriving the super-hedging portfolio, where $\delta(\cdot)$ is Dirac delta function. Then, the payoff is expressed as:

$$(X+Y-\kappa)_+ = \int_0^X \int_{-\infty}^Y K(x,y)dydx. \tag{4.7}$$

Applying Corollary 4, we have

$$\Phi_X(X) = (X-\kappa_X^*)_+ \tag{4.8}$$
$$\Phi_Y(Y) = (Y-\kappa_Y^*)_+, \tag{4.9}$$

where κ_X^* and κ_Y^* are defined by $\kappa_X^* + f^*(\kappa_X^*) = \kappa$ and by $\kappa_Y^* + (f^*)^{-1}(\kappa_Y^*) = \kappa$ respectively. Note that such κ_X^* and κ_Y^* are uniquely determined by $\mathbb{Q}(S_T^X \le \kappa_X^*) = \mathbb{Q}(S_T^Y \ge \kappa_Y^*)$ and $\kappa_X^* + \kappa_Y^* = \kappa$. We obtained the hedging portfolio:

- a call option on S_T^X with strike κ_X^*
- a put option on S_T^Y with strike κ_Y^*,

which is the same as [12] when $\kappa = 0$.

Next, we consider the sub-hedging strategy for the exchange option.

$$\begin{aligned}(X+Y-\kappa)_+ &= -\int_0^X \int_{-\infty}^{-Y} K(x,-y)dydx + \int_0^X \int_{-\infty}^0 K(x,-y)dydx \\ &\ge -\int_0^X 1_{\{0\le x-\kappa\le \tilde{f}(x)\}}dx - \int_0^{-Y} 1_{\{y+\kappa\le \tilde{f}^{-1}(y)\}}dy + \kappa - (\kappa - X)_+,\end{aligned} \tag{4.10}$$

where $\tilde{f}$ is defined by Definition 3 for X and $-Y$. These imply the following sub-hedging portfolio:

- an option on S_T^X whose payoff is $\int_0^{S_T^X} 1_{\{0\le x-\kappa\le \tilde{f}(x)\}}dx - \kappa + (\kappa - S_T^X)_+$
- an option on S_T^Y whose payoff is $\int_0^{S_T^Y} 1_{\{y+\kappa\le \tilde{f}^{-1}(y)\}}dy$.

4.3 Basket Options

A basket option is an exotic option whose underlying is a weighted sum of different assets. [15] derives upper bounds for general n-asset case when prices of call options on each underlying asset with a continuum of strikes or a finite strikes are given. [16] also investigates lower bounds for 2-asset case under the same circumstance.

We consider a basket option whose underlying is a sum of two assets. Our assumption is the same as the continuum case of [15] and [16]. Let the payoff of the basket option be

$$(S_T^X + S_T^Y - \kappa)_+, \tag{4.11}$$

where κ is a positive number.

In order to derive the super-hedging, let us express the payoff with $X = S_T^X$, $Y = S_T^Y$ and $K(x,y) = \delta(x+y-\kappa)$:

$$(X+Y-\kappa)_+ = \int_{-\infty}^{X}\int_{-\infty}^{Y} K(x,y)dydx. \tag{4.12}$$

Then, by applying Corollary 4, the payoff of the super-hedging portfolio is

$$\begin{aligned} \Phi_X(X) &= (X-\kappa_X^*)_+ && (4.13)\\ \Phi_Y(Y) &= (Y-\kappa_Y^*)_+, && (4.14) \end{aligned}$$

where κ_X^* and κ_Y^* are defined by $\kappa_X^* + f^*(\kappa_X^*) = \kappa$ and by $\kappa_Y^* + \tilde{f}^{-1}(\kappa_Y^*) = \kappa$ respectively. Note that such κ_X^* and κ_Y^* are uniquely determined by $\mathbb{Q}(S_T^X \leq \kappa_X^*) = \mathbb{Q}(S_T^Y \leq \kappa_Y^*)$ and $\kappa_X^* + \kappa_Y^* = \kappa$. Finally, we obtained the hedging portfolio:

- a call option on S_T^X with strike κ_X^*
- a call option on S_T^Y with strike κ_Y^*.

Let us extend to an N-asset basket option whose underlying assets are S_T^n $(1 \leq n \leq N)$. By appliying mathematical induction, we have the following inequaility:

$$\left(\sum_{n=1}^{N} S_T^n - \kappa\right)_+ \leq \sum_{n=1}^{N}(S_T^n - \kappa_n^*)_+, \tag{4.15}$$

where κ_n^* are positive numbers such that $\sum_{n=1}^{N}\kappa_n^* = \kappa$ and $\mathbb{Q}(S_T^n \leq \kappa_n^*)$ is common for all of n.

Next, let us consider the sub-hedging for the basket option:

$$\begin{aligned} (X+Y-\kappa)_+ &= -\int_0^X\int_{-\infty}^{-Y} + \int_0^X\int_{-\infty}^{+\infty} + \int_{-\infty}^{0}\int_{-Y}^{+\infty} K(x,-y)dydx \\ &\geq -\int_0^X 1_{\{x\leq\tilde{f}(x)+\kappa\}}dx - \int_{-\infty}^{-Y} 1_{\{0\leq y+\kappa\leq\tilde{f}^{-1}(y)\}}dy + X + (Y-\kappa)_+. && (4.16) \end{aligned}$$

Then, the following portfolio is optimal:

- an option whose payoff is $S_T^X - \int_0^{S_T^X} 1_{\{x\leq\tilde{f}(x)+\kappa\}}dx$
- an option whose payoff is $(S_T^Y - \kappa)_+ - \int_{-\infty}^{-S_T^Y} 1_{\{0\leq y+\kappa\leq\tilde{f}^{-1}(y)\}}dy$.

Sub-hedging an N-asset basket option($N > 2$) is still open to our best knowledge. See [16].

4.4 Forward Starting Options

Forward starting options are options whose strike will be determined at some later date. Let S_t be the underlying asset at time t, T_1 be the date when the strike is determined and T_2 be the date when the payoff is paid. There are two types of payoffs of forward starting options. One is $(S_{T_2} - \kappa S_{T_1})_+$ for a positive number κ. This is equivalent to an exchange option, which has been studied in the previous section and in [17]. Then, in this section, we consider the other type of payoff:

$$\left(\frac{S_{T_2}}{S_{T_1}} - \kappa\right)_+ . \tag{4.17}$$

First, let us consider super-hedging a payoff XY with $X = S_{T_2}$ and $Y = \frac{1}{S_{T_1}}$. Applying the theorem, we have $XY \leq \Phi_X(X) + \Phi_Y(Y)$, where

$$\Phi_X(X) = \int_0^X f^*(x)dx \tag{4.18}$$

$$\Phi_Y(Y) = \int_0^Y (f^*)^{-1}(y)dy. \tag{4.19}$$

Then, the payoff of the forward starting option is satisfied with the following inequality as shown in Section 4.3:

$$(XY - \kappa)_+ \leq (\Phi_X(X) - \kappa_X)_+ + (\Phi_Y(Y) - \kappa_Y)_+ , \tag{4.20}$$

where κ_X and κ_Y are any positive number such that $\kappa_X + \kappa_Y = \kappa$. In order to obtain the cheapest super-hedging, let us define x^* by

$$\Phi_X(x^*) + \Phi_Y((f^*)^{-1}(x^*)) = \kappa. \tag{4.21}$$

Note that x^* is uniquely determined because the left-hand side of (4.21) is increasing with respect to x^* from 0 to $+\infty$. If we take $\kappa_X = \Phi_X(x^*)$ and $\kappa_Y = \kappa - \kappa^*$, the right-hand side of (4.20) implies the cheapest super-hedging portfolio, because $Y = f^*(X)$ gives an equality of the inequality (4.20). Then, the hedging portfolio is as follows:

- an option with maturity T_2 whose payoff is $\left(\int_0^{S_{T_2}} f^*(x)dx - \kappa_X\right)_+$
- an option with maturity T_1 whose payoff is $\left(\int_0^{\frac{1}{S_{T_1}}} (f^*)^{-1}(y)dy - \kappa_Y\right)_+$,

where $\kappa_X = \int_0^{x^*} f^*(x)dx$ and $\kappa_Y = \kappa - \kappa_X$.

Next, let us consider the sub-hedging strategy for the forward starting option. Applying the theorem to XY with $X = S_{T_2}$ and $Y = -\frac{1}{S_{T_1}}$, we have $XY \leq \Phi_X(X) + \Phi_Y(Y)$, where

$$\Phi_X(X) = \int_0^X f^*(x)dx \tag{4.22}$$

$$\Phi_Y(Y) = \int_{-\infty}^Y (f^*)^{-1}(y)dy. \tag{4.23}$$

Then, the payoff of the forward starting option is satisfied with the following inequality:

$$(-XY - \kappa)_+ \geq (-\Phi_X(X) - \Phi_Y(Y) - \kappa)_+ . \tag{4.24}$$

Since the right-hand side of the above inequality is the same as an exchange option considered in Section 4.2, we have

$$\begin{aligned}(-XY-\kappa)_+ &\geq (-\Phi_X(X)-\Phi_Y(Y)-\kappa)_+ \\ &\geq (-\Phi_X(X)-\kappa)_+ - \int_0^{-\Phi_X(X)} 1_{\{0\leq x-\kappa\leq g^*(x)\}}dx \\ &\quad - \int_0^{\Phi_Y(Y)} 1_{\{y+\kappa\leq (g^*)^{-1}(y)\}}dy, \end{aligned} \tag{4.25}$$

where g^* is defined as Definition 3 for $-\Phi_X(X)$ and $\Phi_Y(Y)$. Since $g^*(-\Phi_X(X))$ and $\Phi_Y(f^*(X))$ have the same distribution as $\Phi_Y(Y)$, we have $g^*\circ(-\Phi_X)=\Phi_Y\circ f^*$ almost surely with respect to F and $g^*(-\Phi_X(X))=\Phi_Y(f^*(X))$ almost surely with respect to $\mathbb{Q}$. If we take $Y=f^*(X)$, then we have $\Phi_Y(Y)=\Phi_Y(f^*(X))$, which means that the two equalities in (4.25) occur simultaneously. The hedging portfolio is as follows:

- an option with maturity T_2 whose payoff is the payoff

$$(-\Phi_X(S_{T_2})-\kappa)_+ - \int_0^{-\Phi_X(S_{T_2})} 1_{\{0\leq x-\kappa\leq g^*(x)\}}dx \tag{4.26}$$

- an option with maturity T_1 whose payoff is the payoff

$$-\int_0^{\Phi_Y(-\frac{1}{S_{T_1}})} 1_{\{y+\kappa\leq (g^*)^{-1}(y)\}}dy. \tag{4.27}$$

Remark 8. *Although we assume that the interest rate is zero in this paper, there is a hedging error caused by difference of payment in non-zero interest rate market, because the payoff of the forward starting option is paid at T_2, while one of the components of the hedging portfolio is paid at T_1. It is sufficient that just multiplying a discount factor fills the gap between T_1 and T_2 when interest rate is deterministic.*

Remark 9. *The hedging strategy for an exchange option in Section 4.2 can be applied to forward starting options with payoff $(S_{T_2}-\kappa S_{T_1})_+$. It is different from the strategy considered in [17], which allows to trade forward contracts at time T_1 and requires the martingale condition $\mathbb{E}(S_{T_2}\mid S_{T_1})=S_{T_1}$.*

4.5 Knock-out Options

A knock-out option is a type of an exotic option that provides a payoff only if a certain predetermined event does not occur. Let us first consider an option whose payoff is dependent on an asset price and knock-out event is on another asset:

$$(S_T-\kappa)_+1_A, \tag{4.28}$$

where S_T is an asset price at maturity and A is an event regarding the other. For example, A is an event that the foreign exchange rate reaches or does not reach a predetermined price, namely "barrier level". An option whose payoff is 1_A is called a one-touch option.

Directly applying Corollary 1 to the payoff with $X=(S_T-\kappa)_+$, we obtain the following inequality:

$$\begin{aligned} &-(x_*-(S_T-\kappa)_+)_+ + x_*1_A \\ \leq\ & (S_T-\kappa)_+1_A \\ \leq\ & (S_T-(\kappa+x^*))_+ + x^*1_A, \end{aligned} \tag{4.29}$$

where $x^*, x_*>0$ such that $\mathbb{Q}(A^c)=F(x^*)$ and $\mathbb{Q}(A)=F(x_*)$. This means that the super-hedging portfolio is

- a call option with strike $\kappa + x^*$
- the one touch option

and the sub-hedging portfolio is

- an option whose payoff is $(x_* - (S_T - \kappa)_+)_+$
- the one touch option.

Next, let us consider a knock-out option whose payoff and knock-out event are dependent on a common underlying asset. Suppose that the payoff of the option is:

$$(S_T - \kappa)_+ 1_A, \tag{4.30}$$

where $I := [L, U]$ and $A := \{S_t \in I \mid 0 \leq \forall t \leq T\}$ for some $L, U \in [0, +\infty]$. This is called a single or double barrier call option.

In this case, we cannot obtain the optimal super-hedging and sub-hedging portfolio by directly applying our result. Instead, apply Corollary 1 to $X^* 1_A$ for super-hedging and $X_* 1_A$ for sub-hedging, where $X^* := (S_T - \kappa)_+ 1_{A_T}$ and $X_* := (S_T - \kappa)_+ 1_{A_T} + \infty \cdot 1_{A_T^c}$ (assume $0 \cdot \infty = 0$). Then, we have

$$\begin{aligned} & -(x_* - (S_T - \kappa)_+ 1_{A_T})_+ 1_{A_T} + x_* 1_A \\ \leq \; & (S_T - \kappa)_+ 1_A \\ \leq \; & ((S_T - \kappa)_+ 1_{A_T} - x^*)_+ + x^* 1_A, \end{aligned} \tag{4.31}$$

where

$$x^* := \inf\{x \in [0, U - \kappa] \mid \mathbb{Q}(x + \kappa < S_T \leq U) \leq \mathbb{Q}(A)\} \tag{4.32}$$

$$x_* := \inf\{x \in [0, U - \kappa] \mid \mathbb{Q}(A) \leq \mathbb{Q}(L \leq S_T \leq x + \kappa)\}. \tag{4.33}$$

This means that the super-hedging portfolio is

- an option whose payoff is $((S_T - \kappa)_+ 1_{A_T} - x^*)_+$
- the one touch option

and the sub-hedging portfolio is

- an option whose payoff is $(x_* - (S_T - \kappa)_+ 1_{A_T})_+ 1_{A_T}$
- the one touch option,

which is the same as [28].

Remark 10. *The reason why the same result as [28] cannot be obtained by directly applying our result is that the theorem of this paper does not assume any dependency structure between the two random variables, while both of the random variables $(S_T - \kappa)_+$ and 1_A are dependent on S_T.*

A Copula

Some standard notions and well-known results related to the two dimensional Copula are stated in this section (see [24] for more details).

First of all, let the definition of Copula be introduced.

Definition 4. *A copula is any function $C : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ which has the following properties:*

1. *for every* $x, y \in [0, 1]$

$$C(x, 0) = C(0, y) = 0 \tag{A.1}$$

and

$$C(x, 1) = x, C(1, y) = y; \tag{A.2}$$

2. *for every* $x_1, x_2, y_1, y_2 \in [0, 1]$ *such that* $x_1 \leq x_2$ *and* $y_1 \leq y_2$,

$$C(x_1, y_1) - C(x_1, y_2) - C(x_2, y_1) + C(x_2, y_2) \geq 0. \tag{A.3}$$

Definition 5. *A distribution function* F *of a random variable* X *is defined as* $F(x) := \mathbb{Q}(X \leq x)$. *A distribution function* H *of two random variables* X *and* Y *is defined as* $H(x) := \mathbb{Q}(X \leq x, Y \leq y)$. *Here,* $\mathbb{Q}$ *is a probability under which* X *and* Y *are defined.*

The following is Sklar's theorem, which elucidates the role that copulas play in the relationship between multivariate distribution functions and their univariate margins.

Lemma 3. *Let* H *be a joint distribution function with marginal distribution functions* F *and* G. *Then there exists a copula function* C *such that for every* $x, y \in \mathbb{R}$

$$H(x, y) = C(F(x), G(y)). \tag{A.4}$$

Moreover, if F *and* G *are continuous, then* C *is unique. Otherwise,* C *is uniquely determined on* $F(\mathbb{R}) \times G(\mathbb{R})$.

Conversely, if C *is a copula and* F *and* G *are distribution functions, then the function* H *defined by Eq.(A.4) is a joint distribution function with margins* F *and* G.

Next, we introduce Fréchet-Hoeffding copula boundaries. There are two special copula functions: Minimum copula and Maximum copula, which are defined as follows.

Definition 6. *Minimum copula* W *is defined by*

$$W(x, y) := \max(x + y - 1, 0). \tag{A.5}$$

Maximum copula M *is defined by*

$$M(x, y) := \min(x, y). \tag{A.6}$$

It is easily verified that these functions are copula functions. In addition, these are boundaries in the meanings of the next lemma.

Lemma 4. *For every copula function* C *and for every* $x, y \in [0, 1]$,

$$W(x, y) \leq C(x, y) \leq M(x, y). \tag{A.7}$$

and for every joint distribution function H *with marginal distribution functions* F *and* G *and for every* $x, y \in [0, 1]$,

$$W(F(x), G(y)) \leq H(x, y) \leq M(F(x), G(y)). \tag{A.8}$$

Remark 11. *The copula theory holds for* $n \in \mathbb{N}$. *Note that* W *is not a copula function for* $n > 2$.

B Proof of Lemma 2

proof of Lemma 2. First, we show $\mathbb{Q}(f^*(X) \le y) = \mathbb{Q}(X \le g^*(y))$, where $g^*(y) := \inf\{x \in \mathbb{R} \mid G(y) < F(x)\}$. $f^*(X) \le y$ means that $f^*(X) < y + \epsilon$ for any $\epsilon > 0$, which is equivalent with $F(X) < G(y+\epsilon)$ for any $\epsilon > 0$. On the other hand, $X \le g^*(y)$ means that $X - \epsilon < g^*(y)$ for any $\epsilon > 0$, which is equivalent with $F(X - \epsilon) \le G(y)$ for any $\epsilon > 0$. By continuity of F, this is equivalent with $F(X) \le G(y)$. Then, we have $\mathbb{Q}(f^*(X) \le y) = \mathbb{Q}(X \le g^*(y))$ because of $\mathbb{Q}(F(X) = G(y)) = 0$.

Next, we have $F(g^*(y) - \epsilon) \le G(y) < F(g^*(y) + \epsilon)$ for any $\epsilon > 0$, when $0 < G(y) < 1$. Then, $F(g^*(y)) = G(y)$ by continuity of F. This is also true when $G(y) = 0$ or $G(y) = 1$. Finally, we obtain $\mathbb{Q}(f^*(X) \le y) = \mathbb{Q}(X \le g^*(y)) = F(g^*(y)) = G(y)$. □

C Proof of Theorem 1

proof of Theorem 1. Eq(3.5) is followed by Lemma 1. Let x, y be real numbers and $H(x,y) := \mathbb{Q}(X \le x, f^*(X) \le y)$. (i) Suppose that $F(x) < G(y)$. Then we have $f^*(x) \le y$ and $H(x,y) = \mathbb{Q}(X \le x) = F(x)$. (ii) Suppose that $F(x) > G(y)$. Then, we have $f^*(x) \ge y+\epsilon$ for some $\epsilon > 0$ and $H(x,y) = \mathbb{Q}(f^*(X) \le y) = G(y)$. (iii) Suppose that $F(x) = G(y)$. If $f^*(x) = y$, we have $H(x,y) = \mathbb{Q}(X \le x) = F(x)$. Otherwise, we have $H(x,y) = F(x)$ for $f^*(x) < y$ and $H(x,y) = G(y)$ for $f^*(x) > y$. Therefore, the function H is the same as Maximum copula M. □

D Proof of Corollary 1

proof of Corollary 1. Under the assumptions, we have

$$G(y) = \begin{cases} 0 & (y < 0) \\ p & (0 \le y < 1) \\ 1 & (1 \le y) \end{cases}, \tag{D.1}$$

$$f^*(x) = \begin{cases} 0 & (0 \le F(x) < p) \\ 1 & (p \le F(x) \le 1) \end{cases} \tag{D.2}$$

and

$$(f^*)^{-1}(y) = \begin{cases} -\infty & (y < 0) \\ x^* & (0 \le y < 1) \\ +\infty & (1 \le y) \end{cases}. \tag{D.3}$$

Theorem 1 leads to

$$\begin{aligned} X 1_A &\le \int_0^X f^*(\xi) d\xi + \int_0^{1_A} (f^*)^{-1}(\xi) d\xi \\ &= (X - x^*)_+ + x^* 1_A. \end{aligned} \tag{D.4}$$

Using the upper bound, we have

$$\begin{aligned} -X 1_{A^c} = X 1_A - X &\le (X - x^*)_+ - X + x^* 1_A \\ &= (x^* - X)_+ - x^* + x^* 1_A \end{aligned} \tag{D.5}$$

If we view A^c as A, we obtain the lower bound. □

References

[1] Albrecher, H., Dhaene, J., Goovaerts, M. and Schoutens, W., Static Hedging of Asian Options under Levy Models: The Comonotonicity Approach. *Journal of Derivatives* (2005) 12 (3), 63 - 72.

[2] A. d'Aspremont and L. El-Ghaoui, Static arbitrage bounds on basket option prices, *Journal Mathematical Programming* Series A 106(3) (2006) 467-489

[3] J. Baldeaux and M. Rutkowski, Static replication of forward-start claims and realized variance swaps, *Applied Mathematical Finance* 17(2) (2010) 99-131

[4] M. Baxter and A. Rennie, *Financial Calculus: An Introduction to Financial Derivatives*, Cambridge University Press (1996)

[5] M. N. Bennett and J. E. Kennedy, Quanto pricing with copulas, *Journal of Derivatives* 12(1) (2004) 26-45

[6] D. Breeden and R. Litzenberger, Prices of state contingent claims implicit in option prices, *Journal of Business* 51 (1978) 621-651

[7] H. Brown, D. Hobson and L. C. G. Rogers, Robust hedging of barrier options, *Mathematical Finance* 11(3) (2001) 285 - 314

[8] P. Carr and D.H. Madan, Towards a theory of volatility trading, *Volatility*, Risk Publications (1998) 417-427

[9] X. Chen, G. Deelstra, J. Dhaene and M. Vanmaele, Static super-replicating strategies for a class of exotic options. *Insurance: Mathematics and Economics*, (2008). 42(3), 1067-1085

[10] K.C. Cheung, J. Dhaene, A. Kukush and D. Linders, Ordered random vectors and equality in distribution. Research report AFI-1377 FEB, KU Leuven, (2013).

[11] U. Cherubini and E. Luciano, Multivariate Option Pricing With Copulas, *Applied Mathematical Finance* 9 (2002) 69-86

[12] S-L. Chung and Y-H. Wang, Bounds and prices of currency cross-rate options, *Journal of Banking & Finance* 32(5) (2008) 631-642

[13] J. Dhaene, M. Denuit, M. Goovaerts R. Kaas and D. Vyncke, The concept of comonotonicity in actuarial science and finance: theory, *Insurance: Mathematics & Economics*, 31(1) (2002) 3-33

[14] D. Hobson, Robust hedging of the lookback option, *Finance and Stochastics* 11(3) (1998) 285-314

[15] D. Hobson, P. Laurence and T.-H. Wang, Static-arbitrage upper bounds for the prices of basket options, *Quantitative Finance* 5(4) (2005) 329-342

[16] D. Hobson, P. Laurence and T.-H. Wang, Static arbitrage optimal sub-replicating strategies for basket options, *Insurance Mathematics and Economics* 37, (2005), 553-572

[17] D. Hobson and A. Neuberger, Robust bounds for forward start options, *Mathematical Finance* 22(1) (2012) 31-56

[18] P. Laurence and T.-H. Wang, Sharp upper and lower bounds for basket options, *Applied Mathematical Finance* 12(3) (2003)

[19] P. Laurence and T.-H. Wang, Sharp distribution free lower bounds for spread options and the corresponding optimal subreplicating portfolios, *Insurance Mathematics and Economics* 44(1) (2009) 35-47

[20] P. Laurence and T.-H. Wang, Sharp distribution free lower bounds for spread options and the corresponding optimal subreplicating portfolios, *Insurance Mathematics and Economics* 44(1) (2009) 35-47

[21] D. Linders, J. Dhaene, H. Hounnon and M. Vanmaele, Index options: a model-free approach. Research report AFI-1265, FEB, KU Leuven, (2012).

[22] W. Margrabe, The value of an option to exchange one asset for another, *Journal of Finance* 33 (1978) 177-186

[23] F. Mitroi and C. P. Niculescu, An extension of Young's inequality, *Abstract and Applied Analysis* (2011)

[24] R.B. Nelson, *An Introduction to Copulas*, Springer-Verlag, Lecture Notes In Statistics, Berlin (1998)

[25] A. Neuberger and S. Hodges, Rational bounds on the prices of barrier options, *Technical Report, London Business School* (2000)

[26] M. Schmutz and T. Zürcher, Static replication with traffic light options, Preprint (2010)

[27] Y. Tsuzuki, Pricing bounds on quanto options , Preprint, http://papers.ssrn.com/abstract=1755275

[28] Y. Tsuzuki, Pricing bounds on barrier options , Preprint, http://papers.ssrn.com/abstract=1984953, forthcoming in *Journal of Futures Markets*