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# LIQUIDITY PREFERENCE AND KNIGHTIAN UNCERTAINTY*

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**Abstract**

We consider an infinite-horizon model of a risk-neutral fund-manager who contemplates in each period whether or not to make an irreversible investment which, if made, generates some return under a stochastic environment. Here, the fund-manager evaluates uncertainty by the Choquet expected utility with respect to a convex capacitary kernel and hence she exhibits uncertainty aversion. We provide the exact solution to this problem and show that it takes the form of a reservation strategy: There exists the reservation function such that if the current return exceeds the value of this function, the fund-manager should invest all the money subject to a cash-in-advance constraint; if it does not, she should not make any investment. We also conduct some sensitivity analyses to show that if risk increases in the sense of mean-preserving spread, then the reservation function is raised and that if uncertainty increases in the sense that the set of priors expands, then the reservation function is lowered.

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# 1. Introduction

## 1.1. Motivation

Without doubt, money is the most liquid asset. To convert money to other assets is immediate and costless, whereas to convert non-money assets to other assets including money involves time and substantial transaction costs. Thus, money enables prompt moves among various forms of investment, both financial and real. In a sense, money offers liquidity services. It is natural to assume that these liquidity services are one of the most important determinants of money demand. In fact, this is the heart of the speculative demand for money, as opposed to transaction and precautionary demand for money. Unfortunately, however, there are relatively few examinations of the liquidity or speculative motive of holding money, as compared with the transaction and precautionary motives.

Among existing literature of the liquidity motive of holding money, Jones and Ostroy (1984)'s formulation has attracted much attention. They argue that money, as an asset of the least transaction cost, offers flexibility to its holder, which other assets cannot provide. Under the presence of liquidation (transaction) costs on other assets, money is held to enable the option of waiting for tomorrow to resolve uncertainty rather than investing today under uncertainty. Thus, their formulation of liquidity services of money can be considered as an enabler of options.

In the current paper, we extend the idea of Jones and Ostroy in two ways. First, we put their idea into a truly dynamic framework of infinite horizon. Second, we consider not only the case in which a decision maker is reasonably confident about the nature of uncertainty she faces and her view about her probabilistic environment is summarized by a particular probability measure, but also the case in which her confidence about her probabilistic environment is much lower and she is not certain about even relative plausibility of possible probability measures. Specifically, we consider a behavior of a risk-neutral fund-manager who contemplates in each period whether or not to make an irreversible investment which, if made, generates some returns. Furthermore, we assume that the fund-manager faces true uncertainty, rather than mere risk, about the realization of these returns.

In the tradition of Frank Knight, the "uncertainty" that is reducible to a single distribution with known parameters, is *risk*, while "uncertainty," that is irreducible, is true *uncertainty* (see Knight (1921), and also see Keynes (1921, 1936)). While risk and uncertainty are clearly distinct concepts, they have not been treated separately and adequately in economics in an explicit way, at least until recently. This may be due to the celebrated theorem of Savage (1954) which shows that if the decision maker's behavior complies to certain axioms, her preference is represented by the expectation of some utility function which is computed by means of some *single* probability measure. Uncertainty that the decision maker faces is thus reduced to risk with some probability measure. However, Ellsberg (1961) presented an example of preference under uncertainty that cannot be justified by Savage's expected utility framework. The decision maker's behavior described in Ellsberg's paradox, which is not at all irrational, clearly violates some of Savage's axioms.

In his seminal paper, Schmeidler (1989) weakens Savage-type axioms to settle debates caused by Ellsberg's paradox. Within Anscombe and Aumann's (1963) framework with a randomizing device, he weakens the independence axiom to his comonotonic-independence axiom to characterize the preference which is represented by the Choquet integral of utility numbers with respect to a probability capacity.[1] This preference is called the *Choquet expected utility* (CEU). He then adds an axiom of *uncertainty aversion* to further characterize the capacity to be convex. The Choquet integral with respect to a convex capacity is well-known to be equal to the minimum among the expectations each of which is calculated by an element of the *set* of probability measures called the capacity's *core*. Gilboa (1989) extends the CEU preference to Savage's framework with no randomizing devise. Also, Gilboa and Schmeidler (1989) axiomatize the closely related preference which is represented by the minimum among the expected utilities each of which is calculated by an element of some *set* of probability measures. The sets of probability measures in this model are not necessarily cores of convex capacities. This preference is called the *maximin expected utility* (MMEU). These two models are natural extensions of preference under uncertainty to the case in which the information is too imprecise to summarize it by a single probability measure. This type of uncertainty is often called *Knightian*

[1]The Appendix contains the definitions of the probability capacity, the Choquet integral and other related concepts.

*uncertainty* or *ambiguity*. In this paper, we simply call it *uncertainty*.

We set up a fund-manager's problem so that she should maximize the sum of discounted future gross returns from an investment by tactically choosing its amount in each period, where the uncertainty about the returns is evaluated according to the CEU preference with a convex capacity in a manner that time consistency holds. This time consistency is crucial since we apply dynamic programming techniques to solve the problem. The contribution of this paper includes proving the existence of the optimal investment strategy and presenting that strategy in a closed form under the assumption of stochastic nondecrease of a convex capacitary kernel. Here, a *convex capacitary kernel* is a convex extension of a transition probability and the *stochastic nondecrease* means that the state space is ordered in an appropriate manner.[2] It then turns out that the optimal investment strategy takes a form of reservation strategy: There exists the reservation function, a function of a current return, such that if a current return exceeds the value of this function, the fund-manager should invest all the money subject to a cash-in-advance constraint; if it does not, she should not conduct any investment. Further, we present a condition on the convex capacitary kernel under which the reservation function becomes a constant.

The existence of cash reserve provides the fund-manager with an option not to invest in the current period but to wait until next period. The results mentioned in the previous paragraph show that the fund-manager has a call option when she has cash in hand and exercise this option when the return exceeds some reservation level. This facilitates to interpret cash as an "enabler" of call option and substantiates the meaning of flexibility according to Jones and Ostroy (1984).

### 1.2. Increase in Risk versus Increase in Uncertainty

Given that the optimal investment strategy is characterized by the reservation function, we are next interested in the behavior of this function when the stochastic environment changes. As was claimed in the previous subsection, we distinguish a risky situation from an uncertain situation and hence we need to analyze these two cases separately.

[2]The formal definitions of the capacitary kernel and the stochastic nondecrease appear in Section 2 and in the Appendix.

First, suppose that uncertainty is summarized by a single probability measure, that is, the uncertainty is reduced to risk. Suppose further that the situation becomes riskier in the sense of mean-preserving spread according to Rothschild and Stiglitz (1970). We then prove that the reservation function is raised in response to this increase in risk. Intuitively, an increase in risk tends to raise the reservation function for the fund-manager in order to exploit potentially more favorable future opportunities. An increase in risk or volatility thus increases the value of a waiting option. Since money as liquidity is an enabler of this option, the value of money as liquidity increases when the risk increases.

Second, we analyze the effects of an increase in uncertainty on the reservation function. We call that uncertainty increases if the capacity which describes the uncertain situation becomes more convex. To understand this rather mathematical definition in a more intuitive way, it should be noted that this "more convexity" leads to the expansion of the core of the capacity. Since the CEU with a convex capacity equals the minimum among the expected utilities each of which is calculated by an element of the core of the capacity, the expansion of the core implies more uncertainty *and* more uncertainty aversion at the same time (due to the characteristics of "min" operator). We then prove that an increase in uncertainty in the said sense *lowers* the reservation function. This seems to be quite intuitive. When the uncertainty increases, the fund-manager who hates uncertainty prefers to make the gross return determinate rather than to keep it indeterminate as a random variable depending on the future states' realization. Therefore, she accelerates investment by lowering the reservation function. That is, the fund-manager more prefers to resolve uncertainty by making an unambiguous investment now. (Note that given the current-period's return, the gross return due to an investment is also determined.)

These results present a stark contrast between the risk and uncertainty with respect to the effects on decision making. In a job search context, Nishimura and Ozaki (2004) showed that an unemployed worker, who seeks to maximize her life-time income, uses the reservation strategy: There exists the reservation wage such that she accepts the wage offer if it exceeds this reservation wage and she does not otherwise. Then, they showed that this reservation wage increases if risk increases and it decreases if uncertainty increases. There, both the increase of risk and that of uncertainty are defined in the same manner as this paper. One can see a clear resemblance between their paper and this paper. However, there are important differences between them. First, the action space is discrete in Nishimura and Ozaki (2004). It consists of the two alternatives: to accept the job offer or to reject it. In this paper, the action space is a continuum. The action is the amount of investment and it takes on any nonnegative real number as long as it satisfies the cash-in-advance constraint. Second, the decision is once and for all in Nishimura and Ozaki (2004). Once the unemployed worker accepts the offer, she is supposed to keep working at that wage level from that time on. On the other hand, in this paper, the decision is made in each period. Even if the investment is once made, the fund-manager needs to contemplate whether to invest or not in the next period. The fact that we obtain similar results in spite of these differences suggests a robust and contrasting difference between risk and uncertainty.[3]

As a closing comment of this subsection, we point out one implication of our analyses on the money demand. An increase in risk raises the reservation function and increases the average cash-holdings. And hence, it increases the demand for money. On the contrary, an increase in uncertainty lowers the reservation function and hence decreases the demand for money. Thus if the uncertainty (or more precisely speaking, Knightian uncertainty) is prevalent in the market, and if stimulating the demand for money is desirable from the policy perspective, then the policy maker may be advised to reduce the (Knightian) uncertainty in the market as much as possible, by providing more uncertainty-reducing information.

### 1.3. Some Technical Aspects

In order to obtain the results stated in the preceding two subsections, we develop some mathematical tools in this paper. In particular, we prove that Bellman's principle of optimality holds for our model by developing some dynamic programming techniques. That is, we prove that any solution to Bellman's equation is the value function and that recursive optimality implies optimality. When our Choquet integral framework is reduced to the usual (Lebesgue) integral framework *and* when our Markovian stochastic environment is reduced to the *i.i.d.*

[3] To be fair, the risk-neutrality of the fund-manager is crucial for us to obtain that the optimal strategy takes the form of the reservation strategy. Without it, this property of the optimal strategy vanishes. In the discrete action space model of Nishimura and Ozaki (2004), the reservation strategy continues to be optimal even when the fund-manager is risk-averse. This may be thought of as another difference between two models.

stochastic environment, then our model turns out to be equal to Stokey and Lucas' (1989, p.401) model of pure-currency economy.[4] While their model assumes that money is required to buy commodities, we assume that money is required to make investment. Although interpretations of both models are quite different, we may say that their model is a very special case of ours in terms of technicality. We prove that Bellman's principle of optimality holds if we extend Stokey and Lucas' framework of an *i.i.d.* probability measure to that of a convex capacitary kernel, which is a convex "transformation" of a transition function or a stochastic kernel in the terminology of Stokey and Lucas (1989).

To this end, we develop a new assumption of upper semi-continuity (u.s.c.) of a capacitary kernel. The u.s.c. is concerned with some continuity property of a capacitary kernel when both a set and a current state change simultaneously and mainly used when we prove that the Bellman operator is well-defined. We provide a sufficient condition for a capacitary kernel to be u.s.c. and present a family of capacitary kernels which satisfies this condition. These results are collected in the Appendix.

Given Bellman's principle of optimality, we need to solve Bellman's equation to find the exact solution of the fund-manager's problem. We do this by assuming that a capacitary kernel is stochastically nondecreasing. The stochastic nondecrease appears in Topkis (1998) for a stochastic kernel and appears in Ozaki and Streufert (2001) for a capacitary kernel with a finite state space. This assumption amounts to say that the Choquet integral of a nondecreasing function is nondecreasing as a function of a current state. Because of the stochastic nondecrease, all relevant functions become nondecreasing and hence mutually "co-monotonic." By the fact that the Choquet integral of the sum of mutually co-monotonic functions is equal to the sum of their Choquet integrals, we can interchange the sum and the integral freely, which enables us to solve Bellman's equation. Similarly to the case of u.s.c., we provide a sufficient condition for a capacitary kernel to be stochastically nondecreasing and present a family of capacitary kernels which satisfies this condition. These results are collected in the Appendix.

The organization of the paper is as follows. The next section formulates the stochastic

[4] Their model is quite similar to Taub's (1988) model, which itself specifies Lucas' (1980) model by assuming that a decision-maker is risk-neutral.

environment underlying the model and formally presents the fund-manager's problem. Section 3 gives the exact solution of the fund-manager's problem and shows that the solution has the form of reservation strategy, Section 4 conducts sensitivity analyses both when risk increases and when uncertainty increases. Section 5 contains some lemmas and all proofs. Some definitions and results related to the capacity and the Choquet integral are collected in the Appendix. The Appendix also contains some new results concerning the capapcitary kernel.

## 2. The Model

This section defines the stochastic environment of our model and introduces the fund-manager's problem.

### 2.1. Stochastic Environment

Let $Z := [\underline{z}, \bar{z}]$ be a compact and connected subset of $\mathbb{R}_+$ and let $\mathcal{B}_Z$ be the Borel $\sigma$-algebra on $Z$. An element $z_t$ of $Z$ represents the gross rate of return on investment made in period $t$. In our model, $z_t$ also serves as a state variable. Construct the $t$-fold self-product measurable space from $(Z, \mathcal{B}_Z)$ and denote it by $(Z^t, \mathcal{B}_{Z^t})$, that is, $(Z^t, \mathcal{B}_{Z^t}) = (Z \times \cdots \times Z, \mathcal{B}_Z \otimes \cdots \otimes \mathcal{B}_Z)$, where the products are $t$-fold. A generic element of $(Z^t, \mathcal{B}_{Z^t})$, which is denoted by $(z_1, \ldots, z_t)$ or ${}_1\boldsymbol{z}_t$, is a history of states' realized up to period $t$.

We assume that $z_t$ is "distributed" according to a convex and continuous capacitary kernel $\theta$. Here, a *convex and continuous capacitary kernel* is a function $\theta(\cdot|\cdot) : \mathcal{B}_Z \times Z \to [0, 1]$ such that $(\forall z)$ $\theta(\cdot|z)$ is a convex and continuous capacity on $\mathcal{B}_Z$ and $(\forall E)$ $\theta(E|\cdot)$ is a $\mathcal{B}_Z$-measurable function. (Basic definitions related to the capacity are collected in the Appendix.) A capacitary kernel $\theta$ is *stochastically nondecreasing* if for each nondecreasing function $h : Z \to \mathbb{R}$, the mapping defined by

$$z \mapsto \int_Z h(z')\theta(dz'|z)$$

is nondecreasing, where the integral is the Chuquet integral.[5] This is very closely related to

[5]This definition of stochastic nondecrease here extends Topkis (1998, p.159) to a non-additive case. For an additive case, Topkis adopts as definition a property which turns out to be equivalent to the one in this paper. (Such an equivalence holds also for a non-additive case. See Lemma A9 in the Appendix.) We follow the convention in the text for an expository ease.

the concept of stochastically-ordered columns of a finite Markov chain and their monotonic transformation developed by Ozaki and Streufert (2001), and plays a crucial role for us to obtain the exact solution to the fund-manager's problem. A capacitary kernel $\theta$ is *upper semi-continuous (u.s.c.)* if for any sequence of $\mathcal{B}_Z$-measurable subsets of $Z$, $\langle A_n \rangle_{n=1}^{\infty}$, such that $A_n \supseteq A_{n+1} \supseteq \cdots$ and for any sequence $\langle z_n \rangle_{n=1}^{\infty} \subseteq Z$ such that $z_n \to z_0$, it holds that

$$\limsup_{n\to\infty} \theta\left(A_n | z_n\right) \leq \theta\left(\lim_{n\to\infty} A_n \middle| z_0\right) .$$

We can show that there certainly exists a class of convex and continuous capacitary kernels which satisfies both the stochastic nondecrease and the u.s.c. See right after Lemma A13 in the Appendix. *Throughout the paper (except for the Appendix), we maintain the assumption that $\theta$ is stochastically nondecreasing and u.s.c.*

### 2.2. The Fund-manager's Problem

In this subsection, we consider a problem facing a risk-neutral fund-manager who contemplates in each period whether or not to make an irreversible investment which, if made, generates some return.

An *investment strategy* is any $\mathbb{R}_+$-valued, $\langle \mathcal{B}_{Z^t} \rangle$-adapted stochastic process and denoted by ${}_0\boldsymbol{x}$ or $\langle x_t \rangle_{t=0}^{\infty}$. Here, the $\langle \mathcal{B}_{Z^t} \rangle$-adaptedness requires that $x_0 \in \mathbb{R}_+$ and $(\forall t \geq 1)\ x_t : Z^t \to \mathbb{R}_+$ should be $\mathcal{B}_{Z^t}$-measurable. A *money-holding strategy*, denoted ${}_1\boldsymbol{m}$ or $\langle m_t \rangle_{t=1}^{\infty}$, is any $\mathbb{R}_+$-valued, $\langle \mathcal{B}_{Z^{t-1}} \rangle$-adapted stochastic process. That is, $m_1 \in \mathbb{R}_+$ and $(\forall t \geq 2)\ m_t$ is $\mathcal{B}_{Z^{t-1}}$-measurable.

Let $y > 0$ be an income which is given in each period. Given $m_0 \geq 0$, an investment strategy ${}_0\boldsymbol{x}$ is *feasible from* $m_0$ if there exists a money-holding strategy ${}_1\boldsymbol{m}$ such that the *budget constraint* :

$$(\forall t \geq 0) \quad x_t + m_{t+1} \leq y + m_t \,, \tag{1}$$

and the *liquidity constraint in investment* (or, the *cash-in-advance constraint in investment*) :

$$(\forall t \geq 0) \quad x_t \leq m_t \tag{2}$$

are both met.

Let $\beta = 1/(1+r)$, where $r > 0$ is the net rate of interest. The expected present value of all the future gross returns on investment is given by

$$I_{z_0}({}_0\boldsymbol{x}) := \lim_{T\to+\infty} x_0 z_0 + \beta \int_Z \cdots$$
$$\beta \int_Z \left( x_{T-1} z_{T-1} + \beta \int_Z x_T z_T \, \theta(dz_T|z_{T-1}) \right) \theta(dz_{T-1}|z_{T-2}) \cdots \theta(dz_1|z_0)\,, \quad (3)$$

when the initial state is $z_0$ and the investment strategy ${}_0\boldsymbol{x}$ is chosen. Since each component of the sequence is well-defined by the Fubini property (Lemma A8 in the Appendix) and the sequence is non-decreasing, the limit exists (allowing $+\infty$).

The monotone convergence theorem (Lemma A6 in the Appendix) shows that this objective function satisfies *Koopmans' equation* :

$$(\forall z_0)(\forall {}_0\boldsymbol{x}) \quad I_{z_0}({}_0\boldsymbol{x}) = x_0 z_0 + \beta \int_Z I_{z_1}({}_1\boldsymbol{x}) \, \theta(dz_1|z_0)\,,$$

where ${}_1\boldsymbol{x}$ is a continuation of ${}_0\boldsymbol{x}$ after the realization of $z_1$. This recursive structure of the objective function enables us to apply dynamic programming techniques.

The fund-manager maximizes the objective (3) given $z_0$ by choosing an investment strategy ${}_0\boldsymbol{x}$ and a money-holding strategy ${}_1\boldsymbol{m}$ under the budget constraint (1) and the liquidity constraint in investment (2). More formally, let a function $v^* : \mathbb{R}_+ \times Z \to \mathbb{R}$ be called the *value function* for the fund-manager's problem if it satisfies

$$(\forall m, z) \quad v^*(m, z) = \max \{\, I_z({}_0\boldsymbol{x}) \mid {}_0\boldsymbol{x} \text{ is feasible from } m \,\} \,.$$

Note that the existence of the value function (that is, the existence of the maximum) needs to be proven. An investment strategy ${}_0\boldsymbol{x}$ is *optimal from* $(m, z) \in \mathbb{R}_+ \times Z$ if it is feasible from $m$ and satisfies

$$I_z({}_0\boldsymbol{x}) = v^*(m, z)$$

when the value function exists. In the next section, we prove that the value function certainly exists, and then we characterize the optimal investment strategy for the fund-manager.

## 3. The Exact Solution

This section gives the solution of the fund-manager's problem presented in Section 2. We show in the first subsection some dynamic programming results which justify the exactness of our solution to the problem. We then present the exact solution to the problem in the next subsection. In the final subsection of this section, we show that the solution may be further simplified with an additional assumption of stochastic convexity.

### 3.1. Dynamic Programming

The value function for the fund-manager's problem is (possibly) unbounded. And hence, the contraction-mapping theorem cannot be invoked to prove the existence of the value function. In this section, we develop the dynamic programming techniques for this problem and show the existence of the value function.

We start with a series of definitions. Define the *feasibility correspondence* $\Gamma : \mathbb{R}_+ \twoheadrightarrow \mathbb{R}_+^2$ by

$$(\forall m) \quad \Gamma(m) = \left\{ \, (x, m') \in \mathbb{R}_+^2 \;\middle|\; x + m' \leq y + m \text{ and } x \leq m \, \right\} .$$

When $v^*$ exists, we define the *policy correspondence* $g : \mathbb{R}_+ \times Z \twoheadrightarrow \mathbb{R}_+^2$ by

$$(\forall m, z) \quad g(m, z) = \arg\max \left\{ \, xz + \beta \int_Z v^*(m', z') \, \theta(dz'|z) \;\middle|\; (x, m') \in \Gamma(m) \, \right\} . \tag{4}$$

For the policy correspondence to be well-defined, $v^*$ must be such that $(\forall m')$ $v^*(m', \cdot)$ is $\mathcal{B}_Z$-measurable and the right-hand side of (4) is nonempty. We show the existence of the policy correspondence later. An investment strategy ${}_0\boldsymbol{x}$ is *recursively optimal* from $(m, z) \in \mathbb{R}_+ \times Z$ if there exists a money-holding strategy ${}_1\boldsymbol{m}$ such that

$$(x_0, m_1) \in g(m, z) \quad \text{and} \quad (\forall t \geq 1) \;\; (x_t, m_{t+1}) \in g(m_t, z_t) \, .$$

Among the requirements of recursive optimality is the existence of a measurable selection of $g$.

Define the function $v^+ : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\begin{aligned}
(\forall m) \quad v^+(m) &= \lim_{T \to +\infty} m\bar{z} + \cdots + \beta^{T-1} \left( (T-1)y + m \right) \bar{z} + \beta^T (Ty + m)\bar{z} \\
&= \sum_{t=0}^{\infty} \beta^t m\bar{z} + \sum_{t=0}^{\infty} t\beta^t y\bar{z}
\end{aligned}$$

$$= \frac{m\bar{z}}{1-\beta} + \frac{\beta y\bar{z}}{(1-\beta)^2} =: B^+m + A^+ .$$

The function $v^+$ may be called the *overly-optimistic value function* since for any investment strategy ${}_0\boldsymbol{x}$ which is feasible from $m$, it holds that

$$(\forall z) \quad I_z({}_0\boldsymbol{x}) \leq v^+(m).$$

A function $v : \mathbb{R}_+ \times Z \to \mathbb{R}$ is *admissible* if it is upper semi-continuous (u.s.c.), nondecreasing in its 1st argument and satisfies

$$(\forall m, z) \quad 0 \leq v(m, z) \leq v^+(m).$$

Obviously, $v^+$ is admissible.

Define the *Bellman operator*, which maps an admissible function $v$ to another function $Bv$, by

$$(\forall v)(\forall m, z) \quad Bv(m, z) = \max \left\{ xz + \beta \int_Z v(m', z')\, \theta(dz'|z) \;\middle|\; (x, m') \in \Gamma(m) \right\} . \tag{5}$$

The next lemma shows that when $v$ is admissible, the right-hand side of (5) is well-defined.

**Lemma 1.** *The Bellman operator is well-defined. That is, the maximum exists.*

By this lemma, we know that when $v^*$ exists and is admissible (which is among the conclusions of Theorem 1 below), the policy correspondence exists.

**Lemma 2.** $Bv^+ \leq v^+$ *and for any admissible function* $v$*,* $Bv$ *is admissible.*

This lemma shows that the Bellman operator maps the space of admissible functions into itself. Finally, an admissible function $v$ *solves Bellman's equation* if $v = Bv$ .

Our main result of this section is the following.

**Theorem 1.** *The value function exists, it is the unique admissible solution to Bellman's equation, and recursive optimality implies optimality.*

We rely on this theorem to characterize the optimal investment strategy, which will be conducted in the next subsection.

### 3.2. Finding the Exact Solution

This subsection gives the solution of the fund-manager's problem presented in Section 2. Suppose that $R : Z \to \mathbb{R}$ is a nondecreasing (and hence $\mathcal{B}_Z$-measurable) function and define the operator $T$ which maps such a function $R$ to another nondecreasing function $TR$ by

$$(\forall R)(\forall z) \quad TR(z) = \beta \int_Z \max\left\{z', R(z')\right\} \theta(dz'|z)\,. \tag{6}$$

To see that $TR$ is certainly nondecreasing, note that $\max\{z', R(z')\}$ is nondecreasing in $z'$ and that $\theta$ is assumed to be stochastically nondecreasing. Lemma 3 below proves that there exists a fixed point of $T$ which satisfies $(\forall z)\ 0 \leq R(z) \leq \bar{z}$. Lemma 3 also shows that such a function $R$ is unique and we denote it by $R^*$.

**Lemma 3.** *There exists a unique fixed point $R^*$ which satisfies $(\forall z)\ 0 \leq R^*(z) \leq \bar{z}$ to the operator $T$ defined by (6). Furthermore, $R^*$ is u.s.c., nondecreasing and given by*

$$R^* = \lim_{n\to\infty} T^n \bar{z} = \lim_{n\to\infty} T^n 0\,,$$

*where $T^n$ denotes the $n$-fold self-composition of $T$, $T \circ \cdots \circ T$.*

Given $t \geq 1$, $z \in Z$ and a nondecreasing function $h : Z \to \mathbb{R}$, we denote by $E^t[h|z]$ the $t$-fold iterated expectation of $h$ with respect to $\theta$ :

$$E^t[h|z] = \int_Z \cdots \int_Z \int_Z h(z_t)\, \theta(dz_t|z_{t-1})\, \theta(dz_{t-1}|z_{t-2}) \cdots \theta(dz_1|z)\,.$$

We define $E^0$ by $(\forall h, z)\ E^0[h|z] = h(z)$ and we often write $E^1$ as $E$. Clearly $E^t[h|z]$ is well-defined and nondecreasing in $z$ since $\theta$ is stochastically nondecreasing. We then define a function $A : Z \to \mathbb{R}_+$ by

$$(\forall z) \quad A(z) = y \sum_{s=0}^{+\infty} \beta^s E^s[R^*|z] = y \lim_{t\to\infty} \sum_{s=0}^{t} \beta^s E^s[R^*|z]\,. \tag{7}$$

Note that $A$ is well-defined and finite-valued since $R^*$ is a $\mathcal{B}_Z$-measurable bounded function and $\beta \in (0,1)$. Furthermore, $A$ is nondecreasing since the sum of nondecreasing functions is nondecreasing and since $A$ is the limit of a sequence of such nondecreasing functions.

We find the value function by solving Bellman's equation and then find the policy correspondence. Such a procedure is justified by Theorem 1. As a result, we have the next theorem, whose proof can be found in Section 5.

**Theorem 2.** *The value function $v^*$ exists and is given by*

$$(\forall m, z) \quad v^*(m,z) = \begin{cases} R^*(z)m + A(z) & \textit{if} \quad z \le R^*(z) \\ zm + A(z) & \textit{if} \quad z > R^*(z) \end{cases} \tag{8}$$

*and the policy correspondence $g$ exists and is given by*

$$(\forall m, z) \quad g(m,z) = \begin{cases} \{\,(0, m+y)\,\} & \textit{if} \quad z < R^*(z) \\ \{\,(x, m') \in \Gamma(m) \,|\, x + m' = y + m\,\} & \textit{if} \quad z = R^*(z) \\ \{\,(m, y)\,\} & \textit{if} \quad z > R^*(z)\,. \end{cases} \tag{9}$$

*Furthermore, recursive optimality implies optimality.*

We construct an investment strategy ${}_0\boldsymbol{x}^*$ (and its associated money-holding strategy ${}_1\boldsymbol{m}$) which is recursively optimal from $(m,z)$ as follows:

$$(\forall t \ge 0) \quad (x_t^*, m_{t+1}) = \begin{cases} (0, m_t + y) & \text{if} \quad z_t \le R^*(z_t) \\ (m_t, y) & \text{if} \quad z_t > R^*(z_t) \end{cases} \tag{10}$$

where $m_0 := m$ and $z_0 := z$. The stochastic process ${}_0\boldsymbol{x}^*$ thus defined is $\langle \mathcal{B}_{Z^t} \rangle$-adapted since $R^*$ is $\mathcal{B}_Z$-measurable. Therefore, it is certainly an investment strategy and recursively optimal from $(m,z)$ by (9) and the definition of recursive optimality. By the last statement of Theorem 2, we know that ${}_0\boldsymbol{x}^*$ is an optimal investment strategy from $(m,z)$.

The existence of cash reserve provides the fund-manager with an option not to invest in the current period but to wait until next period. This shows that the fund-manager has a call option when she has cash in hand. Here, cash is an "enabler" of this call option, or flexibility in terms of Jones and Ostroy (1984). Cash is endowed with this function by the liquidity services it provides, and ultimately by transaction costs implicit in the irreversibility of investment.

### 3.3. Reservation Property

This subsection (and subsection 4.1) assumes that $\theta$ is stochastically convex. Here, a capacitary kernel $\theta$ is *stochastically convex* if for each nondecreasing function $h : Z \to \mathbb{R}$, the mapping defined on $Z$ by

$$z \mapsto \int_Z h(z')\, \theta(dz'|z)$$

is convex. An example of stochastically convex capacitary kernel is provided in the Appendix (see right after Lemma A13). Under this assumption, it can be shown that the trigger function $R^*$ is constant.

More formally, the optimal investment strategy ${}_0\boldsymbol{x}^*$ *has a reservation property* if there exists a *constant* $z^* \geq 0$ such that

$$(\forall t \geq 0) \quad (x_t^*, m_{t+1}) = \begin{cases} (0, m_t + y) & \text{if} \quad z_t \leq z^* \\ (m_t, y) & \text{if} \quad z_t > z^* . \end{cases}$$

We can prove the following result.

**Theorem 3.** *Suppose that the capacitary kernel $\theta$ is stochastically convex. Then, the optimal investment strategy has a reservation property. Furthermore, if $\beta E[z'|\underline{z}] \geq \underline{z}$, then there exists a unique $z^* \in Z$ such that $z^* = R^*(z^*)$ and the reservation level equals $z^*$.*

## 4. Sensitivity Analyses

This section conducts two sensitivity analyses. First, we consider a case where there does not exist uncertainty and assume that the risk increases in the sense of a mean-preserving spread. We then show that such an increase in risk raises the reservation function. Second, we assume that uncertainty exists and further assume that uncertainty increases in the sense that the core of a capacity expands. We then show that an increase in uncertainty lowers the reservation function in contrast to an increase in risk. These results show that an increase either in risk or in uncertainty affects the money demand in the opposite directions.

### 4.1. An Increase in Risk

In this subsection, we assume that there exists no uncertainty. Given a probability measure $P$, we denote by $F$ the (cumulative) distribution function derived from $P$, that is,

$(\forall z)\ F(z) = P([\underline{z}, z])$. Let $P_0$ and $P_1$ be two probability measures. We denote by $F_0$ and $F_1$ the distribution functions associated with $P_0$ and $P_1$, respectively. We say that $P_1$ *is obtained from* $P_0$ *by a mean-preserving spread* if it holds that

$$\int_Z z\, dF_0(z) = \int_Z z\, dF_1(z) \quad \text{and} \quad (\forall x \in \mathbb{R}) \ \int_{-\infty}^{x} F_0(z)\, dz \leq \int_{-\infty}^{x} F_1(z)\, dz\,.$$

A capacitary kernel $\theta : \mathcal{B}_Z \times Z \to [0,1]$ is called *stochastic kernel* (Stokey and Lucas, 1989, p.226) if $(\forall z)\ \theta(\cdot|z)$ is countably additive (that is, a probability measure). Clearly, a stochastic kernel is a convex and continuous capacitary kernel. We say that a stochastic kernel $P_1$ *is obtained from* $P_0$ *by a mean-preserving spread* if $(\forall z \in Z)\ P_1(\cdot|z)$ is obtained from $P_0(\cdot|z)$ by a mean-preserving spread.

**Theorem 4.** *Let $P_0$ be a stochastic kernel which is stochastically nondecreasing, u.s.c. and stochastically convex and let $P_1$ be a stochastic kernel which is u.s.c. Furthermore, assume that $P_1$ is obtained from $P_0$ by the mean-preserving spread. Then, $(\forall z \in Z)$ $R_1^*(z) \geq R_0^*(z)$, where $R_i^*$ is the fixed point of $T_i$ corresponding to $P_i$ for each $i = 0, 1$ .*

**Corollary 1.** *Let $P_0$ and $P_1$ be as in Theorem 4. Also, suppose that $z_t \leq z_0^*$, where $z_0^*$ is the reservation value corresponding to $P_0$ whose existence is guaranteed by Theorem 3. Then, it holds that $z_t \leq R_1^*(z_t)$.*

The corollary shows that if making investment is not an optimal strategy before the risk increases, it cannot be so after the risk has increased. Therefore, an increase in risk tends to increase cash balances to be carried over to the next period in order to exploit potentially more favorable future opportunities. Money cash balances work as a provider of this option, which is more favorable under more risk.

### 4.2. An Increase in Uncertainty

Let $\theta_0$ and $\theta_1$ be two capacitary kernels. We say that $\theta_1$ *represents more (Knightian) uncertainty than* $\theta_0$ if

$$(\forall A)(\forall z) \quad \theta_0(A|z) \geq \theta_1(A|z)\,. \tag{11}$$

Under the assumption that $\theta_0$ and $\theta_1$ are convex, it turns out that (11) is equivalent to

$$(\forall z) \quad \text{core}(\theta_0(\cdot|z)) \subseteq \text{core}(\theta_1(\cdot|z)), \tag{12}$$

which, together with Lemma A5(a), substantiates our definition of more uncertainty. Ghirardato and Marinacci (2002) develop a notion of comparative ambiguity aversion and relate it to (11). They also provides some behavioral foundation of our notion of more uncertainty.

In the theorem below, we assume that $\theta_0$ and $\theta_1$ are convex and continuous capacitary kernels which are stochastically nondecreasing and u.s.c. (like "$\theta$" in the previous sections) and we let $R_0$ and $R_1$ be a reservation function associated with $\theta_0$ and $\theta_1$, respectively. When a capacitary kernel $\theta_1$ is defined by

$$(\forall A)(\forall z) \quad \theta_1(A|z) := g \circ \theta_0(A|z) \tag{13}$$

with some $\theta_0$ and some continuous and convex function $g : [0,1] \to [0,1]$, then $\theta_1$ satisfies all the requirements if so does $\theta_0$ (Lemma A12 in the Appendix). Furthermore, if $\theta_1$ is defined by (13), it satisfies (11) and hence represents more uncertainty than $\theta_0$.[6]

The next result shows that the reservation function is lowered if uncertainty increases in the sense of (11).

**Theorem 5.** *Suppose that $\theta_0$ and $\theta_1$ be convex and continuous capacitary kernels which are stochastically nondecreasing and u.s.c. Also, suppose that $\theta_1$ represents more uncertainty than $\theta_0$. Then, $(\forall z)$ $R_1^*(z) \leq R_0^*(z)$, where $R_i^*$ is the fixed point of $T_i$ corresponding to $\theta_i$ for each $i = 0, 1$ .*

## 5. Lemmas and Proofs

**Proof of Lemma 1.** First, we show that for any admissible function $v$, the mapping defined by

$$(m', z) \mapsto \int_Z v(m', z')\, \theta(dz'|z)$$

[6]Epstein and Zhang (1999) adopt the notion (13) as a definition of more uncertainty (and more uncertainty aversion) when the capacities are convex.

is u.s.c. To do this, let $v$ be an admissible function and let $\langle(m'_n, z_n)\rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{R}_+ \times Z$ which converges to $(m'_0, z_0)$. Then, it holds that, for each $t \geq 0$

$$\begin{aligned}
\limsup_{n\to\infty} \left\{z' \,\middle|\, v(m'_n, z') \geq t\right\} &= \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \left\{z' \,\middle|\, v(m'_k, z') \geq t\right\} \\
&\subseteq \left\{z' \,\middle|\, \limsup_{n\to\infty} v(m'_n, z') \geq t\right\} \\
&\subseteq \left\{z' \,\middle|\, v(m'_0, z') \geq t\right\} \qquad (14)
\end{aligned}$$

where the equality is definitional; the first inclusion follows from the definition of lim sup of a sequence of both sets and real numbers; and the second inclusion follows from the u.s.c. of $v$. Hence, it holds that, for each $t \geq 0$

$$\begin{aligned}
\limsup_{n\to\infty} \theta\left(\left\{z' \,\middle|\, v(m'_n, z') \geq t\right\} \middle|\, z_n\right) &\leq \limsup_{n\to\infty} \theta\left(\bigcup_{k=n}^{\infty} \left\{z' \,\middle|\, v(m'_k, z') \geq t\right\} \middle|\, z_n\right) \\
&\leq \theta\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \left\{z' \,\middle|\, v(m'_k, z') \geq t\right\} \middle|\, z_0\right) \\
&= \theta\left(\limsup_{n\to\infty} \left\{z' \,\middle|\, v(m'_n, z') \geq t\right\} \middle|\, z_0\right) \\
&\leq \theta\left(\left\{z' \,\middle|\, v(m'_0, z') \geq t\right\} \middle|\, z_0\right), \qquad (15)
\end{aligned}$$

where the first inequality holds by the monotonicity of $\theta$; the second inequality holds by the u.s.c. of $\theta$ since the sequence of sets in the parentheses is nonincreasing; the next equality is definitional; and the last inequality holds by (14). Finally, we arrive at

$$\begin{aligned}
\limsup_{n\to\infty} \int_Z v(m'_n, z')\, \theta(dz'|z_n) &= \limsup_{n\to\infty} \int_0^{\infty} \theta\left(\left\{z' \,\middle|\, v(m'_n, z') \geq t\right\} \middle|\, z_n\right) dt \\
&\leq \int_0^{\infty} \limsup_{n\to\infty} \theta\left(\left\{z' \,\middle|\, v(m'_n, z') \geq t\right\} \middle|\, z_n\right) dt \\
&\leq \int_0^{\infty} \theta\left(\left\{z' \,\middle|\, v(m'_0, z') \geq t\right\} \middle|\, z_0\right) dt \\
&= \int_Z v(m'_0, z')\, \theta(dz'|z_0),
\end{aligned}$$

where the both equalities follow from definition of the Choquet integral; the first inequality follows from Fatou's lemma since there exists $N \geq 1$ such that

$$(\forall n \geq N)(\forall z') \quad v(m'_n, z') \leq v^+(m'_n) < v^+(m'_0) + 1$$

by the admissibility of $v$ and the u.s.c. of $v^+$; and the second inequality follows from (15).

Second, we completes the proof by showing that for any admissible function $v$, $Bv$ is well-defined. However, this follows immediately because the maximand in (5) is u.s.c. in $(x, m')$ by the fact proven in the previous paragraph and because $\Gamma$ is compact-valued. ■

**Proof of Lemma 2.** The first half of the lemma follows because

$$
\begin{aligned}
(\forall m, z) \quad Bv^+(m,z) \ &= \ \max\left\{ xz + \beta B^+ m' + \beta A^+ \,\middle|\, (x, m') \in \Gamma(m) \right\} \\
&\leq \ mz + \beta B^+(m+y) + \beta A^+ \\
&\leq \ m\bar{z} + \beta B^+(m+y) + \beta A^+ \\
&= \ \frac{m\bar{z}}{1-\beta} + \frac{\beta y \bar{z}}{(1-\beta)^2} \ = \ v^+(m,z)\,.
\end{aligned}
$$

To show the latter half of the lemma, let $v$ be an admissible function. Then, the admissibility of $v$, the fact that $B$ is monotonically non-decreasing in $v$ and the inequality proven in the previous paragraph show that $0 \leq B0 \leq Bv \leq Bv^+ \leq v^+$. Furthermore, $Bv$ is u.s.c. by the maximum theorem (Berge, 1963) because the maximand in (5) is u.s.c. by the proof of Lemma 1 and because $\Gamma$ is continuous. Finally, the nondecrease is immediate. ■

**Lemma 4.** *For any $m \geq 0$, any investment strategy ${}_0\boldsymbol{x}$ which is feasible from $m$ and its associated money-holding strategy ${}_1\boldsymbol{m}$ and any admissible function $v$, it holds that*

$$
\begin{aligned}
(\forall z) \quad I_z({}_0\boldsymbol{x}) = \lim_{T \to +\infty} x_0 z + \beta \int_Z \cdots \\
\beta \int_Z \left( x_{T-1} z_{T-1} + \beta \int_Z v(m_T, z_T)\, \theta(dz_T | z_{T-1}) \right) \theta(dz_{T-1}|z_{T-2}) \cdots \theta(dz_1|z)
\end{aligned}
$$

**Proof.** Let $(m,z) \in \mathbb{R}_+ \times Z$, let ${}_0\boldsymbol{x}$ be an investment strategy which is feasible from $m$ and let $v$ be an admissible function. The iterated applications of Koopmans' equation to ${}_0\boldsymbol{x}$ shows that

$$
\begin{aligned}
(\forall T \geq 1) \quad I_z({}_0\boldsymbol{x}) = x_0 z + \beta \int_Z \cdots \\
\beta \int_Z \left( x_{T-1} z_{T-1} + \beta \int_Z I_{z_T}({}_T\boldsymbol{x})\, \theta(dz_T | z_{T-1}) \right) \theta(dz_{T-1}|z_{T-2}) \cdots \theta(dz_1|z)\,,
\end{aligned}
$$

where ${}_T\boldsymbol{x}$ is a continuation of ${}_0\boldsymbol{x}$ after the realization of ${}_1z_T$. Therefore, for any $T \geq 1$, it follows that

$$
\Bigg| I_z({}_0\boldsymbol{x}) - \Bigg[ x_0 z + \beta \int_Z \cdots
$$

$$
\begin{aligned}
& \beta \int_Z \left( x_{T-1} z_{T-1} + \beta \int_Z v(m_T, z_T) \theta(dz_T | z_{T-1}) \right) \theta(dz_{T-1}|z_{T-2}) \cdots \theta(dz_1|z) \Big] \Big| \\
= \; & \beta \Big| \int_Z \cdots \beta \int_Z \left( x_{T-1} z_{T-1} + \beta \int_Z I_{z_T}({}_T\boldsymbol{x}) \theta(dz_T|z_{T-1}) \right) \theta(dz_{T-1}|z_{T-2}) \cdots \theta(dz_1|z) \\
& - \int_Z \cdots \beta \int_Z \left( x_{T-1} z_{T-1} + \beta \int_Z v(m_T, z_T) \theta(dz_T|z_{T-1}) \right) \theta(dz_{T-1}|z_{T-2}) \cdots \theta(dz_1|z) \Big| \\
\leq \; & \beta \int_Z \Big| \left( \cdots \beta \int_Z \left( x_{T-1} z_{T-1} + \beta \int_Z I_{z_T}({}_T\boldsymbol{x}) \theta(dz_T|z_{T-1}) \right) \theta(dz_{T-1}|z_{T-2}) \cdots \right) \\
& - \left( \cdots \beta \int_Z \left( x_{T-1} z_{T-1} + \beta \int_Z v(m_T, z_T)\, \theta(dz_T|z_{T-1}) \right) \theta(dz_{T-1}|z_{T-2}) \cdots \right) \Big| \theta'(dz_1|z) \\
\leq \; & \cdots \\
\leq \; & \beta^T \int_Z \cdots \int_Z \int_Z |I_{z_T}({}_T\boldsymbol{x}) - v(m_T, z_T)| \, \theta'(dz_T|z_{T-1}) \theta'(dz_{T-1}|z_{T-2}) \cdots \theta'(dz_1|z) \\
\leq \; & \beta^T \int_Z \cdots \int_Z \int_Z \max \{ I_{z_T}({}_T\boldsymbol{x}), v(m_T, z_T) \} \, \theta'(dz_T|z_{T-1}) \theta'(dz_{T-1}|z_{T-2}) \cdots \theta'(dz_1|z) \\
\leq \; & \beta^T \int_Z \cdots \int_Z \int_Z v^+(Ty + m) \theta'(dz_T|z_{T-1}) \theta'(dz_{T-1}|z_{T-2}) \cdots \theta'(dz_1|z) \\
= \; & \beta^T (B^+ Ty + B^+ m + A^+) \,,
\end{aligned}
$$

where the first to the third inequalities hold by Lemma A5(d) and the fifth inequality holds since $(\forall z_T)\ I_{z_T}({}_T\boldsymbol{x}) \leq v^+(Ty + m)$ by the fact that for any investment strategy ${}_0\boldsymbol{x}$ which is feasible from $m$, $x_T \leq Ty + m$. Since the last term of the above inequalities goes to 0 as $T \to +\infty$, we have

$$
\begin{aligned}
\lim_{T \to \infty} \Big| I_z({}_0\boldsymbol{x}) - \Big[ & x_0 z + \beta \int_Z \cdots \\
& \beta \int_Z \left( x_{T-1} z_{T-1} + \beta \int_Z v(m_T, z_T)\, \theta(dz_T|z_{T-1}) \right) \theta(dz_{T-1}|z_{T-2}) \cdots \theta(dz_1|z) \Big] \Big| = 0 \,,
\end{aligned}
$$

which completes the proof. ■

**Lemma 5.** *Any admissible solution to Bellman's equation is the value function.*

**Proof.** Let $v$ be an admissible function which solves Bellman's equation and let $(m, z) \in \mathbb{R}_+ \times Z$. This paragraph shows that for any investment strategy ${}_0\boldsymbol{x}$ which is feasible from $m$, it holds that $v(m, z) \geq I_z({}_0\boldsymbol{x})$. Let ${}_0\boldsymbol{x}$ be such an investment strategy and let ${}_1\boldsymbol{m}$ be its associated money-holding strategy. Then,

$$
v(m, z) \;\geq\; x_0 z + \beta \int_Z v(m_1, z_1)\, \theta(dz_1|z)
$$

$$
\begin{aligned}
&\geq \; x_0 z + \beta \int_Z \left( x_1 z_1 + \beta \int_Z v(m_2, z_2)\, \theta(dz_2|z_1) \right) \theta(dz_1|z) \\
&\geq \; \cdots \\
&\geq \; x_0 z + \beta \int_Z \left( x_1 z_1 + \cdots \beta \int_Z v(m_T, z_T)\, \theta(dz_T|z_{T-1}) \cdots \right) \theta(dz_1|z)
\end{aligned}
$$

where the first inequality holds since $v$ solves Bellman's equation and $(x_0, m_1) \in \Gamma(m)$ by the feasibility, the second inequality holds since $v$ solves Bellman's equation and $(x_1, m_2) \in \Gamma(m_1)$ by the feasibility, and so on. Since the whole inequality holds for any $T \geq 1$, Lemma 4 proves the claim.

This paragraph completes the proof by showing that there exists an investment strategy ${}_0\boldsymbol{x}$ which is feasible from $m$ and satisfies $v(m, z) = I_z({}_0\boldsymbol{x})$. Define the investment strategy ${}_0\boldsymbol{x}$ and the money-holding strategy ${}_1\boldsymbol{m}$ recursively by

$$
\begin{aligned}
&(x_0, m_1) \in \arg\max \left\{ xz + \beta \int_Z v(m', z')\, \theta(dz'|z) \,\middle|\, (x, m') \in \Gamma(m) \right\} \quad \text{and} \\
(\forall t \geq 1) \quad &(x_t, m_{t+1}) \in \arg\max \left\{ xz_t + \beta \int_Z v(m', z')\, \theta(dz'|z_t) \,\middle|\, (x, m') \in \Gamma(m_t) \right\}.
\end{aligned}
$$

Such strategies are well-defined by the measurable selection theorem (Wagner, 1977, p.880, Theorem 9.1(ii)). Then,

$$
\begin{aligned}
v(m, z) \; &= \; x_0 z + \beta \int_Z v(m_1, z_1)\, \theta(dz_1|z) \\
&= \; x_0 z + \beta \int_Z \left( x_1 z_1 + \beta \int_Z v(m_2, z_2)\, \theta(dz_2|z_1) \right) \theta(dz_1|z) \\
&= \; \cdots \\
&= \; x_0 z + \beta \int_Z \left( x_1 z_1 + \cdots \beta \int_Z v(m_T, z_T)\, \theta(dz_T|z_{T-1}) \cdots \right) \theta(dz_1|z)
\end{aligned}
$$

where the equalities hold by the definition of ${}_0\boldsymbol{x}$ and ${}_1\boldsymbol{m}$ and because $v$ solves Bellman's equation. Since the whole inequality holds for any $T \geq 1$, Lemma 4 proves the claim. ■

**Lemma 6.** *A function $v^\infty$ defined by $v^\infty := \lim_{n\to\infty} B^n v^+$ is an admissible solution to Bellman's equation, where $B^n$ denotes the n-fold self-composition of $B$, $B \circ \cdots \circ B$.*

**Proof.** By Lemma 2 and the fact that $B$ is nondecreasing in $v$, $\langle B^n v^+ \rangle_{n=1}^\infty$ is a nonincreasing sequence of u.s.c. functions which are bounded from below by 0 and hence its limit exists and is u.s.c. Therefore, $v^\infty$ is a well-defined admissible function. In the rest of this proof, we show that $v^\infty$ solves Bellman's equation.

Note that $(\forall n \geq 1)$ $B^{n+1}v^+ = B \circ B^n v^+ \geq B \circ \lim_{n\to\infty} B^n v^+ = Bv^\infty$. Therefore, we have $v^\infty = \lim_{n\to\infty} B^{n+1}v^+ \geq Bv^\infty$.

To show the opposite inequality, let $(m, z) \in \mathbb{R}_+ \times Z$ and let $\langle (x_n, m'_n) \rangle_{n=1}^\infty$ be a sequence in $\mathbb{R}_+^2$ such that

$$(\forall n \geq 1) \quad (x_n, m'_n) \in \arg\max \left\{ xz + \beta \int_Z B^n v^+(m', z')\, \theta(dz'|z) \,\middle|\, (x, m') \in \Gamma(m) \right\} .$$

Such a sequence exists since the right-hand side is nonempty by Lemma 1 and the admissibility of $B^n v^+$. Since $\Gamma(m)$ is compact, there exists a subsequence $\langle (x_{n(i)}, m'_{n(i)}) \rangle_{i=1}^\infty$ which converges to $(x_0, m'_0) \in \Gamma(m)$. Then,

$$\begin{aligned}
Bv^\infty(m, z) &= \max \left\{ xz + \beta \int_Z v^\infty(m', z')\, \theta(dz'|z) \,\middle|\, (x, m') \in \Gamma(m) \right\} \\
&\geq x_0 z + \beta \int_Z v^\infty(m'_0, z')\, \theta(dz'|z) \\
&= x_0 z + \beta \int_Z \lim_{n\to\infty} B^n v^+(m'_0, z')\, \theta(dz'|z) \\
&= x_0 z + \beta \int_Z \lim_{i\to\infty} B^{n(i)} v^+(m'_0, z')\, \theta(dz'|z) \\
&\geq x_0 z + \beta \int_Z \lim_{i\to\infty} \limsup_{j\to\infty} B^{n(i)} v^+(m'_{n(j)}, z')\, \theta(dz'|z) \\
&\geq x_0 z + \beta \int_Z \lim_{i\to\infty} \limsup_{j\to\infty} B^{n(j)} v^+(m'_{n(j)}, z')\, \theta(dz'|z) \\
&= x_0 z + \beta \int_Z \limsup_{j\to\infty} B^{n(j)} v^+(m'_{n(j)}, z')\, \theta(dz'|z) \\
&\geq x_0 z + \beta \limsup_{j\to\infty} \int_Z B^{n(j)} v^+(m'_{n(j)}, z')\, \theta(dz'|z) \\
&= \limsup_{j\to\infty} \left( x_{n(j)} z + \beta \int_Z B^{n(j)} v^+(m'_{n(j)}, z')\, \theta(dz'|z) \right) \\
&= \limsup_{j\to\infty} B^{n(j)+1} v^+(m, z) \\
&= \lim_{j\to\infty} B^{n(j)+1} v^+(m, z) \\
&= v^\infty(m, z) ,
\end{aligned}$$

where the second inequality holds by the u.s.c. of $B^{n(i)}v^+$. To show the fourth inequality, let $J \geq 1$ be such that $(\forall j \geq J)$ $m'_{n(j)} < m'_0 + 1$. Then, it follows that

$$(\forall j \geq J)(\forall z' \in Z) \quad B^{n(j)} v^+(m'_{n(j)}, z') \leq v^+(m'_{n(j)}, z') = B^+ m'_{n(j)} + A^+ < B^+(m'_0 + 1) + A^+ .$$

Therefore, the desired inequality holds by Fatou's lemma (Lemma A7 in the Appendix). ■

**Proof of Theorem 1.** Lemmas 5 and 6 show that $v^\infty$ is a value function, and hence, the value function certainly exists. Suppose that $v$ and $v'$ are two admissible solutions to Bellman's equation. Then, it must be that $v = v'$ because both $v$ and $v'$ must be the value function by Lemma 5 and because the value function is unique by its definition. Therefore, the admissible solution to Bellman's equation is unique and equals $v^\infty$ since $v^\infty$ is admissible by Lemma 6. Finally, the second paragraph of the proof of Lemma 5 shows that recursive optimality implies optimality. ■

**Proof of Lemma 3.** First, define $R^+$ by $(\forall z)\ R^+(z) = \bar{z}$. Then, it follows that

$$(\forall z) \quad TR^+(z) = \beta \int_Z \max\left\{z', \bar{z}\right\}\, \theta(dz'|z) = \beta \int_Z \bar{z}\, \theta(dz'|z) = \beta\bar{z} \leq \bar{z} = R^+(z)\,.$$

Since $T$ is monotonic in the sense that $(\forall R, R')\ R \geq R' \Rightarrow TR \geq TR'$, $\langle T^n R^+ \rangle_{n=1}^\infty$ (where $T^n$ denotes the $t$-fold self-composition of $T$) is a nonincreasing sequence of functions. Hence, its limit exists and is nondecreasing since each $T^n R^+$ is nondecreasing. We denote it by $R^\infty$. We now see that $R^\infty$ is a fixed point of $T$ because

$$\begin{aligned}
(\forall z) \quad TR^\infty(z) &= \beta \int_Z \max\left\{z', R^\infty(z')\right\}\, \theta(dz'|z) \\
&= \beta \int_Z \max\left\{z', \lim_{n\to\infty} T^n R^+(z')\right\}\, \theta(dz'|z) \\
&= \beta \int_Z \lim_{n\to\infty} \max\left\{z', T^n R^+(z')\right\}\, \theta(dz'|z) \\
&= \lim_{n\to\infty} \beta \int_Z \max\left\{z', T^n R^+(z')\right\}\, \theta(dz'|z) \\
&= \lim_{n\to\infty} T^{n+1} R^+(z) \\
&= R^\infty(z)\,,
\end{aligned}$$

where the fourth inequality holds by the monotone convergence theorem (Lemma A6 in the Appendix).

Second, define $R^-$ by $(\forall z)\ R^-(z) = 0$. Then, it follows that

$$(\forall z) \quad TR^-(z) = \beta \int_Z \max\left\{z', 0\right\}\, \theta(dz'|z) \geq 0 = R^-(z)\,.$$

Since $(\forall R, R')\ R \geq R' \Rightarrow TR \geq TR'$, $\langle T^n R^- \rangle_{n=1}^{\infty}$ is a nondecreasing sequence of functions. Hence, its limit exists and is nondecreasing since each $T^n R^-$ is nondecreasing. We denote it by $R_\infty$. We now see that $R_\infty$ is a fixed point of $T$ because

$$\begin{aligned}
(\forall z) \quad TR_\infty(z) &= \beta \int_Z \max\left\{z', R_\infty(z')\right\} \theta(dz'|z) \\
&= \beta \int_Z \max\left\{z', \lim_{n\to\infty} T^n R^-(z')\right\} \theta(dz'|z) \\
&= \beta \int_Z \lim_{n\to\infty} \max\left\{z', T^n R^-(z')\right\} \theta(dz'|z) \\
&= \lim_{n\to\infty} \beta \int_Z \max\left\{z', T^n R^-(z')\right\} \theta(dz'|z) \\
&= \lim_{n\to\infty} T^{n+1} R^-(z) \\
&= R_\infty(z)\,,
\end{aligned}$$

where the fourth inequality holds by the monotone convergence theorem (Lemma A6 in the Appendix).

This paragraph shows that $R^\infty = R_\infty$. To this end, let $z \in Z$ and let $n \geq 1$. Then, we have

$$\begin{aligned}
0 &\leq \left|T^n R^+(z) - T^n R^-(z)\right| \\
&= \beta \left| \int_Z \max\left\{z', T^{n-1}R^+(z')\right\} \theta(dz'|z) - \int_Z \max\left\{z', T^{n-1}R^-(z')\right\} \theta(dz'|z) \right| \\
&\leq \beta \int_Z \left|\max\left\{z_1, T^{n-1}R^+(z_1)\right\} - \max\left\{z_1, T^{n-1}R^-(z_1)\right\}\right| \theta'(dz_1|z) \\
&\leq \beta \int_Z \left|T^{n-1}R^+(z_1) - T^{n-1}R^-(z_1)\right| \theta'(dz_1|z) \\
&\leq \beta^2 \int_Z \int_Z \left|T^{n-2}R^+(z_2) - T^{n-2}R^-(z_2)\right| \theta'(dz_2|z_1)\theta'(dz_1|z) \\
&\leq \cdots \\
&\leq \beta^n \int_Z \cdots \int_Z \int_Z \left|R^+(z_n) - R^-(z_n)\right| \theta'(dz_n|z_{n-1})\theta'(dz_{n-1}|z_{n-2}) \cdots \theta'(dz_1|z) \\
&= \beta^n \int_Z \cdots \int_Z \int_Z \bar{z}\, \theta'(dz_n|z_{n-1})\theta'(dz_{n-1}|z_{n-2}) \cdots \theta'(dz_1|z) \\
&= \beta^n \bar{z}\,,
\end{aligned}$$

where the second inequality holds by Lemma A5(d). Since the whole inequality holds for any $n$, taking the limit proves the claim.

Let $R$ be any fixed point of $T$ such that $R^- = 0 \leq R \leq \bar{z} = R^+$. Then, it holds that

$TR^- \leq TR = R \leq TR^+$ by the monotonicity of $T$ and the assumption that $R$ is a fixed point of $T$. By iterating this procedure, we have $(\forall n)\ T^n R^- \leq R \leq T^n R^+$. Therefore, it follows that $R_\infty = \lim_{n\to\infty} T^n R^- \leq R \leq \lim_{n\to\infty} T^n R^+ = R^\infty$. This and the fact proven in the previous paragraph show that $R = R_\infty = R^\infty$, and hence, $R^* := R^\infty$ is the unique fixed point of $T$ satisfying $0 \leq R^* \leq \bar{z}$.

That $R^*$ is nondecreasing is immediate. Finally, we show that $R^*$ is u.s.c. By the first paragraph of Proof of Lemma 1, we know that $(\forall n)\ T^n R^+$ is u.s.c. in $z$ under the assumpiton of u.s.c. of $\theta$. Therefore, $R^*$ is u.s.c. since it is the infimum of u.s.c. functions by $R^* = \lim_{n\to\infty} T^n R^+ = \inf_{n\geq 1}\langle T^n R^+\rangle$. ■

**Lemma 7.** *The function $A$ defined by (7) is u.s.c. and satisfies*

$$(\forall z) \quad A(z) = R^*(z)y + \beta E[A|z]\,. \tag{16}$$

**Proof.** (U.s.c.) Since $R^*$ is u.s.c. (by Lemma 3) and bounded from above (by $\bar{z}$), $(\forall s \geq 0)$ $E^s[R^*|z]$ is u.s.c. in $z$ by the first paragraph of Proof of Lemma 1. Let $z_0 \in Z$ and let $\varepsilon > 0$. Since $E^s[R^*|z]$ is uniformly bounded from above in $s$ and $z$ and since $\beta < 1$, there exists $S \geq 1$ such that $y\sum_{s=S+1}^{+\infty} \beta^s E^s[R^*|z] < \varepsilon/2$. Furthermore, since $\sum_{s=0}^{S} \beta^s E^s[R^*|z]$ is u.s.c. in $z$ (because it is a *finite* sum of u.s.c. functions), there exists a neighborhood $N$ of $z_0$ such that $(\forall z \in N)\ y\sum_{s=0}^{S} \beta^s E^s[R^*|z] < A(z_0) + \varepsilon/2$. Finally, we have $(\forall z \in N)\ A(z) < A(z_0) + \varepsilon$, which completes the proof.

(Equation (16)) The equation holds because

$$\begin{aligned}
& A(z) \\
= \;& y\sum_{s=0}^{+\infty} \beta^s E^s[R^*|z] \\
= \;& R^*(z)y + \beta y \sum_{s=0}^{+\infty} \beta^s E^{s+1}[R^*|z] \\
= \;& R^*(z)y + \beta y \sum_{s=0}^{+\infty} \beta^s E\left[E^s[R^*|z_1]|z\right] \\
= \;& R^*(z)y + \beta y \lim_{T\to\infty} \sum_{s=0}^{T} \beta^s E\left[E^s[R^*|z_1]|z\right]
\end{aligned}$$

$$
\begin{aligned}
&= R^*(z)y + \beta y \lim_{T\to\infty} E\left[R^*(z_1)\right| z\right] + \beta E\left[E\left[R^*\right| z_1\right]\left| z\right] + \cdots + \beta^T E\left[E^T\left[R^*\right| z_1\right]\Big| z\Big] \\
&= R^*(z)y + \beta y \lim_{T\to\infty} E\left[R^*(z_1)\right| z\right] + E\left[\beta E\left[R^*\right| z_1\right]\left| z\right] + \cdots + E\left[\beta^T E^T\left[R^*\right| z_1\right]\Big| z\Big] \\
&= R^*(z)y + \beta y \lim_{T\to\infty} E\left[R^*(z_1) + \beta E\left[R^*\right| z_1\right] + \cdots + \beta^T E^T\left[R^*\right| z_1\right]\Big| z\Big] \\
&= R^*(z)y + \beta y \lim_{T\to\infty} \beta y E\left[\sum_{s=0}^{T} \beta^s E^s[R^*|z_1]\right| z\Bigg] \\
&= R^*(z)y + \beta y E\left[\sum_{s=0}^{+\infty} \beta^s E^s[R^*|z_1]\right| z\Bigg] \\
&= R^*(z)y + \beta E\left[y\sum_{s=0}^{+\infty} \beta^s E^s[R^*|z_1]\right| z\Bigg] \\
&= R^*(z)y + \beta E[A|z]\,,
\end{aligned}
$$

where the third equality holds by the definition of $E^s$; the seventh equality holds by the co-monotonic additivity of $E[\cdot|z]$ (Lemma A4(d) in the Appendix) since each $E^t\left[R^*\right| z_1]$ $(0 \le t \le T)$ is nondecreasing in $z_1$ and hence mutually co-monotonic and the ninth equality holds by the monotone convergence theorem (Lemma A6 in the Appendix). ■

**Proof of Theorem 2.** First, we show that the function $\hat{v} : \mathbb{R}_+ \times Z \to \mathbb{R}$ defined by

$$
(\forall m, z) \quad \hat{v}(m, z) := \max\big\{z, R^*(z)\big\}m + A(z)
$$

is admissible, that is, $\hat{v}$ is u.s.c., nondecreasing in $m$ and satisfies

$$
(\forall m, z) \quad 0 \le \hat{v}(m, z) \le \frac{m\bar{z}}{1-\beta} + \frac{\beta y \bar{z}}{(1-\beta)^2}\,.
$$

That $\hat{v}$ is u.s.c. follows since $R^*$ is u.s.c. (by Lemma 3) and $A$ is u.s.c. (by Lemma 7). The nondecrease is obvious. To show the inequalities, note that $R^* \le TR^+ = \beta\bar{z}$ by (6). Therefore,

$$
\begin{aligned}
0 \le \hat{v}(m, z) &= \max\big\{z, R^*(z)\big\}m + y\sum_{s=0}^{+\infty} \beta^s E^s[R^*|z] \\
&\le \bar{z}m + y\sum_{s=0}^{+\infty} \beta^s \beta\bar{z} \\
&= \bar{z}m + y\frac{\beta\bar{z}}{1-\beta} \\
&\le \frac{m\bar{z}}{1-\beta} + \frac{\beta y\bar{z}}{(1-\beta)^2}\,.
\end{aligned}
$$

Second, we show that $\hat{v}$ is the solution to Bellman's equation:

$$
(\forall m, z) \quad v(m, z) = \max\left\{ xz + \beta \int_Z v(m', z')\,\theta(dz'|z) \,\middle|\, (x, m') \in \Gamma(m) \right\}. \tag{17}
$$

We have

$$
\begin{aligned}
& \max\left\{ xz + \beta \int_Z \hat{v}(m', z')\, \theta(dz'|z) \,\middle|\, (x, m') \in \Gamma(m) \right\} \\
= \;& \max\left\{ xz + m'\beta \int_Z \max\left\{z', R^*(z')\right\} \theta(dz'|z) + \beta \int_Z A(z')\, \theta(dz'|z) \,\middle|\, (x, m') \in \Gamma(m) \right\} \\
= \;& \max\left\{ xz + m'TR^*(z) + \beta E[A|z] \,\middle|\, (x, m') \in \Gamma(m) \right\} \\
= \;& \max\left\{ xz + m'R^*(z) + \beta E[A|z] \,\middle|\, (x, m') \in \Gamma(m) \right\} \\
= \;& \begin{cases} R^*(z)(m+y) + \beta E[A|z] & \text{if} \quad z \le R^*(z) \\ mz + R^*(z)y + \beta E[A|z] & \text{if} \quad z > R^*(z) \end{cases} \\
= \;& \begin{cases} R^*(z)m + R^*(z)y + \beta E[A|z] & \text{if} \quad z \le R^*(z) \\ zm + R^*(z)y + \beta E[A|z] & \text{if} \quad z > R^*(z) \end{cases} \\
= \;& \begin{cases} R^*(z)m + A(z) & \text{if} \quad z \le R^*(z) \\ zm + A(z) & \text{if} \quad z > R^*(z) \end{cases} \\
= \;& \hat{v}(m, z)\,,
\end{aligned}
$$

where the first equality holds by the co-monotonic additivity of the Choquet integral (Lemma A4(d) in the Appendix) since both $\max\{z', R^*(z')\}$ and $A(z')$ are nondecreasing in $z'$ and hence co-monotonic; the third equality holds by the fact that $R^*$ is the fixed point of $T$; and sixth equality holds by Equation (16).

Finally, since $\hat{v}$ is an admissible solution to Bellman's equation as shown in the preceding paragraphs, we conclude that $v^*$ defined by (8), which equals $\hat{v}$, is the value function by Theorem 1 Furthermore, the second paragraph of this proof shows that $g$ defined by (9) is the policy correspondence. Finally, that recursive optimality implies optimality is among the conclusions of Theorem 1 ■

**Lemma 8.** *Suppose that the capacitary kernel $\theta$ is stochastically convex. Then, $R^*$ is a convex function.*

**Proof.** First, we show that for each $n \ge 1$, $T^n R^+$ is nondecreasing and convex in $z$. We prove this by induction. The statement holds true when $n = 0$ since $T^0 R^+ = R^+ = \bar{z}$ is constant and hence both nondecreasing and convex in $z$. Suppose that $T^{n-1}R^+$ is nondecreasing and convex in $z$. Then, $\max\{z', T^{n-1}R^+(z')\}$ is nondecreasing in $z'$. Therefore, $T^nR^+$ is nondecreasing and convex by the definition of $T$ since $\theta$ is stochastically increasing and stochastically convex.

Since $R^*$ is a pointwise limit of a sequence of convex functions by Lemma 3 and by the fact proven in the previous paragraph, $R^*$ is convex. ■

**Proof of Theorem 3.** By Lemma 8, $R^*$ is convex and hence continuous on $(\underline{z}, \bar{z})$. Since $R^*$ is u.s.c. and nondecreasing by Lemma 3, it is continuous on $[\underline{z}, \bar{z})$ with only possible discontinuity occurring at $z = \bar{z}$. Furthermore, note that $R^*(\bar{z}) \leq TR^+(\bar{z}) = \beta\bar{z} < \bar{z}$. Therefore, the graph of $R^*$ crosses the 45-degree line from above if and only if $R^*(\underline{z}) \geq \underline{z}$. First, suppose that $R^*(\underline{z}) < \underline{z}$. Then, any $z^*$ such that $0 \leq z^* < \underline{z}$ serves as a reservation level and the optimal strategy clearly has a reservation property. Second, suppose that $R^*(\underline{z}) \geq \underline{z}$. Then, there exists a unique $z^* \in Z$ such that $z^* = R^*(z^*)$ and the optimal strategy has a reservation property since $\{\, z \in Z \,|\, z \leq R^*(z) \,\} = \{\, z \in Z \,|\, z \leq z^* \,\}$.

We complete the proof by showing that when $\beta E[z'|\underline{z}] \geq \underline{z}$, it holds that $R^*(\underline{z}) \geq \underline{z}$. To see this, suppose that it does not. Then, since $R^*$ solves $R^* = TR^*$, it follows from (6) that $R^*(\underline{z}) = TR^*(\underline{z}) \geq \beta E[z'|\underline{z}] \geq \underline{z} > R^*(\underline{z})$, which is a contradiction. ■

**Proof of Theorem 4.** For each $i = 0, 1$, let $T_i$ be the operator defined from $P_i$ by (6). We show that for each $n \geq 1$, $T_1^nR^+ \geq T_0^nR^+$, which completes the proof since $(\forall i)\ R_i^* = \lim_{n\to\infty} T_i^nR^+$. We prove the claim by induction. The statement clearly holds true when $n = 0$ since $T_1^0R^+ = R^+ = T_0^0R^+$. Suppose that $T_1^{n-1}R^+ \geq T_0^{n-1}R^+$. Then,

$$
\begin{aligned}
T_1^nR^+ &= T_1 \circ T_1^{n-1}R^+ \\
&\geq T_1 \circ T_0^{n-1}R^+ \\
&= \beta \int_Z \max\left\{z', T_0^{n-1}R^+(z')\right\} P_1(dz'|z) \\
&\geq \beta \int_Z \max\left\{z', T_0^{n-1}R^+(z')\right\} P_0(dz'|z) \\
&= T_0 \circ T_0^{n-1}R^+ \\
&= T_0^nR^+\,,
\end{aligned}
$$

where the first inequality holds by the induction hypothesis. To see that the second inequality holds, note that $\max\left\{z', T_0^{n-1}R^+(z')\right\}$ is convex in $z'$ by the argument similar to the proof of Lemma 8 and by the fact that the maximum of two convex functions is convex. Therefore, the inequality holds true by Rothschild and Stiglitz (1970, page 237, Theorem 2) since $P_1(\cdot|z)$ is obtained from $\theta(\cdot|z)$ by the mean-preserving spread for each $z$. ■

**Proof of Corollary 1.** This follows because $z_t \leq R_0^*(z_t) \leq R_1^*(z_t)$ where the first inequality holds since $z_t \leq z_0^*$ if and only if $z_t \leq R_0^*(z_t)$ by Theorem 3 and the second inequality holds by Theorem 4. ■

**Proof of Theorem 5.** For each $i = 0, 1$, let $T_i$ be the operator defined from $\theta_i$ by (6). We show that for each $n \geq 1$, $T_1^n R^+ \leq T_0^n R^+$, which completes the proof since $(\forall i)$ $R_i^* = \lim_{n\to\infty} T_i^n R^+$. We prove the claim by induction. The statement clearly holds true when $n = 0$ since $T_1^0 R^+ = R^+ = T_0^0 R^+$. Suppose that $T_1^{n-1}R^+ \leq T_0^{n-1}R^+$. Then,

$$
\begin{aligned}
T_1^n R^+ &= T_1 \circ T_1^{n-1} R^+ \\
&\leq T_1 \circ T_0^{n-1} R^+ \\
&= \beta \int_Z \max\left\{z', T_0^{n-1}R^+(z')\right\} \theta_1(dz'|z) \\
&= \beta \min\left\{ \int_Z \max\left\{z', T_0^{n-1}R^+(z')\right\} P(dz') \,\middle|\, P \in \mathrm{core}(\theta_1(\cdot|z)) \right\} \\
&\leq \beta \min\left\{ \int_Z \max\left\{z', T_0^{n-1}R^+(z')\right\} P(dz') \,\middle|\, P \in \mathrm{core}(\theta_0(\cdot|z)) \right\} \\
&= \beta \int_Z \max\left\{z', T_0^{n-1}R^+(z')\right\} \theta_0(dz'|z) \\
&= T_0 \circ T_0^{n-1} R^+ \\
&= T_0^n R^+\,,
\end{aligned}
$$

where the first inequality holds by the induction hypothesis and by the monotonicity of the Choquet integral (Lemma A4(a) in the Appendix); the third equality holds by Lemma A5(a) in the Appendix; the second inequality holds because $\mathrm{core}(\theta_0(\cdot|z)) \subseteq \mathrm{core}(\theta_1(\cdot|z))$ since $\theta_1$ represents more uncertainty than $\theta_0$ by the assumption and since (11) and (12) are equivalent; and the fourth equality holds by Lemma A5(a) in the Appendix again. ■

## APPENDIX

This appendix provides some mathematics for the theory of Choquet capacity, which we rely upon in the text. We omit the proof whenever it is easily available somewhere in the literature (see, for example, Dellacherie (1970), Shapley (1971) and Schmeidler (1972, 1986) among others).

**Probability Capacity and Probability Charge** Let $(S, \mathcal{F})$ be a measurable space, where $\mathcal{F}$ is a $\sigma$-algebra on $S$. A *probability capacity* on $(S, \mathcal{F})$ is a function $\theta : \mathcal{F} \to [0, 1]$ which satisfies

$$\theta(\phi) = 0$$

$$\theta(S) = 1$$

$$\text{and} \quad (\forall A, B \in \mathcal{F}) \;\; A \subseteq B \;\Rightarrow\; \theta(A) \leq \theta(B)\,.$$

A probability capacity is *convex* if

$$(\forall A, B \in \mathcal{F}) \quad \theta(A \cup B) + \theta(A \cap B) \geq \; \theta(A) + \theta(B) \tag{18}$$

while it is *concave* if the inequality in (18) is reversed. A probability capacity is a *probability charge* if the inequality in (18) holds with an equality.

A capacity $\theta$ is *continuous from below* if

$$(\forall \langle A_i \rangle_i \subseteq \mathcal{F}) \quad A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots \;\Rightarrow\; \theta(\cup_i A_i) = \lim_{i \to \infty} \theta(A_i)\,.$$

A capacity $\theta$ is *continuous from above* if

$$(\forall \langle A_i \rangle_i \subseteq \mathcal{F}) \quad A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots \;\Rightarrow\; \theta(\cap_i A_i) = \lim_{i \to \infty} \theta(A_i)\,.$$

A capacity $\theta$ is *continuous* if it is continuous both from below and from above. Note that any finite *measure* is continuous, and that continuity and finite additivity (that is, (18) with the inequality replaced by the equality) together imply countable additivity.

The *conjugate* of a probability capacity $\theta$ is the function $\theta' : \mathcal{F} \to [0, 1]$ defined by

$$(\forall A \in \mathcal{F}) \quad \theta'(A) = 1 - \theta(A^c)$$

where $A^c$ denotes the complement of $A$ in $S$. The *core* of a probability capacity $\theta$, $\text{core}(\theta)$, is defined by

$$\text{core}(\theta) = \{\, P \in \mathcal{M} \,|\, (\forall A \in \mathcal{F})\ P(A) \geq \theta(A) \,\},$$

where $\mathcal{M}$ is the set of all probability charges on $(S, \mathcal{F})$. Note that a probability charge in the core of a continuous capacity is countably additive and hence a probability measure.

**Lemma A1.** *Given a probability capacity $\theta$ on $(S, \mathcal{F})$ and a nondecreasing function $f : [0,1] \to [0,1]$ such that $f(0) = 0$ and $f(1) = 1$, define a mapping $f \circ \theta : \mathcal{F} \to [0,1]$ by*

$$(\forall A \in \mathcal{F}) \quad f \circ \theta(A) = f\left(\theta(A)\right) .$$

*Then $f \circ \theta$ is a probability capacity. Furthermore, $f \circ \theta$ is convex (resp. concave, continuous) when both $f$ and $\theta$ are convex (resp. concave, continuous).*

**Proof.** We show that when both $f$ and $\theta$ are convex, $f \circ \theta$ is convex. The other claims follow similarly or immediately. Let $f$ and $\theta$ be convex and let $A, B \in \mathcal{F}$. Without loss of generality, assume that $\theta(A) \geq \theta(B)$. Note that the convexity of $\theta$ implies

$$\theta(A \cup B) - \theta(A) \geq \theta(B) - \theta(A \cap B) .$$

First, suppose that $\theta(A \cup B) = \theta(A)$. Then, it holds that $\theta(A \cap B) = \theta(B)$, from which it follows that $f \circ \theta(A \cup B) + f \circ \theta(A \cap B) = f \circ \theta(A) + f \circ \theta(B)$. Second, suppose that $\theta(B) = \theta(A \cap B)$. Then, it clearly holds that $f \circ \theta(A \cup B) + f \circ \theta(A \cap B) \geq f \circ \theta(A) + f \circ \theta(B)$. Third and finally, suppose that $\theta(A \cup B) - \theta(A) \geq \theta(B) - \theta(A \cap B) > 0$. Then, the convexity of $f$ implies that

$$\begin{aligned}
\frac{f \circ \theta(A \cup B) - f \circ \theta(A)}{\theta(B) - \theta(A \cap B)} &\geq \frac{f \circ \theta(A \cup B) - f \circ \theta(A)}{\theta(A \cup B) - \theta(A)} \\
&\geq \frac{f \circ \theta(B) - f \circ \theta(A \cap B)}{\theta(B) - \theta(A \cap B)} ,
\end{aligned}$$

which completes the proof. ■

**Lemma A2.** *Suppose $\theta$ is a probability capacity. Then $\theta$ is concave (resp. convex) if and only if $\theta'$ is convex (resp. concave).*

**Lemma A3.** *If $\theta$ is a convex probability capacity, then* $\text{core}(\theta)$ *is non-empty.*

**Choquet Integral** Let $L(S,\bar{\mathbb{R}})$ be the space of $\mathcal{F}$-measurable functions from $S$ into $\bar{\mathbb{R}}$, and let $B(S,\mathbb{R})$ be the subspace of $L(S,\bar{\mathbb{R}})$ which consists of the bounded functions. Then, the *Choquet integral* of $u \in L(S,\bar{\mathbb{R}})$ with respect to a probability capacity $\theta$ is defined by

$$\int u\,d\theta \equiv \int_S u(s)\theta(ds) \equiv \int_{-\infty}^{0} (\theta(\{s|u(s) \geq x\}) - 1)dx + \int_0^{+\infty} \theta(\{s|u(s) \geq x\})dx$$

unless the expression is $-\infty + \infty$.

Two functions $u, v \in L(S,\bar{\mathbb{R}})$ are said to be *co-monotonic* if $(\forall s,t \in S)\ (u(s)-u(t))(v(s)-v(t)) \geq 0$.

**Lemma A4.** *Let $\theta$ be a probability capacity. (a) (Monotonicity)*

$$(\forall u, v \in B(S,\mathbb{R})) \quad u \leq v \ \Rightarrow \int u\,d\theta \leq \int v\,d\theta\,;$$

*(b) (Positive Homogeneity)*

$$(\forall u \in B(S,\mathbb{R}))(\forall a \in \mathbb{R})(\forall b \in \mathbb{R}_+) \quad \int (a+bu)d\theta = a + b\int u\,d\theta$$

*where $a$ in the left-hand side is understood to be a constant function; (c)*

$$(\forall u \in B(S,\mathbb{R})) \quad \int u\,d\theta' = -\int -u\,d\theta\,;$$

*and (d) (Co-monotonic Additivity) if $u, v \in B(S,\mathbb{R})$ are co-monotonic, then*

$$\int (u+v)d\theta = \int u\,d\theta + \int v\,d\theta\,.$$

**Lemma A5.** *Let $\theta$ be a convex probability capacity. (a)*

$$(\forall u \in B(S,\mathbb{R})) \quad \int u\,d\theta = \min\left\{ \int u\,dP \,\middle|\, P \in \mathrm{core}(\theta) \right\}\,;$$

*(b) (Super-additivity)*

$$(\forall u, v \in B(S,\mathbb{R})) \quad \int (u+v)d\theta \ \geq \ \int u\,d\theta + \int v\,d\theta$$

*(the inequality is reversed when $\theta$ is concave); (c)*

$$(\forall u \in B(S,\mathbb{R})) \quad \int u\,d\theta \ \leq \ \int u\,d\theta'\,;$$

*and (d)*

$$(\forall u \in B(S,\mathbb{R})) \quad \left|\int u\,d\theta - \int v\,d\theta\right| \leq \int |u-v|\,d\theta'\,.$$

**Proof.** (a)-(c) are well-known. For (d), see Nishimura and Ozaki (2004). ■

**Lemma A 6** (Monotone Convergence Theorem)**.** *(a) Let $\theta$ be a probability capacity which is continuous from below and let $\langle u_n \rangle_{n=0}^{\infty}$ be a sequence of $\mathcal{F}$-measurable functions such that $u_0 \leq u_1 \leq u_2 \leq u_3 \leq \cdots$ and $\int u_0 \, d\theta > -\infty$. Then,*

$$\lim_{n\to\infty} \int u_n \, d\theta = \int \lim_{n\to\infty} u_n \, d\theta \, .$$

*(b) Let $\theta$ be a probability capacity which is continuous from above and let $\langle u_n \rangle_{n=0}^{\infty}$ be a sequence of $\mathcal{F}$-measurable functions such that $u_0 \geq u_1 \geq u_2 \geq u_3 \geq \cdots$ and $\int u_0 \, d\theta < +\infty$. Then,*

$$\lim_{n\to\infty} \int u_n \, d\theta = \int \lim_{n\to\infty} u_n \, d\theta \, .$$

**Proof.** See Nishimura and Ozaki (2004). ■

Note that by the monotone convergence theorem (Lemma A6), all of the above lemmas concerning the Choquet integral hold true for any continuous capacity $\theta$ and for any function $u \in L(S, \bar{\mathbb{R}})$ whenever the integral is well-defined.

**Lemma A 7** (Fatou's Lemma)**.** *Let $\theta$ be a probability capacity which satisfies that, for any sequence of $\mathcal{F}$-measurable subsets of $S$, $\langle A_n \rangle_{n=1}^{\infty}$, such that $A_1 \supseteq A_2 \supseteq \cdots$, $\limsup_{n\to\infty} \theta(A_n) \leq \theta\left(\limsup_{n\to\infty} A_n\right)$. Also, let $\langle u_n \rangle_{n=1}^{\infty}$ be a sequence of non-negative $\mathcal{F}$-measurable functions which is uniformly bounded from above. Then, it holds that*

$$\limsup_{n\to\infty} \int u_n d\theta \leq \int \limsup_{n\to\infty} u_n d\theta \, .$$

**Proof.** Let $\langle u_n \rangle$ be given as such. Then,

$$\begin{aligned}
\limsup_{n\to\infty} \int u_n(s)\, \theta(ds) &= \limsup_{n\to\infty} \int_0^{\infty} \theta\left(\{s \,|u_n(s) \geq t\}\right) dt \\
&\leq \int_0^{\infty} \limsup_{n\to\infty} \theta\left(\{s \,|u_n(s) \geq t\}\right) dt \\
&\leq \int_0^{\infty} \limsup_{n\to\infty} \theta\left(\bigcup_{k=n}^{\infty} \{s \,|u_k(s) \geq t\}\right) dt \\
&\leq \int_0^{\infty} \theta\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \{s \,|u_n(s) \geq t\}\right) dt
\end{aligned}$$

$$\leq \int_0^\infty \theta\left(\left\{s \,\middle|\, \limsup_{n\to\infty} u_n(s) \geq t\right\}\right) dt$$
$$= \int \limsup_{n\to\infty} u_n(s)\,\theta(ds)\,,$$

where the first inequality holds by "usual" Fatou's lemma since $\langle u_n \rangle$ is uniformly bounded from above; the second inequality is trivial; the third inequality holds by the assumption on $\theta$; and the fourth inequality follows from the definition of lim sup of sequence of both sets and real numbers. ■

**Capacitary Kernel and Its Properties** A mapping $\theta : \mathcal{F} \times S \to [0,1]$ is a *capacitary kernel* (from $S$ to $S$) if it satisfies

$$(\forall s \in S) \quad \theta(\cdot|s) \text{ is a probability capacity on } (S,\mathcal{F}) \text{ and}$$
$$(\forall B \in \mathcal{F}) \quad \theta(B|\cdot) \text{ is } \mathcal{F}\text{-measurable.}$$

A capacitary kernel is *convex* (resp. *continuous*) if $(\forall s)$ $\theta(\cdot|s)$ is convex (resp. continuous).

**Lemma A8** (Fubini Property). *Let $\theta$ be a capacitary kernel such that $(\forall s)$ $\theta(\cdot|s)$ is continuous. Then for any $(\mathcal{F}\otimes\mathcal{F})$-measurable function $u$, the mapping*

$$s \mapsto \int u(s,s_+)\,\theta(ds_+|s)$$

*is $\mathcal{F}$-measurable.*

**Proof.** See Nishimura and Ozaki (2004). ■

In the rest of the Appendix, we set $(S,\mathcal{F}) = (Z,\mathcal{B}_Z)$, where $Z := [\underline{z},\bar{z}]$ with $0 \leq \underline{z} \leq \bar{z}$ and $\mathcal{B}_Z$ is the Borel $\sigma$-algebra on $Z$. A capacitary kernel $\theta$ is *stochastically nondecreasing* if for each nondecreasing function $h : Z \to \mathbb{R}$, the mapping defined by

$$z \mapsto \int_Z h(z')\theta(dz'|z) \tag{19}$$

is nondecreasing.

**Lemma A9** (Stochastic Nondecrease)**.** *A continuous capacitary kernel $\theta$ is stochastically nondecreasing if and only if a mapping defined by*

$$z \mapsto \theta(\{z' \in Z | z' \geq t\} | z) \tag{20}$$

*is nondecreasing for each $t \geq 0$.*

**Proof** ($\Rightarrow$) Suppose that $\theta$ is stochastically nondecreasing and let $t \geq 0$. Also let $\chi_{\{w \in Z | w \geq t\}}$ be the characteristic function[7] of the set $\{w \in Z | w \geq t\}$. Since $\chi_{\{w \in Z | w \geq t\}}(z')$ is nondecreasing in $z'$ on $Z$, the stochastic nondecrease of $\theta$ implies that

$$\int_Z \chi_{\{w \in Z | w \geq t\}}(z') \theta(dz' | z) = \theta(\{z' \in Z | z' \geq t\} | z)$$

is nondecreasing in $z$, which proves the claim.

($\Leftarrow$) Suppose that $\theta(\{z' \in Z | z' \geq t\} | z)$ is nondecreasing in $z$ for each $t \geq 0$. Let $h : Z \to \mathbb{R}_+$ be a nondecreasing function. For $k \geq 1$, let $i(k) := k2^k + 1$. For $i \in \{1, \ldots, i(k) - 1\}$ and $k \geq 1$, let $S(i, k) := \{z | (i-1)/2^k \leq h(z) \leq i/2^k\}$ and $a_{i,k} := (i-1)/2^k$. Let $S(i(k), k) := \{z | (i(k) - 1)/2^k \leq h(z)\}$ and let $a_{i(k),k} := (i(k) - 1)/2^k$. Finally, define $h_k$ by

$$h_k(z) := \sum_{i=1}^{i(k)} a_{i,k} \chi_{S(i,k)}(z)$$

for each $z \in Z$ and $k \geq 1$. Note that the sequence of functions $\langle h_k \rangle_{k=1}^{\infty}$ converges upwards to $h$ pointwise. Then, by definition of $h_k$ and the Choquet integral,

$$\begin{aligned}
& \int_Z h_k(z') \theta(dz' | z) \\
= \; & \frac{1}{2^k} \theta \left( \bigcup_{i=2}^{i(k)} S(i,k) \middle| z \right) + \frac{1}{2^k} \theta \left( \bigcup_{i=3}^{i(k)} S(i,k) \middle| z \right) + \cdots + \frac{1}{2^k} \theta \left( S(i(k), k) \middle| z \right) \\
= \; & \frac{1}{2^k} \Big[ \theta(\{z' | h_k(z') \geq 1/2^k\} | z) + \theta(\{z' | h_k(z') \geq 2/2^k\} | z) + \\
& \cdots + \theta(\{z' | h_k(z') \geq (k2^k - 1)/2^k\} | z) \Big].
\end{aligned}$$

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[7]The characteristic function $\chi_A$ of $A \in \mathcal{B}_Z$ is defined by

$$(\forall z') \quad \chi_A(z') = \begin{cases} 1 & \text{if} \quad z' \in A \\ 0 & \text{if} \quad z' \notin A. \end{cases}$$

Note that $\theta(\{z'|h_k(z') \geq i/2^k\}|z)$ is nondecreasing in $z$ for each $i \in \{1, \ldots, k2^k - 1\}$ by the supposition of the theorem since $h_k$ is nondecreasing. Since $\int_Z h_k(z')\theta(dz'|z)$ is a sum of nondecreasing functions, it is nondecreasing in $z$. Finally, $\int_Z h(z')\theta(dz'|z)$ is nondecreasing in $z$ since it is the limit of the sequence of nondecreasing functions by the monotone convergence theorem (Lemma A6). ■

A capacitary kernel $\theta$ is *stochastically convex* if for each nondecreasing function $h : Z \to \mathbb{R}$, the mapping defined by (19) is convex. An analogous result to Theorem A9 holds for stochastic convexity.

**Lemma A10** (Stochastic Convexity)**.** *A continuous capacitary kernel $\theta$ is stochastically convex if and only if a mapping defined by (20) is convex for each $t \geq 0$.*

**Proof** The proof is the same as that of Theorem A9 except that we use the fact that a sum of convex functions is convex and the fact that the limit of the sequence of convex functions is convex. ■

A capacitary kernel is *upper semi-continuous (u.s.c.)* if for any sequence of $\mathcal{B}_Z$-measurable subsets of $Z$, $\langle A_n \rangle_{n=1}^{\infty}$, such that $A_n \supseteq A_{n+1} \supseteq \cdots$ and for any sequence $\langle z_n \rangle_{n=1}^{\infty} \subseteq Z$ such that $z_n \to z_0$, it holds that

$$\limsup_{n\to\infty} \theta\left(A_n|z_n\right) \leq \theta\left(\lim_{n\to\infty} A_n \middle| z_0\right).$$

We say that a capacitary kernel $\theta$ is *strongly continuous* if for any sequence $\langle z_n \rangle_{n=1}^{\infty} \subseteq Z$ which converges to $z_0$, it holds that

$$\lim_{n\to\infty} \sup_{A\in\mathcal{B}_Z} |\theta(A|z_n) - \theta(A|z_0)| \to 0\,.$$

**Lemma A 11** (U.s.c.)**.** *Assume that a stochastic kernel $\theta$ is strongly continuous and that $(\forall z)$ $\theta(\cdot|z)$ is continuous from above. Then, $\theta$ is u.s.c.*

**Proof** To prove the claim, let $\langle A_n \rangle_{n=1}^{\infty} \subseteq \mathcal{B}_Z$ be such that $A_n \supseteq A_{n+1} \supseteq \cdots$, let $\langle z_n \rangle_{n=1}^{\infty} \subseteq Z$ be such that $z_n \to z_0$ and let $\varepsilon > 0$ be arbitrarily chosen. Then, there exists $N_1 \in \mathbb{N}$ such that

$n > N_1$ implies

$$|\theta(A_n|z_n) - \theta(A_n|z_0)| < \frac{\varepsilon}{2}$$

because $\theta$ is strongly continuous by assumption. Furthermore, there exists $N_2 \in \mathbb{N}$ such that $n > N_2$ implies

$$|\theta(A_n|z_0) - \theta(\lim_{n\to\infty} A_n|z_0)| < \frac{\varepsilon}{2}$$

because $\theta(\cdot|z_0)$ is continuous from above by assumption and because $A_n \downarrow \lim_{n\to+\infty} A_n$ as $n \to +\infty$. Let $N := \max\{N_1, N_2\}$. Then for any $n > N$, it holds that

$$\begin{aligned} & |P(A_n|z_n) - P(\lim_{n\to\infty} A_n|z_0)| \\ \leq\ & |P(A_n|z_n) - P(A_n|z_0)| + |P(A_n|z_0) - P(\lim_{n\to\infty} A_n|z_0)| \\ <\ & \varepsilon\,, \end{aligned}$$

which in turn implies that for any $n > N$,

$$P(A_n|z_n) < P(\lim_{n\to\infty} A_n|z_0) + \varepsilon\,.$$

Since $\varepsilon > 0$ is arbitrary, the proof is complete. ■

**Lemma A12.** *Assume that $f : [0,1] \to [0,1]$ is a convex and continuous function satisfying $f(0) = 0$ and $f(1) = 1$. Also, assume that $\theta$ is a convex and continuous capacitary kernel which is stochastically nondecreasing (resp. stochastically convex, u.s.c.). Then, a mapping $f \circ \theta : \mathcal{B}_Z \times Z \to [0,1]$ defined by*

$$(\forall A)(\forall z) \quad (f \circ \theta)(A|z) = f(\theta(A|z))$$

*is a convex and continuous capacitary kernel which is stochastically nondecreasing (resp. stochastically convex, u.s.c.).*

**Proof** That $f \circ \theta$ is a convex and continuous capacitary kernel follows by Lemma A1. The stochastic nondecrease of $f \circ \theta$ follows by Lemma A9 and by the fact that $f$ is nondecreasing. Its stochastic convexity follows by Lemma A10 and by the convexity of $f$. Finally, its u.s.c. follows by the definition of u.s.c. and by the nondecrease and the continuity of $f$. ■

A capacitary kernel $\theta : \mathcal{B}_Z \times Z \to [0,1]$ is called *stochastic kernel* (Stokey and Lucas, 1989, p.226) if $(\forall z)$ $\theta(\cdot|z)$ is countably additive (that is, a probability measure). Clearly, a stochastic kernel is a convex and continuous capacitary kernel. However, a stochastic kernel need not be stochastically nondecreasing nor u.s.c.

The next lemma (Lemma A13) provides an example of a stochastic kernel $P$ which is stochastically nondecreasing, strongly continuous (and hence, u.s.c. by Lemma A11) as well as stochastically convex.

**Lemma A13.** *Let $Z = [0,1]$ and let $P$ be a stochastic kernel defined by*

$$(\forall z, t \in Z) \quad F(t|z) = P(\,[0,t]\,|\,z\,) = \int_0^t (2-z)\,d\mu\,,$$

*where $F$ is the associated distribution function and $\mu$ is the Lebesgue measure. That is, $P(\cdot|z)$ is the uniform distribution on $[0, 1/(2-z)]$. Then, $P$ is stochastically nondecreasing, strongly continuous and stochastically convex.*

**Proof** (a) Stochastic nondecrease. Note that

$$(\forall z, t) \quad P(\{z' \in Z | z' \geq t\}|z) = 1 - F(t|z) = \max\{0, 1-(2-z)t\}\ , \tag{21}$$

where the first equality holds because the Legesgue measure is atomless. Since $1 - F(t|z)$ is nondecreasing in $z$ for each $t$, the stochastic nondecrease follows from Lemma A9. (b) Strong continuity. Scheffé's theorem (Billingsley, 1986, p.218, Theorem 16.11) shows that $\| P(\cdot|z_n) - P(\cdot|z_0) \| \to 0$ as $n \to +\infty$ since the density function of $P(\cdot|z_n)$ exists and converges to that of $P(\cdot|z_0)$ except at $1/(2-z_0)$, where $\| \cdot \|$ is the total variation norm, which implies the strong continuity of $P$. (c) Stochastic convexity. From (21), we see that $1 - F(t|z)$ is convex in $z$ for each $t$. Hence, the stochastic convexity follows from Lemma A10. ■

Suppose that $P$ is a stochastic kernel which is stochastically nondecreasing, u.s.c. and stochastically convex such as the one in Lemma A13. Also, suppose that $f : [0,1] \to [0,1]$ is a convex and continuous function satisfying $f(0) = 0$ and $f(1) = 1$. Then, by Lemma A12, $\theta := f \circ P$ is convex and continuous capacitary kernel which is stochastically nondecreasing and u.s.c. (as well as stochastically convex) and hence satisfies all the assumpitons in the main text.

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