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Abstract

Central banks of major market economies have recently adopted QE (quantitative easing), allowing excess reserves to build up while maintaining the policy rate at very low levels. We develop a regime-switching SVAR (structural vector autoregression) in which the monetary policy regime, chosen by the central bank responding to economic conditions, is endogenous and observable. The model can incorporate the exit condition for terminating QE. We then apply the model to Japan, a country that has accumulated, by our count, 130 months of QE as of December 2012. Our impulse response analysis yields two findings about QE. First, an increase in reserves raises inflation and output. Second, terminating QE is not necessarily deflationary.

Keywords: quantitative easing, structural VAR, observable regimes, Taylor rule, impulse responses, Bank of Japan.

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1. Introduction and Summary

Since the recent global financial crisis, central banks of major market economies have adopted quantitative easing, or QE, which is to allow reserves held by depository institutions far above the required level while keeping the policy rate very close to zero. This paper uses an SVAR (structural vector autoregression) to evaluate the macroeconomic effects of QE. Estimating such a time-series model with any accuracy is difficult because only several years have passed since the crisis. We are thus led to examine Japan, a country that has already accumulated a history of, by our count, 130 months of QE as of December 2012. Those 130 QE months come in three installments, which allows us to evaluate the effect of exiting from QE as well.

Our SVAR has two monetary policy regimes: the zero-rate regime in which the policy rate is very close to zero, and the normal regime. In Section 2, we document for Japan that bank reserves are greater than required reserves (and often several times greater) when the policy rate is below 0.05% (5 basis points) per year. We say that the zero-rate regime is in place if and only if the policy rate is below this critical rate. Therefore, the regime is observable and, since reserves are substantially higher than the required level for all months under the zero-rate regime in data, the zero-rate regime and QE are synonymous. There are three spells of the zero-rate/QE regime: March 1999 - July 2000, March 2001 - June 2006, and December 2008 to date. (They are indicated by the shades in the time-series plot of the policy rate in Figure 1.) They account for the 130 months. Also documented in Section 2 is that for most of those months the BOJ (Bank of Japan) made a stated commitment of not exiting from the zero-rate regime unless inflation is above a certain target level. That is, the exit condition in Japan is about inflation. Our SVAR model incorporates this exit condition.

The model is a natural extension of standard SVAR models. There are four variables, inflation, output (measured by the output gap), the policy rate, and excess reserves, in that order. We do not impose any structure on inflation and output dynamics, so the first two equations of the four-variable system are reduced-form equations. The third equation is the Taylor rule that specifies a shadow policy rate. The central bank picks the normal regime if the shadow rate is positive. The fourth equation specifies the central bank's supply of excess reserves in the zero-rate regime. We incorporate the exit condition by assuming that the central bank ends the zero-rate regime only if the shadow rate is positive *and* the inflation rate is above a certain target. The implied evolution of the regime is a Markov chain. The regime is endogenous because its transition probabilities depend on inflation and output. The model parameters are estimated by ML (maximum likelihood) that properly takes into account the regime endogeneity arising from the zero lower bound and the exit condition.

We utilize the IRs (impulse responses) to describe the macroeconomic effects of various monetary policies, including those of a change in the monetary policy regime. To describe the effect of, for example, a cut in the policy rate in date t, we compare the the path of inflation and output projected by the model given the baseline history up to t with the projected path given an alternative history that differs from the baseline history only with respect to the policy rate in t. The IR to a rate cut is the difference between those two projections (if the model were linear, the IR would be independent of the history and proportional to the size of the policy rate change). From our estimated IRs, we find the following.

- When the regime is the normal regime in both the baseline and alternative histories so that there is room for rate cuts, the IR of inflation to a policy rate cut is negative for many periods. Thus, consistent with the finding of the literature to be cited below, we observe the price puzzle for Japan.

- Under the zero-rate regime, excess reserves increases are expansionary. That is, the IR of inflation and output to an increase in excess reserves is positive. This, too, is consistent with the literature's finding.

- Exiting from the zero-rate regime (a "lift-off") can be expansionary. That is, take the baseline history to be a history up to t in which the regime in t is zero-rate/QE with some positive level of excess reserves, and consider an alternative history in which the regime is normal, not QE, with the same policy rate (of zero) but with zero excess reserves (as required by the regime). If the excess reserve level is not too large, the IR to exiting from QE is slightly higher inflation and higher output. In particular, if the baseline history is the observed history up to t = June 2006 when the excess reserves were only about 58% of (about 1.6 times) the required level, the effect of terminating the zero-rate regime is *expansionary*.

Turning to the relation of our paper to the literature, there is a rapidly expanding literature on the recent QE programs (called large-scale asset purchases (LSAPs)) by the U.S. Federal Reserve. Given the small sample sizes, researchers wishing to study the macroeconomic effect of LSAPs proceed in two steps, first documenting that LSAPs lowered longer-term interest rates and then evaluating the effect of lower interest rates using macroeconomic models. In a recent review of the literature, Williams (2012) notes that there is a lot of uncertainty surrounding the existing estimates. One reason he cites is that LSAP-induced interest rate declines may be atypical.

Were it not for the small-sample problem, time-series analysis of LSAPs would complement nicely those model-based analyses. There are several SVAR studies about Japan's QE that exploit the many QE months noted above. They can be divided into three groups: (a) those assuming the regime is observable and exogenous, (b) those with exogenous but unobservable regimes, and (c) those (like our paper) with endogenous and observable regimes. All those studies assume the block-recursive structure of Christiano, *et. al.* (1999), which orders variables by placing non-financial variables (such as inflation and output) first, followed by monetary policy instruments (such as the policy rate and measures of money), and financial variables (such as stock prices and long-term interest rates). Honda *et. al.* (2007) and Kimura and Nakajima (2013) fall in category (a). Using Japanese monthly data covering only the zero-rate period of 2001 through 2006 and based on SVARs that exclude the policy rate (because it is zero), Honda *et. al.* (2007) find that the IR of inflation and output to an increase in reserves is positive. Kimura and Nakajima (2013) use quarterly data from 1981 and assume two spells of the QE regime (2001:Q1 - 2006:Q1 and 2010:Q1 on). They too find the expansionary effect of excess reserve under QE.[1] Falling in category (b) are Fujiwara (2006) and Inoue and Okimoto (2008). Both papers apply the hidden-stage Markov Switching SVAR model to Japanese monthly data.[2] They find that the probability of state 2 was very high in most of the months since the late 1990s. For those months, the IR of output to an increase in base money is positive and persistent. In contrast to those papers, Iwata and Wu (2006) and Iwata (2010) treat the regime as observable and endogenous, thus falling in category (c). In these two papers, the regime is necessarily endogenous because the zero lower bound for the policy rate is imposed for all periods. Like the other papers, they find that money is expansionary: the IR of inflation and output to the monetary base is positive. They also find, as in some of the papers already cited, the price puzzle under the normal regime.

Because the regime is chosen by the central bank to honor the zero lower bound, or more

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[1]Within each regime, they use the TVP-VAR (time-varying parameter VAR) model to allow coefficients and error variances to change stochastically. There are studies on the macroeconomic effect of monetary policy of recent years in Japan that utilize TVP-VAR. They include Nakajima, Shiratsuka, and Teranishi (2010) and Nakajima and Watanabe (2011). They do not allow for discrete regime changes, though. For example, when the central bank enters the zero-rate regime, the TV-VAR, ignorant of the regime change, does not shrink the coefficients in the policy rate equation immediately to zero. This sort of shrinking is enforced in Kimura and Nakajima (2013) cited in the text.

[2]A precurser to these two papers is the VAR study by Miyao (2002), which, using the conventional likelihood-ratio method, finds a structural break in 1995.

generally, to respond to inflation and output, it seems clear that the regime should be treated as endogenous. And, as already argued above and will be argued more fully in the next section, a case can be made for the observability of the monetary policy regime. Our paper differs from Iwata and Wu (2006) and Iwata (2010), both of which treat the regime as observable and endogenous, in several respects. First, our SVAR incorporates the exit condition as well as the zero lower bound. Second, we extend the IR analysis to accommodate regime changes. This allows us to examine the macroeconomic effect of exiting from QE, which should be of great interest to policymakers. As already mentioned, our paper has a surprising result on this issue. Third, the interest rate equation in our SVAR is the Taylor rule. Most existing estimates of the Taylor rule in Japan end the sample period at 1995 because there is little movements in the policy rate since then. Our estimation of the Taylor rule, utilizing the full sample subject to the zero lower bound and the exit condition, should be of independent interest.

The rest of the paper is organized as follows. In Section 2, we present the case for the monetary policy regime observability. Section 3 describes our four-variable SVAR. Section 4 derives the ML estimator of the model, describes the monthly data, and reports our parameter estimates. Section 5 defines IRs for our regime-switching SVAR and then displays estimated IRs including regime-change IRs. Section 6 concludes.

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2. Identifying the Zero Rate Regime

Identification by the "L"

To address the issue of whether the regime of zero policy rates can be observed, we examine the relation between excess reserves and the policy rate. Figure 2a plots the policy rate measured by the overnight interbank rate (called the "Call rate" in Japan) against m, the *excess reserve rate* defined as the log of the ratio of the actual to required levels of reserves. The actual reserve level for the month is defined as the average of daily balances over the reserve maintenance period (between the 16th day of the month and the 15th day of the following month), not over the calendar month, because that is how the amount of required reserves is calculated. Accordingly, the policy rate for the month, to be denoted r, is the average of daily rates over the same reserve maintenance period. Because the BOJ (Bank of Japan) recently started paying interest on reserves, the vertical axis in the figure is not the policy rate r itself but the *net* policy rate $r - \overline{r}$ where $\overline{r}$ is the rate paid on reserves (0.1% since November 2008), which is the cost of holding reserves for banks.

The plot in Figure 2a shows a distinct L shape. There are excess reserves (i.e., the excess reserve rate m is positive) for all months for which the net policy rate $r - \bar{r}$ is below some very low critical rate (below 0.05% or 5 basis points, as will be seen from Figure 2b), and no excess reserves for most, but not all, months for which the net rate is above the critical rate. The plot shows a few dots corresponding to low but not very low net policy rates. To examine those exceptions more closely, Figure 2b magnifies the plot near the origin. The red horizontal line is the critical rate of $r - \bar{r} = 0.05\%$. The triangle dots in the magnified plot, off the vertical axis, come from two periods between spells of very low net policy rates and high levels of m. One is a brief period of August 2000 - February 2001. The other is July 2006 - November 2008. In addition to those triangles, there are circle dots above the red line that too are off the vertical axis. They all come from the late 1990s when the Japanese financial system was under stress. For example, $(m_t, r_t - \bar{r}_t) = (8.9\%, 0.22\%)$ in October 1998 when the Long-Term Credit Bank went bankrupt. We interpret those triangle and circle dots off the vertical axis and above the red line as representing the demand for excess reserves. Banks wanted to hold reserves above the required level either for precautionary reasons or because they were reluctant to reduce excess reserves from high levels achieved during preceding months. On the other hand, those dots below the red line represents the *supply* of excess reserves chosen by the central bank, as banks are indifferent between any two levels of excess reserves as the cost of holding them is essentially zero.

We say that the *zero-rate regime* is in place if and only if the net policy rate $r - \bar{r}$ is below the critical rate of 0.05%. Since there are no incidents of near-zero excess reserves when the net rate is below the critical rate (see Figure 2b), the zero-rate regime is synonymous with QE (quantitative easing), which is for the central bank to supply reserves beyond the level of required reserves. Under our definition, there are three periods of the zero-rate regime in Japan, indicated by the shades in Figure 1. They are:

I: March 1999 - July 2000,

II: March 2001 - June 2006 (commonly known in Japan as the "quantitative easing period"),

III: December 2008 to date.

It should be noted that the regime is defined by the *net* policy rate. Because the rate paid on reserves ($\bar{r}$) rose from 0% to 0.1% in November 2008, the policy rate (r) itself is above the critical rate of 0.05% in zero-rate period III. Conceivably, a central bank can supply excess reserves in the zero-rate regime while maintaining the policy rate at high levels very close to $\bar{r}$, although it would not be clear why the central bank wants to do so.

The same "L" shape can be observed for the U.S, as shown in Figure 3. The difference from Japan is that there are far fewer observations near the horizontal axis.

Are All Zero-Regime Periods Alike?

Central banks around the world take various policy actions under near-zero interest rates. It is now standard to divide those actions into three categories: (i) setting an *exit condition* for terminating the policy of maintaining very low interest rates, (ii) changing the composition of the central bank's balance sheet, and (iii) QE.[3] We have already noted the equivalence between our definition of the zero-rate regime and QE at least for Japan. Regarding (i), the exit condition adopted by the BOJ concerns the inflation rate. We argue below that the BOJ made an inflation commitment for most, if not all, of the months covered by the above three zero-rate periods. We will also indicate, briefly, how policy actions in category (ii) can be incorporated in our model of the next section.

Table 1 has the official statements by the BOJ's about the exit condition that we were able to assemble. All of those statements were made during the zero-rate periods we identified above and they are invariably about inflation. However we were unable to find clear statements of the exit condition for the following months in the above three ZIRP periods: (a) the very first month of period I, (b) March 2006 - June 2006 (last four months of period II), and (c) December 2008 - November 2009 (first several months of period III).

We could divide the zero-rate regime into two, one with and the other without the exit condition, but doing so would greatly complicate the analysis in the rest of the paper. In our regime-switching model of the next section, there are two regimes — a single zero-rate regime and a "normal" regime with positive net policy rates. The model assumes that the central bank imposed on itself the exit condition in *all* months during the zero-rate regime, which will, with no further assumptions, generate a two-state Markov chain whose transition probabilities are known functions of inflation and output.

Regarding policy actions in category (ii) above, Ueda (2012, Table 2) has a chronology of actions (such as purchases of common stocks) taken by the BOJ. Their effect could be incorporated in our regime-switching SVAR by including a third policy variable (in addition to the policy rate r and the excess reserve rate m) that measures the riskiness of the central bank's

[3]See, e.g., Bernanke *et. al.* (2004).

balance sheet. We decided not to pursue this because of the small sample size.

3. The regime-switching SVAR

This section presents our four-variable SVAR (structural vector autoregression). A more formal exposition of the model is in Appendix 2.

The Standard Three-Variable SVAR

As a point of departure, consider the standard three-variable SVAR in the review paper by Stock and Watson (2001). The three variables are the monthly inflation rate from month $t-1$ to t (p_t), the output gap (x_t), and the policy rate (r_t).[4] The inflation and output equations are reduced-form equations where the regressors are (the constant and) lagged values of all three variables. The third equation is the Taylor rule that relates the policy rate to the contemporaneous values of inflation and the output gap. The error term in this policy rate equation is assumed to be uncorrelated with the errors in the reduced-form equations. This error covariance structure, standard in the recursive VAR literature (see, e.g., Christiano, Eichenbaum, and Evans (1999)), is a plausible restriction to make, given that our measure of the policy rate for the month is for the reserve maintenance period from the 16th of the month to the 15th of the next month.

As is standard in the literature (see, e.g., Clarida *etl. al.* (1998)), we consider the Taylor rule with interest rate smoothing. That is,

$$\text{(Taylor rule)} \qquad r_t = r_t^e + v_{rt},\ r_t^e \equiv \rho_r r_t^* + (1-\rho_r) r_{t-1},\ \ r_t^* \equiv \alpha_r^* + \underset{(1\times 2)}{\boldsymbol{\beta}_r^{*\prime}} \begin{bmatrix} \pi_t \\ x_t \end{bmatrix},\ \ v_{rt} \sim \mathcal{N}(0, \sigma_r^2). \qquad (3.1)$$

Here, π_t, defined as $\pi_t \equiv \frac{1}{12}(p_t + \cdots + p_{t-11})$, is the inflation rate over the past 12 months (this version of Taylor rule is sometimes described as "backward-looking"). If the adjustment speed parameter ρ_r equals unity, then this equation reduces to the desired Taylor rule that $r_t = r_t^* + v_{rt}$. In Taylor's (1993) original formulation, the vector of inflation and output coefficients $\boldsymbol{\beta}_r^*$ is $(1.5, 0.5)$, and the intercept α_r^* equals 1%, which is the difference between the equilibrium real interest rate of 2% and half times the target inflation rate of 2%. We will call r_t^* the *desired Taylor rate*.

[4] In Stock and Watson (2001), the three variables are inflation, the unemployment rate, and the policy rate. We have replaced the unemployment rate by the output gap, because Okun's law does not seem to apply for Japan.

Introducing Regimes

The three-variable SVAR just described does not take into account the non-negativity constraint on the policy rate. Given the interest rate $\bar{r}_t$ (≥ 0) paid on reserves, the lower bound is not zero but $\bar{r}_t$. The Taylor rule with the lower bound, which we call the *censored Taylor rule*, is

$$\text{(censored Taylor rule)} \qquad r_t = \max\left[r_t^e + v_{rt}, \bar{r}_t\right], \;\; v_{rt} \sim \mathcal{N}(0, \sigma_r^2). \tag{3.2}$$

It will turn out useful to rewrite this in an equivalent form by introducing the regime indicator s_t as

$$\text{(censored Taylor rule)} \qquad r_t = \begin{cases} r_t^e + v_{rt}, \;\; v_{rt} \sim \mathcal{N}(0, \sigma_r^2) & \text{if } s_t = \mathrm{P}, \\ \bar{r}_t & \text{if } s_t = \mathrm{Z}, \end{cases} \tag{3.3}$$

where the two regimes, P (call it the *normal regime*) and Z (the *zero-rate regime*), are defined as

$$s_t = \begin{cases} \mathrm{P} & \text{if } \; r_t^e + v_{rt} \geq \bar{r}_t, \\ \mathrm{Z} & \text{otherwise.} \end{cases} \tag{3.4}$$

We have thus obtained a simple regime-switching three-variable SVAR by replacing the Taylor rule (3.1) by its censored version (3.3) (with the regime s_t defined by (3.4)).[5]

Adding m and the Exit Condition

We expand this model to capture the two aspects of the zero-rate regime discussed in the previous section. First, the m_t (defined, recall, as the log of actual to required reserve ratio) in the zero-rate regime is supply-determined, thus qualifying as an additional monetary policy variable. Second, the (trivial) regime evolution (3.4) needs to be modified to reflect the exit condition.

Excess reserves are set to zero by the market in the normal regime (P) and set by the central bank in the zero-rate regime (Z). We assume that the supply of reserves is given by $\max[m_t^e + v_{mt}, 0]$, where

$$m_t^e \equiv \rho_m \left(\alpha_m^* + \boldsymbol{\beta}_m^{*\prime} \begin{bmatrix} \pi_t \\ x_t \end{bmatrix} \right) + (1 - \rho_m) m_{t-1}. \tag{3.5}$$

That is,

$$m_t = \begin{cases} 0 & \text{if } s_t = \mathrm{P}, \\ \max\left[m_t^e + v_{mt}, 0\right], \;\; v_{mt} \sim \mathcal{N}(0, \sigma_m^2) & \text{if } s_t = \mathrm{Z}, \end{cases} \tag{3.6}$$

[5] If the policy rate equation is not the Taylor rule but a reduced form, the model becomes one estimated in Iwata and Wu (2004).

The "max" operator in the supply function is needed because excess reserves cannot be negative given the system of required reserves. We expect the function's inflation and output coefficients to be negative, i.e., $\boldsymbol{\beta}_m^* < \mathbf{0}$, since the central bank should increase excess reserves when deflation worsens or when output declines.

The second extension of the model concerns the continuation of the zero-rate regime even if the Taylor rule instructs otherwise, unless a certain exit condition is met. As was documented in the previous section, that certain condition for the BOJ is that the twelve-month inflation rate be above some target rate. We allow the target rate to be time-varying. More formally,

$$\text{If } s_{t-1} = \mathrm{Z}, \; s_t = \begin{cases} \mathrm{P} & \text{if } r_t^e + v_{rt} \geq \overline{r}_t \text{ and } \pi_t \geq \underbrace{\overline{\pi} + v_{\overline{\pi}t}}_{\text{period } t \text{ target inflation rate}}, \; v_{\overline{\pi}t} \sim \mathcal{N}(0, \sigma_{\overline{\pi}}^2), \\ \mathrm{Z} & \text{otherwise.} \end{cases} \tag{3.7}$$

We assume that the stochastic component of the target rate ($v_{\overline{\pi}t}$) is i.i.d. over time. If $s_{t-1} = \mathrm{P}$, the inflation exit condition is mute and the central bank picks the current regime s_t by (3.4).

To Recapitulate

This completes our exposition of the regime-switching SVAR on four variables, p_t (monthly inflation), x_t (the output gap), r_t (policy rate), and m_t (the excess reserve rate). The underlying sequence of events leading up to the determination of the two policy instruments (r_t, m_t) can be described as follows. At the beginning of period t, the nature draws two reduced-form errors, one for inflation and the other for output, from a bivariate distribution. This determines (p_t, x_t). The central bank then calculates r_t^e (defined in (3.1)) and m_t^e (defined in (6.1)), and draws *three* policy shocks $(v_{rt}, v_{mt}, v_{\overline{\pi}t})$ from $\mathcal{N}(\underset{(3\times1)}{\mathbf{0}}, \begin{bmatrix} \sigma_r^2 & 0 & 0 \\ 0 & \sigma_m^2 & 0 \\ 0 & 0 & \sigma_{\overline{\pi}}^2 \end{bmatrix})$. The central bank can now calculate $r_t^e + v_{rt}$ (the policy rate dictated by the Taylor rule), $m_t^e + v_{mt}$ (its quantity counterpart), and $\overline{\pi} + v_{\overline{\pi}t}$ (the current target inflation rate). Suppose the previous regime was the normal regime (i.e., suppose that $s_{t-1} = \mathrm{P}$). The bank picks $s_t = \mathrm{P}$ if $r_t^e + v_{rt} \geq \overline{r}_t$, and $s_t = \mathrm{Z}$ otherwise. Suppose, on the other hand, that $s_{t-1} = \mathrm{Z}$. Then the bank terminates the zero-rate regime and picks $s_t = \mathrm{P}$ only if $r_t^e + v_{rt} \geq \overline{r}_t$ *and* $\pi_t \geq \overline{\pi} + v_{\overline{\pi}t}$. If $s_t = \mathrm{P}$, the bank sets r_t by the Taylor rule and the market determines m_t; if $s_t = \mathrm{Z}$, the bank sets r_t at $\overline{r}_t$ and m_t by the reserve supply function.

The implied regime transition matrix can be easily calculated from (3.4) and (3.7). Define

$$P_{rt} \equiv \text{Prob}\,(r_t^e + v_{rt} \geq \bar{r}_t) = \Phi\left(\frac{r_t^e - \bar{r}_t}{\sigma_r}\right), \tag{3.8}$$

$$P_{\pi t} \equiv \text{Prob}\,(\pi_t \geq \overline{\pi} + v_{\overline{\pi}t}) = \Phi\left(\frac{\pi_t - \overline{\pi}}{\sigma_{\overline{\pi}}}\right), \tag{3.9}$$

where $\Phi(.)$ is the cdf of $\mathcal{N}(0,1)$. The matrix is displayed in Table 2.

4. Estimating the Model

This section has three parts. It summarizes the derivation in Appendix 2 of the model's likelihood function and the data description of Appendix 1, followed by a discussion of the estimation results.

The Likelihood Function (Summary of Appendix 2)

The model's likelihood function has a convenient separability property. That is, the log likelihood can be written as the sum of three parts:

$$\text{log likelihood} = L_A(\boldsymbol{\theta}_A) + L_B(\boldsymbol{\theta}_B) + L_C(\boldsymbol{\theta}_C), \tag{4.1}$$

where $(\boldsymbol{\theta}_A, \boldsymbol{\theta}_B, \boldsymbol{\theta}_C)$ form the model's parameter vector. Therefore, the ML (maximum likelihood) estimator of each group of parameters can be obtained by maximizing the corresponding part of the log likelihood function. The first group of parameters, $\boldsymbol{\theta}_A$, is the reduced-form parameters for inflation and output. The second group, $\boldsymbol{\theta}_B$, is the parameters of the Taylor rule with the exit condition appearing in (3.1) and (3.7), while the third group $\boldsymbol{\theta}_C$ describes the reserve supply function $\max[m_t^e + v_{mt}, 0]$ appearing in (3.6). More precisely,

$$\boldsymbol{\theta}_B = \left(\alpha_r^*,\ \underset{(2\times1)}{\boldsymbol{\beta}_r^*},\ \rho_r, \sigma_r, \overline{\pi}, \sigma_{\overline{\pi}}\right) \text{ (7 parameters)}, \quad \boldsymbol{\theta}_C = \left(\alpha_m^*,\ \underset{(2\times1)}{\boldsymbol{\beta}_m^*},\ \rho_m, \sigma_m\right) \text{ (5 parameters)}. \tag{4.2}$$

The first part, $L_A(\boldsymbol{\theta}_A)$, being the log likelihood of the two reduced-form equations, is entirely standard, with the ML estimator of $\boldsymbol{\theta}_A$ given by OLS (ordinary least squares). The ML estimator of $\boldsymbol{\theta}_C$, which maximizes $L_C(\boldsymbol{\theta}_C)$, is Tobit on sample Z, thanks to the censoring implicit in the "max" operator in the reserve supply function. However, as noted in Section 2, there are no observations with $m_t = 0$ in the zero-rate regime (which makes the zero-rate regime synonymous to QE), so the Tobit estimator of $\boldsymbol{\theta}_C$ reduces to OLS. Since the evolution of the regime does not depend on the reserve supply shock (v_{mt}), there is no need to correct for the endogeneity of the regime.

The regime endogeneity becomes relevant for the second part $L_B(\boldsymbol{\theta}_B)$, because the shocks in the Taylor rule and the exit condition $(v_{rt}, v_{\overline{\pi}t})$ affect the regime evolution. If the exit condition were absent so that the censored Taylor rule (3.2) is applicable, then the ML estimator of $\boldsymbol{\theta}_B$ that controls the regime endogeneity is Tobit. With the exit condition, the ML estimation becomes slightly more complicated because, as seen from Table 2, both the lower bound and the exit condition affect to the evolution of the regime.

The Data (Summary of Appendix 1)

The model's variables are p_t (monthly inflation), x_t (output gap), r_t (the policy rate), and m_t (the excess reserve rate, defined as 100 times the log of the ratio of actual to required reserves). Data on actual and required reserves for monthly reserve maintenance periods are available from the BOJ's website, way back to as early as 1960. Figure 4a has m_t since August 1985.

The output measure underlying the monthly output gap (x_t) is the Index of Industrial Production. We apply the HP (Hodrick-Prescott) filter to the log of seasonally adjusted series to generate the trend log output. The log seasonally adjusted output and its HP-filtered series are in Figure 4b. It clearly shows the well-documented decline in the trend growth rate, often described as the (ongoing) lost decade(s). The output gap is defined as 100 times the difference between the two log series. There is a steep decline in the output gap after the Lehman crisis.

The policy rate r_t for month t is the average of daily values, over the reserve maintenance period from the 16th day of month t to the 15th day of month $t + 1$, of the overnight "Call" (i.e., interbank) rate. The Call rate used is the uncollateralized Call rate from August 1985 (when the market was established). For months before then, we use the collateralized Call rate. The graph of r_t thus defined since January 1970 is in Figure 4c.

The inflation rate is constructed from the CPI (consumer price index). The relevant CPI component is the so-called "core CPI" (the CPI excluding fresh food), which, as seen in Table 1, is the price index most often mentioned in BOJ announcements. (Confusingly, the "core CPI" in the U.S. sense, which excludes food and energy, is called the "core-core CPI".) We made some adjustments to remove the effect of the increase in the consumption tax rate in 1989 and 1997 before performing a seasonal adjustment (the official CPI series are not seasonally adjusted). We also adjusted for large movements in the energy component of the CPI in 2007 and 2008. The monthly inflation rate p_t is at annual rates, 1200 times the log difference between month t and month $t - 1$ values of the adjusted CPI. The twelve-month inflation rate π_t is calculated as 100 times the log difference between month t and $t - 12$ values of the CPI, so $\pi_t = \frac{1}{12}(p_t + \cdots + p_{t-11})$.

Simple statistics of the variables mentioned here are shown Table 3.

Parameter Estimates

Having described the estimation method and the data, we are ready to report the parameter estimates. We start with $\boldsymbol{\theta}_B$.

Taylor rule with exit condition ($\boldsymbol{\theta}_B$).

Most estimates of the Taylor rule for Japan take the sample period to be between around 1980 and 1995.[6] The speed of adjustment (ρ_r in (3.1)) is estimated to be about 7% per month and the inflation and output gap coefficients in the desired Taylor rate ($\underset{(2\times1)}{\boldsymbol{\beta}_r^*}$ in (3.1)) are not far from Taylor's original formulation of $(1.5, 0.5)$.[7] The sample typically ends in 1995 because the policy rate shows very little movements near the lower bound since then. In our ML estimation, which can incorporate the lower bound for the policy rate, the sample period includes all the recent months of very low policy rates.

Before reporting our estimates, we mention several issues that turned out to affect the Taylor rule estimates. One is the choice of the first month of the sample period. This issue will be illustrated in a moment. There are two other issues.

- (Treatment of the equilibrium real interest rate) Recall that, in the original Taylor formulation ($r_t^* = 1\% + 1.5\pi_t + 0.5x_t$), the intercept of 1% is the equilibrium real rate of 2% minus half times the assumed target inflation rate of 2%. The assumption of a constant real interest rate may not be appropriate for Japan, given the growth slowdown since the early 1990s. Indeed, the Taylor rule in Braun and Waki (2006) takes into account the decline in the real interest rate caused by the decline in the TFP (total factor productivity) growth documented in Hayashi and Perscott (2002). To account for the decline, we follow Okina and Shiratsuka (2002) and assume that the equilibrium real rate equals the trend growth rate, defined as the twelve-month growth rate of the HP-filtered output (shown in Figure 4b). The desired Taylor rate r_t^* in (3.1) is now

$$r_t^* = \text{trend growth rate in } t + \alpha_r^* + \underset{(1\times2)}{\boldsymbol{\beta}_r^{*\prime}} \begin{bmatrix} \pi_t \\ x_t \end{bmatrix}. \tag{4.3}$$

  The reason why the treatment of the real rate affects the speed of adjustment can be seen from Figure 5. The green line is the original desired Taylor rate $r_t^* = 1\% + 1.5\pi_t + 0.5x_t$. Take, for

[6] See Miyazawa (2010) for a survey.

[7] See, e.g., Clarida *et. al.* (1998). Their estimate of ρ_r, based on the "forward-looking" version of the Taylor rule and the sample pdriod of April 1979 - December 1994, is 7%.

example, the normal period between the zero-regime period II (ending June 2006) and period III (starting December 2008). Despite the original desired Taylor rate being almost as high as it was during the "bubble" period around 1991, the policy rate is far lower. The ML interprets it as extreme interest rate smoothing. The ML estimate of the speed of adjustment for the case of constant real rates is -0.7%.

- (The quasi zero-rate regime) There is a stretch of the nearly constant policy rate at a low, but not very low, level from September 1995 to July 1998. By our definition in Section 2, this period is not in the zero-rate regime. Yet the policy rate was nearly constant despite the rise of the desired Taylor rate shown in Figure 5. We decided to exclude this quasi zero-rate regime from the sample. Perhaps the BOJ, being concerned about the very low inflation rate (see Figure 4c), refrained from raising the policy rate dictated by the Taylor rule. The exclusion of this period raises the ML estimate of ρ_r from 4.9% to 6.0%.

Table 4 has our parameter estimates. Several features are worth noting.

- (the baseline estimate) Line 1 of Table 4 reports the ML estimate of our preferred specification. The first month of the sample is September 1985, the earliest month for which the policy rate can be calculated from the daily data on the uncollateralized Call rate. As just mentioned, the speed of adjustment per month is 6.0%. The inflation and output coefficients in the desired Taylor rate are estimated to be $(1.68, 0.23)$, not very far from Taylor's original formulation of $(1.5, 0.5)$. The t-values indicate that those three parameters are sharply estimated by ML. The mean of the time-varying target inflation rate affecting the exit condition is mere 0.22%. The desired Taylor rate implied by the ML estimate in Line 1 is shown in red in Figure 5.
- (sensitivity to the starting month) The next four lines, lines 2-5, show the sensitivity (or lack thereof) of the estimates to the starting month. The two starting months in line 2 and 3, November 1978 and May 1980, are chosen deliberately. As can be seen from Figure 4c, the month of November 1978 is shortly before the rise in the inflation rate brought about by the second oil crisis. The other starting month (May 1980) is when the inflation rate peaked. The inflation coefficient for this starting month is negative. The point of line 4 and 5 estimates is that the choice of the starting month does not greatly affect the parameter estimates as long as it is after the second oil shock.
- (effect of ignoring the exit condition) It is instructive to compare the ML estimates, which incorporates the exit condition, to the Tobit estimate shown in Line 6 of the table, which

doesn't. Going back to Figure 5, focus, for example, on the zero-rate period II (March 2001 - June 2006). The desired Taylor rate implied by the Line 1 ML estimates, shown in red, crosses the horizontal line from below in the middle of the period. The zero-rate regime was not terminated when the desired rate r_t^*, hence the shadow Taylor rate r_t^e, turned from negative to positive then.[8] This is of course due to the exit condition, but Tobit, not being informed of the condition, take it to be interest rate smoothing. Hence the Tobit estimate of the speed of adjustment ρ_r of 4.3%, shown in Line 6, is lower than the ML estimate of 6.0% shown in Line 1.

- (comparison with existing estimates) For comparison with the existing Taylor rule estimates, Line 7 and 8 display OLS estimates for the sample period of November 1978 - August 1995 which is similar to the sample periods used in existing studies. Line 7 is for the same Taylor rule specification of the variable real rate (set equal to the trend growth rate), while Line 8 assumes a constant real rate. Comparing these two OLS estimates, we see that the treatment of the real rate is not as influential as when the zero-rate periods are included in the sample. The speed of adjustment in Line 8, at 4.3%, is lower than the consensus estimate of around 7%, perhaps because the Taylor rule here is "backward-looking".

Inflation and output equations ($\boldsymbol{\theta}_A$).

Before looking at the estimates of the reduced-form parameters ($\boldsymbol{\theta}_A$) for inflation (p_t) and output (x_t), we need to address two issues on the specification of the reduced-form equations .

- (regime dependence of the reduced form) We take the Lucas critique seriously and allow the reduced-form equations to depend on the regime. If the private sector in period t sets (p_t, x_t) in full anticipation of the period's regime to be chosen by the central bank, the period t reduced form should depend on the date t regime. Since we view this to be a very remote possibility, we assume that the reduced-form coefficients and error variance and covariances in period t depend, if at all, on the lagged regime s_{t-1}. This means that there is no need to correct for the selectivity bias and OLS for each regime is the appropriate estimation method.

- (exclusion restrictions) In order to preserve the degrees of freedom given modest sample sizes, we constrain the lagged policy rate (r) coefficient in the inflation and output equations to be zero when the lagged regime is the zero-rate regime (Z). This should not be controversial. We do the same for the lagged excess reserve rate (m) coefficient under the normal regime (P). As

[8] Recall from (3.1): $r_t^e = \rho_r r_t^* + (1-\rho_r) r_{t-1}$. In the middle of a zero-rate regime, r_{t-1}=0, so $r_t^e = \rho_r r_t^*$.

we noted in Section 2, m is not necessarily zero under P because banks chose to hold excess reserves in the time of financial distress. We nevertheless exclude m from the set of reduced-form regressors for two reasons. First, the incidents of a positive m are rare, as can be seen from Figure 2 and Figure 4a. Second, if m were to be admitted, the VAR system for regime P would have to include an m equation representing the demand for excess reserves under financial distress.

Table 5 reports the OLS estimates of the reduced-form parameters $\boldsymbol{\theta}_A$. The lag length is set to 1 here in order to facilitate interpretation. As might be expected from the break in the trend growth rate shown in Figure 4b, there appears to be a structural break around 1991. That is, for the inflation equation but not for the output equation, and for the whole sample period of 1970-2012 or 1980-2012, the Chow test detects a structural change when the break point is any month before 1992. For this reason, we take the sample period to be January 1992 - December 2012. Several notable features are:

- Under P, the lagged r coefficient in the inflation equation is highly statistically significant. It is quantitatively large as well — a 1 percentage point cut in the policy rate *lowers* inflation by about 0.4 percentage point in the next period.[9] This will be seen as the primal source of the *price puzzle* in the next section's estimated impulse response of p to r. The lagged r coefficient in the output equation, while negative, is relatively small in absolute value and statistically insignificant.

- Under Z, the lagged m coefficients in the inflation and output equations have the sign we would expect: an increase in the supply of excess reserves raises inflation and output in the next period. They are quantitatively small and not sharply determined though. The lagged m coefficient of 0.0028 for the output equation implies that an increase of, say, 20% of reserves raises the output gap by about 0.05 percentage points.

- In either regime, monthly inflation exhibits low persistence, as the lagged p coefficient in the inflation equation is small in absolute value. This will explain why the IR of p exhibits little persistence.

[9]The positive r_{t-1} coefficient may be due to the fact that r_{t-1} is the average over the period of the 16th of month $t-1$ and the 15th of month t. If the central bank can respond to price increases of the month by raising the policy rate in the first 15 days of the month, there will be a positive correlation between p_t and r_{t-1}. To check this, we replaced r_{t-1} by r_{t-2} in the equation for p_t and found a very similar coefficient estimate (the estimate is 0.38, $t = 3.7$).

Panel B shows parameter estimates when the sample ends in 1991, with and without the 1970s. Since they are before the arrival of the first period of the zero-rate regime, the regime is P. The quantitatively large and statistically significant lagged r coefficient in the inflation equation remains for the pre-1991 periods as well. In contrast, the persistence of monthly inflation, while substantial if the 1970s are included, declines as the sample becomes more recent.

Reserve supply equation (θ_C).

Table 6 reports the estimated reserve supply equation in (3.6). The sample period is the 130 months comprising the three zero-rate periods. As alerady mentioned, the ML estimator is OLS. The inflation and output coefficients, alhough not sharply estimated, pick up the expected sign. The estimated inflation coefficient implies that a one percentage point decrease in the twelve-month inflation rate brings about a 117% point increase in the desired excess reserve rate. The speed of adjustment is slow, at 3.4% per month, so the actual monthly increase in the supply of bank reserves is about 4.0% of required reserves.

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5. Impulse Responses (IRs)

With the estimates of our model parameters in hand, we turn IRs (impulse responses). For linear models, the IR analysis is well known since Sims (1980). Our model, however, is nonlinear because the dynamics depends on the regime and also because the reserve supply function, $\max[m_t^e + v_{mt}, 0]$, is a nonlinear function. In this section, we state the definition of IRs for our model and calculate IRs to changes in monetary policy variables including changes in regimes.

IRs for Nonlinear Processes in General

To motivate our definition of IRs, consider for a moment a strictly stationary process $\underset{(n\times 1)}{\mathbf{y}_t}$ in general. Gallant, Rossi, and Tauchen (1993, particularly pp. 876-877) and Potter (2000) proposed to define an IR as the difference in conditional expectations under two alternative possible histories with one history being a perturbation of the other. The IR of the i-th variable to the j-th variable k-period ahead is defined as

$$\mathrm{E}\Big(y_{i,t+k} \mid \underbrace{y_{jt} + \delta, y_{j-1,t}, \ldots, y_{1t}, \mathbf{y}_{t-1}, \mathbf{y}_{t-2}, \ldots}_{\text{alternative history}}\Big) - \mathrm{E}\Big(y_{i,t+k} \mid \underbrace{y_{jt}, y_{j-1,t}, \ldots, y_{1t}, \mathbf{y}_{t-1}, \mathbf{y}_{t-2}, \ldots}_{\text{baseline history}}\Big), \tag{5.1}$$

where δ is the size of perturbation. The perturbation is to $\mathbf{y}$, not to innovations, thus avoiding the issue of how to define innovations for nonlinear processes. It is shown in Hamilton (1994, see his equation [11.4.19]) that this definition, when applied to linear processes, reduces to the standard orthogonalized IR. Thus for the linear case the IR defined above is history-independent and proportional to the size of perturbation.

Adaptation to Our Model: m-IR (IRs to Changes in m) and r-IR (IRs to Changes in r)

Turning to our model of Section 3, it provides a mapping from $(s_t, \mathbf{y}_t, \mathbf{y}_{t-1}, ..., \mathbf{y}_{t-10})$ (the initial condition) and $\boldsymbol{\xi}_{t+1}$ (the shock vector) to $(s_{t+1}, \mathbf{y}_{t+1})$, where the shock vector $\boldsymbol{\xi}_{t+1}$ collects shocks dated $t+1$ of the system (consisting of the reduced-form shocks for (p, x) and policy shocks $(v_r, v_m, v_{\overline{\pi}})$). Ten lags are needed because the Taylor rule in period $t+1$ involves the twelve-month inflation rate $\pi_{t+1} = \frac{1}{12}(p_{t+1} + \cdots + p_{t-10})$.

The adaptation of the IR defined above to our model is easy to see for the last variable of the system, m_t. Since the central bank has control over m only under the normal regime, we assume $s_t = \mathrm{Z}$ and define the IR to a change in m (denoted as m-IR) as

$$(m\text{-IR}) \qquad \mathrm{E}_t\Big(y_{t+k} \,|\, s_t = \mathrm{Z},\ \underbrace{(m_t + \delta_m, \overline{r}_t, p_t, x_t),}_{\mathbf{y}_t \text{ in the alternative history}}\ \mathbf{y}_{t-1}, ..., \mathbf{y}_{t-10}\Big)$$
$$- \mathrm{E}_t\Big(y_{i,t+k} \,|\, s_t = \mathrm{Z},\ \underbrace{(m_t, \overline{r}_t, p_t, x_t),}_{\mathbf{y}_t \text{ in the baseline history}}\ \mathbf{y}_{t-1}, ..., \mathbf{y}_{t-10}\Big),\ \ y = p, x, r, m. \tag{5.2}$$

In both the baseline and alternative histories, $r_t = \overline{r}_t$ because that is what is implied by the regime $s_t = \mathrm{Z}$.[10] Several comments are in order.

- (Monte Carlo integration) The conditional expectation in the above IR definition can be computed by simulating a large number of sample paths from $t+1$ on and then taking the average of those simulated paths. Each sample path given a draw of the shock sequence $(\boldsymbol{\xi}_{t+1}, \boldsymbol{\xi}_{t+2}, ...)$ can be generated by the mapping provided by the model of Section 3. The initial

[10]Stating the definition of m-IR equivalently in terms of the shocks is more complicated, but it can be done. The shocks to the reduced-form equations for (p_t, x_t) are the same across the two histories. Let $(v_{rt}^{(i)}, v_{\overline{\pi}t}^{(i)}, v_{mt}^{(i)})$ be the vector of policy shocks in the baseline history (for $i = b$) and in the alternative history ($i = a$). Recall that the current regime s_t is determined by (3.4) if $s_{t-1} = \mathrm{P}$ and by (3.7) if $s_{t-1} = \mathrm{Z}$ and that $m_t = \max[m_t^e + v_{mt}, 0]$ when $s_t = \mathrm{Z}$. So if $s_{t-1} = \mathrm{P}$, then $(v_{rt}^{(b)}, v_{\overline{\pi}t}^{(b)}, v_{mt}^{(b)})$ satisfies: $r_t^e + v_{rt}^{(b)} < \overline{r}_t$ and $m_t = \max[m_t^e + v_{mt}^{(b)}, 0]$. If $s_{t-1} = \mathrm{Z}$, then it satisfies: $(r_t^e + v_{rt}^{(b)} < \overline{r}_t$ or $\pi_t < \overline{\pi} + v_{\overline{\pi}t}^{(b)})$ and $m_t = \max[m_t^e + v_{mt}^{(b)}, 0]$. Thus he baseline history in the definition of m-IR is the union of two histories differing with respect to s_{t-1}: one with the $(v_{rt}^{(b)}, v_{\overline{\pi}t}^{(b)}, v_{mt}^{(b)})$ for $s_{t-1} = \mathrm{P}$ and the other with the $(v_{rt}^{(b)}, v_{\overline{\pi}t}^{(b)}, v_{mt}^{(b)})$ for $s_{t-1} = \mathrm{Z}$. Because the mapping underlying the conditional expectation is from $(s_t, \mathbf{y}_t, \mathbf{y}_{t-1}, ..., \mathbf{y}_{t-10})$ (the initial condition) and $\boldsymbol{\xi}_{t+1}$ (the shock vector) to $(s_{t+1}, \mathbf{y}_{t+1})$, the conditional expectation is the same across those two baseline histories. Turning to the alternative history, if $s_{t-1} = \mathrm{P}$, then $(v_{rt}^{(a)}, v_{\overline{\pi}t}^{(a)}, v_{mt}^{(a)})$ satisfies: $r_t^e + v_{rt}^{(a)} < \overline{r}_t$ and $m_t + \delta_m = \max[m_t^e + v_{mt}^{(a)}, 0]$. If $s_{t-1} = \mathrm{Z}$, then it satisfies: $(r_t^e + v_{rt}^{(a)} < \overline{r}_t$ or $\pi_t < \overline{\pi} + v_{\overline{\pi}t}^{(a)})$ and $m_t + \delta_m = \max[m_t^e + v_{mt}^{(a)}, 0]$.

condition for generating the sample paths is given by the conditioning information in the conditional expectation. In the IRs to be shown below, 1000 simulations are generated.

- ("E_t" rather than "E") The conditional expectation operator is subscripted by t because the sample path depends on the sequence from $t+1$ on of the interest rate paid on reserves ($\overline{r}$) and the trend growth rate (which appears in (4.3)). This sequence is common to all the simulations in the Monte Carlo integration. The value of sequence at $t+k$ is set equal to its historical value if $t+k$ is in the sample period and to the value at the end of the sample period if $t+k$ is after the sample period.

A change in the policy rate is possible only under regime P. The IR to a policy rate change, denoted r-IR, then, is

$$\begin{aligned}(r\text{-IR}) \qquad & \mathrm{E}_t\Big(y_{t+k}\,|\,s_t = \mathrm{P},\ \ \underbrace{(0, r_t + \delta_r, p_t, x_t),}_{\mathbf{y}_t \text{ in the alternative history}}\ \ \mathbf{y}_{t-1}, ..., \mathbf{y}_{t-10}\Big) \\ & - \mathrm{E}_t\Big(y_{i,t+k}\,|\,s_t = \mathrm{P},\ \ \underbrace{(0, r_t, p_t, x_t),}_{\mathbf{y}_t \text{ in the baseline history}}\ \ \mathbf{y}_{t-1}, ..., \mathbf{y}_{t-10}\Big),\ \ y = p, x, r, m.\end{aligned} \tag{5.3}$$

In both the baseline and alternative histories, we have $m_t = 0$ because it is implied by regime P.[11]

ZP-IR (IRs to a Change in Regime from Z to P)

So far, the perturbed variables are continuous. The above definition of the IR can also be applied to the discrete state variable of the system, the regime s_t. Take as the baseline history a history whose current regime is $s_t = \mathrm{Z}$ (so $r_t = \overline{r}_t$) with some level of excess reserve rate m_t, and consider a central bank contemplating on terminating the zero-rate policy without raising the policy rate.

$$\begin{aligned}(\text{ZP-IR}) \qquad & \mathrm{E}_t\Big(y_{t+k}\,|\,s_t = \mathrm{P},\ \ \underbrace{(0, \overline{r}_t, p_t, x_t),}_{\mathbf{y}_t \text{ in the alternative history}}\ \ \mathbf{y}_{t-1}, ..., \mathbf{y}_{t-10}\Big) \\ & - \mathrm{E}_t\Big(y_{t+k}\,|\,s_t = \mathrm{Z},\ \ \underbrace{(m_t, \overline{r}_t, p_t, x_t),}_{\mathbf{y}_t \text{ in the baseline history}}\ \ \mathbf{y}_{t-1}, ..., \mathbf{y}_{t-10}\Big),\ \ y = p, x, r, m.\end{aligned} \tag{5.4}$$

---

[11] The equivalent definition r-IR in terms of the shocks is as follows. As in footnote 10, let $(v_{rt}^{(i)}, v_{\overline{\pi}t}^{(i)}, v_{mt}^{(i)})$ be the vector of policy shocks in the baseline history (for $i = b$) and in the alternative history ($i = a$). If $s_{t-1} = \mathrm{P}$, then $(v_{rt}^{(b)}, v_{\overline{\pi}t}^{(b)}, v_{mt}^{(b)})$ satisfies: $r_t^e + v_{rt}^{(b)} \geq \overline{r}_t$ and $r_t^e + v_{rt}^{(b)} = r_t$, and $(v_{rt}^{(a)}, v_{\overline{\pi}t}^{(a)}, v_{mt}^{(a)})$ satisfies: $r_t^e + v_{rt}^{(a)} \geq \overline{r}_t$ and $r_t^e + v_{rt}^{(a)} = r_t + \delta_r$. If $s_{t-1} = \mathrm{Z}$, then $(v_{rt}^{(b)}, v_{\overline{\pi}t}^{(b)}, v_{mt}^{(b)})$ satisfies: $r_t^e + v_{rt}^{(b)} \geq \overline{r}_t$, $\pi_t \geq \overline{\pi} + v_{\overline{\pi}t}^{(b)}$, and $r_t^e + v_{rt}^{(b)} = r_t$, and $(v_{rt}^{(a)}, v_{\overline{\pi}t}^{(a)}, v_{mt}^{(a)})$ satisfies: $r_t^e + v_{rt}^{(a)} \geq \overline{r}_t$, $\pi_t \geq \overline{\pi} + v_{\overline{\pi}t}^{(a)}$, and $r_t^e + v_{rt}^{(a)} = r_t + \delta_r$.

This can be rewritten as

$$
\begin{aligned}
&\underbrace{\Big[\mathrm{E}\Big(y_{t+k}\,|\,s_t=\mathrm{P},(0,\bar{r}_t,p_t,x_t),\ldots\Big)-\mathrm{E}\Big(y_{t+k}\,|\,s_t=\mathrm{Z},(0,\bar{r}_t,p_t,x_t),\ldots\Big)\Big]}_{\text{pure regime-change effect}}\\
&-\underbrace{\Big[\mathrm{E}\Big(y_{t+k}\,|\,s_t=\mathrm{Z},(m_t,\bar{r}_t,p_t,x_t),\ldots\Big)-\mathrm{E}\Big(y_{t+k}\,|\,s_t=\mathrm{Z},(0,\bar{r}_t,p_t,x_t),\ldots\Big)\Big]}_{m\text{-IR}}.
\end{aligned}
\tag{5.5}
$$

Here, the second bracketed term is none other than the m-IR to an increase in the excess reserve rate from 0 to m_t. The first bracketed term is what should be called the *pure regime-change effect*.

Some Analytics on the Impact Effect

In general, the IR thus defined satisfies neither state independence nor the proportionality to the perturbation size. The exception is the impact effect on (p,x), namely the IR for $k=1$ (one period ahead). This is because (p_{t+1},x_{t+1}) depends linearly on $\mathbf{y}_t$ and the relevant state is the lagged state s_t. To provide analytical expression for the impact effect, write the reduced-form equations (with L lags) for period $t+1$ as

$$
\begin{bmatrix} p_{t+1} \\ x_{t+1} \end{bmatrix} = \underset{(2\times1)}{\mathbf{c}(s_t)} + \sum_{\ell=1}^{L} \underset{(2\times1)}{\boldsymbol{\phi}_p^{(\ell)}(s_t)} p_{t-\ell+1} + \sum_{\ell=1}^{L} \underset{(2\times1)}{\boldsymbol{\phi}_x^{(\ell)}(s_t)} x_{t-\ell+1} + \sum_{\ell=1}^{L} \underset{(2\times1)}{\boldsymbol{\phi}_r^{(\ell)}(s_t)} r_{t-\ell+1} + \sum_{\ell=1}^{L} \underset{(2\times1)}{\boldsymbol{\phi}_m^{(\ell)}(s_t)} m_{t-\ell+1} + \underset{(2\times1)}{\boldsymbol{\varepsilon}_{t+1}}. \tag{5.6}
$$

Clearly, for the m-IRs of (5.2) and r-IRs of (5.3),

$$
\begin{bmatrix} m\text{-IR of } p_{t+1} \\ m\text{-IR of } x_{t+1} \end{bmatrix} = \boldsymbol{\phi}_m^{(1)}(\mathrm{Z})\,\delta_m, \quad \begin{bmatrix} r\text{-IR of } p_{t+1} \\ r\text{-IR of } x_{t+1} \end{bmatrix} = \boldsymbol{\phi}_r^{(1)}(\mathrm{P})\,\delta_r. \tag{5.7}
$$

Regarding the pure regime-change component of the ZP-IR, since the only difference is in s_t and since the reduced-form coefficients for (p_{t+1},x_{t+1}) depend on s_t, the pure regime-change effect on impact is

$$
\begin{aligned}
\begin{bmatrix} \text{ZP-IR of } p_{t+1} \\ \text{ZP-IR of } x_{t+1} \end{bmatrix} = [\mathbf{c}(\mathrm{P})-\mathbf{c}(\mathrm{Z})] &+ \sum_{\ell=1}^{L}\Big[\boldsymbol{\phi}_p^{(\ell)}(\mathrm{P})-\boldsymbol{\phi}_p^{(\ell)}(\mathrm{Z})\Big]p_{t-\ell+1} + \sum_{\ell=1}^{L}\Big[\boldsymbol{\phi}_x^{(\ell)}(\mathrm{P})-\boldsymbol{\phi}_x^{(\ell)}(\mathrm{Z})\Big]x_{t-\ell+1} \\
&+ \sum_{\ell=1}^{L}\Big[\boldsymbol{\phi}_r^{(\ell)}(\mathrm{P})-\boldsymbol{\phi}_r^{(\ell)}(\mathrm{Z})\Big]r_{t-\ell+1} + \sum_{\ell=1}^{L}\Big[\boldsymbol{\phi}_m^{(\ell)}(\mathrm{P})-\boldsymbol{\phi}_m^{(\ell)}(\mathrm{Z})\Big]m_{t-\ell+1}.
\end{aligned}
\tag{5.8}
$$

For the case of $L=1$ (just one lag), Table 5 has the estimated reduced-form coefficients. The estimated $\boldsymbol{\phi}_m^{(1)}(\mathrm{Z})$ is $(0.0016, 0.0028)'$, so the impact effect of an increase in excess reserves (so $\delta_m>0$) is positive for both inflation (p) and output (x). The estimated $\boldsymbol{\phi}_r^{(1)}(\mathrm{P})$ is $(0.39,-0.10)'$, so the impact effect of a cut in the policy rate (with $\delta_r<0$) for inflation is to *lower* it. Turning to the impact effect of ZP-IR, the first term in brackets on the right-hand-side of (5.8), $\mathbf{c}(\mathrm{P})-\mathbf{c}(\mathrm{Z})$, is

$(-0.23 + 0.60, -0.00 + 0.23)' = (0.37, 0.23)'$. Thus the effect of terminating the zero-rate regime on impact can be positive for both inflation and output if the rest of the terms on the right-hand-side of (5.8) are small relative to $\mathbf{c}(\mathrm{P}) - \mathbf{c}(\mathrm{Z})$.

Estimated IRs

We now report our estimate of IRs. To keep the number of parameters at a minimum, we set the lag length L to one, so the estimated reduced-form equations are as in Table 5.

m-**IRs**

Before displaying the IRs to an increase in m under Z given by (5.2), we first show the conventional IRs that assume no regime change from Z. That is, the inflation and output dynamics are given by the reduced-form coefficients shown in Panel A of Table 5 for regime Z and m is given by the reserve supply equation $m_{t+k} = \max[m^e_{t+k} + v_{m,t+k}, 0]$ (whose parameter estimates are in Table 6) for all horizon k. Additional features are:

- The base period when the perturbation occurs is taken to be February 2004 (when the excess reserve rate m is largest), although the IR does not depend very much on the choice of the base period (it would not depend on the choice of base period at all if the reserve supply equation for $t + k$ were $m_{t+k} = m^e_{t+k} + v_{m,t+k}$).

- The perturbation size δ_m is chosen so that its ratio to the estimated standard deviation of v_{mt} equals the ratio of δ_r to the estimated standard deviation of v_{rt}. We will set $\delta_r = -1\%$ for the r-IRs below, so $\delta_m = 73.3\%$.

- We obtain the error band for the IR by a Monte Carlo method. That is, we first generate a sample (of size 100) of the parameter vector from the estimated asymptotic distribution.[12] For each draw of the parameter vector, we calculate the IR by the Monte Carlo integration described earlier. Finally, we pick the 84 and 16 percentiles for each horizon (so the coverage rate is 68%, corresponding to one-standard error bands).[13]

[12]Let $\mathrm{Avar}(\widehat{\boldsymbol{\theta}}_T)$ be the asymptotic variance of the estimator and let $\widehat{\mathrm{Avar}(\widehat{\boldsymbol{\theta}}_T)}$ its consistent estimator. Each draw is done by generating a random vector from $\mathcal{N}\left(\mathbf{0}, \frac{1}{T}\widehat{\mathrm{Avar}(\widehat{\boldsymbol{\theta}}_T)}\right)$ and adding the vector to $\widehat{\boldsymbol{\theta}}_T$.

[13]For a fraction of the parameter draws, the IRs didn't converge to zero as k increases. Let $IR(i, k)$ be the k-period ahead IR of variable i and let n be the IR horizon. For each i, define $v_{1i} \equiv \sum_{k=1}^{\ell}(IR(i, k))^2$ and $v_{2i} \equiv \sum_{k=\ell+1}^{n}(IR(i, k))^2$ where ℓ is the largest integer not exceeding $0.8n$. We exclude the IRs for which $\min v_{2i}/v_{1i} > 0.1$, before identifying the 84% and 16% percentiles. We set $n = 120$.

Figure 6a displays the m-IRs of (p, x, r, m) to an increase in m_t of 73.3 percentage points for t = February 2004 when m_t = 185%, corresponding to the actual reserve amount of 27.7 trillion yen (about \$280 billion), about 6.4 times the required amount of 4.4 trillion yen. Thus a perturbation of 73.3 percentage points implies increasing actual reserves from 27.7 trillion to 57.7 (= 27.7 × exp(0.733)) trillion yen. As shown of the panel for r of the figure, the m-IR of r (to m) is zero for all k because the regime is assumed to be Z for all future periods. The response of the output gap (x) builds on the impact effect, peaking at about 1.3% at k = 15 months. In contrast, for monthly inflation (p), because of the lack of persistence noted in reference to Table 5, the peak response is 0.13 (percent per year) at k = 3. These features are also found in Honda *et. al.* (2007, Figure 2).

The m-IRs shown in Figure 6b differ from those in Figure 6a in that the regime is endogenous. Because the regime can change from Z to P, the IR of the policy rate (r) gradually rises from 0. The average duration of the initial regime of Z is 25 months with the perturbation and 32 months without. The duration is shorter with the increase in m because the expansionary monetary policy raises future inflation and output, making the zero-rate regime less likely. This is why the m-IR turns negative after several months. The response of x is substantially weaker than when the regime is fixed at Z: the peak response of 0.8% occurs at k = 12. As implied by the analytics above, the impact effect on (p_{t+1}, x_{t+1}) is the same as in Figure 6a because the lagged regime at $t + 1$ is Z in all sample paths.

r-IRs

We now turn the r-IRs given by (5.3) for a 1 percentage point policy rate cuts (with $\delta_r = -1\%$). Figure 6c is for the base period of t = January 1993, when the actual policy rate was 3.6% and hence the 1 percentage point rate cut would not have hit the zero lower bound. The price puzzle emerges: the r-IR of p to the rate cut is negative. It is significantly negative (the error band does not include 0) for nearly 3 years. In contrast, the output effect of the rate cut is positive. The peak of 0.6% of the output response occurs in about 1 and half years, at k = 15 months. The r-IRs of m rises from 0 eventually (because the regime is endogenous), but not immediately. Because of the strength of the price puzzle, the average duration of the regime (P) in the base period is shorter with the rate cut (32 months) than without (38 months).

ZP-IRs

In the next figure, Figure 6d, the base period is t = February 2004, the same as in Figures 6a and 6b. The question we ask now is, what would have happened if the BOJ terminated the zero-rate regime at the peak of QE? As shown in the panel for m in Figure 6d, there is a precipitous decline in m in the base period from $m_t = 185\%$ to zero. The m-IR component of ZP-IR shown in (5.5) is large accordingly. The policy rate starts to rise immediately because the zero-regime has been terminated. Due to the depressing effect of the precipitous decline in m and also the rate increase, the output gap steadily declines, with a trough of -2.0% at $k = 15$. Except for the impact effect, the response of inflation (p) is negative, as might be expected.

To see to what extent the PZ-IR shown in Figure 6d is due to the deflationary effect of m-IR to the precipitous decline in m, we change the base period to June 2006, the last month of the second spell of the zero-rate regime. By then m_t declined from the peak in February 2004 of $m_t = 185\%$ (6.4 times excess reserves) to $m_t = 46\%$ (1.6 times excess reserves). The estiamted ZP-IRs are in Figure 6e. The panel for m reflects the much smaller decline from 46% to zero. As in Figure 6d, the policy rate rises. Very different are the output and inflation response. Instead of sinking eventually to about -2.0%, the output gap *rises* and remains in the positive territory for the first 19 months. The error band is wide enough to include the horizontal line, so the output effect is not significant though. The impact effect of inflation (at $k = 1$) is positive (at 0.34%), but unlike in Figure 6d, inflation ramains positive for most of the 60-month horizon. The deflationary effect on output and inflation (except for the impact effect) shown in Figure 6d is due to the m-IR to the precipitous decline in m.

6. Conclusion

We have constructed a regime-dependent SVAR model in which the regime is determined by the central bank responding to economic conditions. The model was used to study the dynamic effect of not only the policy rate changes but also changes in the reserve supply. It can further be used to study the effect of regime changes engineered by the central bank. Several conclusions emerge.

- We have estimated the Taylor rule for Japan on the sample period including the recent period of low interest rates. It indicates that the desired policy rate satisfies the Taylor principle, provided that the low equilibrium real interest rate during the (ongoing) lost decades is taken into account.
- The inflation and output dynamics in the lost decades (since 1992) seems different from previous periods, with far less persistence in the inflation rate.

- Our IR (impulse response) analysis indicates that a cut in the policy rate lowers inflation and raises output. Thus, consistent with the existing Japanese literature, the price puzzle is observed for Japan as well.

- An increase in the reserve supply under QE raises both inflation and output. The significance of this result relative to the existing Japanese literature is that this conclusion is obtained while the regime endogeneity is taken into account. The effect is substantially smaller if the regime endogeneity is taken into account.

- The effect of exiting from the zero-rate regime on inflation and output depends on the size of excess reserves at the time of regime change. If the exit took place in February 2004, when the ratio of excess to required reserves was 6.4, the output gap would have eventually declined by 2 percentage points, while the inflation rate would have declined only slightly. If the exit took place in June 2006, when the ratio is only 1.6, both inflation and output would have risen.

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Table 1: Bank of Japan's Statements of Exit Condition, 1999-2012

date	quotes and URLs
1999.4.13	"(The Bank of Japan will) continue to supply ample funds as long as deflationary concern remains." (A remark by governor Hayami in a Q & A session with the press. Translation by authors.) http://www.boj.or.jp/announcements/press/kaiken_1999/kk9904a.htm/
1999.9.21	"The Bank of Japan has been pursuing an unprecedented accommodative monetary policy and is explicitly committed to continue this policy until deflationary concerns subside. The Bank views the current state of the Japanese economy as having stopped deteriorating with some bright signs, though a clear and sustainable recovery of private demand has yet to be seen." http://www.boj.or.jp/announcements/release_1999/k990921a.htm/
2000.7.17	"...the majority of the Policy Board views that Japan's economy is coming to a stage where deflationary concerns are dispelled, which the Board have clearly stated as the condition for lifting the zero interest rate policy. At the Meeting, however, some views were expressed that before reaching a final decision to lift the zero interest rate policy, it was desirable to ensure the judgment on the firmness of economic conditions including employment and household income." http://www.boj.or.jp/announcements/release_2000/k000717b.htm/
2000.8.11	"At present, Japan's economy is showing clearer signs of recovery, and this gradual upturn, led mainly by business fixed investment, is likely to continue. Under such circumstances, the downward pressure on prices stemming from weak demand has markedly receded. Considering these developments, the Bank of Japan feels confident that Japan's economy has reached the stage where deflationary concern has been dispelled, the condition for lifting the zero interest rate policy." http://www.boj.or.jp/en/announcements/release_2000/k000811.htm/
2001.3.19	"The new procedures for money market operations continue to be in place until the consumer price index (excluding perishables, on a nationwide statistics) registers stably a zero percent or an increase year on year." http://www.boj.or.jp/en/announcements/release_2001/k010319a.htm/
2003.10.10	"The Bank of Japan is currently committed to maintaining the quantitative easing policy until the consumer price index (excluding fresh food, on a nationwide basis) registers stably a zero percent or an increase year on year. The Bank emphasizes that it is firmly committed to this policy at this juncture where there are signs of improvement in the Japanese economy." http://www.boj.or.jp/en/announcements/release_2003/k031010.htm/
2006.3.9	"Concerning prices, year-on-year changes in the consumer price index turned positive. Meanwhile, the output gap is gradually narrowing. Unit labor costs generally face weakening downward pressures as wages began to rise amid productivity gains. Furthermore, firms and households are shifting up their expectations for inflation. In this environment, year-on-year changes in the consumer price index are expected to remain positive. The Bank, therefore, judged that the conditions laid out in the commitment are fulfilled." http://www.boj.or.jp/en/announcements/release_2006/k060309.htm/
2009.12.18	"The Policy Board does not tolerate a year-on-year rate of change in the CPI equal to or below 0 percent." http://www.boj.or.jp/en/announcements/release_2009/un0912c.pdf
2010.10.5	"The Bank will maintain the virtually zero interest rate policy until it judges, on the basis of the "understanding of medium- to long-term price stability" that price stability is in sight, on condition that no problem will be identified in examining risk factors, including the accumulation of financial imbalances." http://www.boj.or.jp/en/announcements/release_2010/k101005.pdf
2012.2.14	"The Bank will continue pursuing the powerful easing until it judges that the 1 percent goal is in sight on the condition that the Bank does not identify any significant risk, including the accumulation of financial imbalances, from the view point of ensuring sustainable economic growth." http://www.boj.or.jp/en/announcements/release_2012/k120214a.pdf

Table 2: Transition Matrix

s_{t-1} \ s_t	P	Z
P	P_{rt}	$1 - P_{rt}$
Z	$P_{rt}P_{\pi t}$	$1 - P_{rt}P_{\pi t}$

Note: P_{rt} defined in (3.8). $P_{\pi t}$ defined in (3.9).

Table 3: Simple Statistics

sample period is Sept. 1985 - Dec. 2012						
	p (monthly inflation rate, % per year)	π (12-month inflation rate, %)	x (output gap, %	r (policy rate, % per year)	$r - \bar{r}$ (net policy rate)	m (excess reserve rate, %)
subsample P (sample size=198)						
mean	0.8%	0.9%	1.1%	2.93%	2.93%	1.5%
std. dev.	1.7%	1.0%	4.2%	2.55%	2.55%	3.3%
max	6.0%	3.2%	10.9%	8.26%	8.26%	20.6%
min	-3.9%	-0.9%	-8.0%	0.08%	0.08%	0.0%
subsample Z (sample size=130)						
mean	-0.5%	-0.5%	-1.8%	0.04%	0.00%	105.3%
std. dev.	1.4%	0.5%	6.1%	0.04%	0.02%	61.7%
max	4.8%	0.2%	6.8%	0.14%	0.04%	184.9%
min	-4.8%	-1.7%	-29.5%	0.00%	-0.04%	4.1%

Table 4: Taylor Rule Estimates

Line	first month, last month	sample size	equilibrium real rate includes trend growth?	estimation method	inflation coefficient	output coefficient	speed of adjustment (% per month)	std. dev. of error (σ_r, % per year)	mean of target inflation ($\overline{\pi}$, % per year)	std. dev. of target inflation ($\sigma_{\overline{\pi}}$, % per year)
1	198509, 201212	293	yes	ML	1.68 [5.8]	0.23 [3.0]	6.0% [4.5]	0.22% [18]	0.21% [1.6]	0.15% [1.7]
2	197811, 201212	375	yes	ML	0.55 [2.8]	0.37 [3.3]	4.9% [4.7]	0.26% [22]	0.22% [1.6]	0.15% [1.7]
3	198005, 201212	357	yes	ML	−0.47 [−0.7]	0.48 [2.0]	2.6% [2.3]	0.24% [21]	0.20% [1.6]	0.15% [1.7]
4	198301, 201212	325	yes	ML	1.48 [4.9]	0.26 [3.0]	5.0% [4.4]	0.22% [20]	0.21% [1.5]	0.15% [1.7]
5	199201, 201212	217	yes	ML	1.07 [2.8]	0.11 [1.6]	5.4% [2.6]	0.13% [13]	0.24% [1.7]	0.16% [1.7]
6	198509, 201212	293	yes	Tobit	3.12 [6.3]	0.32 [3.0]	4.3% [4.0]	0.22% [18]		
7	197811, 199508	202	yes	OLS	0.34 [1.1]	0.69 [3.0]	5.2% [3.9]	0.33% [20]		
8	197811, 199508	202	no	OLS	0.35 [0.8]	0.95 [2.2]	4.3% [2.1]	0.34% [20]		

Note: *t*-values in brackets. The Taylor rule estimated here is displayed in (3.1). The inflation and output coefficients are the first and second element ofs $\underset{(2\times 1)}{\boldsymbol{\beta}_r^*}$. The speed of adjustment is ρ_r in the Taylor rule.

Table 5: Inflation and Output Reduced Form

Panel A: sample period is January 1992 - December 2012

subsample P (set of t's such that s_{t-1} = P, sample size = 123)

dep. var.	const.	$p(-1)$	$x(-1)$	$r(-1)$	$m(-1)$	R^2	SER
p	−0.23	0.12	0.021	0.39		0.16	1.43
	[−1.1]	[1.3]	[0.7]	[3.6]			
x	−0.00	0.05	0.95	−0.10		0.87	1.73
	[−0.0]	[0.4]	[24]	[−0.8]			

subsample Z (set of t's such that s_{t-1} = Z, sample size = 129)

dep. var.	const.	$p(-1)$	$x(-1)$	$r(-1)$	$m(-1)$	R^2	SER
p	−0.60	0.08	0.02		0.0016	0.03	1.37
	[−2.3]	[0.8]	[1.2]		[0.8]		
x	−0.23	0.16	0.90		0.0028	0.84	2.42
	[−0.5]	[1.0]	[25]		[0.8]		

Panel B: various sample periods, all under regime P

March 1970 - December 1991 (sample size = 262)

dep. var.	const.	$p(-1)$	$x(-1)$	$r(-1)$	R^2	SER
p	−0.67	0.62	0.16	0.37	0.57	3.95
	[−0.9]	[12]	[2.7]	[3.0]		
x	1.03	0.02	0.95	−0.16	0.91	1.28
	[4.1]	[1.3]	[49]	[−4.1]		

January 1980 - December 1991 (sample size = 144)

dep. var.	const.	$p(-1)$	$x(-1)$	$r(-1)$	R^2	SER
p	−1.6	0.33	0.07	0.47	0.40	2.29
	[−2.1]	[4.0]	[1.0]	[3.7]		
x	0.30	0.04	0.91	−0.06	0.83	1.32
	[0.7]	[0.9]	[23]	[−0.7]		

Note: t values in brackets. p is the monthly inflation rate stated at annual rates, x is the output gap in percents. "SER" is the standard error of the regression, defined as the square root of the sum of squares divided by the sample size.

Table 6: Reserve Supply Equation

subsample Z (sample size=130)			
inflation coefficient	output coefficient	speed of adjut-ment (% per month)	std. of dev. of error
−117 [−0.8]	−11 [−0.9]	3.4% [1.4]	16.3% [260]

Note: *t* values in brackets. The intercept is not reported. The estimated equation is $m_t = \max\left[m_t^e + v_{mt}, 0\right]$, $v_{mt} \sim \mathcal{N}(0, \sigma_m^2)$ where

$$m_t^e \equiv \rho_m \left(\alpha_m^* + \boldsymbol{\beta}_m^{*\prime} \begin{bmatrix} \pi_t \\ x_t \end{bmatrix}\right) + (1 - \rho_m) m_{t-1}. \tag{6.1}$$

The estimation method is Tobit. However, since $m_t > 0$ for all months, Tobit reduces to OLS.

Appendix 1: Data Description

This appendix describes how the variables used in the paper — p (monthly inflation), π (twelve-month inflation), x (output gap), r (the policy rate), $\bar{r}$ (the interest rate paid on reserves), m (the excess reserve rate), and the trend growth rate — are derived from various data sources.

Monthly and Twelve-Month Inflation Rates (p and π)

The monthly series on the monthly inflation rate (appearing in the inflation and output reduced-form equations) and the twelve-month inflation rate (in the Taylor rule) are constructed from the CPI (consumer price index). The Japanese CPI is compiled by the Ministry of Internal Affairs and Communications of the Japanese government. The overall CPI and its various subindexes can be downloaded from the portal site of official statistics of Japan called"e-Stat". The URL for the CPI is
`http://www.e-stat.go.jp/SG1/estat/List.do?bid=000001033702&cycode=0`.
The CSV file listed first on this page has the CPI excluding fresh food from January 1970. This is a seasonally *un*adjusted series and combines different base years from January 1970. For how the Ministry combines different base years, see Section III-6 of the document (in Japanese) downloadable from
`http://www.stat.go.jp/data/cpi/2010/kaisetsu/index.htm#p3`
Briefly, to combine base years of 2005 and 2010, say, the Ministry multiplies one of the series by a factor called the "link factor" whose value is such that the two series agree on the average of monthly values for the year 2005.

The above URL also provides seasonally adjusted series for various subindexes, but they are available only for periods from January 2005. As explained below, we use the CPI excluding food and energy from January 2005 that is seasonally adjusted, along with the seasonally unadjusted index excluding fresh food, in order to construct p (monthly inflation) and π (12-month inflation). The construction involves three steps. .

Adjustment for Consumption Tax Hikes. The consumption tax rate rose from 0% from 3% in April 1989 and to 5% in April 1997. We compute the 12-month inflation rate from the seasonally unadjusted index (as the log difference between the current value of the index and the value 12 months ago) and subtract 1.2% for t = April 1989,..., March 1990 (to remove the effect of the April 1989 tax hike) and 1.5% for t = April 1997,..., March 1998 (to remove the effect of the April 1997 tax hike). We then calculate the index such that its implied 12-month inflation agrees with the tax-adjusted 12-month inflation.

Seasonal Adjustment. We apply Kitagawa and Gersch's (1984) seasonal adjustment method that uses a state-space model, known as Decomp, via its online system (`http://ssnt.ism.ac.jp/inets2/title.html`). We apply this method on the seasonally unadjusted (but tax-adjusted) index excluding fresh food from January 1970 through

December 2012 (42 years). We do not use the U.S. Census's X12-ARIMA for seasonal adjustment because it provides a poor fit in earlier periods of the 42 years.

Adjustment for the 2007-2008 Energy Prices. Let CPI_{1t} be the seasonally adjusted CPI excluding fresh food obtained from this operation for t = January 1970,..., December 2012. Let CPI_{2t} be the seasonally adjusted CPI excluding food and energy for t = January 2005,..., December 2012 that is directly available from the above official URL. Our CPI measure (call it CPI) is CPI_1, except that we switch from CPI_1 to CPI_2 in October 2007 to remove the large movement in the energy component of the CPI in 2007 and 2008. More precisely,

$$CPI_t = \begin{cases} CPI_{1t} & \text{for } t = \text{January 1970}, \ldots, \text{October 2007}, \\ CPI_{2t} \times \dfrac{CPI_{1,\text{October 2007}}}{CPI_{2,\text{October 2007}}} & \text{for } t = \text{November 2007}, \ldots, \text{December 2012}. \end{cases} \tag{A1.1}$$

The monthly inflation rate for month t, p_t, is calculated as $p_t \equiv 1200 \times [\log(CPI_t) - \log(CPI_{t-1})]$. The 12-month inflation rate for month t, π_t, is $\pi_t \equiv 100 \times [\log(CPI_t) - \log(CPI_{t-12})]$.

Excess Reserve Rate (m)

Monthly series on actual and required reserves from September 1959 on are available from the BOJ's portal site `http://www.stat-search.boj.or.jp/index_en.html/`. The value for month t is the average of daily balances over the reserve maintenance period of the 16th day of month t to the 15th day of month $t+1$. The excess reserve rate for month t (m_t) is defined as

$$m_t \equiv 100 \times [\log(\text{actual reserve balance for month } t) - \log(\text{required reserve balance for month } t)].$$

The Policy Rate (r)

The monthly time series on the policy rate from January 1970 is a concatenation of three series.

August 1985 - December 2012. We obtained daily data on the uncollateralized overnight "Call" rate (the Japanese equivalent of the U.S. Federal Funds rate) since immediately after the inception of the market (which is July 1985) from *Nikkei* (a data vendor maintained by a subsidiary of *Nihon Keizai Shinbun* (the Japan Economic Daily)). The policy rate for month t, r_t, for t = August 1985,...,December 2012 is the average of the daily values over the reserve maintenance period of the 16th of month t to the 15th of month $t+1$.

October 1978 - July 1985. Daily data on the collateralized overnight "Call" rate from October 1978 are available from *Nikkei*. The policy rate for month t, r_t, for t = October 1978,..., July 1985 is the average of the daily values over the reserve maintenance period of the 16th of month t to the 15th of month $t+1$ plus a risk premium of 7.5 basis points. The risk premium estimate of 7.5 basis points is the difference in the August 1985 reserve maintenance period average between the uncollateralized call rate (6.305%) and the collateralized call rate (6.230%).

January 1970 - September 1978. Monthly averages (over calendar months, not over reserve maintenance periods) of the collateralized rate are available from the above BOJ portal from January 1960. The policy rate for month t in this period of January 1970 - September 1978 is this monthly average for month t plus the risk premium of 7.5 basis points.

Interest Rate paid on Reserves ($\bar{r}$)

$\bar{r}_t$ is 0% before October 2008 and 0.1% since October 2008.

Output Gap (x)

Data on the Index of Industrial Production (IP), compiled by the Ministry of Economy, Trade and Industry of the Japanese government (METI), are available from METI's website. The IP series for the base year of 2005 from January 2003 on can be found in
`http://www.meti.go.jp/statistics/tyo/iip/result-2.html#menu2`.
The IP series that combines different base years (1980, 1985, 1990, 1995, 2000, and 2005) is available from the same URL for January 1978 - December 2007. To link the 2005 series with the 2000 series for example, the two series are adjusted so that they agree on the average of monthly values for January-March of 2003, not for January-December of 2005 as would be the case for the CPI.

The two IP series, one for the base year of 2005 and the other that combines various base years, have the same values for the common period of January 2003 - December 2007, because for both series the value is normalized to 100 for 2005. So we simply concatenate the two series to create the IP series for January 1978 - December 2012. For the earlier period from January 1970, we obtained a monthly IP series from *Datastream* for January 1970 - December 2012. Its values are identical to the METI series for the common period of January 1978 - December 2012. So we decided to use the Datastream IP series for the whole period of January 1970 - December 2012. All these series are seasonally adjusted, so there is no need for us to perform seasonal adjustment of our own. For an official description of the IP index, go to
`http://www.meti.go.jp/statistics/tyo/iip/result/pdf/ha23000j.pdf`. See Section 9 of this document for seasonal adjustment and Section 10 for how to combine the IP series with different base years. The seasonal adjustment uses X12-ARIMA and includes adjustment for holidays and leap years.

Let IP_t be the month t value of the IP series thus obtained. We applied the HP (Hodrick-Prescott) filter to $\log(IP_t)$ (t = January 1970,..., December 2012) to obtain the log output trend (call it $\log(IP_t^*)$) for t = January 1970,..., December 2012. The smoothness parameter for the HP filter is 1600×3^4, which is the value recommended by Ravin and Uhlig (2002) for monthly series. The output gap for month t, x_t, is defined as $x_t \equiv 100 \times [\log(IP_t) - \log(IP_t^*)]$.

Trend Growth Rate

An estimate of the trend growth rate is used as the equilibrium real interest rate in our Taylor rule estimation. Our estimate of the trend growth rate for month t is defined as
$100 \times [\log(IP_t^*) - \log(IP_{t-12}^*)]$.

Appendix 2: The Model and Derivation of the Likelihood Function

The Model

The state vector of the model consists of a vector of continuous state variables $\mathbf{y}_t$ and a discrete state variable s_t (= P, Z). The continuous state $\mathbf{y}_t$ has the following elements:

$$\underset{(4\times1)}{\mathbf{y}_t} \equiv \begin{bmatrix} \underset{(2\times1)}{\mathbf{y}_{1t}} \\ r_t \\ m_t \end{bmatrix}, \quad \underset{(2\times1)}{\mathbf{y}_{1t}} \equiv \begin{bmatrix} p_t \\ x_t \end{bmatrix}, \tag{A2.1}$$

where p = monthly inflation rate, x = output gap, r = policy rate, and m = excess reserve rate.

The model is a mapping from $(s_{t-1}, \mathbf{y}_{t-1}, ..., \mathbf{y}_{t-11})$ to $(s_t, \mathbf{y}_t)$. (We need to include 11 lags of $\mathbf{y}$ because of the appearance of the 12-month inflation rate in the model, see (A2.3) below.) The mapping depends on: (i) an exogenous sequence $\{\overline{r}_t\}$ (the interest rate paid on reserves) and the trend growth rate (appearing in the Taylor rule), (ii) the model parameters listed in (A2.7) below, and (iii) a shock vector $(\boldsymbol{\varepsilon}_t, v_{rt}, v_{\overline{\pi}t}, v_{mt})$ (to be defined below) that are mutually and serially independent. The mapping itself can be described recursively as follows.

(a) ($\mathbf{y}_{1t}$ determined) $\boldsymbol{\varepsilon}_t$ is drawn from $\mathcal{N}(\mathbf{0}, \boldsymbol{\Omega}(s_{t-1}))$ and $\mathbf{y}_{1t}$ (the first two elements of $\mathbf{y}_t$) is given by

$$\underset{(2\times1)}{\mathbf{y}_{1t}} = \underset{(2\times1)}{\mathbf{c}(s_{t-1})} + \underset{(2\times4)}{\boldsymbol{\Phi}(s_{t-1})}\underset{(4\times1)}{\mathbf{y}_{t-1}} + \underset{(2\times1)}{\boldsymbol{\varepsilon}_t}, \quad \mathrm{Var}(\boldsymbol{\varepsilon}_t) = \boldsymbol{\Omega}(s_{t-1}) \tag{A2.2}$$

Here, only one lag is allowed, strictly for expositional purposes; more lags can be included without any technical difficulties.

(b) (s_t determined) Given $\mathbf{y}_{1t}$ and $(\mathbf{y}_{t-1}, ..., \mathbf{y}_{t-11})$, the central bank calculates (through $(p_t, ..., p_{t-11}, x_t, r_{t-1})$)

$$\pi_t \equiv \frac{1}{12}(p_t + \cdots + p_{t-11}), \quad r_t^e \equiv \alpha_r + \boldsymbol{\beta}_r' \begin{bmatrix} \pi_t \\ x_t \end{bmatrix} + \gamma_r r_{t-1}. \tag{A2.3}$$

The central bank draws $(v_{rt}, v_{\overline{\pi}t})$ from $\mathcal{N}(\mathbf{0}, \begin{bmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_{\overline{\pi}}^2 \end{bmatrix})$ (by the central bank) and s_t is determined as

$$\text{If } s_{t-1} = \text{P}, \quad s_t = \begin{cases} \text{P} & \text{if } r_t^e + v_{rt} \geq \overline{r}_t, \\ \text{Z} & \text{otherwise.} \end{cases} \tag{A2.4a}$$

$$\text{If } s_{t-1} = \text{Z}, \quad s_t = \begin{cases} \text{P} & \text{if } r_t^e + v_{rt} \geq \overline{r}_t \text{ and } \pi_t \geq \overline{\pi} + v_{\overline{\pi}t}, \\ \text{Z} & \text{otherwise.} \end{cases} \tag{A2.4b}$$

(c) (r_t determined) Given s_t, r_t is determined as

$$\text{If } s_t = \text{P, then } r_t = r_t^e + v_{rt}. \tag{A2.5a}$$

$$\text{If } s_t = \text{Z, then } r_t = \overline{r}_t. \tag{A2.5b}$$

Note that r_t in (A2.5a) is guaranteed to be $\geq \bar{r}_t$ because by (A2.4a) and (A2.4b) $r_t^e + v_{rt} \geq \bar{r}_t$ if $s_t = \text{P}$.

(d) (m_t determined) Finally, the central bank draws v_{mt} from $\mathcal{N}(0, \sigma_m^2)$ (by the central bank) and m_t is determined as

$$\text{If } s_t = \text{P}, \text{ then } m_t = 0. \tag{A2.6a}$$

$$\text{If } s_t = \text{Z}, \text{ then } m_t = \max\left[m_t^e + v_{mt}, 0\right]. \tag{A2.6b}$$

Here,

$$m_t^e \equiv \alpha_m + \boldsymbol{\beta}_m' \begin{bmatrix} \pi_t \\ x_t \end{bmatrix} + \gamma_m m_{t-1}.$$

Note that m_t^e can be calculated from $(\mathbf{y}_{t-1}, ..., \mathbf{y}_{t-11})$ and $\mathbf{y}_{1t}$ through $(p_t, ..., p_{t-11}, x_t, m_{t-1})$.

Let $\boldsymbol{\theta}$ be the model's parameter vector. It will turn out to be useful to divide into 3 groups:

$$\begin{cases} \boldsymbol{\theta}_A &= \left(\underset{(2\times 1)}{\mathbf{c}(s)}, \underset{(2\times 4)}{\boldsymbol{\Phi}(s)}, \underset{(2\times 2)}{\boldsymbol{\Omega}(s)}, s = \text{P}, \text{Z} \right) \quad (13 \times 2 = 26 \text{ parameters}), \\ \boldsymbol{\theta}_B &= \left(\alpha_r, \underset{(2\times 1)}{\boldsymbol{\beta}_r}, \gamma_r, \sigma_r, \overline{\pi}, \sigma_{\overline{\pi}} \right) \quad (7 \text{ parameters}), \\ \boldsymbol{\theta}_C &= \left(\alpha_m, \underset{(2\times 1)}{\boldsymbol{\beta}_m}, \gamma_m, \sigma_m \right) \quad (5 \text{ parameters}). \end{cases} \tag{A2.7}$$

There is a one-to-one mapping between the $(\boldsymbol{\theta}_B, \boldsymbol{\theta}_C)$ in the text and the $(\boldsymbol{\theta}_B, \boldsymbol{\theta}_C)$ here. The mapping is given by

$$\rho_r = 1 - \gamma_r,\ \alpha_r^* = \alpha_r/\rho_r,\ \boldsymbol{\beta}_r^* = \boldsymbol{\beta}_r/\rho_r,\ \ \rho_m = 1 - \gamma_m,\ \alpha_m^* = \alpha_m/\rho_m,\ \boldsymbol{\beta}_m^* = \boldsymbol{\beta}_m/\rho_m. \tag{A2.8}$$

Derivation of the Likelihood Function

With the mapping from $(s_{t-1}, \mathbf{y}_{t-1}, ..., \mathbf{y}_{t-11})$ to $(s_t, \mathbf{y}_t)$ in hand, we proceed to derive the likelihood function. The likelihood of the data is

$$\mathcal{L} \equiv p\left(s_1, ...,\ s_T, \mathbf{y}_1, ..., \mathbf{y}_T \,|\, s_0, \mathbf{y}_0, \mathbf{y}_{-1}, ..., \mathbf{y}_{-10}\right). \tag{A2.9}$$

Here, $p(.)$ is the joint density-distribution function of $(s_1, ..., s_T)$ and $(\mathbf{y}_1, ..., \mathbf{y}_T)$ conditional on $(s_0, \mathbf{y}_0, \mathbf{y}_{-1}, ..., \mathbf{y}_{-10})$. Since the distribution of $\{s_t, \mathbf{y}_t\}$ depends on the history up to $t-1$ only through $(s_{t-1}, \mathbf{y}_{t-1}, ..., \mathbf{y}_{t-11})$, the usual sequential factorization yields

$$\mathcal{L} = \prod_{t=1}^{T} p\left(s_t, \mathbf{y}_t \,|\, s_{t-1}, \mathbf{y}_{t-1}, ..., \mathbf{y}_{t-10}\right), \ \text{ where } \mathbf{x}_{t-1} \equiv \left(\mathbf{y}_{t-1}, ..., \mathbf{y}_{t-11}\right). \tag{A2.10}$$

The likelihood for period t, $p\left(s_t, \mathbf{y}_t \,|\, s_{t-1}, \mathbf{x}_{t-1}\right)$, can be rewritten as (recall: $\mathbf{y}_t = \left(\mathbf{y}_{1t}', r_t, m_t\right)'$)

$$p\left(s_t, \mathbf{y}_t \,|\, s_{t-1}, \mathbf{x}_{t-1}\right) = f\left(m_t, r_t \,|\, s_t, \mathbf{y}_{1t}, s_{t-1}, \mathbf{x}_{t-1}\right) \times \text{Prob}\left(s_t \,|\, \mathbf{y}_{1t}, s_{t-1}, \mathbf{x}_{t-1}\right) \times f\left(\mathbf{y}_{1t} \,|\, s_{t-1}, \mathbf{x}_{t-1}\right). \tag{A2.11}$$

In what follows, we rewrite each of the three terms on the right hand side of this equation in terms of the model parameters.

The Third Term, $f(\mathbf{y}_{1t} \,|\, s_{t-1}, \mathbf{x}_{t-1})$

This term is entirely standard:

$$f(\mathbf{y}_{1t} \,|\, s_{t-1}, \mathbf{x}_{t-1}) = b\Big(\mathbf{y}_{1t} - \big(\mathbf{c}(s_{t-1}) + \mathbf{\Phi}(s_{t-1})\mathbf{y}_{t-1}\big); \mathbf{\Omega}(s_{t-1})\Big), \tag{A2.12}$$

where $b(.;\mathbf{\Omega})$ is the density of the bivariate normal with mean $\underset{(2\times 1)}{\mathbf{0}}$ and variance-covariance matrix $\underset{(2\times 2)}{\mathbf{\Omega}}$.

The Second Term, $\text{Prob}\,(s_t \,|\, \mathbf{y}_{1t}, s_{t-1}, \mathbf{x}_{t-1})$

This is the transition probability matrix for $\{s_t\}$. The probabilities depend on (r_t^e, π_t) (which in term can be calculated from $(\mathbf{y}_{1t}, \mathbf{x}_{t-1})$). They are easy to derive and are displayed in the text. To reproduce,

s_{t-1} \ s_t	P	Z
P	P_{rt}	$1 - P_{rt}$
Z	$P_{rt}P_{\pi t}$	$1 - P_{rt}P_{\pi t}$

Here,

$$P_{rt} \equiv \text{Prob}\,(r_t^e + v_{rt} \geq \bar{r}_t \,|\, r_t^e) = \Phi\left(\frac{r_t^e - \bar{r}_t}{\sigma_r}\right), \tag{A2.13}$$

$$P_{\pi t} \equiv \text{Prob}\,(\pi_t \geq \overline{\pi} + v_{\overline{\pi} t} \,|\, \pi_t) = \Phi\left(\frac{\pi_t - \overline{\pi}}{\sigma_{\overline{\pi}}}\right), \tag{A2.14}$$

where $\Phi(.)$ is the cdf of $\mathcal{N}(0,1)$.

The First Term, $f(m_t, r_t \,|\, s_t, \mathbf{y}_{1t}, s_{t-1}, \mathbf{x}_{t-1})$

- Case: $s_t = \text{P}$. Since $m_t = 0$ with probability 1 by (A2.6a), we have $f(m_t, r_t \,|\, s_t = \text{P}, \mathbf{y}_{1t}, s_{t-1}, \mathbf{x}_{t-1}) = f(r_t \,|\, s_t = \text{P}, \mathbf{y}_{1t}, s_{t-1}, \mathbf{x}_{t-1})$. We can rewrite $f(r_t \,|\, s_t = \text{P}, \mathbf{y}_{1t}, s_{t-1}, \mathbf{x}_{t-1})$ as follows.
  - For $s_{t-1} = \text{P}$,

$$\begin{aligned}
&f(r_t \,|\, s_t = \text{P}, \mathbf{y}_{1t}, s_{t-1} = \text{P}, \mathbf{x}_{t-1}) \\
&\quad = f(r_t^e + v_{rt} \,|\, r_t^e + v_{rt} \geq \bar{r}_t, r_t^e) \quad \text{(by (A2.4a) and (A2.5a))} \\
&\quad = \frac{f\left(r_t^e + v_{rt} \,|\, r_t^e\right)}{\text{Prob}\left(r_t^e + v_{rt} \geq \bar{r}_t \,|\, r_t^e\right)} \quad \text{(see, e.g., Hayashi, p. 512)} \\
&\quad = \frac{\frac{1}{\sigma_r}\phi\left(\frac{v_{rt}}{\sigma_r}\right)}{\text{Prob}\left(r_t^e + v_{rt} \geq \bar{r}_t \,|\, r_t^e\right)} \quad \text{(b/c } r_t^e + v_{rt} \sim \mathcal{N}\left(r_t^e, \sigma_r^2\right)) \\
&\quad = \frac{\frac{1}{\sigma_r}\phi\left(\frac{r_t - r_t^e}{\sigma_r}\right)}{P_{rt}} \quad \text{(with } P_{rt} \equiv \text{Prob}\,(r_t^e + v_{rt} \geq \bar{r}_t | r_t^e))
\end{aligned} \tag{A2.15}$$

– For $s_{t-1} = \mathrm{Z}$,

$$
\begin{aligned}
& f(r_t \,|\, s_t = \mathrm{P}, \mathbf{y}_{1t}, s_{t-1} = \mathrm{Z}, \mathbf{x}_{t-1}) \\
& \quad = f\left(r_t^e + v_{rt} \,|\, r_t^e + v_{rt} \geq \overline{r}_t, \pi_t \geq \overline{\pi} + v_{\overline{\pi}t}, r_t^e, \pi_t\right) \quad \text{(by (A2.4b) and (A2.5a))} \\
& \quad = f\left(r_t^e + v_{rt} \,|\, r_t^e + v_{rt} \geq \overline{r}_t, r_t^e\right) \quad \text{(b/c } v_{rt} \text{ and } v_{\overline{\pi}t} \text{ are independent)} \\
& \quad = \frac{\frac{1}{\sigma_r}\phi\left(\frac{r_t - r_t^e}{\sigma_r}\right)}{P_{rt}} \qquad \text{(as above).} \qquad \text{(A2.16)}
\end{aligned}
$$

- Case: $s_t = \mathrm{Z}$. Since $r_t = \overline{r}_t$ with probability 1 by (A2.5b), we have $f(m_t, r_t \,|\, s_t = \mathrm{Z}, \mathbf{y}_{1t}, s_{t-1}, \mathbf{x}_{t-1}) = f(m_t \,|\, s_t = \mathrm{Z}, \mathbf{y}_{1t}, s_{t-1}, \mathbf{x}_{t-1})$. By (A2.6b), $m_t = \max\left[m_t^e + v_{mt}, 0\right]$. So $f(m_t \,|\, s_t = \mathrm{Z}, \mathbf{y}_{1t}, s_{t-1}, \mathbf{x}_{t-1})$ is the standard Tobit likelihood:

$$
f(m_t \,|\, s_t = \mathrm{Z}, \mathbf{y}_{1t}, s_{t-1}, \mathbf{x}_{t-1}) = \left[\frac{1}{\sigma_m}\phi\left(\frac{m_t - m_t^e}{\sigma_m}\right)\right]^{1(m_t>0)} \times \left[1 - \Phi\left(\frac{m_t^e}{\sigma_m}\right)\right]^{1(m_t=0)}, \qquad \text{(A2.17)}
$$

where 1(.) is the indicator function, $\phi(.)$ and $\Phi(.)$ are the density and the cdf of $\mathcal{N}(0,1)$.

Putting All Pieces Together

Putting all those pieces together, the likelihood for date t, $p(s_t, \mathbf{y}_t \,|\, s_{t-1}, \mathbf{x}_{t-1}) = f(m_t, r_t \,|\, s_t, \mathbf{y}_{1t}, s_{t-1}, \mathbf{x}_{t-1}) \times \mathrm{Prob}(s_t \,|\, \mathbf{y}_{1t}, s_{t-1}, \mathbf{x}_{t-1}) \times f(\mathbf{y}_{1t} \,|\, s_{t-1}, \mathbf{x}_{t-1})$, can be written as

$s_t \vert s_{t-1}$	$f(m_t, r_t \vert s_t, \mathbf{y}_{1t}, s_{t-1}, \mathbf{x}_{t-1})$	$\mathrm{Prob}(s_t \vert \mathbf{y}_{1t}, s_{t-1}, \mathbf{x}_{t-1})$	$f(\mathbf{y}_{1t} \vert s_{t-1}, \mathbf{x}_{t-1})$
P|P	$\frac{g_t}{P_{rt}}$	P_{rt}	$b(\mathbf{y}_{1t} - \mathbf{c}(\mathrm{P}) - \mathbf{\Phi}(\mathrm{P})\mathbf{y}_{t-1}; \mathbf{\Omega}(\mathrm{P}))$
P|Z	$\frac{g_t}{P_{rt}}$	$P_{rt}P_{\pi t}$	$b(\mathbf{y}_{1t} - \mathbf{c}(\mathrm{Z}) - \mathbf{\Phi}(\mathrm{Z})\mathbf{y}_{t-1}; \mathbf{\Omega}(\mathrm{Z}))$
Z|P	h_t	$1 - P_{rt}$	$b(\mathbf{y}_{1t} - \mathbf{c}(\mathrm{P}) - \mathbf{\Phi}(\mathrm{P})\mathbf{y}_{t-1}; \mathbf{\Omega}(\mathrm{P}))$
Z|Z	h_t	$1 - P_{rt}P_{\pi t}$	$b(\mathbf{y}_{1t} - \mathbf{c}(\mathrm{Z}) - \mathbf{\Phi}(\mathrm{Z})\mathbf{y}_{t-1}; \mathbf{\Omega}(\mathrm{Z}))$

Here,

$$
g_t \equiv \frac{1}{\sigma_r}\phi\left(\frac{r_t - r_t^e}{\sigma_r}\right), \;\; P_{rt} \equiv \Phi\left(\frac{r_t^e - \overline{r}_t}{\sigma_r}\right), \;\; P_{\pi t} \equiv \Phi\left(\frac{\pi_t - \overline{\pi}}{\sigma_{\overline{\pi}}}\right),
$$
$$
h_t \equiv \left[\frac{1}{\sigma_m}\phi\left(\frac{m_t - m_t^e}{\sigma_m}\right)\right]^{1(m_t>0)} \times \left[1 - \Phi\left(\frac{m_t^e}{\sigma_m}\right)\right]^{1(m_t=0)},
$$

and (recall) $b(.;\mathbf{\Omega})$ is the density function of the bivariate normal distribution with mean $\underset{(2\times 1)}{\mathbf{0}}$ and variance-covariance matrix $\underset{(2\times 2)}{\mathbf{\Omega}}$.

Dividing it into Pieces

Taking the log of both sides of (A2.10) with (A2.11), we obtain the log likelihood of the sample:

$$L \equiv \log(\mathcal{L}) = \sum_{t=1}^{T} \log\left[p\left(s_t, \mathbf{y}_t \,|\, s_{t-1}, \mathbf{x}_{t-1}\right)\right]$$
$$= \underbrace{\sum_{t=1}^{T} \log\left[f\left(\mathbf{y}_{1t} \,|\, s_{t-1}, \mathbf{x}_{t-1}\right)\right]}_{=L_A} + \underbrace{\sum_{t=1}^{T} \log\left[\mathrm{Prob}\left(s_t \,|\, \mathbf{y}_{1t}, s_{t-1}, \mathbf{x}_{t-1}\right)\right]}_{=L_1} + \underbrace{\sum_{t=1}^{T} \log\left[f\left(m_t, r_t \,|\, s_t, \mathbf{y}_{1t}, s_{t-1}, \mathbf{x}_{t-1}\right)\right]}_{=L_2}. \tag{A2.18}$$

Taking into account the structure shown in the above table, we can rewrite (L_A, L_1, L_2) of the log likelihood L as

$$L_A = \sum_{t=1}^{T} \log\left[b\left(\mathbf{y}_{1t} - \mathbf{c}(s_{t-1}) - \mathbf{\Phi}(s_{t-1})\mathbf{y}_{t-1}; \mathbf{\Omega}(s_{t-1})\right)\right], \tag{A2.19}$$

$$L_1 = \sum_{s_t=\mathrm{P}} \log\left[P_{rt}\right] + \sum_{s_t\,|\,s_{t-1}=\mathrm{P}\,|\,\mathrm{Z}} \log\left[P_{\pi t}\right] + \sum_{s_t\,|\,s_{t-1}=\mathrm{Z}\,|\,\mathrm{P}} \log\left[1 - P_{rt}\right] + \sum_{s_t\,|\,s_{t-1}=\mathrm{Z}\,|\,\mathrm{Z}} \log\left[1 - P_{rt}P_{\pi t}\right], \tag{A2.20}$$

$$L_2 = \sum_{s_t=\mathrm{P}} \left[\log\left(g_t\right) - \log\left(P_{rt}\right)\right] + \sum_{s_t=\mathrm{Z}} \log\left[h_t\right]. \tag{A2.21}$$

The terms in $L_1 + L_2$ can be regrouped into L_B and L_C, as in

$$L = L_A + \underbrace{L_B + L_C}_{=L_1+L_2}, \tag{A2.22}$$

where L_A is as defined above in (A2.19), and

$$L_B = \sum_{s_t=\mathrm{P}} \log\left[g_t\right] + \sum_{s_t\,|\,s_{t-1}=\mathrm{Z}\,|\,\mathrm{P}} \log\left[1 - P_{rt}\right] + \sum_{s_t\,|\,s_{t-1}=\mathrm{P}\,|\,\mathrm{Z}} \log\left[P_{\pi t}\right] + \sum_{s_t\,|\,s_{t-1}=\mathrm{Z}\,|\,\mathrm{Z}} \log\left[1 - P_{rt}P_{\pi t}\right], \tag{A2.23}$$

$$L_C = \sum_{s_t=\mathrm{Z}} \log\left[h_t\right] \tag{A2.24}$$

L_A, L_B, L_C can be maximized separately, because L_j depends only on $\boldsymbol{\theta}_j$ $(j = A, B, C)$ $((\boldsymbol{\theta}_A, \boldsymbol{\theta}_B, \boldsymbol{\theta}_C)$ was defined in (A2.7) above).

As a special case, consider simplifying step (b) of the mapping above by replacing (A2.4a) and (A2.4b) by

$$s_t = \begin{cases} \mathrm{P} & \text{if } r_t^e + v_{rt} \geq \bar{r}_t, \\ \mathrm{Z} & \text{otherwise.} \end{cases} \tag{A2.25}$$

Namely, drop the exit condition. This is equivalent to constraining $P_{\pi t}$ to be 1, so L_B becomes

$$L_B = \sum_{s_t=\mathrm{P}} \log\left[g_t\right] + \sum_{s_t=\mathrm{Z}} \log\left[1 - P_{rt}\right], \tag{A2.26}$$

which is the Tobit log likelihood function.

Figure 1: Policy Rate in Japan, September 1985 - December 2012

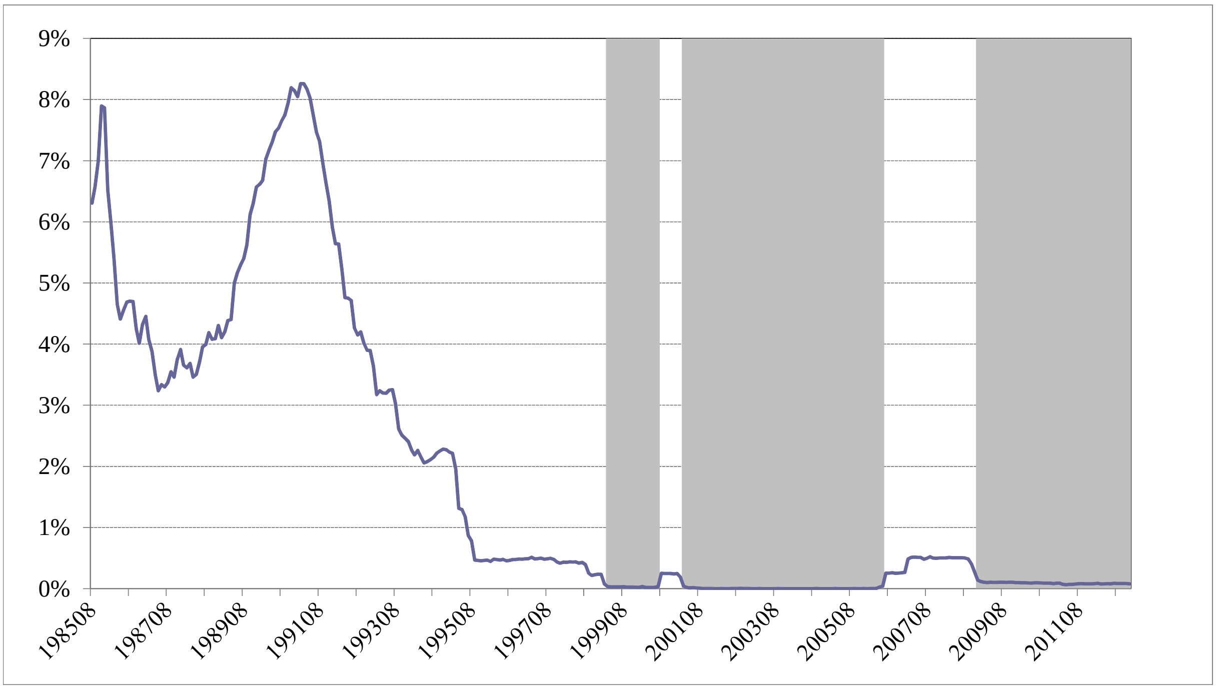

Figure 2a: Plot of Net Policy Rate against Excess Reserve Rate

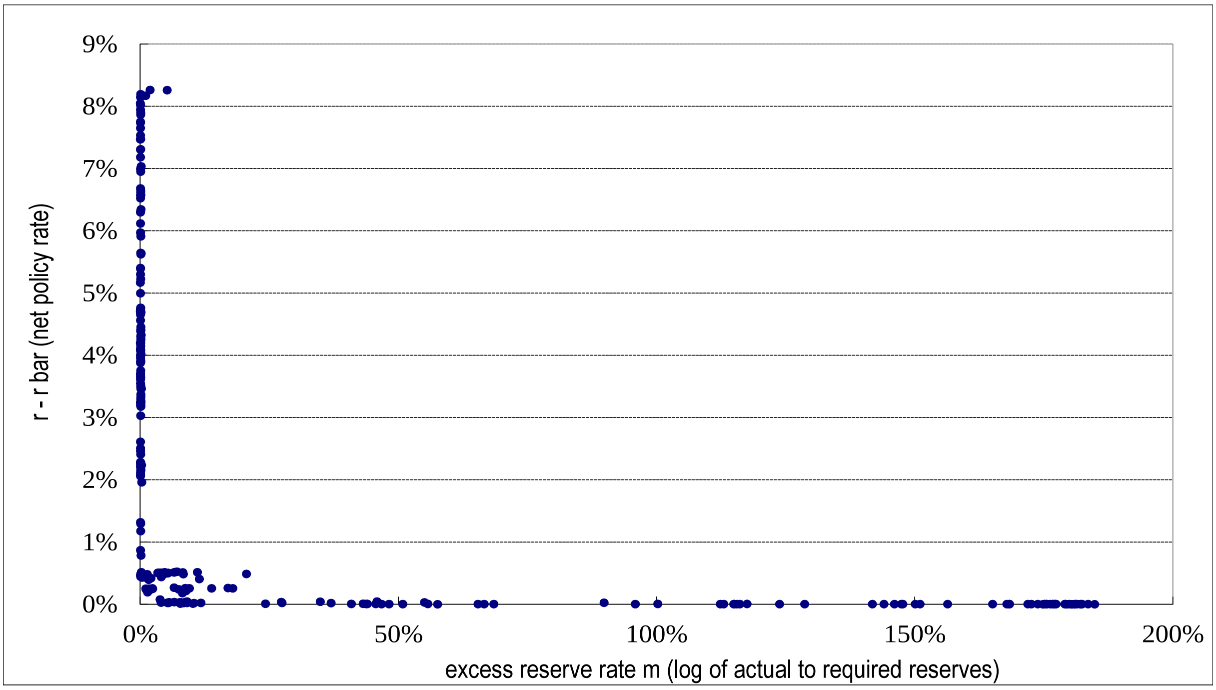

Figure 2b: Plot of Net Policy Rate against Excess Reserve Rate, Near Origin

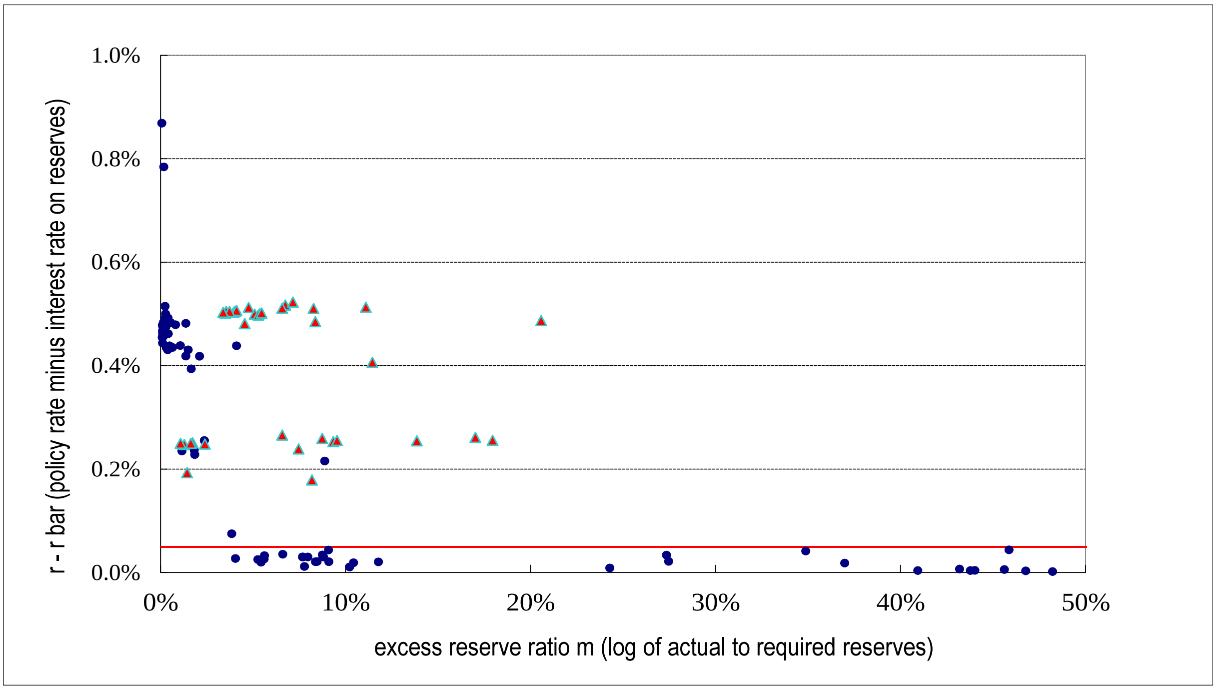

Figure 3: Plot of Net Policy Rate against Excess Reserve Rate, U.S., January 1959 - December 2012

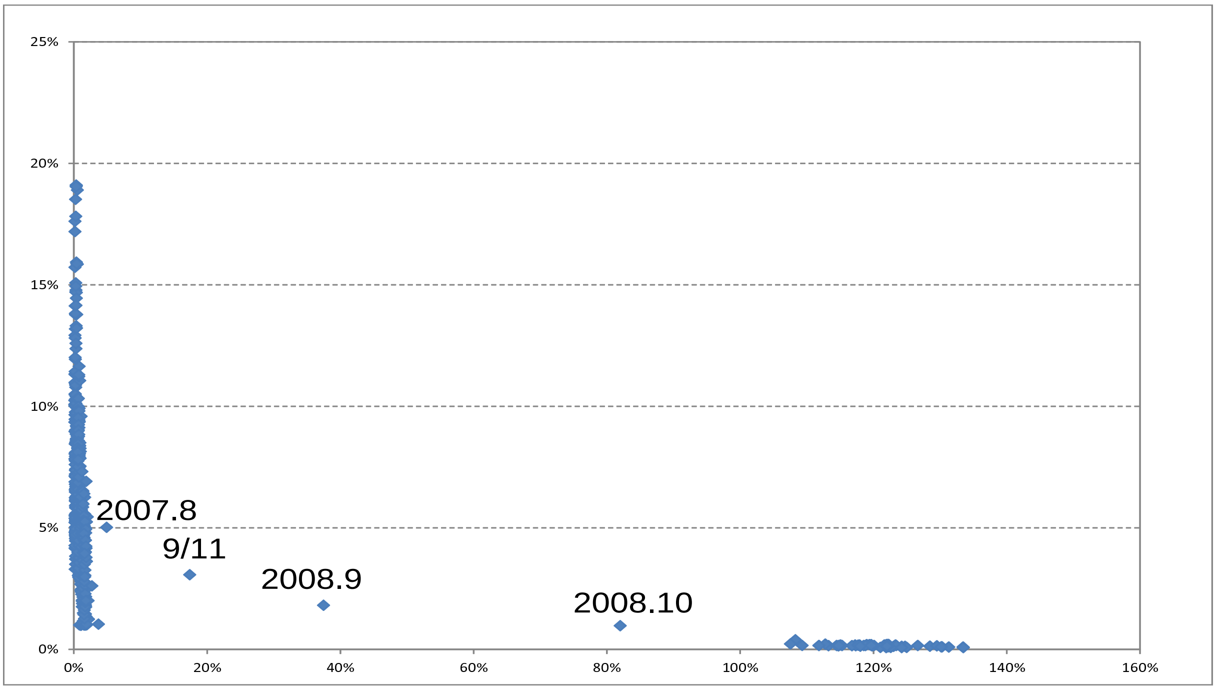

Figure 4a: Excess Reserve Rate, August 1985 - December 2012

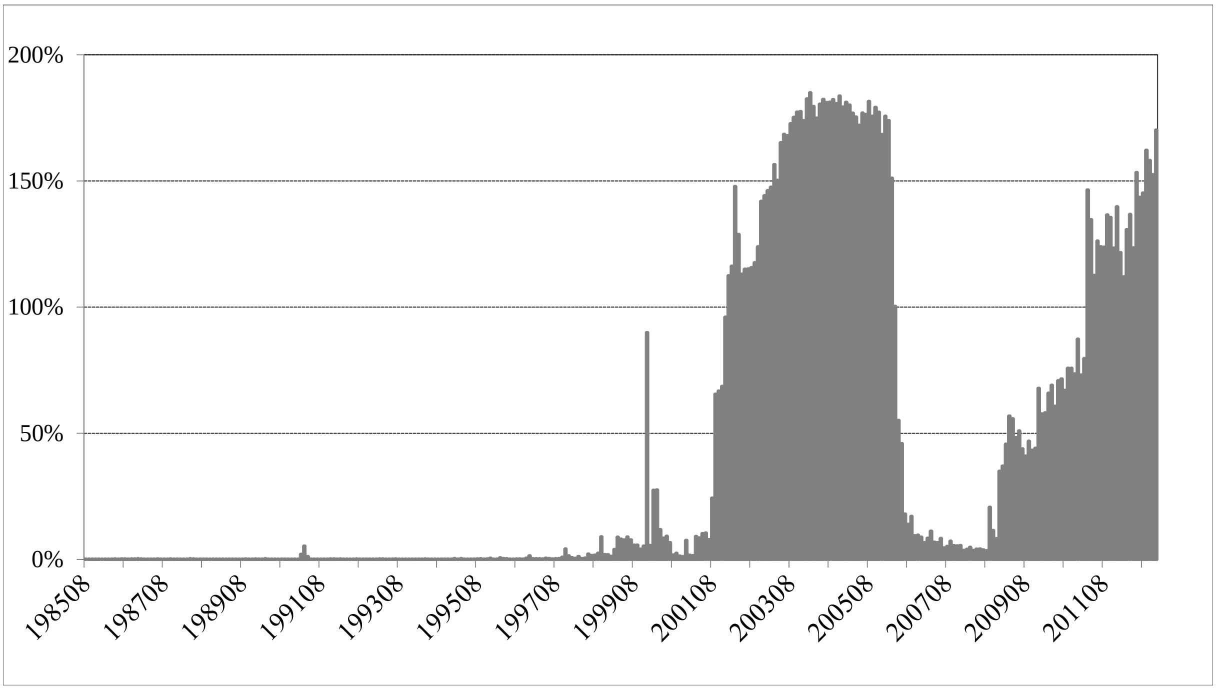

Figure 4b: Log Output and Its Trend, January 1970 - December 2012

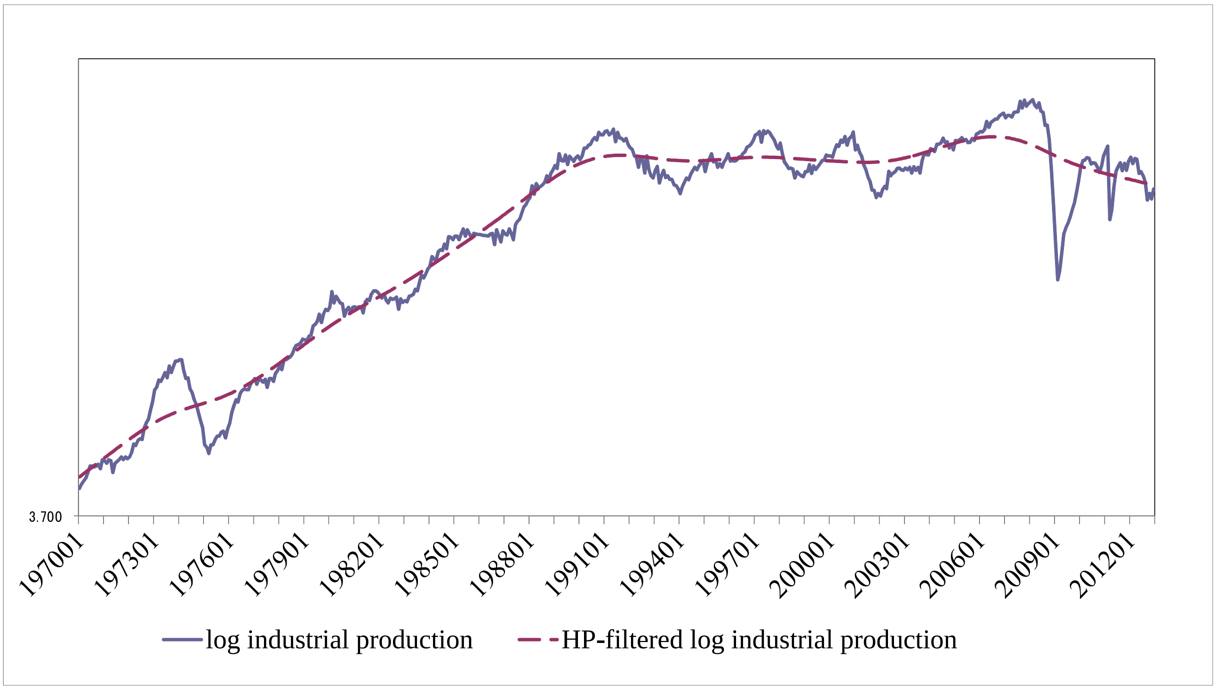

Figure 4c: Policy Rate and 12-Month Inflation Rate, January 1970 - December 2012

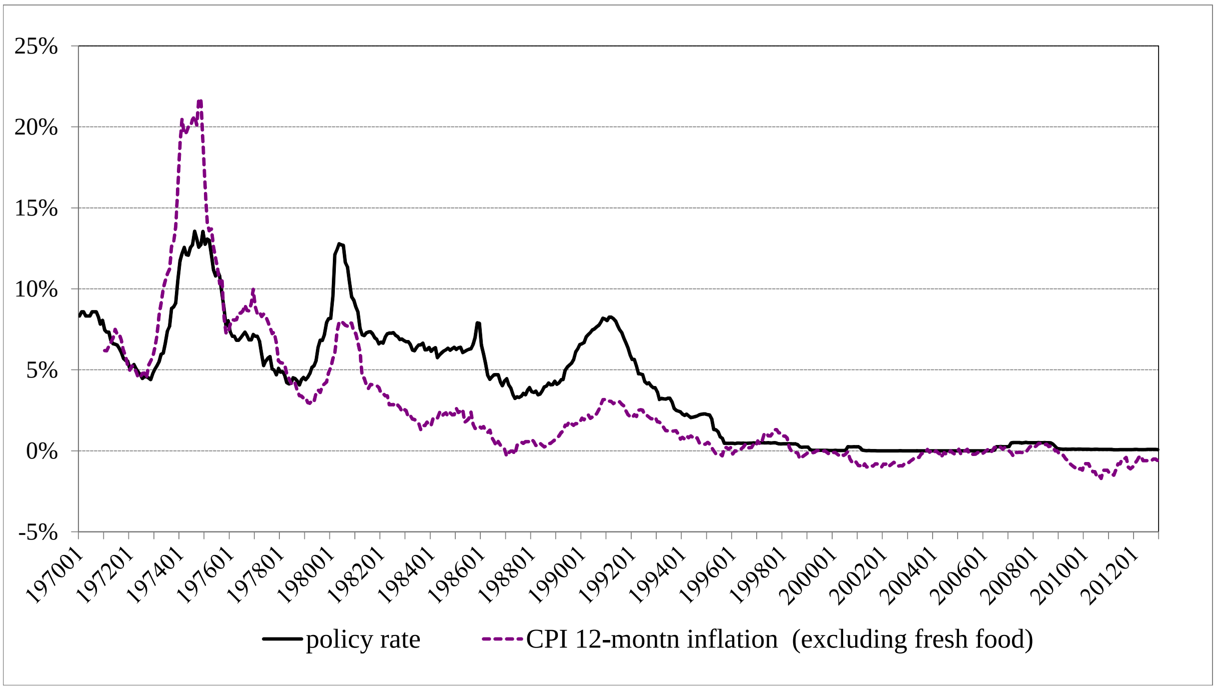

Figure 5: Policy Rate and Desired Taylor Rates, September 1985 - December 2012

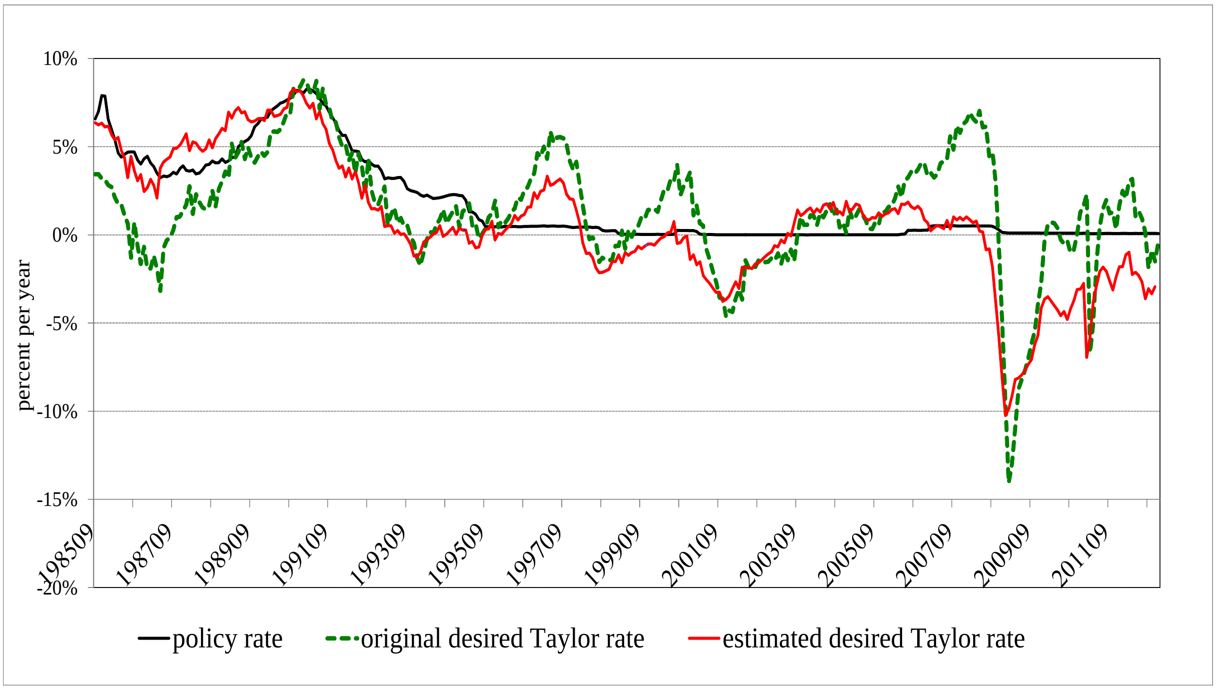

Figure 6a: m-IR (Impulse Response to m), the base period is February 2004, regime fixed at Z

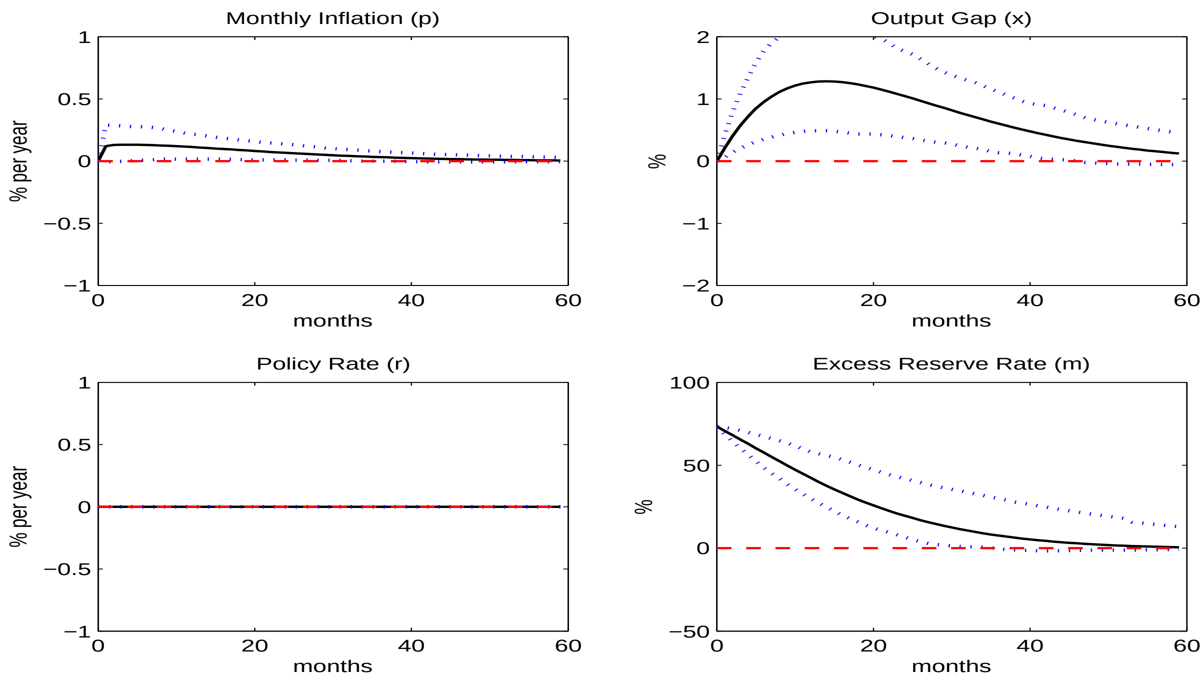

Figure 6b: m-IR (Impulse Response to m), the base period is February 2004

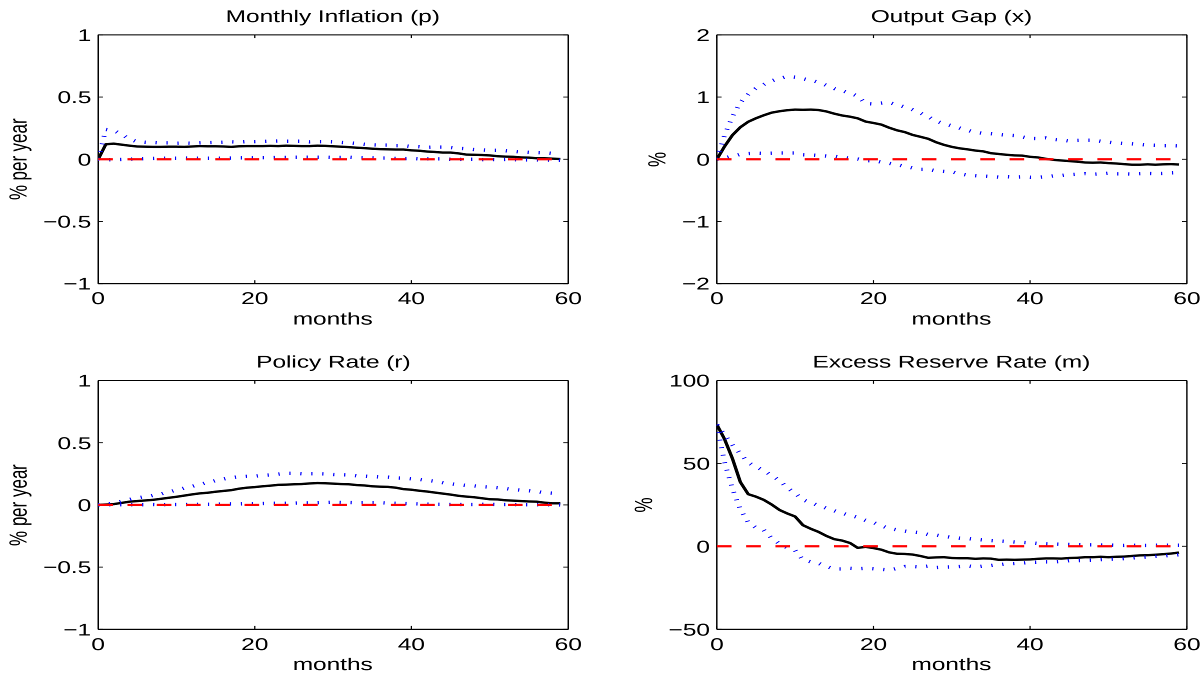

Figure 6c: r-IR (Impulse Response to r), the base period is January 1993

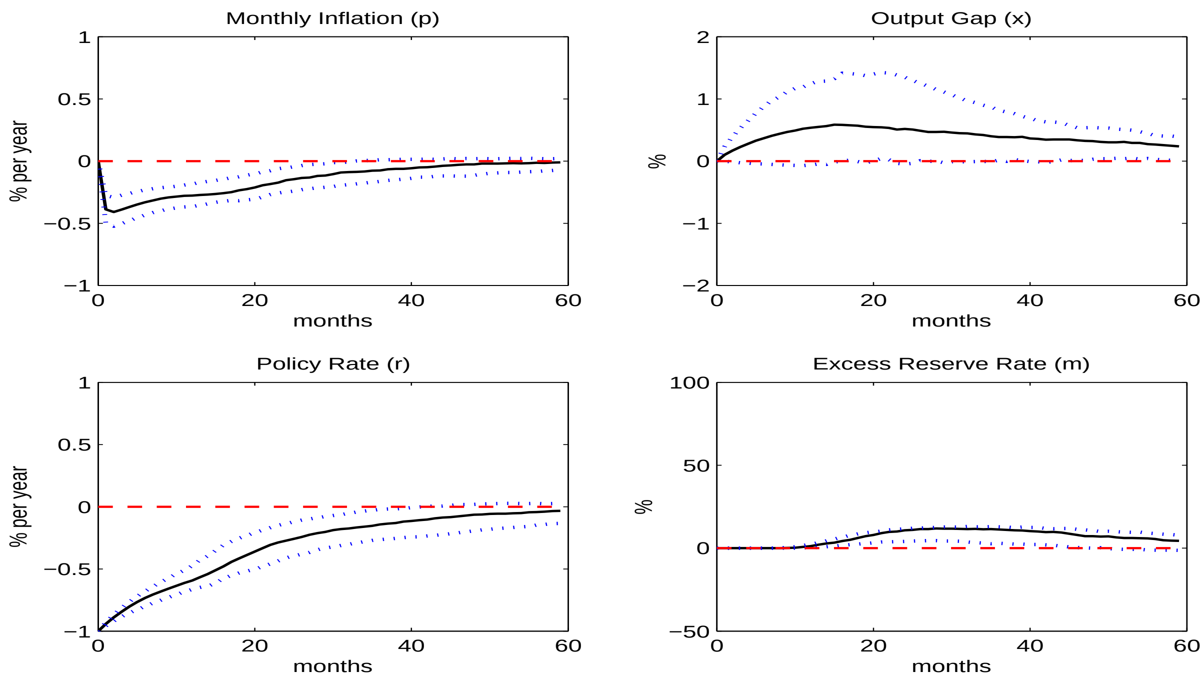

Figure 6d: ZP-IR (Impulse Response to Z $\rightarrow$ P), the base period is February 2004

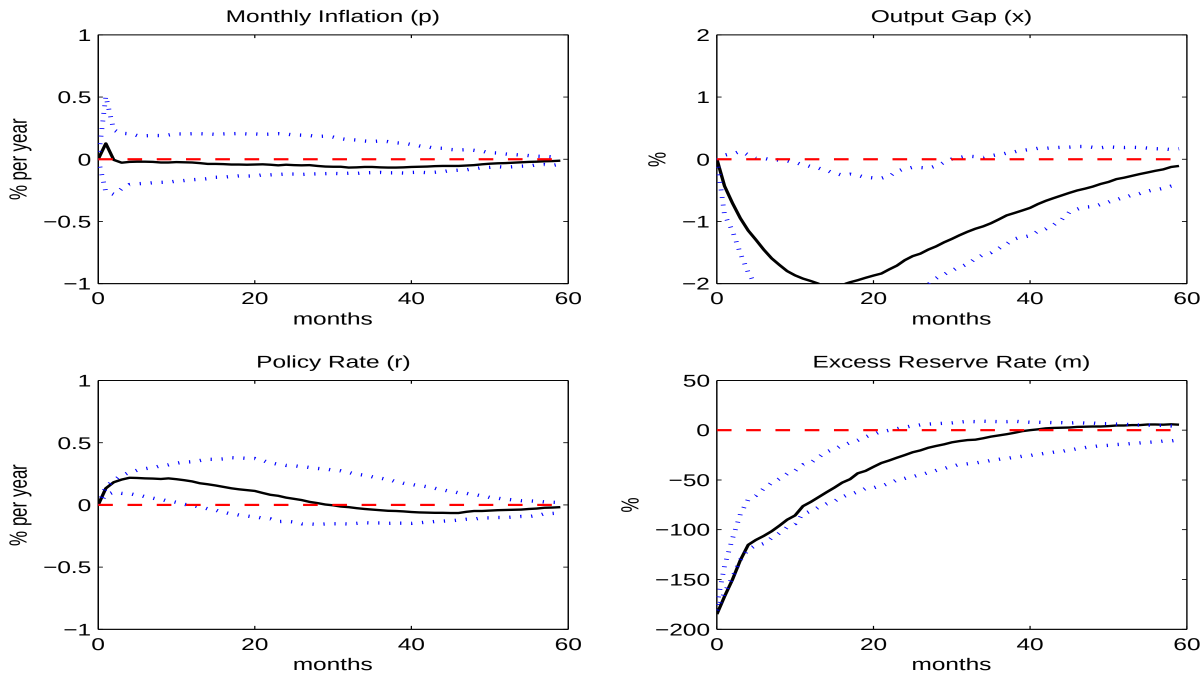

Figure 6e: ZP-IR (Impulse Response to Z → P), the base period is June 2005