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Collective Risk Control And Group Security:
The Unexpected Consequences of Differential Risk Aversion*

Toshihiro Ihori[a] and Martin C. McGuire[b]

Abstract

We provide an analysis of odds-improving self-protection for when it yields collective benefits to groups, such as alliances of nations, for whom risks of loss are public bads and prevention of loss is a public good. Our analysis of common risk reduction shows how diminishing returns in risk improvement can be folded into income effects. These income effects then imply that whether protection is inferior or normal depends on the risk aversion characteristics of underlying utility functions, and on the interaction between these, the level of risk, and marginal effectiveness of risk abatement. We demonstrate how public good inferiority is highly likely when the good is "group risk reduction." In fact, we discover a natural or endogenous limit on the size of a group and of the amount of risk controlling outlay it will provide under Nash behavior. We call this limit an "Inferior Goods Barrier" to voluntary risk reduction. For the paradigm case of declining risk aversion, increases in group size/wealth will cause provision of more safety to change from a normal to an inferior good thereby creating such a barrier.

Key words: Risk Control, Collective Action, Public Goods, Voluntary Provision, International Security

JEL classification numbers: D74, D8, G11, H41, H56 , H87

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1: Introduction

The classic of economic analysis for how an individual expected-utility maximizer should deal with threat of loss is based on Ehrlich and Becker, (1972) "EB". EB's rational agents can choose to allocate resources among risk reducing self-protection, preparations to curtail losses if they happen, or insurance contracts with others to compensate for losses that occur.

EB's model also should apply to policy choices of groups of countries considered as monolithic rational decision units facing common international risks, a topical subject nowadays as in global warming, pollution, security, or finance. There is of course a giant literature on collective provision of insurance, wherein the risks of loss are assumed to be fixed (e.g. Genicot and Ray, 2003). But application of the EB model to collective probability improving, "self-protection" is sparse, except for some work on terrorism such as Lapan and Sandler (1988) or more recently Sandler (1992, 1997, 2005), and Arce, Daniel, and Sandler, (2005). In particular, economists' models of voluntary public good provision with many agents (VPG) have not been extended to understand the consequences of differences in risk aversion in this risk management context where common defense will reduce common hazard. The object of this paper is to make that extension, with the VPG model applied to groups of risk adverse agents who share common chances of loss and common benefit when those risks are curtailed. These agents can protect or defend themselves, but such measures, we assume, necessarily spill over to the benefit of other agents in the group or coalition equally improving their odds as well. Thus, all have incentives to free ride, and their individual actions are influenced by the reciprocal free spill-in benefits they receive from others. Evidently, many international problems in the world today resemble this common threat/loss management problem.

The VPG model (Olson, 1965; Olson and Zeckhauser, 1966), has come to have a standard stylized structure and format that has yielded especially striking properties with respect to Cournot-Nash equilibria among members of a public good consuming group (Bergstrom Blume and Varian, 1986; Andreoni, 1988, 1989; McGuire and Shrestha, 2003; Cornes and Sandler, 1986, 1996; Warr, 1983).

A limit of the original model, however, was that it assumed a linear or summation aggregator for the consumption technology ("summation in consumption"), and also a summation aggregator for group contributions or finance ("summation in finance"). Diminishing marginal returns in provision of the public good were, therefore, assumed away; only constant cost linear production was addressed. Later beginning with the "weakest-link/best-shot" contributions of Hirshleifer (1983, 1985), subsequent literature including Cornes (1993),Vicary (1990), Sandler and Vicary (2002), recognized a variety of ways that individual consumption or enjoyment of the public good depended on individual own and other contributions. These variations in the "aggregation of consumption technology," have become standard in such treatises as Cornes and Sandler (1996) or Mueller (2003).

However diminishing returns are essential in risk reduction. With risk of loss defined as 1-p(M), ($0 \leq p \leq 1$) and M as total expenditures on deterrence or defense to reduce the chance of loss, then assumedly p=p(M) shows diminishing returns, $p' > 0$, $p''(M) < 0$. Thus, to handle collective probability improvement we must cope with "non-summation consumption aggregation" in public good provision. Although this difference may represent only the simplest beginning departure from the standard VPG model and although it is implicit in more advanced studies where individual contributions are not perfect substitutes such as Cornes (1993), nevertheless it has not been explicitly treated elsewhere. (Actually, the fact that our "innovation" here is so minimal makes the perversity of our results all the more notable!)

Thus our simple solution to the problem of diminishing returns and distribution of infra marginal costs/gains will be to assume a "summation *finance* aggregator," $M = \sum m$, in the provision of public good p, even though p(M) represents a "non-summation *consumption* aggregator" (i.e. $p(M) \neq \sum p_i(m_i)$) Then, importing an idea from contest theory[1] we take primitive preferences as being over *contributions* to risk reduction, rather than risk reduction itself. This allows the differences caused by diminishing returns in p(M) and the effects of positive and/or variable differences in risk aversion to be folded into VPG income effects. It will confirm that a "summation aggregator in consumption" is not necessary and that "summation finance" alone is sufficient to maintain the Warr neutrality properties of the original model (as implicit in the treatments of Cornes and Sandler,1996, and Mueller 2003) And we will learn whether Olson's "exploitation of the great by the small," property obtains when the public good is risk reduction under "summation finance", and how heterogeneous attitudes as to risk influence equilibrium.

2. Preliminaries: Preview of Our Argument

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[1] We thank a referee for this reference to contest theory.

Most especially we will learn that the degree of common risk, relative wealth, and variability of risk aversion all interact in a novel and hitherto unrecognized fashion. In particular we demonstrate that when risk aversion increases with income and risk is low (high) then self protection (expenditure thereon) tends to be an inferior good for low (high) income agents. On the other hand, if risk aversion decreases with income and risk is low (high) then self protection tends to be inferior for agents with high (low) income. Now consider the fact that when agents form a group for public good provision then *ipso facto* the full income of each agent increases (possibly dramatically). It follows, that these properties can have major, and possibly quite unwelcome effects on the nature and stability of group behavior and thus of the Nash VPG solution.

As an intuitive preview to motivate interest, consider this problem in a world of two contingencies, 0 and 1, consumption C with $C^0 < C^1$, and utility U(C) with $U^0(C^0) < U^1(C^1)$. Suppose that an agent has allocated resources to probability improvement as per EB to an expected-utility optimum --- designated p*, E*[U]--- where expected marginal benefits (MB) of probability improvement equal expected marginal costs (MC). Suppose his risk aversion is low to nil.

Now let this agent's resource allocations be fixed (same C^0*, C^1*, U^0*, U^1*, p*etc.) but let his utility function become more risk averse (say, due to income growth); and just for illustration assume that it coincides with the first utility function at both outcomes 0 and 1. For this new configuration with higher risk aversion will the old allocation continue to be optimal? Since C^0*, C^1*, U^0*, U^1* and p*, are unchanged there is no change in expected marginal benefit, (MB). However, expected marginal costs (MC) do change --- with MC at the bad contingency becoming greater (assuming U" < 0) and MC at the good contingency becoming smaller due to the higher risk aversion. Thus, when risk is low (p is high), expected MC tends to decline, and vice versa. Expected MC, therefore, will increase or decrease depending on the weights p*, (1-p*). In short, whether greater risk aversion induces more (less) outlay on self-protection depends on whether expected MC decreases (or increases) and therefore on status quo probabilities.

As we will demonstrate, this risk/risk-aversion interaction-effect profoundly influences the behavior of self-protecting groups especially when group size/wealth changes. It can cause risk improvement to become inferior under income growth, can endogenously limit the degree of protection a group will provide for itself, and it introduces an endogenous limit on voluntary group size.

3: Analytical Framework

Let the world consist of two countries; country 1 and country 2, and have two states, a good state "1" and a bad state, "0". Ignoring all insurance and compensation possibilities, expected utility for a single country i (i = 1, 2) is given as:

$$W_i = pU^1(C_i) + (1-p)U^0(C_i - L_i) \tag{1}$$

$$\text{or} \qquad W_i = W_i(C_i, p) \tag{2}$$

where W_i is expected utility for Country i, C_i is i's private consumption, L_i is i's loss in the bad state, and p is the chance of a good state. Our analysis will focus on the first Ehrlich-Becker (EB) modality of defense --- raising p and reducing (1-p) --- EB's "self-protection;" we take L_i to be fixed, (eliminating EB's "self-insurance"). The variable "p" might be risk of war, shared indivisibly by two coalition members. Utility function U( ) is assumed the same whether luck is good or bad. U^1 denotes realized utility if the good event happens, and U^0 if the bad event happens. We assume $U_Y \equiv \partial U/\partial Y > 0, U_{YY} \equiv \partial^2 U/\partial Y^2 < 0$.

The individual country's budget constraint is given as

$$Y_i = C_i + m_i \tag{3}$$

where Y_i is a fixed national income and m_i denotes allocations to risk reduction; that is, m_i gives the voluntary input to the public good by Country i. Here "p" is the public good (as conventionally defined) for countries 1 and 2, since we assume that protection "p" benefits both countries in a non-rival non-excluded fashion. That is, both m_i spent on self-protection reduce the chance of a bad event or decrease what we later call "baseline risk of [1-p(0)]," increasing "p" the probability of a good event for both parties. So we can account for the collective summation technology quality of the inputs to p by writing

$$p = p(M), \tag{4}$$

with summation finance

$$M = m_1 + m_2. \tag{5}$$

Protective expenditures by countries 1 and 2 are equally effective in reducing the common risks. Specifically, m_1 and m_2 are perfect substitutes for each other. M, therefore, is the aggregate voluntary expenditure on the public good, giving the uncoordinated group's total amount of resources devoted to reducing the probability of bad state or the risk of loss. *A priori*, many risk reduction or self protection functions are plausible: quadratic, exponential, logistic etc. One expects p' > 0 throughout for all of these; while p" may vary, we assume here that $p'' < 0$ throughout.

4: Individual Optimizations

For improvement effected through changes in p, expected utility (1) is maximized with respect to m_i subject to constraints (3), (4), and (5). This gives (6) and (7) as first and the second order conditions:

$$\text{FOC} \quad p'(U^1 - U^0) - [pU_Y^1 + (1-p)U_Y^0] = 0 \tag{6}$$

$$\text{SOC} \quad p''(U^1 - U^0) - 2p'(U_Y^1 - U_Y^0) + [pU_{YY}^1 + (1-p)U_{YY}^0] < 0 \tag{7}$$

We will introduce simplifying notation W_Y, W_{YY}, Δ, Δ_Y, MC_M, and MB_M to interpret (6) and (7)

TABLE I HERE

Thus, the term $p'(U^1 - U^0) = p'\Delta$ in (6) gives the marginal benefit --- MB_M --- of expending m_1 in country 1, (or m_2 by its partner country), that is the marginal gain in expected utility from increasing public good M and, therefore, p. The term $pU_Y^1 + (1-p)U_Y^0 = W_Y$ in (6) gives marginal expected cost of providing the public good --- MC_M --- i.e. the marginal loss of expected utility from reducing private consumption by m_1.

TABLE II HERE

Apparently, the second order condition (7) requires some combination of large absolute value of $p''(U^1 - U^0) < 0$ or of $[pU_{YY}^1 + (1-p)U_{YY}^0] < 0$, and/or small absolute value of $p'(U_Y^1 - U_Y^0) < 0$. If this SOC obtains, then FOC (6) will represent a maximum rather than a minimum. Table III extends the compact notation of Tables I and II, with the second subscripts indicating second derivatives of MB_M and of MC_M.

TABLE III HERE

5. Interior Nash Equilibria

Treatment of Cost-Input as the Public Good

Now, to exploit the advantages of "summation financing" instead of the conventional expression for expected welfare, i.e. $W_i(C_i, p)$, we propose --- as described elsewhere in the literature (e.g. Cornes and Hartley, 2003) --- to work with "induced" preferences over C_i and M and adjust terminology slightly so that M is called a "public good." An increase in M at given C_i changes the welfare of both parties (in the relevant range raises welfare) even though M is not itself directly an object of consumption per se; nor is it an argument in the conventional direct expected utility function $W_i(C_i\ , p)$. But providing this "public good" does instrumentally raise expected utility of consuming the private good for both parties in a non-rival non-excluded fashion. Moreover, so long as the Nash equilibrium is interior so that both parties make positive contributions, M is effectively chosen by and agreed on by both as noted first by Becker (1974). We want to introduce this innovation, extending the term "public good' to the indirect productive input, M, because doing so permits us to use conventional geometric properties of the VPG model to derive and illustrate the unconventional insights of this article. These relate to connections among (a) risk aversion in the utility function, (b) status quo risk, (c) normality/inferiority of the public good M, (d) stability and therefore attainability of Nash solutions, and (e) effects of group size on equilibrium. Thus, in place of (2) --- i.e. $W_i(C_i\ , p)$ ---we will write a country's expected welfare objective function as:

$$W_i = W_i(C_i\ , M) \tag{8}$$

First of all, just for completeness we show in footnote 2 that Cournot-Nash solutions and their known properties continue to obtain for the induced or primitive utility functions W(C, M).[2]

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[2] *Nash Equilibria When the Public Good is Measured by Summation of Finance*

Eq. (8) gives expected utility W_i for country i for given L_i. From expressions (3) and (5) we obtain the effective full income budget constraint.

$$C_1 + M = Y_1 + m_2 \tag{8a}$$

Therefore, utility maximizing behavior may be expressed in terms of the expenditure function, $E_i(\ )$

$$E_1(W_1, L_1) = Y_1 + m_2 \tag{8b}$$

and

$$E_2(W_2, L_2) = Y_2 + m_1 \tag{8c}$$

The right-hand side defines full income, Y_i*, and the left-hand side defines the expenditure function, which depends on expected utility and loss in the bad state (taken as a parameter). Adding these two equations gives as the world-wide feasibility condition

$$E_1(W_1, L_1) + E_2(W_2, L_2) = Y_1 + Y_2 + M_1(W_1, L_1) \tag{8d}$$

where $M_1(\ )$ denotes the compensated demand function for M in country 1. If the equilibrium solution to E_1 and E_2 is interior then both countries must have chosen the same value of M. Thus in equilibrium

$$M_1(W_1, L_1) = M_2(W_2, L_2) \tag{8e}$$

Nash Reaction Functions When the Public Good is Total Cost

More important is to show the Nash equilibrium with reaction functions in space (m_1, m_2). Figure 1 has curve N_1 for country 1 and N_2 for 2. As with any good if M is normal ($dM/dY^* > 0$), the absolute value of the slope of N_1 with respect to the m_2-axis is less than 1 ($-1 < dm_1/dm_2 < 0$, (as in Cornes and Sandler, 1986, 1996). Here, as shown in Figure 1, the equilibrium point K is stable. If the public good M is inferior ($dM/dY^* < 0$), then $dm_1/dm_2 < -1$. The (absolute) slope of country 1's reaction curve (with respect to the m_2-axis) is greater than 1, and the Nash equilibrium point would be unstable (not drawn**).**

FIGURE 1 HERE

6: Income Effects: Normal and Inferior

In the standard VPG model all interactions among participants are propagated by income effects. Thus our adaptation of Ehrlich and Becker must explore the effects of income change or differences between individuals on how public good provision is shared, and how free rider incentives operate. Doing this will reveal a surprising insight into the influence of risk aversion on these income effects, which are crucial to the stability and dynamic attainability of Cournot-Nash equilibria. Here we will see how taking M as the public good (rather than p) has facilitated this task.

For comparative static results it is the sign of the income effect we must investigate. Specifically, we want to determine whether M is an inferior or a normal "good". Taking total differentiation of FOC (6) gives:

$$\frac{dM}{dY^*} = -\frac{p'(U_Y^1 - U_Y^0) - [pU_{YY}^1 + (1-p)U_{YY}^0]}{p''(U^1 - U^0) - 2p'(U_Y^1 - U_Y^0) + [pU_{YY}^1 + (1-p)U_{YY}^0]} = -\frac{p'\Delta_Y - W_{YY}}{p''\Delta - 2p'\Delta_Y + W_{YY}} \tag{9}$$

where Y* is the effective individual "full" income that obtains at an interior solution. Assuming identical preferences and wealth to illustrate would give: $Y^* \equiv Y_1 + m_2 (\equiv Y_2 + m_1)$.

Equations (8d) and (8e) then determine expected welfare of each country, (W_1, W_2) as a function of incomes Y_1, Y_2 and losses in the bad state, (L_1, L_2). A diagram in the space of (W_1, W_2) would show loci for the two conditions (8d) and (8e) intersecting where and if Nash equilibria exist.

Based on equations (8d) and (8e) we also see that the equilibrium conditions are independent of redistribution of income. Equilibrium is dependent on total income, $Y_1 + Y_2$ but not on Y_1 or Y_2 separately. Thus, the neutrality result holds as in the conventional VPG model. See Cornes and Sandler (1984, 1996), Warr (1983), and Bergstrom, Blume, and Varian (1986). Provided redistribution of income between members does not change the set of positive contributors, it will not affect the real equilibrium.

FIGURE 2 HERE

Condition (7), the SOC[3], determines the sign of the denominator in (9) as negative at an optimum (as shown in Figure 2 at E), if the second order condition actually obtains. But the sign of the numerator is ambiguous, and the normality or inferiority of M depends on this numerator. Hence, if the numerator is negative, given the SOC, the sign of (9) is negative, and M becomes an inferior good. *Intuitively when M is inferior, an increase in income reduces marginal benefit more than it reduces marginal cost.*

To see this consider the numerator of (9). The term $p'\Delta_Y$ gives the change in marginal benefit or MB_{MY} (of providing the public good, M) that would be caused by an increase in Y; this is negative. When Y* increases, U^0 rises more than U^1 and hence Δ_Y, the difference between U_Y^0 and U_Y^1 declines. Thus, an increase in Y* reduces the marginal benefit of enjoying M measured from any initial optimal level of that public good. Next consider the second term of (9). W_{YY} measures the change (due to an increase in Y) in the expected marginal utility cost of providing M, or MC_{MY}. This also is negative when Y* increases since both components of U_Y decline with an increase in Y*. That is, when Y* increases and a country becomes richer, U_Y^0 and U_Y^1 (the utility cost components of providing the public good) both decline. Thus, an increase in Y* reduces the marginal cost of providing public good, M. So to summarize from (9), if marginal benefit of providing M declines in absolute amount more than marginal cost, --- i.e. if $p'(U_Y^1 - U_Y^0) < [pU_{YY}^1 + (1-p)U_{YY}^0]$ --- then the

[3] *Corner Solutions*

For an interior solution to the problem of risk reduction, the SOC's of equation (7) require

$$p''/p' - 2\Delta_Y/\Delta + W_{YY}/W_Y < 0$$

But at a tangency point, where FOC's obtain, the curvature of M=M(C, W) is given as

$$d^2M/dC^2 = -W_{YY}/W_Y \quad + 2\Delta_Y/\Delta \quad - p''/p'$$

This means that for an interior maximum the curvature of the indifference curve must be positive

$$d^2M/dC^2 > 0$$

As illustrated in Figure 2 this is consistent with tangency between budget line (with slope -1 given that the unit cost of M is 1) and the maximum of indifference curves M = M(C, W). If the SOC is not satisfied and hence d^2M/dC^2 is negative, then the curvature of the indifference curve is "wrong" and no interior solution is possible. In this case one member of the group will provide all the public good (M or p) and others will free ride. Obviously the first and last terms of d^2M/dC^2 are positive, tending to give indifference curves of the "right" curvature. However if these terms are sufficiently small then the middle negative term can dominate and the curvature of indifference curves be negative, in which case and corner solutions become likely.

numerator of (9) becomes negative, so that $dM/dY^* = -(-/-) < 0$. Here an increase in Y depresses the demand for M, meaning that M is inferior.

We can depict such negative income effects by defining a family of indifference curves W(C, M); these are constant expected utility contours in the space (C, M) for a given value of L. We write these as $M = M(C, W)$ shown in Figure 2. For a given expected utility and given loss, L, as in equation (1), we write the absolute value of the slope of an indifference curve:

$$MRS = \left.\frac{dM}{dC}\right|_{W\ const} = -M'(C) = \frac{pU_Y^1 + (1-p)U_Y^0}{p'(M)(U^1 - U^0)} = \frac{W_Y}{p'\Delta} = \frac{MC_M}{MB_M} > 0 \qquad (10)$$

with W_Y, Δ, MB_M, and MC_M as defined above. Now consider an increase in Y* shown by an outward shift in the budget line of Figure 2. For given level of M, let C increase from E to E'. If MRS increases in absolute value at E' compared with E (curve a in Figure 2) the new equilibrium will be to south-east of point E' and desired M will decline in value. This is the case of a negative income effect on M. But if MRS declines in absolute value at E' compared with E (curve b in Figure 2), the new equilibrium point is to north-west of E' and desired M rises, showing a positive income effect on the public good.

To evaluate how MRS varies at an optimum when income changes[4] in (10), partial differentiation of MRS with respect to C and insertion of FOC (6) yields:

$$MRS_C = \frac{\partial MRS}{\partial C} = \frac{\{[pU_{YY}^1 + (1-p)U_{YY}^0] - p'(M)(U_Y^1 - U_Y^0)\}}{p'(M)(U^1 - U^0)} = \frac{W_{YY} - p'\Delta_y}{p'\Delta} = \frac{W_{YY}}{p'\Delta} - \frac{\Delta_Y}{\Delta} \overset{sign}{=} \frac{W_{YY}}{p'} - \Delta_Y \qquad (11)$$

Note that $dM/dY^* < 0$ in (9) and M is inferior if and only if $MRS_C = \partial MRS/\partial C > 0$ in (11). Then, given that p' > 0, from (11) if $\partial(W_Y - p'\Delta)/\partial Y = 0$ i.e. $\partial(MC_M - MB_M)/\partial Y = 0$ then MRS_C is zero, so that M then is borderline normal. Using our compact notation we can summarize succinctly by writing:

- The first order condition requires $MB_M = MC_M$,
- The second order condition requires $MB_{MM} < MC_{MM}$,
- Non-inferiority requires $MB_{MY} > MC_{MY}$,

[4]Evaluation of MRS_C = ∂MRS/∂C is important because it will establish a formula, eq. (11) for seeing if M is inferior or normal, depending on the sign of the formula. We will see presently that its sign depends in turn on risk plus the risk aversion properties of U. Thus it establishes the connection between inferiority and risk-plus-risk-aversion. Since all interaction among group members are propagated as income effects, MRS_C = ∂MRS/∂C captures or displays these interactions.

The sign of MRS_C and thus normality or inferiority of M is seen from (11) to depend on three factors: p', U_{YY}, and p. Details of these effects, item by item, are discussed in the Appendix. Taken together, however, in the aggregate, they imply the hitherto unrecognized connection to risk aversion that we have mentioned above, and to which we now turn.

7: Normality, Inferiority and Risk Aversion: The Inferior Goods Barrier to Public Good Provision

It was shown above that for commonly beneficial expenditures allocated to risk-reduction, the sign of income effects depended on the sign of the numerator in equation (9). But, as we now demonstrate, this numerator depends crucially and systematically on the risk aversion properties of the underlying utility function <u>and on the interaction of these with p</u>. Absolute risk aversion (R) is defined as

$$\mathrm{R} = -U_{YY}/U_Y \ or -U_{YY} = R \cdot U_Y \tag{12}$$

Then the numerator of (9) --- with C_0 and C_1 as consumption in each contingency and R_0 and R_1 as the associated absolute risk aversion ---can be written as

$$H = \underset{(-)}{p'(U_Y^1 - U_Y^0)} + \underset{(+)}{[pR_1U_Y^1 + (1-p)R_0U_Y^0]} \tag{13}$$

So normality or inferiority of M now depends on the sign of (13): for H < 0, M is inferior. But with the sign of each part of (13) indicated in parentheses, the overall sign is ambiguous depending on magnitude and properties of R interacting with (p, [1-p]). To see this ambiguity multiply FOC (6) by R_1 to obtain

$$R_1 p'(U^1 - U^0) = [pR_1U_Y^1 + (1-p)R_1U_Y^0] \tag{14}$$

Then re-write Eq (13) as:

$$H = p'(U_Y^1 - U_Y^0) + [pR_1U_Y^1 + (1-p)R_1U_Y^0] + (1-p)R_0U_Y^0 - (1-p)R_1U_Y^0 \tag{15}$$

Substituting (14) into (15) gives:

$$\mathrm{H} = \begin{array}{l} p'\,[(U_Y^1 + R_1U^1) - (U_Y^0 + R_0U^0)]\{defined\ as = Q\} \\ +[(1-p)U_Y^0 + p'U^0][R_0 - R_1]\{defined\ as = T\} \end{array} \tag{16a}$$

i.e.

$$\mathrm{H} = \mathrm{Q} + \mathrm{T} \tag{16b}$$

As it is demonstrable that $d[U_Y + R(Y)U]/dY = R'U$ Eqs. (17) and (18) follow.

$$R'>0 \rightarrow \{Q>0; T<0\}, \quad R'<0 \rightarrow \{Q<0; T>0\} \quad \therefore (Q+T) \gtreqless 0 \tag{17}$$

$$R'=0 \rightarrow \{Q=0; T=0\}, \text{ in which case } N=Q+T=0 \tag{18}$$

Risk Aversion along Indifference Curves M= M (W, C)

Constant Risk Aversion: We can now dissect the relation shown by (13) if we begin with the knife-edge case of constant risk aversion. When R = R* a constant or R' = 0, expression (18) conclusively implies that H = 0, that good M, therefore, is on the borderline between normality and inferiority and the income effect is zero. This case represents a break-even case of no interdependence between risk aversion and the magnitude of risk as determinants of optimal allocations to defense/deterrence. The negative exponential utility function $U = 1 - e^{-R^*Y}$ generates this case. The reason we call it 'breakeven" is that for all other preference functions other than that of constant R, there is a variable, fluctuating relation between p and desired M, with potential for wobbling between normality and inferiority. We examine this next.

Increasing Risk Aversion: If $R_0 < R_1$ then eq. (13) shows that low risk and high risk aversion interact. When risk is low, and (1-p) small, high risk aversion combines with high p to weight expression (13) positively --- toward normality. Here chance favors the outcome where risk aversion is greater. When R increases with income rational agents will insure against low probability events, low (1-p), and the richer they are the more will they so insure. The opposite risk profile causes (13) to be negative and M to tend toward inferiority. Thus, if risk is high so that (1-p) is great, the positive part of eq. (13) weighs less because R_0 is small; therefore, (13) tends to be negative so that M is an inferior good. Here chance gives the less risk averse outcome more weight. Expenditure on M resembles a gamble, not insurance, and rational agents will gamble by wagering on improvement to p by expending on M. But the richer they are the less will they so gamble.

Decreasing Risk Aversion: On the other hand, if risk aversion is decreasing, $R_0 > R_1$, then high risk and low risk aversion reinforce each other. Now high risk aversion, i.e. large R_0 which obtains at lower wealth, interacts with high risk (1-p) to weight expression (13) positive and the indifference curve toward normality. Here the rational, expected utility maximizing agent will gamble on improving the less likely event and the richer he is the more will he gamble. But the opposite combination of lower risk (meaning high p) and a greater weight, therefore, on lesser risk aversion R_1 leads to inferiority.[5] That is when p is big, low risk aversion correlates with the higher wealth outcome, and due to the higher weight on small R_1 (13) tends to be negative and provision of M inferior. So here the rational agent will insure, but the richer he is the less.

Critical risk p*: We can give a heuristic summary of these forces by introducing the idea of a critical crossover value of p in eq. (13). That is for given R_0 and R_1 and assuming $R_0 < R_1$ there is a critical value of p* (with $0 \leq p^* \leq 1$) such that for p> p* good M is normal, and for p<p* the good is inferior. Thus when $R_0 < R_1$ if p* = 1, M is necessarily inferior and for p* = 0, good M is necessarily normal. Correspondingly if $R_0 > R_1$ there is a critical risk p* such that for p> p* the good is inferior, while for p< p* M is normal. Here if p* = 1, p ≤ p* and M is necessarily normal, while if p* = 0, p ≥ p* necessarily and M is inferior.

Generally we expect absolute risk aversion to decrease with wealth and the taste for risk correspondingly to rise, so that the amount of insurance purchase declines, *ceteris paribus*, as the rational agent grows richer and his propensity to gamble increases. But in our analysis, expenditure on odds-improving self protection resembles insurance when the chance of a bad outcome is low. So, decreasing absolute risk aversion is congruent with inferiority of M ---a negative income effect --- when risk is low leading to lower "insurance" purchase as income increases.

But if risk is high, and thus the weight on high R_0 is great, the decision to improve p(M) by expending M resembles a gamble rather than insurance, and a propensity to gamble correlates with low risk aversion. Here when risk is high with $R_1 < R_0$ risk aversion is also high, so that an increase in income increases the propensity to gamble, and thus here expenditure on M is a normal good. Still we cannot exclude the possibility that risk aversion will increase with income in which case we would anticipate a reversal of the negative/positive income effect described above. So to summarize the whole situation, including the effects of ΔY on ΔM we must include the increasing risk aversion case. We present this summary in Table IVa and Table IVb derived from 4a. [Corresponding cells are labeled A, B, C, and D].

TABLE IVa HERE

Solutions for Nash Equilibria Using the Cornes-Hartley "Replacement Function"

[5] The same idea that risk averse agents insure against unlikely events while risk tolerant agents gamble on them is covered in detail in McGuire, Pratt, and Zeckhauser (1991).

Cornes and Hartley (2000, 2003) have suggested an elegant construction that allows a direct visualization of which members of a public good group will make positive contributions in Nash equilibrium, how much will be supplied, and how this outcome changes with group size and composition (See also Andreoni and McGuire, 1993). We can use this method to demonstrate the consequences of public good inferiority for group provision and its stability[6]. They define a "replacement function" (or more generally replacement correspondence) as follows. Let the Nash reaction function be given as

$$m_i = N^i(Y_i, M_{-i}) = N^i(Y_i, M - m_i); \tag{19a}$$

$$M_{-i} = \Sigma_{j \neq i} m_j; \tag{19b}$$

$$N^i_{M-i} \equiv \partial m_i / \partial M_{-i} \tag{19c}$$

where M_{-i} indicates public good provided by all agents except agent i. Then the replacement function is:

$$m_i = r^i(Y_i, M) \tag{20a}$$

FIGURE 3 HERE

The geometric derivation of this function in the space of (M, m_i) is shown in Figure 3. It follows that

$$r^i_M \equiv \partial r^i / \partial M \quad = N^i_{M-i} / (1 + N^i_{M-i}) \tag{20b}$$

This immediately yields the Cournot-Nash equilibrium where the aggregate of individual replacement functions $\Sigma r = \mathcal{R}$ (different from the "R" of risk aversion) crosses the 45°through the origin, i.e. at

$$\Sigma m_i = M = \Sigma[r_i(M)] \equiv \mathcal{R} \tag{21}$$

To use r^i and $\mathcal{R}$ it will be helpful first to relate their properties to the underlying Nash reaction functions. Thus, if M is a normal good, then -1< N^i_{M-i} < 0, and r^i_M <0. Then the individual replacement function is decreasing with M and we designate the function as "Normal." On the other hand, if M is an inferior good, then $1+N^i_{M-i}$ <0, whence N^i_{M-i} <-1 and r^i_M >1>0. Now the individual replacement function is increasing with M with its slope greater than 1, and we call this function "Inferior."

[6] We owe thanks to a referee for an extraordinarily generous and insightful review that has led to our use of the replacement function.

The replacement function for individual i, of course, incorporates the effects of i's own income level Y_i on i's contribution to the group provision of the public good M. To see these effects consider the individual replacement function $m_i = r^i(Y_i, M)$. Note that an increase in income Y_i will raise m_i whether M is normal or inferior. That is, regardless of income effects

$$\frac{\partial m_i}{\partial Y_i} \equiv r_Y = \frac{\partial N^i / \partial Y_i}{1 + \partial N^i / \partial M_{-i}} \equiv \frac{N_Y^i}{1 + N_{M-i}^i} \geq 0 \tag{22}$$

where $N_Y \equiv \partial N^i / \partial Y_i$ gives the incremental effect on m_i (given M_{-i}) of an increase in Y_i along agent i's Nash reaction function; this is negative for M an inferior good, positive if M is normal. The denominator is also negative when M is inferior, and positive if normal. Thus, if M is inferior, $\partial m_i / \partial Y \equiv r_Y$ is positive --- the same as in the normal good case where both numerator and denominator are positive.

To analyze the group equilibrium as in (21) we focus on the case when all agents are identical, and therefore omit individual i specific notation. In this case of homogeneous agents and symmetric equilibrium the equilibrium requires

$$nr(Y, M) = nm = M, \tag{23}$$

with

$$\frac{dM}{dY} = \frac{nr_Y}{1 - nr_M(M)} \tag{24}$$

Here dM/dY indicates the incremental change in aggregate equilibrium provision of M when the income of each and every individual increases incrementally. If the public good is normal, both numerator and denominator of (24) are positive. An increase in income raises the provision of public good. On the other hand, if the public good is inferior, the numerator is positive, while the denominator is negative. Hence in this case the sign of (24) is negative. Thus, the direction of change in the aggregate equilibrium provision of M correlates with negative or positive income effects[7]. However, in the case of homogeneous agents and symmetric inferiority at the new equilibrium the solution is unstable, and thus most probably unattainable.[8] We turn next to these details.

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[7] Note the indeterminacy here of Nash equilibrium when M is borderline normal. Reaction functions of identical countries have 45^{O} slope all overlap throughout.

[8] Further relations between $r^i(Y_i, M)$ and $N^i(Y_i, M)$ as given by

Multiple Equilibria and Stability

The collective provision of risk reduction as a public good seems beset by effects of changing risk aversion interacting with risk itself. Table IVa shows this, and so suggests the likelihood of multiple equilibria, instabilities, and corner solutions (see footnote 3). Fortunately, the replacement function construct is well suited to analysis of such effects. To illustrate we confine our attention to two identical agents, Mr. 1 and Mr. 2, assuming for both that risk aversion and status quo p are such that M is inferior for low incomes but normal at high incomes. The individual replacement function for either agent is then shown in Figure 4a by curve AB; section AC applies when M is normal and BC when it is inferior. For each section we have the replacement correspondences identified by superscripts A and B:

FIGURE 4a HERE

$$\{m^A, m^B\} = r(M) \tag{26}$$

or

$$m^A = r^A(M) \tag{27}$$

and

$$m^B = r^B(M) \tag{28}$$

$$r_Y^i = N_Y^i / (1 + N_{M-i}^i) \tag{25a}$$

and

$$r_M^i = N_{M-i}^i / (1 + N_{M-i}^i)\,. \tag{25b}$$

allow us to express (24) in terms of the reaction functions:

$$\frac{dM}{dY} = \frac{nr_Y}{1 - nr_{\mathrm{M}}} = \frac{n[N_Y / (1 + N_{M-i})]}{1 - n(\mathrm{N_{M\text{-}i}}/(1+\mathrm{N_{M\text{-}i}})\,)}\,; \tag{25c}$$

that is

$$\frac{dM}{dY} = \frac{nN_Y}{1 + N_{M-i} - nN_{M-i}} \tag{25d}$$

Note that when income effects are negative and M inferior, even though as per eq. (22) each individual agent would contribute more when only his income increases, nevertheless when every agent's income increases and M is an inferior good the aggregate equilibrium provision declines. The reason for the difference lies in the interaction between income effects. The denominators of (25c) or (25d) indicates how in a group interaction the negative effects of reciprocal reductions in M_{-i} when M is inferior outweigh the positive individual effects of eq. (22).

where $m^A < m^B$ and m^A belongs to curve AC, while m^B belongs to curve BC. Thus, for M normal in region AC, m is decreasing with M, and for M inferior in region BC m is increasing with M. Again, as in eq. (5) the Nash equilibrium condition is:. $m_1 + m_2 = M$

This condition gives multiple and/or unstable equilibria depending on the mix of replacement functions -- normal and inferior--- among agents. Figure 4a gives one example showing the vertical sum of two ACB curves, one for 1 and one for 2 where like sections of ACB are summed for 1 and for 2. Denoted A'C'B' this shows the vertical addition of two AC sections together and then two BC sections (not AC + BC). The drawing shows a case where A'C'B' intersects the 45^O line once, at S, so that equilibrium occurs when each agent is on the M-normal or AC section of his replacement function. When agents are homogeneous and their equilibrium positions symmetrically identical

FIGURE 4b HERE

$$R = 2r(M) = M \tag{29}$$

and each provides the same m; therefore, $m_1 = m_2$. This is a "standard" case: with r_1 and r_2 normal, equilibrium is stable as indicated by the direction of the arrows in Figure 4b. However, if replacement functions are such that at the intersection of ACB and the 45^O the inferior section (BC) obtains symmetrically for both 1 and 2 then although still $m_1 = m_2$ equilibrium now becomes unstable. This case is shown in Figure 5a, with arrows pointing to the instability drawn in Figure 5b.

FIGURES 5a AND 5b HERE

Note, however, that even if agents are homogeneous with identical individual replacement functions, the equilibria may not be symmetric if it occurs on the normal section of one agent's replacement function but simultaneously on the inferior section of the other's.[9] To illustrate, go back to Figure 4a and construct the aggregate replacement function from *dissimilar* sections of ACB (i.e. from AC+BC). Suppose equilibrium occurs where M is normal for Mr. 1 but an inferior good for Mr. 2. Then for the individual replacement functions we write

$$m_1 = m^A, \quad m_2 = m^B \tag{30}$$

[9] We thank a referee for pointing out the multiple and diverse possibilities for asymmetric equilibrium to us.

with equilibrium condition

$$R \equiv m^A + m^B = M \tag{31}$$

We illustrate this equilibrium as S* in Figure 4a**,** where the vertical sum of AC curve and CB (shown as A'C'B") curve intersects the 45^{o} line. But even for such limited asymmetry (population still homogeneous), Nash equilibrium may be stable or unstable and depending on the exact shape and positioning of individual replacement functions ACB, and numerous stable/unstable sequences are entirely possible. For example, in Figure 5a even though the sum of identical inferior sections of individual replacement functions gives an equilibrium at "S," we could aggregate AC for one agent plus CB for the other. This would give another aggregate replacement function (A'C'B" not shown in Figure 5a) with another Nash equilibrium, S*, this time <u>stable</u>. The figures therefore illustrate that at a symmetric equilibrium, if the aggregate replacement curve is upward-sloping (downward-sloping), M is inferior (normal) and the equilibrium is unstable (stable). But if the equilibrium is asymmetric, and the slope of aggregate replacement curve is greater (smaller) than 1, then it is stable (unstable) .[10]. Of course if agents are not homogeneous there can be a multitude of other, asymmetric equilibria.

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[10] Note from (20b) we have

$$N^i_{M-i} = r^i_M \big/ (1 - r^i_M) \tag{20c}$$

In a two person model suppose M is normal for person 1 but inferior for person 2. Then, stability requires is that the (absolute) slope of person 1's reaction function

$$-\frac{dm_1}{dm_2} = -N^1_{m2} = -\frac{r^1_M}{1 - r^1_M} \tag{20d}$$

to be less than the inverse of the (absolute) slope of person2's reaction function

$$-\frac{dm_1}{dm_2} = -1/N^2_{m1} = -\frac{1 - r^2_M}{r^2_M} \tag{20e}$$

Therefore if

$$-\frac{dm_1}{dm_2} \text{ given by (20d)} < -\frac{dm_1}{dm_2} \text{ given by (20e)} \tag{20f}$$

then, the equilibrium is stable. Condition (20f) reduces to

$$\frac{r^1_M}{r^1_M - 1} < \frac{r^2_M - 1}{r^2_M} \tag{20g}$$

or

$$r^1_M + r^2_M > 1 \tag{20h}$$

In other words, an asymmetric equilibrium is stable (unstable) if the slope of aggregate replacement curve is greater (smaller) than 1. In Figure 5a, at S* the slope of aggregate replacement curve is greater than 1, and hence S* is stable. In Figure 4a, at S* the slope of aggregate replacement curve is smaller than 1, and hence it is unstable.

8. Effects of Change in Group Membership: Size and the Inferior-Good Barrier to Public Good Supply

Effects of Y on M are of special interest, and emphasized in Table IVa because of the association of greater income with larger group size that we know obtains under Nash-Cournot behavior. Specifically we know from Becker (1974) Atkinson and Stiglitz (1980) and Bergstrom Blume and Varian (1986) that formation of a group for public good provision --- assuming interior solutions--- always increases “full income” for each and every member of the group, and the larger the group of positive contributors, *ceteris paribus*, the larger is this full income. Therefore, group formation or increase in group size by augmenting each agent's "full income" would have a tendency to change the choice of M for each agent depending on the cell in the table.

Thus Table IVa implies certain rather surprising effects when hitherto disconnected countries form a group and react (in a Nash-Cournot manner) to the income effects they confer upon one another. It suggests we might compare static interior Nash equilibria before and after a new entry when all members of a group (new and old) are identical. Since the table implies that many Nash "equilibria" would be unstable we say little about the dynamics nor likely end result of enlarging group size. However, we do know that strong incentives exist for corner solutions to arise when interior Nash outcomes are unstable. In light of this fact generalizations to be derived from our analysis are arresting. To see these effects, consider the case where at an interior equilibrium (stable or unstable) all countries make positive contributions to the public good.

TABLE IVb HERE

First, it is clear that the degree of status quo protection, i.e. the initial value of p, and the direction of change in risk aversion, basically will determine the qualitative result of enlarging a Nash group. For example, if an agent's risk aversion declines with increases in wealth (as expected) and if a group is basically unsafe and p therefore is low, [Cell D in Tables IVa-b] then adding a new member will increase M (normal good, stable equilibrium). However if more members continue to be added so that more and more is spent on M, the value of p will increase and the situation will migrate toward Cell C, where the public good is inferior and equilibrium is unstable, as depicted in the previous diagrams. This suggests a natural or endogenous limit on the size of a group and of the amount of M and therefore of p(M) --- a new conclusion markedly different from the standard VPG model. Once a region where M is inferior is reached and the Nash equilibrium becomes unstable new agents will probably induce chaotic adjustments leading to a corner solution. And even if the new unstable equilibrium were somehow reached, since M becomes inferior on transition from Cell to C to D, the total voluntary provision of M as we prove below will decline, notwithstanding that incentives reward all parties for enlarging the group.

On the other hand, returning to Tables IVa-b if the initial status quo is very hazardous (p is low) but risk aversion is increasing for all agents [cell B] then adding new members actually *reduces* the (comparative static) equilibrium provision of M and therefore of p. So if risk aversion is increasing --- absent some global agreement to collaborate in the universal provision of p --- addition of members who behave by Nash-Cournot rules, can NEVER achieve a high level of protection, crossing the critical level of risk that separates Cell A from Cell B. This phenomenon also has never before been identified. It calls for a more organized rigorous definition of the connections between critical risk and group membership that separates Cells A/B or Cells C/D. The replacement function gives us a tool to do this.

The Replacement Function: Effects of Change in Size of Group

To pursue the analytics of an increase in the number of countries, start with an interior Nash equilibrium in an identical agent model with n members. Using the replacement function the Nash equilibrium for a homogeneous identical membership is given as

$$R = nr(M) = M \tag{32}$$

With comparative statics

$$dM/dn = r(M)/[1 - nr_M(M)] \tag{33}$$

It follows if M is normal, r_M<0 and hence the sign of (33) is positive: an increase in the size of group n raises the total level of public goods M. However, when M increases, p rises and hence sooner or later M becomes inferior. But once M becomes inferior, r_M>1 and hence the sign of (33) is negative: an increase in the size of group n reduces the total level of public goods M. Thus inferiority limits the ability of a group to increase public good provision by means of membership expansion --- an "endogenous barrier" to public good provision.

Next we also derive the effect of an increase in group size on the welfare of the initial membership. We show this for comparative static Nash equilibria when agents are homogeneous and identical irrespective of the stability or instability of such equilibria To see this, the worldwide feasibility condition (8d) (where "F" designates full income[11]) gives:

$$E(W) = Y + [(n-1)/n]M(W) = \text{ F} \tag{34}$$

Differentiating gives:

$$\frac{dW}{dn} = \frac{1}{n^2} M \Big/ [E_w - \frac{n-1}{n} M_w] > 0 \tag{35}$$

Thus, an increase in n always raises welfare, independently of the sign of M_w. Since welfare is increasing with full income F, (35) shows that F(n+1)>F(n). If M is inferior, M_w<0. In such a case, we also have M(n+1)<M(n). Thus, the combination of M(n+1) > M(n) and F(n+1) < F(n) --- or decreasing full income combined with increasing public good provision --- is excluded. Accordingly, adding new members is always beneficial but it eventually makes the public good inferior and hence it cannot indefinitely raise the total provision of the public good.

To sum up, when at the initial state the public good is normal, adding new members would tend to change the nature of public good to inferior, and hence cause the total provision of public good to decline. On the other hand, when at the initial state the public good is inferior, adding new members could not make the nature of public good normal, and hence the total provision of public good will still decline. Thus in a sense, the effect of risk aversion on the desire to insure and/or gamble creates "an inferior good barrier" that obstructs the ordinary consequences of adding new members to an alliance of states or other relevant group.

9. Changes in Magnitude and Distribution of Losses

Of special interest is how the magnitude of loss in adversity affects incentives to self-protect and to form protective alliances with others. To proceed with this problem total differentiation of (6) with respect to L now gives:

$$\frac{dM}{dL} = -\frac{p'U_Y^0 + (1-p)U_{YY}^0}{p''(U^1 - U^0) - 2p'(U_Y^1 - U_Y^0) + [pU_{YY}^1 + (1-p)U_{YY}^0]} \tag{36}$$

[11] Y* represents full income throughout this paper. Here we introduce F as notation for full income to emphasize its functional dependence on size of group.

From (9) and (36), derive

$$\frac{dM}{dY} = -\frac{dM}{dL} - \frac{p'U_Y^1 - pU_{YY}^1}{D} \tag{37}$$

Given the second order condition, the sign of the denominator D would be negative. The second term $-[p'U_Y^1 - pU_{YY}^1]/D$ therefore is positive. It follows that

$$\text{if } M_L \equiv dM/dL \leq 0\text{, then } M_Y \equiv dM/dY > 0: \tag{38}$$

$$\text{if } M_Y \leq 0\text{, then } M_L > 0. \tag{39}$$

although $M_L > 0$ is consistent with positive or negative M_Y.

Now define $M_W = dM/dW$. Since M_W is signed the same as M_Y we can use (38) and (39) to investigate our problem --- welfare effects of changes in magnitude and distribution of losses --- using M_W interchangeably with M_Y to obtain qualitative results. To facilitate this assume two countries are identical except for $L_1 > L_2$. Next we write M_1 and M_2 for country 1's or 2's desired total of M. Then considering the equilibrium condition $M_1(W_1, L_1) = M_2(W_2, L_2)$, we infer following results:

- If $M_L<0$ (then M is normal and $M_W>0$), the inequality $L_1 > L_2$ implies $W_2 < W_1$ in equilibrium; that is, paradoxically in Nash-Cournot equilibrium, as between two otherwise identical countries <u>the country with a smaller loss is worse off</u> than the country with a relatively high loss after taking public good interaction effects into account.
- Correspondingly, if $M_W < 0$ (then $M_L > 0$), the inequality $L_1 > L_2$ again implies $W_2 < W_1$; again the paradoxical relation that a country with a smaller loss ends up worse off. However, because equilibria here are unstable we cannot interpret comparative static results as a transitional process toward a new equilibrium. When an increase in L_1 raises M_1 --- and hence m_1 --- dynamic interactions might lead to a corner solution with $m_2 = 0$. See arrow in Figure 6.

FIGURE 6 HERE

Of course for an increase in L_1 both W_1 and W_2 decline due to a negative wealth effect. But the above analysis implies that the relative impact in equilibrium is greater for the country with the smaller loss.

The intuition here is the same as when a single country's income decreases. For example, let ($M_Y >$ 0) $M_W > 0$ and $M_L < 0$. Then from an initial equilibrium an increase in L_1 reduces m_l ($M_L < 0$), and with it the full income of Country 2 causing a decline in M_2. But to restore equilibrium and a commonly chosen amount of M --- as in (8e) --- Country 2 must compensate in part for the loss of m_1, consuming less C_2. With a decline in both C_2 and M_2 Country 2 is worse off. In equilibrium the greater loss L_1 harms country 2, therefore, more than it does country 1. See Figure 7.

FIGURE 7 HERE

The replacement function also shows how loss L affects the provision of M. In particular this construct shows that a<u>n increase in loss L_i is likely to reduce</u> m_i. If all agents are identical (symmetry case) the equilibrium condition is

$$R \equiv nr(M;Y,L) = M \tag{40}$$

Then

$$\frac{dM}{dL} = \frac{nr_L}{1 - nr_M(M)} \tag{41}$$

If the public good is normal, both numerator and denominator are positive, and increases in L raises the provision of public good. But if the public good is inferior, the numerator is positive, and the denominator negative so that overall (41) is negative. Qualitatively this effect on the Nash outcome is the same as for an increase in n. (But, in the case of inferiority the equilibrium is not stable.)

10: Conclusions

We have employed the VPG model for analysis of group behavior when risk is a collective bad, indivisibly shared by all members of a group, and its control therefore a collective good. Such analysis requires including the effects of increasing-costs/diminishing-returns in public good provision. We have incorporated this effect here by taking preferences over cost inputs as primitive objects --- which allows exploitation of the "summation finance" features of the problem. Effectively this innovative definition of the public good as the aggregate of costs contributed by all agents or countries taken together allows infra-marginal effects of increasing costs to be folded into income effects.

Since all interactions between agents in the VPG world are mediated through income effects, we have focused on these to show when public good M is inferior and when it is normal. This important property of group interaction we show follows from properties of agents' preferences with respect to risk. When it is risk control that is the public good we show that interactions between preferences --- characterized as high vs. low absolute risk aversion[12], and increasing, constant or decreasing risk aversion --- and objective risk levels will decisively influence the VPG interaction among group members. This leads to surprising new properties of Cournot behavior and equilibria in risk control.

Adding new members and/or economic growth may produce "an inferior good barrier" and, if this occurs, further increases in an alliance's membership will not reduce the probability of a bad outcome by providing the public good M. Moreover, systematic patterns of change to and from normality/inferiority are to be expected. In fact, for any configuration there will be a critical risk that together with other inputs determines a crossover point from normality to inferiority or vice versa. Such crossover values moreover, will define barriers to risk improvement and even make growth in group membership a cause for decline in public good provision.

Our analysis along this line shows that goods inferiority is much less unlikely when collective risk control is at stake than in the run-of-the-mill VPG example. Accordingly, instabilities in Nash-Cournot outcomes and absence of interior solutions are altogether more likely than in the received textbook case. Reflection suggests that this analysis is bad news for managing multi-country interactions in risk reduction. There are indeed many world risk problems where collective action is needed, with voluntary provision being the minimal level of such "cooperation." Multinational disease control, coping with terror threats, and environmental risk management to a major degree have as their goals improvements in *risk profiles* (notwithstanding the fact that the public good of risk reduction may be imperfect, mixed, or impure, so that the pure public goods model must be modified). But if this analysis is correct, any complacency which our old friend the VPG model can induce is quite out of place here, since stable VPG behavior is highly vulnerable to breakdown when the object is risk control.

[12] Since the body of the paper deals only with the implications of absolute risk aversion for public goods inferiority, we have conducted an excursion, testing the validity of these results for the case of relative risk aversion. Using the CRRA utility function for decreasing *relative* risk aversion i.e. $U(Y) = Y^{1-\alpha}/(1-\alpha)$,we show that when p=1 the public good is likely to be inferior, while when p=0 it is likely to be normal. This, analysis, therefore, indicates that just as in the canonical case developed in the text, in the case of declining relative risk aversion, when $R_0 > R_1$ there is a critical risk p* such that for p> p* the good is inferior, while for p< p* M is normal . Results are available on request.

Appendix

Properties of Interior Nash Equilibria When
The Public Good is Defined by Summation Financing

For the public good as a common improvement in probabilities our transformation of the problem to take primitive preferences over cost inputs has produced surprising new results. Even though the consumption technology for public good p is "non-summation" the feature we have called "summation finance" leads to rather striking results. Because this technique is so advantageous for managing the problematic nature of Nash equilibrium it is of all the more interest to inquire whether this success is only achieved at the cost of giving up the other desirable properties of Nash equilibrium in the standard "goods summation" of the VPG model. Reassuringly, as this appendix affirms, our unconventional treatment of M as the public good in W(C, M), is compatible with many established properties of interior Cournot Nash solutions. It applies more widely to any VPG problem where costs of public good provision are increasing rather than constant and where the public good depends only on the summation of input contributions by group members..

Neutrality of Wealth Redistributions and Economic Growth

The neutrality of wealth redistributions[13] within a group, follows from equations 8(d) and 8(e) as per footnote 2. Using this fact to analyze the effects of growth suppose both countries are identical except for income; preferences and loss in the bad state are the same: $E_1() = E_2(), L_1 = L_2$. This implies $W_1 = W_2$. Then from (8b) and (8c) it follows that in equilibrium $Y_1 + m_2 = Y_2 + m_1$=Y*, i.e. individual full income is identical for all interior Nash equilibria. If $Y_1 > Y_2$, then $m_1 > m_2$ and a higher wealth implies greater relative contribution to public good M. From Eq.(34) it is easy to see that an increase in Y (economic growth) has qualitatively the same effect as an increase in n (bigger group size) Thus, economic growth may give rise to an “inferior good barrier” to public good provision just as an expansion of group size can.

Effects of Productivity Differentials in Nash Equilibrium

[13] Thus "normality" of good M and "summation finance" are consistent with “exploitation of the great by the small” (Olson, 1965) since the richer country has a disproportionately stronger incentive to provide for security. But if M is inferior, Olson's idea fails since the richer country now has a weaker incentive to provide for security (aside from instability and corner solution problems associated with inferiority).

Now consider differences in technology of providing the pubic good between countries. In place of (5), (with ε_i denoting the relative productivity of providing M for country i) we have

$$M = \varepsilon_1 m_1 + \varepsilon_2 m_2 \tag{42 }$$

From (3) and (8a), the effective budget constraint (c_i = i's consumption) for country 1 becomes:

$$\varepsilon_1 c_1 + M = \varepsilon_1 Y_1 + \varepsilon_2 m_2 \tag{43}$$

Then, equations (8b) and (8c) may be rewritten as

$$E_1(W_1, L_1, \varepsilon_1) = \varepsilon_1 Y_1 + \varepsilon_2 m_2 \tag{44a}$$

$$E_2(W_2, L_2, \varepsilon_2) = \varepsilon_2 Y_2 + \varepsilon_1 m_1 \tag{44b}$$

And equation (8d) will be rewritten as

$$E_1(W_1, L_1, \varepsilon_1) + E_2(W_2, L_2, \varepsilon_2) = \varepsilon_1 Y_1 + \varepsilon_2 Y_2 + M_1(W_1, L_1, \varepsilon_1) \tag{45}$$

Finally, equation (8e) becomes

$$M_1(W_1, L_1, \varepsilon_1) = M_2(W_2, L_2, \varepsilon_2) \tag{46}$$

In the effective budget constraint ε_i may be regarded as the relative price of private consumption in terms of the public good for country i. That is, $\partial M_i / \partial \varepsilon_i$ denotes the substitution effect of an increase in the relative price of private consumption on the public good, and is positive. Therefore, the effect of an increase in ε_i on relative welfare is qualitatively the same as of a decrease in L_i so long as M is normal ($M_W > 0$). And with M normal, in equilibrium $\varepsilon_1 > \varepsilon_2$ entails $W_1 < W_2$; a country of higher productivity benefits less from group formation than a country of low productivity (see Jack, 1991; and Ihori, 1996). Here b<u>oth countries can gain</u> by transferring income from country 2 to 1. However, if M is inferior, the inequality $\varepsilon_1 > \varepsilon_2$ implies $W_2 < W_1$; low productivity is worse for a country, and high productivity is better since $m_1 < m_2$. However, since effective aggregate income is always given by $\varepsilon_1 Y_1 + \varepsilon_2 Y_2$, it remains true that both countries gain by transferring income from country 2 to country 1 even when M is inferior.

Normal and Inferior Income Effects

Peculiarities of common risk control as we have identified them derive from interactions between risk aversion, status quo risk, and wealth. But underlying these are certain prior factors that lead to further surprising insight into the structure of Nash equilibrium. We explore these briefly. Qualitatively , the sign of eq. (11) and therefore normality or inferiority of M can be seen to depend on three factors, p', U_{YY} which influences both W_{YY} and Δ_Y, and p, (1-p) which influences W_{YY} as well.

Effect of p' : Marginal Productivity of Security Expenditures:

Note first that the overall sign of (11) depends on the relative importance of (W_{YY} / p') ---which is negative --- and $-\Delta_Y$ ---which is positive. If the first term is negligible then with the remainder $-\Delta_Y > 0$ (11) will tend to be positive and M therefore inferior. Now with diminishing marginal returns to risk reduction, p" < 0, and p' will be greatest when p is least. *Accordingly, there is an inherent tendency for expenditures on risk improvement to be inferior when risk (1-p) is great, and for these expenditures to be normal or superior when risk is small.* In other words and most paradoxically, the more is security needed the more likely, *ceteris paribus*, is greater wealth a counter-indicator of provision.

Effect of (1-p): Baseline Risk

Should $U_{YY} = 0$ throughout, then (11) is identically zero and M is borderline normal. But if $U_{YY} \neq \gamma$ (γ = 0 or any other constant), then for any given value of p' the sign of $[(W_{YY} / p') - \Delta_Y]$ in (11) will vary systematically with p, *independently of p'*, since p is a component of W_{YY}. We know $\partial W_{YY} / \partial p = U_{YY}^1 - U_{YY}^0$ is positive if and only if $U_{YYY} > 0$. Hence, $[(W_{YY} / p') - \Delta_Y]$ is increasing with p if $U_{YYY} > 0$, which is the standard case where U_{YY} is smoothly declining and approaches the x-axis. Hence, for low p *ceteris paribus* the expression $[(W_{YY} / p') - \Delta_Y]$ will be negative with the first term dominating, while for high values of p with the second term dominating, the overall expression tends to be positive. For any given value of p' then, this analysis implies the existence of a cross-over value of p = p* where $MRS_C = 0$ and M switches from normal to inferior.

Quantitatively, all these tendencies become less important as $U_{YY} \to 0$ for then both parts of (11) vanish and risk reduction expenditures will approach borderline normal. For a wide class of utility functions, lower values for the term $|U_{YY}|$ correlate with the agent being rich or close to wealth satiation (given $U_{YYY} > 0$). So in this case (11) implies that the sensitivity to p of expenditure on M declines as wealth increases. Conversely, as $|U_{YY}|$ and the differential Δ_Y increase --- corresponding to small value of L and/or lower wealth

--- the sign of (11) and therefore the inferiority or normality of M becomes more sensitive to the factors [p, (1-p)] and p' as analyzed above. Such paradoxical effects will be buried, but always to some degree operational, in the other elements which go to make up inferiority/ normality of the public good M.

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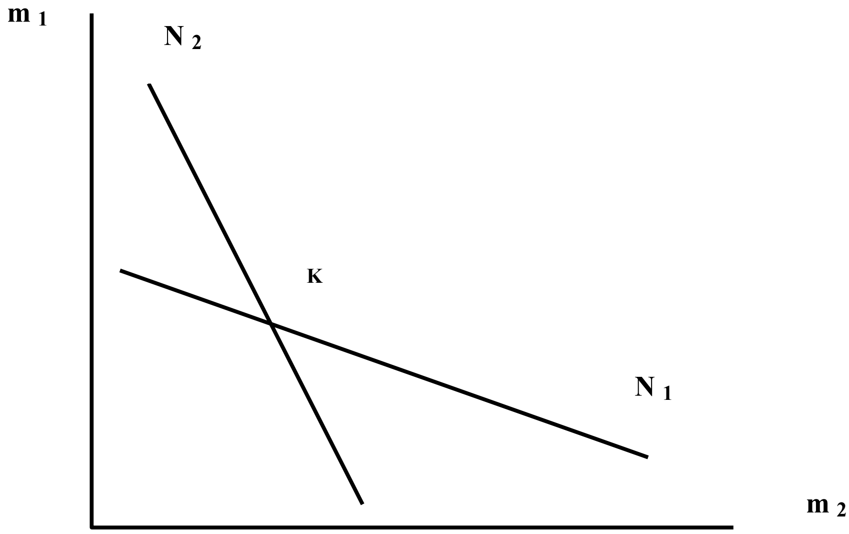

Figure 1
Nash-Cournot Reaction Functions: Stable Equilibrium

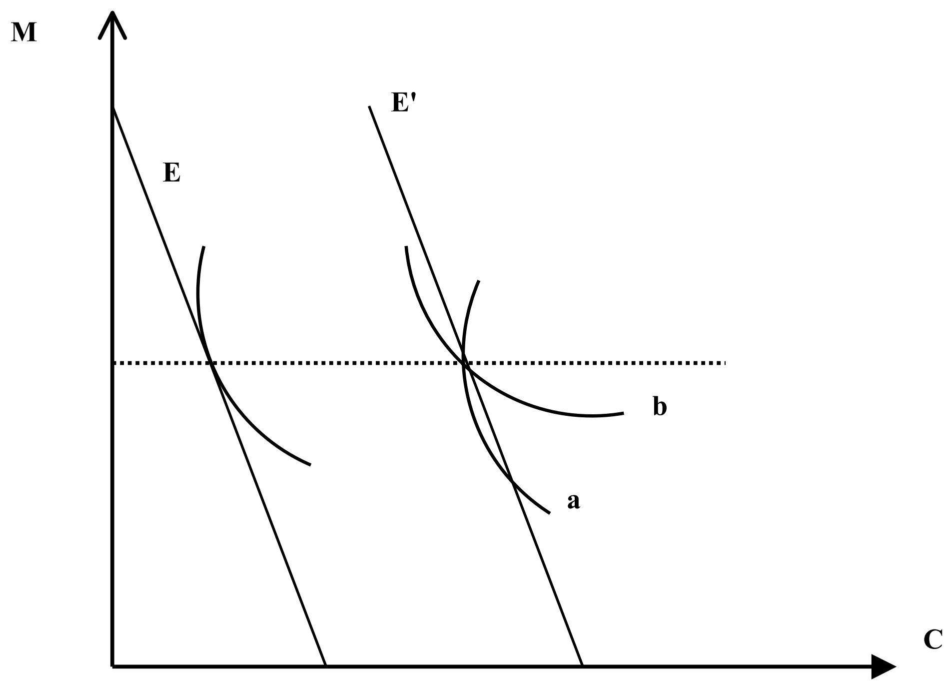

Figure 2
Effect of Increase in Income from E to E' on Expenditure on Public Good M: With Indifference Curve Like "a" if M is Inferior, Like "b" M is Normal

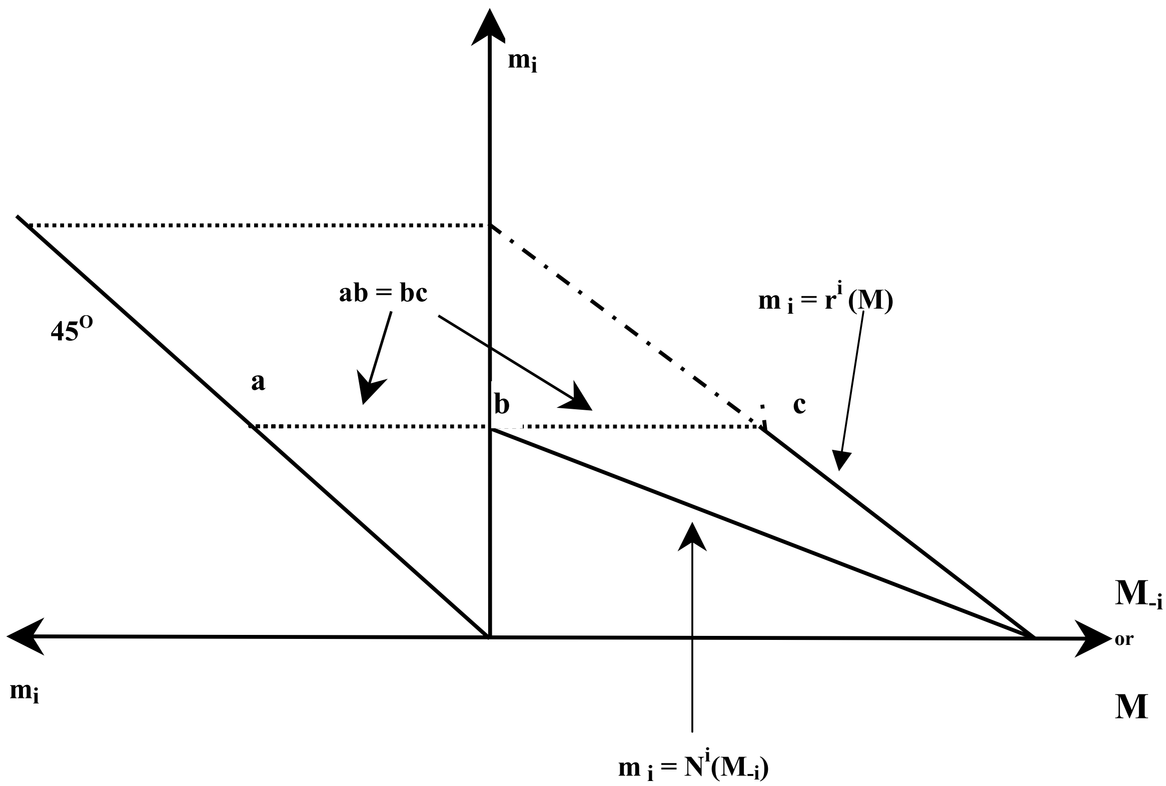

Figure 3

Construction of Individual i's Replacement Function $m_i = r^i(M)$
From its Cournot Reaction Function $m_i = N^i(M_{-i})$:

For any m_i the corresponding M is the horizontal distance
Between the 45^O and $N^i(M_{-i})$ also plotted as $r^i(M)$

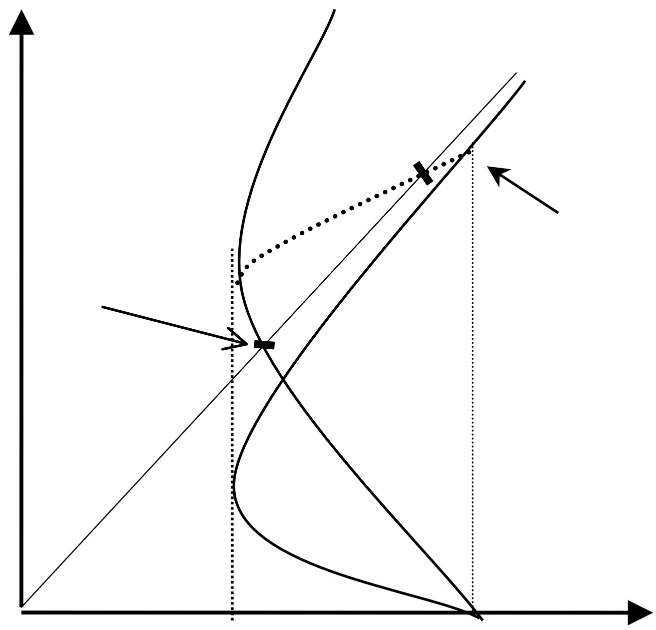

Figure 4a
<u>Multiple Equilibria When Public Good Is Normal/Inferior</u>:
At Point S Where Aggregate Replacement Function A'C'B'
(AC+AC = A'C') Intersects 45^O M is Normal for Both Agents

At Point S* Where Aggregate Replacement Function A'C'B"
(AC+CB = C'B") Intersects 45^O
M is Normal for One Agent but Inferior for the Other

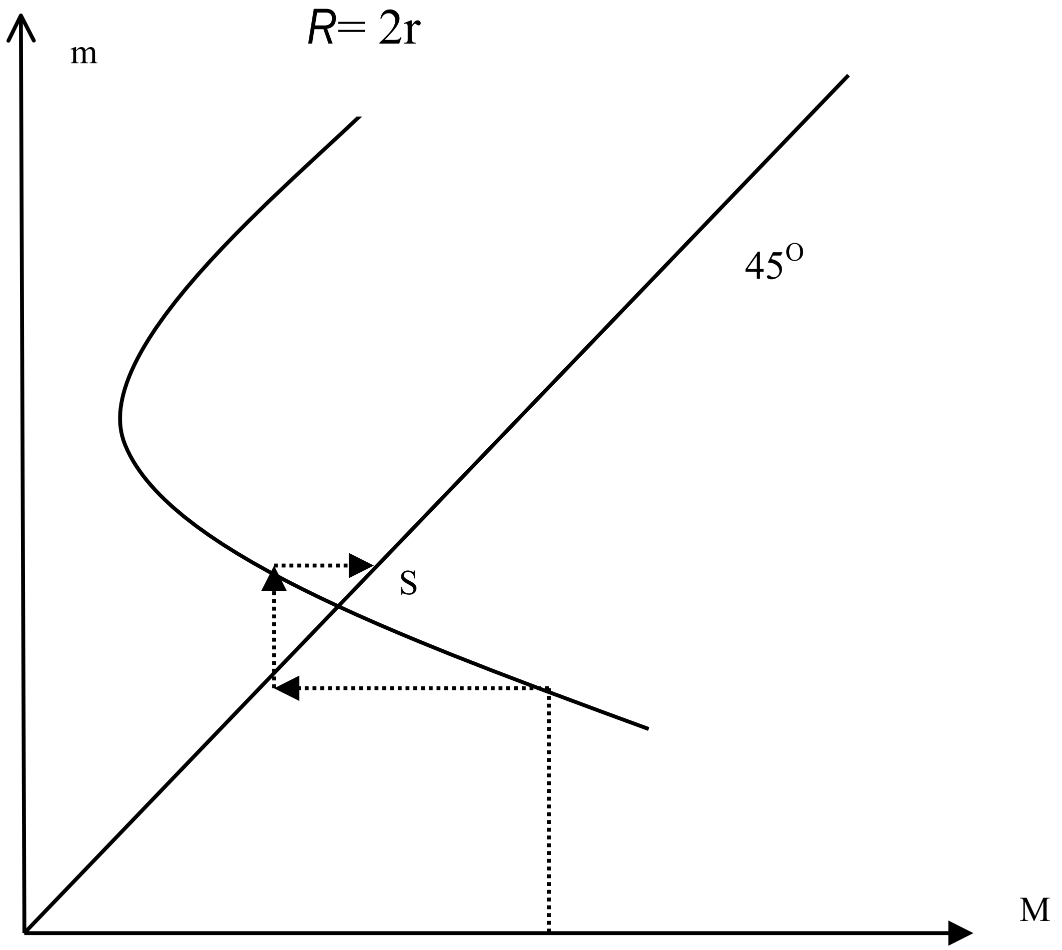

Figure 4b
Intersection of Homogeneous and Symmetric Aggregate Replacement Function *R* and 45^{O} Gives a Stable Nash Equilibrium at point S

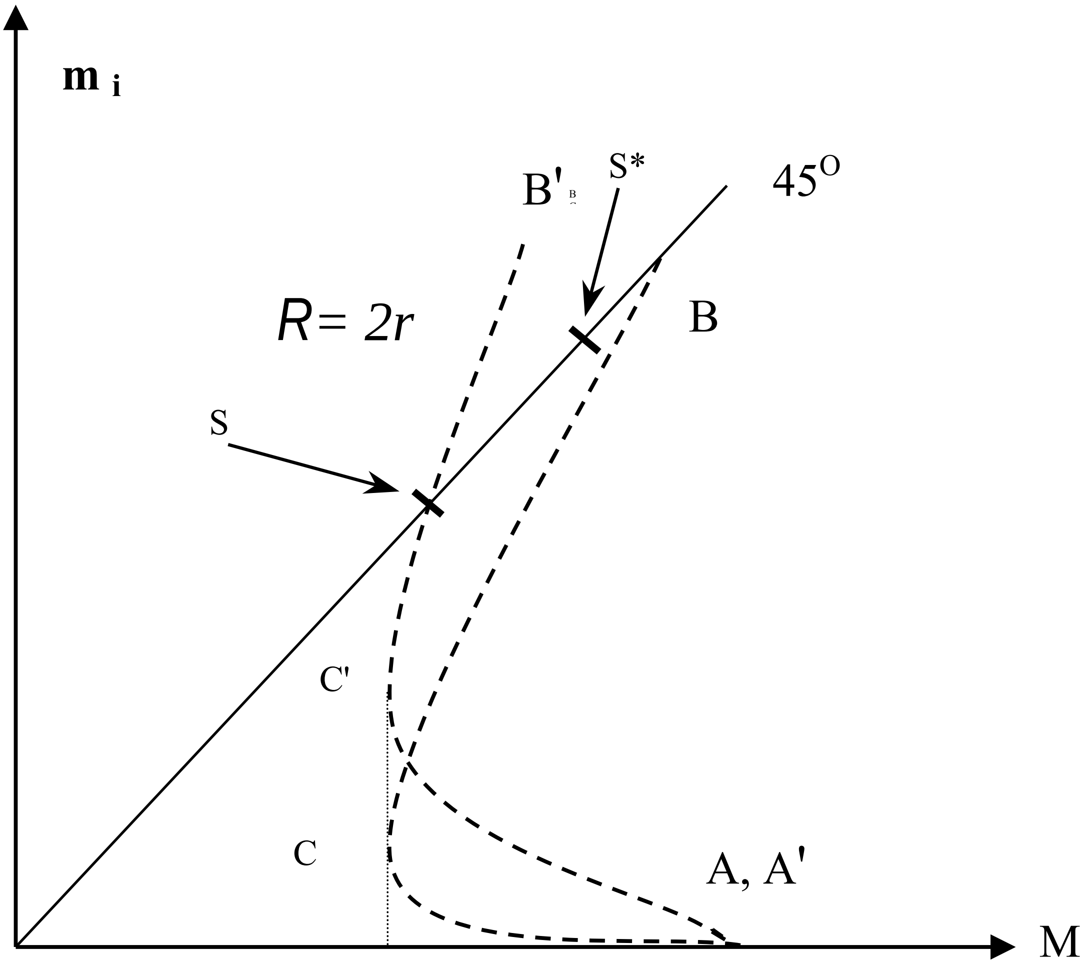

Figure 5a
Symmetric (But Unstable) Equilibrium at Point S Where Public Good Is Inferior:
At Intersection With 45^{O} Slope of Aggregate Replacement Function A'C'B' Greater than 1.

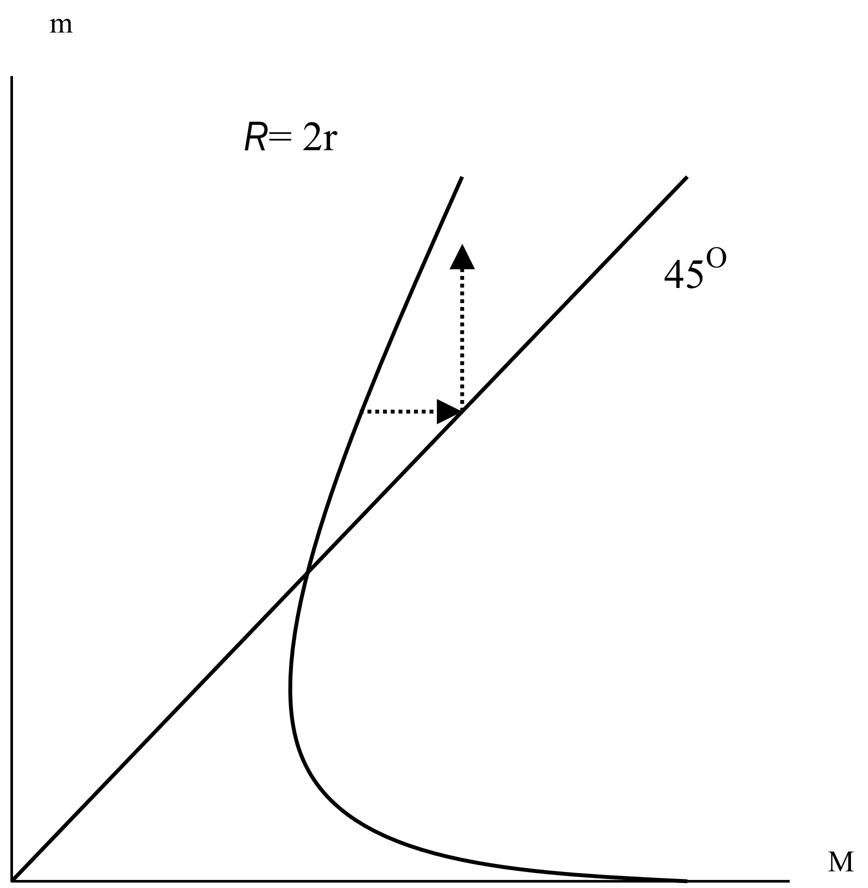

Figure 5b
Intersection of Symmetric and Homogeneous Aggregate Replacement Function *R* from below 45^{O} Gives Unstable Nash Equilibrium

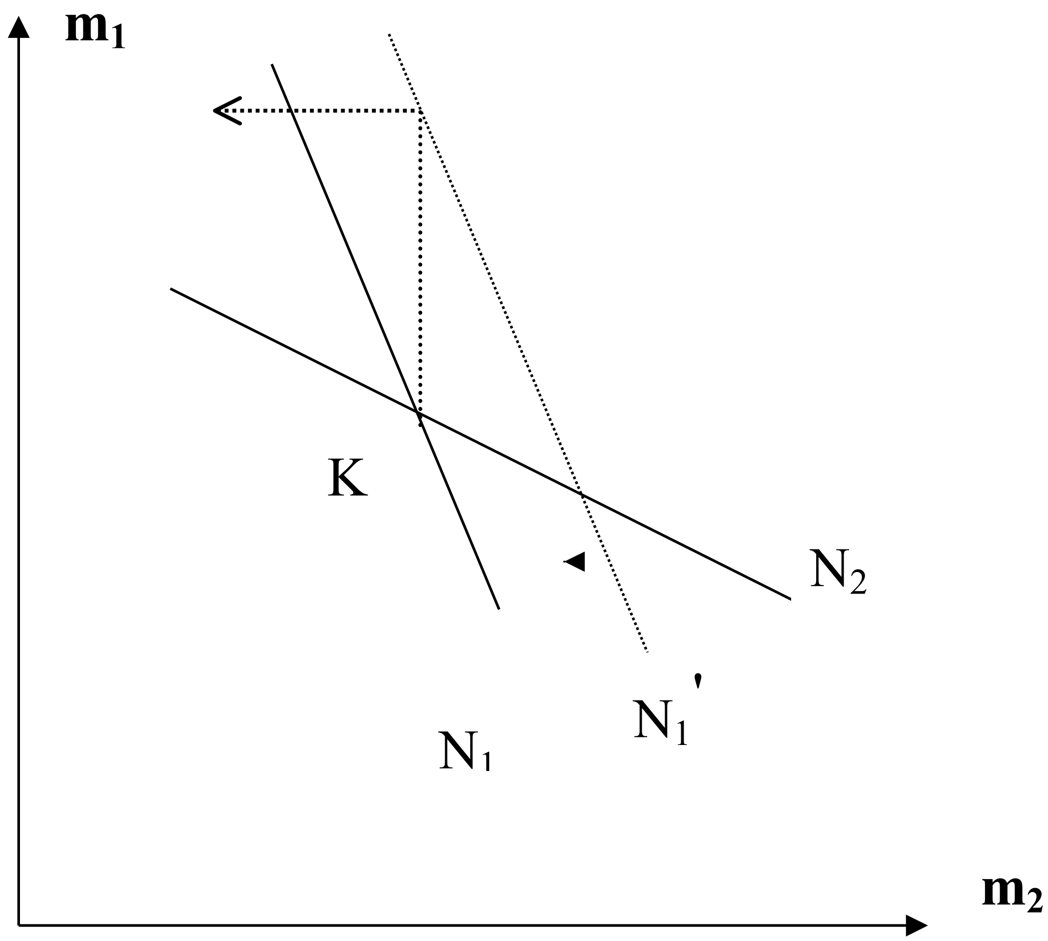

Figure 6
Effects of an Increase in the Loss Under Adversity on Expenditures to Reduce Risk When Risk Improvement is an Inferior Good:
An Increase in loss L_1 Shifts 1's Reaction Function from N_1 to N_1'

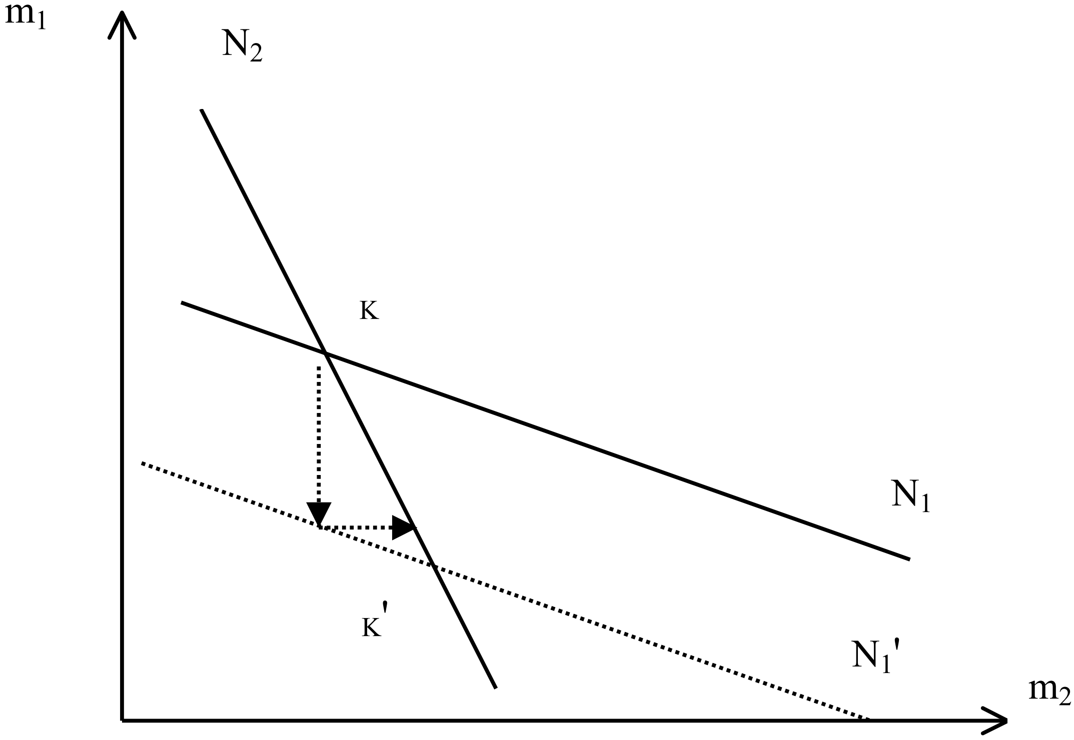

Figure 7
Effects of a Decline in Wealth on Expenditures to Reduce Risk When Risk Improvement is Normal

Table I

$W_Y = pU_Y^1 + (1-p)U_Y^0 > 0$	$W_{YY} = pU_{YY}^1 + (1-p)U_{YY}^0 < 0$
$\Delta = (U^1 - U^0) > 0$	$\Delta_Y = (U_Y^1 - U_Y^0) < 0$

Table II

$pU_Y^1 + (1-p)U_Y^0 = W_Y$ = $\mathbf{MC_M}$ = **Marginal Cost**	$p'(U^1 - U^0) = p'\Delta$ = $\mathbf{MB_M}$ = **Marginal Benefit**

Table III

$p''(U^1 - U^0) = p''\Delta = MB_{MM} < 0$	$[pU_{YY}^1 + (1-p)U_{YY}^0] = W_{YY} = MC_{MY} < 0$	$p'(U_Y^1 - U_Y^0) = p'\Delta_Y = MB_{MY} < 0$

Table IVa: Relation between Risk, Risk Aversion, and Inferiority/Normality of M*

In Eq (13)	Additional M raises p(M)	Effect on Optimal Choice of M when Income, Y, Increases	
		Case of Increasing R: $R_1 > R_0$	Case of Decreasing R: $\rightarrow R_1 < R_0$
Low Risk p is high: (1-p) is low R_1 dominates	Insures against unlikely event (1-p)	A Agent wants to insure and can do so. R_1 is big. Eq (13) >0: normal. Therefore, greater Y makes R_1 and R_0 bigger and leads to more insurance. **M increases**	C Agent wants to gamble but must insure R_1 is small. Eq (13)< 0: inferior. Therefore, greater Y makes R_1 and R_0 smaller and leads to less insurance **M declines**
High Risk p is low: (1-p) is high R_0 dominates	Gambles on unlikely event p	B Agent wants to gamble and can do so. R_0 is small. Eq (13) <0: inferior .Therefore, greater Y makes R_1 and R_0 bigger and leads to less gambling. **M declines**	D Agent wants to insure but must gamble. R_0 is big. Eq (13) >0: normal. Therefore, greater Y makes R_1 and R_0 smaller and leads to more gambling. **M increases**

* This assumes that variables in eq. (13) are such that H will change sign as p varies from 0 to 1. And just to repeat, low risk aversion correlates with a propensity to gamble but not insure; high risk version correlates with a propensity to insure but not gamble.

Table IVb

Effect of Increasing Full Income of Members of a Group by Adding New Participants

Increasing Risk Aversion: $R_0 < R_1$

Initial Level of Self-Protection	Income Effect on M of Increasing Full Income	
p_{LOW}	Inferior	B
p_{HIGH}	Normal	A

Decreasing Risk Aversion: $R_0 > R_1$

Initial Level of Self-Protection	Income Effect on M of Increasing Full Income	
p_{LOW}	Normal	D
p_{HIGH}	Inferior	C