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A New Improvement Scheme for Approximation Methods of Probability Density Functions *

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Abstract

This paper develops a new scheme for improving an approximation method of a probability density function, which is inspired by the idea in *best approximation in an inner product space*. Moreover, we apply “Dykstra’s cyclic projections algorithm” for its implementation. Numerical examples for application to an asymptotic expansion method in option pricing demonstrate the effectiveness of our scheme under SABR model.

Keywords: Density approximation, Probability density function, Asymptotic expansion, Best approximation in inner product spaces, Dykstra’s algorithm, Option pricing, SABR model

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1 Introduction

An approximation for a probability density function is a very interesting topic in various research fields. In fact, it seems so useful that a precise analytical approximation for a density would lead to substantial reduction of computational burden so that the subsequent analyses could be very easily implemented. Particularly, in finance the approximations for the densities of the asset prices have drawn much attention for at least more than two decades since fast and precise computation is so important in terms of competition and risk management, which is crucial in the derivatives business.

Examples among a large number of the related researches are Bayer and Laurence [1], Fouque, Papanicolaou and Sircar [5], Hagan, Kumar, Lesniewski, and Woodward [7], Gatheral, Hsu, Laurence, Ouyang, and Wang [6], Siopacha and Teichmann [11] and an asymptotic expansion approach ([12], [9], [10], [13]). Although those approximation methods have been successfully applied in practice, there exist at least implicitly some problems. For example, it is well known that while the density of the approximation formula for SABR model ([7]) provides sufficiently accurate approximations for option pricing, it has the negative values for the left tail, which could create an arbitrage opportunity in option trading.

This paper develops a new scheme for improving density approximation methods, which also contributes to precise approximation of option values efficiently. Specifically, our scheme is inspired by the idea in *a best approximation method in an inner product space*, and so-called “Dykstra’s cyclic projections algorithm” is applied for its implementation. We also remark that our scheme can be easily implemented in practice, where we need only market data for usual calibration such as option prices with strikes.

Furthermore, numerical experiments for vanilla option pricing under SABR model demonstrate the validity of our scheme. In fact, our scheme improves the third and fifth order asymptotic expansion preserving the required conditions such as nonnegative densities under an appropriate forward measure.

We finally remark that our scheme is general and flexible enough to include a set of conditions and information as one would like to put on an approximate density, and it can be applied to approximation methods other than the asymptotic expansion method. For example, a number of researches have been going on in order to extend SABR model with fixing the problem of the negative densities in the method of [7]. (For instance, see Doust[4].) We note that our scheme is also a candidate for handling this issue. Also, the estimate of the absorption probability based on Monte Carlo simulations as in [4] can be consistently reflected in our scheme.

The organization of the paper is as follows: After the next section describes the setup of the problem, Section 3 provides a concrete formulation of our method as well as the algorithm for the implementation. Section 4 shows numerical examples under SABR model.

2 Setup

Let S_t be the spot price of the underlying asset at time $t \in [0, T]$ and consider a density f of S_T , where S_T takes a value in $I \subseteq \mathbb{R}$, such as $I = \mathbb{R}$, $[0, +\infty)$ or $(0, +\infty)$. Clearly, the density function f of the price S_T must satisfy the following property. Hereafter, η stands for a density function under a risk-neutral or an appropriate forward probability measure.

Property 1. (*Density Condition*) : for a function η on $I \subseteq \mathbb{R}$,

(1) $\int_I \eta(x) dx = 1$

(2) $\eta \geq 0$.

Suppose that we have an approximation \tilde{f} of the density function f based on a certain method. Note that the approximation \tilde{f} does not necessarily have Property 1. Also, the forward price is usually given independently of models, and hence the average value of the underlying asset price at T should be equal to the given forward price with maturity T . Moreover, it is known that some approximation formulas (e.g.

the asymptotic expansion method) provides rather precise approximations for the values close to At-The-Money(ATM) options. Thus, it is reasonable that the option prices around ATM under a new approximate density function are calibrated to those computed based on an initial approximation formula, and that a new density is equal to the one obtained by the approximation formula for a certain range of the underlying asset price around ATM. We call those properties by *Calibration Condition*:

Property 2. (*Calibration Condition*)

- (3) $\int_I x\eta(x)dx = S_0$
- (4) $\int_I (x - K_n)_+\eta(x)dx = C_{K_n}$ for some given strikes $\{K_n\}_{n=1}^N$
- (5) $\eta = \tilde{f}$ on some subset I_0 of I

Here, the risk-free interest rate as well as the dividend rate of the underlying asset are assumed to be zero for simplicity. C_{K_n} denotes the option price with strike K_n and maturity T computed by the initial approximation formula.

In contrast to the accuracy around ATM, the values of the approximated density \tilde{f} may not be reliable around deep out of the money. However, how fast a density decreases to zero is known under some models or through a moment formula for the implied volatility. Namely, the following quantities are known:

$$\tilde{p} := \sup\{p > 0 : \mathbb{E}S_T^p < +\infty\} \quad (2.1)$$

$$\tilde{q} := \sup\{q > 0 : \mathbb{E}S_T^{-q} < +\infty\} \quad (2.2)$$

under some models or through the moment formula derived by Lee [8]:

$$\tilde{p} = \frac{1}{2\beta_R} + \frac{\beta_R}{8} + \frac{1}{2}, \quad \tilde{q} = \frac{1}{2\beta_L} + \frac{\beta_L}{8} - \frac{1}{2}, \quad (2.3)$$

where

$$\beta_R := \limsup_{x \rightarrow +\infty} \frac{IV^2(x)}{|x|/T}, \quad (2.4)$$

$$\beta_L := \limsup_{x \rightarrow -\infty} \frac{IV^2(x)}{|x|/T}. \quad (2.5)$$

Here, $IV(x)$ is an implied volatility function in terms of the log-moneyness that is, $x = \log(S_0/K)$.

Now, let us assume that \tilde{p} and \tilde{q} are known, and suppose $\chi : I \rightarrow (0, +\infty)$ be a density function which has the same order of the tail condition as f :

$$\int_{S_0}^{+\infty} x^p \chi(x)dx < +\infty \quad (p < \tilde{p}), \quad \int_0^{S_0} x^{-q} \chi(x)dx < +\infty \quad (q < \tilde{q}). \quad (2.6)$$

Then, it seems natural to impose the following condition:

Property 3. (*Tail Condition*)

- (6) η has the same tail slopes as χ .

However, for ease of computation, the condition may be replaced with the following:

Property 4. (*Weak Tail Condition*)

- (6-1) $\eta \leq \chi$ on $(0, K_L]$ for some positive number K_L
- (6-2) $\eta \leq \chi$ on $[K_R, +\infty)$ for some positive number K_R

Thus, we state our problem formally as follows:

Definition 1 (Problem). *Find a new approximate density function f^* for the target density f such that it satisfies the properties 1, 2 and 4, and*

$$\|f - f^*\| \leq \|f - \tilde{f}\|, \quad (2.7)$$

where the norm $\|\cdot\|$ will be defined based on the inner product (3.1) in the next section.

3 Formulation and Algorithm

This section concretely formulates the previous discussion and provides an algorithm for the implementation.

3.1 Formulation

Firstly, suppose a probability space $(\mathbb{R}, \mathcal{M}, \mu)$, where the measure μ is assumed to have a density which is equal to χ given in the previous section on I and to 0 on I^c . Next, we define the set of square integrable functions on $(\mathbb{R}, \mathcal{M}, \mu)$ denoted by $\mathcal{H} := L^2(\mathbb{R}, \mathcal{M}, \mu)$, and introduce an inner product for $f, g \in \mathcal{H}$ by

$$\langle f, g \rangle = \int_I f(x)g(x)\chi(x)dx. \quad (3.1)$$

Let us define some subsets of \mathcal{H} as \mathcal{K}_D , \mathcal{K}_C and \mathcal{K}_T which stand for the properties (1)-(2), (3)-(5) and (6-1)-(6-2) in Section 2, respectively:

$$\mathcal{K}_D := \{\varphi \in \mathcal{H} \mid \langle \varphi, 1_I \rangle = 1\} \cap \bigcap_{x \in I} \{\varphi \in \mathcal{H} \mid \langle \varphi, \delta_x \rangle \geq 0\}, \quad (3.2)$$

$$\mathcal{K}_C := \{\varphi \in \mathcal{H} \mid \langle \varphi, id_I \rangle = S_0\} \cap \bigcap_{n \leq N_C} \{\varphi \in \mathcal{H} \mid \langle \varphi, g_{K_n} \rangle = C_{K_n}\} \cap \bigcap_{x \in I_0} \{\varphi \in \mathcal{H} \mid \langle \varphi, \delta_x \rangle = \tilde{f}(x)\} \quad (3.3)$$

and

$$\mathcal{K}_T := \bigcap_{x \in (0, K_L]} \{\varphi \in \mathcal{H} \mid \langle \eta/\chi, \delta_x \rangle \leq \chi(x)\} \cap \bigcap_{x \in [K_R, +\infty)} \{\varphi \in \mathcal{H} \mid \langle \eta/\chi, \delta_x \rangle \leq \chi(x)\}, \quad (3.4)$$

where 1_I , id_I and g_{K_n} are elements of \mathcal{H} such that $1_I(x) = 1$, $id_I(x) = x$ and $g_{K_n}(x) = (x - K_n)_+$. Moreover, define \mathcal{K} as the intersection of \mathcal{K}_D , \mathcal{K}_C and \mathcal{K}_T :

$$\mathcal{K} := \mathcal{K}_D \cap \mathcal{K}_C \cap \mathcal{K}_T, \quad (3.5)$$

which is assumed to be nonempty.

Then, let $\tilde{\varphi} := \tilde{f}/\chi \notin \mathcal{K}$, and the best approximation set from $\tilde{\varphi}$ to \mathcal{K} is defined as

$$P_{\mathcal{K}}(\tilde{\varphi}) := \{\varphi^* \in \mathcal{K} \mid \|\tilde{\varphi} - \varphi^*\| = \inf_{\eta \in \mathcal{K}} \|\tilde{\varphi} - \eta\|\}. \quad (3.6)$$

Note that the set $P_{\mathcal{K}}(\tilde{\varphi})$ has the only one element since \mathcal{K} is a closed convex set in a Hilbert space. Hereafter, we may use the notation $P_{\mathcal{K}}(\tilde{\varphi})$ for the unique element of $P_{\mathcal{K}}(\tilde{\varphi})$ without any confusion.

Thus, it is easily shown that $P_{\mathcal{K}}(\tilde{\varphi})$ is a better approximation for $f/\chi \in \mathcal{K}$ than $\tilde{\varphi} := \tilde{f}/\chi \notin \mathcal{K}$, because we have

$$\begin{aligned} \|f/\chi - P_{\mathcal{K}}(\tilde{\varphi})\|^2 &= \|f/\chi - \varphi^\perp\|^2 + \|\varphi^\perp - P_{\mathcal{K}}(\tilde{\varphi})\|^2 \\ &\leq \|f/\chi - \varphi^\perp\|^2 + \|\varphi^\perp - \tilde{\varphi}\|^2 \\ &= \|f/\chi - \tilde{\varphi}\|^2, \end{aligned} \quad (3.7)$$

where φ^\perp is the foot of a perpendicular line through $\tilde{\varphi}$ and $P_{\mathcal{K}}(\tilde{\varphi})$ from f/χ .

Finally, we are able to obtain a better approximated density function as

$$f^* := P_{\mathcal{K}}(\tilde{\varphi})\chi. \quad (3.8)$$

This is actually better than the original one \tilde{f} in the following sense:

$$\int_I |f(x) - f^*(x)|^2 \frac{1}{\chi(x)} dx \leq \int_I |f(x) - \tilde{f}(x)|^2 \frac{1}{\chi(x)} dx. \quad (3.9)$$

3.2 Algorithm

In order to compute (3.6), we apply an iterative algorithm called *Dijkstra's algorithm*. (See pp. 207-214 of Deutsch [3] for the detail of the algorithm and its convergence discussion.)

In particular, let $\tilde{\mathcal{K}}$ be a convex set obtained by discretization of \mathcal{K} , which is an intersection of finite many closed convex sets \mathcal{K}_i ($i = 1, 2, \dots, r$) in the Hilbert space \mathcal{H} :

$$\tilde{\mathcal{K}} = \bigcap_{i=1}^r \mathcal{K}_i. \quad (3.10)$$

Here, we assume $\tilde{\mathcal{K}}$ to be nonempty.

First, for each $n \in \mathbb{N}$, let $[n]$ denote $n \bmod r$; that is,

$$[n] := \{1, 2, \dots, r\} \cap \{n - kr : k = 0, 1, 2, \dots\}. \quad (3.11)$$

For instance, $[1] = 1, [2] = 2, \dots, [r] = r, [r+1] = 1, \dots, [2r] = r, \dots$.

Next, for $\tilde{\varphi} \in \mathcal{H}$, set

$$\begin{aligned} \varphi_0 &:= \tilde{\varphi}, \quad e_{-(r-1)} = \dots = e_{-1} = e_0 = 0, \\ \varphi_n &:= P_{\mathcal{K}_{[n]}}(\varphi_{n-1} + e_{n-r}), \\ e_n &:= \varphi_{n-1} + e_{n-r} - \varphi_n \\ &= \varphi_{n-1} + e_{n-r} - P_{\mathcal{K}_{[n]}}(\varphi_{n-1} + e_{n-r}). \end{aligned} \quad (3.12)$$

Then, we have

$$\lim_{n \rightarrow +\infty} \|\varphi_n - P_{\mathcal{K}}(\tilde{\varphi})\| = 0. \quad (3.13)$$

See Theorem 9.24 (Boyle-Dijkstra Theorem) in p. 213 of Deutsch [3] for the proof.

3.3 Implementation

In this section, we describe how to implement our scheme numerically. For more details specific to models, see Section 4.

Choice of a Hilbert space Our scheme works on every Hilbert space theoretically. However, how to choose a Hilbert space, namely how to choose χ as a kernel of a Hilbert space, is crucial from a viewpoint of numerical implementation.

We choose the kernel function χ of the Hilbert space as a function close to the target density function f as much as possible. Particularly, we set χ as follows:

$$\chi := f_L 1_{(0, K_L)} + \tilde{f} 1_{[K_L, K_R)} + f_R 1_{[K_R, +\infty)}, \quad (3.14)$$

where \tilde{f} is an original approximation for the density function f , while f_L and f_R are some functions which have the similar tail behaviors as f . Here, K_L and K_R are chosen such that \tilde{f} is accurate on $[K_L, K_R)$. On this Hilbert space, the best approximation $P_{\mathcal{K}}(\tilde{\varphi})$ is expected to be close to 1_I , which makes the algorithm more stable. This is because if $\tilde{\varphi}$ is far from 1_I , an orthogonal projection of $\tilde{\varphi}$ onto the set $\{\varphi \in \mathcal{H} \mid \langle \varphi, 1_I \rangle = 1\}$ can violate the condition $\bigcap_{x \in I} \{\varphi \in \mathcal{H} \mid \langle \varphi, \delta_x \rangle \geq 0\}$. This can be easily understood, if a two-dimensional case is considered: compare two projections from points $(1, 1)$ and $(2, 0)$ onto a plane $\{(x, y) \in \mathbb{R}^2 \mid x + y = 1\}$.

In this framework, the norm to measure the accuracy is given by Eq.(3.9), which implies that more weights are put on tails. This norm is reasonable, because an original approximation is usually rather accurate around ATM and the purpose for using our scheme is to improve accuracy on tails.

Discretization For numerical implementation, we have to make the problem finite dimensional. Let us take a finite increasing sequence $\{x_i\}_{i=0}^d \subseteq \mathbb{R}$. Then, we can regard the Hilbert space \mathcal{H} as a d -dimensional vector space with an inner product:

$$\langle f, g \rangle = \sum_{i=1}^d f(x_i)g(x_i)\chi(x_i)(x_i - x_{i-1}). \quad (3.15)$$

If we choose $\{1_{[x_{i-1}, x_i]}\}_{i=1}^d$ as an orthogonal basis, an element $f \in \mathcal{H}$ can be identified with $(f(x_1), \dots, f(x_d)) \in \mathbb{R}^d$. The properties (1) to (5), (6-1) and (6-2) in Section 2 are conditions that an element $\eta/\chi \in \mathcal{H}$ is on a hyper-plane or on a half-space in a d -dimensional vector space. Projection of an element φ of \mathcal{H} onto a hyper-plane or a half-space \mathcal{M} is given by

$$P_{\mathcal{M}}(\varphi) = \varphi - \frac{1}{\|\varphi^*\|^2} (\langle \varphi, \varphi^* \rangle - c) \varphi^*, \quad (3.16)$$

if \mathcal{M} is a hyper-plane defined by $\mathcal{M} := \{\varphi \in \mathcal{H} \mid \langle \varphi, \varphi^* \rangle = c\}$ for some $\varphi^* \in \mathcal{H}$ and $c \in \mathbb{R}$, or

$$P_{\mathcal{M}}(\varphi) = \varphi - \frac{1}{\|\varphi^*\|^2} (\langle \varphi, \varphi^* \rangle - c)_+ \varphi^*, \quad (3.17)$$

if \mathcal{M} is a half space defined by $\mathcal{M} := \{\varphi \in \mathcal{H} \mid \langle \varphi, \varphi^* \rangle \leq c\}$. In both cases, these calculations are easy to implement and take little computational cost. After repeating the algorithm (3.12), we obtain the best approximation value at each point: $\{P_{\mathcal{K}}(\tilde{\varphi})(x_i)\chi(x_i)\}_{i=1}^d$. Finally, in order to calculate an option value with payoff g_K , we resort to a numerical integration:

$$\sum_{i=1}^d g_K(x_i)P_{\mathcal{K}}(\tilde{\varphi})(x_i)\chi(x_i)(x_i - x_{i-1}). \quad (3.18)$$

Procedure In summary, we have the following procedures:

1. Choose an increasing sequence $\{x_i\}_{i=0}^d \subseteq \mathbb{R}$ for discretization.
2. Calculate values of the original approximated function \tilde{f} at each point $\{x_i\}_{i=0}^d$.
3. Set a density function χ of a Hilbert space:
 - Choice of K_L and K_R is dependent on the domain where the original approximated function \tilde{f} is accurate. See Section 4 for the detail.
 - For a left/right-tail behavior, it is natural to impose constrains such that the density function decreases to zero as $x \rightarrow 0$ and $x \rightarrow +\infty$. More concretely, the tail of χ is set as $e^{-\frac{1}{2}(\log x)^2}$ or x^p for some p according to its decreasing speed. Also, χ should be a natural extension of \tilde{f} so that χ is continuous and decreases so fast to zero as the function \tilde{f} does around K_L and K_R .
4. Execute Dykstra's algorithm:
 - The algorithm (3.12) with an initial value \tilde{f}/χ^1 is iterated for $n = 1, \dots, N$ (see Section 4 on how to determine N). Each projection is calculated by Eq.(3.16) for a hyper-plane or Eq.(3.17) for a half space. It consists of finite-many operations of a finite dimensional vector space.
 - As a result, a set of values for the best approximation $\{P_{\mathcal{K}}(\tilde{\varphi})(x_i)\}_{i=0}^d$ is obtained.
5. Calculate option values:
 - Computation is executed numerically based on Eq.(3.18).

¹Alternatively, $\chi/\chi(\equiv 1)$ can be used in order not to take ill behaviors of \tilde{f} outside $[K_L, K_R]$ into account.

4 Numerical Example

4.1 Preparation

This section examines the validity of our scheme through numerical experiments by applying it to an asymptotic expansion method [13] under SABR model, where the dynamics of the underlying price process under a forward measure is expressed as follows:

$$dS_t = \sigma_t S_t^c dW_t^1 \quad (4.1)$$

$$d\sigma_t = \varepsilon \sigma_t dW_t^2. \quad (4.2)$$

Here, $\varepsilon > 0$, $c \in (0, 1]$ and W^1 and W^2 are Brownian motions with a constant correlation ρ . In the numerical experiment, let us concentrate on the case of $0 < c < 1$.² In this case, it is well known that S_T can reach 0 with positive probability. According to the result by Benaim et al. [2], it holds that

$$\tilde{p} = +\infty, \quad \tilde{q} = 0. \quad (4.3)$$

Then, while the density decreases so fast to zero on the right tail, the left tail is so fat that any moment is infinite: $\mathbb{E}(S_T^p) < +\infty$ for $p > 0$ and $\mathbb{E}(S_T^{-q}) = +\infty$ for $q > 0$. Thus, taking this observation into account, we may specify χ for $I := [0, +\infty)$ in the following:

$$\chi(x) = \begin{cases} A_L & x = 0, \\ \gamma x & x \in (0, K_L) \\ \tilde{f}(x) & x \in [K_L, K_R) \\ A_R \frac{1}{\sqrt{2\pi}} \frac{1}{x\tilde{\sigma}\sqrt{T}} e^{-\frac{(\log x - m)^2}{2\tilde{\sigma}^2 T}} & x \in [K_R, +\infty), \end{cases} \quad (4.4)$$

where A_L is a positive constant and $m = \log S_0 - \frac{1}{2}\tilde{\sigma}^2 T$ with a positive value $\tilde{\sigma}$.

With this setup, we calculate option prices by the following methods:

- (a) Monte Carlo simulation (Benchmark)
- (b) asymptotic expansion up to the fifth order
- (c) asymptotic expansion up to the third order
- (d) our algorithm with *Calibration Condition* for K_L and K_R and with $f^* = \tilde{f}$ on $[K_L, K_R)$, where \tilde{f} is equivalent to (b) the fifth or (c) third order expansion.

The choices of K_R and K_L are based on the discrepancies between (a) the benchmark prices and the approximated ones, that is (b) or (c)³.

We examine three cases of the parameter setting listed in Table 1, where Case 3. is the same as in Figure 4 of [4]. We set $K_R = 120$ and $K_L = 0$ with $A_L = \tilde{f}(0)$ for Case 1., while $K_R = 132$ and $K_L = 72$ for Case 2. In Case 3., we set $K_R = 5.856$ and $K_L = 3.904$ for the third order expansion, and $K_R = 7.2224$ and $K_L = 2.3424$ for the fifth order expansion. In addition, for Case 2 and Case 3, A_L is approximately equal to the absorption density obtained by the third or fifth asymptotic expansion.

Moreover, in $\chi(x)$ for $[K_R, +\infty)$, we set $\tilde{\sigma} = 0.45$ for Case 1. or $\tilde{\sigma} = 0.12$ for Case 2. and Case 3., which is the same level as the implied volatility around K_R . γ and A_R is determined so that χ is continuous. In both cases, we discretize the domain $[0, 4S_0)$ uniformly with 400(or 100) grids, and set N , the number of the iteration in 4. of **Procedure** in Section 3.3 as 100 (or 50) \times (the number of the conditions r in Eq.(3.10)).

²The result for case of $c = 1$ with $\rho < 0$ will be given upon request.

³The frequency of the calibration of K_L and K_R is the same as how often the original approximation method should be examined, for instance when the market drastically changes and otherwise once a day. We do not need any additional timing specific to the calibration of K_L and K_R .

4.2 Result

In Table 2 - 4, we report the results for options values expressed as the implied volatilities based on the three parameter sets in Table 1.

Firstly, in Table 2 of Case 1., that is the shorter maturity case $T = 1$, we note that comparing to the third and fifth order expansions ((b), (c)), our method improves the accuracies for the deep OTM prices, especially in the third order expansion. Moreover, the examination of the left tail of the corresponding density in Figure 1 shows that our nonnegative density approximation is able to take the absorption barrier at $S_T = 0$ into consideration, at least to a certain extent, which cannot be achieved by the original asymptotic expansion method only: the asymptotic expansion itself puts nonzero densities on some negative values of the underlying asset prices.

Next, we report the longer maturity case $T = 10$ in Table 3. Again, our method whose results appears as the columns of (d1)-(a), (d2)-(a), (e1)-(a), (e2)-(a) in the table generally improves the accuracies for the option prices at the deep OTM. This fact is due to the improvement of the densities, which is observed in Figure 2: for example, our scheme corrects the negative densities as well as the nonzero densities for the negative prices in the fifth order expansion.

Furthermore, let us make some comments on the direct comparison to [4] by using Table 4, the result for the same parameter set as in Figure 4 of [4]. Here, in the columns of (f1)-(a), (f2)-(a), (g1)-(a), (g2)-(a), we also report the result by our method that includes the information on the estimate of the absorption probability of $S_T = 0$ to \mathcal{K} in (3.5) through its inner product expression.

Although the precise comparison is not possible since the result of [4] is reported only as the graph, it is observed that at least our method with the fifth order expansion provides a better result from $K=1.0\%$ to $K=12\%$, especially, much better between $K=2.0\%$ and $K=10\%$ including $K=ATM=4.88\%$, possibly because the adjustment with (2.13) and (2.14) in [4] does not work very well. On the other hand, according to Figure 3 in [4], its method seems to work very well for the shorter maturity $T = 1$ as ours. Hence, for the longer maturity our method has an advantage in terms of accuracy. We remark that while in this experiment it is successful to correct the negative densities and the nonzero densities for the negative rates in the asymptotic expansion, our scheme can be applied to the approximate density of [4] for its further improvement, too.

Moreover, the comparison between the result of 400 grids with 100 iterations and that of 200 grids with 50 iterations in Table 3 and Table 4 shows that our scheme is robust to the coarser discretization and less number of iterations. In fact, the accuracies are almost the same for the parameter sets.

Next, let us report the examples of the computational speed for our scheme. The most part of the computational cost comes from the implementation of the Dykstra's algorithm. For instance, in Case 3., based on the machine, Intel(R) Core(TM) i7-3517U @ CPU 1.90GHz \times 2, RAM 4GB, the total computational time with 400 grids and 100 iterations is 0.0845 seconds for one option, 0.0864 seconds for 10 options and 0.1053 seconds for 100 options, while the computational time with 200 grids and 50 iterations is 0.0170 seconds for one option, 0.0175 seconds for 10 options and 0.0224 seconds for 100 options. It is almost the same in the other cases. Thus, combined with the robust result for the coarser discretization and less number of iterations, it seems that the computational speed of our scheme has no problem for practical purpose.

Finally, we remark on a practical application of our scheme. A typical procedure is as follows: firstly, by using an appropriate analytic approximation formula, we determine the model parameters through calibration to a (reliable) set of market option prices. Given the parameters, we apply our improvement scheme to obtain a no-arbitrage density, which generates reasonable option prices of all the strikes with preserving the previous calibration result. Then, we could use this density to evaluate options whose market prices are unquoted or unreliable (as for deep-OTM/ITM options). We note that the computational time in our scheme is fast enough for this purpose. In this example, as the asymptotic expansion method itself is a closed form approximation, the computational time is only 2.4×10^{-6} seconds per option, that is 0.0024 ms cpu time for 1 option and 0.24 ms cpu time for 100 options, which is fast enough for the calibration purpose. As reported above, our scheme adds very few computational costs.

Case	S_0	σ_0	ε	ρ	T
1.	100	5.00	0.3	-0.5	1
2.	100	1.58	0.3	-0.5	10
3.	4.88%	0.026	0.4	-0.1	10

Table 1: Parameter Set

strike	(a) IV %	(b)-(a)	(c)-(a)	(d1)-(a)	(e1)-(a)
20	86.2	0.8	5.5	0.6	0.8
40	69.7	0.2	1.7	0.0	0.2
60	60.5	0.1	0.6	-0.1	0.0
80	54.4	0.0	0.2	-0.0	0.0
100	49.8	0.0	0.1	0.0	0.1
120	46.3	0.0	-0.0	0.0	-0.0
140	43.6	-0.0	-0.1	0.1	-0.0
160	41.4	-0.1	-0.3	0.1	-0.1
180	39.6	-0.1	-0.5	0.0	-0.2
200	38.1	-0.0	-0.7	-0.0	-0.4
220	36.9	0.0	-0.9	-0.1	-0.6
240	36.0	0.1	-1.2	-0.1	-0.9
260	35.3	-0.1	-1.7	-0.2	-1.3
280	34.7	-0.4	-2.3	-0.2	-1.8
300	34.3	-0.9	-3.0	-0.1	-2.3

Table 2: (Case 1) Implied Volatility(IV) (%) under SABR model with $c = \frac{1}{2}$ and $T = 1$
(a) Monte Carlo, (b) 5th order Asymptotic Expansion (5th AE), (c) 3rd order Asymptotic Expansion (3rd AE),
(d1) with 5th AE, (400,100),
(e1) with 3rd AE, (400,100),

(Remark)

- “(b)-(a)”stands for the deviation % of IV by (b) from IV by (a).
- “with 5th (3rd) AE, (400,100)” means that the fifth (third) order asymptotic expansion is used in our scheme with 400 grids and 100 iterations.

Strike	(a) IV%	(b)-(a)	(c)-(a)	(d1)-(a)	(d2)-(a)	(e1)-(a)	(e2)-(a)
20	34.5	1.8	10.8	0.4	0.4	3.0	2.9
40	26.4	0.5	4.6	0.1	0.1	2.3	2.3
60	21.5	0.3	2.2	0.2	0.2	1.8	1.8
80	18.1	0.2	1.0	0.3	0.3	1.0	1.1
100	15.6	0.2	0.5	0.3	0.4	0.6	0.7
120	14.0	0.2	0.2	0.3	0.5	0.3	0.4
140	13.2	0.3	-0.1	0.5	0.7	0.1	0.3
160	12.8	0.5	-0.4	1.0	1.3	-0.0	0.4
180	12.8	0.3	-0.4	1.6	1.8	0.4	0.9
200	13.0	-0.9	-0.3	1.9	2.0	1.0	1.3
220	13.2	-3.9	-0.4	1.9	2.0	1.3	1.5
240	13.4	-	-0.7	1.7	1.8	1.3	1.5
260	13.6	-	-1.3	1.5	1.5	1.2	1.3
280	13.8	-2.0	-2.0	1.1	1.1	0.9	1.0
300	14.0	-1.3	-2.8	0.6	0.6	0.5	0.5

Table 3: (Case 2) Implied Volatility (IV) (%) under SABR model with $c = \frac{1}{2}$ and $T = 10$
(a) Monte Carlo, (b) 5th order Asymptotic Expansion (5th AE), (c) 3rd order Asymptotic Expansion (3rd AE),
(d1) with 5th AE, (400,100), (d2) with 5th AE, (200,50),
(e1) with 3rd AE, (400,100), (e2) with 3rd AE, (200,50)

(Remark)

1. “(b)-(a)” stands for the deviation % of IV by (b) from IV by (a).
2. “with 5th (3rd) AE, (400,100) ((200,50))” means that the fifth (third) order asymptotic expansion is used in our scheme with 400 (200) grids and 100 (50) iterations.
3. “-” in the column “(b)-(a)” means a failure in calculation of an implied volatility for the 5th AE due to a negative option price.

Strike	(a) IV%	(b)-(a)	(c)-(a)	(d1)-(a)	(d2)-(a)	(e1)-(a)	(e2)-(a)	(f1)-(a)	(f2)-(a)	(g1)-(a)	(g2)-(a)
0.1%	52.9	-	40.7	-1.6	-2.1	-3.9	-3.3	-0.1	-0.1	-0.1	-0.1
1.0%	29.9	11.5	8.0	-0.5	-0.8	-2.2	-1.8	0.0	0.0	0.1	0.1
2.0%	22.0	0.7	5.0	0.2	0.2	0.0	0.2	0.2	0.2	1.0	1.0
3.0%	17.2	-0.0	2.7	0.1	0.1	1.3	1.4	0.1	0.1	1.6	1.6
4.0%	14.1	0.4	1.2	0.5	0.6	1.2	1.2	0.5	0.6	1.2	1.2
5.0%	12.6	0.3	0.6	0.8	1.1	0.7	0.8	0.8	1.1	0.7	0.8
6.0%	12.5	0.3	0.6	0.6	0.8	0.8	0.9	0.6	0.8	0.8	0.9
7.0%	13.1	0.1	0.8	0.4	0.5	0.1	0.2	0.4	0.5	0.1	0.2
8.0%	13.9	1.0	0.2	0.1	0.2	-0.9	-0.8	0.1	0.2	-0.9	-0.8
9.0%	14.7	2.9	-1.3	-1.0	-1.0	-1.9	-1.8	-1.0	-1.0	-1.9	-1.8
10.0%	15.4	3.1	-3.2	-2.1	-2.1	-2.7	-2.7	-2.1	-2.1	-2.7	-2.7
11.0%	16.0	1.1	-5.0	-3.0	-3.0	-3.5	-3.5	-3.0	-3.0	-3.5	-3.5
12.0%	16.5	-1.6	-6.6	-3.7	-3.7	-4.2	-4.2	-3.7	-3.7	-4.2	-4.2
13.0%	17.1	-4.1	-8.0	-4.4	-4.4	-4.8	-4.8	-4.4	-4.4	-4.8	-4.8
14.0%	17.5	-6.2	-9.2	-5.1	-5.1	-5.4	-5.4	-5.1	-5.1	-5.4	-5.4

Table 4: (Case 3) Implied Volatility (IV) (%) under SABR model with $c = \frac{1}{2}$ and $T = 10$
(a) Monte Carlo, (b) 5th order Asymptotic Expansion (5th AE), (c) 3rd order Asymptotic Expansion (3rd AE),
(d1) with 5th AE, (400,100), (d2) with 5th AE, (200,50),
(e1) with 3rd AE, (400,100), (e2) with 3rd AE, (200,50),
(f1) with 5th AE, absorption probability, (400,100), (f2) with 5th AE, absorption probability, (200,50),
(g1) with 3rd AE, absorption probability, (400,100), (g2) with 3rd AE, absorption probability, (200,50)

(Remark) This parameter set is the same as in Figure 4 of [4].

1. "(b)-(a)" stands for the deviation % of IV by (b) from IV by (a).
2. "with 5th (3rd) AE, (400,100) ((200,50))" means that the fifth (third) order asymptotic expansion is used in our scheme with 400 (200) grids and 100 (50) iterations.
3. "with 5th (3rd) AE, absorption probability, (400,100) ((200,50))" means that the fifth (third) order asymptotic expansion and the absorption probability at $S_T = 0$ estimated in advance by Monte Carlo are used in our scheme with 400 (200) grids and 100 (50) iterations.
4. The total computational time with 400 grids and 100 iterations is 0.0845 seconds for one option, 0.0864 seconds for 10 options and 0.1053 seconds for 100 options, while the computational time with 200 grids and 50 iterations is 0.0170 seconds for one option, 0.0175 seconds for 10 options and 0.0224 seconds for 100 options. (The result is based on the machine, Intel(R) Core(TM) i7-3517U @ CPU 1.90GHz $\times 2$, RAM 4GB.)
5. "-" in the column "(b)-(a)" means a failure in calculation of an implied volatility for the 5th AE due to a negative option price.

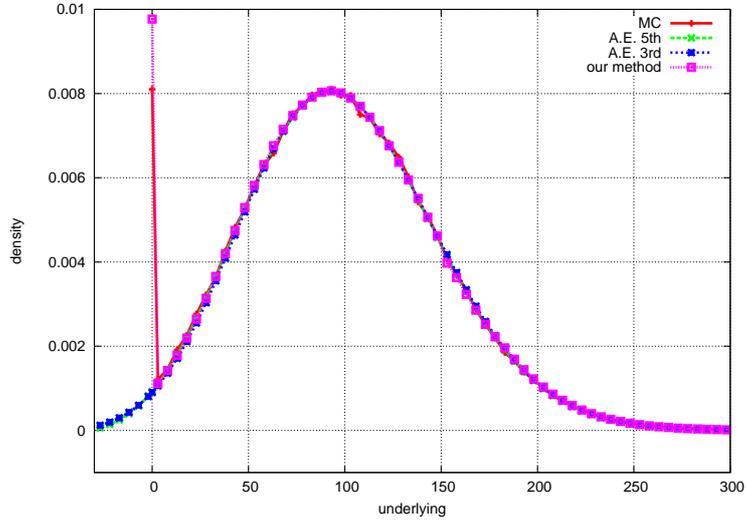


Figure 1: (Case 1) Densities under SABR model with $c = \frac{1}{2}$ and $T = 1$ (our method: (d1) with 5th AE)

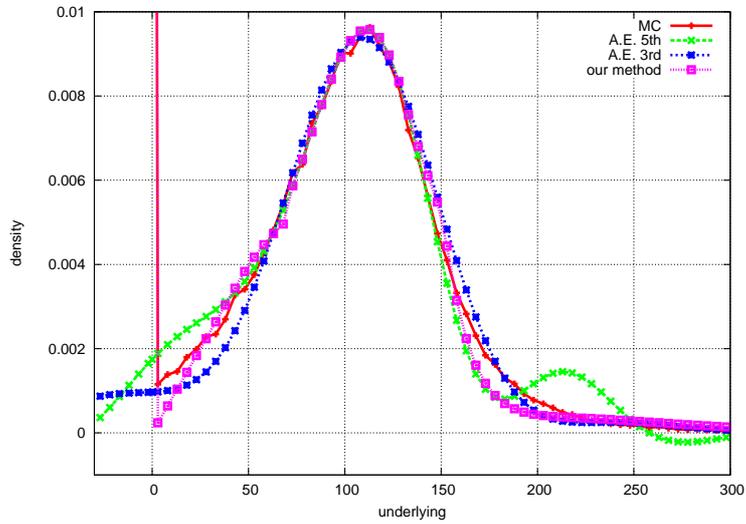


Figure 2: (Case 2) Densities under SABR model with $c = \frac{1}{2}$ and $T = 10$ (our method: (d1) with 5th AE)

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