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Asymptotic Expansion Approach in Finance *

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Abstract

This paper provides a survey on an asymptotic expansion approach to valuation and hedging problems in finance. The asymptotic expansion is a widely applicable methodology for analytical approximations of expectations of certain Wiener functionals. Hence not only academic researchers but also practitioners have been applying the scheme to a variety of problems in finance such as pricing and hedging derivatives under high-dimensional stochastic environments. The present note gives an overview of the approach.

Keywords: Asymptotic Expansion, Derivatives, Option Pricing, hedge, Greeks, Stochastic Volatility, Interest Rate, Term Structure Model, Malliavin Calculus, Watanabe Theory

1 Introduction

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ denote a probability space with filtration, on which a r-dimensional standard Wiener process W is defined, where P is an appropriate pricing measure (a risk neutral measure) in finance, and T denotes some positive constant. Now, let $F(\omega)$ be a Wiener functional and then $\mathbf{V}$, the security or portfolio value can be expressed as $\mathbf{V} = \mathbf{E}[F(\omega)]$ under certain conditions. Evaluating this expectation is one of the main issues in finance. Moreover, if F depends on the parameter θ, computation of $\frac{\partial \mathbf{V}}{\partial \theta} = \frac{\partial}{\partial \theta}\mathbf{E}[F(\omega;\theta)]$, the sensitivity of the security value with respect to the change in this parameter (so called *Greeks*) is also an important task in practice.

As an example, let us consider a d-dimensional diffusion process $X^{(\epsilon)}$ which is obtained as a strong solution to the stochastic differential equation;

$$dX_t^{(\epsilon)} = V_0(X_t^{(\epsilon)}, \epsilon)dt + V(X_t^{(\epsilon)}, \epsilon)dW_t,\ t \in [0,T];\ X_0^{(\epsilon)} = x_0,$$

where $\epsilon \in [0,1]$ is a known parameter. Here, the coefficients are assumed to satisfy some regularity conditions. In finance, many problems of pricing derivatives and evaluating the portfolios in investment theories are reduced to the problems of computing $\mathbf{E}[f(X_T^{(\epsilon)})]$, the expectation of $f(X_T^{(\epsilon)})$, that is a function of $X_T^{(\epsilon)}$.

In finance applications, it is important to deal with not only a smooth function $f(x)$ but also non-smooth one. For example, when various options are evaluated, f is expressed as $f = T \circ g$, where $T(x) = \max\{x, 0\}$ and g stands for a smooth function of $\mathbf{R}^d \mapsto \mathbf{R}$. In general, it is difficult to represent this expectation explicitly except for special cases. Hence, numerical methods such as Monte Carlo simulations or numerical solutions of partial differential equations (PDEs) are employed and various speeding up techniques are developed, since fast and precise computation is required in practice.

As a different approach, an approximation of the expectation by an asymptotic expansion of the stochastic differential equation around $\epsilon = 0$ may be considered. Furthermore, because $\frac{\partial}{\partial x_0}\mathbf{E}[f(X_T^{(\epsilon)})]$ and $\frac{\partial}{\partial \epsilon}\mathbf{E}[f(X_T^{(\epsilon)})]$, the sensitivities of the security value with respect to the changes in the initial value

*I dedicate this note to the late Koji Takahashi.

x_0 and in the parameter ϵ are important indicators for practical purposes, the approximations with high accuracies are so valuable. Moreover, some schemes that combine Monte Carlo simulations with asymptotic expansions with low orders are developed, since the asymptotic expansion up to the first or second order can be easily evaluated. Those schemes are able to improve the efficiencies of Monte Carlo simulations and the accuracies of approximations obtained by the asymptotic expansions.

An asymptotic expansion approach in finance has been developed for the past two decades, which is mathematically justified by Watanabe theory (Watanabe [111]) in Malliavin calculus (e.g. Malliavin [64], Chapter V-8 in Ikeda. and Watanabe [39], Nualart [73]). To the best of our knowledge, the asymptotic expansion technique is firstly applied to finance for evaluation of average options that are popular derivatives in commodity markets. Kunitomo and Takahashi [48] and [85] derive approximation formulas for average options by an asymptotic expansion method based on log-normal approximations for average prices distributions, when the underlying asset prices follow geometric Brownian motions. Yoshida [119] derives an asymptotic expansion of an average option price around a normal distribution for a general diffusion model, which is a byproduct of his result in statistics [118] based on the Watanabe theory.

Thereafter, the asymptotic expansion approach have been applied to a broad class of valuation problems in finance, which includes pricing options with stochastic volatility models, pricing options under Heath-Jarrow-Morton (HJM) models ([37]) or Libor market models (LMM) (Brace, Gatarek and Musiela [7], Jamshidian [43]) of interest rates, and pricing so called exotic-type options such as basket and barrier options in addition to average options.

For instance, please see Kawai [44], Kobayashi, Takahashi and Tokioka [45], Kunitomo and Takahashi [49], [50], [51], Li [59] Matsuoka, Takahshi and Uchida [66], Muroi [67], Nishiba [71], Osajima [75], Shiraya and Takahashi [78], [79], [80], Shiraya, Takahashi and Toda [81], Shiraya, Takahashi and Yamada [83], Shiraya, Takahashi and Yamazaki [82], Takahashi and Matsushima [88], Takahashi and Saito [89], Takahashi and Takehara [90], [91], [92], [93], [94], Takahashi, Takehara and Toda [90], [91], Takahashi and Tsuzuki[98], Takahashi and Uchida [99], Takahashi and Yamada [100], [101], [102], [103], [104], Takahashi and Yoshida [106], [107], Takehara, Takahashi and Toda[92], [93], Violante[110], Xu and Zheng [112], [113], and [86], [87].

We briefly introduce some of above works in Section 3.6. Moreover, we remark that the asymptotic expansion approach is employed by Yamanobe [116], [117] in physics for analyses of the impulse-driven stochastic biological oscillator and global dynamics of a stochastic neuronal oscillator.

We also note that there exist many other types of the expansion/perturbation methods which have turned out to be so useful for applications in finance. For example, see Bayer and Laurence [2], Ben Arous and Laurence [3], Benaim, Friz and Lee [4], Col, Gnoatto and Grasselli [9], Davydov and Linetsky [11], Deuschel, Friz, Jacquier and Violante [12], [13], Forde and Jacquier [18], Forde, Jacquier and Lee [17], Foschi, Pagliarani, Pascucci [19], Fouque, Papanicolaou and Sircar [20], [21], Fujii [24], Fujii and Takahashi [25], [26], [27], [29], Gatheral, Hsu, Laurence, Ouyang, and Wang [30], Gnoatto and Grasselli [31], Gulisashvili [32], Hagan, Kumar, Lesniewski and Woodward [33], Henry-Labordere [38], Kato Takahashi and Yamada [46], [47], Kusuoka and Osajima [57], Lee [58], Lipton [60], Linetsky[61], Osajima [76], Pagliarani and Pascucci [77], Siopacha and Teichmann [84], Yamamoto, Sato and Takahashi [114], Yamamoto and Takahashi [115], and references therein.

The organization of the paper is as follows. The next section describes the outline of the asymptotic expansion approach in a general diffusion setting. Then, Section 3 explains a computational scheme for the expansion method. Section 4 provides an extension of the general computational scheme in the previous section, and Section 5 briefly introduces two improvement scheme for the expansion method. Section 6 extends the approach to non-diffusion Wiener functionals by using an instantaneous forward rates model as an example. Section 7 and Section 8 introduce an asymptotic expansion in jump-diffusion models and a perturbation scheme in forward backward stochastic differential equations (FBSDEs). Section 9 concludes.

2 Asymptotic Expansion in General Diffusion Setting

Following [87] and [96], this section briefly describes an asymptotic expansion method in a general diffusion setting.

Let us consider a d-dimensional diffusion process $X_t^{(\epsilon)} = (X_t^{(\epsilon),1}, \cdots, X_t^{(\epsilon),d})^\top$ which is the solution to the following stochastic differential equation:

$$
\begin{aligned}
dX_t^{(\epsilon),j} &= V_0^j(X_t^{(\epsilon)},\epsilon)dt + \epsilon V^j(X_t^{(\epsilon)})dW_t \quad (j=1,\cdots,d) \\
X_0^{(\epsilon)} &= x_0 \in \mathbf{R}^d,
\end{aligned} \tag{1}
$$

where $W=(W^1,\cdots,W^r)^\top$ is a r-dimensional standard Wiener process, and $\epsilon\in(0,1]$ is a known parameter. Here, $x^\top$ denotes the transpose of x. Next, let us define $V_0=(V_0^1,\cdots,V_0^d)^\top:\mathbf{R}^d\times(0,1]\mapsto\mathbf{R}^d$ and V: $\mathbf{R}^d\mapsto\mathbf{R}^d\otimes\mathbf{R}^r$ whose j-th row is V^j, $j=1,\cdots,d$. Suppose also that V_0 and V satisfy some regularity conditions. (For example, V_0 and V are smooth functions with bounded derivatives of all orders.)

Next, let a function $g:\mathbf{R}^d\mapsto\mathbf{R}$ be smooth and all of its derivatives have polynomial growth. Then, a smooth Wiener functional $g(X_T^{(\epsilon)})$ has its asymptotic expansion:

$$
g(X_T^{(\epsilon)}) \sim g_{0T}+\epsilon g_{1T}+\epsilon^2 g_{2T}+\cdots
$$

in $\mathbf{D}^\infty$ as $\epsilon\downarrow 0$ where $g_{0T},g_{1T},g_{2T},\cdots\in\mathbf{D}^\infty$. For any $k\in\mathbf{N}$, $q\in(1,\infty)$ and $s>0$, this expansion means that

$$
\frac{1}{\epsilon^k}\|g(X_T^{(\epsilon)})-(g_{0T}+\epsilon g_{1T}+\cdots+\epsilon^{k-1}g_{k-1,T})\|_{q,s}=O(1)\ (as\ \epsilon\downarrow 0),
$$

where $\|G\|_{q,s}$ represents the sum of L^q -norms of Malliavin derivatives of a Wiener functional G up to the s-th order. Further, a Banach space $\mathbf{D}_{q,s}=\mathbf{D}_{q,s}(\mathbf{R})$ can be regarded as the totality of random variables bounded with respect to (q,s)-norm $\|\cdot\|_{q,s}$, and $\mathbf{D}^\infty=\cap_{s>0}\cap_{1<q<\infty}\mathbf{D}_{q,s}$. The coefficients $g_{nT}\in\mathbf{D}^\infty(n=0,1,\cdots)$ in the expansion can be obtained by Taylor's formula and represented based on multiple Wiener-Itô integrals. For the details of definitions and proofs above, please consult Watanabe [111], chapter V of Ikeda and Watanabe [39], Malliavin [64], or Chapter 7 of Malliavin and Thalmaier [65].

Remark 1. *As an example of applications in finance, $X^{(\epsilon)}$ consists of n stocks, $X^{(\epsilon)}=(S_1^{(\epsilon)},\cdots,S_n^{(\epsilon)})^\top$ and $g(\cdot)$ is those weighted sum $g(x)=w_1x_1+\cdots+w_nx_n$ for $x=(x_1,\cdots,x_n)^\top$ with constant weights $w_i(i=1,\cdots,n)$. Then, $g(x)$ would represent the spread, the average or the basket price of the stock prices.*

As another example, we can set $X^{(\epsilon)}$ is a vector of N forward Libor rates, $X^{(\epsilon)}=(L_1^{(\epsilon)},\cdots,L_N^{(\epsilon)})^\top$, and

$$
g(X_T^{(\epsilon)})=SR_T^{(\epsilon)}=\frac{1-\prod_{j=0}^{N-1}\frac{1}{1+\tau L_{jT}^{(\epsilon)}}}{\tau\sum_{i=0}^{N-1}\prod_{j=0}^{i}\frac{1}{1+\tau L_{jT}^{(\epsilon)}}},
$$

that is a swap rate with inception date T and maturity date $T_N=T+N\tau$. Here, L_{jT} stands for the forward Libor rate at T fixing at $T+j\tau$ with tenor τ.

Let $A_{kt}=\frac{1}{k!}\frac{\partial^k X_t^{(\epsilon)}}{\partial\epsilon^k}|_{\epsilon=0}$ and A_{kt}^j, $j=1,\cdots,d$ denote the j-th elements of A_{kt}. In particular, A_{1t} is represented by

$$
A_{1t}=\int_0^t Y_tY_u^{-1}\left(\partial_\epsilon V_0(X_u^{(0)},0)du+V(X_u^{(0)})dW_u\right), \tag{2}
$$

where Y denotes the solution to the ordinary differential equation:

$$
dY_t=\partial V_0(X_t^{(0)},0)Y_tdt;\ Y_0=I_d.
$$

Here, ∂V_0 denotes the $d\times d$ matrix whose (j,k)-element is $\partial_k V_0^j=\frac{\partial V_0^j(x,\epsilon)}{\partial x_k}$, V_0^j is the j-th element of V_0, and I_d denotes the $d\times d$ identity matrix.

For $k \geq 2$, A^j_{kt}, $j = 1, \cdots, d$ is recursively determined by the following equation:

$$\begin{aligned} A^j_{kt} &= \frac{1}{k!}\int_0^t \partial^k_\epsilon V^j_0(X^{(0)}, 0)du \\ &+ \sum_{l=1}^{k}\sum_{\vec{l}_\beta,\vec{d}_\beta}^{(l)} \frac{1}{(k-l)!}\frac{1}{\beta!}\int_0^t \left(\prod_{j=1}^{\beta} A^{d_j}_{l_j u}\right)\partial^\beta_{\vec{d}_\beta}\partial^{k-l}_\epsilon V^j_0(X^{(0)}_u, 0)du \\ &+ \sum_{\vec{l}_\beta,\vec{d}_\beta}^{(k-1)} \frac{1}{\beta!}\int_0^t \left(\prod_{j=1}^{\beta} A^{d_j}_{l_j u}\right)\partial^\beta_{\vec{d}_\beta} V^j(X^{(0)}_u)dW_u, \end{aligned} \tag{3}$$

where $\partial^l_\epsilon = \frac{\partial^l}{\partial \epsilon^l}$, $\partial^\beta_{\vec{d}_\beta} = \frac{\partial^\beta}{\partial x_{d_1}\cdots\partial x_{d_\beta}}$,

$$\sum_{\vec{l}_\beta,\vec{d}_\beta}^{(l)} := \sum_{\beta=1}^{l}\sum_{\vec{l}_\beta \in L_{l,\beta}}\sum_{\vec{d}_\beta \in \{1,\cdots,d\}^\beta} \tag{4}$$

for $l \geq 1$,

$$L_{l,\beta} := \left\{\vec{l}_\beta = (l_1, \cdots, l_\beta); \sum_{j=1}^{\beta} l_j = l; (l, l_j, \beta \in \mathbf{N})\right\}, \tag{5}$$

and for $l = 0$,

$$\sum_{\vec{l}_\beta,\vec{d}_\beta}^{(0)} = \sum_{\beta=0}\sum_{\vec{l}_0=(\emptyset)}\sum_{\vec{d}_0=(\emptyset)}.$$

Then, g_{0T} and g_{1T} can be written as

$$\begin{aligned} g_{0T} &= g(X^{(0)}_T), \\ g_{1T} &= \sum_{j=1}^{d}\partial_j g(X^{(0)}_T)A^j_{1T}, \end{aligned}$$

where $\partial_j g(x) = \frac{\partial}{\partial x_j}g(x)$, $j = 1, \cdots, d$.

For $n \geq 2$, g_{nT} is expressed as follows:

$$g_{nT} = \sum_{\vec{l}_\beta,\vec{d}_\beta}^{(n)} \frac{1}{\beta!}\partial^\beta_{\vec{d}_\beta} g(X^{(0)}_T)A^{d_1}_{l_1 T}\cdots A^{d_\beta}_{l_\beta T}. \tag{6}$$

Here, we note that each $A^i_{lt}(i = 1, \cdots, d, l = 1, 2, \cdots, k, 0 \leq t \leq T)$ has all finite moments due to a *grading* structure. We describe the definition of the grading structure by following pp.45-47 in Bichteler, Gravereaux and Jacod [5]: Consider the stochastic differential equation of the form:

$$dS_t = \mu(S_t, t)dt + \sigma(S_t, t)dW_t;\ S_0 = s_0 \in \mathbf{R}^d, \tag{7}$$

where $\mu : \mathbf{R}^d \times \mathbf{R}^+ \to \mathbf{R}^d$ and $\sigma : \mathbf{R}^d \times \mathbf{R}^+ \to \mathbf{R}^d \otimes \mathbf{R}^r$.

Definition 1. *A grading of $\mathbf{R}^d$ is a decomposition $\mathbf{R}^d = \mathbf{R}^{d_1} \times \cdots \times \mathbf{R}^{d_q}$ with $d = d_1 + \cdots + d_q$. The coordinates of a point in $\mathbf{R}^d$ are always arranged in an increasing order along the subspace $\mathbf{R}^{d_i}$, and we set $M_0 = 0$ and $M_l = d_1 + \cdots + d_l$ for $1 \leq l \leq q$. We say that the coefficients μ and σ are graded according to the grading $\mathbf{R}^d = \mathbf{R}^{d_1} \times \cdots \times \mathbf{R}^{d_q}$ if $\mu^i(x, t)$ and $\sigma^i_j(x, t), j = 1, \cdots, r$ depend upon only through the coordinates $(x^k)_{1\leq k\leq M_p}$ when $M_{p-1} \leq i \leq M_p$.*

Theorem 1. *We assume that the coefficients μ and σ in (7) are graded according to $\mathbf{R}^d = \mathbf{R}^{d_1} \times \cdots \times \mathbf{R}^{d_q}$. Moreover for $F(x, t) = \mu(x, t)$ or $\sigma_j(x, t), j = 1, \cdots, r$, we assume that F is differentiable in x on $\mathbf{R}^d$ and*

1. $|F^i(0,t)| \leq Z_t$ *for* $i = 1, \cdots, d$

2. $|\frac{\partial}{\partial x^j} F^i(x,t)| \leq \hat{Z}_t(1+|x|^\theta)$ *for all* i, j

3. $|\frac{\partial}{\partial x^j} F^i(x,t)| \leq \zeta$ *if* $M_{p-1} \leq i, j \leq M_p$ *for some* $p \leq q$,

where $\zeta, \theta \geq 0$ *are constants, and* Z, $\hat{Z}$ *are predictable processes such that* $\|Z\|_p$ *and* $\|\hat{Z}\|_p$ *are finite for all* $p \geq 1$ *where* $\|Z\|_p = \left\{\int_0^T E[|Z_t|^p]dt\right\}^{1/p}$. *Then (7) have a unique solution* S, *and for every* $p \geq 1$ *there are constants* c_p *and* γ_p *depending only upon* $(\zeta, \theta, \{||\hat{Z}||_{p'}\}_{p' \geq 1})$, *such that*

$$|| \sup_{0 \leq t \leq T} S_t||_{L^p} \leq c_p(s_0 + ||Z||_{\gamma_p}).$$

For the detail of the definition and theorem above, see pp.45-47 in Bichteler, Gravereaux and Jacod [5].

Applying Theorem 1 to the system of stochastic differential equations consisting of $A^i_{lt}(i = 1, \cdots, d, l = 1, \cdots, k, 0 \leq t \leq T)$ as well as any products of them, we obtain the following lemma.

Lemma 1. *Each coefficient in the expansion,* $A^i_{lt}(i = 1, \cdots, d, l = 1, \cdots, k, 0 \leq t \leq T)$ *has all finite moments.*

(Proof) We consider the system of stochastic differential equations (SDEs) for $A^1_1, \cdots, A^d_1$, $A^1_1 A^1_1, \cdots, A^d_1 A^d_1$, $A^1_2, \cdots, A^d_2, \cdots$. Then, the coefficients of the SDEs are represented by the derivatives at $\epsilon = 0$ of $\tilde{V}_0(X^{(\epsilon)}_u, \epsilon)$ and $\tilde{V}(X^{(\epsilon)}_u)$, which are bounded in $[0, T]$. Moreover, it is easily shown that the coefficients of the equation are graded and satisfy the conditions in Theorem 1. Hence each coefficient in the expansion, A^i_{kt} has all finite moments.□

Next, let normalize $g(X^{(\epsilon)}_T)$ to

$$G^{(\epsilon)} = \frac{g(X^{(\epsilon)}_T) - g_{0T}}{\epsilon}$$

for $\epsilon \in (0, 1]$. Then, we have

$$G^{(\epsilon)} \sim g_{1T} + \epsilon g_{2T} + \cdots$$

in $\mathbf{D}^\infty$.

Next, for $h \in H$, where H denotes the Cameron-Martin subspace of the r-dimensional Wiener space, the H-derivative of $G^{(\epsilon)}$ is expressed as

$$\mathbf{D}_h G^{(\epsilon)} = \frac{1}{\epsilon}\sum_{i=1}^d \partial_i g(X^{(\epsilon)}_T)\mathbf{D}_h X^{(\epsilon),i}_T = \sum_{i=1}^d \partial_i g(X^{(\epsilon)}_T) \int_0^T [Y^{(\epsilon)}_T (Y^{(\epsilon)}_t)^{-1} V(X^{(\epsilon)}_t)\dot{h}_t]_i \, dt,$$

where $Y^{(\epsilon)}$ is the $\mathbf{R}^d \otimes \mathbf{R}^d$-valued stochastic process which is the solution to the stochastic differential equation:

$$dY^{(\epsilon)}_t = \partial V_0(X^{(\epsilon)}_t, \epsilon)Y^{(\epsilon)}_t dt + \epsilon \sum_{i=1}^m \partial V^i(X^{(\epsilon)}_t)Y^{(\epsilon)}_t dw_{it}; \; Y^{(\epsilon)}_0 = I_d,$$

In fact, $Y_t = Y^{(0)}_t$. Here, $\partial V^i(i = 0, 1, \cdots, m)$ denotes the $d \times d$ matrix whose (j, k)-element is $\partial_k V^i_j$. $(\partial_k = \frac{\partial}{\partial x_k}.)$

Moreover, with a notation $\hat{V}^{(\epsilon)}_t$ that is defined by

$$\hat{V}^{(\epsilon)}_t = \left(\partial g(X^{(\epsilon)}_T)\right)^\top \left[Y^{(\epsilon)}_T (Y^{(\epsilon)}_t)^{-1} V(X^{(\epsilon)}_t)\right],$$

where $\left(\partial g(X^{(\epsilon)}_T)\right)^\top = (\partial_1 g(X^{(\epsilon)}_T), \cdots, \partial_d g(X^{(\epsilon)}_T))$, the Malliavin (co)variance of $G^{(\epsilon)}$ is given by

$$\sigma_{G^{(\epsilon)}} = \int_0^T \hat{V}^{(\epsilon)}_t (\hat{V}^{(\epsilon)}_t)^\top dt. \tag{8}$$

Moreover, let

$$\hat{V}_t := \hat{V}_t^{(0)} = \left(\partial g(X_T^{(0)})\right)^\top \left[Y_T Y_t^{-1} V(X_t^{(0)})\right]$$

and make the following assumption:

$$\text{(Assumption 1)} \quad \Sigma_T = \int_0^T \hat{V}_t \hat{V}_t^\top dt > 0.$$

Note that g_{1T} follows a normal distribution with variance Σ_T, and the density function of g_{1T} denoted by $f_{g_{1T}}(x)$ is given as

$$f_{g_{1T}}(x) = \frac{1}{\sqrt{2\pi\Sigma_T}} \exp\left(-\frac{(x-C)^2}{2\Sigma_T}\right)$$

where

$$C := \left(\partial g(X_T^{(0)})\right)^\top \int_0^T Y_T Y_t^{-1} \partial_\epsilon V_0(X_t^{(0)}, 0) dt. \tag{9}$$

Since Σ_T is the variance of the random variable g_{1T}, which follows a normal distribution, (Assumption 1) means the condition that the distribution of g_{1T} does not degenerate. In application, as it is easy to check this condition in most cases, it plays an important role for practical purposes.

Next, let us briefly introduce a truncated version of the Watanabe theory ([111]) based on Yoshida [118], [119]. Under (Assumption 1), $\sigma_{G^{(\epsilon)}}$ is uniformly non-degenerate for $\{|\eta_c^{(\epsilon)}| \leq 1\}$; that is, it can be shown that there exists a positive real number $c_0 > 0$ such that for any $c > c_0$ and $p > 1$,

$$\sup_{\epsilon \in (0,1]} \mathbf{E}[1_{\{|\eta_c^{(\epsilon)}| \leq 1\}} (|\sigma_{G^{(\epsilon)}}|)^{-p}] < \infty, \tag{10}$$

where $\eta_c^{(\epsilon)} = c\int_0^T |\hat{V}_t^{(\epsilon)} - \hat{V}_t| dt$.

Let $\mathcal{S}$ be the real Schwartz space of rapidly decreasing C^∞-functions on $\mathbf{R}$ and $\mathcal{S}'$ be its dual space. Then, for $\Phi : \mathbf{R} \mapsto \mathbf{R}$, $\Phi \in \mathcal{S}'$, a composite function $\psi(\eta_c^{(\epsilon)})\Phi \circ G^{(\epsilon)} = \psi(\eta_c^{(\epsilon)})\Phi(G^{(\epsilon)})$ is well-defined as an element of $\tilde{\mathbf{D}}^{-\infty} = \cup_{s<0} \cap_{1<p<\infty} \mathbf{D}_{p,s}$. Here, $\psi(x)$, $x \in R$ denotes a smooth function $0 \leq \psi(x) \leq 1$, defined as $\psi(x) = 1$ for $|x| \leq 1/2$ and $\psi(x) = 0$ for $|x| \geq 1$. Here, a Banach space $\mathbf{D}_{p,s}$, $s < 0$ is the dual space of $\mathbf{D}_{q,-s}(\mathbf{R})(q = p/(p-1))$.

Moreover, the coupling with the function 1 is well-defined, which is called as generalized expectation and is written as $\mathbf{E}[\psi(\eta_c^{(\epsilon)})\Phi \circ G^{(\epsilon)}]$. Further, $\psi(\eta_c^{(\epsilon)})\Phi \circ G^{(\epsilon)}$ can be expanded in $\tilde{\mathbf{D}}^{-\infty}$.

In addition, it can be shown that $\{\eta_c^{(\epsilon)}(w); \epsilon \in (0,1]\} \subset \mathbf{D}^\infty$, $\eta_c^{(\epsilon)}(w)$ is $O(1)$ in $\mathbf{D}^\infty$ as $\epsilon \downarrow 0$, and that for any $a_0 > 0$ there exist positive constants $a_i, i = 1,2,3$ such that $P(\{|\eta_c^{(\epsilon)}| > a_0\}) \leq a_1 \exp(-a_2 \epsilon^{-a_3})$. Hence, for any $k = 1, 2, \cdots$, we have

$$\lim_{\epsilon \downarrow 0} \frac{P(|\eta_c^{(\epsilon)}| > \frac{1}{2})}{\epsilon^k} < \infty.$$

This means that the probability of the events truncated by $\psi(\eta_c^{(\epsilon)})$ is smaller than any polynomial orders of ϵ. Then, in the expansion of $\psi(\eta_c^{(\epsilon)})\Phi \circ G^{(\epsilon)}$, the coefficients expressed as generalized Wiener functionals belonging to $\tilde{\mathbf{D}}^{-\infty}$ can be written by applying Taylor's formula to $\Phi(g_{0T} + \epsilon g_{1T} + \epsilon^2 g_{2T} + \cdots)$. Therefore, the asymptotic expansion of the expectation $\mathbf{E}[\Phi(G^{(\epsilon)})]$ can be obtained relatively easily. For the details of Watanabe theory and its truncated version above, please consult Watanabe [111] and Yoshida [118], [119]. For its application to valuation problems in finance, please also see [50].

In particular, if we take the delta function at $y \in \mathbf{R}$, δ_y as Φ, that is $\Phi(x) = \delta_y(x)$, we obtain an asymptotic expansion of the density function of $G^{(\epsilon)}$. Moreover, because functions such as $\Phi(x) = \max\{x, 0\}$ that is measurable but not smooth, frequently appear in finance, the framework mentioned above is necessary for the asymptotic expansion.

For instance, when we take $\max\{x,0\}$, $\min\{x,0\}$ or $\delta_y(x)$ as $\Phi(x)$ for a useful application in finance, the expectation of $\Phi(G^{(\epsilon)})$ is expanded as follows: for $N = 0, 1, 2, \cdots$,

$$\begin{aligned}
\mathbf{E}[\Phi(G^{(\epsilon)})] &= \sum_{n=0}^{N} \epsilon^n \sum_{\vec{k}_m}^{(n)} \frac{1}{m!} \mathbf{E}\left[\Phi^{(m)}(g_{1T}) \left(\prod_{j=1}^{m} g_{(k_j+1)T} \right) \right] + o(\epsilon^N) \\
&= \sum_{n=0}^{N} \epsilon^n \sum_{\vec{k}_m}^{(n)} \frac{1}{m!} \mathbf{E}\left[\Phi^{(m)}(g_{1T}) X^{\vec{k}_m} \right] + o(\epsilon^N) \\
&= \sum_{n=0}^{N} \epsilon^n \sum_{\vec{k}_m}^{(n)} \frac{1}{m!} \int_{-\infty}^{\infty} \Phi^{(m)}(x) \mathbf{E}[X^{\vec{k}_m} | g_{1T} = x] f_{g_{1T}}(x) dx + o(\epsilon^N) \\
&= \sum_{n=0}^{N} \epsilon^n \sum_{\vec{k}_m}^{(n)} \frac{1}{m!} \int_{-\infty}^{\infty} \Phi(x)(-1)^m \frac{d^m}{dx^m} \left\{ \mathbf{E}[X^{\vec{k}_m} | g_{1T} = x] f_{g_{1T}}(x) \right\} dx + o(\epsilon^N)
\end{aligned} \tag{11}$$

where $\Phi^{(m)}(g_{1T}) = \left.\frac{d^m \Phi(x)}{dx^m}\right|_{x=g_{1T}}$, $\sum_{\vec{k}_m}^{(n)} = \sum_{m=1}^{n} \sum_{\vec{k}_m \in L_{n,m}}$, and

$$X^{\vec{k}_m} := \prod_{j=1}^{m} g_{(k_j+1)T}. \tag{12}$$

In order to compute the asymptotic expansion (11), we need to evaluate the conditional expectations of the form:

$$E\left[\tilde{X}^{\vec{k}_m} \middle| g_{1T} = x \right],$$

where $\tilde{X}^{\vec{k}_m}$ is represented by a product of multiple Wiener-Itô integrals.

In the preceding works on application of the asymptotic expansion, the conditional expectations in (11) were directly computed with some formulas including multi-dimensional ones given for example, in [85] and [86]. Recently, while the formulas up to the third order are given in the works, [95] has developed a high-order computation scheme for the conditional expectations by using the fact that each of these $\{A^j_{k,t}\}_{j,k}$, $\{g_{nT}\}_n$ and also $\{X^{\vec{k}_m}\}_{\vec{k}_m}$ can be decomposed into a finite sum of iterated multiple Wiener-Itô integrals by applications of the Itô's formula with certain properties of iterated multiple Wiener-Itô integrals. (Please see Section 4 of [95] for the detail.)

On the other hand, as shown in the next section, we can develop an alternative method which does not evaluate the conditional expectations directly.

3 Computational Scheme

This section follows [96] to introduce a computational scheme for the asymptotic expansion, which is an alternative to the direct calculation method for the conditional expectations given in [95].

3.1 Preparation

To compute the conditional expectations on the right hand side of (11), we use the following lemma which can be derived from a property of Hermite polynomials and leads us to compute the unconditional expectations instead of the conditional ones.

Lemma 2. *Let (Ω, F, P) be a probability space. Suppose that $X \in L^2(\Omega, P)$ and Z is a random variable with Gaussian distribution with mean 0 and variance Σ. Then, the conditional expectation $E[X|Z = x]$*

has the following expansion in $L^2(\mathbf{R},\mu)$ where μ is the Gaussian measure on $\mathbf{R}$ with mean 0 and variance Σ:

$$E[X|Z=x]=\sum_{n=0}^{\infty}\frac{a_n}{\Sigma^n}H_n(x;\Sigma) \tag{13}$$

where $H_n(x;\Sigma)$ is the Hermite polynomial of degree n which is defined as

$$H_n(x;\Sigma)=(-\Sigma)^n e^{x^2/2\Sigma}\frac{d^n}{dx^n}e^{-x^2/2\Sigma}$$

and the coefficients a_n are given by

$$a_n=\frac{1}{n!}\frac{1}{i^n}\frac{\partial^n}{\partial\xi^n}\bigg|_{\xi=0}\left\{e^{\frac{\xi^2}{2}\Sigma}\mathbf{E}[e^{i\xi Z}X]\right\},\ (i=\sqrt{-1}). \tag{14}$$

(Proof) Since the system of Hermite polynomials $\{H_n(x;\Sigma)\}$ is an orthogonal basis of $L^2(\mathbf{R},\mu)$, and $E[X|Z=x]\in L^2(\mathbf{R},\mu)$, we have the following unique expansion of $E[X|Z=x]$ in $L^2(\mathbf{R},\mu)$:

$$E[X|Z=x]=\sum_{n=0}^{\infty}\frac{a_n}{\Sigma^n}H_n(x;\Sigma).$$

Since we have another Taylor expansion

$$e^{i\xi x}=e^{-\frac{\xi^2}{2}\Sigma}\sum_{n=0}^{\infty}\frac{H_n(x;\Sigma)}{n!}(i\xi)^n,$$

then,

$$\begin{aligned} e^{\frac{\xi^2}{2}\Sigma}\mathbf{E}[e^{i\xi Z}X] &= e^{\frac{\xi^2}{2}\Sigma}\int_{\mathbf{R}}e^{i\xi x}\mathbf{E}[X|Z=x]\mu(dx)\\ &= \int_{\mathbf{R}}\sum_{m=0}^{\infty}\frac{H_m(x;\Sigma)}{m!}(i\xi)^m\sum_{n=0}^{\infty}a_n\frac{H_n(x;\Sigma)\mu(dx)}{\Sigma^n}\\ &= \sum_{n=0}^{\infty}a_n(i\xi)^n. \end{aligned}$$

Comparing to the coefficients of the Taylor series of $e^{\frac{\xi^2}{2}\Sigma}\mathbf{E}[e^{i\xi Z}X]$ around 0 with respect to ξ, we see that a_n can be written as (14).□

Next, we write $\hat{V}_t=(\partial g(X_T^{(0)}))^{\top}Y_TY_t^{-1}V(X_t^{(0)})$ as $\hat{V}(X_t^{(0)})$. Then, we define $\hat{g}_1=\{\hat{g}_{1t};t\in\mathbf{R}^+\}$ and $Z^{\langle\xi\rangle}=\{Z_t^{\langle\xi\rangle};t\in\mathbf{R}^+\}$ as the stochastic processes

$$\hat{g}_{1t}=\int_0^t\hat{V}(X_u^{(0)})dW_u$$

and

$$Z_t^{\langle\xi\rangle}=\exp\left(i\xi\hat{g}_{1t}+\frac{\xi^2}{2}\Sigma_t\right),$$

respectively, where $\Sigma_t:=\int_0^t\hat{V}(X_u^{(0)})\hat{V}(X_u^{(0)})^{\top}du$.

Then, from Lemma 2, the conditional expectations appearing on the right hand side of the equation (11) is expressed as

$$\begin{aligned} \mathbf{E}[X^{\vec{k}_m}|g_{1T}=x] &= \mathbf{E}[X^{\vec{k}_m}|\hat{g}_{1T}=x-C]\\ &= \sum_{l=0}^{\infty}\frac{a_l^{\vec{k}_m}}{\Sigma_T^l}H_l(x-C;\Sigma_T) \end{aligned} \tag{15}$$

where

$$a_l^{\vec{k}_m} = \frac{1}{l!}\,\frac{1}{i^l}\,\frac{\partial^l}{\partial \xi^l}\bigg|_{\xi=0}\left\{\mathbf{E}[X^{\vec{k}_m} Z_T^{\langle\xi\rangle}]\right\}. \tag{16}$$

Here it is noted that with this expression we now need to compute unconditional expectations $\mathbf{E}\left[X^{\vec{k}_\delta} Z_T^{\langle\xi\rangle}\right]$ instead of the conditional expectations.

3.2 Asymptotic Expansion of Density Function

In this subsection, we explain a new computational method through deriving a general formula for the expansion (11) with an arbitrary specification of its order N. In particular, we show that the coefficients in the expansion are obtained through a system of ordinary differential equations that is solved easily.

First, we define $\eta_{\vec{l}_\beta}^{\vec{d}_\beta}(t;\xi)$ for $\vec{l}_\beta \in L_{n,\beta}$ and $\vec{d}_\beta \in \{1,\cdots,d\}^\beta$ $(n \geq \beta \geq 1)$ as

$$\eta_{\vec{l}_\beta}^{\vec{d}_\beta}(t;\xi) = \mathbf{E}\left[\left(\prod_{j=1}^{\beta} A_{l_j t}^{d_j}\right) Z_t^{\langle\xi\rangle}\right], \tag{17}$$

and for $n=0$ as

$$\eta_{(\emptyset)}^{(\emptyset)}(t;\xi) = \mathbf{E}\left[Z_t^{\langle\xi\rangle}\right]. \tag{18}$$

Then, by using (6) we write the unconditional expectations $\mathbf{E}[X^{\vec{k}_m} Z_T^{\langle\xi\rangle}]$ in (16) in terms of η as follows:

$$\begin{aligned}
\mathbf{E}[X^{\vec{k}_m} Z_T^{\langle\xi\rangle}] &= \mathbf{E}\left[\left(\prod_{j=1}^{m} g_{(k_j+1)T}\right) Z_T^{\langle\xi\rangle}\right] \\
&= \mathbf{E}\left[\left(\prod_{j=1}^{m}\left\{\sum_{\vec{l}^j_{\beta_j},\vec{d}^j_{\beta_j}}^{(k_j+1)} \frac{1}{\beta_j!}\partial^{\beta_j}_{\vec{d}^j_{\beta_j}} g(X_T^{(0)}) A_{l_1^j T}^{d_1^j}\cdots A_{l^j_{\beta_j} T}^{d^j_{\beta_j}}\right\}\right) Z_T^{\langle\xi\rangle}\right] \\
&= \sum_{\vec{l}^1_{\beta_1},\vec{d}^1_{\beta_1}}^{(k_1+1)}\cdots\sum_{\vec{l}^m_{\beta_m},\vec{d}^\delta_{\beta_m}}^{(k_m+1)}\left(\prod_{j=1}^{m}\frac{1}{\beta_j!}\partial^{\beta_j}_{\vec{d}^j_{\beta_j}} g(X_T^{(0)})\right)\eta_{\vec{l}^1_{\beta_1}\otimes\cdots\otimes\vec{l}^\delta_{\beta_\delta}}^{\vec{d}^1_{\beta_1}\otimes\cdots\otimes\vec{d}^m_{\beta_m}}(T;\xi)
\end{aligned} \tag{19}$$

where

$$\begin{aligned}
\vec{d}^i_{\beta_i}\otimes\vec{d}^j_{\beta_j} &:= (d_1^i,\cdots,d^i_{\beta_i},d_1^j,\cdots,d^j_{\beta_j}), \\
\vec{l}^i_{\beta_i}\otimes\vec{l}^j_{\beta_j} &:= (l_1^i,\cdots,l^i_{\beta_i},l_1^j,\cdots,l^j_{\beta_j}).
\end{aligned}$$

So, we have to calculate $\eta_{\vec{l}_\beta}^{\vec{d}_\beta}(T;\xi)$ to evaluate the asymptotic expansion (11).

In the following, we derive a system of ODEs satisfied by these $\{\eta_{\vec{l}_\beta}^{\vec{d}_\beta}\}$. Before showing a general result, we first derive the ODEs for a few leading-low-order terms explicitly to give a better intuition of a key idea of our method. Particularly, let us consider the evaluation of $\eta_{(2)}^{j}(T;\xi) = E[A_{2T}^j Z_T^{\langle\xi\rangle}]$ which appears in the ϵ-order. Here, for simplicity, we assume that V_0 does not depend on ϵ, and write $V_0(x,\epsilon)$ as $V_0(x)$. In this case, we first note that the SDEs of A^j_{1t} and A^j_{2t} $(j = 1, \cdots, d)$ are given as follows:

$$
\begin{aligned}
dA^j_{1t} &= \sum_{j'=1}^{d} A^{j'}_{1t} \partial_{j'} V^j_0(X^{(0)}_t) dt + V^j(X^{(0)}_t) dW_t && (20) \\
dA^j_{2t} &= \left[\sum_{j'=1}^{d} A^{j'}_{2t} \partial_{j'} V^j_0(X^{(0)}_t) + \frac{1}{2} \sum_{j',k'=1}^{d} A^{j'}_{1t} A^{k'}_{1t} \partial_{j'} \partial_{k'} V^j_0(X^{(0)}_t) \right] dt \\
&+ \sum_{j'=1}^{d} A^{j'}_{1t} \partial_{j'} V^j(X^{(0)}_t) dW_t. && (21)
\end{aligned}
$$

Also, the SDE of $Z^{\langle\xi\rangle}_t$ is expressed as:

$$
dZ^{\langle\xi\rangle}_t = (i\xi)\hat{V}(X^{(0)}) Z^{\langle\xi\rangle}_t dW_t. \tag{22}
$$

Then, applying Itô's formula to $A^j_{2t} Z^{\langle\xi\rangle}_t$, we have

$$
\begin{aligned}
d(A^j_{2t} Z^{\langle\xi\rangle}_t) &= A^j_{2t} dZ^{\langle\xi\rangle}_t + Z^{\langle\xi\rangle}_t dA^j_{2t} + d\langle A^j_2, Z^{\langle\xi\rangle} \rangle_t \\
&= \left\{ (i\xi) \sum_{j'=1}^{d} A^{j'}_{1t} Z^{\langle\xi\rangle}_t \hat{V}(X^{(0)}_t) \partial_{j'} V^j(X^{(0)}_t)' + \sum_{j'=1}^{d} A^{j'}_{2t} Z^{\langle\xi\rangle}_t \partial_{j'} V^j_0(X^{(0)}_t) \right. \\
&\quad \left. + \frac{1}{2} \sum_{j',k'=1}^{d} A^{j'}_{1t} A^{k'}_{1t} Z^{\langle\xi\rangle}_t \partial_{j'} \partial_{k'} V^j_0(X^{(0)}_t) \right\} dt \\
&\quad + \left\{ (i\xi) A^j_{2t} Z^{\langle\xi\rangle}_t \hat{V}(X^{(0)}_t) + \sum_{j'=1}^{d} A^{j'}_{1t} Z^{\langle\xi\rangle}_t \partial_{j'} V^j(X^{(0)}_t) \right\} dW_t.
\end{aligned}
$$

Since the last term is a martingale, taking expectation on both sides, we have the following ordinary differential equation for $\eta^j_{(2)}$:

$$
\begin{aligned}
\frac{d}{dt} \eta^j_{(2)}(t;\xi) &= (i\xi) \sum_{j'=1}^{d} \eta^{j'}_{(1)}(t;\xi) \hat{V}(X^{(0)}_t) \partial_{j'} V^j(X^{(0)}_t)^\top \\
&\quad + \sum_{j'=1}^{d} \eta^{j'}_{(2)}(t;\xi) \partial_{j'} V^j_0(X^{(0)}_t) + \frac{1}{2} \sum_{j',k'=1}^{d} \eta^{j',k'}_{(1,1)}(t;\xi) \partial_{j'} \partial_{k'} V^j_0(X^{(0)}_t).
\end{aligned}
$$

Here, $\eta^j_{(1)}(j = 1, \cdots, d)$ appearing in the right hand side of the above ODE are evaluated in the similar manner:

$$
\begin{aligned}
d(A^j_{1t} Z^{\langle\xi\rangle}_t) &= A^j_{1t} dZ^{\langle\xi\rangle}_t + Z^{\langle\xi\rangle}_t dA^j_{1t} + d\langle A^j_1, Z^{\langle\xi\rangle} \rangle_t \\
&= \left\{ (i\xi) Z^{\langle\xi\rangle}_t \hat{V}(X^{(0)}_t) V^j(X^{(0)}_t)^\top + \sum_{j'=1}^{d} A^{j'}_{1t} Z^{\langle\xi\rangle}_t \partial_{j'} V^j_0(X^{(0)}_t) \right\} dt \\
&\quad + \left\{ (i\xi) A^j_{1t} Z^{\langle\xi\rangle}_t \hat{V}(X^{(0)}_t) + Z^{\langle\xi\rangle}_t V^j(X^{(0)}_t) \right\} dW_t,
\end{aligned}
$$

hence, we have

$$
\frac{d}{dt} \eta^j_{(1)}(t;\xi) = (i\xi) \hat{V}(X^{(0)}_t) V^j(X^{(0)}_t)^\top + \sum_{j'=1}^{d} \eta^{j'}_{(1)}(t;\xi) \partial_{j'} V^j_0(X^{(0)}_t).
$$

$\eta^{j,k}_{(1,1)}$ and other higher-order terms can be evaluated in the same way. The key observation is that each ODE does not involve any higher-order terms, and only lower- or the same order- terms appear in the right hand side of the ODE. So, one can easily solve (analytically or numerically) the system of ODEs and evaluate the expectations.

The following proposition provides a way to calculate general $\eta_{\vec{l}_\beta}^{\vec{d}_\beta}(T;\xi)$ as a solution to the system of the ordinary differential equations:

Proposition 1. *For $\eta_{\vec{l}_\beta}^{\vec{d}_\beta}(t;\xi)$ defined in (17), the following system of ordinary differential equations is satisfied:*

$$\begin{aligned}
\frac{d}{dt}\left\{\eta_{\vec{l}_\beta}^{\vec{d}_\beta}(t;\xi)\right\} &= \sum_{k=1}^{\beta}\frac{1}{l_k!}\left\{\eta_{\vec{l}_{\beta/k}}^{\vec{d}_{\beta/k}}(t;\xi)\right\}\left\{\partial_\epsilon^{l_k}V_0^{d_k}(X_t^{(0)},0)\right\} \\
&+ \sum_{k=1}^{\beta}\sum_{l=1}^{l_k}\sum_{\vec{m}_\gamma,\vec{\tilde{d}}_\gamma}^{(l)}\frac{1}{(l_k-l)!}\frac{1}{\gamma!}\left\{\eta_{(\vec{l}_{\beta/k})\otimes\vec{m}_\gamma}^{(\vec{d}_{\beta/k})\otimes\vec{\tilde{d}}_\gamma}(t;\xi)\right\}\left\{\partial_{\vec{\tilde{d}}_\gamma}^{\gamma}\partial_\epsilon^{l_k-l}V_0^{d_k}(X_t^{(0)},0)\right\} \\
&+ \sum_{\substack{k,m=1\\k<m}}^{\beta}\sum_{\vec{m}_\gamma,\vec{\tilde{d}}_\gamma}^{(l_k-1)}\sum_{\vec{m}_\delta,\vec{\tilde{d}}_\delta}^{(l_m-1)}\frac{1}{\gamma!\delta!}\left\{\eta_{(\vec{l}_{\beta/k,m})\otimes\vec{m}_\gamma\otimes\vec{m}_\delta}^{(\vec{d}_{\beta/k,m})\otimes\vec{\tilde{d}}_\gamma\otimes\vec{\tilde{d}}_\delta}(t;\xi)\right\}\left\{\partial_{\vec{\tilde{d}}_\gamma}^{\gamma}V^{d_k}(X_t^{(0)})\right\}\left\{\partial_{\vec{\tilde{d}}_\delta}^{\delta}V^{d_m}(X_t^{(0)})\right\} \\
&+ (i\xi)\sum_{k=1}^{\beta}\sum_{\vec{m}_\gamma,\vec{\tilde{d}}_\gamma}^{(l_k-1)}\frac{1}{\gamma!}\left\{\eta_{(\vec{l}_{\beta/k})\otimes\vec{m}_\gamma}^{(\vec{d}_{\beta/k})\otimes\vec{\tilde{d}}_\gamma}(t;\xi)\right\}\left\{\partial_{\vec{\tilde{d}}_\gamma}^{\gamma}V^{d_k}(X_t^{(0)})\right\}\hat{V}(X_t^{(0)}) \qquad (23)
\end{aligned}$$

where $\sum_{\vec{m}_\gamma,\vec{\tilde{d}}_\gamma}^{(l)}$ is defined in (4), and

$$\begin{aligned}
\vec{l}_{\beta/k} &:= (l_1,\cdots,l_{k-1},l_{k+1},\cdots,l_\beta) \\
\vec{l}_{\beta/k,n} &:= (l_1,\cdots,l_{k-1},l_{k+1},\cdots,l_{n-1},l_{n+1},\cdots,l_\beta),\ 1\le k<n\le\beta \\
\vec{l}_\beta\otimes\vec{m}_\gamma &:= (l_1,\cdots,l_\beta,m_1,\cdots,m_\gamma)
\end{aligned}$$

for $\vec{l}_\beta=(l_1,\cdots,l_\beta)$ and $\vec{m}_\gamma=(m_1,\cdots,m_\gamma)$.

(Proof) We firstly apply Itô's formula to $\left(\prod_{j=1}^{\beta} A_{l_j t}^{d_j}\right)$ by using (3) to obtain the following:

$$
\begin{aligned}
d\left(\prod_{j=1}^{\beta} A_{l_j t}^{d_j}\right) &= \sum_{k=1}^{\beta}\left(\prod_{\substack{j=1\\ j\neq k}}^{\beta} A_{l_j t}^{d_j}\right) dA_{l_k t}^{d_k} + \sum_{\substack{k,m=1\\ k<m}}^{\beta}\left(\prod_{\substack{j=1\\ j\neq k,m}}^{\beta} A_{l_j t}^{d_j}\right) d\langle A_{l_k}^{d_k}, A_{l_m}^{d_m}\rangle_t \\
&= \sum_{k=1}^{\beta}\left(\prod_{\substack{j=1\\ j\neq k}}^{\beta} A_{l_j t}^{d_j}\right) \frac{1}{l_k!}\partial_{\epsilon}^{l_k} V_0^{d_k}(X_t^{(0)},0)dt \\
&+ \sum_{k=1}^{\beta}\left(\prod_{\substack{j=1\\ j\neq k}}^{\beta} A_{l_j t}^{d_j}\right) \sum_{l=1}^{l_k}\sum_{\vec{m}_\gamma,\vec{\tilde{d}}_\gamma}^{(l)} \frac{1}{(l_k-l)!}\frac{1}{\gamma!}\left(\prod_{j'=1}^{\gamma} A_{m_{j'} t}^{\tilde{d}_{j'}}\right)\partial_{\tilde{d}_\gamma}^{\gamma}\partial_{\epsilon}^{l_k-l} V_0^{d_k}(X_t^{(0)},0)dt \\
&+ \sum_{k=1}^{\beta}\left(\prod_{\substack{j=1\\ j\neq k}}^{\beta} A_{l_j t}^{d_j}\right) \sum_{\vec{m}_\gamma,\vec{\tilde{d}}_\gamma}^{(l_k-1)} \frac{1}{\gamma!}\left(\prod_{j'=1}^{\gamma} A_{m_{j'} t}^{\tilde{d}_{j'}}\right)\partial_{\tilde{d}_\gamma}^{\gamma} V^{d_k}(X_t^{(0)})dW_t \\
&+ \sum_{\substack{k,m=1\\ k<m}}^{\beta}\left(\prod_{\substack{j=1\\ j\neq k,m}}^{\beta} A_{l_j t}^{d_j}\right) \sum_{\vec{m}_\gamma,\vec{\tilde{d}}_\gamma}^{(l_k-1)}\sum_{\vec{m}_\delta,\vec{\hat{d}}_\delta}^{(l_m-1)} \frac{1}{\gamma!\delta!} \\
&\times \left(\prod_{j'=1}^{\gamma} A_{m_{j'} t}^{\tilde{d}_{j'}}\right)\partial_{\tilde{d}_\gamma}^{\gamma} V^{d_k}(X_t^{(0)}) \left(\prod_{j'=1}^{\delta} A_{m_{j'} t}^{\hat{d}_{j'}}\right)\partial_{\hat{d}_\delta}^{\delta} V^{d_m}(X_t^{(0)})dt.
\end{aligned}
\tag{24}
$$

Note also that $dZ_t^{\langle\xi\rangle} = (i\xi)\hat{V}(X_t^{(0)})Z_t^{\langle\xi\rangle}dW_t$. Then, applying Itô's formula again to $\left(\prod_{j=1}^{\beta} A_{l_j t}^{d_j} Z_t^{\langle\xi\rangle}\right)$ and take expectations on both sides to obtain the result. □

Remark 2. *Due to $\eta_{(\emptyset)}^{(\emptyset)}(t;\xi) = \mathbf{E}[Z_t^{\langle\xi\rangle}] = 1$, and the hierarchical structure of the ODEs with respect to $n = \sum_{j=1}^{\beta} l_j$, one can easily solve these ODEs successively from lower-order terms to higher-order terms with initial conditions $\eta_{\vec{l}_\beta}^{\vec{d}_\beta}(0;\xi) = 0$ for $(\vec{l}_\beta, \vec{d}_\beta) \neq (\emptyset,\emptyset)$.*

Remark 3. *Further, due to the structure of the system of the differential equations, it is easily shown by induction that each $\eta_{\vec{l}_\beta}^{\vec{d}_\beta}(t;\xi)$ is expressed as a polynomial of degree $n = \sum_{j=1}^{\beta} l_j$ with respect to $(i\xi)$. Then, we can also show that $\mathbf{E}[X^{\vec{k}_m} Z_T^{\langle\xi\rangle}]$ is a polynomial of degree $(n+m)$ with respect to $(i\xi)$, and thus $a_l^{\vec{k}_m} = 0 (l > n+m)$ for $\vec{k}_m \in L_{n,m}$. This ensures a convergence of the infinite sum in (15).*

Then, from Lemma 2 and (11), we have the following expression of $\mathbf{E}[\Phi(G^{(\epsilon)})]$:

$$
\begin{aligned}
\mathbf{E}[\Phi(G^{(\epsilon)})] &= \sum_{n=0}^{N}\epsilon^n \sum_{\vec{k}_m}^{(n)} \frac{1}{m!}\int_{\mathbf{R}} \Phi(x)(-1)^m \frac{d^m}{dx^m}\left\{\sum_{l=0}^{n+m}\frac{a_l^{\vec{k}_m}}{\Sigma_T^l} H_l(x-C;\Sigma_T) f_{g_{1T}}(x)\right\}dx + o(\epsilon^N) \\
&= \sum_{n=0}^{N}\epsilon^n \sum_{\vec{k}_m}^{(n)} \frac{1}{m!}\int_{\mathbf{R}} \Phi(x)\left\{\sum_{l=0}^{n+m}\frac{a_l^{\vec{k}_m}}{\Sigma_T^{l+m}} H_{l+m}(x-C;\Sigma_T) f_{g_{1T}}(x)\right\}dx + o(\epsilon^N)
\end{aligned}
$$

Here we have used the well-known property of the Hermite polynomial:

$$
\frac{d^m}{dx^m}\left\{H_l(x-C;\Sigma_T)f_{g_{1T}}(x)\right\} = \left(\frac{-1}{\Sigma_T}\right)^m H_{l+m}(x-C;\Sigma_T)f_{g_{1T}}(x).
$$

In particular, let Φ be the delta function at $x \in \mathbf{R}$, δ_x, we obtain the asymptotic expansion of the density of $G^{(\epsilon)}$:

$$\begin{aligned} f_{G^{(\epsilon)}}(x) &= \mathbf{E}[\delta_x(G^{(\epsilon)})] \\ &= \sum_{n=0}^{N} \epsilon^n \sum_{\vec{k}_m}^{(n)} \frac{1}{m!} \sum_{l=0}^{n+m} \frac{a_l^{\vec{k}_m}}{\Sigma_T^{l+m}} H_{l+m}(x - C; \Sigma_T) f_{g_{1T}}(x) + o(\epsilon^N). \end{aligned} \tag{25}$$

We summarize the discussion above as the following theorem:

Theorem 2. *Let $X^{(\epsilon)}$ be the solution to the stochastic differential equation (1). Suppose a function $g : \mathbf{R}^d \mapsto \mathbf{R}$ is smooth and all of its derivatives have polynomial growth. Then, the asymptotic expansion of the density function of $G^{(\epsilon)} = \frac{g(X_T^{(\epsilon)}) - g(X_T^{(0)})}{\epsilon}$ up to ϵ^N-order is given by*

$$\begin{aligned} f_{G^{(\epsilon)}}(x) &= f_{g_{1T}}(x) \\ &\quad + \sum_{n=1}^{N} \epsilon^n \left(\sum_{m=0}^{3n} C_{nm} H_m(x - C; \Sigma_T) \right) f_{g_{1T}}(x) + o(\epsilon^N), \end{aligned} \tag{26}$$

where

$$f_{g_{1T}}(x) = \frac{1}{\sqrt{2\pi\Sigma_T}} \exp\left(-\frac{(x - C)^2}{2\Sigma_T} \right) \tag{27}$$

with

$$\begin{aligned} C &= \left(\partial g(X_T^{(0)}) \right)^\top \int_0^T Y_T Y_t^{-1} \partial_\epsilon V_0(X_t^{(0)}, 0) dt, \\ \Sigma_T &= \int_0^T \hat{V}(X_t^{(0)}) \hat{V}(X_t^{(0)})^\top dt > 0, \\ \hat{V}(X_t^{(0)}) &= (\partial g(X_T^{(0)}))^\top Y_T Y_t^{-1} V(X_t^{(0)}). \end{aligned}$$

$H_n(x; \Sigma)$ is the Hermite polynomial of degree n with parameter Σ, which is defined as

$$H_n(x; \Sigma) = (-\Sigma)^n e^{x^2/2\Sigma} \frac{d^n}{dx^n} e^{-x^2/2\Sigma}, \tag{28}$$

and

$$\begin{aligned} C_{nm} &= \frac{1}{\Sigma_T^m} \sum_{\vec{k}_\delta}^{(m)} \sum_{\vec{l}^1_{\beta_1}, \vec{d}^1_{\beta_1}}^{(k_1+1)} \cdots \sum_{\vec{l}^\delta_{\beta_\delta}, \vec{d}^\delta_{\beta_\delta}}^{(k_\delta+1)} \frac{1}{\delta!(m-\delta)!} \\ &\quad \times \left(\prod_{j=1}^{\delta} \frac{1}{\beta_j!} \partial^{\beta_j}_{\vec{d}^j_{\beta_j}} g(X_T^{(0)}) \right) \frac{1}{i^{m-\delta}} \left. \frac{\partial^{m-\delta}}{\partial \xi^{m-\delta}} \right|_{\xi=0} \left\{ \eta^{\vec{d}^1_{\beta_1} \otimes \cdots \otimes \vec{d}^\delta_{\beta_\delta}}_{\vec{l}^1_{\beta_1} \otimes \cdots \otimes \vec{l}^\delta_{\beta_\delta}}(T; \xi) \right\}, \ \left(i = \sqrt{-1} \right). \end{aligned} \tag{29}$$

$\eta_{\vec{l}_\beta}^{\vec{d}_\beta}(T;\xi)$ *are obtained as a solution to the following system of ODEs:*

$$
\begin{aligned}
\frac{d}{dt}\left\{\eta_{\vec{l}_\beta}^{\vec{d}_\beta}(t;\xi)\right\} &= \sum_{k=1}^{\beta}\frac{1}{l_k!}\eta_{\vec{l}_{\beta/k}}^{\vec{d}_{\beta/k}}(t;\xi)\partial_\epsilon^{l_k}V_0^{d_k}(X_t^{(0)},0)\\
&\quad+\sum_{k=1}^{\beta}\sum_{l=1}^{l_k}\sum_{\vec{m}_\gamma,\vec{\tilde{d}}_\gamma}^{(l)}\frac{1}{(l_k-l)!}\frac{1}{\gamma!}\eta_{(\vec{l}_{\beta/k})\otimes\vec{m}_\gamma}^{(\vec{d}_{\beta/k})\otimes\vec{\tilde{d}}_\gamma}(t;\xi)\partial_{\vec{\tilde{d}}_\gamma}^{\gamma}\partial_\epsilon^{l_k-l}V_0^{d_k}(\tilde{X}_t^{(0)},0)\\
&+\sum_{\substack{k,m=1\\k<m}}^{\beta}\sum_{\vec{m}_\gamma,\vec{\tilde{d}}_\gamma}^{(l_k-1)}\sum_{\vec{m}_\delta,\vec{\tilde{d}}_\delta}^{(l_m-1)}\frac{1}{\gamma!\delta!}\eta_{(\vec{l}_{\beta/k,m})\otimes\vec{m}_\gamma\otimes\vec{m}_\delta}^{(\vec{d}_{\beta/k,m})\otimes\vec{\tilde{d}}_\gamma\otimes\vec{\tilde{d}}_\delta}(t;\xi)\\
&\quad\times\partial_{\vec{\tilde{d}}_\gamma}^{\gamma}V^{d_k}(X_t^{(0)})\partial_{\vec{\tilde{d}}_\delta}^{\delta}V^{d_m}(X_t^{(0)})\\
&+(i\xi)\sum_{k=1}^{\beta}\sum_{\vec{m}_\gamma,\vec{\tilde{d}}_\gamma}^{(l_k-1)}\frac{1}{\gamma!}\eta_{(\vec{l}_{\beta/k})\otimes\vec{m}_\gamma}^{(\vec{d}_{\beta/k})\otimes\vec{\tilde{d}}_\gamma}(t;\xi)\partial_{\vec{\tilde{d}}_\gamma}^{\gamma}V^{d_k}(X_t^{(0)})\hat{V}(X_t^{(0)},t)\\
\eta_{\vec{l}_\beta}^{\vec{d}_\beta}(0;\xi) &= 0 \text{ for } (\vec{l}_\beta,\vec{d}_\beta)\neq(\emptyset,\emptyset),\ \eta_{(\emptyset)}^{(\emptyset)}(t;\xi)=1 \text{ for } (\vec{l}_\beta,\vec{d}_\beta)=(\emptyset,\emptyset).
\end{aligned}\tag{30}
$$

Here, we use the following notations:

$$
\begin{aligned}
\vec{l}_{\beta/k} &:= (l_1,\cdots,l_{k-1},l_{k+1},\cdots,l_\beta)\\
\vec{l}_{\beta/k,n} &:= (l_1,\cdots,l_{k-1},l_{k+1},\cdots,l_{n-1},l_{n+1},\cdots,l_\beta),\ 1\le k<n\le\beta\\
\vec{l}_\beta\otimes\vec{m}_\gamma &:= (l_1,\cdots,l_\beta,m_1,\cdots,m_\gamma)
\end{aligned}
$$

for $\vec{l}_\beta=(l_1,\cdots,l_\beta)$ *and* $\vec{m}_\gamma=(m_1,\cdots,m_\gamma)$.

Remark 4. *Particularly, in order to calculate the expansion above up to the* ϵ^2*-order, we need the Hermite polynomials* $H_n(x;\Sigma)$ *up to* $n=6$*, which are given as follows:*

$$
\begin{aligned}
H_0(x;\Sigma) &= 1,\\
H_1(x;\Sigma) &= x,\\
H_2(x;\Sigma) &= x^2-\Sigma,\\
H_3(x;\Sigma) &= x^3-3\Sigma x,\\
H_4(x;\Sigma) &= x^4-6\Sigma x^2+3\Sigma^2,\\
H_5(x;\Sigma) &= x^5-10\Sigma x^3+15\Sigma^2 x,\\
H_6(x;\Sigma) &= x^6-15\Sigma x^4+45\Sigma^2x^2-15\Sigma^3.
\end{aligned}
$$

3.3 Remarks on the Asymptotic Expansion for Multi-dimensional Density Functions

We can also apply the conditional expectation formulas for the multi-dimensional case in Lemma 1.1' of [85] and Lemma 2.1 of [86] to derive an asymptotic expansion up to the third order of the multi-dimensional density functions. This is particularly useful for pricing exotic-type options such as barrier options with discrete monitoring (e.g. [83]), and pricing Bermudan-type or approximate American-type derivatives (e.g. Nishiba [71]).

Moreover, we obtain the following result as an extension of Lemma 2, which easily leads to an asymptotic expansion of a multi-dimensional density function in the similar manner as in the one dimensional case in Theorem 2.

Lemma 3. *Let* $(\Omega,\mathcal{F},P)$ *be a probability space. Suppose that* $X\in L^2(\Omega,P)$ *and* $\vec{Z}$ *is a d-dimensional random variable with Gaussian distribution with mean* $\vec{0}$*(d-dimensional zero vector) and variance-covariance* *matrix $\underline{\Sigma}$. Then, the conditional expectation $\mathbf{E}[X|\vec{Z}=\vec{x}]$ for $\vec{x}\in\mathbf{R}^d$ has the following expansion in $L^2(\mathbf{R}^d,\mu)$ where μ is the Gaussian measure on $\mathbf{R}^d$ with mean $\vec{0}$ and variance-covariance matrix $\underline{\Sigma}$:*

$$\mathbf{E}[X|\vec{Z}=\vec{x}]=\sum_{|\vec{n}|=0}^{\infty} a_{\vec{n}!}H_{\vec{n}}(\vec{x};\underline{\Sigma}), \tag{31}$$

where $\vec{n}=(n_1,n_2,\cdots,n_d)$, $|\vec{n}|=n_1+n_2+\cdots+n_d$, $\vec{n}!=n_1!n_2!\cdots n_d!$ and

$$a_{\vec{n}}=\frac{1}{\vec{n}}\frac{1}{i^{|\vec{n}|}}\frac{\partial^{\vec{n}}}{\partial\vec{\xi}}\bigg|_{\vec{\xi}=\vec{0}}\left\{e^{\frac{1}{2}(\vec{\xi})^\top\underline{\Sigma}\vec{\xi}}\mathbf{E}\left[e^{i(\vec{\xi})^\top\vec{Z}}X\right]\right\},\ \left(i=\sqrt{-1}\right). \tag{32}$$

Here, $(\vec{\xi})^\top$ denotes the transpose of $\vec{\xi}$. $H_{\vec{n}}(\vec{x};\underline{\Sigma})$ stands for the d-dimensional multiple Hermite polynomial of degree $|\vec{n}|$ with $\vec{n}=(n_1,n_2,\cdots,n_d)$:

$$H_{\vec{n}}(\vec{x};\underline{\Sigma}) = \frac{1}{n[\vec{x}:\underline{\Sigma}]}\left(-\frac{\partial}{\partial_{x_1}}\right)\left(-\frac{\partial}{\partial_{x_2}}\right)\cdots\left(-\frac{\partial}{\partial_{x_d}}\right)n[\vec{x}:\underline{\Sigma}];\ \vec{x}=(x_1,x_2,\cdots,x_d) \tag{33}$$

where

$$n[\vec{x}:\underline{\Sigma}] = \frac{1}{(2\pi)^{d/2}|\underline{\Sigma}|^{1/2}}\exp\left\{-\frac{1}{2}\vec{x}^\top\underline{\Sigma}^{-1}\vec{x}\right\}. \tag{34}$$

(Proof) Basically, we can make a similar discussion as in the proof of Lemma 2. Indeed we first note that the system of the following Hermite polynomials is a complete biorthogonal system in $L^2(\mathbf{R}^d,\mu)$:

$$\{H_{\vec{n}}(\vec{x}:\underline{\Sigma}):\vec{n}=(n_1,n_2,\cdots,n_d);n_i=0,1,2\cdots,(i=1,2,\cdots,d)\},$$
$$\{\tilde{H}_{\vec{n}}(\vec{x}:\underline{\Sigma}):\vec{n}=(n_1,n_2,\cdots,n_d);n_i=0,1,2\cdots,(i=1,2,\cdots,d)\},$$

where $H_{\vec{n}}(\vec{x}:\underline{\Sigma})$ is given by (33) and $\tilde{H}_{\vec{n}}(\vec{x}:\underline{\Sigma})$ is defined as follows:

$$\begin{aligned}\tilde{H}_{\vec{n}}(\vec{x};\underline{\Sigma}) &= \frac{1}{n[\vec{x}:\underline{\Sigma}]}\left(-\frac{\partial}{\partial_{y_1}}\right)\left(-\frac{\partial}{\partial_{y_2}}\right)\cdots\left(-\frac{\partial}{\partial_{y_d}}\right)n[\vec{x}:\underline{\Sigma}], \\ \vec{y} &= (y_1,y_2,\cdots,y_d)^\top=\underline{\Sigma}^{-1}\vec{x}.\end{aligned} \tag{35}$$

Thus, we have the following expansion of $E[X|\vec{Z}=\vec{x}]$ in $L^2(\mathbf{R}^d,\mu)$:

$$\mathbf{E}[X|\vec{Z}=\vec{x}]=\sum_{|\vec{n}|=0}^{\infty} a_{\vec{n}}H_{\vec{n}}(\vec{x};\underline{\Sigma}).$$

On the other hand, we know the relation:

$$\sum_{|\vec{j}|=0}^{\infty}\frac{(i\vec{\xi})^{\vec{j}}}{\vec{j}!}\tilde{H}_{\vec{j}}(\vec{x};\underline{\Sigma})=e^{i\vec{\xi}^\top\vec{x}}e^{\frac{1}{2}\vec{\xi}^\top\underline{\Sigma}\vec{\xi}}, \tag{36}$$

where $(i\vec{\xi})^{\vec{j}}=(i\xi_1)^{j_1}(i\xi_2)^{j_2}\cdots(i\xi_d)^{j_d}$. Hence,

$$e^{i\vec{\xi}^\top\vec{x}}=e^{-\frac{1}{2}\vec{\xi}^\top\underline{\Sigma}\vec{\xi}}\sum_{|\vec{j}|=0}^{\infty}\frac{(i\vec{\xi})^{\vec{j}}}{\vec{j}!}\tilde{H}_{\vec{j}}(\vec{x}:\underline{\Sigma}).$$

It is also well known that

$$\int_{\mathbf{R}^d}H_{\vec{m}}(\vec{x}:\underline{\Sigma})\tilde{H}_{\vec{n}}(\vec{x}:\underline{\Sigma})n[\vec{x}:\underline{\Sigma}]d\vec{x}=\begin{cases}\vec{m}!\ (\text{if } \vec{m}=\vec{n}),\\ 0\ (\text{if } \vec{m}\neq\vec{n}).\end{cases} \tag{37}$$

Therefore,

$$
\begin{aligned}
e^{\frac{1}{2}\vec{\xi}^\top \Sigma \vec{\xi}} \mathbf{E}\left[e^{i\vec{\xi}^\top \vec{Z}} X\right] &= e^{\frac{1}{2}\vec{\xi}^\top \Sigma \vec{\xi}} \mathbf{E}\left[e^{i\vec{\xi}^\top \vec{Z}} \mathbf{E}\left[X|\vec{Z}\right]\right] \\
&= \int_{\mathbf{R}^d} \left\{\sum_{|\vec{j}|=0}^{\infty} \tilde{H}_{\vec{j}}(\vec{x}:\underline{\Sigma}) \frac{(i\vec{\xi})^{\vec{j}}}{\vec{j}!}\right\} \left\{\sum_{|\vec{n}|=0}^{\infty} a_{\vec{n}} H_{\vec{n}}(\vec{x}:\underline{\Sigma})\right\} \mu(d\vec{x}) \qquad (38) \\
&= \sum_{|\vec{n}|=0}^{\infty} a_{\vec{n}} i^{|\vec{n}|} (\vec{\xi})^{\vec{n}};\ ((\vec{\xi})^{\vec{n}} = \xi_1^{n_1}\xi_2^{n_2}\cdots\xi_d^{n_d}), \qquad (39)
\end{aligned}
$$

and making $\vec{n} = (n_1, \cdots, n_d)$-th order differentiation of both sides in the equation above with respect to $\vec{\xi} = (\xi_1, \cdots, \xi_d)$ at $\vec{\xi} = \vec{0}$, we obtain (32) and hence the result, (31) - (34).

3.4 Expansion of Option Prices

Now, we apply the approximate density function in Theorem 2 obtained by the asymptotic expansion technique to option pricing.

In particular, we consider a plain vanilla option on the underlying asset price process $(g(X_t^{(\epsilon)}))_{t\in[0,T]}$, where $(X_t^{(\epsilon)})_{t\in[0,T]}$ is the solution to the stochastic differential equation expressed as the equation (1). As an example, we obtain an approximation of a call option price as follows.

Theorem 3. *An asymptotic expansion up to the $\epsilon^{(N+1)}$-order of a call option price at time 0 with maturity T and strike price K where $K = g(X_T^{(0)}) - \epsilon y$ for arbitrary $y \in \mathbf{R}$ is given as follows:*

$$
\begin{aligned}
C(K,T) &= \epsilon P(0,T)\left[\sqrt{\Sigma_T}\ n\left(\frac{y+C}{\sqrt{\Sigma_T}}\right) + CN\left(\frac{y+C}{\sqrt{\Sigma_T}}\right) + yN\left(\frac{y+C}{\sqrt{\Sigma_T}}\right)\right] \qquad (40) \\
&+ \sum_{n=1}^{N} \epsilon^{n+1} P(0,T) C_{n0}\left[\sqrt{\Sigma_T}\ n\left(\frac{y+C}{\sqrt{\Sigma_T}}\right) + CN\left(\frac{y+C}{\sqrt{\Sigma_T}}\right)\right] \\
&+ \sum_{n=1}^{N} \epsilon^{n+1} P(0,T) C_{n1}\left[\Sigma_T\ N\left(\frac{y+C}{\sqrt{\Sigma_T}}\right) - \sqrt{\Sigma_T}\ y\ n\left(\frac{y+C}{\sqrt{\Sigma_T}}\right)\right] \\
&+ \sum_{n=1}^{N} \epsilon^{n+1} P(0,T) \sum_{m=2}^{3n} C_{nm}\left[-y\sqrt{\Sigma_T}\ H_{m-1}(-(y+C);\Sigma_T)\ n\left(\frac{y+C}{\sqrt{\Sigma_T}}\right) + \Sigma_T^{\frac{3}{2}}\ H_{m-2}(-(y+C);\Sigma_T)\ n\left(\frac{y+C}{\sqrt{\Sigma_T}}\right)\right] \\
&+ y\sum_{n=1}^{N} \epsilon^{n+1} P(0,T)\ C_{n0} N\left(\frac{y+C}{\sqrt{\Sigma_T}}\right) \\
&+ y\sum_{n=1}^{N} \epsilon^{n+1} P(0,T) \sum_{m=1}^{3n} C_{nm}\quad \sqrt{\Sigma_T} H_{m-1}(-(y+C);\Sigma_T)\ n\left(\frac{y+C}{\sqrt{\Sigma_T}}\right) + o(\epsilon^{(N+1)}).
\end{aligned}
$$

Here, C_{nm} is given by (29), and $H_n(x;\Sigma)$ is the Hermite polynomial of degree n with parameter Σ, which is defined as

$$
H_n(x;\Sigma) = (-\Sigma)^n e^{x^2/2\Sigma} \frac{d^n}{dx^n} e^{-x^2/2\Sigma}.
$$

C and Σ_T are given respectively by

$$
C = \left(\partial g(X_T^{(0)})\right)^\top \int_0^T Y_T Y_t^{-1} \partial_\epsilon V_0(X_t^{(0)}, 0) dt
$$

and

$$
\Sigma_T = \int_0^T \hat{V}(X_t^{(0)}) \hat{V}(X_t^{(0)})^\top dt,
$$

where

$$\hat{V}(X_t^{(0)}) = (\partial g(X_T^{(0)}))^{\top} Y_T Y_t^{-1} V(X_t^{(0)}).$$

Also, $P(0,T)$ denotes the price at time 0 of a zero coupon bond with maturity T. $N(x)$ stands for the standard normal distribution function, and its density function is given by $n(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

(Proof) We firstly note that the call price is expanded as follows:

$$\begin{aligned} C(K,T) &= P(0,T)\mathbf{E}[\max\{g(X_T^{(\epsilon)}) - K, 0\}] \\ &= \epsilon P(0,T)\mathbf{E}\left[\max\left\{\left(\frac{g(X_T^{(\epsilon)}) - g(X_T^{(0)})}{\epsilon}\right) + \left(\frac{g(X_T^{(0)}) - K}{\epsilon}\right), 0\right\}\right] \\ &= \epsilon P(0,T)\mathbf{E}\left[\max\left\{G^{(\epsilon)} + y, 0\right\}\right] \\ &= \epsilon P(0,T)\int_{-y}^{\infty} (x+y) f_{G^{(\epsilon)},N}(x)dx + o(\epsilon^{(N+1)}). \end{aligned} \tag{41}$$

Here, $f_{G^{(\epsilon)},N}$ is the asymptotic expansion of the density of $G^{(\epsilon)}$ up to ϵ^N-order, which is given by the first two terms on the right hand side of (26) in Theorem 2:

$$f_{G^{(\epsilon)},N}(x) = f_{g_{1T}}(x) + \sum_{n=1}^{N} \epsilon^n \left(\sum_{m=0}^{3n} C_{nm} H_m(x - C; \Sigma_T)\right) f_{g_{1T}}(x), \tag{42}$$

where

$$f_{g_{1T}}(x) = \frac{1}{\sqrt{2\pi\Sigma_T}} \exp\left(-\frac{(x-C)^2}{2\Sigma_T}\right).$$

Next, we note the well-known properties of the Hermite polynomials:

$$\begin{aligned} &\frac{d}{dx} H_n(x;\Sigma) = n H_{n-1}(x;\Sigma) \\ &\frac{d^m}{dx^m}\{H_n(x;\Sigma) n(x;\Sigma)\} = \left(\frac{-1}{\Sigma}\right)^m H_{n+m}(x;\Sigma) n(x;\Sigma) \\ &H_{n+1}(x;\Sigma) = x H_n(x;\Sigma) - \Sigma n H_{n-1}(x;\Sigma), \end{aligned} \tag{43}$$

where $n(x;\Sigma) = \frac{1}{\sqrt{2\pi\Sigma}} e^{-\frac{x^2}{2\Sigma}}$.

Then, we can obtain the following expressions for the Integrals appearing on the right hand side of (41):

$$\begin{aligned} &\int_{-y}^{\infty} f_{g_{1T}}(x)dx = N\left(\frac{y+C}{\sqrt{\Sigma_T}}\right), \\ &\int_{-y}^{\infty} x f_{g_{1T}}(x)dx = \sqrt{\Sigma_T}\; n\left(\frac{y+C}{\sqrt{\Sigma_T}}\right) + CN\left(\frac{y+C}{\sqrt{\Sigma_T}}\right), \\ &\int_{-y}^{\infty} H_m(x - C;\Sigma_T) f_{g_{1T}}(x)dx = \sqrt{\Sigma_T} H_{m-1}(-(y+C);\Sigma_T)\; n\left(\frac{y+C}{\sqrt{\Sigma_T}}\right);\; m \geq 1, \\ &\int_{-y}^{\infty} x H_m(x - C;\Sigma_T) f_{g_{1T}}(x)dx = -\sqrt{\Sigma_T} y\; H_{m-1}(-(y+C);\Sigma_T)\; n\left(\frac{y+C}{\sqrt{\Sigma_T}}\right) \\ &+\Sigma_T^{\frac{3}{2}}\; H_{m-2}(-(y+C);\Sigma_T)\; n\left(\frac{y+C}{\sqrt{\Sigma_T}}\right);\; m \geq 2. \end{aligned} \tag{44}$$

□

Remark 5. *In practical applications, usually the underlying model is given as a non-perturbed form:*

$$\begin{aligned} d\hat{X}_t^j &= \hat{V}_0^j(\hat{X}_t)dt + \hat{V}^j(\hat{X}_t)dW_t \quad (j = 1, \cdots, d) \\ \hat{X}_0 &= x_0 \in \mathbf{R}^d. \end{aligned} \tag{45}$$

Then, in order to apply the asymptotic expansion method, we may rewrite the model for instance, as

$$\begin{aligned} dX_t^{(\epsilon),j} &= V_0^j(X_t^{(\epsilon)})dt + \epsilon V^j(X_t^{(\epsilon)})dW_t \quad (j = 1, \cdots, d) \\ X_0^{(\epsilon)} &= x_0 \in \mathbf{R}^d, \end{aligned} \tag{46}$$

where by rescaling $\hat{V}^j(x)$ we set $V^j(x)$ so that $\hat{V}^j(x) = \epsilon V^j(x)$ for some $\epsilon \in (0, 1]$. Consequently, an approximate call price under the original model (45) is obtained by (40) without $o(\epsilon^{N+1})$.

3.5 Application to Computation of Greeks

We already have a so called closed form approximate formula (40) for the option price, and hence are able to obtain approximations of its Greeks (that is, sensitivities to the changes in parameters in a model) as closed forms as well (or at least with easy numerical method such as the difference quotient method with the approximate option pricing formula).

For instance, [68] implements direct differentiations of the approximate formulas for option values under a time-homogeneous general local volatility model, and obtains closed form approximate formulas for the Deltas and Vegas. Moreover, [68] applies the similar technique to computing the Deltas and Vegas for average options with continuous monitoring, and gets their closed form approximate formulas as well. They also confirms the validity of the approximations through numerical experiments in the CEV model.

By deriving asymptotic expansions of characteristic functions of option values, [93] and [94] propose a new expansion scheme for pricing options on long-term currencies under a Libor market model (LMM) and a general diffusion stochastic volatility model with jump of spot exchange rates. Furthermore, applying the approximate formulas, they provide analytical (closed form) approximations for the Deltas and Gammas of the options. Please see [93] and [94] for the detail.

Alternatively, for a parameter θ, the sensitivity of a call price $C(K, T)$ with respect to the change in θ is expressed as follows:

$$\begin{aligned} \frac{\partial}{\partial \theta} C(K,T) &= P(0,T)\mathbf{E}[\max\{g(X_T^{(\epsilon)}) - K, 0\}] \\ &= \frac{\partial}{\partial \theta}\left(\epsilon P(0,T)\mathbf{E}\left[\max\left\{G^{(\epsilon)} + y, 0\right\}\right]\right) \\ &= \left(\frac{\partial}{\partial \theta}\{\epsilon P(0,T)\}\right)\mathbf{E}\left[\max\left\{G^{(\epsilon)} + y, 0\right\}\right] + \epsilon P(0,T)\left(\frac{\partial}{\partial \theta}\mathbf{E}\left[\max\left\{G^{(\epsilon)} + y, 0\right\}\right]\right) \\ &= \left(\frac{\partial\{\epsilon P(0,T)\}}{\partial \theta}\right)\frac{C_{AE}(K,T)}{\epsilon P(0,T)} + \epsilon P(0,T)\mathbf{E}\left[\left(\frac{\partial G^{(\epsilon)}}{\partial \theta} + \frac{\partial y}{\partial \theta}\right)1_{\{G^{(\epsilon)} > -y\}}\right], \end{aligned} \tag{47}$$

where $C_{AE}(K, T)$ stands for the approximate call price with strike K and maturity T, which is obtained by the asymptotic expansion.

Then, we are able to obtain an approximation of the sensitivity by a direct application of the asymptotic expansion to the above equation, particularly, the second term in the last equation. For example, under one dimensional diffusion setting, that is a general time homogeneous local volatility model, [66] successfully applies the expansion technique to computation of the Deltas and the Vegas with numerical experiments.

More generally, we note that the similar method as in option pricing in the previous subsection can be applied in Greeks, since we can take $\Phi \in \mathcal{S}'$ for $\mathbf{E}[\Phi(G^{(\epsilon)})]$ in (11) and apply the integration-by-parts method in Malliavin calculus. Recently, [103] takes this approach and derives asymptotic expansions of Greeks around the Black-Scholes model in stochastic volatility environment, and develop a unified method for precise estimates of the expansion errors. Particularly, they make use of the so called Kusuoka-Stroock functions introduced by Kusuoka [52], which is a powerful tool to clarify the order of a Wiener functional with respect to the time parameter t in a unified manner. Then, they estimate the error bounds for the Malliavin weights of both the coefficient and the residual terms in the expansions.

3.6 Approximations of Asset Values under Diffusion Processes

The framework of the asymptotic expansion can be applied not only to the simple cases mentioned above, but also to evaluation of much broader range of asset and security values. In particular, there are many cases where the asymptotic expansion can be applied to approximate their values when the underlying asset prices of financial securities, cash flows and interest rates are expressed as some functions of a random vector $X^{(\epsilon)}$ that follows a diffusion process. The method is almost the same as the one illustrated above and hence it is omitted. In this subsection, we only review how to represent the values of financial assets.

First, just as in the previous subsections, we consider a d-dimensional diffusion process $X^{(\epsilon)}$ defined as the strong solution to the stochastic differential equation (1). As an example, the present value V of a financial asset which generates a cash flow at the maturity date T is represented as

$$V = \mathbf{E}\left[e^{-\int_0^T R_2(X_u^{(\epsilon)})du} F(g(X_T^{(\epsilon)}))\right], \tag{48}$$

where g denotes the underlying asset price and F is the cash flow which characterizes the asset to be evaluated. Note that the underlying asset price g follows a diffusion process, whose drift term (the coefficient of the dt term) is $R_1(X_t^{(\epsilon)})g - D(X_t^{(\epsilon)})$ under an equivalent martingale measure. Moreover, R_1 at time $t \in [0,T]$ is represented as

$$R_1(X_t^{(\epsilon)}) = r(X_t^{(\epsilon)}) + \sum_{j=1}^{J_1} s_{1j}(X_t^{(\epsilon)}),$$

where r denotes the risk-free interest rate and s_{1j}, $j = 1, \cdots, J_1$ stand for various spreads (the differences from the risk-free rate) such as credit spreads and liquidity spreads. Suppose also that those are expressed as functions of the variable $X^{(\epsilon)}$. Further, $D(X_t^{(\epsilon)})$ denotes a payoff generated by the underlying asset such as a dividend or an interest rate and is also represented as a function of the variable $X^{(\epsilon)}$. Meanwhile, the discount rate at time t that is, $R_2(X_t^{(\epsilon)})$ of the target asset F to be evaluated is also expressed as

$$R_2(X_t^{(\epsilon)}) = r(X_t^{(\epsilon)}) + \sum_{j=1}^{J_2} s_{2j}(X_t^{(\epsilon)}),$$

where s_{2j}, $j = 1, \cdots, J_2$ are various spreads related to the objective asset or security. We again assume that those are expressed as some functions of the variable $X^{(\epsilon)}$.

As an example, let $F = 1$ in (48) for a zero-coupon bond with the face value 1 and the maturity date T. Also, let V_i denote the price of the zero-coupon bond with the maturity T_i. Then, V, the value of a coupon bond with the maturity T_N and coupon (and principal) payments c_i at $T_i (i = 1, \cdots, N,$ $T_1 < \cdots < T_N)$ is represented by the equation $V = \sum_{i=1}^N c_i V_i$. Moreover, the present value of a call option on the coupon bond with the option maturity $T (< T_1)$ can be evaluated if we set $F(x) = (x - K)^+$ and $g(X_T^{(\epsilon)}) = \sum_{i=1}^N c_i g_i(X_T^{(\epsilon)})$ in the equation (48), where $g_i(X_T^{(\epsilon)})$, $i = 1, \cdots, N$ are given by

$$g_i(X_T^{(\epsilon)}) = \mathbf{E}\left[e^{-\int_T^{T_i} R_1(X_u^{(\epsilon)})du} | X_T^{(\epsilon)}\right].$$

Finally, we briefly review applications of the asymptotic expansion technique to numerical problems in finance, which can not be introduced in the present note due to the limitation of the space.

[106] applies an asymptotic expansion to a dynamic investment problem with utility maximization for the asset at the end of the investment period, and derives an approximation formula for evaluating the optimal portfolio. Although the optimal portfolio has been numerically evaluated as a function of derivatives of the solution to some Bellman equation except for special cases, it is a hard task to implement it when the number of assets is large. [106] provides its approximation based on the representation which Ocone-Karatzas [74] derives by using the so called Clark-Ocone formula. Moreover, [45] applies this method to a dynamic bond portfolio problem.

In evaluation of the expectation of a Wiener functional based on Monte Carlo simulations, [107] proposes a new estimator with a control variate which has its expectation explicitly obtained by an asymptotic expansion, and has a high correlation with the target Wiener functional. The convergence of the simulation based on this estimator becomes much faster and the approximation error with the asymptotic expansion up to a low order such as the first or second order is decreased. As for the extension of this method, please see [51], [88], and [99].

For pricing American options, [89] extends a well-know decomposition formula for an American option value by Carr-Jarrow-Myneni [8], and proposes an approximation of the value by making use of the approximate density function of the underlying asset, which is obtained by the asymptotic expansion.

Moreover, because of its generality and unified nature of this approach with analytical (so called closed from) formulas, the asymptotic expansion method has been applied to broad class of valuation models which have become popular recently in practice. Especially, comparing to other numerical approximation schemes such as the Monte Carlo simulations and numerically solving methods for the partial differential equations (PDEs), it has an advantage in high dimensional problems. We list the following works as examples.

Applying the framework described above to default risk models, Muroi [67] derives asymptotic expansions for approximations of CDS (credit default swap) spreads.

[82] applies the expansion technique to obtain an approximation of swaption values under the Libor market model(LMM) of interest rates (Brace, Gatarek and Musiela [7], Jamshidian [43]) with local-stochastic volatility models.

[90], [91] and [92] develop asymptotic expansion formulas for pricing long-term currency options with a Libor market model(LMM) of interest rates and diffusion or jump-diffusion stochastic volatility processes of spot exchange rates. Moreover, [92] presents a new characteristic-function-based Monte Carlo simulation scheme with the asymptotic expansion as a control variate.

[96] develops a general computation scheme for a high-order expansion method explained in this section, and applies it to the SABR model(Hagan,Kumar,Lesniewski,and Woodward [33]). They derives the expansions of the option prices up to the fifth order to show that the higher order expansion improves the approximations.

[108] and [109] also apply this scheme to the long-term currency options such as the 10 year maturity one under a Libor market model(LMM) of interest rates and stochastic volatility processes of spot exchange rates. Again, they confirm that the fourth or the fifth order expansion provides the better approximations than the lower order ones.

Furthermore, we are able to apply the expansion method to pricing the so called *exotic* type options. For instance, [78] derives expansions of average options with discrete monitoring under stochastic volatility models in order to obtain approximate prices of commodities average options. Moreover, they implement calibration to real futures plain-vanilla option prices of the underlying commodities, and evaluate average options based on the parameters obtained by the calibration.

[83] develops new approximation formulas for pricing single and double barrier options with discrete monitoring under stochastic volatility models. In addition, they demonstrate its validity through numerical experiments.

[81] presents a new approximation scheme for pricing continuous barrier options in stochastic volatility environment. Particularly, they make use of a static hedging scheme and the fifth order expansions of the vanilla options to obtain accurate approximate prices. Further, they derives the fifth order expansions for pricing average options with continuous monitoring under stochastic volatility models to achieve very precise approximations.

[79] develops a general scheme for evaluation of the so called multi-asset cross currency options. In particular, they derive the expansions of basket option prices with 100 underlying assets (200 state variables with their stochastic volatilities), and cross currency average/basket options with discrete monitoring under stochastic volatility models to obtain accurate approximations.

[46] and [47] develop a new expansion scheme for solutions of Cauchy-Dirichlet problems for second order parabolic partial differential equations(PDEs) and apply it to pricing down-and-out/up-and-out barrier options with continuous monitoring under stochastic volatility models.

4 Extension

This section follows [97] which presents an extension of the general computational scheme of the asymptotic expansion described in the previous section. In particular, by a change of variable technique and by various ways of setting the perturbation parameters in the expansion, we are able to provide the flexibility of setting the benchmark distribution around which the expansion is made, and an automatic way for computation up to any order in the expansion. For instance we introduce expansions, called the log-normal expansion and the CEV expansion.

4.1 Change of Variable and Perturbation

We consider a d-dimensional diffusion process $X_t = (X_t^1, \cdots, X_t^d)$ which is the solution to the following stochastic differential equation:

$$\begin{aligned} dX_t^j &= V_0^j(X_t)dt + V^j(X_t)dW_t \quad (j = 1, \cdots, d) \\ X_0 &= x_0 \in \mathbf{R}^d \end{aligned} \tag{49}$$

where $W = (W^1, \cdots, W^r)$ is an r-dimensional standard Wiener process; $V_0^j : \mathbf{R}^d \mapsto \mathbf{R}$ and $V^j : \mathbf{R}^d \mapsto \mathbf{R}^d$ are smooth functions with bounded derivatives of all orders.

Next, let $C : \mathbf{R}^d \mapsto \mathbf{R}^d$ be a $\mathbf{C}^2$-function which has the unique inverse function, C^{-1}, and define $\tilde{X}_t$ as $\tilde{X}_t = C(X_t)$. Then, the dynamics of $\tilde{X}$ is given by

$$\begin{aligned} d\tilde{X}_t^j &= \tilde{V}_0^j(\tilde{X}_t)dt + \tilde{V}^j(\tilde{X}_t)dW_t \quad (j = 1, \cdots, d), \\ \tilde{X}_0 &= \tilde{x}_0, \end{aligned} \tag{50}$$

where

$$\begin{aligned} \tilde{V}_0^j(\tilde{x}) &:= \sum_{j'=1}^{d} \partial_{j'} C^j(C^{-1}(\tilde{x}))V_0^{j'}(C^{-1}(\tilde{x})) + \frac{1}{2}\sum_{j',k'=1}^{d} \partial_{j'k'} C^j(C^{-1}(\tilde{x}))V^{j'}(C^{-1}(\tilde{x}))V^{k'}(C^{-1}(\tilde{x}))^\top, \\ \tilde{V}^j(\tilde{x}) &:= \sum_{j'=1}^{d} \partial_{j'} C^j(C^{-1}(\tilde{x}))V^{j'}(C^{-1}(\tilde{x})), \end{aligned}$$

and $\tilde{x}_0 = C(x_0)$. ($(C^{-1}(\tilde{x}))^\top$ denotes the transpose of $(C^{-1}(\tilde{x}))$.)

Next, we introduce a perturbation parameter $\epsilon \in (0, 1]$ as follows:

$$\begin{aligned} &\tilde{X}_t \mapsto \tilde{X}_t^{(\epsilon)} \\ &\tilde{V}_0^j(\tilde{x}) \mapsto \tilde{V}_0^{(\epsilon),j}(\tilde{x}, \epsilon) \\ &\tilde{V}^j(\tilde{x}) \mapsto \epsilon\tilde{V}^j(\tilde{x}), \end{aligned}$$

and hence, the dynamics of $\tilde{X}^{(\epsilon)}$ is expressed as

$$d\tilde{X}_t^{(\epsilon),j} = \tilde{V}_0^{(\epsilon),j}(\tilde{X}_t^{(\epsilon)}, \epsilon)dt + \epsilon\tilde{V}^j(\tilde{X}_t^{(\epsilon)})dW_t \quad (j = 1, \cdots, d). \tag{51}$$

Then, we are able to apply the technique developed in the previous section to the transformed SDE (51).

4.2 Applications to Option Pricing under Local-Stochastic Volatility Model

We assume that the underlying process is the unique solution to the following SDE:

$$\begin{aligned} dS_t &= \sigma(X_t)h(S_t)dW_t \\ dX_t^j &= V_0^j(X_t)dt + V^j(X_t)dW_t \quad (j = 2, \cdots, d) \\ S_0 &= s_0 \in \mathbf{R}, \quad X_0 = x_0 \in \mathbf{R}^{d-1}, \end{aligned} \tag{52}$$

where $\sigma : \mathbf{R}^{d-1} \to \mathbf{R}^r$, $h : \mathbf{R} \to \mathbf{R}$, and W is a r-dimensional Brownian motion. Then, we evaluate a call option with strike K and maturity T, whose underlying price process is given by S. Under the zero discount interest rate for simplicity, the call price $Call(K,T)$ with strike price K and maturity T is obtained by

$$Call(K,T) = \mathbf{E}[(S_T - K)_+]. \tag{53}$$

First, for $x = (x^1, x^2, \cdots, x^d)$, let

$$C(x) = (C_1(x^1), x^2, \cdots, x^d),$$

where $C_1 : \mathbf{R} \to \mathbf{R}$ is an invertible $\mathbf{C}^2$-function. Then, $\tilde{S}_t = C_1(S_t)$, which $\tilde{S}$ follows a process of the solution to the following SDE:

$$d\tilde{S}_t = \frac{1}{2}|\sigma(X_t)|^2 h(C_1^{-1}(\tilde{S}_t))^2 C_1^{('')}(C_1^{-1}(\tilde{S}_t))dt + \sigma(X_t)h(C_1^{-1}(\tilde{S}_t))C_1^{(')}(C_1^{-1}(\tilde{S}_t))dW_t, \quad \tilde{s}_0 = C_1(s_0), \tag{54}$$

where $C_1^{(')}(x) := \frac{d}{dx}C_1(x)$ and $C_1^{('')}(x) := \frac{d^2}{dx^2}C_1(x)$.

Next, we introduce a perturbation parameter ϵ as follows:

$$\begin{aligned} d\tilde{S}_t^{(\epsilon)} &= \frac{\eta(\epsilon)}{2}|\sigma(X_t^{(\epsilon)})|^2 h(C_1^{-1}(\tilde{S}_t^{(\epsilon)}))^2 C_1^{('')}(C_1^{-1}(\tilde{S}_t^{(\epsilon)}))dt + \epsilon\sigma(X_t^{(\epsilon)})h(C_1^{-1}(\tilde{S}_t))C_1^{(')}(C_1^{-1}(\tilde{S}_t^{(\epsilon)}))dW_t, \\ dX_t^{(\epsilon),j} &= V_0^j(X_t^{(\epsilon)}, \epsilon)dt + \epsilon V^j(X_t^{(\epsilon)})dW_t \quad (j = 2, \cdots, d), \end{aligned} \tag{55}$$

where $\eta(\epsilon) = \epsilon^k$ and k is a nonnegative integer such as $k = 0, 1, 2, \cdots$. Note that

$$S_t = C_1^{-1}(\tilde{S}_t) = C_1^{-1}(\tilde{S}_t^{(1)}),$$

where $\tilde{S}_t^{(1)} = \tilde{S}_t^{(\epsilon)}|_{\epsilon=1}$.

According to Theorem 2 in the previous section, we have already an asymptotic expansion of the density function of $G^{(\epsilon)} = \frac{\tilde{S}_T^{(\epsilon)} - \tilde{S}_T^{(0)}}{\epsilon}$ up to ϵ^N-order, denoted by $f_{G^{(\epsilon)},N}(x)$.

Therefore, an approximation formula of the call price is given as follows:

$$\begin{aligned} Call(K,T) &= \mathbf{E}[(S_T - K)_+] = \mathbf{E}\left[\left(C_1^{-1}\left(\tilde{S}_T^{(1)}\right) - K\right)_+\right] \quad (56) \\ &\approx \int_y^\infty \left(C_1^{-1}(x + \tilde{S}_T^{(0)}) - K\right) f_{G^{(1)},N}(x)dx, \quad (57) \end{aligned}$$

where $y = C_1(K) - \tilde{S}_T^{(0)}$.

A simple example is the following. Set the local volatility function to be linear:

$$\begin{aligned} dS_t &= \sigma(X_t)S_t dW_t \\ dX_t^j &= V_0^j(X_t)dt + V^j(X_t)dW_t \quad (j = 2, \cdots, d). \end{aligned} \tag{58}$$

For $x = (x^1, x^2, \cdots, x^d)$, let

$$C(x) = (\log x^1, x^2, \cdots, x^d),$$

and set $\eta(\epsilon) = \epsilon^k$ where k is 0, 1 or 2. Then, we have $\tilde{S}_t^{(\epsilon)} = \log S_t^{(\epsilon)}$, where

$$\begin{aligned} d\tilde{S}_t^{(\epsilon)} &= -\frac{\epsilon^k}{2}\sigma(X_t^{(\epsilon)})^2 dt + \epsilon\sigma(X_t^{(\epsilon)})dW_t, \quad (59) \\ dX_t^{(\epsilon),j} &= V_0^j(X_t^{(\epsilon)}, \epsilon)dt + \epsilon V^j(X_t^{(\epsilon)})dW_t \quad (j = 2, \cdots, d). \end{aligned}$$

This case corresponds to some existing researches. (e.g. [91], [92], [95], [96], [100])

4.3 Examples

This subsection shows more specific examples in the local-stochastic volatility model.

4.3.1 CEV Model

The first example is on the well-known CEV (Constant Elasticity of Variance) model (Cox [10]) :

$$dS_t = \sigma(S_t^\beta S_0^{1-\beta})dW_t,\ \sigma \text{ and } S_0 \text{ are positive constants},\ \beta \in [0,1], \tag{60}$$

where the term $S_0^{1-\beta}$ makes the level of σ is of the same order for different β. For $x > 0$, let us take the change of variable function to be $C(x) = \log(x/S_0)$, that is $x = C^{-1}(\tilde{x}) = S_0 \exp(\tilde{x})$. Hence, $\tilde{S}_t = \log \frac{S_t}{S_0}$ and we have

$$d\tilde{S}_t = -\frac{1}{2}\sigma^2 e^{2(\beta-1)\tilde{S}_t}dt + \sigma e^{(\beta-1)\tilde{S}_t}dW_t;\ \tilde{S}_0 = 0. \tag{61}$$

Next, we introduce a perturbation $\epsilon \in [0,1]$, again as follows:

$$d\tilde{S}_t^{(\epsilon)} = -\frac{\eta(\epsilon)}{2}\sigma^2 e^{2(\beta-1)\tilde{S}_t^{(\epsilon)}}dt + \epsilon\sigma e^{(\beta-1)\tilde{S}_t^{(\epsilon)}}dW_t;\ \tilde{S}_0 = 0. \tag{62}$$

where $\eta(\epsilon) = \epsilon^j$ and j is a nonnegative integer.

Because

$$S_T = C^{-1}\left(\tilde{S}_T^{(1)}\right) = S_0 \exp\left(\tilde{S}_T^{(1}\right) = S_0 \exp\left(G^{(1)} + S_T^{(0)}\right),$$

an approximation formula of the call price with strike K and maturity T is given as follows:

$$\begin{aligned} Call(K,T) &= \mathbf{E}[(S_T - K)_+] = \mathbf{E}\left[\left(S_0 \exp\left(G^{(1)} + \tilde{S}_T^{(0)}\right) - K\right)_+\right] \\ &\approx \int_y^\infty \left(S_0 \exp\left(x + \tilde{S}_T^{(0)}\right) - K\right) f_{G^{(1)},N}(x)dx; \qquad (63) \\ y &= C(K) - \tilde{S}_T^{(0)} = \log\frac{K}{S_0} - \tilde{S}_T^{(0)}. \qquad (64) \end{aligned}$$

Note that $f_{g_{1T}}$, the first term in the asymptotic expansion of the density $f_{G^{(\epsilon)}}$ is a normal density and hence, the underlying asset price is expanded around a log-normal distribution. Thus, we could call this case a log-normal asymptotic expansion. We also remark that the case of $\eta(\epsilon) = \epsilon^0 = 1$ is harder to be evaluated than the other cases, which is essentially due to difficulty in computation of $\tilde{S}_t^{(0)}$ for $\eta(\epsilon) = 1$.

- **On the Validity of the Asymptotic Expansion for CEV model**

  Previous works such as [107], [85] and [86] have considered an asymptotic expansion of (average and vanilla) option prices based on the following type of a perturbed process: For $\beta \in [1/2, 1)$,

  $$dS_t^{(\epsilon)} = \epsilon(S_t^{(\epsilon)} \vee 0)^\beta dW_t;\ S_0^{(\epsilon)} = s_0. \tag{65}$$

  Although the coefficient function in this model is not smooth at 0, the asymptotic expansion method is still applicable. For instance, we could use a smooth modification technique.(e.g. [106], [107]). That is, let us take a modified process $(\tilde{S}_t^{(\epsilon)})_{t\in[0,T]}$ of $(S_t^{(\epsilon)})_{t\in[0,T]}$ as follows:

  $$d\tilde{S}_t^{(\epsilon)} = \epsilon g(\tilde{S}_t^{(\epsilon)})dW_t. \tag{66}$$

  Here, $g(x)$ is a smooth modification of $g(x) = (x \vee 0)^\beta$ such that $g(x) = x^\beta$ when $x \geq a_1$ for some small $a_1 \in (0, a)$ for $a = \frac{1}{2}s_0$ and $g(x) = 0$ when $x \leq a_2$ for some $a_2 \in (0, a_1)$. Specifically, we may set $g(x)$ as follows. For $t \in [0,T]$,

  $$\begin{aligned} g(x) &= h(x)x^\beta \\ h(x) &= \frac{\psi(x-a_2)}{\psi(x-a_2) + \psi(a_1 - x)},\ 0 < a_2 < a_1 \\ \psi(x) &= e^{-1/x} \text{ for } x > 0,\ \psi(x) = 0 \text{ for } x \leq 0. \end{aligned}$$

Suppose that for a **R**-valued function f, $E\left[\left|f(S^{(\epsilon)})\right|^2\right] < \infty$ and $E\left[\left|f(\tilde{S}^{(\epsilon)})\right|^2\right] < \infty$. (e.g. we can take option payoff functions as f in our setting.) Then, we have

$$E\left[\left|f(S^{(\epsilon)}) - f(\tilde{S}^{(\epsilon)})\right| 1_{\{S^{(\epsilon)}\neq\tilde{S}^{(\epsilon)}\}}\right] \leq \left(E\left[|f(S^{(\epsilon)})|^2\right]^{\frac{1}{2}} + E\left[|f(\tilde{S}^{(\epsilon)})|^2\right]^{\frac{1}{2}}\right) P\left(\{S^{(\epsilon)} \neq \tilde{S}^{(\epsilon)}\}\right)^{\frac{1}{2}}.$$

It also holds that

$$\begin{aligned}
&P\left(\{S^{\epsilon} \neq \tilde{S}^{\epsilon}\}\right) = P\left(\{S_t^{\epsilon} \leq a_1 \text{ for some } t \in [0,T]\}\right)\\
&\leq P\left(\{\sup_{0\leq t\leq T} |S_t^{\epsilon} - S_t^0| > a\}\right)\\
&+P\left(\{S_t^{\epsilon} \leq a_1 \text{ for some } t \in [0,T]\} \cap \{\sup_{0\leq t\leq T} |S_t^{\epsilon} - S_t^0| \leq a\}\right).
\end{aligned}$$

We can easily see that the second term after the last inequality is 0. The first term is smaller than any ϵ^n for $n = 1, 2, \cdots$ by the following lemma of a large deviation inequality:

Lemma 4. *Suppose that Z_t^{ϵ}, $t \in [0,T]$ follows a process of the solution to the SDE:*

$$dZ_t^{\epsilon} = \mu(Z_t^{\epsilon})dt + \epsilon\sigma(Z_t^{\epsilon})dW_t.$$

where $\mu(z)$ satisfies the Lipschitz and linear growth conditions, and $\sigma(z)$ satisfies the linear growth condition. We assume that the unique strong solution exists. Then, there exists positive constants c_1 and c_2 independent of ϵ such that

$$P(\{\sup_{0\leq s\leq T} |Z_s^{\epsilon} - Z_s^0| > c\}) \leq c_1 \exp(-c_2\epsilon^{-2}) \tag{67}$$

for all $c > 0$.

The lemma can be proved by slight modification of the lemma 5.3 in [119] or the lemma 7.1 in [50]. Note also that S^{ϵ} and $\tilde{S}^{\epsilon}$ satisfy the conditions in the lemma above.

Hence,

$$E\left[\left|f(S^{(\epsilon)}) - f(\tilde{S}^{(\epsilon)})\right|\right] = o(\epsilon^n),\ n = 1, 2, \cdots. \tag{68}$$

Therefore, the difference between $f(S^{(\epsilon)})$ and $f(\tilde{S}^{(\epsilon)})$ is negligible in a *small disturbance asymptotic theory*, and hence we could apply an asymptotic expansion to $E\left[f(\tilde{S}^{(\epsilon)})\right]$ instead of $E\left[f(S^{(\epsilon)})\right]$.

In particular, [107] considered the case that $\beta = 1/2$ and $f(x) = \left(\frac{1}{T}\int_0^T x_t dt - K\right)^+$, $x = S^{(\epsilon)}, \tilde{S}^{(\epsilon)}$ (an average call option's payoff). The similar modification could be applied to the asymptotic expansions for transformed processes in this section. Please also see [88] for numerical experiments under the smooth and bounded modification of this kind for volatility functions in a HJM-type model of interest rates.

4.3.2 SABR Model

Next, let us consider a stochastic volatility model so called SABR [33] (or λ-SABR [38])) Model:

$$\begin{aligned}
dS_t &= \sigma_t(S_t^{\beta} S_0^{1-\beta})dW_t^1;\ S_0 > 0,\\
d\sigma_t &= \lambda(\theta - \sigma_t)dt + \nu\sigma_t dW_t^2;\ \sigma_0 > 0
\end{aligned} \tag{69}$$

where $\beta \in [0,1]$, $\lambda \geq 0$, $\theta > 0$, $\nu > 0$, and $W = (W^1, W^2)$ is a two dimensional Wiener process with correlation $\rho \in [0,1]$.

- **Log-normal Asymptotic Expansion**

  Let us take a log-normal asymptotic expansion for the underlying asset price S, that is for $x_1 > 0$, set $C(x_1, x_2) = (\log(x_1/S_0), x_2)$ and $\tilde{S}_t = \log \frac{S_t}{S_0}$:

$$\begin{aligned} d\tilde{S}_t &= -\frac{1}{2}\sigma_t^2 e^{2(\beta-1)\tilde{S}_t} dt + \sigma_t e^{(\beta-1)\tilde{S}_t} dW_t^1;\ \tilde{S}_0 = 0 \\ d\sigma_t &= \lambda(\theta - \sigma_t)dt + \nu\sigma_t dW_t^2;\ \sigma_0 > 0. \end{aligned} \tag{70}$$

  Next, we introduce a perturbation $\epsilon \in [0, 1]$, again as follows:

$$\begin{aligned} d\tilde{S}_t^{(\epsilon)} &= -\frac{\eta_1(\epsilon)}{2}\sigma^2 e^{2(\beta-1)\tilde{S}_t^{(\epsilon)}} dt + \epsilon\sigma e^{(\beta-1)\tilde{S}_t^{(\epsilon)}} dW_t;\ \tilde{S}_0 = 0, \\ d\sigma_t^{(\epsilon)} &= \eta_2(\epsilon)\lambda(\theta - \sigma_t^{(\epsilon)})dt + \epsilon\nu\sigma_t^{(\epsilon)} dW_t^2;\ \sigma_0^{(\epsilon)} = \sigma_0, \end{aligned} \tag{71}$$

  where $\eta_i(\epsilon) = \epsilon^{j_i}$, $i = 1, 2$ and j_i is a nonnegative integer.

  For instance, typical cases are $\eta_2(\epsilon) = \epsilon^0 = 1$ with $\eta_2(\epsilon) = \epsilon$ (an extension of *the log-normal asymptotic expansion* in [95] and [100]), or $\eta_2(\epsilon) = \epsilon^2$ (an extension of [90] to the CEV-type local volatility).

  An approximation formula of the call price with strike K and maturity T is given as follows:

$$\begin{aligned} Call(K, T) &= \mathbf{E}[(S_T - K)_+] = \mathbf{E}\left[\left(S_0 \exp\left(G^{(1)} + \tilde{S}_T^{(0)}\right) - K\right)_+\right] \\ &\approx \int_y^\infty \left(S_0 \exp\left(x + \tilde{S}_T^{(0)}\right) - K\right) f_{G^{(1)},N}(x)dx; \end{aligned} \tag{72}$$

$$y = C(K) - \tilde{S}_T^{(0)} = \log\frac{K}{S_0} - \tilde{S}_T^{(0)}. \tag{73}$$

  Again, we note that the case of $\eta(\epsilon) = \epsilon^0 = 1$ is harder to be evaluated than the other cases, which results from difficulty in computation of $\tilde{S}_t^{(0)}$ for $\eta(\epsilon) = 1$.

- **CEV Asymptotic Expansion**

  Let us take change of variable function C as $C(x_1, x_2) = (C_1(x_1), x_2)$ for (x_1, x_2), where for $x > 0$ and $\beta \in [0, 1)$,

$$C_1(x) = \frac{1}{1-\beta}\frac{x^{1-\beta}}{S_0^{1-\beta}} \left(= \int^x \frac{dz}{z^\beta S_0^{1-\beta}}\right). \tag{74}$$

  That is,

$$C_1^{-1}(\tilde{x}) = S_0(1-\beta)^{\frac{1}{(1-\beta)}}\tilde{x}^{\frac{1}{(1-\beta)}}. \tag{75}$$

  Then, as $\tilde{S}_t = C_1(S_t)$, we have

$$\begin{aligned} d\tilde{S}_t &= -\frac{1}{2}\frac{\beta}{1-\beta}\sigma_t^2\frac{1}{\tilde{S}_t}dt + \sigma_t dW_t^1;\ \tilde{S}_0 = \frac{1}{1-\beta} > 0 \\ d\sigma_t &= \lambda(\theta - \sigma_t)dt + \nu\sigma_t dW_t^2;\ \sigma_0 > 0. \end{aligned} \tag{76}$$

  Again, we set a perturbed process as follows:

$$\begin{aligned} d\tilde{S}_t^{(\epsilon)} &= -\frac{\eta_1(\epsilon)}{2}\frac{\beta}{1-\beta}(\sigma_t^{(\epsilon)})^2\frac{1}{\tilde{S}_t^{(\epsilon)}}dt + \epsilon\sigma_t^{(\epsilon)} dW_t^1;\ \tilde{S}_0^{(\epsilon)} = \frac{1}{1-\beta} \\ d\sigma_t^{(\epsilon)} &= \eta_2(\epsilon)\lambda(\theta - \sigma_t^{(\epsilon)})dt + \epsilon\nu\sigma_t^{(\epsilon)} dW_t^2;\ \sigma_0^{(\epsilon)} = \sigma_0, \end{aligned} \tag{77}$$

  where $\eta_i(\epsilon) = \epsilon^{j_i}$, $i = 1, 2$ and j_i is a nonnegative integer.

For illustrative purpose, let us set $\eta_1(\epsilon) = \eta_2(\epsilon) = \epsilon$. That is,

$$d\tilde{S}_t^{(\epsilon)} = -\frac{\epsilon}{2}\frac{\beta}{1-\beta}(\sigma_t^{(\epsilon)})^2\frac{1}{\tilde{S}_t^{(\epsilon)}}dt + \epsilon\sigma_t^{(\epsilon)}dW_t^1;\ \tilde{S}_0^{(\epsilon)} = \frac{1}{1-\beta}, \tag{78}$$
$$d\sigma_t^{(\epsilon)} = \epsilon\lambda(\theta - \sigma_t^{(\epsilon)})dt + \epsilon\nu\sigma_t^{(\epsilon)}dW_t^2;\ \sigma_0^{(\epsilon)} = \sigma_0.$$

In this case, as $\tilde{S}_t^{(0)} = \frac{1}{1-\beta}$ and $\sigma_t^{(0)} = \sigma_0$ for all $t \in [0, T]$, the first two terms in the asymptotic expansion, $\tilde{g}_{1t} = \frac{1}{1-\beta} + \frac{\partial}{\partial\epsilon}\big|_{\epsilon=0}\tilde{S}_t^{(\epsilon)}$ follows a Gaussian process:

$$d\tilde{g}_{1t} = \frac{-\beta\sigma_0^2}{2}dt + \sigma_0 dW_t^1;\ \tilde{g}_{10} = \frac{1}{1-\beta}. \tag{79}$$

Then, by applying *Itô*'s formula to

$$\hat{g}_{1t} := C_1^{-1}(\tilde{g}_{1t}) = S_0(1-\beta)^{\frac{1}{(1-\beta)}}\tilde{g}_{1t}^{\frac{1}{(1-\beta)}}, \tag{80}$$

and using

$$\tilde{g}_{1t} = \frac{1}{1-\beta}\frac{\hat{g}_{1t}^{1-\beta}}{S_0^{1-\beta}}, \tag{81}$$

we formally obtain the SDE of $\hat{g}_{1t}$ though it is generally well-defined only for $\tilde{g}_{1t} \geq 0$:

$$d\hat{g}_{1t} = \frac{\beta\sigma_0^2 S_0^{1-\beta}}{2}\hat{g}_{1t}^{\beta}\left[-1 + S_0^{1-\beta}\hat{g}_{1t}^{\beta-1}\right]dt + \sigma_0 S_0^{1-\beta}\hat{g}_{1t}^{\beta}dW_t^1;\ \hat{g}_{10} = S_0. \tag{82}$$

Here, because the diffusion coefficient of $\hat{g}_{1t}$ is given by $\sigma_0 S_0^{1-\beta}(\hat{g}_{1t})^{\beta}$ and we may think that S is expanded around $\hat{g}_1$, we call this case a CEV asymptotic expansion (though $\hat{g}_1$ is not exactly a CEV process).

In particular, when $\beta = 1/2$,

$$d\hat{g}_{1t} = \frac{\sigma_0^2}{4}\left[-\sqrt{S_0\hat{g}_{1t}} + S_0\right]dt + \sigma_0\sqrt{S_0\hat{g}_{1t}}dW_t^1;\ \hat{g}_{10} = S_0, \tag{83}$$

and because

$$\hat{g}_{1T} = \frac{S_0}{4}\tilde{g}_{1T}^2, \tag{84}$$

$\hat{g}_{1T}/(S_0\sigma_0^2 T/4)$ follows a non-central χ^2 distribution, around which the original underlying asset price S_T is expanded.

Finally, for $\eta_i(\epsilon) = \epsilon^{j_i}$, $i = 1, 2$ and j_i is a nonnegative integer, an approximation formula of the call price with strike K and maturity T is obtained as follows:

$$\begin{aligned}
Call(K,T) &= \mathbf{E}[(S_T - K)_+] = \mathbf{E}\left[\left(C_1^{-1}(\tilde{S}_T) - K\right)_+\right] \\
&= \mathbf{E}\left[\left(\left\{S_0(1-\beta)^{\frac{1}{(1-\beta)}}(\tilde{S}_T)^{\frac{1}{(1-\beta)}}\right\} - K\right)_+\right] \\
&= \mathbf{E}\left[\left(\left\{S_0(1-\beta)^{\frac{1}{(1-\beta)}}(\tilde{S}_T^{(1)})^{\frac{1}{(1-\beta)}}\right\} - K\right)_+\right] \\
&= \mathbf{E}\left[\left(\left\{S_0(1-\beta)^{\frac{1}{(1-\beta)}}(G^{(1)} + \tilde{S}_T^{(0)})^{\frac{1}{(1-\beta)}}\right\} - K\right)_+\right] \\
&\approx \int_y^{\infty}\left(\left\{S_0(1-\beta)^{\frac{1}{(1-\beta)}}(x + \tilde{S}_T^{(0)})^{\frac{1}{(1-\beta)}}\right\} - K\right)f_{G^{(1)},N}(x)dx; \qquad (85)
\end{aligned}$$

$$y = C_1(K) - \tilde{S}_T^{(0)} = \frac{1}{1-\beta}\left(\frac{K}{S_0}\right)^{1-\beta} - \tilde{S}_T^{(0)}. \tag{86}$$

As numerical examples, [97] examines *normal*, *log-normal* and *CEV expansions* up to the third order for approximations of option prices under SABR model, which implies that *CEV expansion* provides the most stable approximations. We also observe that *CEV expansion* becomes more precise with the same level of absolute errors across the whole range of β along the higher order expansions. Thus, we expect a higher order *CEV expansion* will produce the better and more stable approximation than the other expansions, though further investigation seems necessary. Please see the original paper [97] for the detail of the numerical experiment.

Remark 6. *If necessary, applying a similar technique as mentioned in Section 4.3.1, we could use the asymptotic expansion for a model with smooth (and bounded) modification of the underlying processes. For a concrete example please see Remark 3 in [97].*

5 Improvement Scheme for Asymptotic Expansion

Although the asymptotic expansion up to the fifth order is known to be sufficiently accurate for option pricing (e.g. [81], [95], [96], [108], [109]), one of the main criticisms against the method would be that the approximate density function admits negative values typically at its tails that is, some region of the deep Out-of-The-Money (OTM), which could create an arbitrage opportunity in option trading. Also, even if the domain of a true density is restricted to be positive, the domain of its approximation may include negative values unless an appropriate boundary condition is assigned. To overcome the problems, we briefly introduce two recent researches related to the present asymptotic expansion approach.

5.1 New Improvement Scheme for Approximation Methods of Probability Density Functions

[98] develops a new scheme for improving density approximation methods, which also provides precise approximations of option values. Specifically, the scheme is inspired by the idea in in the Hilbert space projection theorem, and so called "Dykstra's cyclic projections algorithm" is applied for its implementation. (Please consult Deutsch [14] for the detail of the algorithm.) We also remark that the scheme can be easily implemented in practice, where we need only market data used for usual calibration such as option prices with strikes.

Furthermore, numerical experiments for vanilla option pricing under SABR model demonstrate the validity of the scheme. In fact, in terms of approximation accuracies this scheme improves the third and fifth order asymptotic expansions preserving the required conditions such as nonnegative densities under an appropriate forward measure.

We finally remark that the scheme is general and flexible enough to include a set of conditions and information as one would like to put on an approximate density, and it can be applied to approximation methods other than the asymptotic expansion method. For example, a number of researches have been going on in order to extend SABR model with fixing the problem of the negative densities in the method of [33]. (For instance, see Doust [15].) We note that the scheme is also a candidate for handling this issue. Also, the estimate of the absorption probability based on Monte Carlo simulations as in [15] can be consistently incorporated in the scheme.

5.2 A Weak Approximation with Asymptotic Expansion and Multidimensional Malliavin Weights

[105] develops a new weak approximation scheme for expectations of functions of the solutions to SDEs. In particular, the scheme connects approximate operators constructed based on the asymptotic expansion. More concretely, a diffusion semigroup is defined as the expectation of an appropriate function of the solution to a certain SDE: for example, $P_t^\epsilon f(x) = E[f(X_t^{x,\epsilon})]$ with the solution $X_t^{x,\epsilon}$ of a SDE with perturbation parameter ϵ and a function f. Then, we approximate P_t^ϵ by an operator $Q_t^{\epsilon,m}$ which is constructed based on the asymptotic expansion up to a certain order m. Thus, given a partition of $[0,T]$, $\pi = \{(t_0, t_1, \cdots, t_n) : 0 = t_0 < t_1 < \cdots < t_n = T\}$, we are able to approximate $P_T^\epsilon f(x)$ by connecting the

expansion-based approximations with the use of multi-dimensional Malliavin weights sequentially: that is, roughly speaking, with $s_k = t_k - t_{k-1}$, $k = 1, \cdots, n$,

$$P_T^{\epsilon} f(x) \quad \simeq \quad Q_{s_n}^{\epsilon,m} Q_{s_{n-1}}^{\epsilon,m} \cdots Q_{s_1}^{\epsilon,m} f(x).$$

The present research justifies this idea by applying Malliavin calculus, particularly, theories developed by Watanabe [111] and Kusuoka [52],[53],[54]. In computation, in order to evaluate the Malliavin weights, the paper makes use of conditional expectation formulas for multi-dimensional asymptotic expansions in [86].

Moreover, the paper shows through numerical examples for option pricing under local and stochastic volatility models that very few partition such as $n = 2$ is mostly enough to substantially improve the errors at deep OTMs of expansions with the first or second order ($m = 1, 2$).

6 Asymptotic Expansion in an Instantaneous Forward Rates Model

Among main stochastic models in finance, there exist models in which the stochastic processes of the underlying variables do not belong to the class of diffusion processes. This section illustrates an instantaneous forward rates model as a typical example.

6.1 Asymptotic Expansion for General Wiener Functionals

Watanabe [111] derives an asymptotic expansion for general Wiener functionals. As an example of the Watanabe's expansion, [100] shows the following result:

Theorem 4. *Let us consider a family of smooth Wiener functionals $F^{\epsilon} = (F_1^{\epsilon}, \cdots, F_n^{\epsilon})$, $F_i^{\epsilon} \in \mathbf{D}^{\infty}$ ($i = 1, \cdots, n$) such that F_i^{ϵ} has an asymptotic expansion in $\mathbf{D}^{\infty}$. Moreover, F^{ϵ} satisfies the uniformly non-degenerate condition:*

$$\limsup_{\epsilon \downarrow 0} \|(\det \sigma_{F^{\epsilon}})^{-1}\|_{L^p} < \infty, \text{ for all } p < \infty, \tag{87}$$

where $\sigma_{F^{\epsilon}}$ stands for the Malliavin covariance matrix of F^{ϵ}. Then, for a Schwartz distribution $T \in \mathcal{S}'(\mathbf{R}^n)$, we have an asymptotic expansion in $\mathbf{R}$:

$$\left| E[T(F^{\epsilon})] - \left\{ \int_{\mathbf{R}^n} T(x) p^{F^0}(x) dx + \sum_{j=1}^{N} \epsilon^j \int_{\mathbf{R}^n} T(x) E\left[\sum_{k}^{(j)} H_{\alpha^{(k)}} \left(F^0, \prod_{l=1}^{k} F_{\alpha_l}^{0,\beta_l} \right) | F^0 = x \right] p^{F^0}(x) dx \right\} \right| = O(\epsilon^{N+1}), \tag{88}$$

Equivalently,

$$\left| E[T(F^{\epsilon})] - \left\{ \int_{\mathbf{R}^n} T(x) p^{F^0}(x) dx + \sum_{j=1}^{N} \epsilon^j \sum_{k}^{(j)} (-1)^k \int_{\mathbf{R}^n} T(x) \partial_{\alpha^{(k)}}^k \left\{ E\left[\prod_{l=1}^{k} F_{\alpha_l}^{0,\beta_l} | F^0 = x \right] p^{F^0}(x) \right\} dx \right\} \right| = O(\epsilon^{N+1}), \tag{89}$$

where $F_i^{0,k} := \frac{1}{k!} \frac{d^k}{d\epsilon^k} F_i^{\epsilon}|_{\epsilon=0}$, $k \in \mathbf{N}$ ($i = 1, \cdots, n$), $\alpha^{(k)}$ denotes a multi-index, $\alpha^{(k)} = (\alpha_1, \cdots, \alpha_k)$ and

$$\sum_{k}^{(j)} \equiv \sum_{k=1}^{j} \sum_{\beta_1 + \cdots + \beta_k = j, \beta_i \geq 1} \sum_{\alpha^{(k)} \in \{1, \cdots, n\}^k} \frac{1}{k!}.$$

$p^{F^0}(x)$ stands for the density function of F^0. The Malliavin weight $H_{\alpha^{(k)}}$ is recursively defined as follows:

$$H_{\alpha^{(k)}}(F, G) = H_{(\alpha_k)}(F, H_{\alpha^{(k-1)}}(F, G)), \tag{90}$$

where

$$H_{(l)}(F,G) = D^* \left(\sum_{i=1}^{n} G\gamma_{li}^F DF_i \right). \tag{91}$$

Here, $F_i \in D^\infty$, $G \in D^\infty$, $D^* \left(\sum_{i=1}^n G\gamma_{li}^F DF_i\right)$ is the divergence of $\sum_{i=1}^n G\gamma_{li}^F DF_i$, DF_i is the Malliavin derivative of F_i, and $\gamma^F = \left(\gamma_{ij}^F\right)_{1\le i,j\le n}$ denotes the inverse matrix of the Malliavin covariance matrix of F. Moreover, we use the notation $\int T(x)g(x)dx$ for $T \in \mathcal{S}'(\mathbf{R}^n)$ and $g \in \mathcal{S}(\mathbf{R}^n)$ meaning that ${}_{\mathcal{S}'}\langle T, g\rangle_{\mathcal{S}}$. (See the section 2 of [100] for the details of those definitions.)

Remark 7. *The asymptotic expansion formula (89) is the formula developed by Watanabe [111]. Hence, this theorem shows the expansion (88) based on push down(conditional expectation) of Malliavin weights (divergences) is equivalent to the Watanabe's formula.*

(Proof) We use α as an abbreviation of $\alpha^{(k)}$ in the proof, and the notation $\langle \cdot, \cdot \rangle_{p^{F^0}(x)dx}$ is defined as follows:

$$\langle T, E^{F^0}[\cdot]\rangle_{p^{F^0}(x)dx} := {}_{\mathcal{S}'}\langle T, E^{F^0}[\cdot]\ p^{F^0}\rangle_{\mathcal{S}}.$$

Under the uniformly non-degenerate condition of $F^\epsilon \in \mathbf{D}^\infty(\mathbf{R}^n)$, the lifting up of $T \in \mathcal{S}'(\mathbf{R}^n)$ that is, $(E^{F^\epsilon})^*T$, has the asymptotic expansion in distributions on the Wiener space $\mathbf{D}^{-\infty}$, that is for $N \in \mathbf{N}$, there exists $s \in \mathbf{N}$ such that

$$\left\| (E^{F^\epsilon})^*T - \left\{ T\circ F^0 + \sum_{j=1}^{N} \epsilon^j \sum_{k}^{(j)} (\partial_\alpha^k T)\circ F^0 \prod_{l=1}^{k} F_{\alpha_l}^{0,\beta_l} \right\} \right\|_{\mathbf{D}_{q,-s}} = O(\epsilon^{N+1}), \quad \epsilon \in (0,1], q < \infty. \tag{92}$$

Then, there exists an asymptotic expansion of $\langle (E^{F^\epsilon})^*T, \mathbf{1}\rangle_{\mathbf{D}^{-\infty}\times\mathbf{D}^\infty}$.

The push-down of the divergence are computed as follows:

$$\begin{aligned}
\left\langle \partial_\alpha^k T(F^0), \prod_{l=1}^{k} F_{\alpha_l}^{0,\beta_l} \right\rangle_{\mathbf{D}^{-\infty}\times\mathbf{D}^\infty} &= \left\langle T(F^0), H_\alpha\left(F^0, \prod_{l=1}^{k} F_{\alpha_l}^{0,\beta_l}\right) \right\rangle_{\mathbf{D}^{-\infty}\times\mathbf{D}^\infty} \\
&= \left\langle T, E^{F^0}\left[H_\alpha\left(F^0, \prod_{l=1}^{k} F_{\alpha_l}^{0,\beta_l}\right)\right] \right\rangle_{p^{F^0}(x)dx} \\
&= \int_{\mathbf{R}^n} T(x) E\left[H_\alpha\left(F^0, \prod_{l=1}^{k} F_{\alpha_l}^{0,\beta_l}\right) | F^0 = x\right] p^{F^0}(x)dx. \qquad (93)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\left\langle \partial_\alpha^k T(F^0), \prod_{l=1}^{k} F_{\alpha_l}^{0,\beta_l} \right\rangle_{\mathbf{D}^{-\infty}\times\mathbf{D}^\infty} &= \left\langle \partial_\alpha^k T, E^{F^0}\left[\prod_{l=1}^{k} F_{\alpha_l}^{0,\beta_l} \right] \right\rangle_{p^{F^0}(x)dx} \\
&= \left\langle T, (\partial^*)_\alpha^k E^{F^0}\left[\prod_{l=1}^{k} F_{\alpha_l}^{0,\beta_l} \right] \right\rangle_{p^{F^0}(x)dx} \\
&= (-1)^k \int_{\mathbf{R}^n} T(x) \partial_\alpha^k \left\{ E\left[\prod_{l=1}^{k} F_{\alpha_l}^{0,\beta_l} | F^0 = x\right] p^{F^0}(x) \right\} dx. \qquad (94)
\end{aligned}$$

Here, $(\partial^*)_\alpha^k$ means $(\partial^*)_\alpha^k = \partial_\alpha^* \cdots \partial_\alpha^* (k$ times), and ∂_α^* denotes the divergence operator on the space $\left(\mathbf{R}^n, p^{F^0}(x)dx\right)$.

□

Corollary 1. *The asymptotic expansion of the density function of F^ϵ, $p^{F^\epsilon}(y)$ is expressed with the push-down of the Malliavin weights as the follows:*

$$p^{F^\epsilon}(y) = p^{F^0}(y) + \sum_{j=1}^{m} \epsilon^j E\left[\sum_{k}^{(j)} H_{\alpha^{(k)}}\left(F^0, \prod_{l=1}^{k} F^{0,\beta_l}_{\alpha_l}\right) |F^0 = y\right] p^{F^0}(y) + O(\epsilon^{m+1}), \tag{95}$$

where $p^{F^0}(y)$ is the density function of F^0. An alternative expression is given as follows:

$$p^{F^\epsilon}(y) = p^{F^0}(y) + \sum_{j=1}^{m} \epsilon^j \sum_{k}^{(j)} (-1)^k \partial^k_{\alpha^{(k)}} \left\{E\left[\prod_{l=1}^{k} F^{0,\beta_l}_{\alpha_l} |F^0 = y\right] p^{F^0}(y)\right\} + O(\epsilon^{m+1}). \tag{96}$$

(*Proof*) Take a delta function $\delta_y \in \mathcal{S}'(\mathbf{R}^n)$ in the theorem above. □

6.2 Instantaneous Forward Rates Model

As a typical stochastic model for pricing the interest rate derivatives, there exists a model developed by Heath-Jarrow-Morton [37], the so called HJM model, which is formulated based on the forward rates with infinitesimal terms of the interest rates, that is the instantaneous forward rates $\{f(s,t) : 0 \leq s \leq t \leq T\}$. Here, s is the time when the forward rate is fixed and t denotes the inception time when the forward rate is applied.

The stochastic processes for the instantaneous forward rates are considered in the framework of the asymptotic expansion by introducing a parameter $\epsilon \in [0,1]$. For example, let W be a m-dimensional standard Wiener process and let $f(0,t)$, $t \in [0,T]$ be a given Lipschitz continuous function of t. Then, under the equivalent martingale measure, the stochastic processes of $\{f^{(\epsilon)}(s,t) : 0 \leq s \leq t \leq T\}$ are solutions to the following stochastic integral equations:

$$\begin{aligned} f^{(\epsilon)}(s,t) = f(0,t) \quad &+ \quad \epsilon^2 \int_0^s \sum_{i=1}^{m} \left[\sigma_i(f^{(\epsilon)}(v,t),v,t) \int_v^t \sigma_i(f^{(\epsilon)}(v,y),v,y)dy\right] dv \\ &+ \quad \epsilon \sum_{i=1}^{m} \int_0^s \sigma_i(f^{(\epsilon)}(v,t),v,t)dW_i(v) \ ; \epsilon \in [0,1], \end{aligned} \tag{97}$$

where the volatility functions $\{\sigma_i(x,s,t); i = 1, \cdots, m\}$ are smooth and satisfy the regularity conditions which guarantee that the equation (97) has its unique strong solution. It is to be noted that the drift term (the coefficient of the dv term) of $f^{(\epsilon)}(s,t)$ depends on $\{f^{(\epsilon)}(v,y); 0 \leq v < s, v \leq y < t\}$. Moreover, the stochastic process of the instantaneous short-term interest rate $r^{(\epsilon)}(t)$ is determined by the relation, $r^{(\epsilon)}(t) = f^{(\epsilon)}(t,t)$.

For this model, the approximations of the values for interest rate derivatives can still be considered in a unified framework with derivation of asymptotic expansions of the instantaneous forward rates when $\epsilon \downarrow 0$ and with use of the relation between the instantaneous forward rates and a zero-coupon bond price:

$$P^{(\epsilon)}(t,T) = \exp\left\{-\int_t^T f^{(\epsilon)}(t,u)du\right\}. \tag{98}$$

As an example, we consider pricing an option on a coupon bond (or a swaption), which is a standard interest rate derivative. The payoff at maturity of a call option is given by

$$V_c(T) \quad = \quad \max\left\{\sum_{i=1}^{n} c_i P^{(\epsilon)}(t,T_i) - K, 0\right\},$$

where $0 \leq T \leq T_1 < \cdots < T_n$, c_i $(i = 1, \cdots, n)$ are positive constants and $K(> 0)$ is a strike price. Then, its present value is given by

$$V_c(0) \quad = \quad \mathbf{E}\left[e^{-\int_0^T r_u^{(\epsilon)} du} V_c(T)\right]. \tag{99}$$

When $\epsilon \downarrow 0$, the forward rate $f^{(\epsilon)}(s,t)$ is expanded around $f(0,t)$ as

$$f^{(\epsilon)}(s,t) \sim f(0,t) + \epsilon f_1(s,t) + \epsilon^2 f_2(s,t) + \cdots \ in\ \mathbf{D}^\infty, \tag{100}$$

where the coefficients of ϵ^n, $n = 1, 2, \cdots$, that is $f_1(t,u), f_2(t,u) \cdots$ are also in $\mathbf{D}^\infty$.

As a result, we obtain an expansion of the zero-coupon bond price $P^{(\epsilon)}(t,T)$ around the current forward bond price $\frac{P(0,T)}{P(0,t)}$, and an expansion of the discount factor $\exp\left\{-\int_0^T r^{(\epsilon)}(t)dt\right\}$ around the current zero-coupon bond price $P(0,T)$ as follows:

$$\begin{aligned} P^{(\epsilon)}(t,T) &\sim \frac{P(0,T)}{P(0,t)}\left[1 - \epsilon\int_t^T f_1(t,u)du - \epsilon^2\int_t^T f_2(t,u)du\right. \\ &+ \left.\epsilon^2\frac{1}{2}\left\{\int_t^T f_1(t,u)du\right\}^2\right] + \cdots \ in\ \mathbf{D}^\infty, \end{aligned} \tag{101}$$

$$\begin{aligned} e^{-\int_0^T r^{(\epsilon)}(s)ds} &\sim P(0,T)\left[1 - \epsilon\int_0^T f_1(t,t)dt - \epsilon^2\int_0^T f_2(t,t)dt\right. \\ &+ \left.\epsilon^2\frac{1}{2}\left\{\int_0^T f_1(t,t)dt\right\}^2\right] + \cdots \ in\ \mathbf{D}^\infty, \end{aligned} \tag{102}$$

where $f_i(s,t)$, $i = 1, 2$ are given by

$$\begin{aligned} f_1(s,t) &= \tfrac{\partial f^{(\epsilon)}(s,t)}{\partial \epsilon}|_{\epsilon=0} = \int_0^s \sum_{i=1}^m \sigma_i^{(0)}(v,t)dW_i(v), \\ f_2(s,t) &= \tfrac{1}{2}\tfrac{\partial^2 f^{(\epsilon)}(s,t)}{\partial^2 \epsilon}|_{\epsilon=0} \\ &= \int_0^s b^{(0)}(v,t)dv + \int_0^s \sum_{i=1}^m \partial\sigma_i^{(0)}(v,t) f_1(v,t)dW_i(v). \end{aligned}$$

Here, $\sigma_i^{(0)}(v,t) = \sigma_i(f^{(0)}(v,t),v,t)$, and $b^{(0)}(v,t)$ and $\partial\sigma_i^{(0)}(v,t)$ are defined as

$$\begin{aligned} b^{(0)}(v,t) &= \sum_{i=1}^n \sigma_i(f^{(0)}(v,t),v,t)\int_v^t \sigma_i(f^{(0)}(v,y),v,y)dy, \\ \partial\sigma_i^{(0)}(v,t) &= \frac{\partial\sigma_i(x,v,t)}{\partial x}|_{x=f(0,t)}. \end{aligned}$$

Therefore, in a similar way as in the framework for diffusion cases in the previous sections, we define $X_t^{(\epsilon),1}$ and $X_t^{(\epsilon),i}$ $(i = 2, \cdots, n)$ as

$$X_t^{(\epsilon),1} = \exp\left\{-\int_0^t r^{(\epsilon)}(u)du\right\} \tag{103}$$

$$X_t^{(\epsilon),i} = P^{(\epsilon)}(t,T_i) = \exp\left\{-\int_t^{T_i} f^{(\epsilon)}(t,u)du\right\},\ i = 2, \cdots, n. \tag{104}$$

Then, the payoff at maturity of the call option on a coupon bond is written as

$$V_c(T) = \max\left\{\sum_{i=2}^n c_i X_t^{(\epsilon),i} - K, 0\right\}. \tag{105}$$

Moreover, let $x = (x_1, x_2, \cdots, x_n)$ and define $g(x)$ as

$$g(x) = x_1\left(\sum_{i=2}^n c_i x_i - K\right). \tag{106}$$

In this way, we are able to employ a similar technique to pricing derivatives as in the case of diffusion processes. For example, with redefinition of variables such as Σ_T, the approximation of the option price $V_c(0)$ in (99) can be obtained based on the almost same asymptotic expansion method as in the previous sections. In fact, by using the above expansions of instantaneous forward rates, zero-coupon bond prices and the discount factor, we can apply the expansion to $\mathbf{E}[\max\{g(X_T^{(\epsilon)}), 0\}]$, where $X_T^{(\epsilon)} = (X_T^{(\epsilon),1}, X_T^{(\epsilon),2} \cdots, X_T^{(\epsilon),n})$.

For the details and numerical examples, please see [49], [50] and [88]. In particular, [88] implements numerical experiments under a smooth and bounded modification of two factor CEV-type volatility functions (as explained in Section 4.3.1), and the variance reduction technique in proposed in [107] to demonstrate the effectiveness of the method. We remark that the boundedness of the volatility functions $\{\sigma_i(x, s, t); i = 1, \cdots, m\}$ for the instantaneous forward rates $f^{(\epsilon)}(s, t)$ is one of the sufficient conditions that guarantee the existence of the unique strong solution of the stochastic integral equation (97).

For evaluation of other various interest rate derivatives, approximations based on the asymptotic expansion approach can be derived in the similar manner. Moreover, an example of an approximate formula for derivative prices dependent on the instantaneous forward rates in the HJM model and other variables following general diffusion processes is given by [85].

7 Asymptotic Expansion in Jump and Jump-Diffusion Models

So far, we have used stochastic models whose randomnesses are generated by only Wiener processes. However, we are also able to apply the asymptotic expansion approach to stochastic processes including jumps in their sample paths. This section provides its very brief review. For the details, please see the cited papers.

In terms of the mathematical viewpoint, Yoshida [120] presented an extension of Watanabe theory to develop a framework for providing a validity of asymptotic expansions in Wiener-Poisson spaces, which can be applied to jump-diffusion models under some regularity conditions. Hayashi [34] applied a Malliaivin calculus of jump-type to prove an asymptotic expansion theorem for functionals of a Poisson random measure, and Hayashi [35] derived the coefficients in the expansion of a call option price under a pure jump model. Moreover, Hayashi and Ishikawa [36] proved an asymptotic expansion formula for the compositions of a smooth Wiener-Poisson functional with Schwartz distributions.

In direct applications to finance problems, [51] and [87] derived asymptotic expansion to approximate bond prices or/and plain-vanilla option prices under jump-diffusion with local volatility models.

Subsequently, [93] and [94] found a new expansion scheme for pricing long-term European currency options under a Libor market model (LMM) and a general diffusion stochastic volatility model with jumps of spot exchange rates. Particularly, thanks to a linear structure of the underlying asset price process in their model, they separated the jump component with a known characteristic function to apply the expansion technique developed in the diffusion models. Also, [100] took a Malliavin calculus approach to derive asymptotic expansions of vanilla option prices in a jump-diffusion with stochastic volatility model.

Recently, [80] has generalized the preceding researches such as [51], [87] and [100] in the asymptotic expansion approach, and developed a new approximation formula for pricing basket options in a local-stochastic volatility model with jumps. In particular, the model admits local volatility functions and jump components in not only the underlying asset price processes, but also the volatility processes. Moreover, they implemented some numerical experiments to confirm the validity of the method. Please see the paper for the details.

As an example of asymptotic expansions of option prices under jump-diffusion models, the next subsection describes the outline of the method by using a simplified version of [80].

7.1 Pricing Basket Options under Local Stochastic Volatility with Jumps

In the first place, we define the model of the underlying asset prices and its volatility processes, which is used for pricing the European type basket options. In particular, suppose that the filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ is given, where P is an equivalent martingale measure and the filtration satisfies the usual conditions. The risk-free interest rate is assumed to be a nonnegative constant r for simplicity. Then, $(S_t^i)_{t\in[0,T]}$ and $(\sigma_t^i)_{t\in[0,T]}$, $i = 1, \cdots, d$ represent the underlying asset prices and their volatilities for $t \in [0,T]$, respectively. Particularly, let us assume that S_T^i and σ_T^i are given by the solutions of the following stochastic integral equations:

$$\begin{aligned} S_T^i &= s_0^i + \int_0^T \alpha^i S_{t-}^i dt + \int_0^T \phi_{S^i}\left(\sigma_{t-}^i, S_{t-}^i\right) dW_t^{S^i} \\ &\quad + \sum_{l=1}^n \left(\sum_{j=1}^{N_{l,T}} h_{S^i,l,j} S_{\tau_{j,l}-}^i - \int_0^T \Lambda_l S_{t-}^i \mathbf{E}[h_{S^i,l,j}] dt \right), \quad (107) \\ \sigma_T^i &= \sigma_0^i + \int_0^T \lambda^i(\theta^i - \sigma_{t-}^i) dt + \int_0^T \phi_{\sigma^i}\left(\sigma_{t-}^i\right) dW_t^{\sigma^i} \\ &\quad + \sum_{l=1}^n \left(\sum_{j=1}^{N_{l,T}} h_{\sigma^i,l,j} \sigma_{\tau_{j,l}-}^i - \int_0^T \Lambda_l \sigma_{t-}^i \mathbf{E}[h_{\sigma^i,l,j}] dt \right), \quad (108) \end{aligned}$$

where s_0^i and σ_0^i, $i = 1, \cdots, d$ are given as some constants. The notations are defined as follows:

- α^i $(i = 1, \cdots, d)$ are constants.
  λ^i and θ^i $(i = 1, \cdots, d)$ are nonnegative constants.
- $\phi_{S^i}(x, y)$ and $\phi_{\sigma^i}(x)$ are some functions with appropriate regularity conditions.
- W^{S^i} and W^{σ^i}, $(i = 1, \cdots, d)$ are correlated Brownian motions.
- Each N_l, $(l = 1, \cdots, n)$ is a Poisson process with constant intensity Λ_l. N_l, $l = 1, \cdots, n$ are independent, and also independent of all W^{S^i} and W^{σ^i}.
- $\tau_{j,l}$ stands for the j-th jump time of N_l.
- For each $l = 1, \cdots, n$ and $i = 1, \cdots, d$, both $\left(\sum_{j=1}^{N_{l,t}} h_{S^i,l,j}\right)_{t \geq 0}$ and $\left(\sum_{j=1}^{N_{l,t}} h_{\sigma^i,l,j}\right)_{t \geq 0}$ are compound Poisson processes. ($\sum_{j=1}^{N_{l,t}} \equiv 0$ when $N_{l,t} = 0$.)
- For each l and x^i, $h_{x^i,l,j}$ $(j \in \mathbf{N})$ are independent and identically distributed random variables, where x^i stands for one of S^i and σ^i $(i = 1, \cdots, d)$.
  - for the log-normal jump case, $h_{x^i,l,j} = e^{Y_{x^i,l,j}} - 1$, where
    $Y_{x^i,l,j}$ is a random variable which follows a normal distribution with mean $m_{x^i,l}$ and variance $\gamma_{x^i,l}^2$ that is, $N(m_{x^i,l}, \gamma_{x^i,l}^2)$ for all j.
- $h_{x^i,l,j}$ and $h_{x^{i'},l',j'}$ $(l \neq l')$ are independent.
  $h_{x^i,l,j}$ and $h_{x^{i'},l',j'}$ $(j \neq j')$ are independent.
  N_l and $h_{x^i,l',j}$ are independent.
  For the same l and j, $h_{S^i,l,j}$ and $h_{\sigma^{i'},l,j}$ $(i, i' = 1, \cdots, d)$ are allowed to be dependent, that is $Y_{S^i,l,j}$ and $Y_{\sigma^{i'},l,j}$ $(i, i' = 1, \cdots, d)$ are generally correlated.

Remark 8. *By specifying the functions ϕ_S and ϕ_σ, we can express various types of local-stochastic volatility models. For example, the model with $\phi_S(\sigma, S) = (aS^2 + bS + c)\sqrt{\sigma}$ and $\phi_\sigma(\sigma) = \sqrt{\sigma}$ corresponds to an extension of the Quadratic Heston model. The model with $\phi_S(\sigma, S) = S^{\beta_S}\sigma$ and $\phi_\sigma(\sigma) = \sigma$ corresponds to an extended SABR (λ-SABR) model, and the one with $\phi_S(\sigma, S) = S^{\beta_S}\sigma$ and $\phi_\sigma(\sigma) = \sigma^{\beta_\sigma}$ corresponds to a local volatility on volatility with jumps model.*

Next, we introduce perturbations to the model (107) and (108). That is, for a known parameter $\epsilon \in [0,1]$ we consider the following stochastic integral equations: for $i = 1, \cdots, d$,

$$\begin{aligned} S_T^{i,(\epsilon)} &= s_0^i + \int_0^T \alpha^i S_{t-}^{i,(\epsilon)} dt + \epsilon \int_0^T \phi_{S^i}\left(\sigma_{t-}^{i,(\epsilon)}, S_{t-}^{i,(\epsilon)}\right) dW_t^{S^i} \\ &\quad + \sum_{l=1}^n \left(\sum_{j=1}^{N_{l,T}} h_{S^i,l,j}^{(\epsilon)} S_{\tau_{j,l}-}^{i,(\epsilon)} - \int_0^T \Lambda_l S_{t-}^{i,(\epsilon)} \mathbf{E}[h_{S^i,l,1}^{(\epsilon)}] dt \right), \end{aligned} \tag{109}$$

$$\begin{aligned} \sigma_T^{i,(\epsilon)} &= \sigma_0^i + \int_0^T \lambda^i(\theta^i - \sigma_{t-}^{i,(\epsilon)}) dt + \epsilon \int_0^T \phi_{\sigma^i}\left(\sigma_{t-}^{i,(\epsilon)}\right) dW_t^{\sigma^i} \\ &\quad + \sum_{l=1}^n \left(\sum_{j=1}^{N_{l,T}} h_{\sigma^i,l,j}^{(\epsilon)} \sigma_{\tau_{j,l}-}^{i,(\epsilon)} - \int_0^T \Lambda_l \sigma_{t-}^{i,(\epsilon)} \mathbf{E}[h_{\sigma^i,l,1}^{(\epsilon)}] dt \right), \end{aligned} \tag{110}$$

where $h_{x^i,l,j}^{(\epsilon)} = e^{\epsilon Y_{x^i,l,j}} - 1$, that is, we assume that the jump size follows a log-normal distribution, $\epsilon Y_{x^i,l,j} \sim N(\epsilon m_{x^i,l}, \epsilon^2 \gamma_{x^i,l}^2)$.

We assume the asymptotic expansions of $S_T^{i,(\epsilon)}$ and $\sigma_T^{i,(\epsilon)}$ around $\epsilon = 0$ as follows:

$$S_T^{i,(\epsilon)} = S_T^{i,(0)} + \epsilon S_T^{i,(1)} + \frac{\epsilon^2}{2!} S_T^{i,(2)} + \cdots, \tag{111}$$

$$\sigma_T^{i,(\epsilon)} = \sigma_T^{i,(0)} + \epsilon \sigma_T^{i,(1)} + \frac{\epsilon^2}{2!} \sigma_T^{i,(2)} + \cdots, \tag{112}$$

$$h_{x^i,l,j}^{(\epsilon)} = h_{x^i,l,j}^{(0)} + \epsilon h_{x^i,l,j}^{(1)} + \frac{\epsilon^2}{2!} h_{x^i,l,j}^{(2)} + \cdots, \tag{113}$$

where $S_t^{i,(\iota)} := \frac{\partial^\iota S_t^{i,(\epsilon)}}{\partial \epsilon^\iota}\Big|_{\epsilon=0}$, $\sigma_t^{i,(\iota)} := \frac{\partial^\iota \sigma_t^{i,(\epsilon)}}{\partial \epsilon^\iota}\Big|_{\epsilon=0}$, $h_{x^i,l,j}^{(\iota)} := \frac{\partial^\iota h_{x^i,l,j}^{(\epsilon)}}{\partial \epsilon^\iota}\Big|_{\epsilon=0}$.

We also suppose that $(W^{S^1}, \cdots, W^{S^d}, W^{\sigma^1}, \cdots, W^{\sigma^d})' = \varrho \cdot Z$ where ϱ is a $2d \times 2d$ correlation matrix, and Z is a $2d$-dimensional (independent) Brownian motion.

For ease of the expressions we introduce the following notations:

- $\Phi_{S^i,j} := \phi_{S^i}(\sigma^i, S^i)(\varrho)_{i,j}$ and $\Phi_{\sigma^i,j} := \phi_{\sigma^i}(\sigma^i)(\varrho)_{d+i,j}$, where $(\varrho)_{i,j}$ denotes the (i,j)-element of ϱ.
- $\Phi_{S^i} := (\Phi_{S^i,1}, \cdots, \Phi_{S^i,2d})$ and $\Phi_{\sigma^i} := (\Phi_{\sigma^i,1}, \cdots, \Phi_{\sigma^i,2d})$ are $2d$-dimensional vectors.
- $\Phi_S := (\Phi_{S^1}, \cdots, \Phi_{S^d})'$ and $\Phi_\sigma := (\Phi_{\sigma^1}, \cdots, \Phi_{\sigma^d})'$ are $d \times 2d$ matricies.
- We define a operator "$*$" as follows: When A and B are $d \times 2d$ matrices,

$$A * B := \begin{bmatrix} (A)_{1,1}(B)_{1,1} & \cdots & (A)_{1,2d}(B)_{1,2d} \\ \vdots & \ddots & \vdots \\ (A)_{d,1}(B)_{d,1} & \cdots & (A)_{d,2d}(B)_{d,2d} \end{bmatrix}. \tag{114}$$

When A is a $d \times 2d$ matrix and B is a d-dimensional vector,

$$A * B = B * A := \begin{bmatrix} (A)_{1,1}(B)_1 & \cdots & (A)_{1,2d}(B)_1 \\ \vdots & \ddots & \vdots \\ (A)_{d,1}(B)_d & \cdots & (A)_{d,2d}(B)_d \end{bmatrix}. \tag{115}$$

When A and B are d-dimensional vectors,

$$A * B := \begin{bmatrix} (A)_1(B)_1 \\ \vdots \\ (A)_d(B)_d \end{bmatrix}. \tag{116}$$

- We also define $\partial_x \Phi_S$ ($x = S$ or σ) as

$$\partial_x \Phi_S := \begin{bmatrix} \frac{\partial}{\partial x^1}(\Phi_S)_{1,1} & \cdots & \frac{\partial}{\partial x^1}(\Phi_S)_{1,2d} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x^d}(\Phi_S)_{d,1} & \cdots & \frac{\partial}{\partial x^d}(\Phi_S)_{d,2d} \end{bmatrix}, \tag{117}$$

  where $(\Phi_S)_{i,j}$ denotes the (i,j)-element of the $d \times 2d$ matrix Φ_S.

- Let us introduce the following notations:
  $S_t = (S_t^1, \cdots, S_t^d)$, $\sigma_t = (\sigma_t^1, \cdots, \sigma_t^d)$,
  $h_{S,l,j}^{(i)} = (h_{S^1,l,j}^{(i)}, \cdots, h_{S^d,l,j}^{(i)})$, $h_{\sigma,l,j}^{(i)} = (h_{\sigma^1,l,j}^{(i)}, \cdots, h_{\sigma^d,l,j}^{(i)})$,
  $e^{\alpha t} = (e^{\alpha^1 t}, \cdots, e^{\alpha^d t})$ and $e^{\lambda t} = (e^{\lambda^1 t}, \cdots, e^{\lambda^d t})$.

Based on these preparations, we obtain the next proposition.

Proposition 2. *The coefficients, $S_T^{(i)}$, $\sigma_T^{(i)}$ and $h_{x,l,j}^{(i)}$ ($x =, S, \sigma$), $i = 0, 1, 2$ in the expansions (111), (112) and (113) are given as follows:*

$$S_T^{(0)} = e^{\alpha T} * s_0, \tag{118}$$

$$\sigma_T^{(0)} = \theta + (\sigma_0 - \theta) * e^{-\lambda T}, \tag{119}$$

$$h_{x,l,j}^{(0)} = 0, \tag{120}$$

$$S_T^{(1)} = \int_0^T e^{\alpha(T-t)} * \Phi_S\left(\sigma_{t-}^{(0)}, S_{t-}^{(0)}\right) dZ_t + \sum_{l=1}^n \left(\sum_{j=1}^{N_{l,T}} h_{S,l,j}^{(1)} \cdot - \Lambda_l T \mathbf{E}\left[h_{S,l,1}^{(1)}\right] \right) * S_T^{(0)}, \tag{121}$$

$$\begin{aligned}\sigma_T^{(1)} = & \int_0^T e^{-\lambda(T-t)} * \Phi_\sigma\left(\sigma_{t-}^{(0)}\right) dZ_t + \sum_{l=1}^n \left(\sum_{j=1}^{N_{l,T}} h_{\sigma,l,j}^{(1)} * e^{-\lambda(T-\tau_{j,l})} * \sigma_{\tau_{j,l}-}^{(0)} \right. \\ & \left. -\Lambda_l \mathbf{E}\left[h_{\sigma,l,1}^{(1)}\right] * e^{-\lambda T} * \int_0^T e^{\lambda t} * \sigma_{t-}^{(0)} dt \right),\end{aligned} \tag{122}$$

$$h_{x,l,j}^{(1)} = Y_{x,l,j} := (Y_{x^1,l,j}, \cdots, Y_{x^d,l,j}), \tag{123}$$

$$\begin{aligned}S_T^{(2)} = & \; 2\int_0^T e^{\alpha(T-t)} * \partial_S \Phi_S\left(\sigma_{t-}^{(0)}, S_{t-}^{(0)}\right) * S_{t-}^{(1)} dZ_t + 2\int_0^T e^{\alpha(T-t)} * \partial_\sigma \Phi_S\left(\sigma_{t-}^{(0)}, S_{t-}^{(0)}\right) * \sigma_{t-}^{(1)} dZ_t \\ & + \sum_{l=1}^n \left(\left\{ \sum_{j=1}^{N_{l,T}} h_{S,l,j}^{(2)} - \Lambda_l T \mathbf{E}\left[h_{S,l,1}^{(2)}\right] \right\} * S_T^{(0)} \right. \\ & \left. +2 \sum_{j=1}^{N_{l,T}} h_{S,l,j}^{(1)} * e^{\alpha(T-\tau_{j,l})} * S_{\tau_{j,l}-}^{(1)} - 2\Lambda_l \mathbf{E}\left[h_{S,l,1}^{(1)}\right] * e^{\alpha T} * \int_0^T e^{-\alpha t} * S_{t-}^{(1)} dt \right),\end{aligned} \tag{124}$$

$$h_{x,l,j}^{(2)} = Y_{x,l,j} * Y_{x,l,j}. \tag{125}$$

Next, let us define the payoff of a basket call option with strike price K as

$$(g(x) - K)^+ \; (:= \max\{g(x) - K, 0\}), \tag{126}$$
$$g(x) := w \cdot x = \sum_{i=1}^d w_i x^i,$$

where $g(x)$ represents a weighted sum of the underlying asset prices of $x_1, \cdots, x_d$ with the constant weights $w_1, \cdots, w_d$. Here, we set $x := (x^1, \cdots, x^d)$ and $w := (w_1, \cdots, w_d)$.

For an approximation of a basket option price, we firstly note that $g\left(S_T^{(\epsilon)}\right)$ is expanded around $\epsilon = 0$ as:

$$g\left(S_T^{(\epsilon)}\right) = g\left(S_T^{(0)}\right) + \epsilon g\left(S_T^{(1)}\right) + \frac{\epsilon^2}{2} g\left(S_T^{(2)}\right) + o(\epsilon^2). \tag{127}$$

Then, for a strike price $K = g(S_T^0) - \epsilon y$ for an arbitrary $y \in \mathbf{R}$, the payoff of the call option with maturity T is expanded as follows:

$$\begin{aligned} \left(g\left(S_T^{(\epsilon)}\right) - K\right)^+ &= \epsilon\left(\frac{g(S_T^{(\epsilon)}) - g(S_T^{(0)})}{\epsilon} + y\right)^+ \\ &= \epsilon\left(g\left(S_T^{(1)}\right) + \frac{\epsilon}{2} g\left(S_T^{(2)}\right) + y + o(\epsilon)\right)^+ \\ &= \epsilon\left(g(S_T^{(1)}) + y\right)^+ + \frac{\epsilon^2}{2} 1_{\left\{g(S_T^{(1)}) > -y\right\}} g\left(S_T^{(2)}\right) + o(\epsilon^2). \end{aligned} \tag{128}$$

We next note that when the number of jumps is k_l $(l = 1, \cdots, n)$, that is on $\{N_l = k_l\} := \{N_{1,T} = k_1, \cdots, N_{n,T} = k_n\}$, $S_T^{(1)}$ in the equation (121) becomes

$$\xi_{\{k_l\}} + \hat{S}_T, \tag{129}$$

where

$$\xi_{\{k_l\}} := \sum_{l=1}^{n} (k_l - \Lambda_l T) m_{S,l} * S_T^{(0)} \tag{130}$$

and

$$\hat{S}_T := \int_0^T e^{\alpha(T-t)} * \Phi_S\left(\sigma_t^{(0)}, S_t^{(0)}\right) dZ_t + \sum_{l=1}^{n}\left(\sum_{j=1}^{k_l} \gamma_{S,l} * \zeta_{S,j,l} * S_T^{(0)}\right). \tag{131}$$

Here, we use the following notations:

- $\gamma_{S,l} = (\gamma_{S^1,l}, \cdots, \gamma_{S^d,l})$
- $\zeta_{S,j,l} = (\zeta_{S^1,j,l}, \cdots, \zeta_{S^d,j,l})$ is a vector of random variables, where $\zeta_{S^i,j,l}$ follows $N(0,1)$, that is the standard normal distribution.

We remark that the distribution of $g(\hat{S}_T)$ is $N\left(0, \Sigma_T^{\{k_l\}}\right)$, that is the normal distribution with mean zero and variance $\Sigma_T^{\{k_l\}}$ whose density function is expressed as

$$n\left(x; 0, \Sigma_T^{\{k_l\}}\right) := \frac{1}{\sqrt{2\pi\Sigma_T^{\{k_l\}}}} \exp\left\{\frac{-x^2}{2\Sigma_T^{\{k_l\}}}\right\}. \tag{132}$$

Here, $\Sigma_T^{\{k_l\}}$ is defined as follows:

$$\begin{aligned} \Sigma_T^{\{k_l\}} := & \int_0^T \left(w * e^{\alpha(T-t)} * \Phi_S\left(\sigma_t^{(0)}, S_t^{(0)}\right)\right)^\top \left(w * e^{\alpha(T-t)} * \Phi_S\left(\sigma_t^{(0)}, S_t^{(0)}\right)\right) dt \\ & + \sum_{l=1}^{n} k_l (w * \gamma_{S,l} * S_T^{(0)})^\top \vartheta_{\zeta_{S,l}} (w * \gamma_{S,l} * S_T^{(0)}), \end{aligned} \tag{133}$$

where $\vartheta_{\zeta_{S,l}}$ stands for the correlation matrix of $\zeta_{S,j,l} = (\zeta_{S^1,j,l}, \cdots, \zeta_{S^d,j,l})$, and $x^\top$ denotes the transpose of x.

Next, we define

$$\eta_2(x,\{k_l\}) = \mathbf{E}\left[g\left(S_T^{(2)}\right)\Big|g(\hat{S}_T)=x,\{N_l=k_l\}\right]. \tag{134}$$

With those preparations, we approximate the expectation of the basket call payoff under an equivalent martingale measure in the following way:

$$\begin{aligned}
&\mathbf{E}\left[\left(g\left(S_T^{(\epsilon)}\right)-K\right)^+\right] \\
&=\epsilon\mathbf{E}\left[\mathbf{E}\left[\left(g(S_T^{(1)})+y\right)^+\Big|g(\hat{S}_T)=x,\{N_l=k_l\}\right]\right] \\
&+\frac{\epsilon^2}{2}\mathbf{E}\left[\mathbf{E}\left[1_{\left\{g(S_T^{(1)})>-y\right\}}g\left(S_T^{(2)}\right)\Big|g(\hat{S}_T)=x,\{N_l=k_l\}\right]\right]+o(\epsilon^2).
\end{aligned} \tag{135}$$

We also note that the probability of $\{N_l=k_l\}:=\{N_{1,T}=k_1,\cdots,N_{n,T}=k_n\}$ is expressed as

$$p_{\{k_l\}}:=\prod_{l=1}^{n}\frac{(\Lambda_l T)^{k_l}e^{-\Lambda_l T}}{k_l!}, \tag{136}$$

which is the product of the k_l times of the jump probabilities of $N_{l,T}$ $(l=1,\cdots,n)$, that is $\prod_{l=1}^{n}P(\{N_{l,T}=k_l\})$, thanks to the independence of $N_{l,T}$ $(l=1,\cdots,n)$.

Then, we calculate the coefficients of ϵ and $\frac{\epsilon^2}{2}$ on the right hand of (135) as follows: The coefficient of ϵ is given by:

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{E}\left[\left(g\left(S_T^{(1)}\right)+y\right)^+\Big|g(\hat{S}_T)=x,\{N_l=k_l\}\right]\right] \\
&=\sum_{k=0}^{\infty}\sum_{\sum_{l=1}^{n}k_l=k}p_{\{k_l\}}\int_{-(g(\xi_{\{k_l\}})+y)}^{\infty}\left(x+g(\xi_{\{k_l\}})+y\right)n(x;0,\Sigma_T^{\{k_l\}})dx,
\end{aligned} \tag{137}$$

and the coefficient of $\frac{\epsilon^2}{2}$ is given by:

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{E}\left[1_{\left\{g(S_T^{(1)})>-y\right\}}g\left(S_T^{(2)}\right)\Big|g(\hat{S}_T)=x,\{N_l=k_l\}\right]\right] \\
&=\sum_{k=0}^{\infty}\sum_{\sum_{l=1}^{n}k_l=k}p_{\{k_l\}}\int_{-(g(\xi_{\{k_l\}})+y)}^{\infty}\eta_2(x,\{k_l\})n(x;0,\Sigma_T^{\{k_l\}})dx.
\end{aligned} \tag{138}$$

Then, the initial value, $C(K,T)$ of the basket call option with maturity T and strike K is expanded around $\epsilon=0$ as follows:

$$\begin{aligned}
&C(K,T)= \\
&\sum_{k=0}^{\infty}\sum_{\sum_{l=1}^{n}k_l=k}p_{\{k_l\}}e^{-rT}\left\{\epsilon\int_{-y_{\{k_l\}}}^{\infty}\left(x+y_{\{k_l\}}\right)n(x;0,\Sigma_T^{\{k_l\}})dx+\epsilon^2\int_{-y_{\{k_l\}}}^{\infty}\eta_2(x,\{k_l\})n(x;0,\Sigma_T^{\{k_l\}})dx\right\}+o(\epsilon^2),
\end{aligned} \tag{139}$$

where $y_{\{k_l\}}:=g(\xi_{\{k_l\}})+y$, and r is a constant risk-free rate.

In order to evaluate $\eta_2(x,\{k_l\})$, that is the conditional expectation defined in (134), we apply some formulas derived in Lemma 3.2 of [80].

Consequently, with $\epsilon=1$ we obtain an approximate pricing formula for a basket call option, which corresponds to an asymptotic expansion of the basket option price up to the ϵ^2-order.

Theorem 5. *An approximation formula for the initial value $C(K,T)$ of an basket call option with maturity T and strike price K is given by the following equation:*

$$\sum_{k=0}^{\infty} \sum_{\sum_{l=1}^{n} k_l = k} p_{\{k_l\}} e^{-rT} \left\{ (y_{k_l} + C_1) N\left(\frac{y_{k_l}}{\sqrt{\Sigma_T^{\{k_l\}}}} \right) + \left(C_2 \Sigma_T^{\{k_l\}} + C_3 \frac{H_1\left(y_{k_l}; \Sigma_T^{\{k_l\}}\right)}{\Sigma_T^{\{k_l\}}} \right.\right.$$
$$\left.\left. + C_4 \frac{H_2\left(y_{k_l}; \Sigma_T^{\{k_l\}}\right)}{\left(\Sigma_T^{\{k_l\}}\right)^2} + C_5 \right) n\left(y_{k_l}; 0, \Sigma_T^{\{k_l\}}\right) \right\}, \tag{140}$$

where $p_{\{k_l\}} = \prod_{l=1}^{n} \frac{(\Lambda_l T)^{k_l} e^{-\Lambda_l T}}{k_l!}$, r is a constant risk-free rate, $y = g(S_T^{(0)}) - K$, $y_{\{k_l\}} = g(\xi_{\{k_l\}}) + y$, $N(x)$ denotes the standard normal distribution function and $n(x;0,\Sigma) = \frac{1}{\sqrt{2\pi\Sigma}} \exp\left(\frac{-x^2}{2\Sigma}\right)$. Here, $\Sigma_T^{\{k_l\}}$ is given by (133), and $\xi_{\{k_l\}}$ is defined by (130). C_1, C_2 C_3, C_4 and C_5 are some constants, which are given with the derivations in Appendix B of [80]. Moreover, $H_k\left(x; \Sigma_T^{\{k_l\}}\right)$ denotes the k-th order Hermite polynomial: particularly, $H_1\left(x; \Sigma_T^{\{k_l\}}\right) = x$ and $H_2\left(x; \Sigma_T^{\{k_l\}}\right) = x^2 - \Sigma_T^{\{k_l\}}$.

8 Perturbation Scheme in Forward Backward Stochastic Differential Equations (FBSDEs)

The FBSDEs have become quite popular in finance community since El Karoui, Peng and Quenez [16], especially after the recent financial crises and the subsequent quite volatile markets, which leads us to recognize the importance of counter party risk management, particularly the credit value adjustments (CVA).

However, an explicit solution for a FBSDE has been known only for a simple linear or quadratic example. Although several techniques have been proposed in the last decade, they seem very limited in practical applications since they rely on numerical methods for non-linear partial differential equations (PDEs) or regression based Monte Carlo simulations, which are generally very difficult to implement or quite time-consuming especially for high-dimensional and long-horizon problems.

Recently, [25] has developed a simple analytical approximation scheme for the nonlinear FBSDEs, notably for not only the so called decoupled cases but also the coupled cases. [25] has introduced a perturbation parameter to the generator of a backward stochastic differential equation (BSDE) to expand recursively the non-linear terms around a relevant linear FBSDE. In the computation of each order, [25] explicitly represents the backward elements as the functions of the forward components and take those expectations. Hence, except the cases that the distributions of the forward process are explicitly known, we need to apply some approximations of the distributions, and so, again, the asymptotic expansion technique for the forward stochastic differential equation (FSDE) is useful in the approximations. Section 8.1 below illustrates the scheme briefly. [25] also provided two numerical examples, where the second-order analytic approximations work quite well compared to numerical techniques such as the finite difference method and the regression-based Monte Carlo simulation. Please see the paper for the detail.

Moreover, their subsequent work [26] has applied this scheme to the optimal portfolio problem in an incomplete market with stochastic volatility, and demonstrated the accurate approximations even for long maturities such as 10 years, as opposed to the regression based Monte Carlo simulation which works well only up to short maturities such as one year.

We also note that the method has a great advantage of deriving explicit expressions of the optimal portfolios and hedging strategies, that is very important in practice. Furthermore, we can employ the method for the general multi-dimensional cases.

In order to achieve further reduction of computational burdens in this method, the scheme with an interacting particle method has been recently developed. Section 8.2 describes the outline. Please also see [29] as an application of the method to American option pricing.

Furthermore, [104] provides a mathematical foundation for the original scheme in the decoupled case proposed in [25]. (The justification for the coupled case seems an important and interesting research topic.)

It mainly consisted of two parts. That is, for the BSDE expansion with a perturbed generator they have obtained the coefficients up to an arbitrary order as the solution to a system of the associated BSDEs with the base FSDE, and present the error estimate of the expansion. Accordingly, they showed a concrete representation for each expansion coefficient of the volatility component, that is the martingale integrand in the BSDE. For the FSDE expansion, they derived an expansion formula with its sharp error estimate for the expectation of the solution to the base FSDE in terms of a small diffusion. Then, they combine the both results, particularly applying the FSDE expansion formula to the BSDE expansion coefficients to obtain a main result, that is an asymptotic expansion of FBSDEs with a perturbed generator. In the proofs, [104] effectively applied the representation results in Ma and Zhang [63] for the BSDE expansion and the properties of the Kusuoka-Stroock functions in Kusuoka [52] for the FSDE expansion.

In a different stream, [102] has proposed a new semi closed-form approximation for the solutions of FBSDEs. In particular, applying the asymptotic expansion method in [100] and [103] to the forward SDEs with a Picard-type iteration scheme for the BSDEs, they have obtained an error estimate for the approximation. Moreover, they demonstrated the effectiveness of the method through numerical examples for pricing options with counter party risk under the local and stochastic volatility models, where the credit value adjustment (CVA) is taken into account. Roughly speaking, considering a perturbed forward SDE X^ε, $\varepsilon \in (0,1]$ and an associated backward SDE $(Y^\varepsilon, Z^\varepsilon)$, they have the following recursive asymptotic expansion around some non-degenerate gaussian model $\bar{X}^0$. That is, for $k \geq 0$, $N \geq 1$

$$Y_t^{\varepsilon,t,x} \simeq u^{\varepsilon,k+1,N}(t,x) = E[g(\bar{X}_T^{0,t,x})] + E\left[\int_t^T f(s, \bar{X}_s^{0,t,x}, Y_s^{\varepsilon,k,N,t,x}, Z_s^{\varepsilon,k,N,t,x})ds\right]$$
$$+\sum_{i=1}^N \varepsilon^i E[g(\bar{X}_T^{0,t,x})\pi_{i,T}^{0,t}] + \sum_{i=1}^N \varepsilon^i E\left[\int_t^T f(s, \bar{X}_s^{0,t,x}, Y_s^{\varepsilon,k,N,t,x}, Z_s^{\varepsilon,k,N,t,x})\pi_{i,s}^{0,t}ds\right], \tag{141}$$

$$Z_t^{\varepsilon,t,x} \simeq (\nabla u^{\varepsilon,k+1,N}\sigma)(t,x) = \left\{E[g(\bar{X}_T^{0,t,x})N_{0,T}^{0,t}] + E\left[\int_t^T f(s, \bar{X}_s^{0,t,x}, Y_s^{\varepsilon,k,N,t,x}, Z_s^{\varepsilon,k,N,t,x})N_{0,s}^{0,t}ds\right]\right.$$
$$\left.+\sum_{i=1}^N \varepsilon^i E[g(\bar{X}_T^{0,t,x})N_{i,T}^{0,t}] + \sum_{i=1}^N \varepsilon^i E\left[\int_t^T f(s, \bar{X}_s^{0,t,x}, Y_s^{\varepsilon,k,N,t,x}, Z_s^{\varepsilon,k,N,t,x})N_{i,s}^{0,t}ds\right]\right\}\varepsilon\sigma(t,x), \tag{142}$$

where $Y_s^{\varepsilon,k,N,t,x} = u^{\varepsilon,k,N}(s, \bar{X}_s^{0,t,x})$ and $Z_s^{\varepsilon,k,N,t,x} = (\nabla_x u^{\varepsilon,k,N}\sigma)(s, \bar{X}_s^{0,t,x})$. Here, the processes $\pi_{i,t}^0$ and $N_{i,t}^0$, $i = 1, \cdots, N$ are the *Malliavin weights* and in particular, $N_{0,t}^0$ corresponds to the weight appeared in a representation theorem in Ma and Zhang [63].

8.1 Expansion with Perturbed Generator in BSDE

This subsection briefly describes the perturbation method following [25]. Firstly, let us consider the following decoupled FBSDE:

$$\begin{aligned} dV_t &= -f(X_t, V_t, Z_t)dt + Z_t \cdot dW_t \qquad (143)\\ V_T &= \Phi(X_T), \end{aligned}$$

where V takes the value in $\mathbf{R}$, W is a r-dimensional Winer process, and X_t valued in $\mathbf{R}$ is assumed to follow a diffusion process, which is the solution to the (forward) SDE:

$$dX_t = \gamma_0(X_t)dt + \gamma(X_t)\cdot dW_t;\ X_0 = x\ . \tag{144}$$

Hereafter, we assume the appropriate regularity conditions that guarantee the mathematical validity. For example, pleases see [104] on this point.

In order to approximate the pair of (V_t, Z_t) in terms of X_t, we extract the linear term from the generator f and treat the residual non-linear term as a perturbation to the linear FBSDE. That is, let us introduce a perturbation parameter ϵ, and then write the equation as

$$\begin{aligned} dV_t^{(\epsilon)} &= c(X_t)V_t^{(\epsilon)}dt - \epsilon g(X_t, V_t^{(\epsilon)}, Z_t^{(\epsilon)})dt + Z_t^{(\epsilon)}\cdot dW_t \qquad (145)\\ V_T^{(\epsilon)} &= \Phi(X_T). \end{aligned}$$

Here, the above equation with $\epsilon = 1$ corresponds to the original model:

$$f(X_t, V_t, Z_t) = -c(X_t)V_t + g(X_t, V_t, Z_t) \ . \tag{146}$$

We remark that as in the previous asymptotic expansion cases, the residual part g should be small for a precise approximation. Hence, one should choose the linear term $c(X_t)V_t^{(\epsilon)}$ in such a way that the residual non-linear term g becomes as small as possible.

Now, we are going to expand the solution of BSDE (145) with respect to ϵ. That is, suppose $V_t^{(\epsilon)}$ and $Z_t^{(\epsilon)}$ are expanded as follows:

$$\begin{aligned} V_t^{(\epsilon)} &= V_t^{(0)} + \epsilon V_t^{(1)} + \epsilon^2 V_t^{(2)} + \cdots && (147)\\ Z_t^{(\epsilon)} &= Z_t^{(0)} + \epsilon Z_t^{(1)} + \epsilon^2 Z_t^{(2)} + \cdots \ . && (148) \end{aligned}$$

For illustrative purpose, let us show a first few steps of the expansion. For the zero-th order of ϵ, it is easily seen that $V_t^{(0)}$ is a solution to the following equation:

$$dV_t^{(0)} = c(X_t)V_t^{(0)}dt + Z_t^{(0)} \cdot dW_t \tag{149}$$

$$V_T^{(0)} = \Phi(X_T) \ . \tag{150}$$

Then, $V_t^{(0)}$ can be represented as follows:

$$V_t^{(0)} = E\left[e^{-\int_t^T c(X_s)ds}\Phi(X_T) \middle| \mathcal{F}_t \right] , \tag{151}$$

which is equivalent to the value of a standard European contingent claim with the terminal payoff $\Phi(X_T)$ and the discount rate $c(X_t)$ under a suitable pricing measure. Clearly, $V_t^{(0)}$ is a function of X_t due to the Markovain nature of the model. Moreover, applying Itô's formula (or the Malliavin derivative), we are able to obtain $Z_t^{(0)}$ as a function of X_t as well.

Next, let us consider the process $V^{(\epsilon)} - V^{(0)}$:

$$\begin{aligned} d\big(V_t^{(\epsilon)} - V_t^{(0)}\big) &= c(X_t)\big(V_t^{(\epsilon)} - V_t^{(0)}\big)dt \\ &- \epsilon g(X_t, V_t^{(\epsilon)}, Z_t^{(\epsilon)})dt + \big(Z_t^{(\epsilon)} - Z_t^{(0)}\big) \cdot dW_t \\ V_T^{(\epsilon)} - V_T^{(0)} &= 0 \ . \end{aligned} \tag{152}$$

Now, by extracting the ϵ-first order term, we can once again recover the linear FBSDE:

$$\begin{aligned} dV_t^{(1)} &= c(X_t)V_t^{(1)}dt - g(X_t, V_t^{(0)}, Z_t^{(0)})dt + Z_t^{(1)} \cdot dW_t \\ V_T^{(1)} &= 0 \ , \end{aligned} \tag{153}$$

which leads to

$$V_t^{(1)} = E\left[\int_t^T e^{-\int_t^u c(X_s)ds} g(X_u, V_u^{(0)}, Z_u^{(0)})du \middle| \mathcal{F}_t \right] . \tag{154}$$

Because $V_u^{(0)}$ and $Z_u^{(0)}$ are some functions of X_u, we obtain $V_t^{(1)}$ as a function of X_t, and also $Z_t^{(1)}$ through Itô's formula (or Malliavin derivative).

In exactly the same way, we are able to derive an arbitrarily higher order correction. Particularly, due to the ϵ in front of the non-linear term g, the system remains to be linear in every order of the approximation. For example, $V_t^{(2)}$ that is the ϵ^2-order's coefficient of the expansion of $V_t^{(\epsilon)}$ is the solution to the following equation:

$$\begin{aligned} dV_t^{(2)} &= c(X_t)V_t^{(2)}dt - \left(\frac{\partial}{\partial v} g(X_t, V_t^{(0)}, Z_t^{(0)})V_t^{(1)} \right. \\ &+ \left. \nabla_z g(X_t, V_t^{(0)}, Z_t^{(0)}) \cdot Z_t^{(1)} \right) dt + Z_t^{(2)} \cdot dW_t \\ V_T^{(2)} &= 0. \end{aligned} \tag{155}$$

In general, suppose that we have succeeded to represent backward components (V_t, Z_t) in terms of X_t up to the $(i-1)$-th order. Then, in order to proceed to a higher order approximation, we need to obtain the following form of expressions with some deterministic function $G(\cdot)$ in terms of the forward components X_t.

$$V_t^{(i)} = E\left[\int_t^T e^{-\int_t^u c(X_s)ds} G(X_u)du \middle| \mathcal{F}_t\right]. \tag{156}$$

Even if it seems impossible to get the exact result, we can still have an analytic approximation for $(V_t^{(i)}, Z_t^{(i)})$. through again, the asymptotic expansion method.

As an example, [26] has explicitly derived an approximation formula for the dynamic optimal portfolio in an incomplete market setting, and confirmed its accuracy comparing with the exact result by the Cole-Hopf transformation (Zariphopoulou [121]).

Finally, let us provide a brief remark on an approximation of coupled FBSDEs. Let us consider the following generic *coupled* non-linear FBSDE:

$$\begin{aligned} dV_t &= -f(t, X_t, V_t, Z_t)dt + Z_t \cdot dW_t \qquad (157)\\ V_T &= \Phi(X_T) \\ dX_t &= \gamma_0(t, X_t, V_t, Z_t)dt + \gamma(t, X_t, V_t, Z_t) \cdot dW_t;\ X_0 = x\ . \end{aligned}$$

We are able to treat this case in the similar way as in the decoupled case by introducing perturbations to the forward SDE in addition to the one in BSDE:

$$\begin{aligned} &dV_t^{(\epsilon)} = c(t, X_t^{(\epsilon)})V_t^{(\epsilon)}dt - \epsilon g\left(t, X_t^{(\epsilon)}, V_t^{(\epsilon)}, Z_t^{(\epsilon)}\right)dt + Z_t^{(\epsilon)} \cdot dW_t \\ &V_T^{(\epsilon)} = \Phi\left(X_T^{(\epsilon)}\right) \\ &dX_t^{(\epsilon)} = \left(r\left(t, X_t^{(\epsilon)}\right) + \epsilon\mu\left(t, X_t^{(\epsilon)}, V_t^{(\epsilon)}, Z_t^{(\epsilon)}\right)\right)dt \\ &\qquad\qquad + \left(\sigma\left(t, X_t^{(\epsilon)}\right) + \epsilon\eta\left(t, X_t^{(\epsilon)}, V_t^{(\epsilon)}, Z_t^{(\epsilon)}\right)\right) \cdot dW_t \end{aligned}$$

We also note that the similar method can be applied to the coupled case under a PDE (partial differential equation) formulation based on the so called *four step scheme* (e.g. Ma-Yong [62].) Please see [25] for the details. Developing a mathematical validity of the scheme for the coupled case will be one of the research topics in the future.

8.2 Perturbation Scheme with Interacting Particle Method

This subsection briefly introduces a new scheme proposed by [27]. Except the cases that we are able to obtain fully closed form expressions, the high orders' expansions of perturbed FBSDEs generally contain multi-dimensional time integrations of expectation values due to a convoluted nature of the scheme, which makes standard Monte Carlo simulations too time consuming. To avoid nested simulations, one can applies a particle representation inspired by the ideas of branching diffusion models(e.g. Fujita [23], Ikeda, Nagasawa and Watanabe ([40], [41], [42]), McKean [69], Nagasawa and Sirao [70]). Then, we are able to provide a straightforward simulation scheme to solve nonlinear FBSDEs at each order of the approximation based on the perturbation. In particular, comparing to the direct application of the branching diffusion method, the method is expected to be less numerically intensive, because thanks to expansions of the perturbed generator, the interested system is already decomposed into a set of linear problems. We illustrate the outline of the method by following [27].

Again, let us introduce a perturbation parameter ϵ in the generator of a BSDE as follows:

$$\begin{cases} dV_s^{(\epsilon)} = -\epsilon f(X_s, V_s^{(\epsilon)}, Z_s^{(\epsilon)})ds + Z_s^{(\epsilon)} \cdot dW_s \\ V_T^{(\epsilon)} = \Psi(X_T), \end{cases} \tag{158}$$

where $X_t \in \mathbf{R}$ is assumed to follow a generic Markovian forward SDE:

$$dX_s = \gamma_0(X_s)ds + \gamma(X_s) \cdot dW_s;\ X_t = x_t. \tag{159}$$

Next, let us fix the initial time as t. We denote the Malliavin derivative of X_u $(u \geq t)$ at time t as

$$\mathcal{D}_t X_u \in \mathbf{R}^{r\times d}. \tag{160}$$

Let us also note that in terms of the future time u, the SDE of $(Y_{t,u})^i_j$ defined by $(Y_{t,u})^i_j = \partial_{x_t^j} X_u^i$ is given in the following:

$$\begin{aligned} d(Y_{t,u})^i_j &= \partial_k \gamma_0^i(X_u)(Y_{t,u})^k_j du + \partial_k \gamma_a^i(X_u)(Y_{t,u})^k_j dW_u^a \\ (Y_{t,t})^i_j &= \delta^i_j, \end{aligned} \tag{161}$$

where ∂_k denotes the partial differentiation with respect to the k-th component of X, and δ^i_j stands for the Kronecker delta. Here, i and j run through $\{1,\cdots,d\}$ and $\{1,\cdots,r\}$ for a, and we adopt the Einstein notation which assumes the summation of all the paired indexes. Then, it is well-known that

$$(\mathcal{D}_t X_u^i)_a = (Y_{t,u}\gamma(x_t))^i_a,$$

where $a \in \{1,\cdots,r\}$ is the index of r-dimensional Winer process.

First, for the ϵ-zeroth order, it is easy to see

$$V_t^{(0)} = \mathbf{E}\Big[\Psi(X_T)\Big|\mathcal{F}_t\Big] \tag{162}$$

$$Z_t^{(0),a} = \mathbf{E}\Big[\partial_i\Psi(X_T)(Y_{tT}\gamma(X_t))^i_a\Big|\mathcal{F}_t\Big]. \tag{163}$$

Then, it is clear that they can be evaluated by standard Monte Carlo simulations. However, for their use in higher order approximations, it is crucial to obtain analytical (closed form) approximate expressions for these two quantities, for example based on the asymptotic expansion technique as before.

In the following, let us suppose that we have obtained the solutions up to a given order of the asymptotic expansion, and write each of them as a function of x_t:

$$\begin{cases} V_t^{(0)} = v^{(0)}(x_t) \\ Z_t^{(0)} = z^{(0)}(x_t). \end{cases} \tag{164}$$

Next, for the ϵ-first order's coefficient $V_t^{(1)}$, we obtain an expression as

$$\begin{aligned} V_t^{(1)} &= \int_t^T \mathbf{E}\Big[f(X_u, V_u^{(0)}, Z_u^{(0)})\Big|\mathcal{F}_t\Big]du \\ &= \int_t^T \mathbf{E}\Big[f\Big(X_u, v^{(0)}(X_u), z^{(0)}(X_u)\Big)\Big|\mathcal{F}_t\Big]du. \end{aligned} \tag{165}$$

Then, we define the new process for $(s > t)$ by introducing a deterministic positive process λ_t as follows:

$$\hat{V}_{ts}^{(1)} = e^{\int_t^s \lambda_u du} V_s^{(1)}, \tag{166}$$

Here, λ_t can be a positive constant for the simplest case. Then, for the fixed initial time t, its SDE is given by

$$d\hat{V}_{ts}^{(1)} = \lambda_s \hat{V}_{ts}^{(1)} ds - \lambda_s \hat{f}_{ts}(X_s, v^{(0)}(X_s), z^{(0)}(X_s))ds + e^{\int_t^s \lambda_u du} Z_s^{(1)}\cdot dW_s\ ,$$

where

$$\hat{f}_{ts}(x, v^{(0)}(x), z^{(0)}(x)) = \frac{1}{\lambda_s} e^{\int_t^s \lambda_u du} f(x, v^{(0)}(x), z^{(0)}(x)).$$

Since we have $\hat{V}_{tt}^{(1)} = V_t^{(1)}$, one can easily see the following relation holds:

$$V_t^{(1)} = \mathbf{E}\left[\int_t^T e^{-\int_t^u \lambda_s ds}\lambda_u \hat{f}_{tu}(X_u, v^{(0)}(X_u), z^{(0)}(X_u))du\right|\mathcal{F}_t\Bigg] \tag{167}$$

Similarly to the cases of the standard credit risk modeling (e.g. Bielecki-Rutkowski [6]), it is the present value of default payment where the default intensity is λ_s with the default payoff at $s(>t)$ as $\hat{f}_{ts}(X_s, v^{(0)}(X_s), z^{(0)}(X_s))$. Thus, we obtain the following proposition.

Proposition 3. *The $V_t^{(1)}$ in (165) can be equivalently expressed as*

$$V_t^{(1)} \quad = \quad 1_{\{\tau>t\}}\mathbf{E}\left[1_{\{\tau<T\}}\hat{f}_{t\tau}\left(X_\tau, v^{(0)}(X_\tau), z^{(0)}(X_\tau)\right)\middle|\,\mathcal{F}_t\right]. \tag{168}$$

Here τ is the interaction time where the interaction is drawn independently from the Poisson distribution with an arbitrary deterministic positive intensity process λ_t. $\hat{f}$ is defined as

$$\hat{f}_{ts}(x, v^{(0)}(x), z^{(0)}(x)) = \frac{1}{\lambda_s}e^{\int_t^s \lambda_u du} f(x, v^{(0)}(x), z^{(0)}(x)) \ . \tag{169}$$

Now, let us consider the ϵ-order's coefficient of $Z^{(\epsilon)}$, that is the component $Z^{(1)}$. It can be expressed as

$$Z_t^{(1)} = \int_t^T \mathbf{E}\left[\mathcal{D}_t f\left(X_u, v^{(0)}(X_u), z^{(0)}(X_u)\right)\middle|\,\mathcal{F}_t\right] du \tag{170}$$

Firstly, we observe that the SDE of the Malliavin derivative of $V^{(1)}$ is given as follows:

$$\begin{aligned} d(\mathcal{D}_t V_s^{(1)}) \quad &= \quad -(\mathcal{D}_t X_s^i)\nabla_i(x, v^{(0)}, z^{(0)}) f(x, v^{(0)}, z^{(0)}) + (\mathcal{D}_t Z_s^{(1)}) \cdot dW_s; \\ \mathcal{D}_t V_t^{(1)} \quad &= \quad Z_t^{(1)}, \end{aligned} \tag{171}$$

where

$$\nabla_i(x, v^{(0)}, z^{(0)}) \equiv \partial_i + \partial_i v^{(0)}(x)\partial_v + \partial_i z^{a(0)}(x)\partial_{z^a}, \tag{172}$$

$$f(x, v^{(0)}, z^{(0)}) \equiv f(x, v^{(0)}(x), z^{(0)}(x)). \tag{173}$$

Then, we define for $(s > t)$, $\widehat{\mathcal{D}_t V_s^{(1)}}$ as

$$\widehat{\mathcal{D}_t V_s^{(1)}} = e^{\int_t^s \lambda_u du}(\mathcal{D}_t V_s^{(1)}), \tag{174}$$

and its SDE can be written as

$$\begin{aligned} d(\widehat{\mathcal{D}_t V_s^{(1)}}) = \lambda_s(\widehat{\mathcal{D}_t V_s^{(1)}})ds - \lambda_s(\mathcal{D}_t X_s^i)\nabla_i(X_s, v^{(0)}, z^{(0)})\hat{f}_{ts}(X_s, v^{(0)}, z^{(0)})ds \\ + e^{\int_t^s \lambda_u du}(\mathcal{D}_t Z_s^{(0)}) \cdot dW_s. \end{aligned} \tag{175}$$

Then, we again have

$$\widehat{\mathcal{D}_t V_t^{(1)}} = Z_t^{(1)}. \tag{176}$$

Hence,

$$Z_t^{(1)} = \mathbf{E}\left[\int_t^T e^{-\int_t^u \lambda_s ds}\lambda_u(\mathcal{D}_t X_u^i)\nabla_i(X_u, v^{(0)}, z^{(0)})\hat{f}_{tu}(X_u, v^{(0)}, z^{(0)})du\middle|\,\mathcal{F}_t\right]. \tag{177}$$

Thus, following the same argument as for the previous proposition, we have the next result:

Proposition 4. *$Z_t^{(1)}$ in (170) is equivalently expressed as*

$$Z_t^{(1),a} = 1_{\{\tau>t\}}\mathbf{E}\left[1_{\{\tau<T\}}(Y_{t,\tau}\gamma(X_\tau))_a^i\nabla_i(X_\tau, v^{(0)}, z^{(0)})\hat{f}_{t\tau}(X_\tau, v^{(0)}, z^{(0)})\middle|\,\mathcal{F}_t\right], \tag{178}$$

where the definitions of random time τ and the positive deterministic process λ are the same as those in the previous proposition.

Now, we are able to obtain a new Monte Carlo scheme. That is, we have a new particle interpretation of $(V^{(1)}, Z^{(1)})$ as follows:

$$V_t^{(1)} = 1_{\{\tau>t\}}\mathbf{E}\left[1_{\{\tau<T\}}\hat{f}_{t\tau}\left(X_\tau, v^{(0)}, z^{(0)}\right)\Big|\,\mathcal{F}_t\right] \tag{179}$$

$$Z_t^{(1)} = 1_{\{\tau>t\}}\mathbf{E}\left[1_{\{\tau<T\}}(Y_{t,\tau}\gamma(X_\tau))^i\nabla_i(X_\tau, v^{(0)}, z^{(0)})\hat{f}_{t\tau}(X_\tau, v^{(0)}, z^{(0)})\Big|\,\mathcal{F}_t\right], \tag{180}$$

which allows an efficient time integration with the following Monte Carlo scheme:

- Run the diffusion processes of X and Y.
- Carry out Poisson draw with probability $\lambda_s\Delta s$ at each time s and if "one" is drawn, set that time as τ.
- Then stores the relevant quantities at τ, or in the case of $(\tau > T)$ stores 0.
- Repeat the above procedures and take their expectation.

Finally, we remark that the higher order coefficients in the expansions are evaluated in the similar way. Please see [27] for the details.

9 Conclusion

The present note has reviewed an asymptotic expansion approach in finance, particularly in terms of computational problems arising in practice of financial derivatives. in finance. However, due to the limitation of the space, we have not provided thorough explanations especially for recent progress such as improvement schemes in Section 5, expansion methods in jump and jump-diffusion models in Section 7 and perturbation schemes in forward backward stochastic differential equations (FBSDEs) in Section 8. Please see the cited papers for the details.

Moreover, we have not introduced an application of the method to mean-variance hedging problems in partially observable markets, which is an interesting topic as an application of stochastic filtering problems in finance. Please see [29] for the detail.

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