CARF Working Paper

CARF-F-340

Speculative Attacks with Multiple Targets

Junichi Fujimoto
The University of Tokyo

January 2014

CARF is presently supported by Bank of Tokyo-Mitsubishi UFJ, Ltd., Dai-ichi Mutual Life Insurance Company, Meiji Yasuda Life Insurance Company, Nomura Holdings, Inc. and Sumitomo Mitsui Banking Corporation (in alphabetical order). This financial support enables us to issue CARF Working Papers.

Speculative Attacks with Multiple Targets

Junichi Fujimoto*
University of Tokyo

First Version: October 2009
This Version: January 2014

Abstract

This paper examines a global games model of speculative attacks in which speculators can choose to attack any one of a number of targets. In the canonical global games model of speculative attacks with a single target, it is well known that there exists a unique equilibrium that survives iterative deletion of dominated strategies, characterized by the threshold values of the private signal and the fundamentals. This paper shows that with two targets, there is again a unique, dominance-solvable equilibrium. In this equilibrium, the threshold value of signal for attacking a given currency is a function of the signal for the other target, and the threshold value of fundamentals that determines the outcome of attack on one currency is a function of the other target's fundamentals. Under certain condition on the noise distribution, the result is shown to extend to environments with any N symmetric targets. This paper then presents a number of numerical examples and shows, among other results, that more accurate private signals have a decoupling effect on the outcomes of attack on different currencies.

KEYWORDS: Currency crises; Speculative attacks; Global games

JEL CLASSIFICATION: F31, D84, D82

*I thank Andrew Atkeson, Toni Braun, Matthias Doepke, Gary Hansen, Christian Hellwig, Selahattin Imrohoroglu, Noritaka Kudoh, Junsang Lee, Daisuke Miyakawa, Stephen Morris, Jaeok Park, Guillaume Plantin, Jakub Steiner, and seminar participants at the 2009 GRIPS Conference on Economic Growth, Dynamics, and Policies, the 2010 Academia Sinica Conference on Growth, Trade and Dynamics, the 2010 Econometric Society World Congress, the 2010 European Economic Association Annual Congress, the 2012 Society for Economic Dynamics Meeting, the Development Bank of Japan, the University of Hokkaido, the National Taiwan University, and Yonsei University, for comments and suggestions. This research was supported by the Japan Securities Scholarship Foundation, the Zengin Foundation for Studies on Economics and Finance, and the Center for Advanced Research in Finance (CARF) at the Graduate School of Economics of the University of Tokyo. E-mail: junichif@e.u-tokyo.ac.jp

1 Introduction

Global games of regime change are "coordination games of incomplete information in which a status quo is abandoned once a sufficiently large fraction of agents attack it" (Angeletos, Hellwig, and Pavan (2007)). In the typical setup, agents receive a noisy private signal of the fundamentals, which represents the strength of the regime, or the status quo. The agents then individually decide whether to attack, or more generally take an action against, the regime, and when the fraction of agents attacking exceeds a certain threshold, which depends on the fundamentals, the attack succeeds and the regime is abandoned. Since the seminal work of Morris and Shin (1998), which applies such games to analyze speculative attacks against a currency peg, these games have been actively applied to model a wide range of crisis situations.

This paper extends the literature on global games of regime change by allowing agents to attack any one of multiple regimes, or targets. Such a situation may arise in a number of real-life environments. In the context of currency crises, speculators may, as during the Asian crisis of 1997-98, face multiple emerging countries with currency pegs, and choose from among these currencies to allocate their limited resources for attack, such as wealth. Similar scenarios apply to sovereign debt crises, including the recent episode in Europe in which the debt and credit default swaps of several countries, rumored to be facing potential default, were subject to speculative trading. Thus, while this paper places the discussion in the context of currency crises, its implications extend to a much broader context.[1]

In the canonical global games model of regime change with a single target, it is well known that iterative deletion of (strictly) dominated[2] strategies yields a unique equilibrium.[3] The equilibrium is a *threshold equilibrium* characterized by a threshold value of the private signal that determines whether an agent participates in an attack, and by a threshold value of the fundamentals that determines whether the regime is abandoned.

The main result established in this paper is that a unique, dominance-solvable equilibrium continues to exist in a variety of environments with multiple targets. The paper first shows that with two targets, the equilibrium is always unique and dominance solvable. This equilibrium is a threshold equilibrium characterized by *threshold signal functions* and *threshold fundamentals functions*—for each country, the threshold value of the private signal is a function of an agent's signal for the other country, and the threshold value of the fundamentals is a function of the fundamentals of the other country. This paper then shows, using the contraction mapping theorem, that with any N targets that are ex

[1]Other possible applications include situations in which investors decide which project to invest in, or rioters choose which government facility to attack, where the success of these activities requires the participation of at least a minimum mass of agents. Using the global games approach, Dasgupta (2007) and Atkeson (2000) analyze similar situations with, respectively, a single project and target of attack.

[2]Throughout this paper, "dominated" implies being strictly, rather than weakly, dominated.

[3]See, e.g., Morris and Shin (1998). Note that multiple equilibria may arise when there is public information about the fundamentals, either exogenous (Hellwig (2002)) or endogenous (Angeletos and Werning (2006), Hellwig, Mukherji, and Tsyvinski (2006)) that is sufficiently precise.

ante homogeneous, there exists a threshold equilibrium. Under some assumption on the noise distribution, this is shown to be the unique, dominance-solvable equilibrium.

The key to proving these results is to extend the iterative deletion procedure to the multiple-target environment. As in the case of $N = 1$, this procedure involves generating sequences of the threshold signal and fundamentals from "below" and "above", or from the lower and upper dominance regions. However, here the objects of interest are functions, and since speculators compare the expected profits from attacking different targets, iterative deletion requires proceeding simultaneously from below and above. A unique equilibrium obtains once the two function sequences converge to the same limit.

In such a unique equilibrium, an agent attacks a country more aggressively when receiving strong signals for other countries, and a country is more vulnerable to an attack when other countries have strong fundamentals.[4] This result is in line with actual crisis episodes, in which speculators, facing multiple potential targets, concentrate their attacks on the "weakest link". The model thus provides a complementary perspective to models of contagion, which also examine attacks on multiple targets, but in a sequential framework.[5]

This paper then explores, mainly focusing on the two-country case, several numerical examples, and obtains interesting results. First, the presence of a second target facilitates the survival of a target, if the total measure of speculators is fixed; if the measure of speculators doubles as does the number of targets, however, the range of fundamentals in which both countries sustain the peg becomes smaller than when the two countries separately face speculative attacks. Second, increased precision in speculators' private signals has a *decoupling effect*; since more accurate signals allow speculators to better discern the country with the weaker fundamentals and to concentrate their attack on that country, the two countries are more likely to face different outcomes from an attack.

Finally, this paper examines the introduction of exogenous public information as an extension. The precision of public and private information turns out to play a key role in guaranteeing a unique equilibrium in environments with multiple targets, as with the known sufficient condition for equilibrium uniqueness with a single target.

An important theoretical feature of this paper's model is that, unlike typical global games models, it does not belong to supermodular games, nor games of strategic complementarities (GSC).[6] The model certainly has elements of strategic complementarities, the key feature in such games, in that each speculator has greater incentive to attack a given currency when others do so. In the context of the model, however, supermodular games also require each speculator to have the freedom to attack multiple currencies at

[4]Throughout this paper, increasing (decreasing) implies strictly increasing (decreasing), and nondecreasing (nonincreasing) implies weakly increasing (decreasing).

[5]The relation of this paper to the issue of contagion of crises is discussed in Section 6.1.

[6]For details on these games, see Milgrom and Roberts (1990), Vives (1990, 2005), and Topkis (2001). This paper follows the usage in Vives (2005), according to which GSC is an intuitive concept referring to games in which the best responses of agents are increasing in the actions of rivals, whereas the technical concept of supermodular games provides sufficient conditions for the best responses to be increasing.

the same time, and to have greater incentive to attack a given currency when attacking another currency or currencies. This is not the case here since speculators can attack at most one target, which technically corresponds to an agent's action set not being a *lattice*.[7] Thus, one cannot apply the known results for these games, such as the existence of the largest and smallest pure strategy equilibria, obtained by lattice theoretic methods.

The model of this paper has similar appearances as the models of Oury (2009, 2013), which extend the general global games environment of Frankel, Morris, and Pauzner (2003) by introducing multidimensional actions and payoff parameters. However, Oury (2009, 2013) do not restrict agents' action sets as this paper does,[8] and pursue multidimensional global games within the framework of GSC. Thus, while the models of Oury (2009, 2013) are quite general, they do not include the present model as a special case, which is why the mathematical approach used in this paper and in these studies differ completely. From the practical standpoint, while the models of Oury (2009, 2013) have many potential applications, the model of this paper has an advantage when analyzing situations in which an agent's overall decisions are naturally constrained by the agent's total available resources. To elaborate, the setup of this paper corresponds to a situation in which wealth constrains one's total amount of short selling in all target currencies. This is more plausible than if wealth constrained one's short position in *each* currency separately, as would be the case if one instead applied the framework of Oury (2009, 2013).

This paper also has some similarities with Steiner (2007), which examines a *mobile game* in which agents in a sector coordinate to make an investment and have the outside option of leaving the current sector for other sectors. However, agents in Steiner (2007) make the binary decision of staying or leaving based on the private signal of the fundamentals of only the current sector, instead of directly choosing one of multiple sectors based on the signals of the fundamentals of all sectors, as do the speculators in this paper. The twin crises model of Goldstein (2005) also addresses multiple regimes, the bank and the currency peg. In Goldstein (2005), however, the two regimes share the same fundamentals and face different groups of agents, depositors and speculators, whereas in the present paper, multiple regimes with different fundamentals face the same group of agents. Therefore, Goldstein (2005) does not pursue the problem of choosing from

[7] A lattice is a partially ordered set in which any two elements have a least upper bound (sup) and a greatest lower bound (inf) in the set. In supermodular games and GSC of incomplete information, each action set is assumed to be at least a *complete* lattice, or a lattice in which every nonempty subset has a sup and an inf. This requirement is satisfied for general unidimensional global games models, since any subset of $\mathbb{R}$ is a lattice, and any finite lattice is complete. Footnote 12 explains why this is not true for the present model. The global games models of Dasgupta (2004) and Goldstein and Pauzner (2005) also fail to be GSC, but for a different reason, namely, lack of global strategic complementarities.

[8] In Oury (2009), each action set is a finite (hence, complete) lattice. In Oury (2013), each action set is a finite linearly ordered set, which is again a complete lattice. Thus, in both studies, the action set has a greatest and a smallest element, which is critical in the analysis. The action set is a finite lattice also in McAdams (2003), which provides a sufficient condition for the existence of an isotone pure strategy equilibrium in a class of games of incomplete information with multidimensional actions and types.

multiple regimes, the key theme of the present paper.

Finally, this paper is related to studies that adopt the contraction mapping approach to examine the equilibrium of games of incomplete information. In particular, Mathevet (2010) considers a finite unidimensional global games model in which the existence of pure strategy equilibria follows from the theory of GSC, and proves the uniqueness of equilibrium by showing that the best response function is weak contraction. The present paper considers a multidimensional global games environment that does not belong to GSC, and applies the contraction mapping theorem to show the existence of a threshold equilibrium in one of the cases analyzed, which differs substantially from Mathevet (2010).[9]

2 Model

The model follows a simplified version of the model of Morris and Shin (1998), except that speculators can choose to attack one of multiple potential targets, creating a new dimension of coordination. There are $N > 1$ countries indexed by $j \in J \equiv \{1, 2, \ldots, N\}$, whose currencies are also referred to as currency j. Each currency is pegged to a foreign currency. Country j's economic fundamentals are denoted as $\theta^j \in \mathbb{R}$, and $(\theta^j)_{j \in J}$ are independently drawn from a uniform distribution over $\mathbb{R}$.[10]

There is a continuum of risk-neutral speculators, indexed by $i \in [0, 1]$. Instead of observing the true values of θ^j, speculators receive noisy private signals of their realization. Speculator i's private signal of θ^j, denoted as x_i^j, is expressed as

$$x_i^j = \theta^j + \epsilon_i^j, \tag{1}$$

where $\left(\epsilon_i^j\right)_{j \in J}$ are independent across i and j, and each ϵ_i^j is drawn from a distribution with the cumulative density function (cdf) Ψ^j and the probability density function (pdf) ψ^j. The cdf Ψ^j is defined on $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$, where $\Psi^j(-\infty) = 0$ and $\Psi^j(\infty) = 1$. The pdf ψ^j is continuous and strictly positive over $\mathbb{R}$.[11]

Based on these N signals $x_i \equiv \left(x_i^1, x_i^2, \ldots, x_i^N\right) = \left(x_i^j, x_i^{-j}\right)$, speculators individually decide whether to attack any, but at most one, of the N currencies. In what follows, superscript $-j$ implies that the variable pertains to $N-1$ countries (or currencies) other than j. Speculators' common action set is $C = \left\{\left(a^1, a^2, \ldots, a^N\right) \in \{0, 1\}^N | \Sigma_{j \in J} a^j \in \{0, 1\}\right\}$, where $a^j = 1$ implies attacking currency j and $a^j = 0$ implies not attacking it. The restriction $\Sigma_{j \in J} a^j \in \{0, 1\}$ is theoretically very important, since it prevents C from being

[9]Levin (2001) and Mason and Valentinyi (2007) also resort to the contraction mapping argument to prove equilibrium uniqueness in classes of games related to, but distinct from, global games.

[10]This assumption on the distribution of θ^j implies that speculators have uninformative (or improper) priors on θ^j, such that with the independent noise assumed below, a speculator's posterior belief on θ^j depends only on the private signal for θ^j. For a discussion on improper priors, see Hartigan (1983).

[11]Normal distribution satisfies this assumption, and is assumed in Sections 5 and 6.2.

a lattice,[12] as required for supermodular games and GSC. Speculator i's pure strategy is a function $s_i : \mathbb{R}^N \to C$. Each component function of s_i is written as s_i^j, that is, $s_i(x_i) = \left(s_i^1(x_i), s_i^2(x_i), \ldots, s_i^N(x_i)\right)$.

While I focus on pure strategy, and do not consider partial attacks, that is inessential for the analysis below. Thus, assuming that a speculator can attack at most one target is equivalent to imposing a constraint on the total amount of each speculator's short selling, which can be justified, for example, by margin requirements for taking a short position.[13]

If the measure of speculators attacking currency j[14] equals or exceeds the realization of θ^j, the attack against j succeeds and country j abandons the peg; currency j then depreciates, providing profits to those who attacked j. Otherwise, the attack against j fails and country j sustains the peg. Thus, country j always abandons the peg for $\theta^j \leq 0$, and never does so for $\theta^j > 1$.

Speculators' payoffs are summarized in Table 1. The payoff from attacking a particular currency depends only on the success of that attack, and not on the outcomes of attacks on other currencies. Speculators attacking currency j receive $1 - c^j > 0$ if j abandons the peg, and $-c^j < 0$ otherwise, where $c^j \in (0, 1)$ is a transaction cost, which may differ across j. Speculators attacking none of the currencies receive 0 for sure.

	Country j abandons the peg	Country j sustains the peg
Attack currency j	$1 - c^j$	$-c^j$
Attack no currency	0	0

Table 1: Payoffs.

3 Analysis

3.1 Threshold Equilibrium

It is well known that the canonical global games model, where $N = 1$, has a unique, dominance-solvable equilibrium. Letting x_i be speculator i's signal and θ be the fun-

[12] To see this, let $N = 2$. Then, $C = \{(0,0), (1,0), (0,1)\}$, which is not a lattice under the product order, since the sup of $(1,0)$ and $(0,1)$ is not in C. If attacking both countries were allowed, C would include $(1,1)$ and become a lattice. This conclusion is unaltered by letting $C \equiv \{0,1,2\}$, where the action $a = j \in \{1,2\}$ implies attacking country j, and $a = 0$ implies attacking none. This time, $a = 1$ and 2 are not ordered, since countries are numbered arbitrarily, so the sup of $\{1\}$ and $\{2\}$ is not in C.

[13] Such a wealth constraint is generally present in global games models of currency crises (see, e.g., Morris and Shin (1998) and Corsetti, Dasgupta, Morris, and Shin (2004)). A similar constraint naturally arises in the context of investment in projects or political riots, to which this paper's model can also be applied.

[14] This equals the fraction of speculators attacking j, since there is a measure one of speculators. One numerical exercise in Section 5.1, however, discusses a case where the measure of speculators differs from one.

damentals, the equilibrium is characterized by thresholds (x_s^*, θ_s^*) such that speculator i attacks if and only if $x_i \leq x_s^*$, and the peg collapses if and only if $\theta \leq \theta_s^*$.[15]

In the present model, the fundamentals of different countries are independent, and so are the noises in the signals, and further, the payoff from attacking a currency depends only on whether that country sustains the peg. Despite such lack of intrinsic link between countries, the fact that speculators can attack at most one target makes their decision to attack a given currency dependent on the signals of all countries. As a result, whether a country sustains the peg depends on the fundamentals of all countries.

It is then natural to consider equilibria in which the thresholds for the signal and the fundamentals of each country are given as functions of, respectively, the signals and the fundamentals of other countries. The key objects in such equilibria are defined below.

Definition 1 Let $x^{j*} : \mathbb{R}^{N-1} \to \bar{\mathbb{R}}$ and $\theta^{j*} : \mathbb{R}^{N-1} \to [0, 1]$ for all $j \in J$. Then,
(a) Speculator i follows a threshold strategy $(x^{j*})_{j \in J}$, where x^{j*} is the threshold signal function for country j, if $s_i^j(x_i) = 1$ for $x_i^j < x^{j*}\left(x_i^{-j}\right)$ and $s_i^j(x_i) = 0$ for $x_i^j > x^{j*}\left(x_i^{-j}\right)$.
(b) θ^{j*} is the threshold fundamentals function for country j, if j abandons the peg if and only if $\theta^j \leq \theta^{j*}(\theta^{-j})$.

A (symmetric) *threshold equilibrium* is an equilibrium in which for some $(x^{j*}, \theta^{j*})_{j \in J}$, all speculators follow the threshold strategy $(x^{j*})_{j \in J}$, and $(\theta^{j*})_{j \in J}$ are the threshold fundamentals functions. I also call such an equilibrium a *threshold equilibrium* $(x^{j*}, \theta^{j*})_{j \in J}$, and the functions $(x^{j*}, \theta^{j*})_{j \in J}$ characterizing it *equilibrium threshold functions.*

For $N = 2$, I show that there exists a unique, dominance-solvable equilibrium, which is a threshold equilibrium. For any N countries with the same cost of attack and noise distribution, I show that there exists a threshold equilibrium, which, under certain condition on the noise distribution, is again unique and dominance solvable.

In the next subsection, I explain the strategy used to establish equilibrium uniqueness.

3.2 Strategy for Proving Equilibrium Uniqueness

As a first step, I show that iterative deletion of dominated strategies yields sequences of functions from "below" and "above", $(\underline{x}_n^{j*}, \underline{\theta}_{n+1}^{j*})_{j \in J}$ and $(\bar{x}_n^{j*}, \bar{\theta}_{n+1}^{j*})_{j \in J}$, such that in the n-th round of deletion, not attacking currency j is dominated by attacking j if $x_i^j < \underline{x}_n^{j*}\left(x_i^{-j}\right)$, and attacking j is dominated by not attacking j if $x_i^j > \bar{x}_n^{j*}\left(x_i^{-j}\right)$. Showing that these sequences both converge to some $(x^{j*}, \theta^{j*})_{j \in J}$ completes the proof, because then the threshold strategy $(x^{j*})_{j \in J}$ uniquely survives iterative deletion of dominated strategies.

Below, I provide the basic insight for each of these two steps.

[15] I denote variables in the canonical global games model by dropping superscript j. Appendix A1 discusses how the iterative deletion procedure yields a unique equilibrium in this environment.

3.2.1 Iterative Deletion of Dominated Strategies

With multiple targets, speculators decide not only whether to attack, but also which country to attack. As a result, unlike in the case of a single target, the iterative deletion procedure requires proceeding simultaneously from below and above, by coupling the most pessimistic (optimistic) belief on the success of an attack toward one currency with the most optimistic (pessimistic) beliefs on the success of attacks toward other currencies.[16]

Throughout, let $j = 0$ correspond to attacking no currency, and $J^{-j} \equiv J \backslash \{j\}$ and $J_0^{-j} \equiv \{0\} \cup J \backslash \{j\}$. For any $j \in J$, let $\underline{\theta}_n^{j*}$, $\bar{\theta}_n^{j*} : \mathbb{R}^{N-1} \to [0,1]$ be such that, at the beginning of the n-th round of deletion, all speculators believe that country j abandons the peg for $\theta^j \leq \underline{\theta}_n^{j*}(\theta^{-j})$, and sustains the peg for $\theta^j > \bar{\theta}_n^{j*}(\theta^{-j})$. Note that $\underline{\theta}_1^{j*} = 0$ and $\bar{\theta}_1^{j*} = 1$, since for $\theta^j \leq 0$, country j abandons the peg even if no speculator attacks j, and for $\theta^j > 1$, country j sustains the peg even if all speculators attack it.

Consider the n-th round of deletion. For any $j \in J$, let $\underline{\Gamma}_n^{j*}(x_i)$ ($\bar{\Gamma}_n^{j*}(x_i)$) be the expected payoff from attacking j, given signals x_i, when the threshold fundamentals function for country j is $\underline{\theta}_n^{j*}$ ($\bar{\theta}_n^{j*}$). Then, given x_i, the expected payoff from attacking j must be at least $\underline{\Gamma}_n^{j*}(x_i)$ and at most $\bar{\Gamma}_n^{j*}(x_i)$. Not attacking yields zero, so let $\underline{\Gamma}_n^{0*}(x_i) = \bar{\Gamma}_n^{0*}(x_i) = 0$.

Now, for any $j \in J$, let $\underline{x}_n^{j*}, \bar{x}_n^{j*} : \mathbb{R}^{N-1} \to \bar{\mathbb{R}}$ be such that $x_i^j < \underline{x}_n^{j*}\left(x_i^{-j}\right)$ implies $\underline{\Gamma}_n^{j*}(x_i) > \max_{k \in J_0^{-j}} \bar{\Gamma}_n^{k*}(x_i)$, and $x_i^j > \bar{x}_n^{j*}\left(x_i^{-j}\right)$ implies $\bar{\Gamma}_n^{j*}(x_i) < \max_{k \in J_0^{-j}} \underline{\Gamma}_n^{k*}(x_i)$. Then, if $x_i^j < \underline{x}_n^{j*}\left(x_i^{-j}\right)$, not attacking j is a dominated action, since the expected payoff from attacking j is at least $\underline{\Gamma}_n^{j*}(x_i)$ and that from attacking k is at most $\bar{\Gamma}_n^{k*}(x_i)$. From similar reasoning, if $x_i^j > \bar{x}_n^{j*}\left(x_i^{-j}\right)$, attacking j is a dominated action.

In the $n+1$-th round, it is taken as given that speculators with signals $x_i^j < \underline{x}_n^{j*}\left(x_i^{-j}\right)$ will attack currency j, and those with signals $x_i^j > \bar{x}_n^{j*}\left(x_i^{-j}\right)$ will not, which gives rise to $\underline{\theta}_{n+1}^{j*}$ and $\bar{\theta}_{n+1}^{j*}$. Iterating this procedure from $\underline{\theta}_1^{j*} = 0$ and $\bar{\theta}_1^{j*} = 1$ yields the sequences of functions, $\left((\underline{x}_n^{j*}, \underline{\theta}_{n+1}^{j*})_{j\in J}\right)_{n=1}^{\infty}$ and $\left((\bar{x}_n^{j*}, \bar{\theta}_{n+1}^{j*})_{j\in J}\right)_{n=1}^{\infty}$. I refer to such functions as $\underline{x}_n^{j*}$ and $\underline{\theta}_{n+1}^{j*}$ as *lower threshold functions*, and $\bar{x}_n^{j*}$ and $\bar{\theta}_{n+1}^{j*}$ as *upper threshold functions.*

3.2.2 Convergence of Threshold Functions

One can show that with adequate choices of $\underline{x}_n^{j*}$ and $\bar{x}_n^{j*}$, lower and upper threshold functions become continuous and monotonic, and $\left((\underline{x}_n^{j*}, \underline{\theta}_{n+1}^{j*})_{j\in J}\right)_{n=1}^{\infty}$ and $\left((\bar{x}_n^{j*}, \bar{\theta}_{n+1}^{j*})_{j\in J}\right)_{n=1}^{\infty}$ become, respectively, nondecreasing and nonincreasing sequences. The proofs use these properties and show that both sequences have the same limit, denoted as $(x^{j*}, \theta^{j*})_{j\in J}$. A key role in this analysis is played by the variable defined below.

Definition 2 For any $j \in J$, $\lambda^j \equiv \|\psi^j\| / (1 + \|\psi^j\|)$, where $\|\psi^j\| \equiv \sup_{\epsilon_i^j \in \mathbb{R}} \left|\psi^j\left(\epsilon_i^j\right)\right|$.[17]

Since the pdf ψ^j is positive and continuous over $\mathbb{R}$, $\|\psi^j\|$ is bounded, and thus $\lambda^j \in (0,1)$. When $\psi^j = \psi$, I drop the superscript on λ^j and write λ.

[16] Formal mathematical formulation of the iterative deletion procedure is set forth in Appendix B.
[17] Throughout, $\|\cdot\|$ denotes a sup norm, and $\|\cdot\|_E$ denotes a Euclidean norm.

Now, for $N = 1$, the sequences of threshold signals and fundamentals generated by the iterative deletion procedure, $(\underline{\theta}_n^*, \underline{x}_n^*)_{n=1}^{\infty}$ and $\left(\bar{\theta}_n^*, \bar{x}_n^*\right)_{n=1}^{\infty}$, are shown to satisfy[18]

$$\bar{x}_n^* - \underline{x}_n^* = \bar{\theta}_n^* - \underline{\theta}_n^*, \tag{2}$$

$$\bar{\theta}_{n+1}^* - \underline{\theta}_{n+1}^* \leq \lambda \left(\bar{x}_n^* - \underline{x}_n^*\right). \tag{3}$$

Combining (2) and (3) yields

$$\bar{\theta}_{n+1}^* - \underline{\theta}_{n+1}^* \leq \lambda \left(\bar{\theta}_n^* - \underline{\theta}_n^*\right), \tag{4}$$

so given $\lambda \in (0,1)$, $\bar{\theta}_n^* - \underline{\theta}_n^* \to 0$, and thus $\bar{x}_n^* - \underline{x}_n^* \to 0$, as $n \to \infty$. Note that the vital force behind this convergence result is the constant λ in (3), which measures the impact of the difference in threshold signals on the difference in resulting threshold fundamentals.

The proofs for $N > 1$ resort to a similar argument as above and show that as $n \to \infty$, $d_n^j \to 0$ for all $j \in J$, where $d_n^j \equiv \left\|\bar{\theta}_n^{j*} - \underline{\theta}_n^{j*}\right\| = \sup_{\theta^{-j} \in \mathbb{R}^{N-1}} \left|\bar{\theta}_n^{j*}\left(\theta^{-j}\right) - \underline{\theta}_n^{j*}\left(\theta^{-j}\right)\right|$. Now the objects of interest are functions, but there is a close analogue of (3),[19] and the constant λ^j in that expression plays an essential role in establishing this convergence result.

4 Equilibrium

Before describing the equilibrium, I introduce some notations. For any $X \subseteq \mathbb{R}^m$ and g, $g' : X \to \mathbb{R}$, let $g \geq g'$ and $g > g'$ imply, respectively, $g(x) \geq g'(x)$ and $g(x) > g'(x)$ for any $x \in X$. Let similar notations apply when g' is a scalar. Also, defining $X_{\mathbb{R}}(\cdot)$ as below facilitates stating the results involving $x^{j*}\left(x_i^{-j}\right) = -\infty$.

Definition 3 For any $f : \mathbb{R}^m \to \bar{\mathbb{R}}$, $X_{\mathbb{R}}(f) \equiv \{z \in \mathbb{R}^m | f(z) \in \mathbb{R}\}$.

Note that by definition, $X_{\mathbb{R}}(f) = \mathbb{R}^m$ for $f : \mathbb{R}^m \to \mathbb{R}$.

4.1 The Equilibrium with $N = 2$ Targets

Let x_s^{j*} and θ_s^{j*} be the threshold values of signal and fundamentals of country j, when country j is the sole target. The equilibrium for $N = 2$ is then described as below.[20]

Proposition 1 *If $N = 2$, there is a unique, dominance-solvable equilibrium. This is a threshold equilibrium $(x^{j*}, \theta^{j*})_{j \in J}$, where $x^{j*} : \mathbb{R} \to [-\infty, x_s^{j*})$ is nondecreasing in $\mathbb{R}$ and is continuous and increasing in $X_{\mathbb{R}}(x^{j*})$, and $\theta^{j*} : \mathbb{R} \to (0, \theta_s^{j*})$ is continuous and increasing.*

The economic intuition for equilibrium uniqueness is as follows. When $N = 1$, $\left(\underline{x}_n^*, \underline{\theta}_{n+1}^*\right)$ increases with n, because as some of the strategies that attack least aggressively

[18]Derivation of these equations is provided in Appendix A1.

[19]This corresponds to Lemma 7 in Appendix C.

[20]The proofs to all propositions are in Appendix C.

are deleted each round, strategic complementarity makes attacking the peg more profitable for any signal, inducing speculators to attack more aggressively. One can similarly explain why $\left(\bar{x}_n^*, \bar{\theta}_{n+1}^*\right)$ decreases, and thus, why $\left(\underline{x}_n^*, \underline{\theta}_{n+1}^*\right)$ and $\left(\bar{x}_n^*, \bar{\theta}_{n+1}^*\right)$ approach one another, with n. When $N = 2$, $\left(\underline{x}_n^{1*}, \underline{\theta}_{n+1}^{1*}\right)$ and $\left(\bar{x}_n^{2*}, \bar{\theta}_{n+1}^{2*}\right)$ evolve together, and independently from $\left(\bar{x}_n^{1*}, \bar{\theta}_{n+1}^{1*}\right)$ and $\left(\underline{x}_n^{2*}, \underline{\theta}_{n+1}^{2*}\right)$.[21] Now, as some of the strategies that attack currency 2 most aggressively are deleted each round, the strategic complementarity in attacking each currency makes attacking currency 1 even more profitable, providing an additional force for $\left(\underline{x}_n^{1*}, \underline{\theta}_{n+1}^{1*}\right)$ to increase with n. In other words, the increase in $\left(\underline{x}_n^{1*}, \underline{\theta}_{n+1}^{1*}\right)$ and the decrease in $\left(\bar{x}_n^{2*}, \bar{\theta}_{n+1}^{2*}\right)$ reinforce each other, and similarly for $\left(\underline{x}_n^{2*}, \underline{\theta}_{n+1}^{2*}\right)$ and $\left(\bar{x}_n^{1*}, \bar{\theta}_{n+1}^{1*}\right)$, facilitating convergence of the lower and upper threshold functions to the same limit.

That the equilibrium threshold functions are increasing implies that a speculator is more willing to attack currency j when receiving a strong signal for the other country, and country j is more likely to abandon the peg when the other country has strong fundamentals. These properties are intuitive, because what links the two countries is not something intrinsic such as correlations in fundamentals, but the fact that speculators attack the relatively more attractive target. That $x^{j*} < x_s^{j*}$ implies that speculators attack each currency less aggressively than in the single-target case. This is because, when deciding on whether to attack currency j, each speculator expects some other speculators to attack the other target, and thus the expected payoff from attacking currency j becomes lower than when j is the only target. As a consequence, $\theta^{j*} < \theta_s^{j*}$, such that the presence of multiple targets makes sustaining the peg easier for each country.

4.2 The Equilibrium with $N > 2$ Targets

When $N > 2$, each country is compared with more than one country, which substantially complicates the environment. Most importantly, all lower and upper threshold functions now evolve jointly, unlike the case of $N = 2$.[22] This is why the intuition for equilibrium uniqueness for $N = 2$ does not carry over to $N > 2$.

Due to this complication, it is difficult to conduct analyses for general $N > 2$ environments. However, when countries are symmetric in the sense formally defined below, one can exploit the symmetry to derive implications on the equilibrium.

Definition 4 Countries are symmetric if $\psi^j = \psi$, $\Psi^j = \Psi$, and $c^j = c$ for all $j \in J$.

The definition below is used to describe the relevant functions of symmetric countries.

[21] This follows, since, as explained in Section 3.2.1, the iterative deletion procedure couples the most pessimistic (optimistic) belief on the success of an attack toward currency 1 with the most optimistic (pessimistic) belief on the success of an attack toward currency 2.

[22] To see this, let $N = 3$. The argument in Section 3.2.1 then implies that evolution of $\left(\underline{x}_n^{1*}, \underline{\theta}_{n+1}^{1*}\right)$ is affected by that of $\left(\bar{x}_n^{2*}, \bar{\theta}_{n+1}^{2*}\right)$ and $\left(\bar{x}_n^{3*}, \bar{\theta}_{n+1}^{3*}\right)$. Evolution of $\left(\bar{x}_n^{2*}, \bar{\theta}_{n+1}^{2*}\right)$ in turn depends on that of $\left(\underline{x}_n^{1*}, \underline{\theta}_{n+1}^{1*}\right)$ and $\left(\underline{x}_n^{3*}, \underline{\theta}_{n+1}^{3*}\right)$, and similarly for the process of $\left(\bar{x}_n^{3*}, \bar{\theta}_{n+1}^{3*}\right)$.

Definition 5 (1) $h^j : \mathbb{R}^N \to \bar{\mathbb{R}}$, $j \in J$, is symmetric, if $h^j(s) = h^j(\tilde{s})$ for any $s = (s^1, s^2, \ldots, s^N)$ and $\tilde{s} = (\tilde{s}^1, \tilde{s}^2 \ldots, \tilde{s}^N)$ such that $s^j = \tilde{s}^j$ and $\tilde{s}^{-j} = (\tilde{s}^1, \ldots, \tilde{s}^{j-1}, \tilde{s}^{j+1}, \ldots, \tilde{s}^N)$ is any permutation of s^{-j}. (2) $(h^j)_{j \in J}$ are symmetric across j, if each $h^j : \mathbb{R}^N \to \bar{\mathbb{R}}$ is symmetric, and for any $j, k \in J$ and $s = (s^1, s^2, \ldots, s^N)$, $h^j(s) = h^k(s')$ where s' is created from s by exchanging its j-th and k-th elements.

When h^j is a function of s^{-j}, Definition 5 can be applied by considering h^j as a function of $s \in \mathbb{R}^N$, whose values do not vary with s^j.[23] In the present context, that $(h^j)_{j \in J}$ are symmetric across j simply implies that the roles played by N countries are identical.

4.2.1 Existence of a Threshold Equilibrium

I first state the result on the existence of a threshold equilibrium.[24] Note that x_s^* and θ_s^* are the equilibrium threshold signal and fundamentals for $N = 1$.

Proposition 2 *Let $N > 2$ and countries be symmetric. Then, there exists a threshold equilibrium $(x^{j*}, \theta^{j*})_{j \in J}$, where $x^{j*} : \mathbb{R}^{N-1} \to (-\infty, x_s^*)$ is continuous, nondecreasing, and $x^{j*}(x_i^{-j}) \leq \min_{k \in J^{-j}} x_i^k$ with equality for $\min_{k \in J^{-j}} x_i^k$ sufficiently small, $\theta^{j*} : \mathbb{R}^{N-1} \to (0, \theta_s^*)$ is continuous and increasing, and $(x^{j*})_{j \in J}$ and $(\theta^{j*})_{j \in J}$ are symmetric across j.*

The basic idea of the proof is to define a mapping, whose fixed point corresponds to equilibrium threshold fundamentals functions, and apply the contraction mapping theorem. However, as expected from the discussion in Section 3.2.1, the relevant mapping is in general one from $(\theta_n^{j*})_{j \in J}$ to $(\theta_{n+1}^{j*})_{j \in J}$, which, due to its complexity, cannot be shown to be a contraction. When countries are symmetric, however, one may impose $(\theta_n^{j*})_{j \in J}$ to be symmetric across j, and consider a mapping from θ_n^{j*} into θ_{n+1}^{j*}, separately for each j. One may show that each such simplified mapping is a contraction, and that the fixed point functions $(\theta^{j*})_{j \in J}$, and the associated functions $(x^{j*})_{j \in J}$, are equilibrium threshold functions. The constant λ is critical also in this proof, as it serves as the contraction constant.

Due to such a priori restrictions on $(\theta_n^{j*})_{j \in J}$, however, the proof does not rule out other forms of equilibria, namely threshold equilibria in which $(x^{j*})_{j \in J}$ and $(\theta^{j*})_{j \in J}$ are not symmetric across j, nor equilibria in non-threshold strategies.

The threshold strategy in this equilibrium has an intuitive property; $x^{j*}(x_i^{-j}) \leq \min_{k \in J^{-j}} x_i^k$ implies that a speculator attacks only the currency with the weakest signal. That $x^{j*} < x_s^*$ and $\theta^{j*} < \theta_s^*$ can be interpreted just as in the $N = 2$ case.

4.2.2 Uniqueness of the Equilibrium

With some restrictions on the noise distribution, the threshold equilibrium above is shown to be the unique, dominance-solvable equilibrium, as stated below.

[23] For example, let $N = 3$ and consider $\underline{\theta}_n^{1*}(\theta^2, \theta^3)$, $\underline{\theta}_n^{2*}(\theta^1, \theta^3)$, $\underline{\theta}_n^{3*}(\theta^1, \theta^2)$. If $(\underline{\theta}_n^{j*})_{j \in J}$ are symmetric across j, then $\underline{\theta}_n^{j*}(a, b) = \underline{\theta}_n^{j*}(b, a)$ for any $a, b \in \mathbb{R}$ and $j \in J$, and the value is independent of j.

[24] Propositions 2 and 3 below also apply to two symmetric countries. However, the result in Proposition 1 is stronger, except for the symmetry of the equilibrium threshold functions.

Proposition 3 *Let $N > 2$, countries be symmetric, and $\|\psi\| < 1$. Then, the threshold equilibrium in Proposition 2 is the unique, dominance-solvable equilibrium.*

The key to the proof is to choose $\underline{x}_n^{j*}$ and $\bar{x}_n^{j*}$ that are consistent with deletion of dominated strategies and that ensure the monotonicity and continuity of the lower and upper threshold functions.[25] Suppose $\left(\underline{\theta}_n^{j*}\right)_{j\in J}$ and $\left(\bar{\theta}_n^{j*}\right)_{j\in J}$ are symmetric across j, which holds for $n = 1$ since $\underline{\theta}_1^{j*} = 0$ and $\bar{\theta}_1^{j*} = 1$ for all j. Then, $d_n^j = \left\|\bar{\theta}_n^{j*} - \underline{\theta}_n^{j*}\right\|$ is common to all j, so it can be denoted as d_n. One can then find $\underline{x}_n^{j*}$ and $\bar{x}_n^{j*}$ that satisfy the requirement above and $\|\bar{x}_n^{j*} - \underline{x}_n^{j*}\| \leq 2d_n$, and that make $\left(\underline{\theta}_{n+1}^{j*}\right)_{j\in J}$ and $\left(\bar{\theta}_{n+1}^{j*}\right)_{j\in J}$ again symmetric across j. Thus, $d_{n+1} \leq 2\lambda d_n$ from the analogue of (3), so $d_n \to 0$ as $n \to \infty$, since $\|\psi\| < 1$ implies $\lambda = \|\psi\| / (1 + \|\psi\|) < 1/2$. Therefore, there is a unique, dominance-solvable equilibrium, which must coincide with the threshold equilibrium shown in Proposition 2.

The intuition for the condition $\|\psi\| < 1$ follows from the discussion for $N = 1$ in Section 3.2.2. When $\|\psi\|$ is large, there is a point at which the cdf Ψ rises sharply. Then, a given difference in the threshold signal, $\bar{x}_n^* - \underline{x}_n^*$, may lead to a relatively large difference in the fraction of speculators attacking, and thus a relatively large $\bar{\theta}_{n+1}^* - \underline{\theta}_{n+1}^*$, as observed from (3) and the fact that λ is increasing in $\|\psi\|$. Nevertheless, given (2), $\bar{\theta}_n^* - \underline{\theta}_n^* \to 0$ is guaranteed for $N = 1$ since $\lambda < 1$. The same conclusion would hold in the present environment if $\|\bar{x}_n^{j*} - \underline{x}_n^{j*}\| \leq d_n$. But since only a weaker condition, $\|\bar{x}_n^{j*} - \underline{x}_n^{j*}\| \leq 2d_n$, can be shown here, a restriction on $\|\psi\|$ is needed to guarantee $d_n \to 0$.

5 Numerical Examples

This section explores several numerical examples to obtain additional insight on the properties of equilibrium. In this section, ϵ_i^j is assumed to follow $N(0, \beta^j)$. The pdf and cdf of the standard normal distribution are denoted by ϕ and Φ, respectively.

5.1 Symmetric and Nonsymmetric $N = 2$ Targets

This subsection discusses the case of $N = 2$, the main focus of this analysis.

5.1.1 Symmetric Targets

I first consider two symmetric countries, where $c^1 = c^2 = c = 0.1$ and $\beta^1 = \beta^2 = \beta = 1$. Results are symmetric across two countries, so the explanations below apply when the roles of country 1 and 2 are reversed. Figures 1–4 depict the equilibrium threshold functions.

As shown in Figure 1, x^{1*} consists of two parts; one that lies on the 45 degree line, where the speculator is indifferent between attacking currency 1 and 2, and a flatter part, where the speculator is indifferent between attacking currency 1 and not attacking. Note that

[25] For $N > 2$, the fact that each country is compared with more than one country adds complications in ensuring the monotonicity of these functions. For details, see footnote 52 in Appendix C.

$x^{1*} < x_s^*$, as shown in Proposition 2, and that $x^{1*}(x_i^2) \to x_s^*$ as $x_i^2 \to \infty$. Since each speculator expects some other speculators to attack currency 2, the expected payoff from attacking currency 1 is, for any x_i^2, lower than with a single target. Thus, speculators require lower x_i^1 for attacking currency 1, hence $x^{1*} < x_s^*$. But as x_i^2 increases, a speculator infers greater values of θ^2 and expects a smaller fraction of other speculators to attack currency 2, so the presence of currency 2 becomes less relevant for the decision to attack currency 1. As $x_i^2 \to \infty$, the speculator behaves as if currency 1 is the only target. Speculators' attacking decisions for given (x_i^1, x_i^2) are shown in Figure 2. Note that x^{1*} and x^{2*} divide the (x_i^1, x_i^2) space into three regions according to the corresponding attacking decision.

Similarly, Figure 3 indicates that $\theta^{1*} < \theta_s^*$, as shown in Proposition 2, and that $\theta^{1*}(\theta^2) \to \theta_s^*$ as $\theta^2 \to \infty$. Thus, sustaining a peg is easier than in the single-target case, which is natural because the presence of multiple targets serves to diversify the attacking pressure. More importantly, $\theta^{1*}(\theta^2)$ is highly dependent on the value of θ^2. This is more clearly observed in Figure 4, in which θ^{1*} and θ^{2*} divide the (θ^1, θ^2) space into four regions according to the outcomes of attack for the two countries. Figure 4 shows that the success of an attack on one country depends critically on the other country's fundamentals, and not just on its own. Put differently, what matters is not necessarily the fundamentals per se, but their value relative to the other target. Such a feature is entirely lacking in the complete information version of the model in which fundamentals are publicly observed.[26]

The comparison between θ^{1*} and θ_s^* above assumes the same total measure of speculators, normalized to one, for single and two-country environments. One may ask what happens if the per-country, not total, measure of speculators is fixed. In other words, the question is whether doubling both the number of targets and the population of speculators, which makes coordination among speculators more difficult but increases the potential size of the attack toward each country, makes collapse of the peg more likely or less so. To answer this question, Figure 5 draws θ^{1*} and θ^{2*}, along with the lines corresponding to $\theta_{s,0.5}^*$, the threshold value of fundamentals in the single-target case with measure 0.5 of speculators. Note that θ^{1*} can be lower than $\theta_{s,0.5}^*$, such that currency 1 may be able to sustain the peg, even with values of fundamentals that force country 1 to abandon the peg in the single-target case with half the population of speculators. However, this can occur only when country 2 falls victim to attack; indeed, the region of (θ^1, θ^2), under which both countries sustain the peg (i.e., the upper right region of Figure 5 in which both θ^1 and θ^2 are above the relevant threshold), is smaller when the number of targets and the population of speculators are doubled. This finding turns out to be robust to changes in β and c.

Another interesting exercise involves varying β. Figure 6 shows the outcomes of attack when $\beta^1 = \beta^2 = 4$, corresponding to more precise private signals than in Figure 4.[27] Comparing Figures 4 and 6, we observe that in Figure 6, θ^{1*} is higher for relatively large values of θ^2, and is lower for relatively small values of θ^2. As a result, the region of (θ^1, θ^2)

[26] Appendix A2 presents a discussion of this complete information model.

[27] As shown in Appendix A1 (eq (12)), θ_s^* is independent of β, so Figures 4 and 6 are directly comparable.

for which both pegs survive, and the region for which both pegs fail, are both smaller in Figure 6. The implication here is that increased precision of signals has a decoupling effect, in the sense that the two countries are less likely to face the same outcomes of attack. With more precise private signals, speculators are able to better discern the country with the weaker fundamentals and to concentrate their attack on that country. Thus, the country with the stronger fundamentals survives the attack for a wider range of its own fundamentals, at the cost of placing the other country in a more vulnerable position.

5.1.2 Nonsymmetric Targets

I now examine the nonsymmetric case, where $c^1 = c^2 = 0.1$, $\beta^1 = 1$, and $\beta^2 = 4$. Figure 7 illustrates the attacking decisions in the (x_i^1, x_i^2) space. Unlike in Figure 2, the part of x^{1*} where the speculator is indifferent between attacking currency 1 and 2 is steeper than the 45 degree line. The intuition here is that country 2's signal, which is more precise, has a stronger impact on the speculator's decision than country 1's signal. For example, when both x_i^1 and x_i^2 are relatively large (say, equal to 1), speculators recognize that θ^2 is more certain to be strong than θ^1, and hence prefer to attack currency 1 over currency 2. The opposite is true when both x_i^1 and x_i^2 are relatively small.

Figure 8 shows the outcomes of attack in the (θ^1, θ^2) space. Note that the outcome of attack for currency 1 is more strongly dependent on θ^2 than the outcome of attack for currency 2 is on θ^1. Again, this follows since country 2's fundamentals are perceived more accurately by speculators. That is, when country 2 has strong fundamentals, speculators well recognize this and tend to shift their target to country 1, placing country 1 in a vulnerable position. Conversely, when country 2 has weak fundamentals, speculators tend to shift their target to country 2, relaxing the attacking pressure on country 1.

This explanation suggests that what really matters for the profile of the threshold fundamentals function of a country (say, θ^{1*}) is the precision of the signal for the other country (β^2), not its own (β^1). Indeed, in Figure 8, θ^{1*} resembles that in Figure 6, where $\beta^1 = \beta^2 = 4$, whereas θ^{2*} is similar to that in Figure 4, where $\beta^1 = \beta^2 = 1$. This finding may appear surprising, but it is natural because θ^{1*} represents how country 1's threshold fundamentals are affected by the variations in country 2's fundamentals, and such an effect depends critically on how accurately speculators perceive θ^2, and thus on β^2.

This observation has the following implication on the issue of decoupling mentioned above. Suppose the precision of private signals reflects the transparency of government policy,[28] such that country j's government can, to some extent, control β^j. Then, what country 1's government can affect is mainly how the outcome of attack for currency 2 depends on country 1's fundamentals; the issue that is probably more important to country 1's government, dependence of the outcome of attack for currency 1 on country 2's fundamentals, is instead at the discretion of country 2's government.

[28] Heinemann and Illing (2002) adopts such an interpretation and discusses the impact of transparency on the probability of successful speculative attack.

5.2 Symmetric $N = 3$ Targets

I conclude this section with an example for the symmetric $N > 2$ case. Figures 9 and 10 depict the equilibrium threshold functions for $N = 3$, where $c^j = c = 0.1$ and $\beta^j = \beta = 1$ for all j. Since $\|\psi\| = \sqrt{\beta}\phi(0) = \sqrt{\beta}/\sqrt{2\pi}$, the condition $\|\psi\| < 1$ in Proposition 3 holds for such β, hence there is a unique equilibrium.

Figure 9 depicts x^{3*}. As for $N = 2$, x^{3*} consists of two parts: one that coincides with $\min\{x_i^1, x_i^2\}$, where the speculator is indifferent between attacking currency 3 and either currency 1 or 2, and the flatter part, where the speculator is indifferent between attacking currency 3 and not attacking. While x_s^* is not shown in the figure to avoid graphical clutter, $x^{3*} < x_s^*$ as shown in Proposition 2, and x^{3*} approaches x_s^* as both x_i^1 and x_i^2 tend to ∞. The intuition here is similar to the $N = 2$ case. Compared to the single-target case, speculators attack country 3 less aggressively, since the attacking pressure is spread out over three countries. But as x_i^1 and x_i^2 increase, speculators expect a smaller fraction of other speculators to attack countries 1 and 2, so the presence of these two countries becomes less important for the decision to attack country 3. As both x_i^1 and x_i^2 approach ∞, the speculator behaves as if currency 3 is the only target. Note that x^{1*}, x^{2*} and x^{3*} divide the (x_i^1, x_i^2, x_i^3) space into four regions according to speculators' attacking decisions.

Figure 10 depicts θ^{3*}. Note that θ^{3*} increases with θ^1 and θ^2, approaches 0 as θ^1 or θ^2 tends to $-\infty$, and approaches θ_s^* as both θ^1 and θ^2 tend to ∞. Thus, if there is one country with very weak fundamentals, speculators target this country, relaxing the attacking pressure on the other countries. The $(\theta^1, \theta^2, \theta^3)$ space is divided by θ^{1*}, θ^{2*}, and θ^{3*} into eight regions according to the outcomes of attack for the three countries.

6 Discussions and Extensions

6.1 Relation to the Contagion of Crises

I now discuss how the implications above relate to the issue of contagion of crises. That the survival of the peg in one country is facilitated by the weak fundamentals of other countries may appear as *negative contagion*, contradicting the existing models of contagion and the empirical observations that these models aim to address. But in fact, these models and the model of this paper shed light on different aspects of actual crises.

A contagion of crises involves sequential, rather than simultaneous, speculative attacks on multiple targets. For example, Branson (2001) describes the EMS crisis in 1992 as "Speculation then focused on the weakest link, the Finmark, and proceeded to Swedish krona, sterling, and so on ...", and the Asian crisis in 1997-98 as "... the attacks began in Thailand. ... Since the markets concentrated on one country at a time, speculative pressure was maximized and devaluations overshot." Such sequential attacks on the *weakest link* applies also to the contagion of crises other than currency crises.[29]

[29]For example, regarding the recent European debt crises, many argued that speculators targeted

The existing global games models of contagion of currency or financial crises[30] take as given the sequential nature of attacks, and focus on explaining the mechanisms of contagion, that is, how a successful attack on the first target facilitates an attack on the second target. Such explanation follows also from the present model if there are multiple rounds of attacks. If $N = 2$, for example, once one country abandons the peg, the remaining country faces all speculators alone, such that it must now abandon the peg for a wider range of its own fundamentals.

However, the emphasis of this paper is not on the mechanism of contagion, but on the theory of how speculators choose the weakest link from among multiple candidates. Such an element is absent in the studies above, in which attacks on two targets occur sequentially with an exogenously given order.[31] To put this in the context of the Asian crisis, existing studies explain how the crisis in Thailand placed Indonesia, for example, in a more vulnerable situation. In contrast, the present paper explains why the Thai baht was the first to be heavily attacked. Importantly, the model's implication is that the weak fundamentals of Thailand helped Indonesia to avoid being speculators' first target, not in ultimately sustaining the peg; thus, there is no conflict with the previous studies.

6.2 Introduction of Public Information

Thus far, speculators' information on fundamentals has been restricted to private signals. I now introduce public information through informative priors of the fundamentals.

In the global games model of speculative attacks with a single target, it is shown[32] that multiple equilibria may exist if public information is sufficiently precise relative to private information. For example, if the common prior for θ is $N(w, 1/\alpha)$ and ϵ_i follows $N(0, 1/\beta)$ in the $N = 1$ version of the model of this paper, there may be multiple equilibria if $\alpha/\sqrt{\beta} > \sqrt{2\pi}$. Thus, the sufficient condition for a unique equilibrium is $\alpha/\sqrt{\beta} \leq \sqrt{2\pi}$.

I provide below the corresponding sufficient condition in the case of symmetric multiple targets, for which the analysis is much simpler than for the general two-country case.[33]

Greece as the weakest link in the Eurozone, with a country like Portugal being the next possible candidate.

[30] In Goldstein and Pauzner (2004), the interaction of wealth and risk aversion creates financial contagion through agents' portfolio decisions. Guimaraes and Morris (2007), which generalizes such portfolio decisions by allowing continuous actions, also provides valuable insights on contagion. The wealth effect generates contagion also in Keister (2009), but through a different channel. In Dasgupta (2004), contagion of financial crises occurs due to capital linkage between banks in two different regions.

[31] Outside the global games framework, Botman and Jager (2002) extends the models of Krugman (1979) and Flood and Garber (1984) into a two-country environment, and analyzes the issue of coordination among speculators. In the model, the reserves of both countries decrease deterministically over time in the same fashion, forcing eventual collapse of the peg, but the timing of collapse for each currency depends on the initial beliefs of speculators on the fraction of speculators attacking each of the two currencies.

[32] See, e.g., Morris and Shin (2004) and Hellwig (2002).

[33] For nonsymmetric $N = 2$ targets, equilibrium uniqueness is guaranteed when private information is sufficiently precise relative to public information, as for $N = 1$. The analysis is available upon request.

Proposition 4 *For $N > 2$, let the common prior for θ^j be $N(w, 1/\alpha)$, $\epsilon_i^j \sim N(0, 1/\beta)$, and $c^j = c \in (0,1)$, for all $j \in J$. Then, (1) the conclusion of Proposition 2 holds if $\alpha/\sqrt{\beta} < \sqrt{2\pi}$, and (2) that of Proposition 3 holds if $\alpha < \sqrt{\beta}\left(\sqrt{2\pi} - \sqrt{\beta}\right)/2$.*

To see the intuition for these conditions, suppose the common prior for θ is $N(w, 1/\alpha)$, and $\epsilon_i \sim N(0, 1/\beta)$, in the argument for $N = 1$ in Section 3.2.2. Then, (4) becomes[34]

$$\bar{\theta}^*_{n+1} - \underline{\theta}^*_{n+1} \leq \lambda \frac{\alpha + \beta}{\beta} \left(\bar{\theta}^*_n - \underline{\theta}^*_n\right), \tag{5}$$

so the term λ in (4) is replaced by $\lambda(\alpha + \beta)/\beta$.

Now, as explained in Section 4.2.1, Proposition 2 is proved by considering a relevant function mapping, which turns out to be a contraction with the contraction constant λ. In line with the argument above for $N = 1$, with public information, this constant is replaced by $\lambda(\alpha + \beta)/\beta$. Thus, for the mapping to be a contraction, it requires $\lambda(\alpha + \beta)/\beta < 1$, or equivalently $\alpha/\sqrt{\beta} < \sqrt{2\pi}$, since $\|\psi\| = \sqrt{\beta}/\sqrt{2\pi}$ and $\lambda = \|\psi\|/(1 + \|\psi\|)$.

Further, recall from the discussion of Proposition 3 that with only private information, there is a unique equilibrium if $2\lambda < 1$. As the comparison of (4) and (5) suggests, with public information, a unique equilibrium is ensured under a stronger condition, $2\lambda(\alpha + \beta)/\beta < 1$, or equivalently $\alpha < \sqrt{\beta}\left(\sqrt{2\pi} - \sqrt{\beta}\right)/2$. This condition requires public information be sufficiently diffuse, just like the condition $\alpha/\sqrt{\beta} \leq \sqrt{2\pi}$ for $N = 1$.

7 Conclusions

This paper has examined a global games model in which speculators can attack any one of multiple currencies. The main result shown is that with two countries, or any N symmetric countries satisfying some condition on the noise distribution, there is a unique, dominance-solvable equilibrium. In equilibrium, a country's fundamentals are evaluated in relation to those of other countries, which contrasts with the complete information version of the model. This paper has then derived, through numerical examples, several implications on how the number of targets and the precision of signals affect survival of the pegs. Finally, the paper has considered the extension of introducing exogenous public information, and has shown that the sufficient condition for a unique equilibrium is given in terms of the precision of public and private information, as in the single-target case.

The environment of this paper is not only of theoretical interest, but has many interesting applications. Formulating the iterative deletion procedure as function iterations turns out to be a powerful approach not only for theoretical analysis of equilibrium, but also for the numerical computations necessary for such applications. As a limitation, this approach yields little information on equilibrium when it fails to show equilibrium uniqueness, as in the case of more than two nonsymmetric targets. Analyzing such environments would seem to require a different approach, and such pursuit is left for future study.

[34]See Appendix A1 for derivation.

Appendix A: Two Related Models

Appendix A discusses two models, mentioned in the main body of this paper, that have environments similar to the model of this paper.

A1. Model with a Single Target

Appendix A1 discusses the canonical global games model, where $N = 1$. The superscript denoting country is now redundant, and hence is removed; otherwise, assumptions are as in Section 2. As is well known, this model has a unique Bayesian Nash equilibrium that survives iterative deletion of dominated strategies, as I briefly describe below.[35]

By assumption, the peg collapses for $\theta \leq 0 = \underline{\theta}_1^*$ even if no speculator attacks the peg. Thus, given signal x_i, the expected payoff from attacking is at least

$$(1-c) \cdot \Pr\left(\theta \leq \underline{\theta}_1^* | x_i\right) - c \cdot \left(1 - \Pr\left(\theta \leq \underline{\theta}_1^* | x_i\right)\right) = \Pr\left(\theta \leq \underline{\theta}_1^* | x_i\right) - c, \tag{6}$$

where $\Pr\left(\theta \leq \underline{\theta}_1^* | x_i\right)$ is the probability that $\theta \leq \underline{\theta}_1^*$, conditional on receiving x_i. Noting (1), $\Pr\left(\theta \leq \underline{\theta}_1^* | x_i\right) = 1 - \Psi\left(x_i - \underline{\theta}_1^*\right)$, which is decreasing in x_i. Thus, not attacking is a dominated action for speculators with signals $x_i < \underline{x}_1^*$, where

$$c = \Pr\left(\theta \leq \underline{\theta}_1^* | \underline{x}_1^*\right) = 1 - \Psi\left(\underline{x}_1^* - \underline{\theta}_1^*\right), \tag{7}$$

which completes the first round of deletion.

In the second round of deletion, it is taken as given that speculators with signals $x_i \leq \underline{x}_1^*$ will attack the currency.[36] Let $\Pr\left(x_i \leq \underline{x}_1^* | \theta\right)$ be the probability that $x_i \leq \underline{x}_1^*$, conditional on a given realization of θ. From the law of large numbers, this is also the fraction of speculators with such signals. Noting (1), $\Pr\left(x_i \leq \underline{x}_1^* | \theta\right) = \Psi\left(\underline{x}_1^* - \theta\right)$, which is decreasing in θ. Then, for $\theta \leq \underline{\theta}_2^*$, where

$$\underline{\theta}_2^* = \Pr\left(x_i \leq \underline{x}_1^* | \underline{\theta}_2^*\right) = \Psi\left(\underline{x}_1^* - \underline{\theta}_2^*\right), \tag{8}$$

the fraction of speculators who receive signals $x_i \leq \underline{x}_1^*$ and attack is at least $\underline{\theta}_2^*$, so speculators must expect the peg to collapse. Thus, given x_i, the expected payoff from attacking is at least $\Pr\left(\theta \leq \underline{\theta}_2^* | x_i\right) - c = 1 - \Psi\left(x_i - \underline{\theta}_2^*\right) - c$, which is decreasing in x_i. Therefore, not attacking is a dominated action for speculators with signals $x_i < \underline{x}_2^*$, where

$$c = \Pr\left(\theta \leq \underline{\theta}_2^* | \underline{x}_2^*\right) = 1 - \Psi\left(\underline{x}_2^* - \underline{\theta}_2^*\right). \tag{9}$$

Clearly $\underline{\theta}_2^* > \underline{\theta}_1^* = 0$, which implies $\underline{x}_2^* > \underline{x}_1^*$. This completes the second round of deletion.

This procedure yields an increasing sequence $\left(\underline{\theta}_n^*, \underline{x}_n^*\right)_{n=1}^{\infty}$ such that in the n-th round of deletion, speculators must expect the peg to collapse for $\theta \leq \underline{\theta}_n^*$, and not attacking is

[35] For a more detailed description, see, e.g., Atkeson (2000).

[36] Having a weak, instead of strict, inequality here presumes that speculators attack when they are indifferent; such a tie-breaking rule is immaterial.

a dominated action for $x_i < \underline{x}_n^*$. A similar procedure from $\bar{\theta}_1 = 1$ yields a decreasing sequence $\left(\bar{\theta}_n^*, \bar{x}_n^*\right)_{n=1}^{\infty}$ such that in the n-th round of deletion, speculators must expect the peg to be sustained for $\theta > \bar{\theta}_n^*$, and attacking is a dominated action for $x_i > \bar{x}_n^*$. Clearly, $\bar{\theta}_n^* > \underline{\theta}_n^*$ and $\bar{x}_n^* > \underline{x}_n^*$ for all n.

The limits of these sequences, $(\underline{\theta}_\infty^*, \underline{x}_\infty^*)$ and $\left(\bar{\theta}_\infty^*, \bar{x}_\infty^*\right)$, must both be solutions to

$$c = \Pr\left(\theta \leq \theta_s^* | x_s^*\right) = 1 - \Psi\left(x_s^* - \theta_s^*\right), \tag{10}$$

$$\theta_s^* = \Pr\left(x_i \leq x_s^* | \theta_s^*\right) = \Psi\left(x_s^* - \theta_s^*\right). \tag{11}$$

These equations yield

$$\theta_s^* = 1 - c, \tag{12}$$

$$x_s^* = \theta_s^* + \Psi^{-1}\left(\theta_s^*\right) = 1 - c + \Psi^{-1}\left(1 - c\right), \tag{13}$$

so $\underline{\theta}_\infty^* = \bar{\theta}_\infty^* = \theta_s^*$ and $\underline{x}_\infty^* = \bar{x}_\infty^* = x_s^*$. Thus, there is a unique, dominance-solvable equilibrium, such that speculator i attacks if and only if $x_i \leq x_s^*$, and the peg collapses if and only if $\theta \leq \theta_s^*$.

From the argument above, one can obtain (2) and (3) in Section 3.2.2 as follows. In the n-th round of deletion, (7) and (8) become $\Psi\left(\underline{x}_n^* - \underline{\theta}_n^*\right) = 1 - c$ and $\underline{\theta}_{n+1}^* = \Psi\left(\underline{x}_n^* - \underline{\theta}_{n+1}^*\right)$. Similarly, $\Psi\left(\bar{x}_n^* - \bar{\theta}_n^*\right) = 1 - c$ and $\bar{\theta}_{n+1}^* = \Psi\left(\bar{x}_n^* - \bar{\theta}_{n+1}^*\right)$. Therefore,

$$\bar{x}_n^* - \bar{\theta}_n^* = \underline{x}_n^* - \underline{\theta}_n^* = \Psi^{-1}\left(1 - c\right), \tag{14}$$

which yields (2). Further, noting $\bar{\theta}_{n+1}^* > \underline{\theta}_{n+1}^*$,

$$\begin{aligned} \bar{\theta}_{n+1}^* - \underline{\theta}_{n+1}^* &= \Psi\left(\bar{x}_n^* - \bar{\theta}_{n+1}^*\right) - \Psi\left(\underline{x}_n^* - \underline{\theta}_{n+1}^*\right) \\ &\leq \|\psi\| \cdot \left[\left(\bar{x}_n^* - \bar{\theta}_{n+1}^*\right) - \left(\underline{x}_n^* - \underline{\theta}_{n+1}^*\right)\right], \end{aligned}$$

which yields (3).

Finally, consider the extension discussed in Section 6.2. This time, conditional on receiving x_i, θ follows $N\left(\left(\alpha w + \beta x_i\right) / \left(\alpha + \beta\right), 1/\left(\alpha + \beta\right)\right)$ such that $\Pr\left(\theta \leq \theta^* | x_i\right) = \Phi\left(\sqrt{\alpha + \beta}\left(\theta^* - \left(\alpha w + \beta x_i\right) / \left(\alpha + \beta\right)\right)\right)$. Thus, in the n-th round of deletion, (7) yields $\sqrt{\alpha + \beta}\left(\left(\alpha w + \beta \underline{x}_n^*\right) / \left(\alpha + \beta\right) - \underline{\theta}_n^*\right) = \Phi^{-1}\left(1 - c\right)$ and similarly, $\sqrt{\alpha + \beta}\left(\left(\alpha w + \beta \bar{x}_n^*\right) / \left(\alpha + \beta\right) - \bar{\theta}_n^*\right) = \Phi^{-1}\left(1 - c\right)$. Therefore, (2) is replaced with $\bar{x}_n^* - \underline{x}_n^* = \frac{\alpha + \beta}{\beta}\left(\bar{\theta}_n^* - \underline{\theta}_n^*\right)$, hence combining with (3) yields (5).

A2. Model with Complete Information

Appendix A2 discusses the complete information version of the model with $N > 1$ targets and publicly observed fundamentals $\left(\theta^j\right)_{j \in J}$. For simplicity, let $c^j = c$ for all j.[37] This can be considered a multi-country extension of Obstfeld (1996), whose equilibrium is

[37]When c^j varies with j, one must examine various cases according to the values of c^j and θ^j; this provides little insight on the difference between complete and incomplete information environments.

as follows: all speculators attack and the peg is abandoned if $\theta \leq 0$, no speculator attacks and the peg is sustained if $\theta > 1$, and both of these are equilibria if $\theta \in (0, 1]$.

In the present environment, for each speculator, attacking currency j is (at least weakly) optimal if and only if j abandons the peg, and attacking no currency is optimal if and only if the peg is sustained for all j. Thus, the Nash equilibrium is as follows.

If $\theta^j > 1$ for all j, all speculators follow the dominant strategy of attacking no currency, and the peg is sustained for all j. If $\theta^k \leq 0$ for some k and $\theta^j > 1$ for all $j \neq k$, all speculators follow the dominant strategy of attacking k, and only k abandons the peg. In all other cases, there are multiple equilibria. If $\theta^k \leq 0$ for some k and $\theta^j \leq 1$ for at least one $j \neq k$, one or more j with $\theta^j \leq 1$ abandon the peg, and each speculator attacks one of such j. This is because attacking no currency is a dominated strategy, since k abandons the peg for sure. Finally, if $\theta^j > 0$ for all j and $\theta^j \leq 1$ for at least one j, there clearly exist, just as in the previous case, equilibria in which one or more j with $\theta^j \leq 1$ abandon the peg, and each speculator attacks one of such j. This time, however, since $\theta^j > 0$ for all j, there is also an equilibrium in which no speculator attacks any currency and the peg is sustained for all j.

Therefore, as in Obstfeld (1996), there may be unique or multiple equilibria, depending on the realization of fundamentals. Moreover, the outcome of attack for each country depends on own fundamentals in a very similar fashion as for $N = 1$: country j abandons the peg for sure if $\theta^j \leq 0$, sustains the peg for sure if $\theta^j > 1$, and either outcome may arise if $\theta^j \in (0, 1]$. Thus, whether a country sustains the peg depends little on the fundamentals of other countries, which contrasts with the incomplete information model of this paper.

Appendix B: Iterative Deletion Procedure

Appendix B provides the formal mathematical formulation of the iterative deletion procedure. As in Section 3.2.1, for any $j \in J$, let $\underline{\theta}_n^{j*}$, $\bar{\theta}_n^{j*} : \mathbb{R}^{N-1} \to [0, 1]$ be such that, at the beginning of the n-th round of deletion, all speculators believe that country j abandons the peg for $\theta^j \leq \underline{\theta}_n^{j*}(\theta^{-j})$, and sustains the peg for $\theta^j > \bar{\theta}_n^{j*}(\theta^{-j})$. Such $\underline{\theta}_n^{j*}$ and $\bar{\theta}_n^{j*}$ are used to eliminate dominated strategies in the n-th round of deletion, which in turn yields $\underline{\theta}_{n+1}^{j*}$ and $\bar{\theta}_{n+1}^{j*}$.

To utilize these expressions later on, I describe below the n-th round of deletion. For any N signals x_i, given the beliefs described by $\underline{\theta}_n^{j*}$ and $\bar{\theta}_n^{j*}$, the expected payoff from attacking currency $j \in J$ must be at least $\underline{\Gamma}_n^{j*}(x_i)$ and at most $\bar{\Gamma}_n^{j*}(x_i)$, where

$$\underline{\Gamma}_n^{j*}(x_i) = \Pr\left(\theta^j \leq \underline{\theta}_n^{j*}\left(\theta^{-j}\right)|x_i\right) - c^j, \tag{15}$$

$$\bar{\Gamma}_n^{j*}(x_i) = \Pr\left(\theta^j \leq \bar{\theta}_n^{j*}\left(\theta^{-j}\right)|x_i\right) - c^j. \tag{16}$$

Not attacking yields zero, so $\underline{\Gamma}_n^{0*}(x_i) = \bar{\Gamma}_n^{0*}(x_i) = 0$. Then, if $\underline{\Gamma}_n^{j*}(x_i) > \max_{k \in J_0^{-j}} \bar{\Gamma}_n^{k*}(x_i)$, not attacking j is dominated by attacking j, since the expected payoff from attacking j is at least $\underline{\Gamma}_n^{j*}(x_i)$, while that from attacking k is at most $\bar{\Gamma}_n^{k*}(x_i)$. From similar reasoning, if $\bar{\Gamma}_n^{j*}(x_i) < \max_{k\in J_0^{-j}} \underline{\Gamma}_n^{k*}(x_i)$, attacking j is dominated by not attacking j.

Now, for any $j \in J$, let $\underline{x}_n^{j*}, \bar{x}_n^{j*} : \mathbb{R}^{N-1} \to \bar{\mathbb{R}}$ be such that not attacking j is a dominated action for $x_i^j < \underline{x}_n^{j*}\left(x_i^{-j}\right)$, and attacking j is a dominated action for $x_i^j > \bar{x}_n^{j*}\left(x_i^{-j}\right)$. There is more than one possible choice for $\underline{x}_n^{j*}$ and $\bar{x}_n^{j*}$, but from the argument above, the most natural ones, which I adopt for $N = 2$ are,[38]

$$\underline{x}_n^{j*}\left(x_i^{-j}\right) = \inf\left\{x_i^j \in \mathbb{R} | \underline{\Gamma}_n^{j*}(x_i) \leq \max_{k\in J_0^{-j}} \bar{\Gamma}_n^{k*}(x_i)\right\}, \tag{17}$$

$$\bar{x}_n^{j*}\left(x_i^{-j}\right) = \sup\left\{x_i^j \in \mathbb{R} | \bar{\Gamma}_n^{j*}(x_i) \geq \max_{k\in J_0^{-j}} \underline{\Gamma}_n^{k*}(x_i)\right\}. \tag{18}$$

In the $n+1$-th round, it is taken as given that speculators with signals $x_i^j < \underline{x}_n^{j*}\left(x_i^{-j}\right)$ will attack currency j, and those with signals $x_i^j > \bar{x}_n^{j*}\left(x_i^{-j}\right)$ will not. Thus, one must believe that the fraction of speculators attacking j is at least $\Pr\left(x_i^j < \underline{x}_n^{j*}\left(x_i^{-j}\right)|\theta\right)$, and at most $\Pr\left(x_i^j \leq \bar{x}_n^{j*}\left(x_i^{-j}\right)|\theta\right)$, which are increasing in θ^j given (1). Then, $\theta^j \leq \underline{\theta}_{n+1}^{j*}\left(\theta^{-j}\right)$ implies $\theta^j \leq \Pr\left(x_i^j < \underline{x}_n^{j*}\left(x_i^{-j}\right)|\theta\right)$, and $\theta^j > \bar{\theta}_{n+1}^{j*}\left(\theta^{-j}\right)$ implies $\theta^j > \Pr\left(x_i^j < \bar{x}_n^{j*}\left(x_i^{-j}\right)|\theta\right)$, where

$$\underline{\theta}_{n+1}^{j*}\left(\theta^{-j}\right) = \Pr\left(x_i^j < \underline{x}_n^{j*}\left(x_i^{-j}\right)|\underline{\theta}_{n+1}^{j*}\left(\theta^{-j}\right), \theta^{-j}\right), \tag{19}$$

$$\bar{\theta}_{n+1}^{j*}\left(\theta^{-j}\right) = \Pr\left(x_i^j \leq \bar{x}_n^{j*}\left(x_i^{-j}\right)|\bar{\theta}_{n+1}^{j*}\left(\theta^{-j}\right), \theta^{-j}\right). \tag{20}$$

Thus, at the beginning of the $n+1$-th round of deletion, all speculators must believe that country j abandons the peg for all $\theta^j \leq \underline{\theta}_{n+1}^{j*}\left(\theta^{-j}\right)$, and sustains the peg for all $\theta^j > \bar{\theta}_{n+1}^{j*}\left(\theta^{-j}\right)$, which is in line with the definitions of $\underline{\theta}_n^{j*}$ and $\bar{\theta}_n^{j*}$.

Iterating this procedure from $\underline{\theta}_1^{j*} = 0$ and $\bar{\theta}_1^{j*} = 1$ yields the sequences of functions, $\left((\underline{x}_n^{j*}, \underline{\theta}_{n+1}^{j*})_{j\in J}\right)_{n=1}^{\infty}$ and $\left((\bar{x}_n^{j*}, \bar{\theta}_{n+1}^{j*})_{j\in J}\right)_{n=1}^{\infty}$.

[38] As I discuss in the proof of Proposition 3, for $N > 2$, I adopt alternative definitions of $\underline{x}_n^{j*}$ and $\bar{x}_n^{j*}$ to ensure that the sequence of functions generated by the iterative deletion procedure are well behaved. Note that $\underline{x}_n^{j*}\left(x_i^{-j}\right) = -\infty$ if the set on the right-hand side (RHS) of (17) is unbounded below, and $\bar{x}_n^{j*}\left(x_i^{-j}\right) = -\infty$ if the set on the RHS of (18) is empty. These may arise if $c^j > c^k$ for some $k \in J^{-j}$.

Appendix C: Proofs

Appendix C contains all proofs.

Lemmas for General Environments

This subsection collects lemmas that hold for general $N>1$ environments. Below, let $\tilde{B}(X)$ be the set of bounded, continuous, and nondecreasing functions from X to $\mathbb{R}$.

Lemmas 1–4 concern how continuity and monotonicity of the relevant functions are conveyed through the iterative deletion procedure.

Lemma 1 *For any $j\in J$ and $\theta^{j*}:\mathbb{R}^{N-1}\to\mathbb{R}$, define $\Gamma^{j*}:\mathbb{R}^N\to\mathbb{R}$ by $\Gamma^{j*}(x_i)=\Pr\left(\theta^j\leq\theta^{j*}\left(\theta^{-j}\right)|x_i\right)-c^j$. If $\theta^{j*}\in\tilde{B}(\mathbb{R}^{N-1})$, then Γ^{j*} is continuous, decreasing in x_i^j, and nondecreasing in x_i^{-j}; further, if θ^{j*} is increasing, then Γ^{j*} is increasing in x_i^{-j}.*

Proof. Since $\theta^j=x_i^j-\epsilon_i^j$, $\Pr\left(\theta^j\leq\theta^{j*}|x_i^j\right)=\Pr\left(\epsilon_i^j\geq x_i^j-\theta^{j*}\right)=1-\Psi^j\left(x_i^j-\theta^{j*}\right)$ for a fixed θ^{j*}. Thus, noting that $\theta^{-j}=x_i^{-j}-\epsilon_i^{-j}$,

$$\Gamma^{j*}(x_i)=1-c^j-\int_{\epsilon_i^{-j}\in\mathbb{R}^{N-1}}\tilde{\psi}^{-j}\left(\epsilon_i^{-j}\right)\Psi^j\left(x_i^j-\theta^{j*}\left(x_i^{-j}-\epsilon_i^{-j}\right)\right)d\epsilon_i^{-j},\tag{21}$$

where $\tilde{\psi}^{-j}:\mathbb{R}^{N-1}\to\mathbb{R}$ is the joint pdf of $N-1$ independent noises, ϵ_i^{-j}. Since the pdf of ϵ_i^k is ψ^k, it follows that $\tilde{\psi}^{-j}\left(\epsilon_i^{-j}\right)=\Pi_{k\in J-j}\psi^k\left(\epsilon_i^k\right)$.

That Γ^{j*} is decreasing in x_i^j is immediate from (21); that Γ^{j*} is nondecreasing (increasing) in x_i^{-j} also follows from (21), as well as the assumption that θ^{j*} is nondecreasing (increasing). For continuity, define $H^j:\mathbb{R}^N\times\mathbb{R}^{N-1}\to\mathbb{R}^+$ by

$$H^j\left(x_i,\epsilon_i^{-j}\right)=\tilde{\psi}^{-j}\left(\epsilon_i^{-j}\right)\Psi^j\left(x_i^j-\theta^{j*}\left(x_i^{-j}-\epsilon_i^{-j}\right)\right).\tag{22}$$

By assumption, Ψ^j and θ^{j*} are continuous, and hence, so is their composition, $\Psi^j\left(x_i^j-\theta^{j*}\left(x_i^{-j}-\epsilon_i^{-j}\right)\right)$. Being a product of continuous pdfs ψ^k, $\tilde{\psi}^{-j}$ is also continuous, and thus, so is H^j.[39] Since $\Psi^j\in[0,1]$ and $\tilde{\psi}^{-j}>0$, it follows that $\left|H^j\left(x_i,\epsilon_i^{-j}\right)\right|\leq\tilde{\psi}^{-j}\left(\epsilon_i^{-j}\right)$ for any $\left(x_i,\epsilon_i^{-j}\right)\in\mathbb{R}^N\times\mathbb{R}^{N-1}$. Moreover, $\tilde{\psi}^{-j}$ is independent of x_i, and being a joint pdf, $\tilde{\psi}^{-j}$ is improperly integrable on $\mathbb{R}^{N-1}$, where the value of the improper integral is 1. Therefore, the improper integral $\int_{\epsilon_i^{-j}\in\mathbb{R}^{N-1}}H^j\left(x_i,\epsilon_i^{-j}\right)d\epsilon_i^{-j}$ converges uniformly on $\mathbb{R}^{N-1}$, and thus $\int_{\epsilon_i^{-j}\in\mathbb{R}^{N-1}}H^j\left(x_i,\epsilon_i^{-j}\right)d\epsilon_i^{-j}$ is continuous in x_i, and hence, so is Γ^{j*}.[40] ■

Lemma 1 can be used to establish the properties of $\underline{\Gamma}_n^{j*}$ and $\bar{\Gamma}_n^{j*}$, by letting $\theta^{j*}=\underline{\theta}_n^{j*}$ and $\theta^{j*}=\bar{\theta}_n^{j*}$. For later use, note that (15) and (16) can be rewritten as

$$\underline{\Gamma}_n^{j*}(x_i)=1-c^j-\int_{\epsilon_i^{-j}\in\mathbb{R}^{N-1}}\tilde{\psi}^{-j}\left(\epsilon_i^{-j}\right)\Psi^j\left(x_i^j-\underline{\theta}_n^{j*}\left(x_i^{-j}-\epsilon_i^{-j}\right)\right)d\epsilon_i^{-j},\tag{23}$$

$$\bar{\Gamma}_n^{j*}(x_i)=1-c^j-\int_{\epsilon_i^{-j}\in\mathbb{R}^{N-1}}\tilde{\psi}^{-j}\left(\epsilon_i^{-j}\right)\Psi^j\left(x_i^j-\bar{\theta}_n^{j*}\left(x_i^{-j}-\epsilon_i^{-j}\right)\right)d\epsilon_i^{-j}.\tag{24}$$

[39] For these properties of continuous functions, see Johnsonbaugh and Pfaffenberger (2010), Theorem 40.4 and Corollary 40.6.

[40] For the continuity of functions defined by integrals (or *parameter-dependent integrals*), see, e.g., Burkill and Burkill (2002), Theorems 6.35 and 8.73, and Bauer (2001), Lemma 16.1.

Since the definitions of $\underline{x}_n^{j*}$ and $\bar{x}_n^{j*}$ differ for $N=2$ and $N>2$, discussions of their properties are deferred to relevant propositions. Instead, I define $\underline{x}_n^{j,k}$ and $\bar{x}_n^{j,k}$, which are closely related to $\underline{x}_n^{j*}$ and $\bar{x}_n^{j*}$, and examine their properties in Lemmas 2 and 3.

Definition 6 *For any* $\left(\underline{\Gamma}_n^{j*}, \bar{\Gamma}_n^{j*}\right)_{j\in J}$, $\underline{\Gamma}_n^{j*}, \bar{\Gamma}_n^{j*} : \mathbb{R}^N \to \mathbb{R}$, *define* $\underline{x}_n^{j,k}, \bar{x}_n^{j,k} : \mathbb{R}^{N-1} \to \bar{\mathbb{R}}$ *by*

$$\underline{x}_n^{j,k}\left(x_i^{-j}\right) \equiv \inf\left\{x_i^j \in \mathbb{R} | \underline{\Gamma}_n^{j*}(x_i) \leq \bar{\Gamma}_n^{k*}(x_i)\right\}, \tag{25}$$

$$\bar{x}_n^{j,k}\left(x_i^{-j}\right) \equiv \sup\left\{x_i^j \in \mathbb{R} | \bar{\Gamma}_n^{j*}(x_i) \geq \underline{\Gamma}_n^{k*}(x_i)\right\}, \tag{26}$$

for any $j \in J$ *and* $k \in J_0^{-j}$, *where* $\underline{\Gamma}_n^{0*}(x_i) \equiv 0$ *and* $\bar{\Gamma}_n^{0*}(x_i) \equiv 0$ *for any* $x_i \in \mathbb{R}^N$.

Note that Definition 6 includes the definitions of $\underline{x}_n^{j,0}$ and $\bar{x}_n^{j,0}$, since $k \in J_0^{-j} = \{0\} \cup J^{-j}$ above. Also note that $\underline{x}_n^{j,k}$ and $\bar{x}_n^{j,k}$ are allowed to take infinite values.

Lemma 2 *Suppose* $\underline{\theta}_n^{j*}$, $\bar{\theta}_n^{j*} \in \tilde{B}(\mathbb{R}^{N-1})$ *for all* $j \in J$, *and define* $\underline{\Gamma}_n^{j*}$, $\bar{\Gamma}_n^{j*} : \mathbb{R}^N \to \mathbb{R}$ *by (15), (16). Then,* $\underline{\Gamma}_n^{j*} - \bar{\Gamma}_n^{k*}$ *is continuous, decreasing in* x_i^j, *and increasing in* x_i^k *for any* $j \in J$ *and* $k \in J^{-j}$, *and for any* $x_i^{-j} \in X_{\mathbb{R}}\left(\underline{x}_n^{j,k}\right)$, $\underline{x}_n^{j,k}\left(x_i^{-j}\right)$ *is the unique* x_i^j *such that* $\underline{\Gamma}_n^{j*}(x_i) = \bar{\Gamma}_n^{k*}(x_i)$. *Moreover,* $\underline{x}_n^{j,k}$ *is continuous in* x_i^{-j} *and increasing in* x_i^k *in* $X_{\mathbb{R}}\left(\underline{x}_n^{j,k}\right)$. *The same statements hold with* $\underline{\Gamma}_n^{j*}$, $\bar{\Gamma}_n^{k*}$, *and* $\underline{x}_n^{j,k}$ *replaced by* $\bar{\Gamma}_n^{j*}$, $\underline{\Gamma}_n^{k*}$, *and* $\bar{x}_n^{j,k}$.

Proof. Without loss of generality, I prove the claim for $j = 1$.[41] Fix any $k \in J^{-1}$. From Lemma 1, $\underline{\Gamma}_n^{j*}$ and $\bar{\Gamma}_n^{j*}$ are continuous, decreasing in x_i^j, and nondecreasing in x_i^{-j} for all $j \in J$. Thus, $\underline{\Gamma}_n^{1*} - \bar{\Gamma}_n^{k*}$ is continuous, decreasing in x_i^1 and increasing in x_i^k. Therefore, if $z \in X_{\mathbb{R}}\left(\underline{x}_n^{1,k}\right)$, then (25) implies that $\underline{x}_n^{1,k}(z)$ is the unique x_i^1 such that $\underline{\Gamma}_n^{1*}\left(\underline{x}_n^{1,k}(z), z\right) = \bar{\Gamma}_n^{k*}\left(\underline{x}_n^{1,k}(z), z\right)$, and moreover, $\underline{x}_n^{1,k}$ is increasing in x_i^k.

Further, for any $z \in X_{\mathbb{R}}\left(\underline{x}_n^{1,k}\right)$ and $\nu > 0$, $\underline{\Gamma}_n^{1*}\left(\underline{x}_n^{1,k}(z) + \nu, z\right) < \bar{\Gamma}_n^{k*}\left(\underline{x}_n^{1,k}(z) + \nu, z\right)$ and $\underline{\Gamma}_n^{1*}\left(\underline{x}_n^{1,k}(z) - \nu, z\right) > \bar{\Gamma}_n^{k*}\left(\underline{x}_n^{1,k}(z) - \nu, z\right)$, since $\underline{\Gamma}_n^{1*} - \bar{\Gamma}_n^{k*}$ is decreasing in x_i^1. Then, since $\underline{\Gamma}_n^{1*} - \bar{\Gamma}_n^{k*}$ is continuous, there exists $\delta > 0$ such that $\|z_\delta - z\|_E \leq \delta$ implies

$$\underline{\Gamma}_n^{1*}\left(\underline{x}_n^{1,k}(z) + \nu, z_\delta\right) < \bar{\Gamma}_n^{k*}\left(\underline{x}_n^{1,k}(z) + \nu, z_\delta\right), \tag{27}$$

$$\underline{\Gamma}_n^{1*}\left(\underline{x}_n^{1,k}(z) - \nu, z_\delta\right) > \bar{\Gamma}_n^{k*}\left(\underline{x}_n^{1,k}(z) - \nu, z_\delta\right). \tag{28}$$

Since $\underline{\Gamma}_n^{1*} - \bar{\Gamma}_n^{k*}$ is decreasing in x_i^1, (25) and (27) imply $\underline{x}_n^{1,k}(z_\delta) \leq \underline{x}_n^{1,k}(z) + \nu$, whereas (25) and (28) imply $\underline{x}_n^{1,k}(z_\delta) \geq \underline{x}_n^{1,k}(z) - \nu$. Thus, $\left|\underline{x}_n^{1,k}(z_\delta) - \underline{x}_n^{1,k}(z)\right| \leq \nu$, so $\underline{x}_n^{1,k}$ is continuous at z. The claims for $\bar{x}_n^{1,k}$ follow from a similar argument. ■

Note that (24) and the boundedness of $\bar{\theta}_n^{k*}$ imply $\bar{\Gamma}_n^{k*}(x_i) \in \left(-c^k, 1-c^k\right)$ so long as x_i^k is finite. Thus, if $c^1 = c^k = c$, then for any $x_i^{-1} \in \mathbb{R}^{N-1}$, (23) yields $\underline{\Gamma}_n^{1*}(x_i) - \bar{\Gamma}_n^{k*}(x_i) \to 1 - c - \lim_{x_i^1 \to -\infty} \bar{\Gamma}_n^{k*}(x_i) > 0$ as $x_i^1 \to -\infty$, and $\underline{\Gamma}_n^{1*}(x_i) - \bar{\Gamma}_n^{k*}(x_i) \to -c - \lim_{x_i^1 \to \infty} \bar{\Gamma}_n^{k*}(x_i) < 0$

[41] This is to simplify notation. For example, I let $\left(\underline{x}_n^{1,k}(z), z\right)$ denote an N-vector with $x_i^1 = \underline{x}_n^{1,k}(z)$ and $\left(x_i^2, x_i^3, \ldots, x_i^N\right) = z$; when x_i^1 changes to $\underline{x}_n^{1,k}(z) + \nu$, the new vector is conveniently expressed as $\left(\underline{x}_n^{1,k}(z) + \nu, z\right)$. Describing a similar situation for general j requires cumbersome notations, so the proof here, and some of the proofs below, consider $j = 1$. This is without loss of generality, since numbering of countries is arbitrary.

as $x_i^1 \to \infty$, so $\underline{x}_n^{1,k}\left(x_i^{-1}\right) \in \mathbb{R}$. More generally, if $c^j = c^k$, then $X_{\mathbb{R}}\left(\underline{x}_n^{j,k}\right) = \mathbb{R}^{N-1}$, and similarly, $X_{\mathbb{R}}\left(\bar{x}_n^{j,k}\right) = \mathbb{R}^{N-1}$. However, $\underline{x}_n^{j,k}$ and $\bar{x}_n^{j,k}$ may equal $-\infty$ or ∞ when $c^j \neq c^k$.

Also note that Lemma 2 shows the monotonicity of $\underline{x}_n^{j,k}$ and $\bar{x}_n^{j,k}$ in x_i^k, but not in all elements of x_i^{-j} (except for $N = 2$, in which case x_i^{-j} is a scalar). As stated below, the implications for $\underline{x}_n^{j,0}$ and $\bar{x}_n^{j,0}$ are stronger in this sense, and also, $X_{\mathbb{R}}\left(\underline{x}_n^{j,0}\right) = X_{\mathbb{R}}\left(\bar{x}_n^{j,0}\right) = \mathbb{R}^{N-1}$.

Lemma 3 *Under the assumptions of Lemma 2, for any* $x_i^{-j} \in \mathbb{R}^{N-1}$, $\underline{x}_n^{j,0}\left(x_i^{-j}\right)$ *is the unique* x_i^j *such that* $\underline{\Gamma}_n^{j*}(x_i) = 0$, *and* $\underline{x}_n^{j,0} \in \tilde{B}(\mathbb{R}^{N-1})$; *further, if* $\underline{\theta}_n^{j*}$ *is increasing, so is* $\underline{x}_n^{j,0}$. *The same statement holds with* $\underline{x}_n^{j,0}$, $\underline{\Gamma}_n^{j*}$, *and* $\underline{\theta}_n^{j*}$ *replaced by* $\bar{x}_n^{j,0}$, $\bar{\Gamma}_n^{j*}$, *and* $\bar{\theta}_n^{j*}$.

Proof. I prove the claim for $j = 1$. For any $x_i^{-1} \in \mathbb{R}^{N-1}$, (23) implies $\underline{\Gamma}_n^{1*}(x_i) \to 1 - c^1 > 0$ as $x_i^1 \to -\infty$, and $\underline{\Gamma}_n^{1*}(x_i) \to -c^1 < 0$ as $x_i^1 \to \infty$. Also, from Lemma 1, $\underline{\Gamma}_n^{1*}$ is continuous, decreasing in x_i^1, and nondecreasing in x_i^{-1}. Thus, (25) implies that for any x_i^{-1}, $\underline{x}_n^{1,0}\left(x_i^{-1}\right)$ is the unique x_i^1 such that $\underline{\Gamma}_n^{1*}(x_i) = 0$, and moreover, $\underline{x}_n^{1,0}$ is nondecreasing. Further, if $\underline{\theta}_n^{1*}$ is increasing, then $\underline{\Gamma}_n^{1*}$ is increasing in x_i^{-1} from Lemma 1, so $\underline{x}_n^{1,0}$ is increasing. The boundedness of $\underline{x}_n^{1,0}$ follows from (23) and the assumption that $\underline{\theta}_n^{1*}$ is bounded. The continuity of $\underline{x}_n^{1,0}$ follows by replacing $\bar{\Gamma}_n^{k*}$ in the proof of Lemma 2 by $\bar{\Gamma}_n^{0*} = 0$. The claim for $\bar{x}_n^{1,0}$ follows similarly. ■

Lemma 4 *For any* $j \in J$ *and* $x^{j*} : \mathbb{R}^{N-1} \to \mathbb{R}$, *define* $\theta^{j*} : \mathbb{R}^{N-1} \to [0,1]$ *by* $\theta^{j*}\left(\theta^{-j}\right) = \Pr\left(x_i^j < x^{j*}\left(x_i^{-j}\right) | \theta^{j*}\left(\theta^{-j}\right), \theta^{-j}\right)$. *If* x^{j*} *is continuous and nondecreasing, then* $\theta^{j*} \in \tilde{B}(\mathbb{R}^{N-1})$; *further, if, for each* $k \in J^{-j}$, *there is some* $x_i^{-j} \in \mathbb{R}^{N-1}$ *at which* x^{j*} *is increasing in* x_i^k, *then* θ^{j*} *is increasing.*

Proof. Noting that $\Pr\left(x_i^j < x^{j*} | \theta\right) = \Psi^j\left(x^{j*} - \theta\right)$ for fixed x^{j*},

$$\theta^{j*}\left(\theta^{-j}\right) = \int_{\epsilon_i^{-j} \in \mathbb{R}^{N-1}} \tilde{\psi}^{-j}\left(\epsilon_i^{-j}\right) \Psi^j\left(x^{j*}\left(\theta^{-j} + \epsilon_i^{-j}\right) - \theta^{j*}\left(\theta^{-j}\right)\right) d\epsilon_i^{-j}. \tag{29}$$

I prove the claim for $j = 1$. The boundedness of θ^{1*} is obvious, since $\theta^{1*} : \mathbb{R}^{N-1} \to [0,1]$. Define $\tilde{H}^1 : \mathbb{R}^N \times \mathbb{R}^{N-1} \to \mathbb{R}^+$ by

$$\tilde{H}^1\left(\theta, \epsilon_i^{-1}\right) = \tilde{\psi}^{-1}\left(\epsilon_i^{-1}\right) \Psi^1\left(x^{1*}\left(\theta^{-1} + \epsilon_i^{-1}\right) - \theta^1\right). \tag{30}$$

Then, $\tilde{H}^1$ is continuous because, by assumption, $\tilde{\psi}^{-1}$, Ψ^1, and x^{1*} are continuous. Thus, $\int_{\epsilon_i^{-1} \in \mathbb{R}^{N-1}} \tilde{H}^1\left(\theta, \epsilon_i^{-1}\right) d\epsilon_i^{-1}$ is continuous in θ from the same argument as for H^j. Also, given $\tilde{\psi}^{-1} > 0$, for any ϵ_i^{-1}, $\tilde{H}^1$ is clearly decreasing in θ^1, and since x^{1*} is by assumption nondecreasing, $\tilde{H}^1$ is nondecreasing in θ^{-1}. Thus, $\theta^1 - \int_{\epsilon_i^{-1} \in \mathbb{R}^{N-1}} \tilde{H}^1\left(\theta, \epsilon_i^{-1}\right) d\epsilon_i^{-1}$ is continuous, increasing in θ^1, nonincreasing in θ^{-1}, and tends to $-\infty$ as $\theta^1 \to -\infty$, and to ∞ as $\theta^1 \to \infty$. So, for any θ^{-1}, there is a unique θ^1 such that $\theta^1 = \int_{\epsilon_i^{-1} \in \mathbb{R}^{N-1}} \tilde{H}^1\left(\theta, \epsilon_i^{-1}\right) d\epsilon_i^{-1}$, which corresponds to $\theta^{1*}\left(\theta^{-1}\right)$. Thus, for any z, $z' \in \mathbb{R}^{N-1}$,[42]

$$\theta^{1*}(z) = \int_{\epsilon_i^{-1} \in \mathbb{R}^{N-1}} \tilde{H}^1\left(\theta^{1*}(z), z, \epsilon_i^{-1}\right) d\epsilon_i^{-1}, \tag{31}$$

$$\theta^{1*}(z') = \int_{\epsilon_i^{-1} \in \mathbb{R}^{N-1}} \tilde{H}^1\left(\theta^{1*}(z'), z', \epsilon_i^{-1}\right) d\epsilon_i^{-1}. \tag{32}$$

[42] The first two arguments in $\tilde{H}^1\left(\theta^{1*}(z), z, \epsilon_i^{-1}\right)$ imply $\theta^1 = \theta^{1*}(z)$ and $\theta^{-1} = z$.

Therefore, if $z \geq z'$, (31) implies

$$\theta^{1*}(z) \geq \int_{\epsilon_i^{-1} \in \mathbb{R}^{N-1}} \tilde{H}^1\left(\theta^{1*}(z), z', \epsilon_i^{-1}\right) d\epsilon_i^{-1}, \tag{33}$$

since $\int_{\epsilon_i^{-1} \in \mathbb{R}^{N-1}} \tilde{H}^1\left(\theta, \epsilon_i^{-1}\right) d\epsilon_i^{-1}$ is nondecreasing in θ^{-1}. Since $\theta^1 - \int_{\epsilon_i^{-1} \in \mathbb{R}^{N-1}} \tilde{H}^1\left(\theta, \epsilon_i^{-1}\right) d\epsilon_i^{-1}$ is increasing in θ^1, (32) and (33) imply $\theta^{1*}(z) \geq \theta^{1*}(z')$, so θ^{1*} is nondecreasing.

Further, suppose, for each $k \in J^{-1}$, there is some x_i^{-1} at which x^{1*} is increasing in x_i^k. Take any $\theta^1 \in \mathbb{R}$ and z, $z' \in \mathbb{R}^{N-1}$ such that $z \geq z'$ and $z \neq z'$. Then, for some $k \in J^{-1}$, the k-th element of z is strictly greater than that of z'. Then, $x^{1*}\left(z + \epsilon_i^{-1}\right) \geq x^{1*}\left(z' + \epsilon_i^{-1}\right)$ for any $\epsilon_i^{-1} \in \mathbb{R}^{N-1}$, with strict inequality in some neighborhood of ϵ_i^{-1} for which $x^{1*}\left(z + \epsilon_i^{-1}\right)$ is increasing in the k-th argument. Thus, $\tilde{H}^1\left(\theta^1, z, \epsilon_i^{-1}\right) \geq \tilde{H}^1\left(\theta^1, z', \epsilon_i^{-1}\right)$ for any ϵ_i^{-1}, with strict inequality for some ϵ_i^{-1}. Then, since $\tilde{H}^1\left(\theta^1, z, \epsilon_i^{-1}\right)$ and $\tilde{H}^1\left(\theta^1, z', \epsilon_i^{-1}\right)$ are continuous in ϵ_i^{-1}, *strict monotonicity of the integral*[43] yields

$$\int_{\epsilon_i^{-1} \in \mathbb{R}^{N-1}} \tilde{H}^1\left(\theta^1, z, \epsilon_i^{-1}\right) d\epsilon_i^{-1} > \int_{\epsilon_i^{-1} \in \mathbb{R}^{N-1}} \tilde{H}^1\left(\theta^1, z', \epsilon_i^{-1}\right) d\epsilon_i^{-1}. \tag{34}$$

Thus, the inequality in (33) becomes strict, hence θ^{1*} is increasing.

For the continuity of θ^{1*}, fix any $z \in \mathbb{R}^{N-1}$, and take any $z' \in \mathbb{R}^{N-1}$, $z' \neq z$. Then, if $\theta^{1*}(z) > \theta^{1*}(z')$,

$$\begin{aligned} 0 &> \theta^{1*}(z') - \theta^{1*}(z) \\ &= \int_{\epsilon_i^{-1} \in \mathbb{R}^{N-1}} \tilde{H}^1\left(\theta^{1*}(z'), z', \epsilon_i^{-1}\right) d\epsilon_i^{-1} - \int_{\epsilon_i^{-1} \in \mathbb{R}^{N-1}} \tilde{H}^1\left(\theta^{1*}(z), z, \epsilon_i^{-1}\right) d\epsilon_i^{-1} \\ &> \int_{\epsilon_i^{-1} \in \mathbb{R}^{N-1}} \tilde{H}^1\left(\theta^{1*}(z), z', \epsilon_i^{-1}\right) d\epsilon_i^{-1} - \int_{\epsilon_i^{-1} \in \mathbb{R}^{N-1}} \tilde{H}^1\left(\theta^{1*}(z), z, \epsilon_i^{-1}\right) d\epsilon_i^{-1}, \end{aligned}$$

where the equality is from (31) and (32), and the second inequality follows since $\int_{\epsilon_i^{-1} \in \mathbb{R}^{N-1}} \tilde{H}^1\left(\theta, \epsilon_i^{-1}\right) d\epsilon_i^{-1}$ is decreasing in θ^1 as argued above. If $\theta^{1*}(z) < \theta^{1*}(z')$, the two inequalities above are reversed, and thus,

$$\begin{aligned} &\left|\theta^{1*}(z') - \theta^{1*}(z)\right| \\ &< \left| \int_{\epsilon_i^{-1} \in \mathbb{R}^{N-1}} \tilde{H}^1\left(\theta^{1*}(z), z', \epsilon_i^{-1}\right) d\epsilon_i^{-1} - \int_{\epsilon_i^{-1} \in \mathbb{R}^{N-1}} \tilde{H}^1\left(\theta^{1*}(z), z, \epsilon_i^{-1}\right) d\epsilon_i^{-1} \right|. \end{aligned} \tag{35}$$

But since $\int_{\epsilon_i^{-1} \in \mathbb{R}^{N-1}} \tilde{H}^1\left(\theta, \epsilon_i^{-1}\right) d\epsilon_i^{-1}$ is continuous in θ as argued, for any $\nu > 0$, there exists $\delta > 0$ such that the RHS of (35) is smaller than ν for any $z' \in \mathbb{R}^{N-1}$, $\|z' - z\|_E < \delta$. Thus, $|\theta^{1*}(z') - \theta^{1*}(z)| < \nu$ for such z'. The choice of z was arbitrary, so θ^{1*} is continuous. ■

[43] The *monotonicity of the integral* holds that if $f, g : \mathbb{R}^m \to \mathbb{R}$ are integrable on $\mathbb{R}^m$ and $f \geq g$, then $\int_{\mathbb{R}^m} f \geq \int_{\mathbb{R}^m} g$ (see, e.g., Hijab (2011), Theorem 4.3.1). The *strict monotonicity of integral* holds that if f, g are also continuous and $f(w) > g(w)$ for some $w \in \mathbb{R}^m$, then $\int_{\mathbb{R}^m} f > \int_{\mathbb{R}^m} g$. To see this, let $\varphi \equiv f - g$. Then, $\varphi \geq 0$ and $\varphi(w) > 0$, and since φ is continuous, there exists $\nu > 0$ such that $|\varphi(w) - \varphi(z)| < \varphi(w)/2$ for all $z \in B_\nu(w) \equiv \{w' \in \mathbb{R}^m | \|w' - w\|_E < \nu\}$. Then, $\int_{z \in \mathbb{R}^m} \varphi(z)\, dz \geq \int_{z \in B_\nu(w)} \varphi(z)\, dz > \int_{z \in B_\nu(w)} \frac{\varphi(w)}{2} dz > 0$.

For later use, note that (19) and (20) can be rewritten as

$$\underline{\theta}_{n+1}^{j*}\left(\theta^{-j}\right)=\int_{\epsilon_i^{-j}\in\mathbb{R}^{N-1}}\tilde{\psi}^{-j}\left(\epsilon_i^{-j}\right)\Psi^j\left(\underline{x}_n^{j*}\left(\theta^{-j}+\epsilon_i^{-j}\right)-\underline{\theta}_{n+1}^{j*}\left(\theta^{-j}\right)\right)d\epsilon_i^{-j}, \tag{36}$$

$$\bar{\theta}_{n+1}^{j*}\left(\theta^{-j}\right)=\int_{\epsilon_i^{-j}\in\mathbb{R}^{N-1}}\tilde{\psi}^{-j}\left(\epsilon_i^{-j}\right)\Psi^j\left(\bar{x}_n^{j*}\left(\theta^{-j}+\epsilon_i^{-j}\right)-\bar{\theta}_{n+1}^{j*}\left(\theta^{-j}\right)\right)d\epsilon_i^{-j}. \tag{37}$$

Lemma 5 is used to show how the ordering of $\underline{x}_n^{j*}$ and $\bar{x}_n^{j*}$ affect that of $\underline{\theta}_{n+1}^{j*}$ and $\bar{\theta}_{n+1}^{j*}$.

Lemma 5 *For any $j\in J$ and $x^{j*},\hat{x}^{j*}:\mathbb{R}^{N-1}\to\mathbb{R}$, define $\theta^{j*},\hat{\theta}^{j*}:\mathbb{R}^{N-1}\to[0,1]$ by $\theta^{j*}\left(\theta^{-j}\right)=\Pr\left(x_i^j<x^{j*}\left(x_i^{-j}\right)|\theta^{j*}\left(\theta^{-j}\right),\theta^{-j}\right)$ and $\hat{\theta}^{j*}\left(\theta^{-j}\right)=\Pr\left(x_i^j<\hat{x}^{j*}\left(x_i^{-j}\right)|\hat{\theta}^{j*}\left(\theta^{-j}\right),\theta^{-j}\right)$. If x^{j*} and $\hat{x}^{j*}$ are continuous and $x^{j*}\geq\hat{x}^{j*}$, then $\theta^{j*}\geq\hat{\theta}^{j*}$; further, if $x^{j*}\left(x_i^{-j}\right)>\hat{x}^{j*}\left(x_i^{-j}\right)$ for some $x_i^{-j}\in\mathbb{R}^{N-1}$, then $\theta^{j*}>\hat{\theta}^{j*}$.*

Proof. I prove the claim for $j=1$. Define $\hat{H}^1:\mathbb{R}^N\times\mathbb{R}^{N-1}\to\mathbb{R}^+$ by $\hat{H}^1\left(\theta,\epsilon_i^{-1}\right)=\tilde{\psi}^{-1}\left(\epsilon_i^{-1}\right)\Psi^1\left(\hat{x}^{1*}\left(\theta^{-1}+\epsilon_i^{-1}\right)-\theta^1\right)$. Since $\hat{x}^{1*}$ is continuous, so is $\hat{H}^1$ from the same argument as for $\tilde{H}^1$, and since $x^{1*}\geq\hat{x}^{1*}$, it follows that $\tilde{H}^1\geq\hat{H}^1$. Further, if $x^{1*}\left(x_i^{-1}\right)>\hat{x}^{1*}\left(x_i^{-1}\right)$ for some $x_i^{-1}\in\mathbb{R}^{N-1}$, then for any $\theta^1\in\mathbb{R}$ and $z\in\mathbb{R}^{N-1}$, $\tilde{H}^1\left(\theta^1,z,\epsilon_i^{-1}\right)\geq\hat{H}^1\left(\theta^1,z,\epsilon_i^{-1}\right)$ for any ϵ_i^{-1}, with strict inequality for some ϵ_i^{-1}. Thus, the claim follows by letting $\hat{H}^1\left(\theta^1,z,\epsilon_i^{-1}\right)$ and $\hat{\theta}^{1*}(z)$ play the role of $\tilde{H}^1\left(\theta^1,z',\epsilon_i^{-1}\right)$ and $\theta^{1*}(z')$ in the proof of Lemma 4. ■

With adequate assumptions, the conclusions of Lemmas 4 and 5 hold when x^{j*} and $\hat{x}^{j*}$ may equal $-\infty$. This issue is discussed in the proof of propositions for the nonsymmetric two-country environment, where such consideration is necessary.

Lemmas 6 and 7 are critical to showing the convergence of the relevant functions.

Lemma 6 *Under the assumptions of Lemma 2,*
$\bar{\Gamma}_n^{j*}\left(x_i\right)\leq\underline{\Gamma}_n^{j*}\left(x_i^1,\ldots,x_i^{j-1},x_i^j-d_n^j,\ldots,x_i^N\right)$ for any $j\in J$ and $x_i\in\mathbb{R}^N$.

Proof. For any $j\in J$ and $x_i^{-j}-\epsilon_i^{-j}\in\mathbb{R}^{N-1}$, $\bar{\theta}_n^{j*}\left(x_i^{-j}-\epsilon_i^{-j}\right)\leq\underline{\theta}_n^{j*}\left(x_i^{-j}-\epsilon_i^{-j}\right)+d_n^j$. Thus, the claim follows from (23), (24), and the monotonicity of the integral. ■

Lemma 7 *For any $j\in J$ and $\underline{x}_n^{j*}$, $\bar{x}_n^{j*}:\mathbb{R}^{N-1}\to\{-\infty\}\cup\mathbb{R}$, define $\underline{\theta}_{n+1}^{j*}$, $\bar{\theta}_{n+1}^{j*}:\mathbb{R}^{N-1}\to[0,1]$ by (19), (20). If there exist $z\in\mathbb{R}^{N-1}$ and $y>0$ such that $\bar{x}_n^{j*}\left(x_i^{-j}-z\right)\leq\underline{x}_n^{j*}\left(x_i^{-j}\right)+y$ for $x_i^{-j}\in X_{\mathbb{R}}\left(\underline{x}_n^{j*}\right)$, and $\bar{x}_n^{j*}\left(x_i^{-j}-z\right)=\underline{x}_n^{j*}\left(x_i^{-j}\right)=-\infty$ for $x_i^{-j}\notin X_{\mathbb{R}}\left(\underline{x}_n^{j*}\right)$, then $\bar{\theta}_{n+1}^{j*}\left(\theta^{-j}-z\right)-\underline{\theta}_{n+1}^{j*}\left(\theta^{-j}\right)\leq\lambda^j y$ for any $\theta^{-j}\in\mathbb{R}^{N-1}$. The same statement holds with both $\leq$ replaced by $<$.*

Proof. Replacing θ^{-j} with $\theta^{-j}-z$ in (37),

$$\bar{\theta}_{n+1}^{j*}\left(\theta^{-j}-z\right)=\int_{\epsilon_i^{-j}\in\mathbb{R}^{N-1}}\tilde{\psi}^{-j}\left(\epsilon_i^{-j}\right)\Psi^j\left(\bar{x}_n^{j*}\left(\theta^{-j}-z+\epsilon_i^{-j}\right)-\bar{\theta}_{n+1}^{j*}\left(\theta^{-j}-z\right)\right)d\epsilon_i^{-j}. \tag{38}$$

Take any $\theta^{-j} \in \mathbb{R}^{N-1}$. If $\bar{\theta}_{n+1}^{j*}(\theta^{-j}-z)-\underline{\theta}_{n+1}^{j*}(\theta^{-j}) \leq 0$, the claim follows since $y>0$. Suppose $\bar{\theta}_{n+1}^{j*}(\theta^{-j}-z)-\underline{\theta}_{n+1}^{j*}(\theta^{-j})>0$, and let

$$F_n^j\left(\epsilon_i^{-j}\right) \equiv \Psi^j\left(\bar{x}_n^{j*}\left(\theta^{-j}-z+\epsilon_i^{-j}\right)-\bar{\theta}_{n+1}^{j*}\left(\theta^{-j}-z\right)\right)-\Psi^j\left(\underline{x}_n^{j*}\left(\theta^{-j}+\epsilon_i^{-j}\right)-\underline{\theta}_{n+1}^{j*}\left(\theta^{-j}\right)\right). \tag{39}$$

Then (36), (38), and (39) imply $\|F_n^j\| = \sup_{\epsilon_i^{-j}\in\mathbb{R}^{N-1}} F_n^j\left(\epsilon_i^{-j}\right) > 0$ and

$$\begin{aligned}
\bar{\theta}_{n+1}^{j*}\left(\theta^{-j}-z\right)-\underline{\theta}_{n+1}^{j*}\left(\theta^{-j}\right) &= \int_{\epsilon_i^{-j}\in\mathbb{R}^{N-1}} \tilde{\psi}^{-j}\left(\epsilon_i^{-j}\right) F_n^j\left(\epsilon_i^{-j}\right) d\epsilon_i^{-j} \\
&\leq \int_{\epsilon_i^{-j}\in\mathbb{R}^{N-1}} \tilde{\psi}^{-j}\left(\epsilon_i^{-j}\right) \left\|F_n^j\right\| d\epsilon_i^{-j} \\
&= \left\|F_n^j\right\|,
\end{aligned} \tag{40}$$

where the second equality follows since $\int_{\epsilon_i^{-j}\in\mathbb{R}^{N-1}} \tilde{\psi}^{-j}\left(\epsilon_i^{-j}\right) d\epsilon_i^{-j} = 1$.

Let $X_{\mathbb{R}}\left(\bar{x}_n^{j*};\theta^{-j}-z\right) \equiv \left\{\epsilon_i^{-j} \in \mathbb{R}^{N-1} | \theta^{-j}-z+\epsilon_i^{-j} \in X_{\mathbb{R}}\left(\bar{x}_n^{j*}\right)\right\}$. If $\epsilon_i^{-j} \notin X_{\mathbb{R}}\left(\bar{x}_n^{j*};\theta^{-j}-z\right)$, then $\bar{x}_n^{j*}\left(\theta^{-j}-z+\epsilon_i^{-j}\right)=-\infty$, hence $F_n^j\left(\epsilon_i^{-j}\right) \leq 0$ from (39). Thus, $F_n^j\left(\epsilon_i^{-j}\right)$ can be positive only if $\epsilon_i^{-j} \in X_{\mathbb{R}}\left(\bar{x}_n^{j*};\theta^{-j}-z\right)$, in which case $\bar{x}_n^{j*}\left(\theta^{-j}-z+\epsilon_i^{-j}\right)$ and $\underline{x}_n^{j*}\left(\theta^{-j}+\epsilon_i^{-j}\right)$ are both finite from the assumption of the lemma. Then, from (39),

$$\begin{aligned}
\left\|F_n^j\right\| \leq \left\|\psi^j\right\| \Big\{ & \sup_{\epsilon_i^{-j}\in X_{\mathbb{R}}\left(\bar{x}_n^{j*};\theta^{-j}-z\right)} \left[\bar{x}_n^{j*}\left(\theta^{-j}-z+\epsilon_i^{-j}\right)-\underline{x}_n^{j*}\left(\theta^{-j}+\epsilon_i^{-j}\right)\right] \\
& -\left[\bar{\theta}_{n+1}^{j*}\left(\theta^{-j}-z\right)-\underline{\theta}_{n+1}^{j*}\left(\theta^{-j}\right)\right] \Big\},
\end{aligned} \tag{41}$$

where the terms in the curly bracket are positive given $\|F_n^j\|>0$. Substituting (41) into (40) and rearranging,

$$\bar{\theta}_{n+1}^{j*}\left(\theta^{-j}-z\right)-\underline{\theta}_{n+1}^{j*}\left(\theta^{-j}\right) \leq \lambda^j \sup_{\epsilon_i^{-j}\in X_{\mathbb{R}}\left(\bar{x}_n^{j*};\theta^{-j}-z\right)} \left[\bar{x}_n^{j*}\left(\theta^{-j}-z+\epsilon_i^{-j}\right)-\underline{x}_n^{j*}\left(\theta^{-j}+\epsilon_i^{-j}\right)\right] \leq \lambda^j y. \tag{42}$$

The second inequality here follows from the assumption of the lemma, and it is a strict inequality if the inequality in the assumption is strict. ■

Proof of Proposition 1

I first prove two lemmas, and then use them to prove Proposition A1, which shows that the iterative deletion procedure defined by (15)–(20) yields monotonic sequences of monotonic and continuous functions from below and above. I then show that both these sequences converge to a common limit, which has the claimed properties.

Lemmas 8 and 9 below complement Lemmas 2 and 3. For the sake of disposition, I summarize below the assumptions made in these two lemmas.

Assumption 1 *Suppose $N=2$, and for all $j \in J=\{1,2\}$, let $\underline{\theta}_n^{j*}$, $\bar{\theta}_n^{j*} \in \tilde{B}(\mathbb{R})$, $\underline{\Gamma}_n^{j*}$, $\bar{\Gamma}_n^{j*} : \mathbb{R}^2 \to \mathbb{R}$ be defined by (15), (16), and $\underline{x}_n^{j*}$, $\bar{x}_n^{j*} : \mathbb{R} \to \bar{\mathbb{R}}$ be defined by (17), (18).*

Note that one may invoke Lemmas 1–3 under Assumption 1.

Lemma 8 *Let Assumption 1 hold. Then, for any $j \in J$, $\underline{\Gamma}_n^{j*} - \max_{k \in J_0^{-j}} \bar{\Gamma}_n^{k*}$ and $\bar{\Gamma}_n^{j*} - \max_{k \in J_0^{-j}} \underline{\Gamma}_n^{k*}$ are continuous, decreasing in x_i^j, and nondecreasing in x_i^{-j}. Moreover, if $x_i^{-j} \in X_{\mathbb{R}}\left(\underline{x}_n^{j*}\right)$, then $\underline{\Gamma}_n^{j*}(x_i) = \max_{k \in J_0^{-j}} \bar{\Gamma}_n^{k*}(x_i)$ at $x_i^j = \underline{x}_n^{j*}\left(x_i^{-j}\right)$, and if $x_i^{-j} \in X_{\mathbb{R}}\left(\bar{x}_n^{j*}\right)$, then $\bar{\Gamma}_n^{j*}(x_i) = \max_{k \in J_0^{-j}} \underline{\Gamma}_n^{k*}(x_i)$ at $x_i^j = \bar{x}_n^{j*}\left(x_i^{-j}\right)$.*

Proof. I prove the claim for $j = 1$. Note that a function obtained by taking the minimum of continuous functions is continuous[44], and that a similar statement holds for monotonic functions.[45] Now, from Lemma 1, $\underline{\Gamma}_n^{j*}$ and $\bar{\Gamma}_n^{j*}$ are continuous, decreasing in x_i^j, and nondecreasing in x_i^{-j} for all $j \in J$. Thus, both $\underline{\Gamma}_n^{1*}$ and $\underline{\Gamma}_n^{1*} - \bar{\Gamma}_n^{2*}$ are continuous, decreasing in x_i^1, and nondecreasing in x_i^2, so the same is true for $\underline{\Gamma}_n^{1*} - \max_{k \in J_0^{-1}} \bar{\Gamma}_n^{k*} = \min\left\{\underline{\Gamma}_n^{1*}, \underline{\Gamma}_n^{1*} - \bar{\Gamma}_n^{2*}\right\}$. The claim for $\bar{\Gamma}_n^{1*} - \max_{k \in J_0^{-1}} \underline{\Gamma}_n^{k*}$ follows similarly.

Therefore, (17) implies that if $x_i^2 \in X_{\mathbb{R}}\left(\underline{x}_n^{1*}\right)$, then $\underline{x}_n^{1*}\left(x_i^2\right)$ is the unique x_i^1 such that $\underline{\Gamma}_n^{1*}(x_i) = \max_{k \in J_0^{-1}} \bar{\Gamma}_n^{k*}(x_i)$. Similarly, (18) implies that if $x_i^2 \in X_{\mathbb{R}}\left(\bar{x}_n^{1*}\right)$, then $\bar{x}_n^{1*}\left(x_i^2\right)$ is the unique x_i^1 such that $\bar{\Gamma}_n^{1*}(x_i) = \max_{k \in J_0^{-j}} \underline{\Gamma}_n^{k*}(x_i)$. ■

Lemma 9 below examines the properties of $\underline{x}_n^{j*}$ and $\bar{x}_n^{j*}$. Part (a)–(d) concern $\underline{x}_n^{j*}$, and (A)–(D) are parallel statements for $\bar{x}_n^{j*}$.

Lemma 9 *Under Assumption 1, the following holds for any $j \in J$.*

(a) $\underline{x}_n^{j*} < \infty$, and $X_{\mathbb{R}}\left(\underline{x}_n^{j*}\right) = \left(\underline{\omega}_n^{j*}, \infty\right)$ for some $\underline{\omega}_n^{j*} < \infty$. (b) $\underline{x}_n^{j*} = \min_{k \in J_0^{-j}} \underline{x}_n^{j,k}$, and for any $k \in J_0^{-j}$, $\underline{x}_n^{j*}\left(x_i^{-j}\right) = \underline{x}_n^{j,k}\left(x_i^{-j}\right)$ at some $x_i^{-j} \in X_{\mathbb{R}}\left(\underline{x}_n^{j*}\right)$. (c) $\underline{x}_n^{j*}$ is nondecreasing in $\mathbb{R}$, continuous in $X_{\mathbb{R}}\left(\underline{x}_n^{j*}\right)$, and increasing at some $x_i^{-j} \in X_{\mathbb{R}}\left(\underline{x}_n^{j*}\right)$; further, if $\underline{\theta}_n^{j*}$ is increasing, then $\underline{x}_n^{j*}$ is increasing in $X_{\mathbb{R}}\left(\underline{x}_n^{j*}\right)$. (d) If $\underline{\omega}_n^{j*} \in \mathbb{R}$, $\lim_{x_i^{-j} \searrow \underline{\omega}_n^{j*}} \underline{x}_n^{j*}\left(x_i^{-j}\right) = -\infty$.

(A) $\bar{x}_n^{j*} < \infty$, and $X_{\mathbb{R}}\left(\bar{x}_n^{j*}\right) = \left(\bar{\omega}_n^{j*}, \infty\right)$ for some $\bar{\omega}_n^{j*} < \infty$. (B) $\bar{x}_n^{j*} = \min_{k \in J_0^{-j}} \bar{x}_n^{j,k}$, and for any $k \in J_0^{-j}$, $\bar{x}_n^{j*}\left(x_i^{-j}\right) = \bar{x}_n^{j,k}\left(x_i^{-j}\right)$ at some $x_i^{-j} \in X_{\mathbb{R}}\left(\bar{x}_n^{j*}\right)$. (C) $\bar{x}_n^{j*}$ is nondecreasing in $\mathbb{R}$, continuous in $X_{\mathbb{R}}\left(\bar{x}_n^{j*}\right)$, and increasing at some $x_i^{-j} \in X_{\mathbb{R}}\left(\bar{x}_n^{j*}\right)$; further, if $\bar{\theta}_n^{j*}$ is increasing, then $\bar{x}_n^{j*}$ is increasing in $X_{\mathbb{R}}\left(\bar{x}_n^{j*}\right)$. (D) If $\bar{\omega}_n^{j*} \in \mathbb{R}$, $\lim_{x_i^{-j} \searrow \bar{\omega}_n^{j*}} \bar{x}_n^{j*}\left(x_i^{-j}\right) = -\infty$.

Proof. Note below that (15), (16) are rewritten as (23), (24), and $J_0^{-1} = \{0, 2\}$.

For (a), note from (17) that $\underline{x}_n^{1*}\left(x_i^2\right) = \infty$ if the set $\left\{x_i^1 \in \mathbb{R} | \underline{\Gamma}_n^{1*}(x_i) \leq \max_{k \in J_0^{-1}} \bar{\Gamma}_n^{k*}(x_i)\right\}$ is empty, and $\underline{x}_n^{1*}\left(x_i^2\right) = -\infty$ if this set is unbounded below. From Lemma 3, $\underline{x}_n^{1,0}\left(x_i^2\right) \in \mathbb{R}$ for any $x_i^2 \in \mathbb{R}$, and $\underline{\Gamma}_n^{1*}(x_i) \leq \bar{\Gamma}_n^{0*}(x_i)$ if and only if $x_i^1 \geq \underline{x}_n^{1,0}\left(x_i^2\right)$. Thus, this set is

[44] Let f and g be continuous. Then, $f+g$, $f-g$, and $|f|$ are continuous (Johnsonbaugh and Pfaffenberger (2010), Theorem 40.4). But then, $\min\{f, g\} = (f + g - |f - g|)/2$ is continuous. By repeating the argument, the same result follows for more than two functions.

[45] Let $f, g : \mathbb{R}^m \to \mathbb{R}$ be decreasing, $h \equiv \min\{f, g\}$, and take any $z, z' \in \mathbb{R}^m$, $z' \geq z$ and $z' \neq z$. Then, $h(z') \leq f(z')$ and $h(z') \leq g(z')$, so $h(z) - h(z') = \min\{f(z), g(z)\} - h(z') \geq \min\{f(z) - f(z'), g(z) - g(z')\} > 0$, so h is decreasing. If f, g are nonincreasing, the last inequality is weak, so h is nonincreasing. By repeating the argument, the same result follows for more than two functions. Minor modifications from above yield similar statements for increasing (nondecreasing) functions.

nonempty for any $x_i^2 \in \mathbb{R}$, and is bounded below if and only if $\underline{\Gamma}_n^{1*}(x_i) > \bar{\Gamma}_n^{2*}(x_i)$ for any sufficiently small x_i^1. Therefore,

$$X_{\mathbb{R}}\left(\underline{x}_n^{1*}\right) = \left\{x_i^2 \in \mathbb{R} | \lim_{x_i^1 \to -\infty} \left[\underline{\Gamma}_n^{1*}(x_i) - \bar{\Gamma}_n^{2*}(x_i)\right] > 0\right\}, \tag{43}$$

where the limit exists for any $x_i^2 \in \mathbb{R}$, since $\underline{\Gamma}_n^{1*} - \bar{\Gamma}_n^{2*}$ is decreasing in x_i^1.

Clearly $X_{\mathbb{R}}(\underline{x}_n^{1*}) \neq \varnothing$, since from (23) and (24), $\underline{\Gamma}_n^{1*}(x_i) - \bar{\Gamma}_n^{2*}(x_i) > 0$ for combinations of sufficiently small x_i^1 and sufficiently large x_i^2. Moreover, since $\underline{\Gamma}_n^{1*} - \bar{\Gamma}_n^{2*}$ is increasing in x_i^2, if $z \in X_{\mathbb{R}}(\underline{x}_n^{1*})$, then $z' \in X_{\mathbb{R}}(\underline{x}_n^{1*})$ for all $z' \geq z$, and if $z \notin X_{\mathbb{R}}(\underline{x}_n^{1*})$, then $z'' \notin X_{\mathbb{R}}(\underline{x}_n^{1*})$ for all $z'' \leq z$. Thus, $X_{\mathbb{R}}(\underline{x}_n^{1*}) = (\underline{\omega}_n^{1*}, \infty)$, where $\underline{\omega}_n^{1*} = -\infty$ if $X_{\mathbb{R}}(\underline{x}_n^{1*}) = \mathbb{R}$. If $X_{\mathbb{R}}(\underline{x}_n^{1*}) \neq \mathbb{R}$, then $\underline{\omega}_n^{1*}$ is the value of $x_i^2 \in \mathbb{R}$ such that $\lim_{x_i^1 \to -\infty}\left[\underline{\Gamma}_n^{1*}(x_i) - \bar{\Gamma}_n^{2*}(x_i)\right] = 0$, where such x_i^2 uniquely exists since $\underline{\Gamma}_n^{1*} - \bar{\Gamma}_n^{2*}$ is continuous and increasing in x_i^2.

For (b), note that for any $x_i^2 \in X_{\mathbb{R}}(\underline{x}_n^{1*})$, Lemma 8 implies

$$\underline{x}_n^{1*}\left(x_i^2\right) = \left\{x_i^1 \in \mathbb{R} | \underline{\Gamma}_n^{1*}(x_i) = \max\nolimits_{k \in J_0^{-1}} \bar{\Gamma}_n^{k*}(x_i)\right\}. \tag{44}$$

Moreover, if $l \in J_0^{-1}$ is such that $\underline{\Gamma}_n^{1*}(\underline{x}_n^{1*}(x_i^2), x_i^2) = \bar{\Gamma}_n^{l*}(\underline{x}_n^{1*}(x_i^2), x_i^2)$, then $\underline{x}_n^{1*}(x_i^2) = \underline{x}_n^{1,l}(x_i^2)$ from Lemma 2. Now, (17) implies that for any $x_i^2 \in \mathbb{R}$ and $k \in J_0^{-1}$, if $x_i^1 < \underline{x}_n^{1*}(x_i^2)$, then $\underline{\Gamma}_n^{1*}(x_i) > \bar{\Gamma}_n^{k*}(x_i)$, so from the definition of $\underline{x}_n^{1,k}$ in (25), $\underline{x}_n^{1*}(x_i^2) \leq \underline{x}_n^{1,k}(x_i^2)$. Combining these results, $\underline{x}_n^{1*}(x_i^2) = \min_{k \in J_0^{-1}} \underline{x}_n^{1,k}(x_i^2)$. For any $x_i^2 \notin X_{\mathbb{R}}(\underline{x}_n^{1*})$, the argument in the proof of (a) implies $\underline{x}_n^{1*}(x_i^2) = \underline{x}_n^{1,2}(x_i^2) = -\infty$, and thus for any $x_i^2 \in \mathbb{R}$,

$$\underline{x}_n^{1*}\left(x_i^2\right) = \min\nolimits_{k \in J_0^{-1}} \underline{x}_n^{1,k}\left(x_i^2\right). \tag{45}$$

For the latter part of the claim, note from (24) and the boundedness of $\bar{\theta}_n^{2*}$ that for sufficiently large x_i^2, $\bar{\Gamma}_n^{2*}(x_i) < 0 = \bar{\Gamma}_n^{0*}(x_i)$ for any x_i^1, hence $\underline{x}_n^{1*}(x_i^2) = \underline{x}_n^{1,0}(x_i^2) \in \mathbb{R}$. Further, since $\bar{\Gamma}_n^{2*}$ is continuous from Lemma 1, (24) and the boundedness of $\bar{\theta}_n^{2*}$ imply that there exists $z \in \mathbb{R}$ such that $\lim_{x_i^1 \to -\infty} \bar{\Gamma}_n^{2*}(x_i^1, z) = 0$. Then, since $\lim_{x_i^1 \to -\infty} \underline{\Gamma}_n^{1*}(x_i^1, z) > 0$ from (17), $\lim_{x_i^1 \to -\infty}\left[\underline{\Gamma}_n^{1*}(x_i^1, z) - \bar{\Gamma}_n^{2*}(x_i^1, z)\right] > 0$, which implies $z \in X_{\mathbb{R}}(\underline{x}_n^{1*})$ from (43). Also, since $\bar{\Gamma}_n^{2*}$ is nondecreasing in x_i^1, $\bar{\Gamma}_n^{2*}(x_i^1, z) \geq 0 = \bar{\Gamma}_n^{0*}(x_i^1, z)$ for any $x_i^1 \in \mathbb{R}$. Thus, $\underline{\Gamma}_n^{1*}(\underline{x}_n^{1*}(z), z) = \bar{\Gamma}_n^{2*}(\underline{x}_n^{1*}(z), z)$ from Lemma 8, hence $\underline{x}_n^{1*}(z) = \underline{x}_n^{1,2}(z)$.

For (c), note from (45) and $X_{\mathbb{R}}(\underline{x}_n^{1,0}) = \mathbb{R}$ that $X_{\mathbb{R}}(\underline{x}_n^{1,2}) \subset X_{\mathbb{R}}(\underline{x}_n^{1*})$, and that $x_i^2 \in X_{\mathbb{R}}(\underline{x}_n^{1*})$ implies $\underline{x}_n^{1,2}(x_i^2) \in \mathbb{R} \cup \{\infty\}$. Now, from Lemmas 2 and 3, $\underline{x}_n^{1,2}$ is continuous and increasing, and $\underline{x}_n^{1,0}$ is continuous and nondecreasing, in $X_{\mathbb{R}}(\underline{x}_n^{1,2})$. Thus, from (45), $\underline{x}_n^{1*}$ is continuous and nondecreasing in $X_{\mathbb{R}}(\underline{x}_n^{1,2})$ (see footnotes 44 and 45). Moreover, for x_i^2 such that $\underline{x}_n^{1,2}(x_i^2) = \infty$, (45) implies $\underline{x}_n^{1*}(x_i^2) = \underline{x}_n^{1,0}(x_i^2)$, which is continuous and nondecreasing. Thus, $\underline{x}_n^{1*}$ is continuous and nondecreasing in $x_i^2 \in X_{\mathbb{R}}(\underline{x}_n^{1*})$, and is increasing if $\underline{x}_n^{1*}(x_i^2) = \underline{x}_n^{1,2}(x_i^2)$, where such $x_i^2 \in X_{\mathbb{R}}(\underline{x}_n^{1,2}) \subset X_{\mathbb{R}}(\underline{x}_n^{1*})$ exists as shown in (b). Further, if $\underline{\theta}_n^{j*}$ is increasing, so is $\underline{x}_n^{1,0}$ from Lemma 3, hence $\underline{x}_n^{1*}$ is increasing in $X_{\mathbb{R}}(\underline{x}_n^{1*})$.

For (d), note that the limit is well defined, since $\underline{x}_n^{1*}$ is nondecreasing. As argued in the proof of (a), $\lim_{x_i^1 \to -\infty}\left[\underline{\Gamma}_n^{1*}(x_i^1, \underline{\omega}_n^{1*}) - \bar{\Gamma}_n^{2*}(x_i^1, \underline{\omega}_n^{1*})\right] = 0$ since $\underline{\omega}_n^{1*} \in \mathbb{R}$. Suppose the claim is false, and $\lim_{x_i^2 \searrow \underline{\omega}_n^{1*}} \underline{x}_n^{1*}(x_i^2) = \kappa \in \mathbb{R}$. Then, since $\underline{\Gamma}_n^{1*} - \bar{\Gamma}_n^{2*}$ is decreasing in x_i^1, $\underline{\Gamma}_n^{1*}(\kappa, \underline{\omega}_n^{1*}) < \bar{\Gamma}_n^{2*}(\kappa, \underline{\omega}_n^{1*})$, and since $\underline{\Gamma}_n^{1*} - \bar{\Gamma}_n^{2*}$ is continuous, $\underline{\Gamma}_n^{1*}(\kappa, \underline{\omega}_n^{1*} + \nu) <$

$\bar{\Gamma}_n^{2*}(\kappa, \underline{\omega}_n^{1*}+\nu)$ for sufficiently small $\nu > 0$. Thus, $\underline{\Gamma}_n^{1*}(\kappa, \underline{\omega}_n^{1*}+\nu) < \max_{k\in J_0^{-1}} \underline{\Gamma}_n^{k*}(\kappa, \underline{\omega}_n^{1*}+\nu)$, and since $\underline{\omega}_n^{1*}+\nu \in X_{\mathbb{R}}(\underline{x}_n^{1*})$ from (a) and $\underline{\Gamma}_n^{1*} - \max_{k\in J_0^{-1}} \underline{\Gamma}_n^{k*}$ is decreasing in x_i^1 from Lemma 8, (44) implies $\underline{x}_n^{1*}(\underline{\omega}_n^{1*}+\nu) < \kappa$. Then, since $\underline{x}_n^{1*}$ is nondecreasing, $\lim_{x_i^2 \searrow \underline{\omega}_n^{1*}} \underline{x}_n^{1*}(x_i^2) < \kappa$. This is a contradiction, so $\lim_{x_i^2 \searrow \underline{\omega}_n^{1*}} \underline{x}_n^{1*}(x_i^2) = -\infty$.

This concludes the proof of Lemma 9(a)–(d), and (A)–(D) follow similarly. ■

For later use, note from above that (18) can be rewritten as

$$\bar{x}_n^{1*}(x_i^2) = \left\{ x_i^1 \in \mathbb{R} \,\middle|\, \bar{\Gamma}_n^{1*}(x_i) = \max_{k\in J_0^{-1}} \underline{\Gamma}_n^{k*}(x_i) \right\} \tag{46}$$

if $x_i^2 \in X_{\mathbb{R}}(\bar{x}_n^{1*})$, and $\bar{x}_n^{1*}(x_i^2) = -\infty$ otherwise.

By combining Lemmas 1–5, 8, and 9, I now prove the following proposition.[46]

Proposition A1 *Let $N = 2$, and $\left((\underline{x}_n^{j*}, \bar{x}_n^{j*}, \underline{\theta}_{n+1}^{j*}, \bar{\theta}_{n+1}^{j*})_{j\in J}\right)_{n=1}^{\infty}$ be defined by (15)–(20), where $\underline{\theta}_1^{j*} \equiv 0$ and $\bar{\theta}_1^{j*} \equiv 1$ for all $j \in J$. Then, for any $n \in \mathbb{N}$ and $j \in J$, (a) $\underline{\theta}_{n+1}^{j*}$ and $\bar{\theta}_{n+1}^{j*}$ are continuous and increasing, and $\underline{x}_{n+1}^{j*}$ and $\bar{x}_{n+1}^{j*}$ are nondecreasing in $\mathbb{R}$ and continuous and increasing, respectively, in $X_{\mathbb{R}}\left(\underline{x}_{n+1}^{j*}\right)$ and $X_{\mathbb{R}}\left(\bar{x}_{n+1}^{j*}\right)$, (b) $\bar{\theta}_n^{j*} > \bar{\theta}_{n+1}^{j*} > \underline{\theta}_{n+1}^{j*} > \underline{\theta}_n^{j*}$, and $\bar{x}_n^{j*} \geq \bar{x}_{n+1}^{j*} \geq \underline{x}_{n+1}^{j*} \geq \underline{x}_n^{j*}$.*

Proof. Note that $\underline{\theta}_1^{j*} = 0$, $\bar{\theta}_1^{j*} = 1$, and $\underline{\theta}_{n+1}^{j*}$, $\bar{\theta}_{n+1}^{j*} \in (0, 1)$ from (19) and (20). Thus, for any $n \in \mathbb{N}$, $\underline{\theta}_n^{j*}$ and $\bar{\theta}_n^{j*}$ are bounded, so Assumption 1 holds if $\underline{\theta}_n^{j*}$ and $\bar{\theta}_n^{j*}$ are continuous and nondecreasing.

For Proposition A1(a), I first show by induction that $\underline{\theta}_{n+1}^{j*}$ and $\bar{\theta}_{n+1}^{j*}$ are continuous and increasing for all $j \in J$ and $n \in \mathbb{N}$. By assumption, $\underline{\theta}_1^{j*}, \bar{\theta}_1^{j*} \in \tilde{B}(\mathbb{R})$, so from Lemma 9$(c)$, $\underline{x}_1^{j*}$ is nondecreasing in $\mathbb{R}$, continuous in $X_{\mathbb{R}}\left(\underline{x}_1^{j*}\right)$, and increasing at some $x_i^{-j} \in X_{\mathbb{R}}\left(\underline{x}_1^{j*}\right)$. Thus, if $X_{\mathbb{R}}\left(\underline{x}_1^{j*}\right) = \mathbb{R}$, then $\underline{\theta}_2^{j*}$ is continuous and increasing from Lemma 4. If $X_{\mathbb{R}}\left(\underline{x}_1^{j*}\right) \neq \mathbb{R}$, then from Lemma 9$(a)(d)$, there exists $\underline{\omega}_1^{j*} \in \mathbb{R}$ such that $\underline{x}_1^{j*}\left(x_i^{-j}\right) = -\infty$ for $x_i^{-j} \leq \underline{\omega}_1^{j*}$ and $\underline{x}_1^{j*}\left(x_i^{-j}\right) \searrow -\infty$ as $x_i^{-j} \searrow \underline{\omega}_1^{j*}$. Now, if one replaces $\tilde{H}^1$ in the proof of Lemma 4 by $\tilde{H}_1^j\left(\theta, \epsilon_i^{-j}\right) \equiv \tilde{\psi}^{-j}\left(\epsilon_i^{-j}\right) \Psi^j\left(\underline{x}_1^{j*}\left(\theta^{-j}+\epsilon_i^{-j}\right) - \theta^j\right)$, then $\tilde{H}_1^j\left(\theta, \epsilon_i^{-j}\right) = 0$ for $\theta^{-j}+\epsilon_i^{-j} \leq \underline{\omega}_1^{j*}$, and $\tilde{H}_1^j\left(\theta, \epsilon_i^{-j}\right) \searrow 0$ as $\theta^{-j}+\epsilon_i^{-j} \searrow \underline{\omega}_1^{j*}$, so $\tilde{H}_1^j$ is continuous just like $\tilde{H}^1$.[47] Thus, the argument in the proof of Lemma 4 goes through, and $\underline{\theta}_2^{j*}$ is continuous and increasing. That $\bar{\theta}_2^{j*}$ is continuous and increasing follows similarly, so the claim holds for $n = 1$. Now, suppose $\underline{\theta}_{n+1}^{j*}$ and $\bar{\theta}_{n+1}^{j*}$ are continuous and increasing for some $n \in \mathbb{N}$. Then, from the same argument, $\underline{\theta}_{n+2}^{j*}$ and $\bar{\theta}_{n+2}^{j*}$ are continuous and increasing, so the claim holds for $n+1$. Thus, by induction, $\underline{\theta}_{n+1}^{j*}$ and $\bar{\theta}_{n+1}^{j*}$ are continuous and increasing for all $n \in \mathbb{N}$. Then, from Lemma 9$(c)(C)$, $\underline{x}_{n+1}^{j*}$ is continuous and increasing in $X_{\mathbb{R}}\left(\underline{x}_{n+1}^{j*}\right)$, and so is $\bar{x}_{n+1}^{j*}$ in $X_{\mathbb{R}}\left(\bar{x}_{n+1}^{j*}\right)$.

For Proposition A1(b), I first show by induction that $\bar{\theta}_n^{j*} > \bar{\theta}_{n+1}^{j*} > \underline{\theta}_{n+1}^{j*} > \underline{\theta}_n^{j*}$ for all $j \in J$ and $n \in \mathbb{N}$. For any $j \in J$, $1 = \bar{\theta}_1^{j*} > \underline{\theta}_1^{j*} = 0$, so $\bar{\Gamma}_1^{j*} > \underline{\Gamma}_1^{j*}$ from (23) and (24).

[46] Throughout, $\mathbb{N}$ denotes the set of positive integers. Thus, Proposition A1(a) excludes $\underline{x}_1^{j*}$ and $\bar{x}_1^{j*}$, which is because $\underline{x}_1^{j*}$ and $\bar{x}_1^{j*}$ turn out to be continuous and nondecreasing, but not necessarily increasing.

[47] Recall the assumption that Ψ^j is defined on $\bar{\mathbb{R}}$, where $\Psi^j(-\infty) = 0$.

Take any $x_i^2 \in \mathbb{R}$. If $x_i^2 \notin X_{\mathbb{R}}\left(\underline{x}_1^{1*}\right)$, then $\bar{x}_1^{1*}\left(x_i^2\right) \geq \underline{x}_1^{1*}\left(x_i^2\right) = -\infty$. If $x_i^2 \in X_{\mathbb{R}}\left(\underline{x}_1^{1*}\right)$, then (44), $\bar{\Gamma}_1^{1*} > \underline{\Gamma}_1^{1*}$, $\bar{\Gamma}_1^{2*} > \underline{\Gamma}_1^{2*}$, and $\bar{\Gamma}_1^{0*} = \underline{\Gamma}_1^{0*} = 0$ imply

$$\bar{\Gamma}_1^{1*}\left(\underline{x}_1^{1*}\left(x_i^2\right), x_i^2\right) > \max_{k \in J_0^{-1}} \underline{\Gamma}_1^{k*}\left(\underline{x}_1^{1*}\left(x_i^2\right), x_i^2\right). \tag{47}$$

From Lemma 8, $\bar{\Gamma}_1^{1*} - \max_{k \in J_0^{-1}} \underline{\Gamma}_1^{k*}$ is decreasing in x_i^1, so (46) and (47) imply $\bar{x}_1^{1*}\left(x_i^2\right) > \underline{x}_1^{1*}\left(x_i^2\right)$. Therefore $\bar{x}_1^{1*} \geq \underline{x}_1^{1*}$, where the inequality is strict except when $x_i^2 \notin X_{\mathbb{R}}\left(\bar{x}_1^{1*}\right)$, in which case $\bar{x}_1^{1*}\left(x_i^2\right) = \underline{x}_1^{1*}\left(x_i^2\right) = -\infty$. Moreover, $X_{\mathbb{R}}\left(\bar{x}_1^{1*}\right) \neq \varnothing$ from Lemma 9(A). Also, from Lemma 9(d)(D), $\lim_{x_i^2 \searrow \underline{\omega}_1^{1*}} \underline{x}_1^{1*}\left(x_i^2\right) = -\infty$ and $\lim_{x_i^2 \searrow \bar{\omega}_1^{1*}} \bar{x}_1^{1*}\left(x_i^2\right) = -\infty$, so $\Psi^1\left(\underline{x}_1^{1*}\left(\theta^2 + \epsilon_i^2\right) - \theta^1\right)$ and $\Psi^1\left(\bar{x}_1^{1*}\left(\theta^2 + \epsilon_i^2\right) - \theta^1\right)$ are continuous for any $\left(\theta, \epsilon_i^2\right) \in \mathbb{R}^2 \times \mathbb{R}$. Thus, the argument in the proof of Lemma 5 goes through, hence $\bar{\theta}_2^{1*} > \underline{\theta}_2^{1*}$. Moreover, $\bar{\theta}_2^{1*}, \underline{\theta}_2^{1*} \in (0,1)$, so $1 = \bar{\theta}_1^{1*} > \bar{\theta}_2^{1*} > \underline{\theta}_2^{1*} > \underline{\theta}_1^{1*} = 0$. The same argument establishes the inequality for $j = 2$, so the claim holds for $n = 1$.

Now, suppose $\bar{\theta}_n^{j*} > \bar{\theta}_{n+1}^{j*} > \underline{\theta}_{n+1}^{j*} > \underline{\theta}_n^{j*}$ for all $j \in J$ and some $n \in \mathbb{N}$. Then, clearly $\bar{\Gamma}_n^{j*} > \bar{\Gamma}_{n+1}^{j*} > \underline{\Gamma}_{n+1}^{j*} > \underline{\Gamma}_n^{j*}$ from (23) and (24). Since $\bar{\Gamma}_{n+1}^{j*} > \underline{\Gamma}_{n+1}^{j*}$, the same argument as above shows $\bar{x}_{n+1}^{j*} \geq \underline{x}_{n+1}^{j*}$, where the inequality is strict for $x_i^{-j} \in X_{\mathbb{R}}\left(\bar{x}_{n+1}^{j*}\right)$.

To show $\underline{x}_{n+1}^{j*} \geq \underline{x}_n^{j*}$ and $\bar{x}_n^{j*} \geq \bar{x}_{n+1}^{j*}$, again, take any $x_i^2 \in \mathbb{R}$. If $x_i^2 \notin X_{\mathbb{R}}\left(\underline{x}_n^{1*}\right)$, then $\underline{x}_{n+1}^{1*}\left(x_i^2\right) \geq \underline{x}_n^{1*}\left(x_i^2\right) = -\infty$. If $x_i^2 \in X_{\mathbb{R}}\left(\underline{x}_n^{1*}\right)$, then (44), $\underline{\Gamma}_{n+1}^{1*} > \underline{\Gamma}_n^{1*}$, $\bar{\Gamma}_n^{2*} > \bar{\Gamma}_{n+1}^{2*}$, and $\bar{\Gamma}_{n+1}^{0*} = \bar{\Gamma}_n^{0*} = 0$ imply

$$\underline{\Gamma}_{n+1}^{1*}\left(\underline{x}_n^{1*}\left(x_i^2\right), x_i^2\right) > \max_{k \in J_0^{-1}} \bar{\Gamma}_{n+1}^{k*}\left(\underline{x}_n^{1*}\left(x_i^2\right), x_i^2\right). \tag{48}$$

From Lemma 8, $\underline{\Gamma}_{n+1}^{1*} - \max_{k \in J_0^{-1}} \bar{\Gamma}_{n+1}^{k*}$ is decreasing in x_i^1, so (44) (for $n+1$) and (48) imply $\underline{x}_{n+1}^{1*}\left(x_i^2\right) > \underline{x}_n^{1*}\left(x_i^2\right)$. Thus, $\underline{x}_{n+1}^{1*} \geq \underline{x}_n^{1*}$, with strict inequality for $x_i^2 \in X_{\mathbb{R}}\left(\underline{x}_{n+1}^{1*}\right)$. Similarly, $\underline{x}_{n+1}^{2*} \geq \underline{x}_n^{2*}$, with strict inequality for $x_i^1 \in X_{\mathbb{R}}\left(\underline{x}_{n+1}^{2*}\right)$.

Likewise, if $x_i^2 \notin X_{\mathbb{R}}\left(\bar{x}_{n+1}^{1*}\right)$, then $\bar{x}_n^{1*}\left(x_i^2\right) \geq \bar{x}_{n+1}^{1*}\left(x_i^2\right) = -\infty$. If $x_i^2 \in X_{\mathbb{R}}\left(\bar{x}_{n+1}^{1*}\right)$, then (46), $\bar{\Gamma}_n^{1*} > \bar{\Gamma}_{n+1}^{1*}$, $\underline{\Gamma}_{n+1}^{2*} > \underline{\Gamma}_n^{2*}$, and $\underline{\Gamma}_{n+1}^{0*} = \underline{\Gamma}_n^{0*} = 0$ imply

$$\bar{\Gamma}_n^{1*}\left(\bar{x}_{n+1}^{1*}\left(x_i^2\right), x_i^2\right) > \max_{k \in J_0^{-1}} \underline{\Gamma}_n^{k*}\left(\bar{x}_{n+1}^{1*}\left(x_i^2\right), x_i^2\right). \tag{49}$$

From Lemma 8, $\bar{\Gamma}_n^{1*} - \max_{k \in J_0^{-1}} \underline{\Gamma}_n^{k*}$ is decreasing in x_i^1, so (46) and (49) imply $\bar{x}_n^{1*}\left(x_i^2\right) > \bar{x}_{n+1}^{1*}\left(x_i^2\right)$. Thus, $\bar{x}_n^{1*} \geq \bar{x}_{n+1}^{1*}$, with strict inequality for $x_i^2 \in X_{\mathbb{R}}\left(\bar{x}_n^{1*}\right)$. Similarly, $\bar{x}_n^{2*} \geq \bar{x}_{n+1}^{2*}$, with strict inequality for $x_i^1 \in X_{\mathbb{R}}\left(\bar{x}_n^{2*}\right)$.

To summarize, if, for some $n \in \mathbb{N}$, $\bar{\theta}_n^{j*} > \bar{\theta}_{n+1}^{j*} > \underline{\theta}_{n+1}^{j*} > \underline{\theta}_n^{j*}$ for all $j \in J$, then $\bar{x}_n^{j*} \geq \bar{x}_{n+1}^{j*} \geq \underline{x}_{n+1}^{j*} \geq \underline{x}_n^{j*}$ for all $j \in J$, where each inequality is strict for some $x_i^{-j} \in \mathbb{R}$. By invoking the continuity argument as when establishing $\bar{\theta}_2^{1*} > \underline{\theta}_2^{1*}$ and applying Lemma 5, $\bar{\theta}_{n+1}^{j*} > \bar{\theta}_{n+2}^{j*} > \underline{\theta}_{n+2}^{j*} > \underline{\theta}_{n+1}^{j*}$ for all $j \in J$, so the claim holds for $n+1$. Thus, by induction, $\bar{\theta}_n^{j*} > \bar{\theta}_{n+1}^{j*} > \underline{\theta}_{n+1}^{j*} > \underline{\theta}_n^{j*}$ for all $j \in J$ and $n \in \mathbb{N}$, hence $\bar{x}_n^{j*} \geq \bar{x}_{n+1}^{j*} \geq \underline{x}_{n+1}^{j*} \geq \underline{x}_n^{j*}$. ■

The remaining proof of Proposition 1 again proceeds through a series of lemmas. Note from the proof of Proposition A1 that under the assumptions of this proposition, for any $j \in J$ and $n \in \mathbb{N}$, $\underline{\Gamma}_{n+1}^{j*}$ and $\bar{\Gamma}_{n+1}^{j*}$ are continuous, decreasing in x_i^j, and increasing in

x_i^{-j}, and further, $\bar{\Gamma}_n^{j*} > \bar{\Gamma}_{n+1}^{j*} > \underline{\Gamma}_{n+1}^{j*} > \underline{\Gamma}_n^{j*}$. Similarly, $\underline{\Gamma}_{n+1}^{j*} - \max_{k\in J_0^{-1}} \bar{\Gamma}_{n+1}^{k*}$ and $\bar{\Gamma}_{n+1}^{j*} - \max_{k\in J_0^{-1}} \underline{\Gamma}_{n+1}^{k*}$ are continuous, decreasing in x_i^j, and increasing in x_i^{-j}. Moreover, $(d_n^j)_{n=1}^{\infty}$ is a nonincreasing sequence, since $\bar{\theta}_n^{j*} > \bar{\theta}_{n+1}^{j*} > \underline{\theta}_{n+1}^{j*} > \underline{\theta}_n^{j*}$.

Lemmas 10 and 11 state how the horizontal and vertical distances between $\bar{x}_n^{j*}$ and $\underline{x}_n^{j*}$ decrease with n.

Lemma 10 *Under the assumptions of Proposition A1, for any $j \in J$ and $n \in \mathbb{N}$, $\bar{x}_{n+1}^{j*}\left(x_i^{-j} - d_{n+1}^{-j}\right) < \underline{x}_{n+1}^{j*}\left(x_i^{-j}\right) + d_{n+1}^j$ if $x_i^{-j} \in X_{\mathbb{R}}\left(\underline{x}_{n+1}^{j*}\right)$, and $\bar{x}_{n+1}^{j*}\left(x_i^{-j} - d_{n+1}^{-j}\right) = \underline{x}_{n+1}^{j*}\left(x_i^{-j}\right) = -\infty$ if $x_i^{-j} \notin X_{\mathbb{R}}\left(\underline{x}_{n+1}^{j*}\right)$.*

Proof. I prove the claim for $j = 1$. Fix any $n \in \mathbb{N}$ and $z \in \mathbb{R}$, and let $y \equiv \underline{x}_{n+1}^{1*}(z)$ to simplify notation. If $z \in X_{\mathbb{R}}\left(\underline{x}_{n+1}^{1*}\right)$, (44) implies $\underline{\Gamma}_{n+1}^{1*}(y,z) = \max_{k\in J_0^{-1}} \bar{\Gamma}_{n+1}^{k*}(y,z)$. Then,

$$\begin{aligned}
\bar{\Gamma}_{n+1}^{1*}\left(y + d_{n+1}^1, z - d_{n+1}^2\right) &< \bar{\Gamma}_{n+1}^{1*}\left(y + d_{n+1}^1, z\right) \\
&\le \underline{\Gamma}_{n+1}^{1*}(y,z) \\
&= \max_{k\in J_0^{-1}} \bar{\Gamma}_{n+1}^{k*}(y,z) \\
&\le \max_{k\in J_0^{-1}} \underline{\Gamma}_{n+1}^{k*}\left(y, z - d_{n+1}^2\right) \\
&\le \max_{k\in J_0^{-1}} \underline{\Gamma}_{n+1}^{k*}\left(y + d_{n+1}^1, z - d_{n+1}^2\right). \qquad (50)
\end{aligned}$$

Here, the first inequality follows since $\bar{\Gamma}_{n+1}^{1*}$ is increasing in x_i^2. The second and third inequalities are from Lemma 6, whereas the last inequality follows since $\underline{\Gamma}_{n+1}^{k*}$, $k \in J_0^{-1} = \{0, 2\}$, is nondecreasing in x_i^1. Then, since $\bar{\Gamma}_{n+1}^{1*} - \max_{k\in J_0^{-1}} \underline{\Gamma}_{n+1}^{k*}$ is decreasing in x_i^1, (46), (50), and $y = \underline{x}_{n+1}^{1*}(z)$ imply

$$\bar{x}_{n+1}^{1*}\left(z - d_{n+1}^2\right) < \underline{x}_{n+1}^{1*}(z) + d_{n+1}^1. \qquad (51)$$

If $z \notin X_{\mathbb{R}}\left(\underline{x}_{n+1}^{1*}\right)$, or equivalently $\underline{x}_{n+1}^{1*}(z) = -\infty$, then $\underline{\Gamma}_{n+1}^{1*}(x_i^1, z) < \max_{k\in J_0^{-1}} \bar{\Gamma}_{n+1}^{k*}(x_i^1, z)$ for any $x_i^1 \in \mathbb{R}$. But then, a similar argument as above shows that

$$\bar{\Gamma}_{n+1}^{1*}\left(x_i^1 + d_{n+1}^1, z - d_{n+1}^2\right) < \max_{k\in J_0^{-1}} \underline{\Gamma}_{n+1}^{k*}\left(x_i^1 + d_{n+1}^1, z - d_{n+1}^2\right) \qquad (52)$$

for any $x_i^1 \in \mathbb{R}$, hence $\bar{x}_{n+1}^{1*}\left(z - d_{n+1}^2\right) = -\infty$. ■

Lemma 11 *Under the assumptions of Proposition A1, for any $j \in J$, $n, \tau \in \mathbb{N}$ and $\tau' \in \{1, 2, \ldots, \tau\}$, $\bar{x}_{n+\tau}^{j*}\left(x_i^{-j} - (\lambda^{-j})^{\tau-\tau'} d_{n+1}^{-j}\right) < \underline{x}_{n+\tau}^{j*}\left(x_i^{-j}\right) + (\lambda^j)^{\tau'-1} d_{n+1}^j$ if $x_i^{-j} \in X_{\mathbb{R}}\left(\underline{x}_{n+\tau}^{j*}\right)$, and $\bar{x}_{n+\tau}^{j*}\left(x_i^{-j} - (\lambda^{-j})^{\tau-\tau'} d_{n+1}^{-j}\right) = \underline{x}_{n+\tau}^{j*}\left(x_i^{-j}\right) = -\infty$ if $x_i^{-j} \notin X_{\mathbb{R}}\left(\underline{x}_{n+\tau}^{j*}\right)$.*

Proof. Fix any $n \in \mathbb{N}$. As shown in Lemma 10, the claim holds for $\tau = 1$. Now, suppose the claim holds for some $\tau \in \mathbb{N}$. Then, for any $j \in J$ and $\tau' \in \{1, 2, \ldots, \tau\}$,

$$\bar{\theta}_{n+\tau+1}^{j*}\left(\theta^{-j} - \left(\lambda^{-j}\right)^{\tau-\tau'} d_{n+1}^{-j}\right) < \underline{\theta}_{n+\tau+1}^{j*}\left(\theta^{-j}\right) + \left(\lambda^j\right)^{\tau'} d_{n+1}^j \qquad (53)$$

from Lemma 7, which, combined with (23) and (24), implies

$$\bar{\Gamma}^{1*}_{n+\tau+1}\left(x^1_i + \left(\lambda^1\right)^{\tau'} d^1_{n+1}, x^2_i - \left(\lambda^2\right)^{\tau-\tau'} d^2_{n+1}\right) < \underline{\Gamma}^{1*}_{n+\tau+1}\left(x^1_i, x^2_i\right), \tag{54}$$

$$\bar{\Gamma}^{2*}_{n+\tau+1}\left(x^1_i - \left(\lambda^1\right)^{\tau-\tau'} d^1_{n+1}, x^2_i + \left(\lambda^2\right)^{\tau'} d^2_{n+1}\right) < \underline{\Gamma}^{2*}_{n+\tau+1}\left(x^1_i, x^2_i\right). \tag{55}$$

Using (54) and (55), I show that the claim holds for $\tau+1$.

Fix any $z \in \mathbb{R}$, and let $y \equiv \underline{x}^{1*}_{n+\tau+1}(z)$ to simplify notation. For now, suppose $z \in X_{\mathbb{R}}\left(\underline{x}^{1*}_{n+\tau+1}\right)$. Then, (44) implies

$$\underline{\Gamma}^{1*}_{n+\tau+1}(y, z) = \max_{k \in J_0^{-1}} \bar{\Gamma}^{k*}_{n+\tau+1}(y, z). \tag{56}$$

The goal here is to show that for any $\tau'' \in \{1, 2, \ldots, \tau+1\}$,

$$\begin{aligned} &\bar{\Gamma}^{1*}_{n+\tau+1}\left(y + \left(\lambda^1\right)^{\tau''-1} d^1_{n+1}, z - \left(\lambda^2\right)^{\tau+1-\tau''} d^2_{n+1}\right) \\ &< \max_{k \in J_0^{-1}} \underline{\Gamma}^{k*}_{n+\tau+1}\left(y + \left(\lambda^1\right)^{\tau''-1} d^1_{n+1}, z - \left(\lambda^2\right)^{\tau+1-\tau''} d^2_{n+1}\right), \end{aligned} \tag{57}$$

which requires separate arguments for $\tau'' = 1$, $\tau'' \in \{2, 3, \ldots, \tau\}$, and $\tau'' = \tau+1$.

First,

$$\begin{aligned} \bar{\Gamma}^{1*}_{n+\tau+1}\left(y + d^1_{n+1}, z - \left(\lambda^2\right)^{\tau} d^2_{n+1}\right) &< \bar{\Gamma}^{1*}_{n+\tau+1}\left(y + d^1_{n+1}, z\right) \\ &\leq \bar{\Gamma}^{1*}_{n+\tau+1}\left(y + d^1_{n+\tau+1}, z\right) \\ &\leq \underline{\Gamma}^{1*}_{n+\tau+1}(y, z) \\ &= \max_{k \in J_0^{-1}} \bar{\Gamma}^{k*}_{n+\tau+1}(y, z) \\ &\leq \max_{k \in J_0^{-1}} \underline{\Gamma}^{k*}_{n+\tau+1}\left(y + d^1_{n+1}, z - \left(\lambda^2\right)^{\tau} d^2_{n+1}\right). \end{aligned} \tag{58}$$

Here, the first inequality follows since $\bar{\Gamma}^{1*}_{n+\tau+1}$ is increasing in x^2_i. The second inequality holds since $d^1_{n+1} \geq d^1_{n+\tau+1}$ and $\bar{\Gamma}^{1*}_{n+\tau+1}$ is decreasing in x^1_i, and the third inequality follows from Lemma 6. The equality is from (56), and the last inequality results from setting $\tau' = \tau$, $x^1_i = y + d^1_{n+1}$, and $x^2_i = z - \left(\lambda^2\right)^{\tau} d^2_{n+1}$ in (55).

Second, for any $\tau'' \in \{2, 3, \ldots, \tau\}$,

$$\begin{aligned} &\bar{\Gamma}^{1*}_{n+\tau+1}\left(y + \left(\lambda^1\right)^{\tau''-1} d^1_{n+1}, z - \left(\lambda^2\right)^{\tau+1-\tau''} d^2_{n+1}\right) \\ &< \underline{\Gamma}^{1*}_{n+\tau+1}(y, z) \\ &= \max_{k \in J_0^{-1}} \bar{\Gamma}^{k*}_{n+\tau+1}(y, z) \\ &\leq \max_{k \in J_0^{-1}} \underline{\Gamma}^{k*}_{n+\tau+1}\left(y + \left(\lambda^1\right)^{\tau''-1} d^1_{n+1}, z - \left(\lambda^2\right)^{\tau+1-\tau''} d^2_{n+1}\right). \end{aligned} \tag{59}$$

Here, the first inequality follows from setting $\tau' = \tau'' - 1$ (hence, $\tau' \in \{1, 2, \ldots, \tau-1\}$), $x^1_i = y$, and $x^2_i = z$ in (54), whereas the equality follows from (56). The last inequality follows from setting $\tau' = \tau + 1 - \tau''$ (hence $\tau' \in \{1, 2, \ldots, \tau-1\}$), $x^1_i = y + \left(\lambda^1\right)^{\tau''-1} d^1_{n+1}$, and $x^2_i = z - \left(\lambda^2\right)^{\tau+1-\tau''} d^2_{n+1}$ in (55).

Third,

$$\begin{aligned}
\bar{\Gamma}^{1*}_{n+\tau+1}\left(y+\left(\lambda^{1}\right)^{\tau} d^{1}_{n+1}, z-d^{2}_{n+1}\right) &< \underline{\Gamma}^{1*}_{n+\tau+1}\left(y,z\right) \\
&= \max_{k\in J_0^{-1}} \bar{\Gamma}^{k*}_{n+\tau+1}\left(y,z\right) \\
&\le \max_{k\in J_0^{-1}} \underline{\Gamma}^{k*}_{n+\tau+1}\left(y, z-d^{2}_{n+\tau+1}\right) \\
&\le \max_{k\in J_0^{-1}} \underline{\Gamma}^{k*}_{n+\tau+1}\left(y, z-d^{2}_{n+1}\right) \\
&\le \max_{k\in J_0^{-1}} \underline{\Gamma}^{k*}_{n+\tau+1}\left(y+\left(\lambda^{1}\right)^{\tau} d^{1}_{n+1}, z-d^{2}_{n+1}\right). \qquad (60)
\end{aligned}$$

Here, the first inequality follows by setting $\tau' = \tau$, $x^1_i = y$, and $x^2_i = z$ in (54). The equality is from (56), and the second inequality uses Lemma 6. The third inequality follows since $d^2_{n+1} \ge d^2_{n+\tau+1}$ and $\underline{\Gamma}^{k*}_{n+\tau+1}$, $k \in J_0^{-1}$, is nonincreasing in x^2_i, whereas the last inequality obtains since $\underline{\Gamma}^{k*}_{n+\tau+1}$, $k \in J_0^{-1}$, is nondecreasing in x^1_i.

Thus, if $z \in X_{\mathbb{R}}\left(\underline{x}^{1*}_{n+\tau+1}\right)$, (58)–(60) imply (57) for any $\tau'' \in \{1, 2, \ldots, \tau+1\}$. Then, since $\bar{\Gamma}^{1*}_{n+\tau+1} - \max_{k\in J_0^{-1}} \underline{\Gamma}^{k*}_{n+\tau+1}$ is decreasing in x^1_i, (46), (57), and $y = \underline{x}^{1*}_{n+\tau+1}(z)$ imply

$$\bar{x}^{1*}_{n+\tau+1}\left(z-\left(\lambda^{2}\right)^{\tau+1-\tau''} d^{2}_{n+1}\right) < \underline{x}^{1*}_{n+\tau+1}(z) + \left(\lambda^{1}\right)^{\tau''-1} d^{1}_{n+1}. \qquad (61)$$

Finally, if $z \notin X_{\mathbb{R}}\left(\underline{x}^{1*}_{n+\tau+1}\right)$, then $\underline{\Gamma}^{1*}_{n+\tau+1}\left(x^1_i, z\right) < \max_{k\in J_0^{-1}} \bar{\Gamma}^{k*}_{n+\tau+1}\left(x^1_i, z\right)$ for any $x^1_i \in \mathbb{R}$. Then, a similar argument as above shows that

$$\begin{aligned}
&\bar{\Gamma}^{1*}_{n+\tau+1}\left(x^1_i+\left(\lambda^{1}\right)^{\tau''-1} d^{1}_{n+1}, z-\left(\lambda^{2}\right)^{\tau+1-\tau''} d^{2}_{n+1}\right) \\
&< \max_{k\in J_0^{-1}} \underline{\Gamma}^{k*}_{n+\tau+1}\left(x^1_i+\left(\lambda^{1}\right)^{\tau''-1} d^{1}_{n+1}, z-\left(\lambda^{2}\right)^{\tau+1-\tau''} d^{2}_{n+1}\right)
\end{aligned}$$

for any $\tau'' \in \{1, 2, \ldots, \tau+1\}$ and $x^1_i \in \mathbb{R}$, hence $\bar{x}^{1*}_{n+\tau+1}\left(z-\left(\lambda^{2}\right)^{\tau+1-\tau''} d^{2}_{n+1}\right) = -\infty$.

A symmetric argument yields the corresponding relation between $\bar{x}^{2*}_{n+\tau+1}$ and $\underline{x}^{2*}_{n+\tau+1}$, so the claim holds for $\tau+1$. Thus, by induction, the claim holds for any $\tau \in \mathbb{N}$. ■

The final piece of the proof of Proposition 1 is Lemma 12, which applies Lemma 11 and shows that the lower and upper threshold functions converge to the common limit. Below, $f_n \nearrow f$ ($f_n \searrow f$) implies that $\{f_n\}_{n=1}^{\infty}$ is a nondecreasing (nonincreasing) sequence of functions that converge pointwise to f.

Lemma 12 *Under the assumptions of Proposition A1, for any $j \in J$, as $n \to \infty$, $\underline{\theta}^{j*}_n \nearrow \theta^{j*}$ and $\bar{\theta}^{j*}_n \searrow \theta^{j*}$, where $\theta^{j*} \in (0, \theta^{j*}_s)$ is continuous and increasing, and $\underline{x}^{j*}_n \nearrow x^{j*}$ and $\bar{x}^{j*}_n \searrow x^{j*}$, where $x^{j*} \in [-\infty, x^{j*}_s)$ is nondecreasing in $\mathbb{R}$ and is continuous and increasing in $X_{\mathbb{R}}(x^{j*})$.*

Proof. From Proposition A1, $(\underline{x}^{j*}_n)_{n=1}^{\infty}$ is a nondecreasing sequence, and $(\bar{x}^{j*}_n)_{n=1}^{\infty}$ is a nonincreasing sequence, and the two sequences are bounded by each other. Thus, they respectively converge pointwise to some function, $\underline{x}_{\infty}^{j*}$ and $\bar{x}_{\infty}^{j*}$, where $\bar{x}_{\infty}^{j*} \geq \underline{x}_{\infty}^{j*}$. Moreover, from Proposition A1, $\underline{x}_{n}^{j*}$ and $\bar{x}_{n}^{j*}$ are nondecreasing in $\mathbb{R}$ for all $n \in \mathbb{N}$, so $\underline{x}_{\infty}^{j*}$ and $\bar{x}_{\infty}^{j*}$ are also nondecreasing in $\mathbb{R}$. Thus, the left- and the right-hand limits of $\underline{x}_{\infty}^{j*}$ and $\bar{x}_{\infty}^{j*}$ exist on all $x_i^{-j} \in \mathbb{R}$, and the set of discontinuity points of $\underline{x}_{\infty}^{j*}$ and $\bar{x}_{\infty}^{j*}$ are at most countable.[48]

Fix any $n \in \mathbb{N}$. As shown in Lemma 11, for all $j \in J$, $\tau \in \mathbb{N}$, and $\tau' \in \{1, 2, \ldots, \tau\}$,

$$\bar{x}_{n+\tau}^{j*}\left(x_i^{-j} - \left(\lambda^{-j}\right)^{\tau-\tau'} d_{n+1}^{-j}\right) < \underline{x}_{n+\tau}^{j*}\left(x_i^{-j}\right) + \left(\lambda^{j}\right)^{\tau'-1} d_{n+1}^{j} \tag{62}$$

if $x_i^{-j} \in X_{\mathbb{R}}\left(\underline{x}_{n+\tau}^{j*}\right)$, and $\bar{x}_{n+\tau}^{j*}\left(x_i^{-j} - \left(\lambda^{-j}\right)^{\tau-\tau'} d_{n+1}^{-j}\right) = \underline{x}_{n+\tau}^{j*}\left(x_i^{-j}\right) = -\infty$ if $x_i^{-j} \notin X_{\mathbb{R}}\left(\underline{x}_{n+\tau}^{j*}\right)$. Below, for each $\tau \in \mathbb{N}$, fix $\tau' = (\tau+1)/2$ if τ is odd, and $\tau' = \tau/2$ if τ is even. Then, as $\tau \to \infty$, $\tau' \to \infty$ and $\tau - \tau' \to \infty$, which imply $\left(\lambda^{-j}\right)^{\tau-\tau'} d_{n+1}^{-j} \to 0$ and $\left(\lambda^{j}\right)^{\tau'-1} d_{n+1}^{j} \to 0$ since λ^j, $\lambda^{-j} < 1$.

First, take $x_i^{-j} \notin X_{\mathbb{R}}\left(\underline{x}_{\infty}^{j*}\right)$. Then, for any τ, $\underline{x}_{n+\tau}^{j*}\left(x_i^{-j}\right) \leq \underline{x}_{\infty}^{j*}\left(x_i^{-j}\right) = -\infty$ and thus $x_i^{-j} \notin X_{\mathbb{R}}\left(\underline{x}_{n+\tau}^{j*}\right)$, so $\bar{x}_{n+\tau}^{j*}\left(x_i^{-j} - \left(\lambda^{-j}\right)^{\tau-\tau'} d_{n+1}^{-j}\right) = \underline{x}_{n+\tau}^{j*}\left(x_i^{-j}\right) = -\infty$. Letting $\tau \to \infty$, $\bar{x}_{\infty}^{j*}\left(x_i^{-j}-\right) = -\infty$, where $\bar{x}_{\infty}^{j*}\left(x_i^{-j}-\right)$ is the left-hand limit of $\bar{x}_{\infty}^{j*}$ at x_i^{-j}. Since this holds for any $x_i^{-j} \notin X_{\mathbb{R}}\left(\underline{x}_{\infty}^{j*}\right)$ and $\underline{x}_{\infty}^{j*}$ is nondecreasing in $\mathbb{R}$, there exists $\underline{\omega}_{\infty}^{j*} \in \{-\infty\} \cup \mathbb{R}$ such that $\bar{x}_{\infty}^{j*}(z) = \underline{x}_{\infty}^{j*}(z) = -\infty$ for $z < \underline{\omega}_{\infty}^{j*}$, and $\bar{x}_{\infty}^{j*}(z) \geq \underline{x}_{\infty}^{j*}(z) > -\infty$ for $z > \underline{\omega}_{\infty}^{j*}$.

Next, take $x_i^{-j} \in X_{\mathbb{R}}\left(\underline{x}_{\infty}^{j*}\right)$, hence $\underline{x}_{\infty}^{j*}\left(x_i^{-j}\right) \in \mathbb{R}$. Then, since $\underline{x}_{n+\tau}^{j*} \nearrow \underline{x}_{\infty}^{j*}$, for τ sufficiently large, $\underline{x}_{n+\tau}^{j*}\left(x_i^{-j}\right) \in \mathbb{R}$, and thus (62) holds. As $\tau \to \infty$, the left-hand side (LHS) and the RHS of (62) converges to $\bar{x}_{\infty}^{j*}\left(x_i^{-j}-\right)$ and $\underline{x}_{\infty}^{j*}\left(x_i^{-j}\right)$, respectively, hence $\bar{x}_{\infty}^{j*}\left(x_i^{-j}-\right) \leq \underline{x}_{\infty}^{j*}\left(x_i^{-j}\right)$. But now, if $\bar{x}_{\infty}^{j*}$ is continuous at x_i^{-j}, then $\bar{x}_{\infty}^{j*}\left(x_i^{-j}-\right) = \bar{x}_{\infty}^{j*}\left(x_i^{-j}\right)$ and thus $\bar{x}_{\infty}^{j*}\left(x_i^{-j}\right) \leq \underline{x}_{\infty}^{j*}\left(x_i^{-j}\right)$, which implies $\bar{x}_{\infty}^{j*}\left(x_i^{-j}\right) = \underline{x}_{\infty}^{j*}\left(x_i^{-j}\right)$ since $\bar{x}_{\infty}^{j*} \geq \underline{x}_{\infty}^{j*}$ as argued above.

Summarizing these results, $\bar{x}_{\infty}^{j*} = \underline{x}_{\infty}^{j*}$ except possibly at $\underline{\omega}_{\infty}^{j*}$ and the set of discontinuity points of $\bar{x}_{\infty}^{j*}$ in $X_{\mathbb{R}}\left(\bar{x}_{\infty}^{j*}\right)$, which are at most countable.

Now, from Proposition A1, $\left(\underline{\theta}_{n}^{j*}\right)_{n=1}^{\infty}$ is an increasing sequence, and $\left(\bar{\theta}_{n}^{j*}\right)_{n=1}^{\infty}$ is a decreasing sequence, and the two sequences are bounded by each other. Thus, they respectively converge pointwise to $\underline{\theta}_{\infty}^{j*}$ and $\bar{\theta}_{\infty}^{j*}$, where

$$\underline{\theta}_{\infty}^{j*}\left(\theta^{-j}\right) = \int_{\epsilon_i^{-j} \in \mathbb{R}} \tilde{\psi}^{-j}\left(\epsilon_i^{-j}\right) \Psi^{j}\left(\underline{x}_{\infty}^{j*}\left(\theta^{-j} + \epsilon_i^{-j}\right) - \underline{\theta}_{\infty}^{j*}\left(\theta^{-j}\right)\right) d\epsilon_i^{-j}, \tag{63}$$

$$\bar{\theta}_{\infty}^{j*}\left(\theta^{-j}\right) = \int_{\epsilon_i^{-j} \in \mathbb{R}} \tilde{\psi}^{-j}\left(\epsilon_i^{-j}\right) \Psi^{j}\left(\bar{x}_{\infty}^{j*}\left(\theta^{-j} + \epsilon_i^{-j}\right) - \bar{\theta}_{\infty}^{j*}\left(\theta^{-j}\right)\right) d\epsilon_i^{-j}. \tag{64}$$

[48] See Kolmogorov and Fomin (1975), Chapter 9, Theorems 2 and 3.

But since $\underline{x}_{\infty}^{j*} = \bar{x}_{\infty}^{j*}$ almost everywhere on $\mathbb{R}$ as shown above, it follows that[49]

$$\int_{\epsilon_i^{-j} \in \mathbb{R}} \tilde{\psi}^{-j}\left(\epsilon_i^{-j}\right) \Psi^j \left(\underline{x}_{\infty}^{j*}\left(\theta^{-j} + \epsilon_i^{-j}\right) - \theta^j\right) d\epsilon_i^{-j}$$
$$= \int_{\epsilon_i^{-j} \in \mathbb{R}} \tilde{\psi}^{-j}\left(\epsilon_i^{-j}\right) \Psi^j \left(\bar{x}_{\infty}^{j*}\left(\theta^{-j} + \epsilon_i^{-j}\right) - \theta^j\right) d\epsilon_i^{-j} \tag{65}$$

for any $\theta \in \mathbb{R}^2$. Therefore, $\underline{\theta}_{\infty}^{j*}\left(\theta^{-j}\right) = \bar{\theta}_{\infty}^{j*}\left(\theta^{-j}\right) \equiv \theta^{j*}\left(\theta^{-j}\right)$ for all $\theta^{-j} \in \mathbb{R}$.

Now, a pointwise limit of an increasing sequence of lower semicontinuous functions is lower semicontinuous, and that of a decreasing sequence of upper semicontinuous functions is upper semicontinuous.[50] From Proposition A1, $\underline{\theta}_{n+1}^{j*}$ and $\bar{\theta}_{n+1}^{j*}$ are continuous for any $n \in \mathbb{N}$, so θ^{j*} is both lower and upper semicontinuous, hence continuous; moreover, since $\underline{\theta}_{n+1}^{j*}$ and $\bar{\theta}_{n+1}^{j*}$ are increasing for any $n \in \mathbb{N}$, θ^{j*} is nondecreasing. Clearly θ^{j*} is bounded, so $\theta^{j*} \in \tilde{B}(\mathbb{R})$. Further, by the definition of θ^{j*}, if $\underline{\theta}_{n}^{j*} = \bar{\theta}_{n}^{j*} = \theta^{j*}$ for all $j \in J$, then $\underline{\theta}_{n+1}^{j*} = \bar{\theta}_{n+1}^{j*} = \theta^{j*}$ for all $j \in J$. As shown in the proof of Proposition A1, if $\underline{\theta}_{n}^{j*}, \bar{\theta}_{n}^{j*} \in \tilde{B}(\mathbb{R})$ for all $j \in J$, then $\underline{\theta}_{n+1}^{j*}$ and $\bar{\theta}_{n+1}^{j*}$ are increasing for all $j \in J$. Thus, θ^{j*} is increasing.

Also, since $\underline{\theta}_{\infty}^{j*} = \bar{\theta}_{\infty}^{j*} = \theta^{j*}$, (23) and (24) imply $\underline{\Gamma}_{\infty}^{j*} = \bar{\Gamma}_{\infty}^{j*} \equiv \Gamma^{j*}$. Thus, from (44) and (46), $\underline{x}_{\infty}^{j*} = \bar{x}_{\infty}^{j*} \equiv x^{j*}$. Since $\theta^{j*} \in \tilde{B}(\mathbb{R})$ and θ^{j*} is increasing, Lemma 9$(a)(c)$ imply that x^{j*} is nondecreasing in $\mathbb{R}$, continuous and increasing in $X_{\mathbb{R}}\left(x^{j*}\right)$, and $x^{j*} < \infty$.

To see $\theta^{j*} < \theta_s^{j*}$, note that an environment where country j is the only target can be expressed as a two-country environment where attacking $k \in J^{-j}$ is never profitable, such that in (17) and (18), the RHS of the inequality is replaced by 0. In that case, speculators care only about whether attacking currency j is profitable, so the value of x_i^{-j} becomes irrelevant. Thus, this modified iterative deletion procedure yields, given $\underline{\theta}_{1,s}^{j*} = 0$ and $\bar{\theta}_{1,s}^{j*} = 1$, sequences of constant functions, $\left(\underline{x}_{n,s}^{j*}, \underline{\theta}_{n+1,s}^{j*}\right)_{n=1}^{\infty}$ and $\left(\bar{x}_{n,s}^{j*}, \bar{\theta}_{n+1,s}^{j*}\right)_{n=1}^{\infty}$, which both converge to functions constant at x_s^{j*} and θ_s^{j*}. Note that for any $x_i^{-j} \in \mathbb{R}$, $\underline{x}_{n,s}^{j*}\left(x_i^{-j}\right)$ is the value of x_i^j such that $\underline{\Gamma}_{n,s}^{j*}\left(x_i\right) \equiv \Pr\left(\theta^j \leq \underline{\theta}_{n,s}^{j*}\left(\theta^{-j}\right) | x_i\right) - c^j = 0$. Since $\underline{\theta}_1^{j*} = \underline{\theta}_{1,s}^{j*} = 0$, it follows that $\underline{\Gamma}_1^{j*} = \underline{\Gamma}_{1,s}^{j*}$. Thus, $\underline{x}_1^{j,0} = \underline{x}_{1,s}^{j*}$, and since $\underline{x}_1^{j*} \leq \underline{x}_1^{j,0}$ from Lemma 9(b), $\underline{x}_1^{j*} \leq \underline{x}_{1,s}^{j*}$. Thus, $\underline{\theta}_2^{j*} \leq \underline{\theta}_{2,s}^{j*}$ from Lemma 5, so from the monotonicity of the integral, $\underline{\Gamma}_2^{j*} \leq \underline{\Gamma}_{2,s}^{j*}$. Since $\underline{\Gamma}_2^{j*}$ and $\underline{\Gamma}_{2,s}^{j*}$ are decreasing in x_i^j from Lemma 1, $\underline{x}_2^{j,0} \leq \underline{x}_{2,s}^{j*}$, and since $\underline{x}_2^{j*} \leq \underline{x}_2^{j,0}$ from Lemma 9(b), $\underline{x}_2^{j*} \leq \underline{x}_{2,s}^{j*}$. Thus, $\underline{\theta}_3^{j*} \leq \underline{\theta}_{3,s}^{j*}$ from Lemma 5. Then, by the induction argument, $\underline{\theta}_{n+1}^{j*} \leq \underline{\theta}_{n+1,s}^{j*}$ for any $n \in \mathbb{N}$, so letting $n \to \infty$, $\theta^{j*} \leq \theta_s^{j*}$. Moreover, $\theta^{j*} < \theta_s^{j*}$, because if $\theta^{j*}(z) = \theta_s^{j*}$ for some $z \in \mathbb{R}$, then $\theta^{j*}(z') > \theta_s^{j*}$ for $z' > z$ since θ^{j*} is increasing, violating $\theta^{j*} \leq \theta_s^{j*}$. Since $\theta^{j*} > \underline{\theta}_1^{j*} = 0$, it follows that $\theta^{j*} \in (0, \theta_s^{j*})$. That $x^{j*} < x_s^{j*}$ follows from the same argument, since x^{j*} is nondecreasing in $\mathbb{R}$ and is increasing in $X_{\mathbb{R}}\left(x^{j*}\right)$.

Accordingly, for all $j \in J$, $\left(\underline{x}_n^{j*}, \underline{\theta}_{n+1}^{j*}\right)_{n=1}^{\infty}$ and $\left(\bar{x}_n^{j*}, \bar{\theta}_{n+1}^{j*}\right)_{n=1}^{\infty}$ monotonically converge to (x^{j*}, θ^{j*}), which have the claimed properties. ■

---

[49] Fix $\theta \in \mathbb{R}^2$ and let Λ^1 and Λ^2 be the integrand on the LHS and RHS of (65), respectively. Clearly, Λ^1 is nonnegative and bounded, and is nondecreasing since $\underline{x}_{\infty}^{j*}$ is nondecreasing. Thus, Λ^1 is Riemann integrable on any closed interval in $\mathbb{R}$, and the value of the integral is bounded by 1. So, the improper Riemann integral of Λ^1 on $\mathbb{R}$ exists and equals the Lebesgue integral of Λ^1 on $\mathbb{R}$ (Apostol (1974), Theorem 10.33), and since $\Lambda^1 = \Lambda^2$ almost everywhere on $\mathbb{R}$, their integrals on $\mathbb{R}$ coincide (Apostol (1974), Theorem 10.21).

[50] See Kaczor and Nowak (2001), Question 1.4.18.

Thus, the threshold equilibrium $(x^{j*}, \theta^{j*})_{j\in J}$ is the unique equilibrium that survives iterative deletion of dominated strategies, which concludes the proof of Proposition 1.$\square$

Proof of Proposition 2

By assumption, $\psi^j = \psi$, $\Psi^j = \Psi$, and $c^j = c$ for all $j \in J$. Then, clearly $\left(\tilde{\psi}^{-j}\right)_{j\in J}$ are symmetric across j.

Now, if there exist $(\theta^{j*})_{j\in J}$, $\theta^{j*} \in \tilde{B}(\mathbb{R}^{N-1})$, such that

$$\Gamma^{j*}(x_i) = 1 - c - \int_{\epsilon_i^{-j}\in\mathbb{R}^{N-1}} \tilde{\psi}^{-j}\left(\epsilon_i^{-j}\right)\Psi\left(x_i^j - \theta^{j*}\left(x_i^{-j} - \epsilon_i^{-j}\right)\right)d\epsilon_i^{-j}, \tag{66}$$

$$x^{j*}\left(x_i^{-j}\right) = \left\{x_i^j \in \mathbb{R} | \Gamma^{j*}(x_i) = \max_{k\in J_0^{-j}} \Gamma^{k*}(x_i)\right\}, \tag{67}$$

$$\theta^{j*}\left(\theta^{-j}\right) = \int_{\epsilon_i^{-j}\in\mathbb{R}^{N-1}} \tilde{\psi}^{-j}\left(\epsilon_i^{-j}\right)\Psi\left(x^{j*}\left(\theta^{-j} + \epsilon_i^{-j}\right) - \theta^{j*}\left(\theta^{-j}\right)\right)d\epsilon_i^{-j}, \tag{68}$$

where $\Gamma^{0*}(x_i) \equiv 0$ for all $x_i \in \mathbb{R}^N$, then $(x^{j*}, \theta^{j*})_{j\in J}$ are equilibrium threshold functions. To see this, suppose such $(\theta^{j*})_{j\in J}$ exist, and further, are threshold fundamentals functions. Then, $\Gamma^{j*}(x_i)$ is the expected payoff from attacking currency j, and $\Gamma^{0*}(x_i) = 0$ is the payoff from not attacking. Since $\theta^{j*} \in \tilde{B}(\mathbb{R}^{N-1})$, Lemma 1 implies that Γ^{j*} is decreasing in x_i^j and nondecreasing in x_i^{-j}. Thus, $\Gamma^{j*} - \max_{k\in J_0^{-j}} \Gamma^{k*}$ is decreasing in x_i^j, so from (67), it is optimal to follow the threshold strategy $(x^{j*})_{j\in J}$. Now, if all speculators follow the threshold strategy $(x^{j*})_{j\in J}$, then for any θ, the fraction of speculators attacking j is

$$\Pr\left(x_i^j \le x^{j*}\left(x_i^{-j}\right)|\theta\right) = \int_{\epsilon_i^{-j}\in\mathbb{R}^{N-1}} \tilde{\psi}^{-j}\left(\epsilon_i^{-j}\right)\Psi\left(x^{j*}\left(\theta^{-j} + \epsilon_i^{-j}\right) - \theta^j\right)d\epsilon_i^{-j}, \tag{69}$$

which is decreasing in θ^j. Then, $\theta^j - \Pr\left(x_i^j \le x^{j*}\left(x_i^{-j}\right)|\theta\right)$ is increasing in θ^j, and from (68), equals zero at $\theta^j = \theta^{j*}(\theta^{-j})$. Thus, country j abandons the peg if and only if $\theta^j \le \theta^{j*}(\theta^{-j})$, hence $(\theta^{j*})_{j\in J}$ are threshold fundamentals functions as conjectured. Therefore, $(x^{j*}, \theta^{j*})_{j\in J}$ are equilibrium threshold functions.

To show the existence of $(\theta^{j*})_{j\in J}$ satisfying (66)–(68), I consider a modified system of equations in which the functions of different countries are independently determined. I then use the contraction mapping theorem to find the solution to this system of equations, and resort to symmetry to show that they also satisfy the original system of equations.

Let $\tilde{B}^s(\mathbb{R}^{N-1})$ be the set of bounded, continuous, nondecreasing, and symmetric functions from $\mathbb{R}^{N-1}$ to $\mathbb{R}$. Equipped with a sup metric, $\tilde{B}(\mathbb{R}^{N-1})$ is a complete metric space, and being its closed subset, so is $\tilde{B}^s(\mathbb{R}^{N-1})$.[51] Fix any $j \in J$. For any $\theta_n^{j*} \in \tilde{B}^s(\mathbb{R}^{N-1})$,

[51] Equipped with a sup metric, the set of bounded, continuous functions from $\mathbb{R}^{N-1}$ to $\mathbb{R}$ is complete, and being its closed subset, so is $\tilde{B}(\mathbb{R}^{N-1})$ (Stokey and Lucas (1989), p80). Clearly $\tilde{B}^s(\mathbb{R}^{N-1}) \subset \tilde{B}(\mathbb{R}^{N-1})$, so $\tilde{B}^s(\mathbb{R}^{N-1})$ is complete if it is closed. Take any convergent sequence $\left\{\theta_n^{j*}\right\}_{n=1}^{\infty}$ in $\tilde{B}^s(\mathbb{R}^{N-1})$, and let θ_∞^{j*} be its limit. Since $\theta_n^{j*} \in \tilde{B}^s(\mathbb{R}^{N-1}) \subset \tilde{B}(\mathbb{R}^{N-1})$ and $\tilde{B}(\mathbb{R}^{N-1})$ is complete, $\theta_\infty^{j*} \in \tilde{B}(\mathbb{R}^{N-1})$. Moreover, θ_n^{j*} is symmetric, so for any $\theta^{-j} \in \mathbb{R}^{N-1}$ and its permutation $\tilde{\theta}^{-j}$, $\theta_n^{j*}\left(\theta^{-j}\right) = \theta_n^{j*}\left(\tilde{\theta}^{-j}\right)$. Due to the uniqueness of the limit, scalar sequences $\left\{\theta_n^{j*}\left(\theta^{-j}\right)\right\}_{n=1}^{\infty}$ and $\left\{\theta_n^{j*}\left(\tilde{\theta}^{-j}\right)\right\}_{n=1}^{\infty}$ converge to the same limit, so $\theta_\infty^{j*}\left(\theta^{-j}\right) = \theta_\infty^{j*}\left(\tilde{\theta}^{-j}\right)$. Therefore, θ_∞^{j*} is symmetric and thus $\theta_\infty^{j*} \in \tilde{B}^s(\mathbb{R}^{N-1})$, so $\tilde{B}^s(\mathbb{R}^{N-1})$ is closed.

define $\Gamma_n^{j*} : \mathbb{R}^N \to \mathbb{R}$, $x_n^{j*} : \mathbb{R}^{N-1} \to \mathbb{R}$, and $\theta_{n+1}^{j*} : \mathbb{R}^{N-1} \to [0,1]$ by

$$\Gamma_n^{j*}(x_i) = 1 - c - \int_{\epsilon_i^{-j} \in \mathbb{R}^{N-1}} \tilde{\psi}^{-j}\left(\epsilon_i^{-j}\right) \Psi\left(x_i^j - \theta_n^{j*}\left(x_i^{-j} - \epsilon_i^{-j}\right)\right) d\epsilon_i^{-j}, \tag{70}$$

$$x_n^{j*}\left(x_i^{-j}\right) = \min\left\{x_n^{j,0}\left(x_i^{-j}\right), \left(x_i^k\right)_{k \in J^{-j}}\right\}, \tag{71}$$

$$\theta_{n+1}^{j*}\left(\theta^{-j}\right) = \int_{\epsilon_i^{-j} \in \mathbb{R}^{N-1}} \tilde{\psi}^{-j}\left(\epsilon_i^{-j}\right) \Psi\left(x_n^{j*}\left(\theta^{-j} + \epsilon_i^{-j}\right) - \theta_{n+1}^{j*}\left(\theta^{-j}\right)\right) d\epsilon_i^{-j}, \tag{72}$$

where for any $x_i^{-j} \in \mathbb{R}^{N-1}$, $x_n^{j,0}\left(x_i^{-j}\right)$ is the value of x_i^j such that $\Gamma_n^{j*}(x_i) = 0$; such x_i^j exists uniquely from Lemma 3. Note that (71) involves only Γ_n^{j*} (through $x_n^{j,0}$), unlike (67) which involves Γ^{k*} for all $k \in J^{-j}$. This enables defining an operator T^j on $\tilde{B}^s(\mathbb{R}^{N-1})$ by $\theta_{n+1}^{j*} = T^j \theta_n^{j*}$, separately for each j. Below, I show that T^j is a contraction mapping.

First, let us show $T^j : \tilde{B}^s(\mathbb{R}^{N-1}) \to \tilde{B}^s(\mathbb{R}^{N-1})$. Take any $\theta_n^{j*} \in \tilde{B}^s(\mathbb{R}^{N-1})$. From Lemma 3, $x_n^{j,0}$ is continuous and nondecreasing. Also, x_i^k is continuous and nondecreasing, viewed as a function of x_i^{-j}. Thus, x_n^{j*} is continuous and nondecreasing, since it is obtained by taking the minimum of continuous and nondecreasing functions (see footnotes 44 and 45). Moreover, from Lemma 3, $x_n^{j,0}$ is bounded. Thus, for each $k \in J^{-j}$, for x_i^{-j} with sufficiently small x_i^k, $x_n^{j*}\left(x_i^{-j}\right) = x_i^k$ and thus x_n^{j*} is increasing in x_i^k. Therefore, from Lemma 4, θ_{n+1}^{j*} is bounded, continuous, and increasing. Finally, if θ_n^{j*} is symmetric, then clearly so is Γ_n^{j*} from (70) since $\tilde{\psi}^{-j}$ is symmetric, and similarly for x_n^{j*} and θ_{n+1}^{j*} from (71) and (72). Thus, $\theta_{n+1}^{j*} \in \tilde{B}^s(\mathbb{R}^{N-1})$, and furthermore, θ_{n+1}^{j*} is increasing.

Next, let us show that T^j satisfies the Blackwell's sufficient conditions for a contraction. For *monotonicity*, take any $\theta_n^{j*}, \hat{\theta}_n^{j*} \in \tilde{B}^s(\mathbb{R}^{N-1})$ such that $\hat{\theta}_n^{j*} \geq \theta_n^{j*}$. Define $\hat{\Gamma}_n^{j*} : \mathbb{R}^N \to \mathbb{R}$, $\hat{x}_n^{j*} : \mathbb{R}^{N-1} \to \mathbb{R}$, and $\hat{\theta}_{n+1}^{j*} : \mathbb{R}^{N-1} \to [0,1]$ by

$$\hat{\Gamma}_n^{j*}(x_i) = 1 - c - \int_{\epsilon_i^{-j} \in \mathbb{R}^{N-1}} \tilde{\psi}^{-j}\left(\epsilon_i^{-j}\right) \Psi\left(x_i^j - \hat{\theta}_n^{j*}\left(x_i^{-j} - \epsilon_i^{-j}\right)\right) d\epsilon_i^{-j}, \tag{73}$$

$$\hat{x}_n^{j*}\left(x_i^{-j}\right) = \min\left\{\hat{x}_n^{j,0}\left(x_i^{-j}\right), \left(x_i^k\right)_{k \in J^{-j}}\right\}, \tag{74}$$

$$\hat{\theta}_{n+1}^{j*}\left(\theta^{-j}\right) = \int_{\epsilon_i^{-j} \in \mathbb{R}^{N-1}} \tilde{\psi}^{-j}\left(\epsilon_i^{-j}\right) \Psi\left(\hat{x}_n^{j*}\left(\theta^{-j} + \epsilon_i^{-j}\right) - \hat{\theta}_{n+1}^{j*}\left(\theta^{-j}\right)\right) d\epsilon_i^{-j}, \tag{75}$$

where for any $x_i^{-j} \in \mathbb{R}^{N-1}$, $\hat{x}_n^{j,0}\left(x_i^{-j}\right)$ is the value of x_i^j such that $\hat{\Gamma}_n^{j*}(x_i) = 0$. Then, (70), (73), and $\hat{\theta}_n^{j*} \geq \theta_n^{j*}$ imply $\hat{\Gamma}_n^{j*} \geq \Gamma_n^{j*}$ from the monotonicity of the integral, and since $\hat{\Gamma}_n^{j*}$ and Γ_n^{j*} are decreasing in x_i^j from Lemma 1, it follows that $\hat{x}_n^{j,0} \geq x_n^{j,0}$. But then, (71) and (74) imply $\hat{x}_n^{j*} \geq x_n^{j*}$. Thus, $\hat{\theta}_{n+1}^{j*} \geq \theta_{n+1}^{j*}$ from Lemma 5, hence *monotonicity* holds.

For *discounting*, take any $y > 0$, and let $\hat{\theta}_n^{j*} = \theta_n^{j*} + y$ in the argument above. Then, (70) and (73) imply $\hat{\Gamma}_n^{j*}\left(x_i^1, \ldots, x_i^j + y, \ldots, x_i^N\right) = \Gamma_n^{j*}(x_i)$ for any $x_i \in \mathbb{R}^N$, such that $\hat{x}_n^{j,0}\left(x_i^{-j}\right) = x_n^{j,0}\left(x_i^{-j}\right) + y$. From (71) and (74), then, $\hat{x}_n^{j*} \leq x_n^{j*} + y$. By replacing $\bar{x}_n^{j*}\left(x_i^{-j} - z\right)$ and $\underline{x}_n^{j*}\left(x_i^{-j}\right)$ in Lemma 7 by $\hat{x}_n^{j*}\left(x_i^{-j}\right)$ and $x_n^{j*}\left(x_i^{-j}\right)$, one obtains $\hat{\theta}_{n+1}^{j*}(\theta^{-j}) - \theta_{n+1}^{j*}(\theta^{-j}) \leq \lambda y$ for any $\theta^{-j} \in \mathbb{R}^{N-1}$. Thus, $\hat{\theta}_{n+1}^{j*} \leq \theta_{n+1}^{j*} + \lambda y$ where $\lambda \in (0,1)$, and since the choice of $y > 0$ was arbitrary, *discounting* also holds.

Thus, T^j is a contraction, so it has a unique fixed point. Let $\tilde{\theta}^{j*}$ be this fixed point, and similarly denote the associated functions by dropping the subscript and placing tilde. Then, $\tilde{\theta}^{j*} \in \tilde{B}^s(\mathbb{R}^{N-1})$, and since T^j maps a nondecreasing function to an increasing function as shown above, $\tilde{\theta}^{j*}(= T^j\tilde{\theta}^{j*})$ is increasing. Then, the same argument as for x_n^{j*} shows that $\tilde{x}^{j*}$ is continuous, nondecreasing, and $\tilde{x}^{j*}\left(x_i^{-j}\right) \leq \min_{k\in J^{-j}} x_i^k$ with equality for $\min_{k\in J^{-j}} x_i^k$ sufficiently small. Moreover, since $\tilde{\theta}^{j*}$ is increasing, so is $\tilde{x}^{j,0}$ from Lemma 3.

To see $\tilde{\theta}^{j*} < \theta_s^*$ and $\tilde{x}^{j*} < x_s^*$, let T_s^j be a modified operator of T^j in which the RHS of (71) is replaced by $x_n^{j,0}\left(x_i^{-j}\right)$, and let $\theta_1^{j*} \in \tilde{B}^s(\mathbb{R}^{N-1})$ be a function constant at θ_s^*. The equilibrium conditions for $N = 1$, (12) and (13), imply that such θ_1^{j*} is a fixed point of T_s^j, and that $x_1^{j,0}$ is a function constant at x_s^*. Thus, $x_1^{j*} \leq x_1^{j,0}$ from (71), which yields $\theta_2^{j*} \leq T_s^j\theta_1^{j*} = \theta_1^{j*}$ from Lemma 5. Then, $\theta_3^{j*} \leq \theta_2^{j*}$ from the *monotonicity* of T^j, and thus, iterating on T^j yields a nonincreasing sequence of functions $\{\theta_n^{j*}\}_{n=1}^{\infty}$. From the contraction mapping theorem, this sequence converges to $\tilde{\theta}^{j*}$, hence $\tilde{\theta}^{j*} \leq \theta_1^{j*} = \theta_s^*$. Moreover, $\tilde{\theta}^{j*} < \theta_s^*$, because if $\tilde{\theta}^{j*}(z) = \theta_s^*$ for some $z \in \mathbb{R}^{N-1}$, then $\tilde{\theta}^{j*}(z') > \theta_s^*$ for $z' > z$ since $\tilde{\theta}^{j*}$ is increasing, violating $\tilde{\theta}^{j*} \leq \theta_s^*$. Now, since $\{\theta_n^{j*}\}_{n=1}^{\infty}$ is a nonincreasing sequence, clearly so is $\{x_n^{j*}\}_{n=1}^{\infty}$ from (70) and (71), and thus $\tilde{x}^{j*} \leq x_1^{j*} \leq x_1^{j,0} = x_s^*$. Moreover, $\tilde{x}^{j*} < x_s^*$, because if $\tilde{x}^{j*}(z) = x_s^*$ for some $z \in \mathbb{R}^{N-1}$, then from (71) and the fact that $\tilde{x}^{j,0}$ is increasing, $\tilde{x}^{j*}(z + \delta) > x_s^*$ for any $\delta > 0$, violating $\tilde{x}^{j*} \leq x_s^*$.

Finally, note that if $(\theta^{j*})_{j\in J}$ are symmetric across j, then so are $(\Gamma^{j*})_{j\in J}$, because given that $\left(\tilde{\psi}^{-j}\right)_{j\in J}$ are symmetric across j, the functional form of (66) is common to all j. Thus, $\Gamma^{j*}(x_i) = \Gamma^{k*}(x_i)$ if and only if $x_i^j = x_i^k$, hence (67) can be rewritten as

$$x^{j*}\left(x_i^{-j}\right) = \min\left\{x^{j,0}\left(x_i^{-j}\right), \left(x_i^k\right)_{k\in J^{-j}}\right\}, \tag{76}$$

where for any $x_i^{-j} \in \mathbb{R}^{N-1}$, $x^{j,0}\left(x_i^{-j}\right)$ is the value of x_i^j such that $\Gamma^{j*}(x_i) = 0$. But now, $\left(\tilde{\theta}^{j*}\right)_{j\in J}$ are symmetric across j, because each $\tilde{\theta}^{j*}$ is symmetric, and given that $\left(\tilde{\psi}^{-j}\right)_{j\in J}$ are symmetric across j, the functional forms of (70)–(72) are common to all j. Thus, comparing (71) and (76), one observes that if $\theta^{j*} = \tilde{\theta}^{j*}$ for all $j \in J$, the equilibrium conditions (66)–(68) are satisfied, where $x^{j*} = \tilde{x}^{j*}$ for all $j \in J$.□

Proof of Proposition 3

Again, the proof proceeds through a series of lemmas.

For $N > 2$, $\underline{x}_n^{j*}$ and $\bar{x}_n^{j*}$ defined by (17) and (18) are not necessarily nondecreasing, even when $\underline{\theta}_n^{j*}, \bar{\theta}_n^{j*} \in \tilde{B}(\mathbb{R}^{N-1})$ for all $j \in J$.[52] This issue is overcome as follows. Suppose $\left(\underline{\theta}_n^{j*}\right)_{j\in J}$ and $\left(\bar{\theta}_n^{j*}\right)_{j\in J}$ are symmetric across j, such that $d_n^j = \left\|\bar{\theta}_n^{j*} - \underline{\theta}_n^{j*}\right\|$ is the same for

[52]Let $N = 3$ and $\underline{\theta}_n^{j*}, \bar{\theta}_n^{j*} \in \tilde{B}(\mathbb{R}^2)$ for all $j \in J$. Then from Lemma 1, $\underline{\Gamma}_n^{j*}$ and $\bar{\Gamma}_n^{j*}$ are decreasing in x_i^j and nondecreasing in x_i^{-j} for all $j \in J$. Now, suppose $\underline{x}_n^{1*}$ is defined by (17), and $\left(x_i^2, x_i^3\right)$ is such that $\underline{\Gamma}_n^{1*}\left(\underline{x}_n^{1*}\left(x_i^2, x_i^3\right), x_i^2, x_i^3\right) = \bar{\Gamma}_n^{2*}\left(\underline{x}_n^{1*}\left(x_i^2, x_i^3\right), x_i^2, x_i^3\right)$. As x_i^3 rises, $\underline{\Gamma}_n^{1*}$ and $\bar{\Gamma}_n^{2*}$ both weakly rise, and thus x_i^1 may need to fall to sustain the equality. Thus, $\underline{x}_n^{1*}$ defined by (17) may not be nondecreasing, and similarly for $\bar{x}_n^{1*}$ defined by (18).

all j and thus can be written as d_n, and define $\underline{x}_n^{j*}$ and $\bar{x}_n^{j*}$ as

$$\underline{x}_n^{j*}\left(x_i^{-j}\right) \equiv \min\left\{\underline{x}_n^{j,0}\left(x_i^{-j}\right), \left(x_i^k - d_n\right)_{k\in J^{-j}}\right\}, \tag{77}$$

$$\bar{x}_n^{j*}\left(x_i^{-j}\right) \equiv \min\left\{\bar{x}_n^{j,0}\left(x_i^{-j}\right), \left(x_i^k + d_n\right)_{k\in J^{-j}}\right\}. \tag{78}$$

Recall the expressions $\underline{x}_n^{j*} = \min_{k\in J_0^{-j}} \underline{x}_n^{j,k}$ and $\bar{x}_n^{j*} = \min_{k\in J_0^{-j}} \bar{x}_n^{j,k}$ for $N = 2$. The idea here is to replace $\underline{x}_n^{j,k}$ with $x_i^k - d_n$, and $\bar{x}_n^{j,k}$ with $x_i^k + d_n$, for all $k \in J^{-j}$. Lemma 13 below shows that if $\underline{x}_n^{j*}$ and $\bar{x}_n^{j*}$ are defined by (77) and (78), the iterative deletion procedure generates well-behaved functions, just like for $N = 2$, and for all $n \in \mathbb{N}$, $\left(\underline{\theta}_n^{j*}\right)_{j\in J}$ and $\left(\bar{\theta}_n^{j*}\right)_{j\in J}$ are indeed symmetric across j. The result of Lemma 14 then confirms that defining $\underline{x}_n^{j*}$ and $\bar{x}_n^{j*}$ this way is consistent with iterative deletion of dominated strategies, since it does not cause deletion of undominated strategies.

Lemma 13 *Let $N > 2$ and countries be symmetric, and $\left(\left(\underline{x}_n^{j*}, \bar{x}_n^{j*}, \underline{\theta}_{n+1}^{j*}, \bar{\theta}_{n+1}^{j*}\right)_{j\in J}\right)_{n=1}^{\infty}$ be defined by (15), (16), (77), (78), (19), and (20), where $\underline{\theta}_1^{j*} \equiv 0$ and $\bar{\theta}_1^{j*} \equiv 1$ for all $j \in J$. Then, for any $n \in \mathbb{N}$, (a) $\left(\underline{\theta}_{n+1}^{j*}\right)_{j\in J}$ and $\left(\bar{\theta}_{n+1}^{j*}\right)_{j\in J}$ are symmetric across j, and for any $j \in J$, (b) $\underline{\theta}_{n+1}^{j*}$, $\bar{\theta}_{n+1}^{j*} \in \tilde{B}^s(\mathbb{R}^{N-1})$, and $\underline{\theta}_{n+1}^{j*}$ and $\bar{\theta}_{n+1}^{j*}$ are increasing, (c) $\bar{\theta}_n^{j*} > \bar{\theta}_{n+1}^{j*} > \underline{\theta}_{n+1}^{j*} > \underline{\theta}_n^{j*}$. Similarly, (A) $(\underline{x}_n^{j*})_{j\in J}$ and $(\bar{x}_n^{j*})_{j\in J}$ are symmetric across j, and for any $j \in J$, (B) $\underline{x}_n^{j*}$ and $\bar{x}_n^{j*}$ are continuous and nondecreasing, (C) $\bar{x}_n^{j*} \geq \bar{x}_{n+1}^{j*} > \underline{x}_{n+1}^{j*} \geq \underline{x}_n^{j*}$.*

Proof. Since $\underline{\theta}_1^{j*} = 0$ and $\bar{\theta}_1^{j*} = 1$, it follows that $\underline{\theta}_1^{j*}$, $\bar{\theta}_1^{j*} \in \tilde{B}^s(\mathbb{R}^{N-1})$, and moreover, $\left(\underline{\theta}_1^{j*}\right)_{j\in J}$ and $\left(\bar{\theta}_1^{j*}\right)_{j\in J}$ are symmetric across j. Thus, $d_1^j = \left\|\bar{\theta}_1^{j*} - \underline{\theta}_1^{j*}\right\|$ is the same for all j, and hence can be written as d_1. Then, by letting (15), (77), (19) and (16), (78), (20) play the role of (70)–(72), the same argument as in the proof of Proposition 2 shows that $(a)(b)$ and $(A)(B)$ hold for $n = 1$, and by induction, the same applies to any $n \in \mathbb{N}$. Lemma 1 then implies that for any $j \in J$ and $n \in \mathbb{N}$, $\underline{\Gamma}_n^{j*}$ and $\bar{\Gamma}_n^{j*}$ are continuous, decreasing in x_i^j, and nondecreasing in x_i^{-j}.

Below, I prove $(c)(C)$ for $j = 1$. For (c), note from $1 = \bar{\theta}_1^{1*} > \underline{\theta}_1^{1*} = 0$, (23), and (24) that $\bar{\Gamma}_1^{1*} > \underline{\Gamma}_1^{1*}$. Take any $x_i^{-1} = \left(x_i^2, x_i^3, \dots, x_i^N\right) \in \mathbb{R}^{N-1}$. Then, Lemma 3 implies $\bar{\Gamma}_1^{1*}\left(\bar{x}_1^{1,0}\left(x_i^{-1}\right), x_i^{-1}\right) = \underline{\Gamma}_1^{1*}\left(\underline{x}_1^{1,0}\left(x_i^{-1}\right), x_i^{-1}\right) = 0$, so since $\underline{\Gamma}_1^{1*} < \bar{\Gamma}_1^{1*}$, it follows that $\bar{\Gamma}_1^{1*}\left(\bar{x}_1^{1,0}\left(x_i^{-1}\right), x_i^{-1}\right) < \bar{\Gamma}_1^{1*}\left(\underline{x}_1^{1,0}\left(x_i^{-1}\right), x_i^{-1}\right)$, and since $\bar{\Gamma}_1^{1*}$ is decreasing in x_i^1, $\bar{x}_1^{1,0}\left(x_i^{-1}\right) > \underline{x}_1^{1,0}\left(x_i^{-1}\right)$. Thus, $\bar{x}_1^{1*} > \underline{x}_1^{1*}$ from (77) and (78), hence $\bar{\theta}_2^{1*} > \underline{\theta}_2^{1*}$ from Lemma 5. Moreover, clearly $\bar{\theta}_2^{1*}$, $\underline{\theta}_2^{1*} \in (0, 1)$, hence $1 = \bar{\theta}_1^{1*} > \bar{\theta}_2^{1*} > \underline{\theta}_2^{1*} > \underline{\theta}_1^{1*} = 0$.

Now, suppose $\bar{\theta}_n^{1*} > \bar{\theta}_{n+1}^{1*} > \underline{\theta}_{n+1}^{1*} > \underline{\theta}_n^{1*}$ for some $n \in \mathbb{N}$. Then, $\bar{\Gamma}_n^{1*} > \bar{\Gamma}_{n+1}^{1*} > \underline{\Gamma}_{n+1}^{1*} > \underline{\Gamma}_n^{1*}$ from (23) and (24), and also, $d_n \geq d_{n+1}$. For any $x_i^{-1} \in \mathbb{R}^{N-1}$, $\bar{\Gamma}_n^{1*}\left(\bar{x}_n^{1,0}\left(x_i^{-1}\right), x_i^{-1}\right) = \bar{\Gamma}_{n+1}^{1*}\left(\bar{x}_{n+1}^{1,0}\left(x_i^{-1}\right), x_i^{-1}\right) = \underline{\Gamma}_{n+1}^{1*}\left(\underline{x}_{n+1}^{1,0}\left(x_i^{-1}\right), x_i^{-1}\right) = \underline{\Gamma}_n^{1*}\left(\underline{x}_n^{1,0}\left(x_i^{-1}\right), x_i^{-1}\right) = 0$ from Lemma 3, so it follows from $\bar{\Gamma}_n^{1*} > \bar{\Gamma}_{n+1}^{1*} > \underline{\Gamma}_{n+1}^{1*} > \underline{\Gamma}_n^{1*}$ and the fact that these functions are decreasing in x_i^1 that $\bar{x}_n^{1,0}\left(x_i^{-1}\right) > \bar{x}_{n+1}^{1,0}\left(x_i^{-1}\right) > \underline{x}_{n+1}^{1,0}\left(x_i^{-1}\right) > \underline{x}_n^{1,0}\left(x_i^{-1}\right)$. Then, $\bar{x}_n^{1*} \geq \bar{x}_{n+1}^{1*} > \underline{x}_{n+1}^{1*} \geq \underline{x}_n^{1*}$ from (77), (78), and $d_n \geq d_{n+1}$. Since $\bar{x}_{n+1}^{1,0} \geq \bar{x}_{n+1}^{1*}$ from (78), the first inequality here is strict at least when $\bar{x}_n^{1*}\left(x_i^{-1}\right) = \bar{x}_n^{1,0}\left(x_i^{-1}\right)$, which holds for sufficiently large $\min_{k\in J^{-j}} x_i^k$ because $\bar{x}_n^{1,0}$ is bounded from $\bar{\theta}_n^{j*} \in \tilde{B}^s(\mathbb{R}^{N-1})$ and Lemma 3. A similar statement applies to the third inequality. Therefore, $\bar{\theta}_{n+1}^{1*} > \bar{\theta}_{n+2}^{1*} > \underline{\theta}_{n+2}^{1*} > \underline{\theta}_{n+1}^{1*}$ from Lemma 5. Thus, by induction, the claim holds for any $n \in \mathbb{N}$, which proves (c). Then, from the argument above, $\bar{x}_n^{1*} \geq \bar{x}_{n+1}^{1*} > \underline{x}_{n+1}^{1*} \geq \underline{x}_n^{1*}$ for any $n \in \mathbb{N}$, which proves (C). ■

Lemma 14 bounds the range of functions $\underline{x}_n^{j,k}$, $\bar{x}_n^{j,k}$ for $k \in J^{-j}$ and $k = 0$.

Lemma 14 *Under the assumptions of Lemma 13, for any $j \in J$, $n \in \mathbb{N}$, and $x_i^{-j} \in \mathbb{R}^{N-1}$, a) $\underline{x}_n^{j,k}\left(x_i^{-j}\right) \in [x_i^k - d_n, x_i^k)$ and $\bar{x}_n^{j,k}\left(x_i^{-j}\right) \in (x_i^k, x_i^k + d_n]$ for any $k \in J^{-j}$, (b) $\bar{x}_n^{j,0}\left(x_i^{-j}\right) - \underline{x}_n^{j,0}\left(x_i^{-j}\right) \in (0, d_n]$.*

Proof. Note from Lemma 13 that $\underline{\theta}_n^{j*}$, $\bar{\theta}_n^{j*} \in \tilde{B}^s(\mathbb{R}^{N-1}) \subset \tilde{B}(\mathbb{R}^{N-1})$ for any $j \in J$ and $n \in \mathbb{N}$, such that one may invoke Lemmas 2, 3, and 6. Moreover, the argument in the proof of Lemma 13 implies that for any $j \in J$ and $n \in \mathbb{N}$, $\underline{\Gamma}_n^{j*}$ and $\bar{\Gamma}_n^{j*}$ are continuous, decreasing in x_i^j, and nondecreasing in x_i^{-j}, and further, $\bar{\Gamma}_n^{j*} > \bar{\Gamma}_{n+1}^{j*} > \underline{\Gamma}_{n+1}^{j*} > \underline{\Gamma}_n^{j*}$.

I prove the claim for $j = 1$, and $k = 2$ in (a). Fix any $x_i^{-1} \in \mathbb{R}^{N-1}$. For (a), note from Lemma 2 that

$$\underline{\Gamma}_n^{1*}\left(\underline{x}_n^{1,2}\left(x_i^{-1}\right), x_i^{-1}\right) = \bar{\Gamma}_n^{2*}\left(\underline{x}_n^{1,2}\left(x_i^{-1}\right), x_i^{-1}\right), \tag{79}$$
$$\bar{\Gamma}_n^{1*}\left(\bar{x}_n^{1,2}\left(x_i^{-1}\right), x_i^{-1}\right) = \underline{\Gamma}_n^{2*}\left(\bar{x}_n^{1,2}\left(x_i^{-1}\right), x_i^{-1}\right). \tag{80}$$

Also, from Lemma 13, $\left(\underline{\theta}_n^{j*}\right)_{j\in J}$ and $\left(\bar{\theta}_n^{j*}\right)_{j\in J}$ are symmetric across j, and thus so are $\left(\underline{\Gamma}_n^{j*}\right)_{j\in J}$ and $\left(\bar{\Gamma}_n^{j*}\right)_{j\in J}$, because given that $\left(\tilde{\psi}^{-j}\right)_{j\in J}$ are symmetric across j and $\Psi^j = \Psi$ for all $j \in J$, the functional forms of (23) and (24) are common to all j. Thus, given $x_i^{-1} = \left(x_i^2, x_i^3, \ldots, x_i^N\right)$, $\underline{\Gamma}_n^{1*}(x_i) = \underline{\Gamma}_n^{2*}(x_i)$ and $\bar{\Gamma}_n^{1*}(x_i) = \bar{\Gamma}_n^{2*}(x_i)$ if and only if $x_i^1 = x_i^2$, and thus,

$$\underline{\Gamma}_n^{1*}\left(x_i^2, x_i^{-1}\right) = \underline{\Gamma}_n^{2*}\left(x_i^2, x_i^{-1}\right), \tag{81}$$
$$\bar{\Gamma}_n^{1*}\left(x_i^2, x_i^{-1}\right) = \bar{\Gamma}_n^{2*}\left(x_i^2, x_i^{-1}\right). \tag{82}$$

Therefore,

$$\underline{\Gamma}_n^{1*}\left(x_i^2, x_i^{-1}\right) = \underline{\Gamma}_n^{2*}\left(x_i^2, x_i^{-1}\right) < \bar{\Gamma}_n^{2*}\left(x_i^2, x_i^{-1}\right), \tag{83}$$

where the equality follows from (81), and the inequality follows since $\underline{\Gamma}_n^{2*} < \bar{\Gamma}_n^{2*}$. Moreover,

$$\underline{\Gamma}_n^{1*}\left(x_i^2 - d_n, x_i^{-1}\right) \geq \bar{\Gamma}_n^{1*}\left(x_i^2, x_i^{-1}\right) = \bar{\Gamma}_n^{2*}\left(x_i^2, x_i^{-1}\right) \geq \bar{\Gamma}_n^{2*}\left(x_i^2 - d_n, x_i^{-1}\right), \tag{84}$$

where the first inequality is from Lemma 6, the equality is from (82), and the second inequality follows since $\bar{\Gamma}_n^{2*}$ is nondecreasing in x_i^1. Noting that $\underline{\Gamma}_n^{1*} - \bar{\Gamma}_n^{2*}$ is decreasing in x_i^1 and recalling (79), (83) implies $\underline{x}_n^{1,2}\left(x_i^{-1}\right) < x_i^2$, and (84) implies $\underline{x}_n^{1,2}\left(x_i^{-1}\right) \geq x_i^2 - d_n$.

To establish the desired property of $\bar{x}_n^{1,2}$, note that

$$\bar{\Gamma}_n^{1*}\left(x_i^2, x_i^{-1}\right) > \underline{\Gamma}_n^{1*}\left(x_i^2, x_i^{-1}\right) = \underline{\Gamma}_n^{2*}\left(x_i^2, x_i^{-1}\right), \tag{85}$$

where the inequality follows from $\bar{\Gamma}_n^{1*} > \underline{\Gamma}_n^{1*}$, and the equality is from (81). Moreover,

$$\bar{\Gamma}_n^{1*}\left(x_i^2 + d_n, x_i^{-1}\right) \leq \underline{\Gamma}_n^{1*}\left(x_i^2, x_i^{-1}\right) = \underline{\Gamma}_n^{2*}\left(x_i^2, x_i^{-1}\right) \leq \underline{\Gamma}_n^{2*}\left(x_i^2 + d_n, x_i^{-1}\right), \tag{86}$$

where the first inequality is from Lemma 6, the equality is from (81), and the second inequality follows since $\underline{\Gamma}_n^{2*}$ is nondecreasing in x_i^1. Noting that $\bar{\Gamma}_n^{1*} - \underline{\Gamma}_n^{2*}$ is decreasing in x_i^1 and recalling (80), (85) implies $\bar{x}_n^{1,2}\left(x_i^{-1}\right) > x_i^2$, and (86) implies $\bar{x}_n^{1,2}\left(x_i^{-1}\right) \leq x_i^2 + d_n$.

For (b), note from the proof of Lemma 13 that $\bar{x}_n^{1,0}\left(x_i^{-1}\right) > \underline{x}_n^{1,0}\left(x_i^{-1}\right)$. Moreover,

$$\bar{\Gamma}_n^{1*}\left(\bar{x}_n^{1,0}\left(x_i^{-1}\right), x_i^{-1}\right) = \underline{\Gamma}_n^{1*}\left(\underline{x}_n^{1,0}\left(x_i^{-1}\right), x_i^{-1}\right) \geq \bar{\Gamma}_n^{1*}\left(\underline{x}_n^{1,0}\left(x_i^{-1}\right) + d_n, x_i^{-1}\right), \tag{87}$$

where the equality is from Lemma 3, and the inequality is from Lemma 6. Since $\bar{\Gamma}_n^{1*}$ is decreasing in x_i^1, (87) implies $\bar{x}_n^{1,0}\left(x_i^{-1}\right) \leq \underline{x}_n^{1,0}\left(x_i^{-1}\right) + d_n$. ■

Since Lemma 14 implies $x_i^k - d_n \leq \underline{x}_n^{j,k}\left(x_i^{-j}\right)$ and $x_i^k + d_n \geq \bar{x}_n^{j,k}\left(x_i^{-j}\right)$ for any $k \in J^{-1}$ and $x_i^{-j} \in \mathbb{R}^{N-1}$, the definitions of $\underline{x}_n^{j*}$ and $\bar{x}_n^{j*}$ in (77) and (78) guarantee $\underline{x}_n^{j*} \leq \min_{k \in J_0^{-1}} \underline{x}_n^{j,k}$ and $\bar{x}_n^{j*} \geq \min_{k \in J_0^{-1}} \bar{x}_n^{j,k}$. Therefore, performing iterative deletion using $\underline{x}_n^{j*}$ and $\bar{x}_n^{j*}$ thus defined does not result in deletion of undominated strategies.

Lemma 15 *Under the assumptions of Lemma 13,* $\bar{x}_n^{j*}\left(x_i^{-j}\right) - \underline{x}_n^{j*}\left(x_i^{-j}\right) \leq 2d_n$ *for any* $j \in J$, $n \in \mathbb{N}$, *and* $x_i^{-j} \in \mathbb{R}^{N-1}$.

Proof. Recall (77). If $\underline{x}_n^{j*}\left(x_i^{-j}\right) = \underline{x}_n^{j,0}\left(x_i^{-j}\right)$, then $\bar{x}_n^{j*}\left(x_i^{-j}\right) - \underline{x}_n^{j*}\left(x_i^{-j}\right) \leq \bar{x}_n^{j,0}\left(x_i^{-j}\right) - \underline{x}_n^{j,0}\left(x_i^{-j}\right) \leq d_n$, where the first inequality follows since $\bar{x}_n^{j*}\left(x_i^{-j}\right) \leq \bar{x}_n^{j,0}\left(x_i^{-j}\right)$ from (78), and the second inequality follows from Lemma 14(b). If $\underline{x}_n^{j*}\left(x_i^{-j}\right) = \min_{k \in J^{-j}} x_i^k - d_n$, then $\bar{x}_n^{j*}\left(x_i^{-j}\right) - \underline{x}_n^{j*}\left(x_i^{-j}\right) \leq \left[\min_{k \in J^{-j}} x_i^k + d_n\right] - \left[\min_{k \in J^{-j}} x_i^k - d_n\right] = 2d_n$, where the first inequality follows since $\bar{x}_n^{j*}\left(x_i^{-j}\right) \leq \min_{k \in J^{-j}} x_i^k + d_n$ from (78). ■

To conclude the proof of Proposition 2, let z be a zero vector in $\mathbb{R}^{N-1}$ in Lemma 7. Then, combining with Lemma 15, it follows for all $n \in \mathbb{N}$ that

$$d_{n+1} = \left\|\bar{\theta}_{n+1}^{j*} - \underline{\theta}_{n+1}^{j*}\right\| \leq \lambda \left\|\bar{x}_n^{j*} - \underline{x}_n^{j*}\right\| \leq 2\lambda d_n. \tag{88}$$

Since $2\lambda < 1$ by assumption, as $n \to \infty$, $d_{n+1} \leq (2\lambda)^n d_1 \to 0$, or $\left\|\bar{\theta}_{n+1}^{j*} - \underline{\theta}_{n+1}^{j*}\right\| \to 0$. Then, from Lemma 15, $\left\|\bar{x}_n^{j*} - \underline{x}_n^{j*}\right\| \to 0$. Thus, the lower and upper threshold functions converge uniformly to the equilibrium threshold functions $\left(x^{j*}, \theta^{j*}\right)_{j \in J}$ in Proposition 2.□

Proof of Proposition 4

Given the assumptions on the common prior of θ^j and the distribution of ϵ_i^j, conditional on receiving x_i^j, the posterior distribution of θ^j follows $N\left(\frac{\alpha w + \beta x_i^j}{\alpha + \beta}, 1/(\alpha+\beta)\right)$. Thus, $\Pr\left(\theta^j \leq \theta^{j*} | x_i^j\right) = \Phi\left(\sqrt{\alpha+\beta}\left(\theta^{j*} - \frac{\alpha w + \beta x_i^j}{\alpha+\beta}\right)\right)$ for a fixed θ^{j*}. Now, let $\hat{\epsilon}_i^j \equiv \frac{\alpha w + \beta x_i^j}{\alpha+\beta} - \theta^j$ for any $j \in J$. Conditional on receiving x_i^j, $\hat{\epsilon}_i^j$ follows $N\left(0, 1/(\alpha+\beta)\right)$, hence the pdf is $\sqrt{\alpha+\beta}\phi\left(\sqrt{\alpha+\beta}\hat{\epsilon}_i^j\right)$. Thus, for any $\theta^{j*} : \mathbb{R}^{N-1} \to \mathbb{R}$,

$$\Pr\left(\theta^j \leq \theta^{j*}\left(\theta^{-j}\right) | x_i\right) \tag{89}$$

$$= \int_{\hat{\epsilon}_i^{-j} \in \mathbb{R}^{N-1}} \hat{\phi}^{-j}\left(\hat{\epsilon}_i^{-j}\right) \Phi\left(\sqrt{\alpha+\beta}\left(\theta^{j*}\left(\frac{\alpha w^{-j} + \beta x_i^{-j}}{\alpha+\beta} - \hat{\epsilon}_i^{-j}\right) - \frac{\alpha w + \beta x_i^j}{\alpha+\beta}\right)\right) d\hat{\epsilon}_i^{-j},$$

where $\hat{\phi}^{-j}$ is the pdf of $\hat{\epsilon}_i^{-j}$, given by $\hat{\phi}^{-j}\left(\hat{\epsilon}_i^{-j}\right) = \Pi_{k \in J^{-j}} \sqrt{\alpha+\beta}\phi\left(\sqrt{\alpha+\beta}\hat{\epsilon}_i^k\right)$, and w^{-j} is a $N-1$ vector, $(w, w, \ldots, w)$. Thus,

$$\begin{aligned}
&\underline{\Gamma}_n^{j*}(x_i) && (90)\\
&= \int_{\hat{\epsilon}_i^{-j} \in \mathbb{R}^{N-1}} \hat{\phi}^{-j}\left(\hat{\epsilon}_i^{-j}\right) \Phi\left(\sqrt{\alpha+\beta}\left(\underline{\theta}_n^{j*}\left(\frac{\alpha w^{-j} + \beta x_i^{-j}}{\alpha+\beta} - \hat{\epsilon}_i^{-j}\right) - \frac{\alpha w + \beta x_i^j}{\alpha+\beta}\right)\right) d\hat{\epsilon}_i^{-j} - c,\\
&\bar{\Gamma}_n^{j*}(x_i) && (91)\\
&= \int_{\hat{\epsilon}_i^{-j} \in \mathbb{R}^{N-1}} \hat{\phi}^{-j}\left(\hat{\epsilon}_i^{-j}\right) \Phi\left(\sqrt{\alpha+\beta}\left(\bar{\theta}_n^{j*}\left(\frac{\alpha w^{-j} + \beta x_i^{-j}}{\alpha+\beta} - \hat{\epsilon}_i^{-j}\right) - \frac{\alpha w + \beta x_i^j}{\alpha+\beta}\right)\right) d\hat{\epsilon}_i^{-j} - c.
\end{aligned}$$

Let $\gamma \equiv (\alpha+\beta)/\beta$. Among Lemmas 1–7, only Lemma 6 is affected, and is modified as follows.

Lemma 16 *Under the assumptions of Lemma 2,*
$\bar{\Gamma}_n^{j*}(x_i) \leq \underline{\Gamma}_n^{j*}\left(x_i^1, \ldots, x_i^{j-1}, x_i^j - \gamma d_n^j, \ldots, x_i^N\right)$ *for any* $j \in J$ *and* $x_i \in \mathbb{R}^N$.

Proof. Since $\bar{\theta}_n^{j*}(\theta^{-j}) \leq \underline{\theta}_n^{j*}(\theta^{-j}) + d_n^j$ for any $j \in J$ and $\theta^{-j} \in \mathbb{R}^{N-1}$,

$$\begin{aligned}
&\bar{\theta}_n^{j*}\left(\frac{\alpha w^{-j} + \beta x_i^{-j}}{\alpha+\beta} - \hat{\epsilon}_i^{-j}\right) - \frac{\alpha w + \beta x_i^j}{\alpha+\beta}\\
&\leq \underline{\theta}_n^{j*}\left(\frac{\alpha w^{-j} + \beta x_i^{-j}}{\alpha+\beta} - \hat{\epsilon}_i^{-j}\right) - \frac{\alpha w + \beta\left(x_i^j - \gamma d_n^j\right)}{\alpha+\beta} \qquad (92)
\end{aligned}$$

for any $x_i \in \mathbb{R}^N$ and $\hat{\epsilon}_i^{-j} \in \mathbb{R}^{N-1}$. The claim then follows from (90), (91), and the monotonicity of the integral. ■

I now prove Proposition 4(1) and 4(2) in turn.

Proof of Proposition 4(1)

In the proof of Proposition 2, the expressions for Γ^{j*}, Γ_n^{j*}, and $\hat{\Gamma}_n^{j*}$ (eqs (66), (70), and (73)) are rewritten similarly to (90). The only part of the proof that is affected by this modification is the argument for *discounting*. This time, $\hat{\theta}_n^{j*} = \theta_n^{j*} + y$ implies $\hat{\Gamma}_n^{j*}\left(x_i^1, \ldots, x_i^j + \gamma y, \ldots, x_i^N\right) = \Gamma_n^{j*}(x_i)$, which yields $\hat{x}_n^{j,0}\left(x_i^{-j}\right) = x_n^{j,0}\left(x_i^{-j}\right) + \gamma y$ and thus $\hat{x}_n^{j*} \leq x_n^{j*} + \gamma y$. Thus, continuing the argument by replacing y with γy, *discounting* follows if $\lambda\gamma < 1$. Now, $\lambda = \beta / \left(\sqrt{2\pi}\sqrt{\beta} + \beta\right)$ since $\|\psi\| = \sqrt{\beta}\|\phi\| = \sqrt{\beta}/\sqrt{2\pi}$. Then, since $\alpha < \sqrt{2\pi}\sqrt{\beta}$ by assumption,

$$\lambda\gamma = \frac{\beta}{\sqrt{2\pi}\sqrt{\beta} + \beta}\frac{\alpha+\beta}{\beta} < \frac{\beta}{\sqrt{2\pi}\sqrt{\beta} + \beta}\frac{\sqrt{2\pi}\sqrt{\beta} + \beta}{\beta} = 1.$$

The rest of the proof is as for Proposition 2.□

Proof of Proposition 4(2)

The proof goes through up to proving Lemma 15, once one replaces d_n with γd_n, and Lemma 6 with Lemma 16. Then, (88) becomes

$$d_{n+1} = \left\|\bar{\theta}_{n+1}^{j*} - \underline{\theta}_{n+1}^{j*}\right\| \leq \lambda \left\|\bar{x}_n^{j*} - \underline{x}_n^{j*}\right\| \leq 2\lambda\gamma d_n. \tag{93}$$

Thus, $d_n \to 0$ as before, if $\lambda\gamma < 1/2$. But since $\alpha < \sqrt{\beta}\left(\sqrt{2\pi} - \sqrt{\beta}\right)/2$ by assumption,

$$\lambda\gamma = \frac{\beta}{\sqrt{2\pi}\sqrt{\beta} + \beta}\frac{\alpha + \beta}{\beta} < \frac{\beta}{\sqrt{2\pi}\sqrt{\beta} + \beta}\frac{\sqrt{\beta}\left(\sqrt{2\pi} - \sqrt{\beta}\right) + 2\beta}{2\beta} = \frac{1}{2}, \tag{94}$$

which concludes the proof.□

References

Angeletos, G., C. Hellwig, and A. Pavan (2007): "Dynamic Global Games of Regime Change: Learning, Multiplicity, and the Timing of Attacks," *Econometrica*, 75(3), 711–756.

Angeletos, G., and I. Werning (2006): "Crises and Prices: Information Aggregation, Multiplicity, and Volatility," *American Economic Review*, 96(5), 1720–1736.

Apostol, T. (1974): *Mathematical Analysis*, vol. 2. Reading, MA: Addison-Wesley.

Atkeson, A. (2000): "Discussion of Morris and Shin's 'Rethinking Multiple Equilibria in Macroeconomic Modelling'," in *NBER Macroeconomics Annual*, ed. by B. Bernanke, and K. Rogoff, pp. 162–171. Cambridge, MA: MIT Press.

Bauer, H. (2001): *Measure and Integration Theory*. Berlin, Germany: de Gruyter.

Botman, D. P., and H. Jager (2002): "Coordination of Speculation," *Journal of International Economics*, 58(1), 159–175.

Branson, W. (2001): "Intermediate Exchange Rate Regimes for Groups of Developing Countries," in *Don't Fix, Don't Float*, ed. by J. Braga De Macedo, D. Cohen, and H. Reisen, pp. 55–76. Paris, France: OECD Publications.

Burkill, J., and H. Burkill (2002): *A Second Course in Mathematical Analysis*. Cambridge, UK: Cambridge University Press.

Corsetti, G., A. Dasgupta, S. Morris, and H. S. Shin (2004): "Does One Soros Make a Difference? A Theory of Currency Crises with Large and Small Traders," *Review of Economic Studies*, 71(1), 87–113.

Dasgupta, A. (2004): "Financial Contagion through Capital Connections: A Model of the Origin and Spread of Bank Panics," *Journal of the European Economic Association*, 2(6), 1049–1084.

——— (2007): "Coordination and Delay in Global Games," *Journal of Economic Theory*, 134(1), 195–225.

Flood, R. P., and P. M. Garber (1984): "Collapsing Exchange-Rate Regimes: Some Linear Examples," *Journal of International Economics*, 17(1-2), 1–13.

Frankel, D. M., S. Morris, and A. Pauzner (2003): "Equilibrium Selection in Global Games with Strategic Complementarities," *Journal of Economic Theory*, 108(1), 1–44.

Goldstein, I. (2005): "Strategic Complementarities and the Twin Crises," *Economic Journal*, 115(503), 368–390.

GOLDSTEIN, I., AND A. PAUZNER (2004): "Contagion of Self-Fulfilling Financial Crises Due to Diversification of Investment Portfolios," *Journal of Economic Theory*, 119(1), 151–183.

——— (2005): "Demand–deposit Contracts and the Probability of Bank Runs," *Journal of Finance*, 60(3), 1293–1327.

GUIMARAES, B., AND S. MORRIS (2007): "Risk and Wealth in a Model of Self-fulfilling Currency Crises," *Journal of Monetary Economics*, 54, 2205–2230.

HARTIGAN, J. (1983): *Bayes Theory*. New York, NY: Springer-Verlag.

HEINEMANN, F., AND G. ILLING (2002): "Speculative Attacks: Unique Equilibrium and Transparency," *Journal of International Economics*, 58(2), 429–450.

HELLWIG, C. (2002): "Public Information, Private Information, and the Multiplicity of Equilibria in Coordination Games," *Journal of Economic Theory*, 107(2), 191–222.

HELLWIG, C., A. MUKHERJI, AND A. TSYVINSKI (2006): "Self-fulfilling Currency Crises: The Role of Interest Rates," *American Economic Review*, 96(5), 1769–1787.

HIJAB, O. (2011): *Introduction to Calculus and Classical Analysis*. New York, NY: Springer, 3rd edn.

JOHNSONBAUGH, R., AND W. PFAFFENBERGER (2010): *Foundations of Mathematical Analysis*. Mineola, NY: Dover Publications.

KACZOR, W., AND M. NOWAK (2001): *Problems in Mathematical Analysis II*. Providence, RI: American Mathematical Society.

KEISTER, T. (2009): "Expectations and Contagion in Self-Fulfilling Currency Attacks," *International Economic Review*, 50(3).

KOLMOGOROV, A., AND S. FOMIN (1975): *Introductory Real Analysis*. New York, NY: Dover Publications.

KRUGMAN, P. (1979): "A Model of Balance-of-Payments Crises," *Journal of Money, Credit, and Banking*, 11(3), 311–325.

LEVIN, J. (2001): "A Note on Global Equilibrium Selection in Overlapping Generations Games," mimeo.

MASON, R., AND A. VALENTINYI (2007): "The Existence and Uniqueness of Monotone Pure Strategy Equilibrium in Bayesian Games," University of Southampton Discussion Paper Series in Economics and Econometrics 0710.

MATHEVET, L. (2010): “A Contraction Principle for Finite Global Games,” *Economic Theory*, 42(3), 539–563.

MCADAMS, D. (2003): “Isotone Equilibrium in Games of Incomplete Information,” *Econometrica*, 71(4), 1191–1214.

MILGROM, P., AND J. ROBERTS (1990): “Rationalizability, Learning, and Equilibrium in Games with Strategic Complementarities,” *Econometrica*, 58(6), 1255–1277.

MORRIS, S., AND H. S. SHIN (1998): “Unique Equilibrium in a Model of Self-Fulfilling Currency Attacks,” *American Economic Review*, 88(3), 587–597.

——— (2004): “Coordination Risk and the Price of Debt,” *European Economic Review*, 48(1), 133–153.

OBSTFELD, M. (1996): “Models of Currency Crises with Self-Fulfilling Features,” *European Economic Review*, 40(3-5), 1037–1047.

OURY, M. (2009): “Multidimensional Global Games,” mimeo.

——— (2013): “Noise-independent Selection in Multidimensional Global Games,” *Journal of Economic Theory*, 148, 2638–2665.

STEINER, J. (2007): “Coordination of Mobile Labor,” *Journal of Economic Theory*, 139(1), 25–46.

STOKEY, N., AND R. LUCAS (1989): *Recursive Methods in Economic Dynamics*. Cambridge, MA: Harvard University Press.

TOPKIS, D. (2001): *Supermodularity and Complementarity*. Princeton, NJ: Princeton University Press.

VIVES, X. (1990): “Nash Equilibrium with Strategic Complementarities,” *Journal of Mathematical Economics*, 19(3), 305–321.

——— (2005): “Complementarities and Games: New Developments,” *Journal of Economic Literature*, 43(2), 437–479.

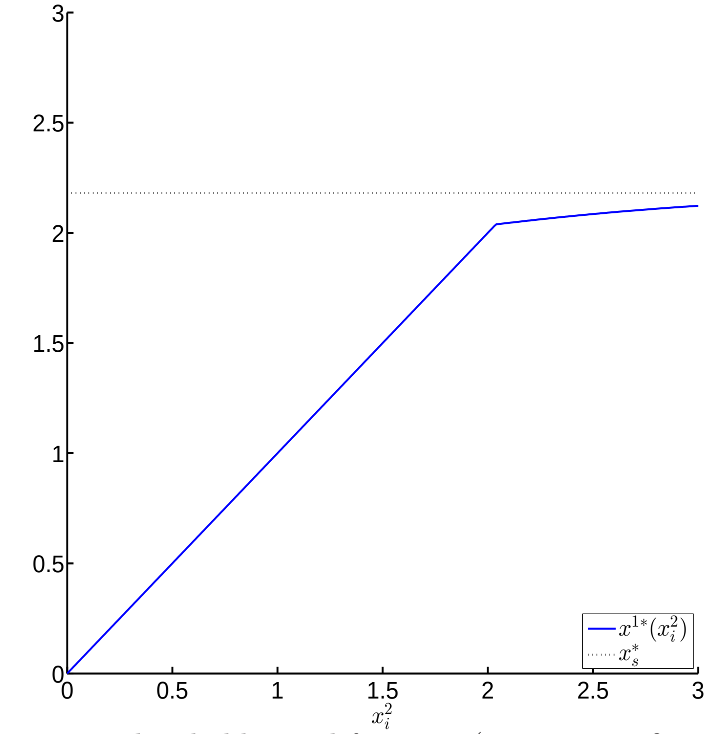

Figure 1: Threshold signal function (symmetric, $\beta = 1$).

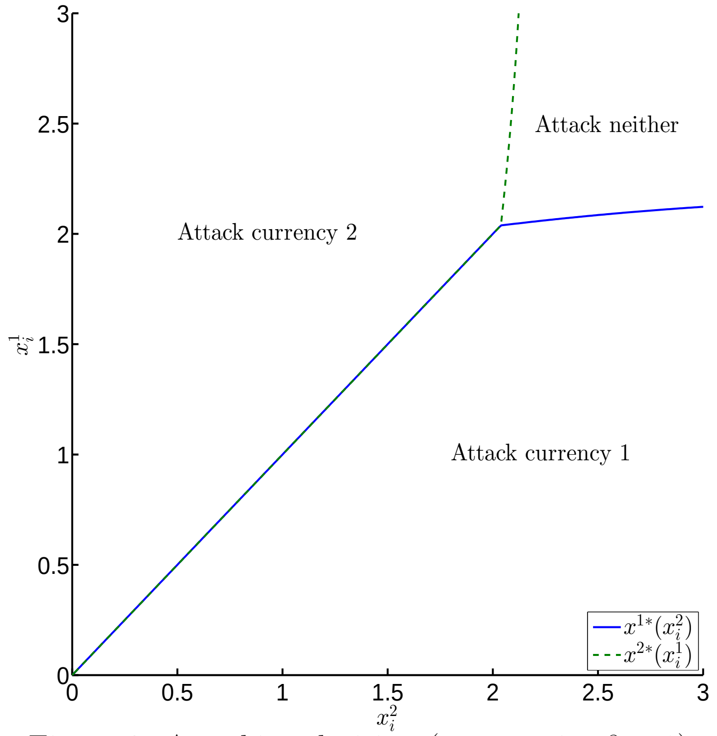

Figure 2: Attacking decision (symmetric, $\beta = 1$).

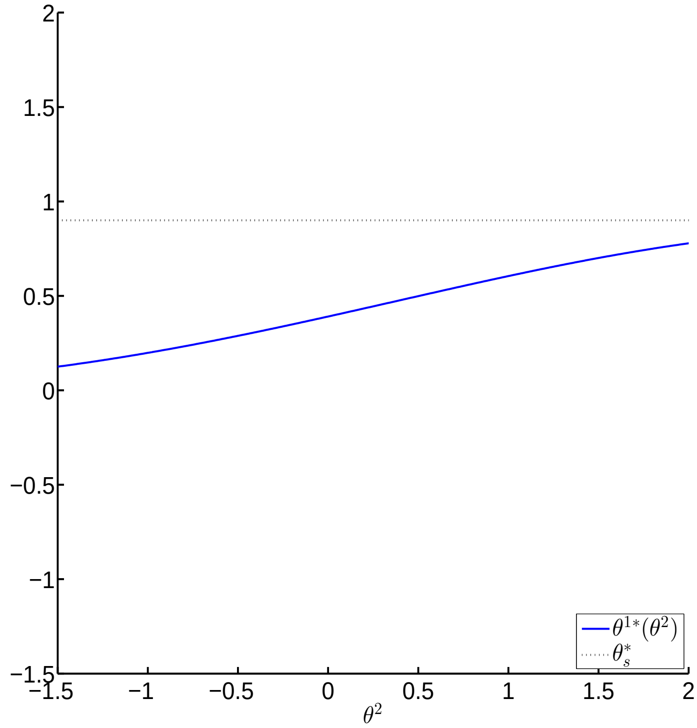

Figure 3: Threshold fundamentals function (symmetric, $\beta = 1$).

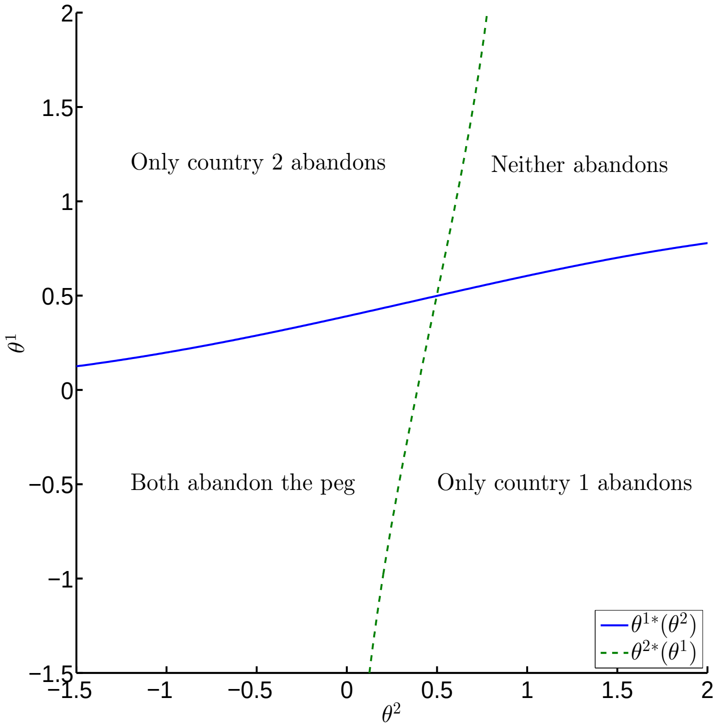

Figure 4: Outcomes of attack (symmetric, $\beta = 1$).

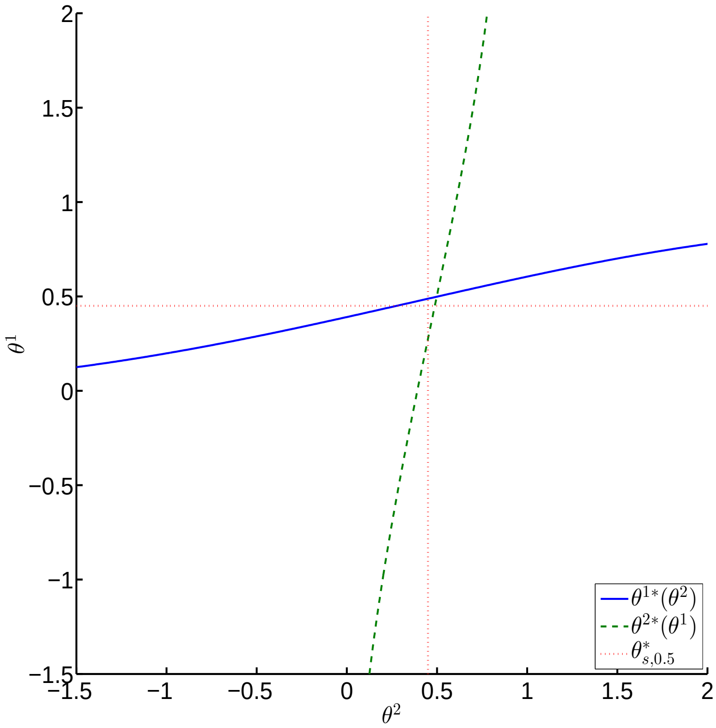

Figure 5: $\theta^{1*}(\theta^2)$ and $\theta^{2*}(\theta^1)$ compared with $\theta^*_{s,0.5}$ (symmetric, $\beta = 1$).

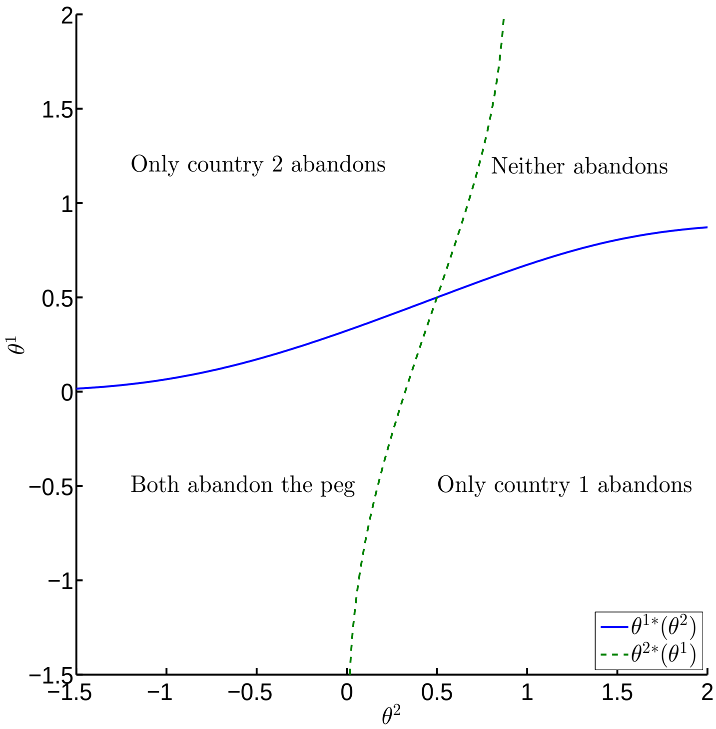

Figure 6: Outcomes of attack (symmetric, $\beta = 4$).

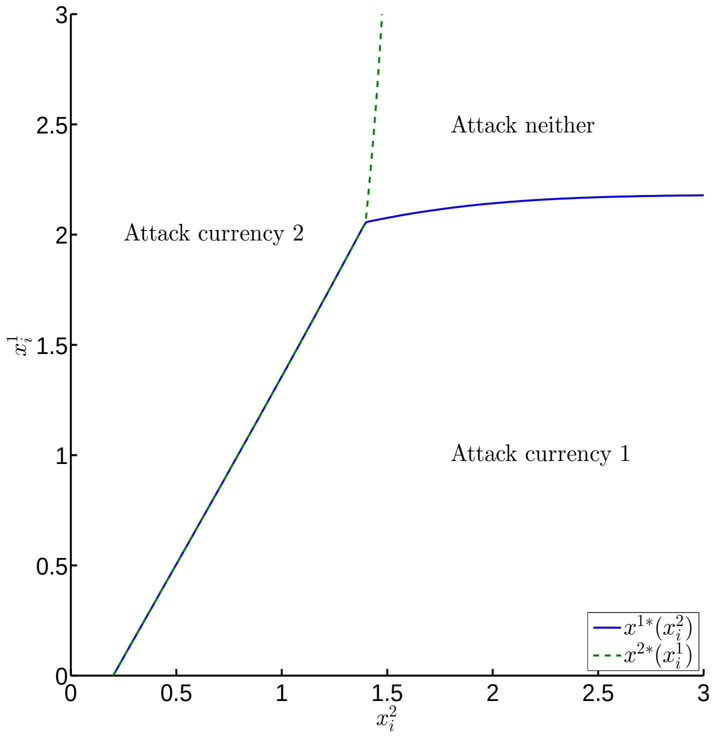

Figure 7: Attacking decisions ($\beta^1 = 1$, $\beta^2 = 4$).

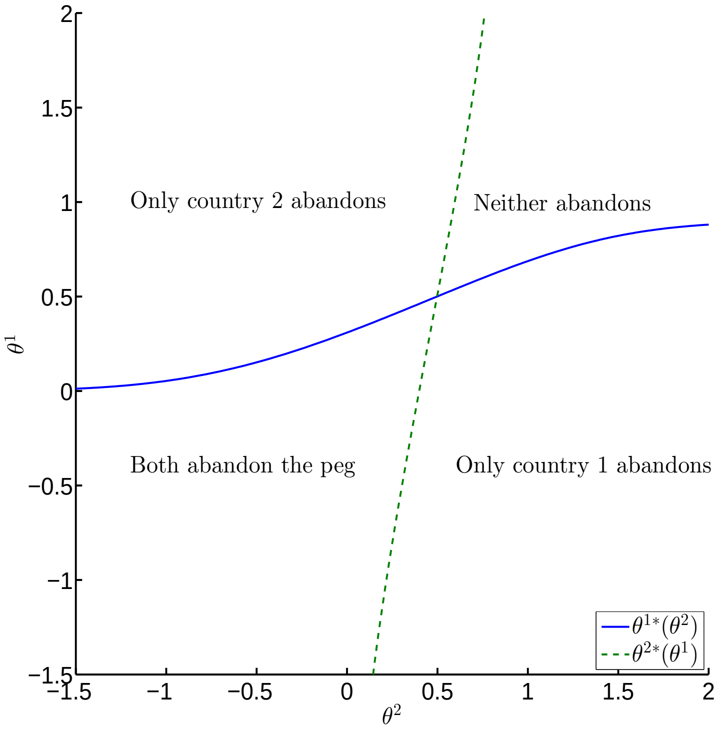

Figure 8: Outcomes of attack ($\beta^1 = 1$, $\beta^2 = 4$).

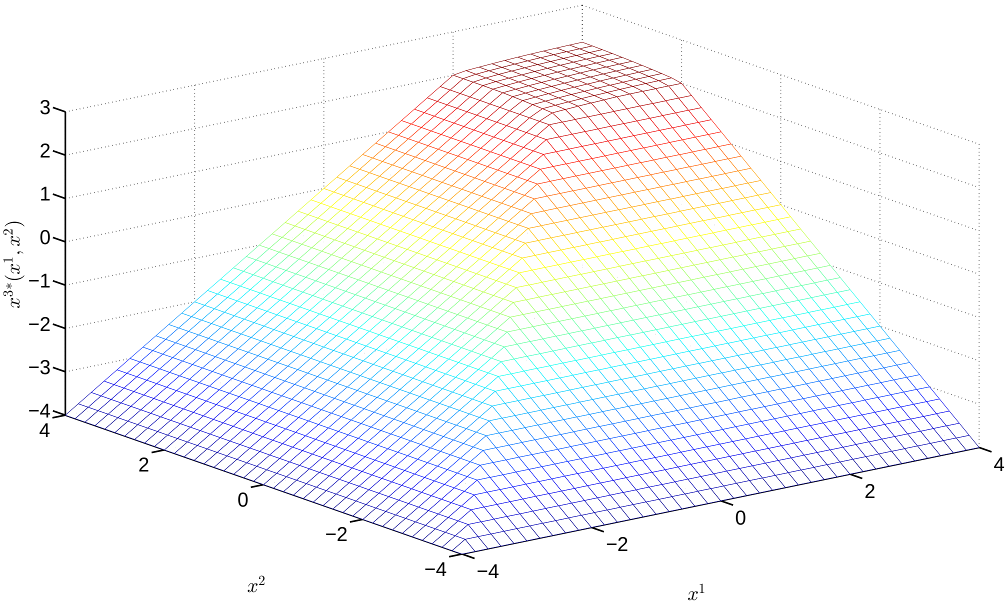

Figure 9: Threshold signal function ($N = 3$).

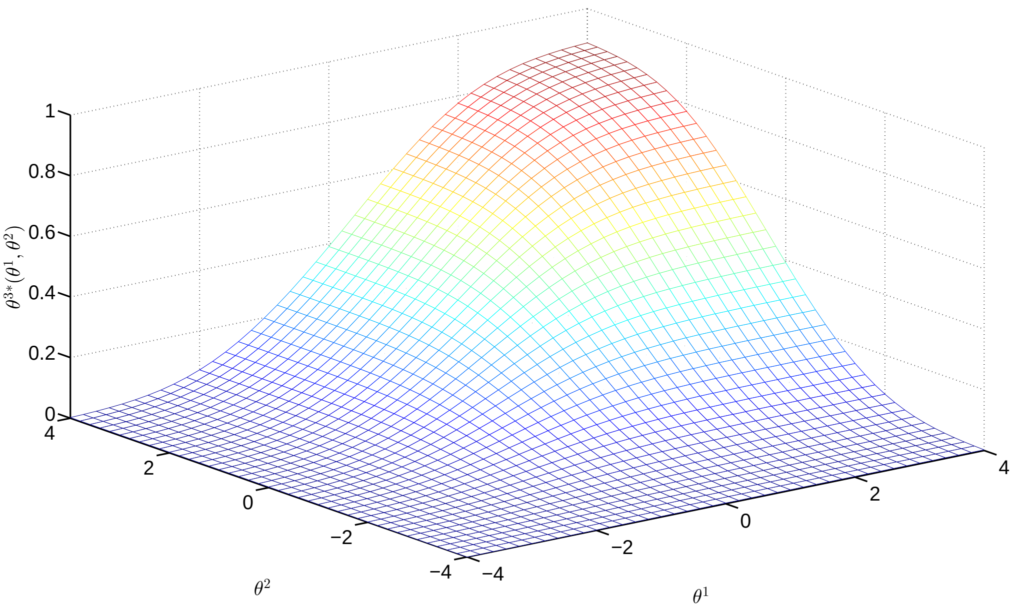

Figure 10: Threshold fundamentals function ($N = 3$).