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Takashi Kato  
Osaka University  
Akihiko Takahashi  
The University of Tokyo  
Toshihiro Yamada  
The University of Tokyo & MTEC

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# An Asymptotic Expansion Formula for Up-and-Out Barrier Option Price under Stochastic Volatility Model

Takashi Kato\*      Akihiko Takahashi†      Toshihiro Yamada‡

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## Abstract

This paper derives a new semi closed-form approximation formula for pricing an up-and-out barrier option under a certain type of stochastic volatility model including SABR model by applying a rigorous asymptotic expansion method developed by Kato, Takahashi and Yamada [1]. We also demonstrate the validity of our approximation method through numerical examples.

**Keywords:** Barrier Option, Up-and-Out Call Option, Asymptotic Expansion, Stochastic Volatility Model

## 1 Introduction

Numerical computation schemes for pricing barrier options have been a topic of great interest in mathematical finance and stochastic analysis. One of the tractable approaches for evaluation of barrier options is to derive an analytical approximation. However, from the mathematical viewpoint, deriving an approximation formula by applying stochastic analysis is not an easy task since the Malliavin calculus approach as in Takahashi and Yamada [3] cannot be directly applied. Recently, Kato, Takahashi and Yamada [1] has provided a new asymptotic expansion method for the Cauchy–Dirichlet problem by developing a rigorous perturbation scheme in a partial differential equation (PDE), and as an example, derived an approximation formula for a down-and-out call option price under a stochastic volatility model. In this paper, we give a new asymptotic expansion formula for an up-and-out call option price under a stochastic volatility model which is widely used in trading practice. Moreover, we show the validity of our formula through numerical experiments.

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\*Osaka University,

†The University of Tokyo,

‡The University of Tokyo & MTEC

## 2 Asymptotic expansion formula for up-and-out barrier option prices

Consider the following stochastic differential equation (SDE) in a stochastic volatility model:

$$\begin{aligned} dS_t^\varepsilon &= (c - q)S_t^\varepsilon dt + \sigma_t^\varepsilon S_t^\varepsilon dB_t^1, \\ S_0^\varepsilon &= S, \\ d\sigma_t^\varepsilon &= \varepsilon\lambda(\theta - \sigma_t^\varepsilon)dt \\ &\quad + \varepsilon\nu\sigma_t^\varepsilon(\rho dB_t^1 + \sqrt{1 - \rho^2}dB_t^2), \\ \sigma_0^\varepsilon &= \sigma, \end{aligned}$$

where  $S, \sigma, c, q > 0$ ,  $\varepsilon \in [0, 1)$ ,  $\lambda, \theta, \nu > 0$ ,  $\rho \in [-1, 1]$  and  $B = (B^1, B^2)$  is a two dimensional standard Brownian motion. This model is motivated by pricing currency options. In this case,  $c$  and  $q$  represent a domestic interest rate and a foreign interest rate, respectively. The process  $S^\varepsilon$  denotes a price of the underlying currency. Our purpose is to evaluate an *up-and-out* barrier option with time-to-maturity  $T - t$  and the upper barrier price  $H(> S)$ , and its initial value is represented under a risk-neutral probability measure as follows:

$$\begin{aligned} C_{\text{Barrier}}^{SV, \varepsilon}(T - t, S) \\ = \mathbb{E} \left[ e^{-c(T-t)} f(S_{T-t}^\varepsilon) 1_{\{\tau_{(0, H)}(S^\varepsilon) > T-t\}} \right], \end{aligned}$$

where  $f$  stands for a call option payoff function  $f(s) = \max\{s - K, 0\}$  for some  $K > 0$ . Here, the stopping time  $\tau_{(0, H)}(S^\varepsilon)$  is defined as

$$\tau_{(0, H)}(S^\varepsilon) = \inf\{t \in [0, T]; S_t^\varepsilon \notin (0, H)\} \quad (\inf \emptyset := \infty).$$

Remark that  $C_{\text{Barrier}}^{SV, \varepsilon}(T - t, S)$  has no closed-form solution and therefore we have to rely on some numerical method such as the Monte-Carlo simulation in order to calculate  $C_{\text{Barrier}}^{SV, \varepsilon}(T - t, S)$ . However, when  $\varepsilon = 0$ ,  $C_{\text{Barrier}}^{SV, 0}(T - t, S)$  corresponds to the up-and-out barrier option price in the Black-Scholes model which is known to be solved explicitly. Then, for  $\varepsilon > 0$ , we are able to derive a *semi closed-form expansion* around  $C_{\text{Barrier}}^{SV, 0}(T - t, S)$  when  $\varepsilon \downarrow 0$ . This is our main result and hereafter we show our approximation method for  $C_{\text{Barrier}}^{SV, \varepsilon}(T - t, S)$ .

Clearly, applying Itô's formula, we can derive the SDE of logarithmic process of  $S_t^\varepsilon$  as

$$\begin{aligned} dX_t^\varepsilon &= (c - q - \frac{1}{2}(\sigma_t^\varepsilon)^2)dt + \sigma_t^\varepsilon dB_t^1, \\ X_0^\varepsilon &= x := \log S. \end{aligned}$$

Then we can rewrite  $C_{\text{Barrier}}^{SV, \varepsilon}(T - t, S)$  as

$$\begin{aligned} C_{\text{Barrier}}^{SV, \varepsilon}(T - t, e^x) \\ = \mathbb{E} \left[ e^{-c(T-t)} \bar{f}(X_{T-t}^\varepsilon) 1_{\{\tau_D(X^\varepsilon) > T-t\}} \right], \end{aligned}$$

where  $\bar{f}(x) = \max\{e^x - K, 0\}$  and  $D = (-\infty, \log H)$ . Note that

$$\tau_D(X^\varepsilon) = \inf\{t \in [0, T]; X_t^\varepsilon \notin D\} = \tau_{(0, H)}(S^\varepsilon).$$

Let  $u^\varepsilon(t, x) = C_{\text{Barrier}}^{SV, \varepsilon}(T - t, e^x)$  for  $t \in [0, T]$  and  $x \in \mathbf{R}$ . Then  $u^\varepsilon(t, x)$  satisfies the following PDE:

$$\begin{cases} \left( \frac{\partial}{\partial t} + \mathcal{L}^\varepsilon - c \right) u^\varepsilon(t, x) = 0, & (t, x) \in (0, T] \times D, \\ u^\varepsilon(T, x) = \bar{f}(x), & x \in \bar{D}, \\ u^\varepsilon(t, \log H) = 0, & t \in [0, T], \end{cases}$$

where

$$\begin{aligned} \mathcal{L}^\varepsilon = & \left( c - q - \frac{1}{2}\sigma^2 \right) \frac{\partial}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} \\ & + \varepsilon \rho \nu \sigma^2 \frac{\partial^2}{\partial x \partial \sigma} + \varepsilon \lambda (\theta - \sigma) \frac{\partial}{\partial \sigma} + \varepsilon^2 \frac{1}{2} \nu^2 \sigma^2 \frac{\partial^2}{\partial \sigma^2}. \end{aligned}$$

As mentioned above, when  $\varepsilon = 0$ , we can obtain the explicit value of  $u^0(t, x)$ . In this case,  $u^0(t, x) = C_{\text{Barrier}}^{BS}(T - t, e^x, \sigma, H)$  represents the price of the up-and-out barrier call option under the Black–Scholes model. We have

$$C_{\text{Barrier}}^{BS} = C_{\text{Vanilla}}^{BS} - C,$$

where

$$\begin{aligned} C_{\text{Vanilla}}^{BS} &= e^x e^{-qT} N(d_1) - K e^{-cT} N(d_2), \\ C &= e^x e^{-qT} N(x_1) - K e^{-cT} N(x_2) \\ &\quad - e^x e^{-qT} \left( \frac{H}{e^x} \right)^{2\lambda} [N(-y) - N(-y_1)] \\ &\quad + K e^{-cT} \left( \frac{H}{e^x} \right)^{2\lambda-2} \\ &\quad \times [N(-y + \sigma\sqrt{T}) - N(-y_1 + \sigma\sqrt{T})] \end{aligned}$$

with

$$\begin{aligned} x_1 &= \frac{x - \log H + (c - q)T + 1/2\sigma^2 T}{\sigma\sqrt{T}}, \\ x_2 &= x_1 - \sigma\sqrt{T}, \\ \lambda &= \frac{(c - q)}{\sigma^2} + \frac{1}{2}, \\ y &= \frac{2 \log H - x - \log K + (c - q)T + 1/2\sigma^2 T}{\sigma\sqrt{T}}, \\ y_1 &= \frac{\log H - x + (c - q)T + 1/2\sigma^2 T}{\sigma\sqrt{T}}. \end{aligned}$$

See Hull [2] for the details.

We can represent  $u^0(t, x) = \bar{P}_t^D \bar{f}(x)$  by using a semi-group  $(\bar{P}_t^D)_t$  defined as

$$\bar{P}_s^D g(x) = \int_{-\infty}^{\log H} e^{-cs} \left( 1 - e^{-\frac{2(\log H - x)(\log H - y)}{\sigma^2 s}} \right)$$

$$\times \frac{1}{\sqrt{2\pi\sigma^2s}} e^{-\frac{(y-x-(c-q-\frac{1}{2}\sigma^2)s)^2}{2\sigma^2s}} g(y) dy \quad (2.1)$$

for a continuous function  $g$  with polynomial growth rate which satisfies  $g(x) = 0$  on  $\partial D$ .

The main result of Kato, Takahashi and Yamada [1] suggests the following approximation formula.

**[Asymptotic expansion formula]**

$$u^\varepsilon(t, x) = C_{\text{Barrier}}^{BS} + \varepsilon e^{-c(T-t)} \int_0^{T-t} \bar{P}_s^D \tilde{\mathcal{L}}_1^0 \bar{P}_{T-t-s}^D \bar{f}(x) ds + O(\varepsilon^2),$$

where

$$\tilde{\mathcal{L}}_1^0 = \frac{\partial}{\partial \varepsilon} \mathcal{L}^\varepsilon|_{\varepsilon=0} = \rho\sigma^2 \frac{\partial^2}{\partial x \partial \sigma} + \lambda(\theta - \sigma) \frac{\partial}{\partial \sigma}.$$

Using (2.1), the term  $\int_0^{T-t} \bar{P}_s^D \tilde{\mathcal{L}}_1^0 \bar{P}_{T-t-s}^D \bar{f}(x) ds$  is expressed as follows:

$$\begin{aligned} & \int_0^{T-t} \bar{P}_s^D \tilde{\mathcal{L}}_1^0 \bar{P}_{T-t-s}^D \bar{f}(x) ds \\ &= \int_0^{T-t} \int_{-\infty}^{\log H} e^{-cs} \left(1 - e^{-\frac{2(\log H-x)(\log H-y)}{\sigma^2s}}\right) \\ & \quad \times \frac{1}{\sqrt{2\pi\sigma^2s}} e^{-\frac{(y-x-(c-q-\frac{1}{2}\sigma^2)s)^2}{2\sigma^2s}} \tilde{\mathcal{L}}_1^0 \bar{P}_{T-t-s}^D \bar{f}(y) dy ds. \end{aligned} \quad (2.2)$$

We are able to compute the integrand of the right hand side of the above formula (2.2) as

$$\begin{aligned} & \tilde{\mathcal{L}}_1^0 \bar{P}_{T-t}^D \bar{f}(x) \\ &= e^{c(T-t)} \left\{ \rho\sigma^2 \frac{\partial^2}{\partial x \partial \sigma} C_{\text{Barrier}}^{BS}(T-t, e^x, \sigma) \right. \\ & \quad \left. + \lambda(\theta - \sigma) \frac{\partial}{\partial \sigma} C_{\text{Barrier}}^{BS}(T-t, e^x, \sigma) \right\}. \end{aligned}$$

Here,  $\frac{\partial}{\partial \sigma} C_{\text{Barrier}}^{BS}(T, e^x)$  and  $\frac{\partial^2}{\partial x \partial \sigma} C_{\text{Barrier}}^{BS}(T, e^x)$  are concretely expressed as follows:

$$\begin{aligned} & \frac{\partial}{\partial \sigma} C_{\text{Barrier}}^{BS}(T, e^x) \\ &= e^{-qT} e^x n(d_1) \sqrt{T} \\ & \quad - e^{-qT} e^x n(x_1) \sqrt{T} - (H-K) e^{-cT} n(x_2) \frac{-x_1}{\sigma} \end{aligned}$$

$$\begin{aligned}
& +e^x e^{-qT} \left(\frac{H}{e^x}\right)^{2\lambda} \\
& \times \left\{ (\log H - x) \frac{-4(c-q)}{\sigma^3} [N(-y) - N(-y_1)] \right. \\
& \left. + [n(y) \frac{y'}{\sigma} - n(y_1) \frac{y'_1}{\sigma}] \right\} \\
& - K e^{-cT} \left(\frac{H}{e^x}\right)^{2\lambda-2} \\
& \times \left\{ (\log H - x) \frac{-4(c-q)}{\sigma^3} [N(-y') - N(-y'_1)] \right. \\
& \left. + [n(y') \frac{y}{\sigma} - n(y'_1) \frac{y_1}{\sigma}] \right\},
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial^2}{\partial x \partial \sigma} C_{\text{Barrier}}^{BS}(T, e^x) = e^{-qT} e^x n(d_1)(-d_2) \frac{1}{\sigma} \\
& - e^{-qT} e^x n(x_1)(-x_2) \frac{1}{\sigma} \\
& - (H - K) e^{-cT} \frac{n(x_2)}{\sigma^2 \sqrt{T}} \{x_1 x_2 - 1\} \\
& + \frac{4(c-q)}{\sigma^3} \{(-1 + 2\lambda)(\log H - x) + 1\} \\
& \times e^x e^{-qT} \left(\frac{H}{e^x}\right)^{2\lambda} [N(-y) - N(-y_1)] \\
& + e^x e^{-qT} \left(\frac{H}{e^x}\right)^{2\lambda} [n(y) \frac{y'}{\sigma} - n(y_1) \frac{y'_1}{\sigma}] \\
& \times \left(1 - 2\lambda \left(\frac{H}{e^x}\right)^{2\lambda}\right) \\
& - e^x e^{-qT} \left(\frac{H}{e^x}\right)^{2\lambda} (\log H - x) \\
& \times \frac{4(c-q)}{\sigma^3} \left(n(y) \left(\frac{1}{\sigma \sqrt{T}}\right) - n(y_1) \left(\frac{1}{\sigma \sqrt{T}}\right)\right) \\
& + e^x e^{-qT} \left(\frac{H}{e^x}\right)^{2\lambda} \\
& \times \left(n(y) \frac{1}{\sigma^2 \sqrt{T}} (yy' - 1) - n(y_1) \frac{1}{\sigma^2 \sqrt{T}} (y_1 y'_1 - 1)\right) \\
& - K e^{-cT} [N(y') - N(y'_1)] \\
& \times \left(\left(\frac{H}{e^x}\right)^{2\lambda-2} \frac{4(c-q)}{\sigma^3} \{(2\lambda - 2)(\log H - x) + 1\}\right) \\
& + K e^{-cT} \left(\frac{H}{e^x}\right)^{2\lambda-2} (\log H - x) \frac{4(c-q)}{\sigma^3}
\end{aligned}$$

$$\begin{aligned}
& \times \left( n(y') \frac{1}{\sigma\sqrt{T}} - n(y'_1) \frac{1}{\sigma\sqrt{T}} \right) \\
& + Ke^{-cT} (2\lambda - 2) \left( \frac{H}{e^x} \right)^{2\lambda-2} \\
& \times \left( n(y') \frac{y}{\sigma} - n(y'_1) \frac{y_1}{\sigma} \right) \\
& - Ke^{-cT} \left( \frac{H}{e^x} \right)^{2\lambda-2} \\
& \times \left( n(y') \frac{1}{\sigma^2\sqrt{T}} (y'y - 1) - n(y'_1) \frac{1}{\sigma^2\sqrt{T}} (y'_1 y_1 - 1) \right),
\end{aligned}$$

where

$$\begin{aligned}
y' &= \frac{2 \log H - x - \log K + (c - q)T - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \\
y'_1 &= \frac{\log H - x + (c - q)T - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}.
\end{aligned}$$

### 3 Numerical Examples

In this section we show numerical examples for pricing European up-and-out barrier call options under SABR volatility model ( $\lambda = 0$ ) as an illustrative purpose. By the asymptotic expansion formula in the previous section, we see

$$\begin{aligned}
C_{\text{Barrier}}^{SV,\varepsilon}(T, S) &\simeq C_{\text{Barrier}}^{BS}(T, S) \\
&+ \varepsilon e^{-cT} \int_0^T \bar{P}_s^D \tilde{\mathcal{L}}_1^0 \bar{P}_{T-s}^D f(S) ds.
\end{aligned}$$

Let us define **AE first** and **AE zeroth** as

$$\begin{aligned}
\mathbf{AE\ first} &= C_{\text{Barrier}}^{BS}(T, S) \\
&+ \varepsilon e^{-cT} \int_0^T \bar{P}_s^D \tilde{\mathcal{L}}_1^0 \bar{P}_{T-s}^D f(S) ds, \\
\mathbf{AE\ zeroth} &= C_{\text{Barrier}}^{BS}(T, S).
\end{aligned}$$

Below we list the numerical examples, [Case 1] – [Case 6], where the numbers in the parentheses show the error rates (%) relative to the benchmark prices of  $C_{\text{Barrier}}^{SV,\varepsilon}(T, S)$  which are computed by Monte–Carlo simulations with 100,000 time steps and 1,000,000 trials (denoted by **MC**). We check the accuracy of our approximation formula by changing the model parameters.

Apparently, our approximation formula **AE first** improves the accuracy for  $C_{\text{Barrier}}^{SV,\varepsilon}(T, S)$ , and it is observed that the approximation term  $\varepsilon e^{-cT} \int_0^T \bar{P}_s^D \tilde{\mathcal{L}}_1^0 \bar{P}_{T-s}^D f(S) ds$  accurately compensates for the difference between  $C_{\text{Barrier}}^{SV,\varepsilon}(T, S)$  and  $C_{\text{Barrier}}^{BS}(T, S)$ , which confirms the validity of our method.

For all cases, we set  $S = 100$ ,  $\sigma = 0.2$ ,  $c = 0.0$ ,  $q = 0.0$ ,  $\rho = -0.5$ ,  $\varepsilon\lambda = 0.0$ ,  $\theta = 0.0$  and  $T = 1.0$ . In Case 1, 2 and 3, given  $\varepsilon\nu = 0.1$ , the upper bound price is set as

$H = 120, 130, 140$ , respectively, while in Case 4, 5 and 6, given  $\varepsilon\nu = 0.2$ ,  $H$  is set as 120, 130, 140, respectively. Particularly, for the case of  $\varepsilon\nu = 0.2$  (that is, higher volatility of volatility case, Case 4, 5 and 6), we remark that the errors of the approximation become slightly larger. However, as observed in comparison between **AE first** and **AE zeroth**, we are convinced that the higher order expansion improves the approximation further, which will be investigated in our next research.

[Case1]

$$S = 100, \sigma = 0.2, c = 0.0, q = 0.0, \varepsilon\nu = 0.1, \\ \rho = -0.5, \varepsilon\lambda = 0.0, \theta = 0.0, H = 120, T = 1.0.$$

Table 1: up-and-out barrier option prices and the relative errors (Case 1)

Strike: $K$	MC	AE first	AE zeroth
100	1.204	1.188 (-1.35%)	1.105 (-8.25%)
102	0.882	0.869 (-1.44%)	0.804 (-8.78%)
105	0.512	0.504 (-1.62%)	0.463 (-9.59%)

[Case2]

$$S = 100, \sigma = 0.2, c = 0.0, q = 0.0, \varepsilon\nu = 0.1, \\ \rho = -0.5, \varepsilon\lambda = 0.0, \theta = 0.0, H = 130, T = 1.0.$$

Table 2: up-and-out barrier option prices and the relative errors (Case 2)

Strike: $K$	MC	AE first	AE zeroth
100	3.216	3.200 (-0.49%)	2.966 (-7.78%)
102	2.621	2.607 (-0.55%)	2.406 (-8.22%)
105	1.869	1.857 (-0.69%)	1.702 (-8.93%)

[Case3]

$$S = 100, \sigma = 0.2, c = 0.0, q = 0.0, \varepsilon\nu = 0.1, \\ \rho = -0.5, \varepsilon\lambda = 0.0, \theta = 0.0, H = 140, T = 1.0.$$



Table 3: up-and-out barrier option prices and the relative errors (Case 3)

Strike: $K$	MC	AE first	AE zeroth
100	5.184	5.186 (0.05%)	4.847 (-6.49%)
102	4.420	4.423 (0.06%)	4.121 (-6.77%)
105	3.420	3.422 (0.06%)	3.174 (-7.19%)

[Case4]

$$S = 100, \sigma = 0.2, c = 0.0, q = 0.0, \varepsilon\nu = 0.2,$$

$$\rho = -0.5, \varepsilon\lambda = 0.0, \theta = 0.0, H = 120, T = 1.0.$$

Table 4: up-and-out barrier option prices and the relative errors (Case 4)

Strike: $K$	MC	AE first	AE zeroth
100	1.317	1.271 (-3.51%)	1.105 (-16.12%)
102	0.971	0.934 (-3.83%)	0.804 (-17.15%)
105	0.569	0.545 (-4.30%)	0.463 (-18.65%)

[Case5]

$$S = 100, \sigma = 0.2, c = 0.0, q = 0.0, \varepsilon\nu = 0.2,$$

$$\rho = -0.5, \varepsilon\lambda = 0.0, \theta = 0.0, H = 130, T = 1.0.$$

Table 5: up-and-out barrier option prices and the relative errors (Case 5)

Strike: $K$	MC	AE first	AE zeroth
100	3.475	3.435 (-1.15%)	2.966 (-14.66%)
102	2.844	2.808 (-1.27%)	2.406 (-15.42%)
105	2.041	2.011 (-1.48%)	1.702 (-16.58%)

[Case6]

$$S = 100, \sigma = 0.2, c = 0.0, q = 0.0, \varepsilon\nu = 0.2, \\ \rho = -0.5, \varepsilon\lambda = 0.0, \theta = 0.0, H = 140, T = 1.0.$$

Table 6: up-and-out barrier option prices and the relative errors (Case 6)

Strike: $K$	MC	AE first	AE zeroth
100	5.483	5.526 (0.78%)	4.847 (-11.59%)
102	4.683	4.725 (0.85%)	4.121 (-12.03%)
105	3.635	3.670 (0.97%)	3.174 (-12.68%)

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