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Patterns of Non-exponential Growth of Macroeconomic Models: Two-parameter Poisson-Dirichlet Models

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Abstract

This paper discusses non-exponential growth patterns of macroeconomic models. More specifically, the paper discusses asymptotic growth patterns of the numbers of clusters and of components of partition vectors, that is, the number of clusters of specific sizes, of one- and two-parameter Poisson-Dirichlet models as the model sizes grow towards infinity.

As the model sizes become large, the coefficients of variaation of the cluster sizes and components of the partition vector tend to zero in one-parameter Poisson-Dirichlet model, but they remain positive in the two-parameter version. Furthermore, the two-parameter version of the model exhibits power-law behavior, while the one-parameter version does not.

The growth behavior of the two-parameter models is shown to be expressed in terms of generalized Mittag-Leffler distributions.

The paper ends with preliminary discussion of the effects of demand pattern management policies on growth patterns of models that endogenize the parameters of the two-parameter Poisson-Dirichlet model.

Key Words: Non-exponential growth patterns, One- and Two-parameter Poisson-Dirichlet distributions; Mittag-Leffler distributions; Non-self averaging phenomena, Power laws.

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Introduction

In a recent paper entitled "What should we mean by growth policy?", Solow has expressed his concerns about a common practice by the mainstream growth economists of focusing almost exclusively on exponential growths, and wondered about their adverse influences on growth policy, Solow (2004).

In short, there are two aspects to his comments. First, he complains that growth economists have centered their attention on steady-state exponential growth, and that they made special assumptions for convenience to gauarantee the existence of exponential steady states. Second, he mentions that the set of their assumptions has become standard as if they have some independent validation for policy makers to speak of their intention of raising the growth rate. According to him, the very vocabulary of growth policy becomes identified with moving the growth rate. He condems this pattern as being unnecessary and dysfunctional both for theory and for policy.

Aoki and Yoshikawa (2002) has mentioned one growth model that escapes his criticism, since it does not have a constant exponential growth rate as time goes to infinity. This point was not developped fully in that paper, however, because this point was not the center-piece of our analysis.

Here we describe two types of models called one- and two-parameter Poisson-Dirichlet models which have non-exponential growth patterns. We show that although both of our models grow at non-exponential rates, their growth patterns are qualitatively different. We make this point using the notion of the coefficient of variation. This notion is commonly used in econometrics but less so in growth literature. We find that this concept is useful in discussing how policies may affect growth patterns, as some preliminary esxamination of examples in Aoki (2002, sec.7.4, sec. 8.6) and Aoki and Yoshikawa (2006, ch.6, ch.7) seem to indicate.

It is shown that the one parameter Poisson-Dirichlet model is well behaved in the sense that its coefficient of variation tend to zero as model size grows unboundedly. The coefficient of variation of the two-parameter model, on the other hand, does not go to zero in the limit. This implies that growth patterns of the two-parameter Poisson-Dirichlet models are more unpredictable, and history-dependent. Effects in changes in growth path, intentional or accidental will affect future growth patterns. We return to this point later in the paper.

In a later section the paper points out the connection of Poisson-Dirichlet models with Mittag-Leffler distributions. Mittag-Leffler functions are generic in problems of occupation times and first-passage problems of Markov processes, see for example Darling and Kac theorem in Bingham, Goldie and Teugels (1987, Ch. 8).

There are models of entries and exits by heterogeneous agents with non Markovian character for which analysis similar to those of waiting time distributions in financial variables are applicable, with fractional master equations. These extend the elementary model sketched in Aoki (1996). The model responses are slower than exponential. Power-laws govern the two-parameter Poisson-Dirichlet models. See Mainardi, Raberto, Gorenflo and Sclals (2000), and Mainardi, Gorenflo and Scalas (2004) for example.

These results indicate how the so-called fractional master equation also arise in some macroeconomic models of cluster formations, entries and exits of heterogeneous agents.

The paper is organized as follows: The coefficient of variation is defined first. Then Poisson-Dirichlet models are introduced. We then describe how point processes of innovations initiate or grow clusters of productive units (agents, firms, or sectors of an economy). The asymptotic behavior of the one- and two-parameter models, called Poisson-Dirichlet models, are described. Then, the coefficients of variation of the number of clusters, suitably normalized, are derived. Two-parameter Poisson-Dirichlet models have power law behavior. This is mentioned in connection with the Mittag-Leffler probability density. Since this density is mostly unknown to economists in general, a short summary of Mittag-Leffler probability density is in appendix. It is a generalizetion of the well-known exponential density, and appear in sluggish reponse patterns.

Some preliminary discussions are then given on endogenizing the models of this paper in a way different from the mainstream endogenous growth model literature in the penultimate section of this paper, with a short sketch on the interactions of growth patterns and macroeconomic policies.

Coefficient of Variation: A Measure of Uncertainty

The coefficient of variation of a random variable, X, denoted by c.v.(X) is defined by

$$c.v.(X) = \frac{\sqrt{variance(X)}}{mean(X)}.$$

In this paper we use the number of clusters as X, which is an extensive random variable[1]

As a simple example of the coefficient of variation, consider a pure-birth model with a constant birth-rate λ and with an initial size n_0.

Its first two cumulants are governed by

$$d\kappa_1(t)/dt = \lambda\kappa_1(t),$$

and

$$d\kappa_2(t)/dt = 2\lambda\kappa_2(t) + \lambda_1(t).$$

The solution is

$$\kappa_1(t) = n_0 e^{\lambda t}$$

and

$$\kappa_2(t) = n_0[e^{2\lambda t} - e^{\lambda t}].$$

The coefficient of variation for the size $n(t)$ of this model is given by

$$c.v.(n(t)) = \frac{(1 - e^{-\lambda t})^{1/2}}{\sqrt{(n_0)}} \sim 1/\sqrt{(n_0)}.$$

See Cox and Miller,(1965, 159) for example.

[1] A variable is extensive if it scales with the "size" of the model.

The larger the initial size, the smaller the coefficient of variation of this growth model. By shifting the origin of time to an instant with a large size for the initial cluster n_0, we see that its coefficient of variation can be made as small as you like.

In the main part of this paper we use the terms self- and non-self averaging. The term "non-self-averaging" is used as in the physics literature, see Sornette (2000) for example. It means that some extensive random variable X of the model, such as the number of sectors, has the coefficient of variation that does not converge to zero as model size goes to infinity.[2]

Why is this important? It is important because non-self-averaging growth models are sample dependent, and some degree of impreciseness or uncertainty remain about the growth path trajectories, even when the sample sizes go to infinity. We develop this point for the one-parameter, and two-parameter Poisson-Dirichlet models introduced in the next section. In one-parameter model, the coefficient of variations of the number of sector sizes, suitably normalized is self-averaging, while the same variable in a two-parameter model is not self-averaging.

Poisson-Dirichlet Models

Agents or factors of production of different characteristics or strategies belong to different types and form separate clusters (firsm, sectors). These clusters jointly affect aggregate behavior.

Kingman invented the one-parameter Poisson-Dirichlet distribution to describe random partitions of populations of heterogeneous agents into distinct clusters. Models of this class are also known as Ewens models, Ewens (1972). See Aoki (2000a, 2000b) for further explanation.

The one-parameter model was extended to two-parameter Poisson-Dirichlet distributions by Pitman. See Kingman (1993), Carlton (1999), Feng and Hoppe(1998), Pitman (1999, 2002), and Pitman and Yor (1996), among others.[3]

If the coefficient of variation of an extensive random variable X does not approach zero but goes to some positive number or tends towards infinity as the size of "clusters" or model becomes very large, then X remains sample-dependent even when the sample size approaches infinity.[4]

This paper shows that variables in the two-parameter Poisson-Dirichlet

[2] This limit is called thermodynamic limit in the physics terminology. The thermodynamic behavior of these two classes of models have been examined in Aoki (2006), and is shown to be qualitatively different betwee these two-classes of models.

[3] In physics literature, Mekjian and Chase (1997) have used two-parameter models. They refer to Pitman (1996). There are other related works in the physics literature, see the papers by Derrida-Flyvbjerg (1987), and Derrida (1994a, 1994b). Higgs (1995) have noted the similarities of some physical distributions and power laws, and mention population genetics papers by Ewens in particular. There are many papers on stick-brekding version of the residual allocation processes, such as Krapivvsky, Grosse, and B. Nadin (2002). They have not touch on connections with the two-parameter Poisson-Dirichlet distributions, however.

[4] The square of the coefficient of variation is called the measure of non-self averaging in the physics literature, Sornette (2000).

model, denoted by $PD(\alpha, \theta)$, with positive α less than 1, and $\theta + \alpha > 0$ have non-vanishing coefficients of variations as the number of samples approaches infinity, while the corresponding variables in the one-parameter Poisson-Dirichlet distribution with $\alpha = 0$, denoted by $PD(\theta)$, also known as Ewens model, does not.

These models are not exponential growth models familiar to economists but they belong to a broader class of models without steady state constant exponential growth rate. Innovations occur to existing clusters (firms, sectors, or goods) as well as innovations create new clusters (sectors or goods). By making arrival rates of innovation endogenous, endogenous growth models result. An elementary example is in Aoki (2002, Sec.8.6) See the discussion section on further comments on this point.

None of the previous works, however, have comparatively examined the asymptotic behavior of the coefficient of variation of these two classes of models.

Clusters in one- and two-parameter Poisson-Dirichlet Distributions

The model consists of several clusters of basic units of production or productive factor.[5] The clusters could be sectors of macroeconomy or firms of a sector, as the case may be. Suppose that there are k clusters of sizes n_i, $i = 1, 2, \ldots, k$. The size of the model is $n = n_1 + n_2 + \cdots + n_k$. A new basic unit (agent) joins one of the existing clusters of size n_i with probability rate

$$\frac{n_i - \alpha}{n + \theta}, \tag{1}$$

where $\theta + \alpha > 0$, and α is between 0 and 1. With $\alpha = 0$ there is a single parameter θ, else we have a two-parameter model. In most part of the paper these parameters are exogenously fixed. In the last section we discuss some examples where these parameters are endogenized.

A unit of new type starts a new cluster of its own with probability rate[6]

$$1 - \sum_{1}^{k} \frac{n_i - \alpha}{n + \theta} = \frac{\theta + k\alpha}{n + \theta}. \tag{2}$$

The above generalize the recurrence relation for the one-parameter $PD(\theta)$. In the one-parameter case, $\theta/(\theta + n)$ is a probability rate that the (n+1)th agent that enter the model is a new type, hence it creates a new cluster, and $n/(\theta + n)$ is the probability that the next agent is one of the types already in the model. In the two-parameter Poisson-Dirichlet distribution the conditional probabilities for the number of clusters in a sample of size n, K_n is given by

$$\Pr(K_{n+1} = k + 1 | K_1, \ldots, K_n = k) = \frac{k\alpha + \theta}{n + \theta}, \tag{3}$$

[5] The model is a re-interpretation of Feng-Hoppe (1998), and the chinese restaurant model in Pitman (2002).

[6] Probabilities of new types entering Ewens model, and the number of clusters have been discussed in Aoki (2002, Sec.10.8, App. A.5), for example.

and

$$\Pr(K_{n+1} = k | K_1, \ldots, K_n = k) = \frac{n - k\alpha}{n + \theta}, \tag{4}$$

where the random variable K_n is the number of clusters, i.e., subsets of agents of different types present in a sample of size n.

Eq.(3) gives the expression for the probability that the $(n+1)$th entrant is a new type that starts a new cluster with initial size one. Eq.(4) expressess the probability that it is one of the previously existing types. Hence the number of clusters does not change.

Let the probability for $K_n = k$ be denoted by $q_{\alpha\theta}(n, k)$. Using (1) through (4) it can be recursively computed by

$$q_{\alpha\theta}(n+1, k) = \frac{(n - k\alpha)}{(n + \theta)} q_{\alpha\theta}(n, k) + \frac{\theta + (k-1)\alpha}{n + \theta} q_{\alpha\theta}(n, k-1), \tag{5}$$

for $1 \leq k \leq n$. The expressions for the boundary $K_n = 1$ for all n, and that of $K_n = n$ are given by the expression

$$q_{\alpha\theta}(n, 1) = \frac{(1-\alpha)(2-\alpha)\cdots(n-1-\alpha)}{(\theta+1)(\theta+2)\cdots(\theta+n-1)},$$

and

$$q_{\alpha\theta}(n, n) = \frac{(\theta+\alpha)(\theta+2\alpha)\cdots(\theta+(n-1)\alpha)}{(\theta+1)(\theta+2))\cdots(\theta+n-1)}.$$

These expressions reduces to the Ewens model expressions when α is set to zero. See Aoki (2002). In the two-parameter $PD(\alpha, \theta)$ case, the rate of forming a new cluster is slightly increased from $\theta/(n+\theta)$ to $(\theta+k\alpha)/(n+\theta)$, where k is the number of existing clusters (firms). This seemingly slight endogenous increase in the rate of new cluster formation turns out to be responsible for a qualitatively difference in the long-run model behavior.

In the one-parameter case, $q_\theta(n, k) := P(K_n = k)$ is governed by the recurrence relation

$$q_\theta(n+1, k) = \frac{n}{n+\theta} q_\theta(n, k) + \frac{\theta}{\theta+n} q_\theta(n, k-1).$$

The solution of this recurrence equation is

$$q_{n,k} = \frac{c(n,k)\theta^k}{\theta^{[n]}},$$

where $\theta^{[n]} := \theta(\theta+1)\cdots(\theta+n-1) = \frac{\Gamma(\theta+n)}{\Gamma(\theta)}$, and where $c(n, k)$ is named as the unsigned (signless) Stirling number of the first kind. It satisfies the recursion

$$c(n+1, k) = nc(n, k) + c(n, k-1).$$

Because $q_\theta(n, k)$ sums to one with respect to k we have

$$\theta^{[n]} = \sum_{k=1}^{n} c(n, k)\theta^k. \tag{6}$$

See Aoki (2002,p.208), for example, on the Stirling numbers, and their combinatorial interpretations.

In the two-parameter version, the number of clusters is given by

$$P_{\alpha,\theta}(K_n = k) = \frac{\theta^{[k,\alpha]}}{\alpha^k \theta^{[n]}} c(n,k;\alpha), \tag{7}$$

where

$$\theta^{[k,\alpha]} := \theta(\theta+\alpha)(\theta+2\alpha)\cdots(\theta+(k-1)\alpha),$$

and the expression $c(n,k;\alpha)$ generalizes the signless Stirling number of the first kind of one-parameter situation. It is called generalized Stirling number of the first kind. See Charalambides (2002).

Let $S_\alpha(n,k) := \frac{1}{\alpha^k} c(n,k;\alpha)$. It satisfies the recursion

$$S_\alpha(n+1,k) = (n-k\alpha)S_\alpha(n,k) + S_\alpha(n,k-1),$$

where we set $S_\alpha(0,0) = 1, S_\alpha(n,0) = 0$, and $S_\alpha(0,k) = 0, k > 0$ to make the recursion valid for all cases.

Some special cases of interest are $S_\alpha(n,n) = 1; S_\alpha(n,1) = (1-\alpha)^{[n-1]}$, and

$$S_\alpha(n,2) = \frac{1}{2\alpha^2}\{(-2\alpha)^{[n]} - 2(-\alpha)^{[n]}\}.$$

Instead of (6) we have

$$\theta^{[n]} = \sum_{k=1}^{n} S_\alpha(n,k)\theta^{[k,\alpha]}. \tag{8}$$

The Ewens distribution is replaced by

$$S_\alpha(n,k) = n! \sum^{*} \frac{1}{a_j!}\{\frac{(1-\alpha)^{[j-1]}}{j!}\}^{a_j},$$

where a_j is the jth component of the partition vector, see Aoki (2002). It is the number of clusters of size j, and the summation $\sum^*$ is over a_j such that $\sum_j j a_j = n$ and $\sum a_j = K_n$. Pitman (1999) obtained its asymptotic expression as

$$S_\alpha(n,k) \sim \frac{\Gamma(n)}{\Gamma(k)} n^{-\alpha} \alpha^{1-k} g_\alpha(x),$$

where $k \sim x n^\alpha$. Here, g_α is the Mittag-Leffler (α)function. This function is discussed in the next section, and in Appendix.

Asymptotic Behavior

The normalized number of clusters K_n/n^α

In $PD(\theta)$ it is known that

$$\frac{K_n - \theta log(n)}{\sqrt{\theta log(n)}} \to N(0,1).$$

that is,

$$E(K_n) = \theta log(n),$$

and

$$var(K_n) = \theta log(n).$$

Hence, $c.v.(K_n) = (\theta log(n))^{-1/2}$, and it goes to zero as n approaches infinity. This model is therefore self-averaging.

Coefficient of variation of $PD(\alpha, \theta)$

Yamato and Sibuya (2000) obtained the expression

$$E(\frac{K_n}{n^\alpha}) = \frac{\Gamma(\theta+1)}{\alpha\Gamma(\alpha+\theta)}. \tag{9}$$

More generally, it is known that $K_n/n^\alpha \to^d \mathcal{L}$, and its first two moments are

$$\mu_1' = E_{\alpha,\theta}(\mathcal{L}) = \Gamma(\theta+1)/\alpha\Gamma(\theta+\alpha),$$

and

$$\mu_2' = E_{\alpha,\theta}(\mathcal{L}^2) = \Gamma(\theta+1)(\theta+\alpha)/\alpha^2\Gamma(\theta+2\alpha).$$

Hence variance of $\mathcal{L}$ is given as $\mu_2' - (\mu_1')^2 = [\Gamma(\theta+1)/\alpha^2]\gamma_{\alpha\theta}$. Hence we have

$$c.v.(K_n/n^\alpha) = \sqrt{\gamma_{\alpha,\theta}}\Gamma(\theta+\alpha)/\sqrt{\Gamma(\theta+1)}.$$

This expression can be simplified to

$$c.v.(K_n/n^\alpha) = \frac{\alpha}{2}[\theta^{-1} + \psi(\theta)], \tag{10}$$

where $\psi(\theta)$ is the digamma function, that is the derivative of logarithm of $\Gamma(\theta)$ with respect to θ.

Recall that

$$\frac{K_n - \theta ln(n)}{\sqrt{\theta ln(n)}} \to N(0,1),$$

in the Ewens model. Hence $(K_n/ln(n))$ is self-averaging in one-parameter model.

For the two-parameter model, Yamato and Sibuya (2000) calculated the expected value of the number of clusters K_n to be given by

$$EK_n = \frac{\theta}{\alpha}[\frac{(\theta+\alpha)^{[n]}}{\theta^{[n]}} - 1],$$

where we note that

$$\frac{(\theta+\alpha)^{[n]}}{\theta^{[n]}} = \frac{\Gamma(\theta)}{\Gamma(\theta+\alpha)}\frac{\Gamma(\theta+\alpha+n)}{\Gamma(\theta+n)}.$$

Mittag-Leffler distributions and the method of moments

In general the fact that all moments of two distributions defined on infinite domain $[0, \infty)$ match does not imply that the distributions are the same. There is, however, a sufficient condition on the moments that the distribution functions are uniquely determined by the equalities of all the moments. This condition is satified for the problem at hand.[7]

Applying the asymptotic expression for the Gamma function for large n

$$\frac{\Gamma(n+a)}{\Gamma(n)} \sim n^a,$$

[7] See Bingham et al. for example.

to the above expression, we have an asymptotic expression,

$$E(\frac{K_n}{n^\alpha}) \sim \frac{\Gamma(\theta+1)}{\alpha\Gamma(\theta+\alpha)}. \tag{11}$$

All other moments such as the asymptotic value of the variance of K_n/n^α . In particular,

$$var(K_n/n^\alpha) \sim \frac{\Gamma(\theta+1)}{\alpha^2}\gamma_{\alpha,\theta} \geq 0, \tag{12}$$

where

$$\gamma_{\alpha,\theta} := \frac{\theta+\alpha}{\Gamma(\theta+2\alpha)} - \frac{\Gamma(\theta+1)}{[\Gamma(\theta+\alpha)]^2}. \tag{13}$$

We thus deduce the thermodynamic limit is

$$c.v.(\frac{K_n}{n^\alpha}) \to \Gamma(\theta+\alpha)\sqrt{\frac{\gamma(\alpha,\theta)}{\Gamma(\theta+1)}}. \tag{14}$$

It is given asymptotically by $\sqrt{\frac{\gamma_{\alpha\theta}}{\Gamma(\theta+1)}}\Gamma(\theta+\alpha)$. This ratio is positive for $\alpha \neq 0$. This is one of the important difference in the asymptotic behaviors of one- and two-parameter Poisson-Dirichlet models.

See also Blumenfeld and Mandelbrot (1997) who credit Feller (1949) as the original source.

The right-hand side of the above equation is approximately equal to $\sqrt{\Gamma(\theta+1)(1+\alpha)/\theta}$.

We calculate the asymptotic behavior of the coefficient of variation next. Expanding the gamma function $\Gamma(\theta+\alpha) \approx \Gamma(\theta)[1+\psi(\theta)\alpha+o(\alpha)]$,where $\psi(\theta)$ is the digamma or psi function given as the derivative of log of $Gamma(\theta)$ with respect to θ, we obtain

$$c.v.(K_n/n^\alpha) \approx \frac{1}{\sqrt{\theta}}[\alpha+o(\alpha)].$$

The partition vector a

For simpler presentation we have just discussed the random variable K_n, even though the components of the partition vector, i.e., the number of clusters of size j, denoted by a_j, and the total size of clusters of size j, ja_j can be analogously treated.

Components of partition vector **a** has expected value

$$E(a_j) = \frac{n!}{j!(n-j)!}\frac{(\theta+\alpha)^{[n-j]}}{(1-\alpha)^{[j-1]}(\theta+1)^{[n-1]}}.$$

We can show that

$$\frac{a_j(n)}{K_n} \to^d P_{\alpha,j},$$

a.s., where

$$P_{\alpha,j} = \frac{\Gamma(j-\alpha)}{\Gamma(1-\alpha)}.$$

Yamato and Sibuya noted that

$$limE(\frac{K_n}{n^\alpha})^r = \mu_r',$$

for $r = 1, 2, \ldots$, where μ_r' is the $r-th$ moment of the generalized Mittag-Leffler distribution with density

$$g_{\alpha,\theta} := \frac{\Gamma(\theta+1)}{\Gamma(\theta/\alpha+1)} x^{\frac{\theta}{\alpha}} g_\alpha(x),$$

where $\theta/\alpha > -1$, and where $g_\alpha(x)$ is the Mittag-Leffler (α) density function. Its moments are given by

$$\int_0^\infty x^p g_\alpha(x) dx = \frac{\Gamma(p+1)}{\Gamma(p\alpha+1)},$$

for all $p > -1$.

See Appendix for the expression of $g_\alpha(\cdot)$.

Local Limit Theorem

Suppose N independent positive random variables X_i, $i = 1, 2, \ldots N$ are normalized by their sum $S_N = X_1 + \cdots + X_N$

$$x_i = X_i/S_N, i = 1, \ldots N,$$

so that

$$Y_1 := \sum_i x_i = 1.$$

Suppose that the probability density of X_i is such that it has a power-law tail,

$$\rho(x) \sim Ax^{-1-\mu},$$

with $0 < \mu < 1$. Then, $S_N/N^{1/\mu}$ has a stable distribution (called Lévy distribution).

Pitman's formula for the probability of $K_n = k$, with $k \sim sn^\alpha$ indicates that the power law n^α which is $2\alpha < 2$ or $2\alpha = 1 + \mu$ with $0 < \mu < 1$, the case in Derrida.

With the 2-parameter PD distribution satisfying the power law condition, Derrida's conclusion that the Hs are non-self averaging applies to this case as well.

Aggregate Demand and Policy and Growth Policy

Here we examine some implications of non-self averaging growth patterns and growth policies. In Aoki (2002, Ch.8) and further in Aoki and Yoshikawa (2006, Ch.6) multi-sector models with non-identical productivity coefficients have been examined for growth patterns and fluctuations that mimic business cycles with slightly differently specified arrival rates of innovations to initiate a new sector and to increase the sizes of existing sectors from the ones adopted in this paper. Sectors are made to have different productivity coefficients. There demand patterns to sectors were varied to see how the fluctuations and growth rates of the aggregate models are affected. Not enough numbers of simulations were run to state the results of this examination with sufficient conficence. However, we observed definite changes in coefficients of variation.

In these models each sector attempts to increase its sizes if it experience excess demand for its goods, and it reduces its output by laying-off a unit of production factors (employees) when excess supply is experienced. In analysis of these models, we have demonstrated that "aggregate demand management of sorts" which differentially allocate demands to more and less productive sectors to affect total outputs and patterns of output fluctuations.

The number of sectors have been fixed at $K = 10$ for all n.

Potential Applications: Waiting time distributions

It is known that Mittag-Leffler functions generically appear in situations where Darling-Kac theorem applies. See Bingham et al (1999). For example waiting time distribution problems in the econo-physics literature are such examples. Waiting time situations arise also in macroeconomics. For example, the entry and exit problem discussed by Dixit (1989) in exchange rate pass-through can be phrased more correctly as waiting time problem.

In view of these results, we conjecture that the model of this paper can be used with minor changes to analyze effects of various growth policies to determine how they affect growth patterns, and characterize their effects in terms of the coefficients of variation, for example. We have shown that the coefficient of variation of normalized cluster sizes increase with α and change with θ as shown in (14).

With a more general specification of the arrival rates of innovations α and θ such as those in Aoki (2002, Sec.8.6) are treated exogenously in this paper could be made to be affected some policy instruments.

Concluding Remarks

In physics phenomena with non-vanishing coefficients of variation abound. In traditional microeconomic foundations of economics, one deals almost exclusively with well-posed optimization problems for the representative agents with well defined peaks and valleys of the cost functions. It is also taken for granted that as the number of agents goes to infinity, any unpleasant fluctuations vanish and well defined deterministic macroeconomic relations prevail. In other words, non-self-averaging phenomena are not in the mental pictures of average macro- or microeconomists.

However, we know that as we go to problems which require agents to solve some combinatorial optimization problems, this nice picture may disappear. In the limit of the number of agents going to infinity some results remain sample-dependent and deterministic results will not follow. Some of this type of phenomena have been reported in Aoki (1996, Sec. 7.1.7) and also in Aoki (1996, p. 225) where Derrida's random energy model was introduced to the economic audience. Unfortunately it did not catch the attention of the economic audiences. See Mertens (2000). This paper is another attempt at exposing non-self-averaging phenomena in economics.

What are the implications if some economic models have non-self averaging property? For one thing, it means that we cannot blindly try for larger size samples in the hope that we obtain better estimates.

The examples above are just a hint of the potential of this approach of using exchangeable random partition methods. It is the opinion of this author that subjects such as in the papers by Fabritiis, Pammolli, and Riccaboni (2003), or by Amaral et al (1998) could be re-examined from the random combinatorial partition approach with profit. Another example is Sutton (2002). He modeled independent business in which the business sizes vary by partitions of integers to discuss the dependence of variances of firm growth rates. He assumed each partition is equally likely, however. Use of random partitions discussed in this paper may provide more realistic or flexible framework for the question he examined.

Finally, the key question in applications to macroeconomic or financial modelings of the random partition approach is "What are the most likely combinations of the values of $K_n = k$, a_j , and ja_j all suitably normalized ?" This question appears too complicated to answer analytically at this time, except for some special cases. Some simulations would help.

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Appendices

Pitman showed that

$$K_n/n^\alpha \to \mathcal{L},$$

in distribution and Pitman (2002, Sec. 3) has stronger result of convergence a.s. See Yamato and Sibuya also. The random variable $\mathcal{L}$ has the density

$$\frac{d}{ds} P_{\alpha,\theta}(\mathcal{L} \in ds) = g_{\alpha,\theta}$$

where letting $\eta = \frac{\theta}{\alpha}$ we define

$$g_{\alpha,\theta}(s) := \frac{\Gamma(\theta+1)}{\Gamma(\eta+1)} s^\eta g_\alpha(s),$$

where $s > 0$, and where $g_\alpha = g_{\alpha,0}$ is the Mittag-Leffler density

$$g_\alpha(s) = \frac{1}{\pi} \sum_{k=1}^{\infty} [\frac{\Gamma(k\alpha)}{\Gamma(k)} sin(k\pi\alpha)(-s)^{k-1}].$$

See Blumenfeld and Mandelbrot (1997), or Pitman (2000) for example.