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Quadratic-exponential growth BSDEs with Jumps and their Malliavin's Differentiability

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Abstract

In this paper, we study a class of quadratic-exponential growth BSDEs with jumps. The quadratic structure was introduced by Barrieu & El Karoui (2013) and yields a very useful *universal bound* on the possible solutions. With the bounded terminal condition as well as an additional local Lipschitz continuity, we give a simple and streamlined proof for the existence and the uniqueness of the solution. The universal bound and the stability result for the locally Lipschitz BSDEs with coefficients in the BMO space enable us to show the strong convergence of a sequence of globally Lipschitz BSDEs. The result is then used to generalize the existing results on the Malliavin's differentiability of the quadratic BSDEs in the diffusion setup to the quadratic-exponential growth BSDEs with jumps.

Keywords : jump, random measure, Lévy, Malliavin derivative, asymptotic expansion

1 Introduction

The backward stochastic differential equations (BSDEs) have been subjects of strong interest of many researchers since they were introduced by Bismut (1973) [5] and generalized later by Pardoux & Peng (1990) [30]. This is particularly because they provide a truly probabilistic approach to stochastic control problems, which has been soon recognized as a very powerful tool both for theoretical as well as numerical issues in many important applications.

More recently, there has appeared an acute interest in quadratic-growth BSDEs because of their various fields of applications such as, risk sensitive control problems, dynamics risk measures and indifference pricing in an incomplete market. The first breakthrough was made by Kobylanski (2000) [24] in a Brownian filtration with a bounded terminal condition. The result was then extended by Briand & Hu (2006, 2008) [7, 8] to unbounded solutions. Direct convergence based on a fixed-point theorem was proposed by Tevzadze (2008) [33]. Various extensions/applications can be found in, for example, Hu, Imkeller & Muller (2005) [18], Mania & Tevzadze (2006) [26], Morlais (2009) [27], Hu & Schweizer (2011) [19], Delbaen, Hu & Richou (2011) [11].

In contrast to the diffusion setup, the number of researches on quadratic BSDEs with jumps has been rather small. Morlais (2010) [28] deals with a particular BSDE appearing in the exponential utility optimization with jumps using the technique of Kobylanski [24],

from which it inherits the complexity in showing the strong convergence of martingale components. Cohen & Elliott (2015) [9] and also Kazi-Tani, Possamai & Zhou (2015) [23] have adopted the fixed-point approach of Tevzadze [33], from which they inherit the requirement of the second order differentiability of the driver.

Recently, Barrieu & El Karoui (2013) [4] have proposed a totally new approach based on a stability of quadratic semimartingales by introducing a so-called quadratic structure condition. They have shown the existence of a solution, without the uniqueness, under the minimal assumption allowing the unbounded terminal condition in a continuous setup. Their result has been extended to the exponential utility optimization in a market with counterparty default risks by generalizing quadratic structure condition to a quadratic-exponential (Q_{exp}) structure condition in Ngoupeyou (2010) [29] (See also Jeanblanc, Matoussi & Ngoupeyou (2013) [20].).

In this paper, we propose a new application of the result [4] to a class of BSDEs satisfying the Q_{exp}-growth as well as the local Lipschitz continuity with a bounded terminal condition in the presence of σ-finite random Poisson measures. In contrast to the existing works, we exploit the universal bound found in [4, 29] and the properties of the BSDEs with the local Lipschitz condition whose coefficients belong to the space of $\mathbb{H}^2_{BMO}$. By deriving a new stability result, we are able to show the uniqueness in a very simple fashion without relying on the comparison principle. Note that, in order for the comparison principle to hold, one needs a stronger assumption than the Lipschitz continuity in the setups with jumps (See, Barles, Buckdahn & Pardoux (1997) [3].). Furthermore, we have shown the existence of a solution by the convergence of a sequence of globally Lipschitz BSDEs in the space $\mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$ by directly applying the stability result used in the proof of the uniqueness. This approach greatly simplifies the classical result of Kobylanski [24] for the martingale components in particular.

Moreover, the strongly converging sequence of Lipschitz BSDEs allows us to obtain the sufficient conditions for the Malliavin's differentiability of the Q_{exp}-growth BSDEs with jumps. This extends the work of Ankirchner, Imkeller & Dos Reis (2007) [1] on the Malliavin's differentiability in the diffusion setup. The obtained representation theorem will be useful for the optimal hedging problems in financial applications, investigations on the path regularity necessary for numerical as well as analytical issues, and also for the development of an asymptotic expansion for the BSDEs [1].

The organization of the paper is as follows: Section 2 gives preliminaries including some important results on the BMO martingales. Section 3 explains the setup of Q_{exp}-growth BSDEs with jumps and gives the uniqueness result. Section 4 constructs a sequence of regularized BSDEs and then shows the existence of a solution by their convergence. Sections 5 deals with the Malliavin's differentiability of the Q_{exp}-growth BSDEs, which is then applied to a forward-backward system to obtain a representation theorem on the martingale components in Section 6. Appendix A is a simple generalization of the results by Ankirchner, Imkeller & Dos Reis (2007) [1] and Briand & Confortola (2008) [6] on the locally Lipschitz BSDEs with BMO coefficients to the setup with jumps. Appendix B gives a detailed proof for the Malliavin's differentiability of the Lipschitz BSDEs with jumps, which generalizes the result of Delong & Imkeller (2010) [13] and Delong (2013) [12] to local (instead of global) Lipschitz continuity for the Malliavin derivative of the driver, which becomes necessary to investigate a forward-backward system driven by a Markovian forward process.

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[1]Recently, we have proposed an analytic approximation method of the Lipschitz BSDEs with jumps in Fujii & Takahashi (2015) [16], which is based on the small-variance asymptotic expansion (See, Takahashi (2015) [34] as a general review.). Its extension to the Q_{exp}-growth BSDEs is now ready to be investigated using the new results obtained here, which will be pursued in a different opportunity.

2 Preliminaries

2.1 General Setting

Let us first state the general setting to be used throughout the paper. $T > 0$ is some bounded time horizon. The space $(\Omega_W, \mathcal{F}_W, \mathbb{P}_W)$ is the usual canonical space for a d-dimensional Brownian motion equipped with the Wiener measure $\mathbb{P}_W$. We also denote $(\Omega_\mu, \mathcal{F}_\mu, \mathbb{P}_\mu)$ as a product of canonical spaces $\Omega_\mu := \Omega_\mu^1 \times \cdots \times \Omega_\mu^k$, $\mathcal{F}_\mu := \mathcal{F}_\mu^1 \times \cdots \times \mathcal{F}_\mu^k$ and $\mathbb{P}_\mu^1 \times \cdots \times \mathbb{P}_\mu^k$ with some constant $k \geq 1$, on which each μ^i is a Poisson measure with a compensator $\nu^i(dz)dt$. Here, $\nu^i(dz)$ is a σ-finite measure on $\mathbb{R}_0 = \mathbb{R}\backslash\{0\}$ satisfying $\int_{\mathbb{R}_0} |z|^2 \nu^i(dz) < \infty$. Throughout the paper, we work on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, where the space $(\Omega, \mathcal{F}, \mathbb{P})$ is the product of the canonical spaces $(\Omega_W \times \Omega_\mu, \mathcal{F}_W \times \mathcal{F}_\mu, \mathbb{P}_W \times \mathbb{P}_\mu)$, and that the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is the canonical filtration completed for $\mathbb{P}$ and satisfying the usual conditions. In this construction, $(W, \mu^1, \cdots, \mu^k)$ are independent. We use a vector notation $\mu(\omega, dt, dz) := (\mu^1(\omega, dt, dz^1), \cdots, \mu^k(\omega, dt, dz^k))$ and denote the compensated Poisson measure as $\widetilde{\mu} := \mu - \nu$. We represent the $\mathbb{F}$-predictable σ-field on $\Omega \times [0, T]$ by $\mathcal{P}$.

2.2 Notation

We use basically the same notation as [16]. We denote a generic constant by C, which may change line by line, is sometimes associated with several subscripts (such as $C_{K,T}$) showing its dependence when necessary. $\mathcal{T}_0^T$ denotes the set of $\mathbb{F}$-stopping times $\tau \in [0, T]$.

Let us introduce a sup-norm for a $\mathbb{R}^r$-valued function $x : [0, T] \to \mathbb{R}^r$ as

$$||x||_{[a,b]} := \sup\{|x_t|, t \in [a, b]\}$$

and write $||x||_t := ||x||_{[0,t]}$. We also use the following spaces for stochastic processes for $p \geq 2$:

• $\mathbb{S}_r^p[s, t]$ is the set of $\mathbb{R}^r$-valued adapted càdlàg processes X such that

$$||X||_{\mathbb{S}_r^p[s,t]} := \mathbb{E}\left[||X(\omega)||_{[s,t]}^p\right]^{1/p} < \infty .$$

• $\mathbb{S}_r^\infty$ is the set of $\mathbb{R}^r$-valued essentially bounded càdlàg processes X such that

$$||X||_{\mathbb{S}_r^\infty} := \left|\left|\sup_{t\in[0,T]} |X_t|\right|\right|_\infty < \infty.$$

• $\mathbb{H}^p[s, t]$ is the set of progressively measurable $\mathbb{R}^d$-valued processes Z such that

$$||Z||_{\mathbb{H}_r^p[s,t]} := \mathbb{E}\left[\left(\int_s^t |Z_u|^2 du\right)^{\frac{p}{2}}\right]^{\frac{1}{p}} < \infty.$$

• $\mathbb{J}^p[s, t]$ is the set of k-dimensional functions $\psi = \{\psi^i, 1 \leq i \leq k\}$, $\psi^i : \Omega \times [0, T] \times \mathbb{R}_0 \to \mathbb{R}$ which are $\mathcal{P} \times \mathcal{B}(\mathbb{R}_0)$-measurable and satisfy

$$||\psi||_{\mathbb{J}^p[s,t]} := \mathbb{E}\left[\left(\sum_{i=1}^k \int_s^t \int_{\mathbb{R}_0} |\psi_u^i(x)|^2 \nu^i(dx)du\right)^{\frac{p}{2}}\right]^{\frac{1}{p}} < \infty.$$

and $\mathbb{J}^\infty$ is the space of functions which are $d\mathbb{P} \otimes \nu(dz)$ essentially bounded i.e.,

$$||\psi||_{\mathbb{J}^\infty} := \left|\left|\sup_{t\in[0,T]} ||\psi_t||_{\mathbb{L}^\infty(\nu)}\right|\right|_\infty < \infty.$$

3

For notational simplicity, we use $(E, \mathcal{E}) = (\mathbb{R}_0^k, \mathcal{B}(\mathbb{R}_0)^k)$ and denote the above maps $\{\psi^i, 1 \le i \le k\}$ as $\psi : \Omega \times [0, T] \times E \to \mathbb{R}^k$ and say ψ is $\mathcal{P} \otimes \mathcal{E}$-measurable without referring to each component. We also use the notation such that

$$\int_s^t \int_E \psi_u(x) \widetilde{\mu}(du, dx) := \sum_{i=1}^k \int_s^t \int_{\mathbb{R}_0} \psi_u^i(x) \widetilde{\mu}^i(du, dx)$$

for simplicity. The similar abbreviation is used also for the integrals with respect to μ and ν. When we use E and $\mathcal{E}$, one should always interpret it in this way so that the integral with the k-dimensional Poisson measure does make sense. On the other hand, when we use the range $\mathbb{R}_0$ with the integrators $(\widetilde{\mu}, \mu, \nu)$, for example,

$$\int_{\mathbb{R}_0} \psi_u(x) \nu(dx) := \left(\int_{\mathbb{R}_0} \psi_u^i(x) \nu^i(dx) \right)_{1 \le i \le k}$$

we interpret it as a k-dimensional vector.

• $\mathcal{K}^p[s, t]$ is the set of functions (Y, Z, ψ) in the space $\mathbb{S}^p[s, t] \times \mathbb{H}^p[s, t] \times \mathbb{J}^p[s, t]$ with the norm defined by

$$||(Y, Z, \psi)||_{\mathcal{K}^p[s,t]} := \left(||Y||_{\mathbb{S}^p[s,t]}^p + ||Z||_{\mathbb{H}^p[s,t]}^p + ||\psi||_{\mathbb{J}^p[s,t]}^p \right)^{\frac{1}{p}}.$$

We frequently omit the subscripts specifying the dimension r and the time interval $[s, t]$ when they are unnecessary or obvious in the context. We use $(\Theta_s, s \in [0, T])$ as a collective argument

$$\Theta_s = \left(Y_s, Z_s, \psi_s \right) \tag{2.1}$$

to lighten the notation. We use the notation of partial derivatives such that for $x \in \mathbb{R}^d$

$$\partial_x = (\partial_{x_1}, \cdots, \partial_{x_d}) = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_d} \right)$$

and for Θ, $\partial_\Theta = (\partial_y, \partial_z, \partial_\psi)$. We use the similar notations for every higher order derivative without a detailed indexing. We suppress the obvious summation of indexes throughout the paper for notational simplicity.

Remark

The contents up to Section 4 can be easily extendable to $\mathcal{P} \otimes \mathcal{E}$-measurable compensator $\nu_t(dx)$ as long as $(W, \mu - \nu)$ is *assumed* to have *the weak property of predictable representation* (See Chapter XIII in [17].).

2.3 BMO-martingale and its properties

The properties of the BMO-martingales play a crucial role throughout this work. This section summarizes the necessary facts used in the following discussions.

Definition 2.1. *Let M be a square integrable martingale. When it satisfies*

$$||M||_{BMO}^2 := \sup_{\tau \in \mathcal{T}_0^T} \left|\left| \mathbb{E}\left[(M_T - M_{\tau-} \mathbf{1}_{\tau > 0})^2 | \mathcal{F}_\tau \right] \right|\right|_\infty < \infty$$

then M is called a BMO-martingale and denoted by $M \in BMO$.

Lemma 2.1. *Suppose M is a square integrable martingale with initial value $M_0 = 0$. If M is a BMO-martingale, then its jump component is essentially bounded $\Delta M \in \mathbb{S}^\infty$. On the other hand, if $\Delta M \in \mathbb{S}^\infty$ and*

$$\sup_{\tau \in \mathcal{T}_0^T} \left\| \mathbb{E}\left[\langle M \rangle_T - \langle M \rangle_\tau | \mathcal{F}_\tau \right] \right\|_\infty < \infty,$$

then M is a BMO-martingale.

Proof. From Lemma 10.7 in [17], we have

$$
\begin{aligned}
||M||_{BMO}^2 &= \sup_{\tau \in \mathcal{T}_0^T} \left\| \mathbb{E}\left[[M]_T - [M]_\tau | \mathcal{F}_\tau \right] + M_0^2 \mathbf{1}_{\tau=0} + (\Delta M_\tau)^2 \right\|_\infty \\
&= \sup_{\tau \in \mathcal{T}_0^T} \left\| \mathbb{E}\left[\langle M \rangle_T - \langle M \rangle_\tau | \mathcal{F}_\tau \right] + (\Delta M_\tau)^2 \right\|_\infty .
\end{aligned}
$$

Thus,

$$\sup_{\tau \in \mathcal{T}_0^T} \left\| \mathbb{E}\left[\langle M \rangle_T - \langle M \rangle_\tau | \mathcal{F}_\tau \right] \right\|_\infty \vee ||\Delta M||_{\mathbb{S}^\infty}^2 \le ||M||_{BMO}^2$$

$$\le \sup_{\tau \in \mathcal{T}_0^T} \left\| \mathbb{E}\left[\langle M \rangle_T - \langle M \rangle_\tau | \mathcal{F}_\tau \right] \right\|_\infty + ||\Delta M||_{\mathbb{S}^\infty}^2$$

and hence the claim is proved. $\qquad\square$

Let us introduce the following spaces. $\mathbb{H}_{BMO}^2$ is the set of progressively measurable $\mathbb{R}^d$-valued function Z satisfying [2]

$$||Z||_{\mathbb{H}_{BMO}^2}^2 \quad := \quad \left\| \int_0^\cdot Z_s dW_s \right\|_{BMO}^2 = \sup_{\tau \in \mathcal{T}_0^T} \left\| \mathbb{E}\left[\int_\tau^T |Z_s|^2 ds | \mathcal{F}_\tau \right] \right\|_\infty < \infty.$$

$\mathbb{J}_{BMO}^2$ and $\mathbb{J}_B^2$ are the sets of $\mathcal{P} \otimes \mathcal{E}$-measurable functions $\psi : \Omega \times [0,T] \times \mathbb{E} \to \mathbb{R}^k$ satisfying

$$
\begin{aligned}
||\psi||_{\mathbb{J}_{BMO}^2}^2 &:= \left\| \int_0^\cdot \int_E \psi_s(x) \widetilde{\mu}(ds, dx) \right\|_{BMO}^2 \\
&= \sup_{\tau \in \mathcal{T}_0^T} \left\| \mathbb{E}\left[\int_\tau^T \int_E |\psi_s(x)|^2 \mu(ds, dx) | \mathcal{F}_\tau \right] \right\|_\infty < \infty ,
\end{aligned}
$$

and

$$||\psi||_{\mathbb{J}_B^2}^2 \quad := \quad \sup_{\tau \in \mathcal{T}_0^T} \left\| \mathbb{E}\left[\int_\tau^T \int_E |\psi_s(x)|^2 \nu(dx) ds | \mathcal{F}_\tau \right] \right\|_\infty < \infty,$$

respectively. Note that $(||\psi||_{\mathbb{J}_B^2}^2 \vee ||\psi||_{\mathbb{J}^\infty}^2) \le ||\psi||_{\mathbb{J}_{BMO}^2}^2 \le ||\psi||_{\mathbb{J}_B^2}^2 + ||\psi||_{\mathbb{J}^\infty}^2$ from the proof of Lemma 2.1.

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[2]We sometimes include a scalar function satisfying the rightmost inequality also in $\mathbb{H}_{BMO}^2$. By multiplying a d-dimensional unit vector, one can always connect to it the BMO norm if necessary.

Lemma 2.2 (energy inequality). *Let $Z \in \mathbb{H}^2_{BMO}$ and $\psi \in \mathbb{J}^2_{BMO}$. Then, for any $n \in \mathbb{N}$,*

$$\mathbb{E}\left[\left(\int_0^T |Z_s|^2 ds\right)^n\right] \leq n!\left(||Z||^2_{\mathbb{H}^2_{BMO}}\right)^n,$$

$$\mathbb{E}\left[\left(\int_0^T \int_E |\psi_s(x)|^2 \mu(ds, dx)\right)^n\right] \leq n!\left(||\psi||^2_{\mathbb{J}^2_{BMO}}\right)^n,$$

$$\mathbb{E}\left[\left(\int_0^T \int_E |\psi_s(x)|^2 \nu(dx)ds\right)^n\right] \leq n!\left(||\psi||^2_{\mathbb{J}^2_B}\right)^n \leq n!\left(||\psi||^2_{\mathbb{J}^2_{BMO}}\right)^n.$$

Proof. See proof of Lemma 9.6.5 in [10]. □

Let $\mathcal{E}(M)$ be a Doléan-Dade exponential of M.

Lemma 2.3 (reverse Hölder inequality). *Let $\delta > 0$ be a positive constant and M be a BMO-martingale satisfying $\Delta M_t \geq -1 + \delta$ $\mathbb{P}$-a.s. for all $t \in [0, T]$. Then, $\big(\mathcal{E}_t(M), t \in [0, T]\big)$ is a uniformly integrable martingale, and for every stopping time $\tau \in \mathcal{T}_0^T$, there exists some $p > 1$ such that*

$$\mathbb{E}\left[\mathcal{E}_T(M)^p | \mathcal{F}_\tau\right] \leq C_{p,M} \mathcal{E}_\tau(M)^p$$

with some positive constant $C_{p,M}$ depending only on p and $||M||_{BMO}$.

Proof. See Kazamaki (1979) [21], and also Remark 3.1 of Kazamaki (1994) [22]. □

Note here that the condition $\delta M_t \geq -1 + \delta$ is the very reason why one needs a stronger assumption than the Lipschitz continuity for the comparison principle to hold for the BSDEs with jumps (See Proposition 2.6 in Barles et.al. (1997) [3].). If one relies on the comparison theorem to show the uniform convergence of the BSDE's solution, the same assumption is required. In the current work, by deriving the new stability result, we can restrict its use only to the continuous martingale part and hence avoid this condition.

The following properties of the *continuous* BMO martingales by Kazamaki [22] are very useful.

Lemma 2.4. *Let M be a square integrable continuous martingale and $\hat{M} := \langle M \rangle - M$. Then, $M \in BMO(\mathbb{P})$ if and only if $\hat{M} \in BMO(\mathbb{Q})$ with $d\mathbb{Q}/d\mathbb{P} = \mathcal{E}_T(M)$. Furthermore, $||\hat{M}||_{BMO(\mathbb{Q})}$ is determined by some function of $||M||_{BMO(\mathbb{P})}$ and vice versa.*

Proof. See Theorem 3.3 and Theorem 2.4 in [22]. □

Remark

For continuous martingales, Theorem 3.1 [22] also tells that there exists some decreasing function $\Phi(p)$ with $\Phi(1+) = \infty$ and $\Phi(\infty) = 0$ such that if $||M||_{BMO(\mathbb{P})}$ satisfies

$$||M||_{BMO(\mathbb{P})} < \Phi(p)$$

then $\mathcal{E}(M)$ satisfies the reverse Hölder inequality with power p. This implies together with Lemma 2.4, one can take a common positive constant $\bar{r}$ satisfying $1 < \bar{r} \leq r^*$ such that both of the $\mathcal{E}(M)$ and $\mathcal{E}(\hat{M})$ satisfy the reverse Hölder inequality with power $\bar{r}$ under the respective probability measure $\mathbb{P}$ and $\mathbb{Q}$. Furthermore, the upper bound r^* is determined only by $||M||_{BMO(\mathbb{P})}$ (or equivalently by $||M||_{BMO(\mathbb{Q})}$).

3 $Q_{\exp}$-growth BSDEs with Jumps

3.1 Universal Bound

We now introduce, for $t \in [0, T]$, the quadratic-exponential ($Q_{\exp}$) growth BSDE;

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, \psi_s) ds - \int_t^T Z_s dW_s - \int_t^T \int_E \psi_s(x) \widetilde{\mu}(ds, dx) , \qquad (3.1)$$

where $\xi : \Omega \to \mathbb{R}$, $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^2(E, \nu; \mathbb{R}^k) \to \mathbb{R}$ and denote Z and ψ as row vectors for simplicity.

Let us introduce the quadratic-exponential structure condition proposed by Barrieu & El Karoui (2013) [4] and extended to a jump diffusion case by Ngoupeyou (2010) [29].

Assumption 3.1. *The map $(\omega, t) \mapsto f(\omega, t, \cdot)$ is $\mathbb{F}$-progressively measurable. For every $(y, z, \psi) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^2(E, \nu; \mathbb{R}^k)$, there exist two constants $\beta \geq 0$ and $\gamma > 0$ and a positive $\mathbb{F}$-progressively measurable process $(l_t, t \in [0, T])$ such that*

$$-l_t - \beta|y| - \frac{\gamma}{2}|z|^2 - \int_E j_\gamma(-\psi(x)) \nu(dx)$$

$$\leq f(t, y, z, \psi) \leq l_t + \beta|y| + \frac{\gamma}{2}|z|^2 + \int_E j_\gamma(\psi(x)) \nu(dx)$$

$d\mathbb{P} \otimes dt$-a.e. $(\omega, t) \in \Omega \times [0, T]$, where $j_\gamma(u) := \frac{1}{\gamma}\left(e^{\gamma u} - 1 - \gamma u\right)$.

This assumption yields a very useful universal bound for the possible solutions of (3.1). The following result is an adaptation of Proposition 3.2 in [4] and Proposition 16 in [29] to our case.

Lemma 3.1. *Assume the driver f of (3.1) satisfies Assumption 3.1 and $\exp\left(\gamma e^{\beta T}|\xi|\right)$ and $\exp\left(\gamma e^{\beta T} \int_0^T l_s ds\right)$ are in $\mathbb{L}^1(\Omega, \mathcal{F}_T; \mathbb{P})$. Then, if there exists a solution to the BSDE (3.1), it satisfies*

$$|Y_t| \leq \frac{1}{\gamma} \ln \mathbb{E}\left[\exp\left(\gamma e^{\beta(T-t)}|\xi| + \gamma \int_t^T e^{\beta(t-s)} l_s ds\right)\Big|\mathcal{F}_t\right] .$$

In particular, when $||\xi||_\infty, ||l||_{\mathbb{S}^\infty} < \infty$, the solution Y is essentially bounded:

$$||Y||_{\mathbb{S}^\infty} \leq e^{\beta T}\left(||\xi||_\infty + T||l||_{\mathbb{S}^\infty}\right) .$$

Proof. A simple application of Itô formula yields

$$d\left(e^{\beta s}|Y_s|\right) = e^{\beta s}\left(\beta|Y_s| ds + d|Y_s|\right)$$

$$= e^{\beta s}\left\{\beta|Y_s| ds + \text{sign}(Y_{s-})\left(-f(s, \Theta_s) ds + Z_s dW_s + \int_E \psi_s(x) \widetilde{\mu}(ds, dx)\right) + dL_s^Y\right\}$$

where L^Y is a non-decreasing process including a local time of Y at the origin. Let us define the process $(B_s, s \in [0, T])$ with $B_0 = 0$ by

$$dB_s = -\text{sign}(Y_s) f(s, \Theta_s) ds + \left(l_s + \beta|Y_s| + \frac{\gamma}{2}|Z_s|^2 + \int_E j_\gamma(\text{sign}(Y_s)\psi_s(x)) \nu(dx)\right) ds$$

which is also a non-decreasing process by Assumption 3.1. Using this process,

$$d(e^{\beta s}|Y_s|) = e^{\beta s}(dB_s + dL_s^Y) + e^{\beta s}\text{sign}(Y_{s-})\Big(Z_s dW_s + \int_E \psi_s(x)\widetilde{\mu}(ds, dx)\Big)$$
$$-e^{\beta s}\Big(l_s + \frac{\gamma}{2}|Z_s|^2 + \int_E j_\gamma(\text{sign}(Y_s)\psi_s(x))\nu(dx)\Big)ds ,$$

which is further transformed as

$$d(e^{\beta s}|Y_s|) = e^{\beta s}\text{sign}(Y_{s-})\Big(Z_s dW_s + \int_E \psi_s(x)\widetilde{\mu}(ds, dx)\Big)$$
$$-\frac{\gamma}{2}\big|e^{\beta s}\text{sign}(Y_s)Z_s\big|^2 ds - \int_E j_\gamma(e^{\beta s}\text{sign}(Y_s)\psi_s(x))\nu(dx)ds - e^{\beta s}l_s ds$$
$$+\frac{\gamma}{2}\Big(e^{2\beta s}|Z_s|^2 - e^{\beta s}|Z_s|^2\Big)ds + \int_E \Big(j_\gamma(e^{\beta s}\text{sign}(Y_s)\psi_s(x)) - e^{\beta s}j_\gamma(\text{sign}(Y_s)\psi_s(x))\Big)\nu(dx)ds$$
$$+e^{\beta s}(dB_s + dL_s^Y) .$$

It is easy to confirm that for $k \geq 1$,

$$j_\gamma(kx) - kj_\gamma(x) = \frac{1}{\gamma}(e^{k\gamma x} - ke^{\gamma x} - 1 + k) \geq 0 .$$

Thus we obtain

$$d(e^{\beta s}|Y_s|) = e^{\beta s}\text{sign}(Y_{s-})\Big(Z_s dW_s + \int_E \psi_s(x)\widetilde{\mu}(ds, dx)\Big)$$
$$-\frac{\gamma}{2}\big|e^{\beta s}\text{sign}(Y_s)Z_s\big|^2 ds - \int_E j_\gamma(e^{\beta s}\text{sign}(Y_s)\psi_s(x))\nu(dx)ds - e^{\beta s}l_s ds + dC_s,$$

where C is a non-decreasing process.

Define the process P by $P_t := \exp\Big(\gamma e^{\beta t}|Y_t| + \gamma \int_0^t e^{\beta s}l_s ds\Big)$. Since

$$dP_t = P_{t-}\Big(\gamma e^{\beta t}\text{sign}(Y_t)Z_t dW_t + \int_E \Big(\exp\big(\gamma e^{\beta t}\text{sign}(Y_{t-})\psi_t(x)\big) - 1\Big)\widetilde{\mu}(dt, dx) + \gamma dC_t\Big),$$

P is a submartingale. Therefore,

$$\exp\Big(\gamma e^{\beta t}|Y_t| + \gamma \int_0^t e^{\beta s}l_s ds\Big) \leq \mathbb{E}\Big[\exp\Big(\gamma e^{\beta T}|\xi| + \gamma \int_0^T e^{\beta s}l_s ds\Big)\Big|\mathcal{F}_t\Big] ,$$

which proves the first claim. The second claim is obvious using the first result. $\qquad\square$

Let us mention the fact that $(e^x - 1)^2 + (e^{-x} - 1)^2 \geq x^2, \quad \forall x \in \mathbb{R}$. In order to prove this, set $g(x) := (e^x - 1)^2 + (e^{-x} - 1)^2 - x^2$. Then,

$$g'(x) = 2(e^x - 1)e^x + 2(1 - e^{-x})e^{-x} - 2x$$

which is an odd function. It is easy to see that $g'(x) \geq 0$ for $x \geq 0$ and $g'(0) = 0$. Thus $g(x) \geq g(0) = 0$. The next lemma connects the bounded solution $||Y||_{\mathbb{S}^\infty} < \infty$ and the BMO-properties of the control variables.

Lemma 3.2. *Assume $||\xi||_\infty, ||l||_{\mathbb{S}^\infty} < \infty$ and Assumption 3.1 hold. If there exists a solution (Y, Z, ψ) to the BSDE (3.1), then $Z \in \mathbb{H}^2_{BMO}$ and $\psi \in \mathbb{J}^2_{BMO}$ (and hence $\psi \in \mathbb{J}^\infty$) and $||Z||_{\mathbb{H}^2_{BMO}} + ||\psi||_{\mathbb{J}^2_{BMO}}$ is bounded by some constant depending only on $(\gamma, \beta, T, ||\xi||_\infty, ||l||_{\mathbb{S}^\infty})$.*

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Proof. By Lemma 3.1, we have $||Y||_{\mathbb{S}^\infty} < \infty$. Since $||\psi||_{J^\infty} \le 2||Y||_{\mathbb{S}^\infty}$, it is clear that $\psi \in \mathbb{J}^\infty$. Applying Itô formula to $e^{2\gamma Y_t}$ and using the equality $2\gamma j_{2\gamma}(x) = (e^{\gamma x} - 1)^2 + 2\gamma j_\gamma(x)$, one obtains

$$
\int_\tau^{\tau_n} e^{2\gamma Y_s} 2\gamma^2 |Z_s|^2 ds + \int_\tau^{\tau_n} e^{2\gamma Y_s} \left(e^{\gamma \psi_s(x)} - 1\right)^2 \nu(dx) ds
$$
$$
= e^{2\gamma Y_{\tau_n}} - e^{2\gamma Y_\tau} + 2\gamma \int_\tau^{\tau_n} e^{2\gamma Y_s} \left(f(s, Y_s, Z_s, \psi_s) - \int_E j_\gamma(\psi_s(x))\nu(dx)\right) ds
$$
$$
- \int_\tau^{\tau_n} e^{2\gamma Y_s} 2\gamma Z_s dW_s - \int_\tau^{\tau_n} \int_E e^{2\gamma Y_{s-}} \left(e^{2\gamma \psi_s(x)} - 1\right) \widetilde{\mu}(ds, dx) ,
$$

where $\tau \in \mathcal{T}_0^T$, and $\{\tau_n\}_{n \in \mathbb{N}}$ is a localizing sequence of the last line.

Taking a conditional expectation and using Assumption 3.1, one obtains

$$
\mathbb{E}\left[\int_\tau^{\tau_n} e^{2\gamma Y_s} \gamma^2 |Z_s|^2 ds + \int_\tau^{\tau_n} e^{2\gamma Y_s} \left(e^{\gamma \psi_s(x)} - 1\right)^2 \nu(dx) ds \Big| \mathcal{F}_\tau\right]
$$
$$
\le \mathbb{E}\left[e^{2\gamma Y_{\tau_n}} + 2\gamma \int_\tau^{\tau_n} e^{2\gamma Y_s}(l_s + \beta|Y_s|) ds \Big| \mathcal{F}_\tau\right]
$$
$$
\le e^{2\gamma ||Y||_{\mathbb{S}^\infty}} + 2\gamma e^{2\gamma ||Y||_{\mathbb{S}^\infty}} T\left(\beta ||Y||_{\mathbb{S}^\infty} + ||l||_{\mathbb{S}^\infty}\right) .
$$

By taking $n \to \infty$,

$$
\mathbb{E}\left[\int_\tau^T \gamma^2 |Z_s|^2 ds + \int_\tau^T \left(e^{\gamma \psi_s(x)} - 1\right)^2 \nu(dx) ds \Big| \mathcal{F}_\tau\right]
$$
$$
\le e^{4\gamma ||Y||_{\mathbb{S}^\infty}} + 2\gamma e^{4\gamma ||Y||_{\mathbb{S}^\infty}} T\left(\beta ||Y||_{\mathbb{S}^\infty} + ||l||_{\mathbb{S}^\infty}\right) . \tag{3.2}
$$

Similar calculation for $e^{-2\gamma Y_t}$ yields

$$
\mathbb{E}\left[\int_\tau^T \gamma^2 |Z_s|^2 ds + \int_\tau^T \left(e^{-\gamma \psi_s(x)} - 1\right)^2 \nu(dx) ds \Big| \mathcal{F}_\tau\right]
$$
$$
\le e^{4\gamma ||Y||_{\mathbb{S}^\infty}} + 2\gamma e^{4\gamma ||Y||_{\mathbb{S}^\infty}} T\left(\beta ||Y||_{\mathbb{S}^\infty} + ||l||_{\mathbb{S}^\infty}\right) . \tag{3.3}
$$

Thus, adding (3.2) and (3.3), and then taking $\sup_\tau || \ ||_\infty$ separately for Z and ψ terms yields

$$
||Z||_{\mathbb{H}^2_{BMO}}^2 + ||\psi||_{\mathbb{J}^2_B}^2 \le \frac{e^{4\gamma ||Y||_{\mathbb{S}^\infty}}}{\gamma^2}\left(3 + 6\gamma T\left(\beta ||Y||_{\mathbb{S}^\infty} + ||l||_{\mathbb{S}^\infty}\right)\right) < \infty.
$$

Since $||\psi||_{\mathbb{J}^\infty} \le 2||Y||_{\mathbb{S}^\infty}$, one also sees $||\psi||_{\mathbb{J}^2_{BMO}} \le ||\psi||_{\mathbb{J}^2_B} + ||\psi||_{\mathbb{J}^\infty} < \infty$. $\qquad \square$

3.2 Stability and Uniqueness

We now introduce local Lipschitz conditions to derive a stability and uniqueness result for a bounded solution.

Assumption 3.2. *(i) $|\xi|$ and $(l_t, t \in [0,T])$ are essentially bounded: $||\xi||_\infty + ||l||_{\mathbb{S}^\infty} < \infty$. (ii) For each $M > 0$, and for every $(y, z, \psi), (y', z', \psi') \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^2(E, \nu; \mathbb{R}^k)$ satisfying*

$$
|y|, |y'|, ||\psi||_{\mathbb{L}^\infty(\nu)}, ||\psi'||_{\mathbb{L}^\infty(\nu)} \le M
$$

there exists some positive constant K_M possibly depending on M such that

$$\left| f(t, y, z, \psi) - f(t, y', z', \psi') \right| \leq K_M \left(|y - y'| + ||\psi - \psi'||_{\mathbb{L}^2(\nu)} \right)$$
$$+ K_M \left(1 + |z| + |z'| + ||\psi||_{\mathbb{L}^2(\nu)} + ||\psi'||_{\mathbb{L}^2(\nu)} \right) |z - z'|$$

$d\mathbb{P} \otimes dt$-*a.e.* $(\omega, t) \in \Omega \times [0, T]$.

Consider the two BSDEs with $i \in \{1, 2\}$ satisfying Assumptions 3.1 and 3.2;

$$Y_t^i = \xi^i + \int_t^T f^i(s, Y_s^i, Z_s^i, \psi_s^i) ds - \int_t^T Z_s^i dW_s - \int_t^T \int_E \psi_s^i(x) \widetilde{\mu}(ds, dx), \qquad (3.4)$$

for $t \in [0, T]$ and let us denote

$$\delta Y := Y^1 - Y^2, \quad \delta Z := Z^1 - Z^2, \quad \delta \psi := \psi^1 - \psi^2,$$
$$\delta f(s) := (f^1 - f^2)(s, Y_s^1, Z_s^1, \psi_s^1) \ .$$

Lemma 3.3. *Suppose Assumptions 3.1 and 3.2 hold for the two BSDEs (3.4) with $i \in \{1, 2\}$. Then, if there exists a solution $(Y^i, Z^i, \psi^i), i \in \{1, 2\}$ to the BSDEs, the following inequality is satisfied;*

(a) $||\delta Z||_{\mathbb{H}_{BMO}^2} + ||\delta \psi||_{\mathbb{J}_{BMO}^2} \leq C \left(||\delta Y||_{\mathbb{S}^\infty} + ||\delta \xi||_\infty + \sup_{\tau \in \mathcal{T}_0^T} \left| \left| \mathbb{E} \left[\int_\tau^T |\delta f(s)| ds \Big| \mathcal{F}_\tau \right] \right| \right|_\infty \right)$

(b) $\left| \left| (\delta Y, \delta Z, \delta \psi) \right| \right|_{\mathcal{K}^p[0,T]}^p \leq C' \left(\mathbb{E} \left[|\delta \xi|^{p \bar{q}^2} + \left(\int_0^T |\delta f(s)| ds \right)^{p \bar{q}^2} \right] \right)^{\frac{1}{\bar{q}^2}}, \forall p \geq 2, \ \forall \bar{q} \geq q_*$

Here, C and $q_* (> 1)$ are positive constants depending only on $(K_M, \gamma, \beta, T, ||\xi||_\infty, ||l||_{\mathbb{S}^\infty})$ and the constant M is chosen such that $||Y^i||_{\mathbb{S}^\infty}, ||\psi^i||_{\mathbb{J}^\infty} \leq M$ for both $i \in \{1, 2\}$. C' is a positive constant depending only on $(p, \bar{q}, K_M, \gamma, \beta, T, ||\xi||_\infty, ||l||_{\mathbb{S}^\infty})$.

Proof. Proof for (a)
Firstly, due to the universal bound, it is obvious that one can choose M such that $||Y^i||_{\mathbb{S}^\infty} \leq M$ and $||\psi^i||_{\mathbb{J}^\infty} \leq M$ for both $i \in \{1, 2\}$. Set a sequence of $\mathbb{F}$-stopping times as

$$\tau_n := \inf \left\{ t \geq 0; \int_0^t |\delta Z_s|^2 ds + \int_0^t \int_E |\delta \psi_s(x)|^2 \mu(ds, dx) \geq n \right\} \wedge T \ .$$

Then, for $\forall \tau \in \mathcal{T}_0^T$, one has

$$|\delta Y_\tau|^2 + \int_\tau^{\tau_n} |\delta Z_s|^2 ds + \int_\tau^{\tau_n} \int_E |\delta \psi_s(x)|^2 \mu(ds, dx)$$
$$= |\delta Y_{\tau_n}|^2 + \int_\tau^{\tau_n} 2\delta Y_s \left(\delta f(s) + f^2(s, \Theta_s^1) - f^2(s, \Theta_s^2) \right) ds$$
$$- \int_\tau^{\tau_n} 2\delta Y_s \delta Z_s dW_s - \int_\tau^{\tau_n} \int_E 2\delta Y_{s-} \delta \psi_s(x) \widetilde{\mu}(ds, dx) \ .$$

Taking the conditional expectation and passing to the limit $n \to \infty$, one obtains

$$|\delta Y_\tau|^2 + \mathbb{E} \left[\int_\tau^T |\delta Z_s|^2 ds + \int_\tau^T \int_E |\delta \psi_s(x)|^2 \mu(ds, dx) \Big| \mathcal{F}_\tau \right]$$
$$= \mathbb{E} \left[|\delta \xi|^2 + \int_\tau^T 2\delta Y_s \left(\delta f(s) + f^2(s, \Theta_s^1) - f^2(s, \Theta_s^2) \right) ds \Big| \mathcal{F}_\tau \right] \ .$$

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Taking $\sup_{\tau \in \mathcal{T}_0^T}$ for each term in the left gives

$$||\delta Z||_{\mathbb{H}^2_{BMO}}^2 + ||\delta \psi||_{\mathbb{J}^2_{BMO}}^2 \leq 2||\delta \xi||_\infty^2$$
$$+ 4||\delta Y||_{\mathbb{S}^\infty} \sup_{\tau \in \mathcal{T}_0^T} \left|\left| \mathbb{E}\left[\int_\tau^T \Big(|\delta f(s)| + K_M\big(|\delta Y|_s + ||\delta \psi_s||_{\mathbb{L}^2(\nu)} + H_s|\delta Z_s|\big)\Big)ds \Big| \mathcal{F}_\tau\right]\right|\right|_\infty ,$$

where the process H is defined by $H_s := 1 + \sum_{i=1}^2 \big(|Z_s^i| + ||\psi_s^i||_{\mathbb{L}^2(\nu)}\big)$. It is clear that $H \in \mathbb{H}^2_{BMO}$ whose norm is dominated by the universal bound given in Lemma 3.2. One can see

$$\sup_{\tau \in \mathcal{T}_0^T} \left|\left| \mathbb{E}\left[\int_\tau^T H_s|\delta Z_s|ds \Big| \mathcal{F}_\tau\right]\right|\right|_\infty \leq \sup_{\tau \in \mathcal{T}_0^T}\left|\left|\mathbb{E}\left[\int_\tau^T |H_s|^2 ds\Big|\mathcal{F}_\tau\right]^{\frac{1}{2}}\right|\right|_\infty \sup_{\tau \in \mathcal{T}_0^T}\left|\left|\mathbb{E}\left[\int_\tau^T |\delta Z_s|^2 ds\Big|\mathcal{F}_\tau\right]^{\frac{1}{2}}\right|\right|_\infty$$
$$\leq ||H||_{\mathbb{H}^2_{BMO}}||\delta Z||_{\mathbb{H}^2_{BMO}} .$$

Thus, with an arbitrary positive constant $\epsilon > 0$,

$$||\delta Z||_{\mathbb{H}^2_{BMO}}^2 + ||\delta \psi||_{\mathbb{J}^2_{BMO}}^2 \leq 2||\delta \xi||_\infty^2 + 2\sup_{\tau \in \mathcal{T}_0^T}\left|\left|\mathbb{E}\left[\int_\tau^T |\delta f(s)|ds \Big| \mathcal{F}_\tau\right]\right|\right|_\infty^2$$
$$+||\delta Y||_{\mathbb{S}^\infty}^2\left(2 + 4K_M T + \frac{4K_M^2}{\epsilon} + \frac{4K_M^2}{\epsilon}||H||_{\mathbb{H}^2_{BMO}}^2\right) + \epsilon\Big(||\delta Z||_{\mathbb{H}^2_{BMO}}^2 + ||\delta \psi||_{\mathbb{J}^2_B}^2\Big).$$

Since $||\delta \psi||_{\mathbb{J}^2_B} \leq ||\delta \psi||_{\mathbb{J}^2_{BMO}}$, choosing $\epsilon < 1$ yields the desired result.

Proof for (b)
Define a d-dimensional $\mathbb{F}$-progressively measurable process $(b_s, s \in [0,T])$ by

$$b_s := \frac{f^2(s, Y_s^1, Z_s^1, \psi_s^1) - f^2(s, Y_s^1, Z_s^2, \psi_s^1)}{|\delta Z_s|^2}\mathbf{1}_{\delta Z_s \neq 0}\delta Z_s$$

and also the map $\widetilde{f} : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{L}^2(E, \nu; \mathbb{R}^k) \to \mathbb{R}$:

$$\widetilde{f}(\omega, s, \bar{y}, \bar{\psi}) := \delta f(\omega, s) - f^2(\omega, s, \Theta_s^2) + f^2\big(\omega, s, \bar{y} + Y_s^2, Z_s^2, \bar{\psi} + \psi_s^2\big) .$$

Then, $(\delta Y, \delta Z, \delta \psi)$ is the solution to the BSDE

$$\delta Y_t = \delta \xi + \int_t^T \Big(\widetilde{f}(s, \delta Y_s, \delta \psi_s) + b_s \cdot \delta Z_s\Big)ds - \int_t^T \delta Z_s dW_s - \int_t^T \int_E \delta \psi_s(x)\widetilde{\mu}(ds, dx). \tag{3.5}$$

Since $|b_s| \leq K_M(1 + |Z_s^1| + |Z_s^2| + 2||\psi_s^1||_{\mathbb{L}^2(\nu)})$, the process b belongs to $\mathbb{H}^2_{BMO}$. Furthermore, $\widetilde{f}$ satisfies the linear growth property $|\widetilde{f}(s, \bar{y}, \bar{\psi})| \leq |\delta f(s)| + K_M(|\bar{y}| + ||\bar{\psi}||_{\mathbb{L}^2(\nu)})$. Thus, the BSDE (3.5) satisfies Assumption A.1 with $g = |\delta f|$. One obtains the desired result by applying Lemma A.1. The dependency of the constants C', q_* is obtained from the universal bound in Lemmas 3.1 and 3.2, as well as the properties of the reverse Hölder inequality in Lemma 2.3 and the remarks that follow. $\qquad \square$

We now gives the uniqueness result:

Proposition 3.1. *Suppose the BSDE (3.1) satisfies Assumption 3.1 and 3.2. Then, if there exists a solution (Y, Z, ψ) to (3.1), it is unique in the space $\mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$.*

Proof. By Lemmas 3.1 and 3.2, if there exists a solution it satisfies $(Y, Z, \psi) \in \mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$. Firstly, by Lemma 3.3 (b), the solution is unique in the space $\mathcal{K}^p[0,T]$

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for $\forall p \geq 2$. Since $Y \in \mathbb{S}^\infty$, the uniqueness of Y in $\mathbb{S}^p$ gives the uniqueness of Y also in the space $\mathbb{S}^\infty$. This can be shown by the argument of contradiction. Suppose that there exist two solution $Y^1, Y^2 \in \mathbb{S}^\infty$ which are equal in the space of $\mathbb{S}^p$ i.e., $||Y^1 - Y^2||_{\mathbb{S}^p}^p = 0$ but not equal in $\mathbb{S}^\infty$. This implies that there exists some constant $a > 0$ such that

$$\left|\left| \sup_{t \in [0,T]} |Y_t^1 - Y_t^2| \right|\right|_\infty = a .$$

Then, for any $0 < b < a$, there exists some positive constant $0 < c \leq 1$ such that

$$\mathbb{P}\left(\sup_{t \in [0,T]} |Y_t^1 - Y_t^2| \geq b \right) = c .$$

This gives $||Y^1 - Y^2||_{\mathbb{S}^p}^p \geq b^p c > 0$ and hence yields a contradiction. Combined with Lemma 3.3 (a), the solution (Y, Z, ψ) is unique in the space $\mathbb{S}^\infty \times \mathbb{H}_{BMO}^2 \times \mathbb{J}_{BMO}^2$. $\quad\square$

4 Existence of solution to a $Q_{\exp}$-growth BSDE

4.1 An approximating sequence of globally Lipschitz BSDEs

In this section, we shall prove the existence of a unique solution to the BSDE (3.1) under Assumptions 3.1 and 3.2. For this purpose, we first consider an approximating sequence of globally Lipschitz BSDEs for which the existence and uniqueness of the solutions are well known.

Let us introduce a sequence of mollifiers $\varphi_m : \mathbb{R} \to \mathbb{R}$ with $m \in \mathbb{N}$ which are continuously differentiable and with the following properties:

$$\varphi_m(x) = \begin{cases} -(m+1) & \text{for } x \leq -(m+2) \\ x & \text{for } |x| \leq m \\ m+1 & \text{for } x \geq m+2 \end{cases}$$

and $|\partial_x \varphi_m(x)| \leq 1$ uniformly in $x \in \mathbb{R}$. We also denote, for any $\psi \in \mathbb{L}^2(\nu)$ and $x \in \mathbb{R}$,

$$(\psi \circ \zeta_m)(x) := \psi(x) \mathbf{1}_{|x| \geq \frac{1}{m}} .$$

We then define the truncated driver $f_m : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^2(E, \nu; \mathbb{R}^k)$ by

$$f_m(\omega, t, y, z, \psi) := f\left(\omega, t, \varphi_m(y), \varphi_m(z), \varphi_m(\psi \circ \zeta_m) \right) , \tag{4.1}$$

and consider a sequence of truncated BSDEs:

$$Y_t^m = \xi + \int_t^T f_m(s, Y_s^m, Z_s^m, \psi_s^m) ds - \int_t^T Z_s^m dW_s - \int_t^T \psi_s^m(x) \widetilde{\mu}(ds, dx) \tag{4.2}$$

for $t \in [0,T]$.

Lemma 4.1. *The truncated driver $f_m(\omega, t, y, z, \psi)$ in (4.1) with $\forall m \in \mathbb{N}$ satisfies the global Lipschitz condition. Furthermore it also satisfies the quadratic-exponential growth condition given in Assumption 3.1 uniformly in m.*

Proof. Let us put $C_m := k \max_{i \in \{1, \cdots, k\}} \left(\int_{\mathbb{R}_0} \mathbf{1}_{|x| \geq \frac{1}{m}} \nu^i(dx) \right) < \infty$ then one sees

$$\left|\left|\varphi_m(\psi_s \circ \zeta_m)\right|\right|^2_{\mathbb{L}^2(\nu)} = \int_E |\varphi_m(\psi_s \circ \zeta_m(x))|^2 \nu(dx) \leq (m+1)^2 C_m.$$

Thus, by taking $M \geq (k \vee d)(m+1)$, the truncated driver satisfies for any $(y, z, \psi), (y', z', \psi') \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^2(E, \nu; \mathbb{R}^k)$,

$$|f_m(t, y, z, \psi) - f_m(t, y', z', \psi')| \leq K_M \Big(|\varphi_m(y) - \varphi_m(y')| + ||\varphi_m(\psi \circ \zeta_m) - \varphi_m(\psi' \circ \zeta_m)||_{\mathbb{L}^2(\nu)} \Big)$$

$$+ K_M \Big(1 + |\varphi_m(z)| + |\varphi_m(z')| + ||\varphi_m(\psi \circ \zeta_m)||_{\mathbb{L}^2(\nu)} + ||\varphi_m(\psi' \circ \zeta_m)||_{\mathbb{L}^2(\nu)} \Big) |\varphi_m(z) - \varphi_m(z')|$$

$$\leq K_M \Big(|y - y'| + ||\psi - \psi'||_{\mathbb{L}^2(\nu)} + \big[1 + 2d(m+1) + 2(m+1)^2 C_m \big] |z - z'| \Big),$$

which proves the global Lipschitz condition.

The truncated driver also satisfies

$$-l_t - \beta|y| - \frac{\gamma}{2}|\varphi_m(z)|^2 - \int_E j_\gamma \Big(-\varphi_m(\psi \circ \zeta_m(x)) \Big) \nu(dx)$$

$$\leq f_m(t, y, z, \psi) \leq l_t + \beta|y| + \frac{\gamma}{2}|\varphi_m(z)|^2 + \int_E j_\gamma \Big(\varphi_m(\psi \circ \zeta_m(x)) \Big) \nu(dx).$$

From the convexity of the positive function $j_\gamma(u)$, we have $\int_E j_\gamma(\pm\varphi_m(\psi \circ \zeta_m(x)))\nu(dx) \leq \int_E j_\gamma(\pm\psi(x))\nu(dx)$. Therefore,

$$-l_t - \beta|y| - \frac{\gamma}{2}|z|^2 - \int_E j_\gamma(-\psi(x))\nu(dx)$$

$$\leq f_m(t, y, z, \psi) \leq l_t + \beta|y| + \frac{\gamma}{2}|z|^2 + \int_E j_\gamma(\psi(x))\nu(dx),$$

which proves the second claim. $\qquad \square$

We are now ready to give the existence result for the $Q_{\exp}$-growth BSDE.

Theorem 4.1. *Under Assumptions 3.1 and 3.2, there exists a solution (Y, Z, ψ) to the BSDE (3.1) which is unique in the space $\mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$.*

Proof. We consider a sequence of the approximating BSDEs (4.2) with $m \in \mathbb{N}$. By the standard result for the Lipschitz BSDEs (See, for example, Lemma B.2 in [16].), there exists a unique solution $(Y^m, Z^m, \psi^m) \in \mathcal{K}^p[0, T]$ for $\forall p \geq 2$. Furthermore, by the second claim of Lemma 4.1 and also by the universal bound given in Lemmas 3.1 and 3.2, one can conclude that $\Theta^m := (Y^m, Z^m, \psi^m) \in \mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$ and also that the norm $||Y^m||_{\mathbb{S}^\infty} + ||Z^m||_{\mathbb{H}^2_{BMO}} + ||\psi^m||_{\mathbb{J}^2_{BMO}}$ is bounded uniformly in m by some constant depending only on $(\gamma, \beta, T, ||\xi||_\infty, ||l||_\infty)$. Thus, one can fix a positive constant M such that $||Y^m||_{\mathbb{S}^\infty}, ||\psi^m||_{\mathbb{J}^\infty} \leq M$ uniformly in m so that K_M becomes m-independent.

Put, for each $m, n \in \mathbb{N}$,

$$\delta Y^{m,n} := Y^m - Y^n, \quad \delta Z^{m,n} := Z^m - Z^n, \quad \psi^{m,n} := \psi^m - \psi^n,$$
$$\delta f^{m,n}(s) := (f_m - f_n)(s, Y^m_s, Z^m_s, \psi^m_s).$$

Since the BSDE (4.2) with $\forall m \in \mathbb{N}$ satisfies Assumptions 3.1 and 3.2, Lemma 3.3 (b) implies that $||(\delta Y^{m,n}, \delta Z^{m,n}, \delta \psi^{m,n})||^p_{\mathcal{K}^p} \leq C' \left(\mathbb{E} \left[\left(\int_0^T |\delta f^{m,n}(s)| ds \right)^{p\bar{q}^2} \right] \right)^{\frac{1}{\bar{q}^2}}$ for $\forall p \geq 2$.

Thanks to the universal bound discussed above, one can take C' and $\bar{q} > 1$ independent of m, n. Assumption 3.2 and the definition of the truncated driver implies

$$
\begin{aligned}
&|\delta f^{m,n}(s)| \\
&= \left| f\big(s, \varphi_m(Y_s^m), \varphi_m(Z_s^m), \varphi_m(\psi_s^m \circ \zeta_m)\big) - f\big(s, \varphi_n(Y_s^m), \varphi_n(Z_s^m), \varphi_n(\psi_s^m \circ \zeta_n)\big) \right| \\
&\le K_M \left(|Y_s^m| \mathbf{1}_{\{|Y_s^m| \ge m \wedge n\}} + \left(\int_E |\psi_s^m(x)|^2 \mathbf{1}_{\{|\psi_s^m(x)| \ge m \wedge n\}} \nu(dx) \right)^{\frac{1}{2}} \right) \\
&\quad + K_M \left(1 + 2|Z_s^m| + 2||\psi_s^m||_{\mathbb{L}^2(\nu)} \right) |Z_s^m| \mathbf{1}_{\{|Z_s^m| \ge m \wedge n\}} .
\end{aligned}
$$

Therefore, by the Cauchy-Schwartz inequality,

$$
\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T |\delta f^{m,n}(s)| ds \right)^{p\bar{q}^2} \right] \le C \mathbb{E}\left[||Y^m||_T^{2p\bar{q}^2} \right]^{\frac{1}{2}} \mathbb{E}\left[\left(\int_0^T \mathbf{1}_{\{|Y_s^m| \ge m \wedge n\}} ds \right)^{2p\bar{q}^2} \right]^{\frac{1}{2}} \\
&\quad + C\mathbb{E}\left[\left(\int_0^T \int_E |\psi_s^m(x)|^2 \mathbf{1}_{\{|\psi_s^m(x)| \ge m \wedge n\}} \nu(dx) ds \right)^{\frac{p\bar{q}^2}{2}} \right] \\
&\quad + C\mathbb{E}\left[\left(\int_0^T |H^m(s)|^2 ds \right)^{p\bar{q}^2} \right]^{\frac{1}{2}} \mathbb{E}\left[\left(\int_0^T |Z_s^m|^2 \mathbf{1}_{\{|Z_s^m| \ge m \wedge n\}} ds \right)^{p\bar{q}^2} \right]^{\frac{1}{2}} , \quad\quad (4.3)
\end{aligned}
$$

where $H^m := 1 + 2|Z^m| + 2||\psi^m||_{\mathbb{L}^2(\nu)}$ and C is some positive constant depending only on $(K_M, T, p, \bar{q})$. We know that $||Y^m||_{\mathbb{S}^\infty}, ||Z^m||_{\mathbb{H}^2_{BMO}}, ||\psi^m||_{\mathbb{J}^2_{BMO}}$ are bounded uniformly in m and thus the energy inequality in Lemma 2.2 allows us to apply the extended Fatou's lemma (See, Theorem 7.5.2 in [2], for example) to (4.3).

This gives

$$
\begin{aligned}
&\lim_{m,n \to \infty} \mathbb{E}\left[\left(\int_0^T \int_E |\psi_s^m(x)|^2 \mathbf{1}_{\{|\psi_s^m(x)| \ge m \wedge n\}} \nu(dx) ds \right)^{\frac{p\bar{q}^2}{2}} \right] \\
&\quad \le \mathbb{E}\left[\limsup_{m,n \to \infty} \left(\int_0^T \int_E |\psi_s^m(x)|^2 \mathbf{1}_{\{|\psi_s^m(x)| \ge m \wedge n\}} \nu(dx) ds \right)^{\frac{p\bar{q}^2}{2}} \right],
\end{aligned}
$$

and also

$$
\begin{aligned}
&\lim_{m,n \to \infty} \mathbb{E}\left[\left(\int_0^T |Z_s^m|^2 \mathbf{1}_{\{|Z_s^m| \ge m \wedge n\}} ds \right)^{p\bar{q}^2} \right] \\
&\quad \le \mathbb{E}\left[\limsup_{m,n \to \infty} \left(\int_0^T |Z_s^m|^2 \mathbf{1}_{\{|Z_s^m| \ge m \wedge n\}} ds \right)^{p\bar{q}^2} \right] ,
\end{aligned}
$$

both of which converge to zero since the integrands go to zero $d\mathbb{P} \otimes ds$-a.e., because otherwise $||\psi^m||_{\mathbb{J}^2_{BMO}}, ||Z^m||_{\mathbb{H}^2_{BMO}}$ must diverge which contradicts the fact. Passing to the limit $m, n \to \infty$ in (4.3) yields $\lim_{m,n \to \infty} \mathbb{E}\left[\left(\int_0^T |\delta f^{m,n}(s)| ds \right)^{p\bar{q}^2} \right] = 0$. Thus one can conclude that $\lim_{m,n \to \infty} ||(\delta Y^{m,n}, \delta Z^{m,n}, \delta \psi^{m,n})||_{\mathcal{K}^p}^p = 0$ and that there exists $(Y, Z, \psi) \in \mathcal{K}^p$ to which (Y^m, Z^m, ψ^m) converges in the space $\mathcal{K}^p$ for $\forall p \ge 2$. By construction of the approximating BSDEs, it is easy to see that (Y, Z, ψ) satisfies the original BSDE (3.1). The uniqueness of the solution is already proved in Proposition 3.1.

One can also see the strong convergence directly in the space $\mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$. Since $\mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$ is a Banach space, the universal bound on (Y^m, Z^m, ψ^m) implies that (Y, Z, ψ) also belongs to this space. By the argument used in the proof in

Proposition 3.1, one can show that $\lim_{m,n\to\infty}||Y^m - Y^n||_{\mathbb{S}^\infty} = 0$. Thus, by Lemma 3.3 (a), one obtains

$$\lim_{m,n\to\infty}\left(||\delta Z^{m,n}||_{\mathbb{H}^2_{BMO}} + ||\delta\psi^{m,n}||_{\mathbb{J}^2_{BMO}}\right) \leq \lim_{m,n\to\infty} C\left(\sup_{\tau\in\mathcal{T}_0^T}\left|\left|\mathbb{E}\left[\int_\tau^T |\delta f^{m,n}(s)|ds|\mathcal{F}_\tau\right]\right|\right|_\infty\right) .$$

Since $\left(\sup_{\tau\in\mathcal{T}_0^T}\left|\left|\mathbb{E}\left[\int_\tau^T |\delta f^{m,n}(s)|ds|\mathcal{F}_\tau\right]\right|\right|_\infty\right)$ is bounded uniformly in m, n due to the universal bound, one can exchange the order of lim and sup operations to get

$$\lim_{m,n\to\infty}\left(||\delta Z^{m,n}||_{\mathbb{H}^2_{BMO}} + ||\delta\psi^{m,n}||_{\mathbb{J}^2_{BMO}}\right) \leq C\left(\sup_{\tau\in\mathcal{T}_0^T}\left|\left|\lim_{m,n\to\infty}\mathbb{E}\left[\int_\tau^T |\delta f^{m,n}(s)|ds|\mathcal{F}_\tau\right]\right|\right|_\infty\right) ,$$

which converges to zero since the convergence $|\varphi_m(\Theta^m) - \varphi_n(\Theta^m)| \to 0$ as $m, n \to \infty$ in $\mathcal{K}^p$ for $\forall p \geq 2$ has already been shown. Thus, one can conclude that (Y^m, Z^m, ψ^m) converges to (Y, Z, ψ) also in the space $\mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$. $\qquad\square$

5 Malliavin Differentiability

In the reminder of the paper, we study the Malliavin differentiability of the quadratic-exponential growth BSDEs. Among the various ways to develop Malliavin's calculus, we follow the conventions based on the chaos expansion used in Delong & Imkeller (2010) [13] and Delong (2013) [12], which were adopted from the work of Solé et.al. (2007) [32]. See also Di Nunno et.al. (2009) [14] for an extension to a multi-dimensional setup and other applications (with only a slight adjustment of conventions).

For the detailed conventions, see Section 3 of [13]. Following the extension given in Section 17 of [14], we denote $(D_{t,0}^i, i \in \{1,\cdots,d\})$ and $(D_{t,z}^i, i \in \{1,\cdots,k\})$ as the Malliavin derivatives with respect to $(W_i(t), i \in \{1,\cdots,d\})$ and $(\widetilde{\mu}^i(dt,dz), i \in \{1,\cdots,k\})$, respectively.

Note that a random variable F is Malliavin differentiable if and only if $F \in \mathbb{D}^{1,2}$. Here, the space $\mathbb{D}^{1,2} \subset \mathbb{L}^2(\mathbb{P})$ is defined by the completion with respect to the norm $||\cdot||_{1,2}$ which is given by

$$||F||_{1,2}^2 := \mathbb{E}\left[|F|^2\right] + \sum_{i=1}^d \mathbb{E}\left[\int_0^T |D_{s,0}^i F|^2 ds\right] + \sum_{i=1}^k \mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} |D_{s,z}^i F|^2 z^2 \nu^i(dz)ds\right] .$$

For notational convenience, let us introduce a finite measure $m^i(dz) = z^2\nu^i(dz)$ with $i \in \{1,\cdots,k\}$ defined on $\mathbb{R}_0$ and also q defined on $\widetilde{E} := [0,T] \times \mathbb{R}^k$ by

$$q(dt,dz) = \mathbf{1}_{z=0}dt + \sum_{i=1}^k \mathbf{1}_{z\neq 0}z^2\nu^i(dz)dt.$$

We also introduce a space $\mathbb{L}^{1,2}(\mathbb{R}^n)$ of product measurable and $\mathbb{F}$-adapted processes $\chi : \Omega \times [0,T] \times \mathbb{R}^k \to \mathbb{R}^n$ satisfying

$$\mathbb{E}\left[\int_{\widetilde{E}} |\chi(s,y)|^2 q(ds,dy)\right] < \infty,$$

$$\chi(s,y) \in \mathbb{D}^{1,2}(\mathbb{R}^n), \quad \text{for } q\text{-a.e. } (s,y) \in \widetilde{E},$$

$$\mathbb{E}\left[\int_{\widetilde{E}}\int_{\widetilde{E}} |D_{t,z}\chi(s,y)|^2 q(ds,dy)q(dt,dz)\right] < \infty.$$

Note that the space $\mathbb{L}^{1,2}$ is a Hilbert space endowed with the norm

$$||\chi||^2_{\mathbb{L}^{1,2}} := \mathbb{E}\left[\int_{\widetilde{E}}|\chi(s,y)|^2 q(ds,dy)\right] + \mathbb{E}\left[\int_{\widetilde{E}}\int_{\widetilde{E}}|D_{t,z}\chi(s,y)|^2 q(ds,dy)q(dt,dz)\right] .$$

The fact that the Malliavin derivative is a closed operator in $\mathbb{L}^{1,2}$ (See, Theorem 12.6 in [14]) plays a crucial role later.

Suppose that (t,z) is a jump of size z at time t in a random measure μ^i. We denote by $\omega^{t,z}_{\mu^i}$ a transformed family of $\omega_{\mu^i} = ((t_1,z_1),(t_2,z_2),\cdots) \in \Omega_{\mu^i}$ into a new family with additional jump at (t,z); $\omega^{t,z}_{\mu^i} = ((t,z),(t_1,z_1),(t_2,z_2),\cdots) \in \Omega_{\mu^i}$. As for an element $\omega = (\omega_W, \omega_{\mu^1}, \omega_{\mu^2}, \cdots, \omega_{\mu^k}) \in \Omega$ in the full canonical product space, we denote $\omega^{t,z} \in \Omega$ as the above transformation only in the corresponding element, such as $\omega^{t,z} = (\omega_W, \omega_{\mu^1}, \cdots, \omega^{t,z}_{\mu^i}, \cdots, \omega_{\mu^k}) \in \Omega$ without specifying the relevant coordinate for notational simplicity. By the same reason, we also frequently omit i denoting the direction of derivative $D^i_{s,z}$ by assuming that we consider each Wiener ($z = 0, i \in \{1, \cdots, d\}$) and jump ($z \neq 0, i \in \{1, \cdots, k\}$)) direction separately (and summing them up whenever necessary, such as when considering integration on $\widetilde{E}$).

In order to apply the *chain* rule of the Malliavin derivative (Theorem 3.5 and Theorem 12.8 in [14] with the division by the jump size in the current convention), we adopt the following explicit form of the driver:

$$Y_t = \xi + \int_t^T f\left(s, Y_s, Z_s, \int_{\mathbb{R}_0}\rho(x)G(s,\psi_s(x))\nu(dx)\right)ds$$

$$- \int_t^T Z_s dW_s - \int_t^T \int_E \psi_x(x)\widetilde{\mu}(ds,dx), \quad t \in [0,T], \tag{5.1}$$

where $\xi : \Omega \to \mathbb{R}$, $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}$, and $\rho^i : \mathbb{R} \to \mathbb{R}$, $G^i : [0,T] \times \mathbb{R} \to \mathbb{R}$ for each $i \in \{1, \cdots, k\}$. The last arguments of the driver denotes a k-dimensional vector whose i-th element is given by

$$\int_{\mathbb{R}_0}\rho^i(x)G^i(s,\psi^i_s(x))\nu^i(dx) .$$

With slight abuse of notation, we adopt

$$\Theta_r := \left(Y_r, Z_r, \int_{\mathbb{R}_0}\rho(z)G(r,\psi_r(z))\nu(dz)\right), \quad r \in [0,T]$$

as a collective argument in this section.

Let us make the following assumptions for ρ and G:

Assumption 5.1. *(i) For every $i \in \{1, \cdots, k\}$, ρ^i is a continuous function satisfying $\int_{\mathbb{R}_0}|\rho^i(x)|^2\nu^i(dx) < \infty$.*
(ii) For every $i \in \{1, \cdots, k\}$, $G^i(s,v)$ is a continuous function in the both arguments and one-time continuously differentiable with respect to v with continuous derivative. Moreover, for every $R > 0$,

$$G_R := \sup_{(s,v)\in[0,T]\times(|v|\leq R)} \sum_{i=1}^k |G^i(s,v)| < \infty,$$

$$G'_R := \sup_{(s,v)\in[0,T]\times(|v|\leq R)} \sum_{i=1}^k |\partial_v G^i(s,v)| < \infty .$$

We put without loss of generality that $G^i(\cdot, 0) = 0$ for every $i \in \{1, \cdots, k\}$.

We modify Assumptions 3.1 and 3.2 according to the current parametrization:

Assumption 5.2. *The map $(\omega, t) \mapsto f(\omega, t, \cdot)$ is $\mathbb{F}$-progressively measurable. For every $(y, z, \psi) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^2(E, \nu; \mathbb{R}^k)$, there exist two constants $\beta \geq 0$ and $\gamma > 0$ and a positive $\mathbb{F}$-progressively measurable process $(l_t, t \in [0, T])$ such that*

$$-l_t - \beta|y| - \frac{\gamma}{2}|z|^2 - \int_E j_\gamma(-\psi(x))\nu(dx) \leq f\left(t, y, z, \int_{\mathbb{R}_0} \rho(x)G(t, \psi(x))\nu(dx)\right)$$

$$\leq l_t + \beta|y| + \frac{\gamma}{2}|z|^2 + \int_E j_\gamma(\psi(x))\nu(dx)$$

$d\mathbb{P} \otimes dt$-a.e. $(\omega, t) \in \Omega \times [0, T]$, where $j_\gamma(u) := \frac{1}{\gamma}\left(e^{\gamma u} - 1 - \gamma u\right)$.

Assumption 5.3. *(i) $|\xi|$ and $(l_t, t \in [0, T])$ are essentially bounded: $||\xi||_\infty + ||l||_{\mathbb{S}^\infty} < \infty$.*
(ii) For each $M > 0$, and for every $(y, z, \psi), (y', z', \psi') \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^2(E, \nu; \mathbb{R}^k)$ satisfying

$$|y|, |y'|, ||\psi||_{\mathbb{L}^\infty(\nu)}, ||\psi'||_{\mathbb{L}^\infty(\nu)} \leq M$$

there exists some positive constant K_M possibly depending on M such that

$$\left|f\left(t, y, z, u_t\right) - f\left(t, y', z', u'_t\right)\right| \leq K_M\left(|y - y'| + |u_t - u'_t|\right)$$

$$+K_M\left(1 + |z| + |z'| + |u_t| + |u'_t|\right)|z - z'|$$

$d\mathbb{P} \otimes dt$-a.e. $(\omega, t) \in \Omega \times [0, T]$, where we have used $u_t := \int_{\mathbb{R}_0} \rho(x)G(t, \psi(x))\nu(dx)$ and $u'_t := \int_{\mathbb{R}_0} \rho(x)G(t, \psi'(x))\nu(dx)$ for notational simplicity.

Remark

In the above assumption, using the fact that

$$|u_t| \leq ||\rho||_{\mathbb{L}^2(\nu)}G'_M||\psi||_{\mathbb{L}^2(\nu)}, \quad |u_t - u'_t| \leq ||\rho||_{\mathbb{L}^2(\nu)}G'_M||\psi - \psi'||_{\mathbb{L}^2(\nu)},$$

one can see the consistency with Assumption 3.2. Therefore, under Assumptions 5.1, 5.2 and 5.3, there exists a unique solution $(Y, Z, \psi) \in \mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$ to the BSDE (5.1).

In order to obtain Malliavin differentiability, we need the following additional assumptions:

Assumption 5.4. *With the notation $u_t = \int_{\mathbb{R}_0} \rho(x)G(t, \psi(x))\nu(dx)$, $u'_t = \int_{\mathbb{R}_0} \rho(x)G(t, \psi'(x))\nu(dx)$,*
(i) The terminal value is Malliavin differentiable; $\xi \in \mathbb{D}^{1,2}$.
(ii) For each $M > 0$, and for every $(y, z, \psi) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^2(E, \nu; \mathbb{R}^k)$ satisfying $|y|, ||\psi||_{\mathbb{L}^\infty(\nu)} \leq M$, the driver $\left(f(t, y, z, u_t), t \in [0, T]\right)$ belongs to $\mathbb{L}^{1,2}(\mathbb{R})$ and its Malliavin derivative is denoted by $(D_{s,z}f)(t, y, z, u_t)$. Furthermore, the driver f is one-time continuously differentiable with respect to its spacial variables with continuous derivatives.
(iii) For every Wiener as well as jump direction, for every $M > 0$ and $d\mathbb{P} \otimes dt$-a.e. $(\omega, t) \in \Omega \times [0, T]$, and for every $(y, z, \psi), (y', z', \psi') \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^2(E, \nu; \mathbb{R}^k)$ satisfying $|y|, |y'|, ||\psi||_{\mathbb{L}^\infty(\nu)}, ||\psi'||_{\mathbb{L}^\infty(\nu)} \leq M$, the Malliavin derivative of the driver satisfies the following local Lipschitz conditions;

$$\left|(D^i_{s,0}f)(t, y, z, u_t) - (D^i_{s,0}f)(t, y', z', u')\right|$$

$$\leq K^{M,i}_{s,0}(t)\left(|y - y'| + |u_t - u'_t| + \left(1 + |z| + |z'| + |u_t| + |u'_t|\right)|z - z'|\right)$$

for ds-a.e. $s \in [0, T]$ with $i \in \{1, \cdots, d\}$, and

$$\left|(D_{s,z}^i f)(t, y, z, u_t) - (D_{s,z}^i f)(t, y', z', u_t')\right|$$
$$\leq K_{s,z}^{M,i}(t)\left(|y - y'| + |u_t - u_t'| + \left(1 + |z| + |z'| + |u_t| + |u_t'|\right)|z - z'|\right)$$

for $m^i(dz)ds$-a.e. $(s, z) \in [0, T] \times \mathbb{R}_0$ with $i \in \{1, \cdots, k\}$. For every $M > 0$ and (s, z), $\left(K_{s,0}^{M,i}(t), t \in [0, T]\right)_{i \in \{1, \cdots, d\}}$ and $\left(K_{s,z}^{M,i}(t), t \in [0, T]\right)_{i \in \{1, \cdots, k\}}$ are $\mathbb{R}_+$-valued $\mathbb{F}$-progressively measurable processes.

(iv) There exists some positive constant $p \geq 2$ such that

$$\int_{\widetilde{E}} \left(\mathbb{E}\left[|D_{s,z}\xi|^{pq} + \left(\int_0^T |(D_{s,z}f)(r, 0)|dr\right)^{pq} + ||K_{s,z}^M||_T^{2pq}\right]\right)^{\frac{1}{q}} q(ds, dz) < \infty$$

hold for $\forall q \geq 1$ and $\forall M > 0$.

(v) For the constant $p \geq 2$ in (iv), the following equality

$$\lim_{\epsilon \downarrow 0} \sum_{i=1}^k \int_0^T \int_{|z| \leq \epsilon} \left(\mathbb{E}\left[|D_{s,z}\xi|^{pq} + \left(\int_0^T |(D_{s,z}f)(r, 0)|dr\right)^{pq} + ||K_{s,z}^M||_T^{2pq}\right]\right)^{\frac{1}{p}} m^i(dz)ds = 0$$

hold for $\forall q \geq 1$ and $\forall M > 0$.

Remark

Assumption 5.4 (iv) implies, for $q(ds, dz)$-a.e. $(s, z) \in \widetilde{E}$,

$$\mathbb{E}\left[|D_{s,z}\xi|^{p'} + \left(\int_0^T |(D_{s,z}f)(r, 0)|dr\right)^{p'} + ||K_{s,z}^M||_T^{2p'}\right] < \infty$$

for $\forall p' \geq 2$. We now give the main result of this section.

Theorem 5.1. *Suppose that Assumptions 5.1, 5.2, 5.3 and 5.4 hold true and denote the solution to the BSDE (5.1) as $(Y, Z, \psi) \in \mathbb{S}^\infty \times \mathbb{H}_{BMO}^2 \times \mathbb{J}_{BMO}^2$. Then, the following statements hold: (a) For each Wiener direction $i \in \{1, \cdots, d\}$ and ds-a.e. $s \in [0, T]$, there exists a unique solution $(Y^{s,0,i}, Z^{s,0,i}, \psi^{s,0,i}) \in \mathcal{K}^{p'}[0, T]$ with $\forall p' \geq 2$ to the BSDE*

$$Y_t^{s,0,i} = D_{s,0}^i \xi + \int_t^T f^{s,0,i}(r)dr - \int_t^T Z_r^{s,0,i}dW_r - \int_t^T \int_E \psi_r^{s,0,i}(x)\widetilde{\mu}(dr, dx) \qquad (5.2)$$

for $0 \leq s \leq t \leq T$, where

$$
\begin{aligned}
f^{s,0,i}(r) &:= (D_{s,0}^i f)(r, \Theta_r) + \partial_\Theta f(r, \Theta_r)\Theta_r^{s,0,i} \\
&= (D_{s,0}^i f)(r, \Theta_r) + \partial_y f(r, \Theta_r)Y_r^{s,0,i} + \partial_z f(r, \Theta_r)Z_r^{s,0,i} \\
&\quad + \partial_u f(r, \Theta_r)\int_E \rho(x)\partial_v G(r, \psi_r(x))\psi_r^{s,0,i}(x)\nu(dx) \ .
\end{aligned}
$$

The solution also satisfies $\int_0^T ||(Y^{s,0,i}, Z^{s,0,i}, \psi^{s,0,i})||_{\mathcal{K}^p[0,T]}^p ds < \infty$.

(b) For each jump direction $i \in \{1, \cdots, k\}$ and $m^i(dz)ds$-a.e $(s, z) \in [0, T] \times \mathbb{R}_0$, there exists a unique solution $(Y^{s,z,i}, Z^{s,z,i}, \psi^{s,z,i}) \in \mathbb{S}^\infty \times \mathbb{H}_{BMO}^2 \times \mathbb{J}_{BMO}^2$ to the BSDE

$$Y_t^{s,z,i} = D_{s,z}^i \xi + \int_t^T f^{s,z,i}(r)dr - \int_t^T Z_r^{s,z,i}dW_r - \int_t^T \int_E \psi_r^{s,z,i}(x)\widetilde{\mu}(dr, dx) \qquad (5.3)$$

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for $0 \le s \le t \le T$ and $z \ne 0$, where

$$f^{s,z,i}(r) := \frac{1}{z}\Big(f(\omega^{s,z}, r, \Theta_r + z\Theta_r^{s,z,i}) - f(\omega, r, \Theta_r)\Big) = \frac{1}{z}\Big\{f\Big(\omega^{s,z}, r, Y_r + zY_r^{s,z,i}$$

$$, Z_r + zZ_r^{s,z,i}, \int_{\mathbb{R}_0} \rho(x)G\big(r, \psi_r(x) + z\psi_r^{s,z,i}(x)\big)\nu(dx)\Big) - f(\omega, r, \Theta_r)\Big\} .$$

The solution also satisfies $\int_0^T \int_{\mathbb{R}_0} ||(Y^{s,z,i}, Z^{s,z,i}, \psi^{s,z,i})||_{\mathcal{K}^p[0,T]}^p m^i(dz)ds < \infty .$

(c) The solution of the BSDE (5.1) is Malliavin differentiable $(Y, Z, \psi) \in \mathbb{L}^{1,2} \times \mathbb{L}^{1,2} \times \mathbb{L}^{1,2}$. Put, for every i, $Y_t^{s,\cdot,i} = Z_t^{s,\cdot,i} = \psi_t^{s,\cdot,i}(\cdot) \equiv 0$ for $t < s \le T$, then $\Big((Y_t^{s,z,i}, Z_t^{s,z,i}, \psi_t^{s,z,i}(x)), 0 \le s, t \le T, x \in \mathbb{R}_0, z \in \mathbb{R}\Big)$ is a version of the Malliavin derivative $\Big((D_{s,z}^i Y_t, D_{s,z}^i Z_t, D_{s,z}^i \psi_t(x)), 0 \le s, t \le T, x \in \mathbb{R}_0, z \in \mathbb{R}\Big)$ for every Wiener and jump direction.

Proof. Firstly, from Assumptions 5.1, 5.2 and 5.3, Theorem 4.1 tells us that there exists a unique solution $(Y, Z, \psi) \in \mathbb{S}^\infty \times \mathbb{H}_{BMO}^2 \times \mathbb{J}_{BMO}^2$. Since $||Y||_{\mathbb{S}^\infty}, ||\psi||_{\mathbb{J}^\infty} < \infty$, one can choose a constant $M > 0$ big enough so that the local Lipschitz conditions hold true for the whole relevant range. We choose one such M and fix it throughout the proof. We also omit the superscript i denoting the direction of derivative by assuming that we always discuss each direction separately.

Proof for (a): Firstly, the continuous differentiability of f with respect to the spacial variables and the local Lipschitz conditions imply that

$$|\partial_y f(t, y, z, u_t)| \le K_M, \quad |\partial_u f(t, y, z, u_t)| \le K_M,$$
$$|\partial_z f(t, y, z, u_t)| \le K_M(1 + 2|z| + 2|u_t|) .$$

Thus, it is easy to check that the BSDE (5.2) satisfies Assumption A.2. Note that

$$|(D_{s,0}f)(r, \Theta_r)| \le |(D_{s,0}f)(r, 0)| + K_{s,0}^M(|Y_r| + ||\rho||_{\mathbb{L}^2(\nu)}G_M'||\psi_r||_{\mathbb{L}^2(\nu)})$$
$$+ K_{s,0}^M(1 + |Z_r| + ||\rho||_{\mathbb{L}^2(\nu)}G_M'||\psi||_{\mathbb{L}^2(\nu)})|Z_r|$$

Thus, by Assumption 5.4 (iv) and Theorem A.1, there exists a unique solution to the BSDE (5.2) satisfying, for $\forall p' \ge 2$,

$$||(Y^{s,0}, Z^{s,0}, \psi^{s,0})||_{\mathcal{K}^{p'}[0,T]}^{p'} \le C\left(\mathbb{E}\left[|D_{s,0}\xi|^{p'\bar{q}^2} + \left(\int_0^T |(D_{s,0}f)(r, \Theta_r)|dr\right)^{p'\bar{q}^2}\right]\right)^{\frac{1}{\bar{q}^2}}$$

$$\le C\left(1 + \mathbb{E}\left[|D_{s,0}\xi|^{p'\bar{q}^2} + \left(\int_0^T |(D_{s,0}f)(r, 0)|dr\right)^{p'\bar{q}^2} + ||K_{s,0}^M||_T^{2p'\bar{q}^2}\right.\right.$$

$$\left.\left. + ||Y||_T^{2p'\bar{q}^2} + \left(\int_0^T |Z_r|^2 dr\right)^{2p'\bar{q}^2} + \left(\int_0^T ||\psi_r||_{\mathbb{L}^2(\nu)}^2 dr\right)^{2p'\bar{q}^2}\right]\right)^{\frac{1}{\bar{q}^2}} < \infty,$$

where C is a positive constant depending only on $(p', \bar{q}, \beta, \gamma, T, ||\xi||_\infty, ||l||_{\mathbb{S}^\infty}, K_M)$ and $\bar{q}$ is a positive constant satisfying $1 < q_* \le \bar{q} < \infty$ where the lower bound q_* is an increasing function of the $\mathbb{H}_{BMO}^2$-norm of $Z^{s,0}$'s coefficient, which is also controlled by the universal bound given by $(\beta, \gamma, T, ||\xi||_\infty, ||l||_{\mathbb{S}^\infty})$. We have also used Lemma 2.2 to obtain the last inequality. When $p' = p$, the above inequality together with Assumption 5.4 (iv) implies

$$\int_0^T ||(Y^{s,0}, Z^{s,0}, \psi^{s,0})||_{\mathcal{K}^p[0,T]}^p ds < \infty .$$

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Proof for (b): Let us first consider the BSDE

$$\mathcal{Y}_t^{s,z} = \xi(\omega^{s,z}) + \int_t^T f\Big(\omega^{s,z}, r, \mathcal{Y}_r^{s,z}, \mathcal{Z}_r^{s,z}, \int_{\mathbb{R}_0} \rho(x)G(r, \Psi_r^{s,z}(x))\nu(dx)\Big)dr$$

$$- \int_t^T \mathcal{Z}_r^{s,z}dW_r - \int_t^T \int_E \Psi_r^{s,z}(x)\widetilde{\mu}(dr, dx) \ . \tag{5.4}$$

Although each path ω has an additional jump, the BSDE satisfies the same local Lipschitz as well as the quadratic-exponential structure conditions (Assumptions 3.1 and 3.2) for $m(dz)ds$-a.e. $(s,z) \in [0,T] \times \mathbb{R}_0$. Thus, by Theorem 4.1, there exists a unique solution $(\mathcal{Y}^{s,z}, \mathcal{Z}^{s,z}, \Psi^{s,z}) \in \mathbb{S}^\infty \times \mathbb{H}_{BMO}^2 \times \mathbb{J}_{BMO}^2$ to the BSDE (5.4) satisfying the universal bound given in Lemma 3.1 and 3.2. Now, let us define,

$$Y^{s,z} := \frac{\mathcal{Y}^{s,z} - Y}{z}, \quad Z^{s,z} := \frac{\mathcal{Z}^{s,z} - Z}{z}, \quad \psi^{s,z} := \frac{\Psi^{s,z} - \psi}{z} \ ,$$

and then $(Y^{s,z}, Z^{s,z}, \psi^{s,z}) \in \mathbb{S}^\infty \times \mathbb{H}_{BMO}^2 \times \mathbb{J}_{BMO}^2$ is a solution to the BSDE (5.3). Note that $D_{s,z}\xi := \frac{1}{z}(\xi(\omega^{s,z}) - \xi(\omega))$. The uniqueness is obvious since otherwise $(zY^{s,z} + Y, zZ^{s,z} + Z, z\psi^{s,z} + \psi)$ gives another solution to to the BSDE (5.4). Let us introduce a new collective argument as $\Xi^{s,z} := \Big(\mathcal{Y}^{s,z}, \mathcal{Z}^{s,z}, \int_{\mathbb{R}_0} \rho(x)G(r, \Psi_r^{s,z}(x))\nu(dx)\Big)$. Then,

$$\begin{aligned} f^{s,z}(r) &= \frac{1}{z}\Big(f(\omega^{s,z}, r, \Xi^{s,z}) - f(\omega, r, \Theta_r)\Big) \\ &= (D_{s,z}f)(r, \Theta_r) + \frac{f(\omega^{s,z}, r, \Xi_r^{s,z}) - f(\omega^{s,z}, r, \Theta_r)}{z} \end{aligned}$$

Let us put a d-dimensional $\mathbb{F}$-progressively measurable process $(b_r^{s,z}, r \in [0,T])$ by

$$\begin{aligned} b_r^{s,z}(\omega) &:= \frac{1}{|\mathcal{Z}_r^{s,z} - Z_r|^2}\Big\{f\Big(\omega^{s,z}, r, Y_r, \mathcal{Z}_r^{s,z}, \int_{\mathbb{R}_0} \rho(x)G(r, \psi_r(x))\nu(dx)\Big) \\ &\quad -f\Big(\omega^{s,z}, r, Y_r, Z_r, \int_{\mathbb{R}_0} \rho(x)G(r, \psi_r(x))\nu(dx)\Big)\Big\}\mathbf{1}_{\mathcal{Z}_r^{s,z} - Z_r \neq 0}(\mathcal{Z}_r^{s,z} - Z_r) \end{aligned}$$

and the map $\widetilde{f}^{s,z} : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{L}^2(E, \nu; \mathbb{R}_k) \to \mathbb{R}$ by,

$$\begin{aligned} \widetilde{f}^{s,z}(\omega, r, \bar{y}, \bar{\psi}) &:= D_{s,z}f(r, \Theta_r) + \frac{1}{z}\Big\{f\Big(\omega^{s,z}, r, z\bar{y} + Y_r, \mathcal{Z}_r^{s,z}, \int_{\mathbb{R}_0} \rho(x)G(r, z\bar{\psi}(x) + \psi_r(x))\nu(dx)\Big) \\ &\quad -f\Big(\omega^{s,z}, r, Y_r, \mathcal{Z}_r^{s,z}, \int_{\mathbb{R}_0} \rho(x)G(r, \psi_r(x))\nu(dx)\Big)\Big\} \ . \end{aligned}$$

Then, $(Y^{s,z}, Z^{s,z}, \psi^{s,z})$ can also be expressed as a solution to the BSDE

$$Y_t^{s,z} = D_{s,z}\xi + \int_t^T \Big(\widetilde{f}^{s,z}(r, Y_r^{s,z}, \psi_r^{s,z}) + b_r^{s,z} \cdot Z_r^{s,z}\Big)dr - \int_t^T Z_r^{s,z}dW_r - \int_t^T \int_E \psi_r^{s,z}(x)\widetilde{\mu}(dr, dx) \ .$$

Note that $|b_r| \leq H_r$, $r \in [0,T]$ where $H_r := K_M\Big(1 + |\mathcal{Z}_r|^{s,z} + |Z_r| + 2||\rho||_{\mathbb{L}^2(\nu)}G_M'||\psi_r||_{\mathbb{L}^2(\nu)}\Big)$ and that $H \in \mathbb{H}_{BMO}^2$. Furthermore, the new driver satisfies a linear growth property $|\widetilde{f}(r, \bar{y}, \bar{\psi})| \leq |(D_{s,z}f)(r, \Theta_r)| + K_M\Big(|\bar{y}| + ||\rho||_{\mathbb{L}^2(\nu)}G_M'||\bar{\psi}||_{\mathbb{L}^2(\nu)}\Big)$. Thus, Lemma A.1 implies

that

$$\||(Y^{s,z}, Z^{s,z}, \psi^{s,z})\||^{p'}_{\mathcal{K}^{p'}[0,T]} \leq C \left(\mathbb{E}\left[|D_{s,z}\xi|^{p'\bar{q}^2} + \left(\int_0^T |(D_{s,z}f)(r,\Theta_r)|dr\right)^{p'\bar{q}^2} \right] \right)^{\frac{1}{\bar{q}^2}}$$

$$\leq C \left(1 + \mathbb{E}\left[|D_{s,z}\xi|^{p'\bar{q}^2} + \left(\int_0^T |(D_{s,z}f)(r,0)|dr\right)^{p'\bar{q}^2} + \||K^M_{s,z}\||^{2p'\bar{q}^2}_T \right.\right.$$

$$\left.\left. + \||Y\||^{2p'\bar{q}^2}_T + \left(\int_0^T |Z_r|^2 dr\right)^{2p'\bar{q}^2} + \left(\int_0^T \||\psi_r\||^2_{\mathbb{L}^2(\nu)}dr\right)^{2p'\bar{q}^2} \right] \right)^{\frac{1}{\bar{q}^2}} < \infty$$

for $\forall p' \geq 2$, where a positive constant C depending only on $(p', \bar{q}, \beta, \gamma, T, \||\xi\||_\infty, \||l\||_{\mathbb{S}^\infty}, K_M)$ and $\bar{q}$ is a positive constant satisfying $q_* \leq \bar{q} < \infty$ where $q_* > 1$ is determined by $\||H\||_{\mathbb{H}^2_{BMO}}$. Choosing $p' = p$ together with Assumption 5.4 (iv) implies

$$\int_0^T \int_{\mathbb{R}_0} \||(Y^{s,z}, Z^{s,z}, \psi^{s,z})\||^p_{\mathcal{K}^p[0,T]} m(dz)ds < \infty \ ,$$

which proves the second claim of (b). Note that, we also have

$$\int_{\widetilde{E}} \||(Y^{s,z}, Z^{s,z}, \psi^{s,z})\||^p_{\mathcal{K}^p[0,T]} q(ds, dz) < \infty$$

by combining the results (a) and (b).

Proof for (c): First step (Approximating sequence of globally Lipschitz BSDEs)
We finally proceed to the proof for (c). Firstly, let us introduce a truncated driver as in Section 4.1. For $m \in \mathbb{N}$, we define

$$G_m(s, \psi(x)) := G\Big(s, \varphi_m(\psi \circ \zeta_m(x))\Big), \quad f_m(s, y, z, u) := f\Big(s, \varphi_m(y), \varphi_m(z), u\Big) \ ,$$

and thus for every $(s, y, z, \psi) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^2(E, \nu; \mathbb{R}^k)$,

$$f_m\Big(s, y, z, \int_{\mathbb{R}_0} \rho(x)G_m(s, \psi(x))\nu(dx)\Big)$$

$$= f\Big(s, \varphi_m(y), \varphi_m(z), \int_{\mathbb{R}_0} \rho(x)G\Big(s, \varphi_m(\psi \circ \zeta_m(x))\Big)\nu(dx)\Big) \ .$$

We now introduce a sequence of approximating BSDEs,

$$Y^m_t = \xi + \int_t^T f_m\Big(r, Y^m_r, Z^m_r, \int_{\mathbb{R}_0} \rho(x)G_m(r, \psi^m_r(x))\nu(dx)\Big)dr$$

$$- \int_t^T Z^m_r dW_r - \int_t^T \int_E \psi^m_r(x)\widetilde{\mu}(dr, dx) \ . \tag{5.5}$$

As in Lemma 4.1, it is not difficult to check that for each m the truncated driver f_m is globally Lipschitz. Furthermore, it also satisfies the quadratic-exponential structure condition in Assumption 5.2 *uniformly* in $m \in \mathbb{N}$. Thus there exists a unique solution $(Y^m, Z^m, \psi^m) \in \mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$ to the BSDE (5.5) satisfying the universal bound given in Lemma 3.1 and 3.2. Thus, there exists some constant $C > 0$ depending only on $(\beta, \gamma, T, \||\xi\||_\infty, \||l\||_{\mathbb{S}^\infty})$ such that

$$\sup_{m \in \mathbb{N}} \Big(\||Y^m\||_{\mathbb{S}^\infty} + \||Z^m\||_{\mathbb{H}^2_{BMO}} + \||\psi^m\||_{\mathbb{J}^2_{BMO}}\Big) \leq C \ .$$

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This also implies $M > 0$ can be chosen such that f_m satisfies the same local Lipschitz property as the original driver f given in Assumption 5.3 with the coefficient K_M independent of m. From the proof for Theorem 4.1, $(Y^m, Z^m, \psi^m) \to (Y, Z, \psi)$ in $\mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$.

One can also check that, for each $m \in \mathbb{N}$, the BSDE (5.5) satisfies Assumptions B.1 as well as B.2. Therefore Theorem B.1 implies that the approximating BSDEs are Malliavin differentiable i.e., $(Y^m, Z^m, \psi^m) \in \mathbb{L}^{1,2} \times \mathbb{L}^{1,2} \times \mathbb{L}^{1,2}$ for $\forall m \in \mathbb{N}$.

Second step (Uniform boundedness of $\mathbb{L}^{1,2}$-norm of the approximating BSDEs)

From the *first step*, one can define the Malliavin derivatives of (Y^m, Z^m, ψ^m) for every $m \in \mathbb{N}$ as the solution to the following BSDEs: For every Wiener direction $i \in \{1, \cdots, d\}$, ds-a.e. $s \in [0, T]$ and $s \leq t \leq T$,

$$D^i_{s,0} Y^m_t = D^i_{s,0} \xi + \int_t^T D^i_{s,0} f_m(r) dr - \int_t^T D^i_{s,0} Z^m_r dW_r - \int_t^T \int_E D^i_{s,0} \psi^m_r(x) \widetilde{\mu}(dr, dx),$$

$$D^i_{s,0} f_m(r) := (D_{s,0} f_m)(r, \Theta^m_r) + \partial_\Theta f_m(r, \Theta^m_r) D^i_{s,0} \Theta^m_r, \tag{5.6}$$

and for jump direction $i \in \{1, \cdots, k\}$, $m^i(dz) ds$-a.e. $(s, z) \in [0, T] \times \mathbb{R}_0$ and $s \leq t \leq T$,

$$D^i_{s,z} Y^m_t = D^i_{s,z} \xi + \int_t^T D^i_{s,z} f_m(r) dr - \int_t^T D^i_{s,z} Z^m_r dW_r - \int_t^T \int_E \psi^m_r(x) \widetilde{\mu}(dr, dx),$$

$$D^i_{s,z} f_m(r) := \frac{1}{z} \Big(f_m(\omega^{s,z}, r, \Theta^m_r + z D^i_{s,z} \Theta^m_r) - f_m(\omega, r, \Theta^m_r) \Big)$$

$$= (D^i_{s,z} f_m)(r, \Theta^m_r) + \frac{1}{z} \Big(f_m(\omega^{s,z}, r, \Theta^m_r + z D^i_{s,z} \Theta^m_r) - f_m(\omega^{s,z}, r, \Theta^m_r) \Big). \tag{5.7}$$

Here, we have defined $\Theta^m_r := \Big(Y^m_r, Z^m_r, \int_{\mathbb{R}_0} \rho(x) G_m(r, \psi^m_r(x)) \nu(dx) \Big)$ for $r \in [0, T]$ and slightly abused its notation in such a way that

$$f_m(\omega^{s,z}, r, \Theta^m_r + z D^i_{s,z} \Theta^m_r) := f_m\Big(\omega^{s,z}, r, Y^m_r + z D^i_{s,z} Y^m_r, \ Z^m_r + z D^i_{s,z} Z^m_r$$

$$, \int_{\mathbb{R}_0} \rho(x) G_m\big(r, \psi^m_r(x) + z D^i_{s,z} \psi^m_r(x)\big) \nu(dx) \Big)$$

to save the space. For $0 \leq t < s$, one has $D_{s,z} \Theta^m_t \equiv 0$. Let us omit the superscript i denoting the direction of derivative in the following discussions.

Let us also define (for each direction $i \in \{1, \cdots, k\}$)

$$\mathcal{Y}^m_{s,z}(t) := Y^m_t + z D_{s,z} Y^m_t, \quad \mathcal{Z}^m_{s,z}(t) := Z^m_t + z D_{s,z} Z^m_t,$$
$$\Psi^m_{s,z}(t, \cdot) := \psi^m_t(\cdot) + z D_{s,z} \psi^m_t(\cdot),$$

for $(s, z) \in [0, T] \times \mathbb{R}_0$ and $t \in [0, T]$, and denote its collective argument as

$$\Xi^m_{s,z}(t) := \Big(\mathcal{Y}^m_{s,z}(t), \mathcal{Z}^m_{s,z}(t), \int_{\mathbb{R}_0} \rho(x) G_m(t, \Psi^m_{s,z}(t, x)) \nu(dx) \Big).$$

By Theorem B.1, we have already known that for $q(ds, dz)$-a.e. $(s, z) \in [0, T] \times \mathbb{R}^k$, the BSDEs (5.6) and (5.7) have a unique solution $(D_{s,z} Y^m, D_{s,z} Z^m, D_{s,z} \psi^m) \in \mathcal{K}^2[0, T]$. Furthermore, the arguments used in the proof for (b) and the fact that f_m satisfies the quadratic-exponential structure condition uniformly in m also imply that

$$\sup_{m \in \mathbb{N}} \Big(||\mathcal{Y}^m_{s,z}||_{\mathbb{S}^\infty} + ||\mathcal{Z}^m_{s,z}||_{\mathbb{H}^2_{BMO}} + ||\Psi^m_{s,z}||_{\mathbb{J}^2_{BMO}} \Big) \leq C,$$

with some positive constant C depending only on $(\beta, \gamma, T, ||\xi||_\infty, ||l||_{\mathbb{S}^\infty})$. It then shows $(D_{s,z}Y^m, D_{s,z}Z^m, D_{s,z}\psi^m) \in \mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$. Note also that, by the proof of Theorem 4.1, we have the convergence $\Xi^m_{s,z} := (\mathcal{Y}^m_{s,z}, \mathcal{Z}^m_{s,z}, \Psi^m_{s,z}) \to \Xi^{s,z} := (\mathcal{Y}^{s,z}, \mathcal{Z}^{s,z}, \Psi^{s,z})$ in the space $\mathbb{S}^\infty \times \mathbb{H}^2_{BMO} \times \mathbb{J}^2_{BMO}$.

By the same arguments used in *Proof for (a), (b)*, one can apply Theorem A.1 to the BSDE (5.6) and Lemma A.1 (or equivalently Lemma 3.3 (b)) to the BSDE (5.7) to obtain,

$$\left|\left|(D_{s,z}Y^m, D_{s,z}Z^m, D_{s,z}\psi^m)\right|\right|^{p'}_{\mathcal{K}^{p'}[0,T]} \le C\left(\mathbb{E}\left[|D_{s,z}\xi|^{p'\bar{q}^2} + \left(\int_0^T |(D_{s,z}f_m)(r, \Theta^m_r)|dr\right)^{p'\bar{q}^2}\right]\right)^{\frac{1}{\bar{q}^2}}$$

$$\le C\left(1 + \mathbb{E}\left[|D_{s,z}\xi|^{p'\bar{q}^2} + \left(\int_0^T |(D_{s,z}f)(r, 0)|dr\right)^{p'\bar{q}^2} + ||K_{s,z}||_T^{2p'\bar{q}^2}\right.\right.$$

$$\left.\left. + ||Y^m||_T^{2p'\bar{q}^2} + \left(\int_0^T |Z^m_r|^2 dr\right)^{2p'\bar{q}^2} + \left(\int_0^T ||\psi^m_r||^2_{\mathbb{L}^2(\nu)} dr\right)^{2p'\bar{q}^2}\right]\right)^{\frac{1}{\bar{q}^2}} \tag{5.8}$$

both for Wiener ($z = 0$) as well as jump ($z \ne 0$) directions. Here, C is a positive constant depending only on $(p', \bar{q}, \beta, \gamma, T, ||\xi||_\infty, ||l||_{\mathbb{S}^\infty}, K_M)$ and $\bar{q}$ is a positive constant satisfying $1 < q_* \le \bar{q} < \infty$ where the lower bound q_* is an increasing function of the $\mathbb{H}^2_{BMO}$-norm of $D_{s,z}Z^m$'s coefficients, which are also controlled by the universal bound given by $(\beta, \gamma, T, ||\xi||_\infty, ||l||_{\mathbb{S}^\infty})$. In particular, they are independent of m. For $\forall p' \ge 2$, due to the energy inequality in Lemma 2.2, together with Lemmas 3.1 and 3.2 imply that the right-hand side of (5.8) is bounded by some positive constant independent of m for $q(ds, dz)$-a.e. $(s, z) \in \widetilde{E}$. In particular, $D_{s,z}\Theta^m \in \mathcal{K}^{p'}[0, T]$ for $\forall p' \ge 2$.

When one chooses $p' = p$, Assumption 5.4 (iv) gives

$$\sup_{m \in \mathbb{N}} \int_{\widetilde{E}} \left|\left|(D_{s,z}Y^m, D_{s,z}Z^m, D_{s,z}\psi^m)\right|\right|^p_{\mathcal{K}^p[0,T]} q(ds, dz) < \infty . \tag{5.9}$$

Assumption 5.4 (v) and the fact that $(\Theta^m)_{m \in \mathbb{N}}$ satisfy the universal bound also imply that the convergence

$$\lim_{\epsilon \downarrow 0} \sum_{i=1}^k \int_0^T \int_{|z| > \epsilon} \left|\left|(D^i_{s,z}Y^m, D^i_{s,z}Z^m, D^i_{s,z}\psi^m)\right|\right|^p_{\mathcal{K}^p[0,T]} m^i(dz)ds$$

$$= \sum_{i=1}^k \int_0^T \int_{\mathbb{R}_0} \left|\left|(D^i_{s,z}Y^m, D^i_{s,z}Z^m, D^i_{s,z}\psi^m)\right|\right|^p_{\mathcal{K}^p[0,T]} m^i(dz)ds$$

is uniform in $m \in \mathbb{N}$. (See the discussion given just below (B.10). Thanks to the universal bound, the arguments are much simpler here.)

Third step (Convergence of $D_{s,0}\Theta^m \to \Theta^{s,0}$ as $n \to \infty$ in $\mathbb{L}^{1,2}$)

For ds-a.e. $s \in [0, T]$ and $m \in \mathbb{N}$, set

$$\Delta^{s,0}Y^m := Y^{s,0} - D_{s,0}Y^m, \quad \Delta^{s,0}Z^m := Z^{s,0} - D_{s,0}Z^m, \quad \Delta^{s,0}\psi^m := \psi^{s,0} - D_{s,0}\psi^m$$

and then $(\Delta^{s,0}Y^m, \Delta^{s,0}Z^m, \Delta^{s,0}\psi^m) \in \mathcal{K}^{p'}[0, T]$ with $\forall p' \ge 2$ is the unique solution to the BSDE

$$\Delta^{s,0}Y^m_t = \int_t^T \left(f^{s,0}(r) - D_{s,0}f_m(r)\right)dr - \int_t^T \Delta^{s,0}Z^m_r dW_r - \int_t^T \int_E \Delta^{s,0}\psi^m_r(x)\widetilde{\mu}(dr, dx) .$$

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Since

- $f^{s,0}(r) - D_{s,0}f_m(r) = f^{s,0}(r) - \left((D_{s,0}f_m)(r, \Theta_r^m) + \partial_\Theta f_m(r, \Theta_r^m) \Theta_r^{s,0} \right)$
$$\qquad\qquad + \partial_\Theta f_m(r, \Theta_r^m)(\Theta_r^{s,0} - D_{s,0}\Theta_r^m) \ ,$$

- $\left| f^{s,0}(r) - \left((D_{s,0}f_m)(r, \Theta_r^m) + \partial_\Theta f_m(r, \Theta_r^m) \Theta_r^{s,0} \right) \right|$
$$\qquad \le |(D_{s,0}f)(r, \Theta_r) - (D_{s,0}f)(r, \Theta_r^m)| + |(D_{s,0}f)(r, \Theta_r^m) - (D_{s,0}f_m)(r, \Theta_r^m)|$$
$$\qquad + |\partial_\Theta f(r, \Theta_r) - \partial_\Theta f_m(r, \Theta_r^m)||\Theta_r^{s,0}| \ ,$$

Lemma A.2 implies that

$$\int_0^T ||(\Delta^{s,0}Y^m, \Delta^{s,0}Z^m, \Delta^{s,0}\psi^m)||_{\mathcal{K}^p[0,T]}^p ds$$

$$\le C \int_0^T \left(\mathbb{E}\left[\left(\int_0^T |(D_{s,0}f)(r, \Theta_r) - (D_{s,0}f)(r, \Theta_r^m)| dr \right)^{p\bar{q}^2} \right. \right.$$

$$+ \left(\int_0^T |(D_{s,0}f)(r, \Theta_r^m) - (D_{s,0}f_m)(r, \Theta_r^m)| dr \right)^{p\bar{q}^2}$$

$$\left. \left. + \left(\int_0^T |\partial_\Theta f(r, \Theta_r) - \partial_\Theta f_m(r, \Theta_r^m)||\Theta_r^{s,0}| dr \right)^{p\bar{q}^2} \right] \right)^{\frac{1}{\bar{q}^2}} ds \qquad (5.10)$$

where, as before, $C > 0$ and $\bar{q} > 1$ are constants independent of m. By (5.9), in order to obtain

$$\lim_{m \to \infty} \int_0^T ||(\Delta^{s,0}Y^m, \Delta^{s,0}Z^m, \Delta^{s,0}\psi^m)||_{\mathcal{K}^p[0,T]}^p ds = 0$$

it suffices to show $\lim_{m \to \infty} ||(\Delta^{s,0}Y^m, \Delta^{s,0}Z^m, \Delta^{s,0}\psi^m)||_{\mathcal{K}^p[0,T]}^p = 0.$ for ds-a.e. $s \in [0,T]$.

Let us check, each term in (5.10). By the local Lipschitz property, the first term yields

$$\mathbb{E}\left[\left(\int_0^T |(D_{s,0}f)(r, \Theta_r) - (D_{s,0}f)(r, \Theta_r^m)| dr \right)^{p\bar{q}^2} \right]$$

$$\le C\mathbb{E}\left[||K_{s,0}^M||^{2p\bar{q}^2} \right]^{\frac{1}{2}} \mathbb{E}\left[||\delta Y^m||_T^{2p\bar{q}^2} + \left(\int_0^T ||\delta\psi_r^m||_{\mathbb{L}^2(\nu)}^2 dr \right)^{p\bar{q}^2} \right]^{\frac{1}{2}}$$

$$+ C\mathbb{E}\left[||K_{s,0}^M||_T^{2p\bar{q}^2} \left(\int_0^T |H^m(r)|^2 dr \right)^{p\bar{q}^2} \right]^{\frac{1}{2}} \mathbb{E}\left[\left(\int_0^T |\delta Z_r^m|^2 dr \right)^{p\bar{q}^2} \right]^{\frac{1}{2}}, (5.11)$$

where the process H^m is defined by $H^m(r) := 1 + |Z_r| + |Z_r^m| + ||\psi_r||_{\mathbb{L}^2(\nu)} + ||\psi_r^m||_{\mathbb{L}^2(\nu)}$ and $(\delta Y^m, \delta Z^m, \delta\psi^m) := (Y - Y^m, Z - Z^m, \psi - \psi^m)$. Since $H^m \in \mathbb{H}_{BMO}^2$ with the norm dominated by constant independent of m, the convergence of $\Theta^m \to \Theta$ in $\mathbb{S}^\infty \times \mathbb{H}_{BMO}^2 \times \mathbb{J}_{BMO}^2$ implies that (5.11) converges to zero as $m \to \infty$. Secondly, by definition of the truncated driver, one has

$$(D_{s,0}f_m)(r, \Theta_r^m) = (D_{s,0}f)\left(r, \varphi_m(Y_r^m), \varphi_m(Z_r^m), \int_{\mathbb{R}_0} \rho(x)G(r, \varphi_m(\psi_r^m \circ \zeta_m)(x))\nu(dx) \right) \ .$$

Since both of $(\Theta^m, \varphi_m(\Theta^m))$ converge to Θ, the convergence of the second term can be shown in the same way as the first term. Let us now consider the third term. By the

Cauchy-Schwartz inequality,

$$\mathbb{E}\left[\left(\int_0^T |\partial_\Theta f(r,\Theta_r) - \partial_\Theta f_m(r,\Theta_r^m)||\Theta_r^{s,0}|dr\right)^{p\bar{q}^2}\right]$$

$$\leq \mathbb{E}\left[\left(\int_0^T |\partial_\Theta f(r,\Theta_r) - \partial_\Theta f_m(r,\Theta_r^m)|^2 dr\right)^{p\bar{q}^2}\right]^{\frac{1}{2}} \mathbb{E}\left[\left(\int_0^T |\Theta_r^{s,0}|^2 dr\right)^{p\bar{q}^2}\right]^{\frac{1}{2}}.$$

Using the extended Fatou's lemma for uniformly integrable variables (Theorem 7.5.2 in [2]), one obtains

$$\lim_{m\to\infty}\mathbb{E}\left[\left(\int_0^T |\partial_\Theta f(r,\Theta_r) - \partial_\Theta f_m(r,\Theta_r^m)|^2 dr\right)^{p\bar{q}^2}\right]$$

$$\leq \mathbb{E}\left[\limsup_{m\to\infty}\left(\int_0^T |\partial_\Theta f(r,\Theta_r) - \partial_\Theta f_m(r,\Theta_r^m)|^2 dr\right)^{p\bar{q}^2}\right] = 0$$

since the integrand goes $d\mathbb{P}\otimes dr$-a.e. to zero by the convergence $\varphi_m(\Theta^m)\to\Theta$. This finishes the proof.

Fourth step (Convergence of $D_{s,z}\Theta^m \to \Theta^{s,z}$ ($z\neq 0$) as $n\to\infty$ in $\mathbb{L}^{1,2}$)

For each direction of jump, let us put

$$\Delta^{s,z}Y^m := Y^{s,z} - D_{s,z}Y^m, \quad \Delta^{s,z}Z^m = Z^{s,z} - D_{s,z}Z^m, \quad \Delta\psi^m = \psi^{s,z} - D_{s,z}\psi^m.$$

Then, $(\Delta^{s,z}Y^m, \Delta^{s,z}Z^m, \Delta^{s,z}\psi^m) \in \mathbb{S}^\infty \times \mathbb{H}_{BMO}^2 \times \mathbb{J}_{BMO}^2$ is the unique solution to

$$\Delta^{s,z}Y_t^m = \int_t^T \left(f^{s,z}(r) - D_{s,z}f_m(r)\right)dr - \int_t^T \Delta^{s,z}Z_r^m dW_r - \int_t^T \int_E \Delta^{s,z}\psi_r^m(x)\widetilde{\mu}(dr,dx),$$

with $t\in[0,T]$.

Let us define a d-dimensional $\mathbb{F}$-progressively measurable process $(b_{s,z}^m(r), r\in[0,T])$ by

$$b_{s,z}^m(\omega,r) := \frac{f_m(\omega^{s,z},r,\check{\Xi}_{s,z}^m(r)) - f_m(\omega^{s,z},r,\Xi_{s,z}^m(r))}{z|\Delta^{s,z}Z_r^m|^2}\mathbf{1}_{\Delta^{s,z}Z_r^m\neq 0}\Delta^{s,z}Z_r^m$$

where $\check{\Xi}_{s,z}^m := (\mathcal{Y}_{s,z}^m, Z^m + zZ^{s,z}, \int_{\mathbb{R}_0} G_m(r,\Psi_{s,z}^m(\cdot,x))\nu(dx))$ which differs from $\Xi_{s,z}^m$ only in the second component. It also converges $\check{\Xi}_{s,z}^m \to \Xi^{s,z}$ in $\mathbb{S}^\infty \times \mathbb{H}_{BMO}^2 \times \mathbb{J}_{BMO}^2$. We also introduce a map $\widetilde{f}_{s,z}^m : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{L}^2(E,\nu;\mathbb{R}^k) \to \mathbb{R}$ by

$$\widetilde{f}_{s,z}^m(\omega,r,\bar{y},\bar{\psi}) := (D_{s,z}f)(r,\Theta_r) - (D_{s,z}f_m)(r,\Theta_r^m) - \frac{1}{z}\Big[f(\omega^{s,z},r,\Theta_r) - f_m(\omega^{s,z},r,\Theta_r^m)\Big]$$

$$+\frac{1}{z}\bigg\{f\Big(\omega^{s,z},r,z\bar{y} + \mathcal{Y}_{s,z}^m(r) + \delta Y_r^m, \mathcal{Z}_r^{s,z}$$

$$, \int_{\mathbb{R}_0} \rho(x)G(r,z\bar{\psi}(x) + \Psi_{s,z}^m(r,x) + \delta\psi_r^m(x))\nu(dx)\Big) - f_m(\omega^{s,z},r,\check{\Xi}_{s,z}^m(r))\bigg\}.$$

Then, one can see that $(\Delta^{s,z}Y^m, \Delta^{s,z}Z^m, \Delta^{s,z}\psi^m)$ is the solution to the BSDE

$$\Delta^{s,z}Y_t^m = \int_t^T \Big(\widetilde{f}_{s,z}^m(r, \Delta^{s,z}Y_r^m, \Delta^{s,z}\psi_r^m) + b_{s,z}^m(r) \cdot \Delta^{s,z}Z_r^m \Big) dr$$

$$- \int_t^T \Delta^{s,z}Z_r^m dW_r - \int_t^T \int_E \Delta^{s,z}\psi_r^m(x)\widetilde{\mu}(dr, dx).$$

By denoting an $\mathbb{F}$-progressively measurable process $H_{s,z}^m$ as

$$H_{s,z}^m(r) := K_M\Big(1 + |\mathcal{Z}_{s,z}^m(r)| + |\mathcal{Z}_r^{s,z}| + |\delta Z^m| + 2\|\rho\|_{\mathbb{L}^2(\nu)}G_M'\|\Psi_{s,z}^m(r, \cdot)\|_{\mathbb{L}^2(\nu)} \Big),$$

one obtains $|b_{s,z}^m(r)| \leq H_{s,z}^m(r)$ for $\forall r \in [0, T]$. Here, $H_{s,z}^m \in \mathbb{H}_{BMO}^2$ and for $m(dz)ds$-a.e. $(s, z) \in [0, T] \times \mathbb{R}_0$, its norm $\|H_{s,z}^m\|_{\mathbb{H}_{BMO}^2}$ is dominated by some constant independent of m thank to the universal bound. Furthermore, the new driver satisfies the linear growth property

$$|\widetilde{f}_{s,z}^m(r, \bar{y}, \bar{\psi})| \leq |\widetilde{f}_{s,z}^m(r, 0, 0)| + K_M\big(|\bar{y}| + \|\rho\|_{\mathbb{L}^2(\nu)}G_M'\|\bar{\psi}\|_{\mathbb{L}^2(\nu)} \big)$$

and

$$\begin{aligned}
|\widetilde{f}^m(s, z)(r, 0, 0)| &\leq |(D_{s,z}f)(r, \Theta_r) - (D_{s,z}f_m)(r, \Theta_r^m)| + \frac{1}{|z|}|f(\omega^{s,z}, r, \Theta_r) - f_m(\omega, r, \Theta_r^m)| \\
&\quad + \frac{1}{|z|}|f(\omega^{s,z}, r, \check{\Xi}_{s,z}^m(r)) - f_m(\omega^{s,z}, r, \check{\Xi}_{s,z}^m(r))| \\
&\quad + CK_M\frac{1}{|z|}\Big(|\delta Y_r^m| + \|\delta\psi_r^m\|_{\mathbb{L}^2(\nu)} + \mathcal{H}_{s,z}^m(r)|\delta Z_r^m| \Big)
\end{aligned} \tag{5.12}$$

where C is a positive constant depending only on $\|\rho\|_{\mathbb{L}^2(\nu)}, G_M'$ and

$$\mathcal{H}_{s,z}^m(r) := 1 + 2|\mathcal{Z}_r^{s,z}| + |\delta Z_r^m| + 2\|\Psi_{s,z}^m(r, \cdot)\|_{\mathbb{L}^2(\nu)} + \|\delta\psi_r^m\|_{\mathbb{L}^2(\nu)} .$$

One sees $\mathcal{H}_{s,z}^m \in \mathbb{H}_{BMO}^2$ and its norm is dominated by m-independent constant $m(dz)ds$-a.e. $(s, z) \in [0, T] \times \mathbb{R}_0$ thanks to the universal bound.

By applying Lemma A.1, one obtains

$$\big\|(\Delta^{s,z}Y^m, \Delta^{s,z}Z^m, \Delta^{s,z}\psi^m)\big\|_{\mathcal{K}^p[0,T]}^p$$

$$\leq C\mathbb{E}\left[\left(\int_0^T |(D_{s,z}f)(r, \Theta_r) - (D_{s,z}f_m)(r, \Theta_r^m)|dr \right)^{p\bar{q}^2} \right]^{\frac{1}{\bar{q}^2}}$$

$$+ \frac{C}{|z|^p}\mathbb{E}\left[\left(\int_0^T |f(\omega^{s,z}, r, \Theta_r) - f_m(\omega^{s,z}, r, \Theta_r^m)|dr \right)^{p\bar{q}^2} \right]^{\frac{1}{\bar{q}^2}}$$

$$+ \frac{C}{|z|^p}\mathbb{E}\left[\left(\int_0^T |f(\omega^{s,z}, r, \check{\Xi}_{s,z}^m(r)) - f_m(\omega^{s,z}, r, \check{\Xi}_{s,z}^m(r))|dr \right)^{p\bar{q}^2} \right]^{\frac{1}{\bar{q}^2}}$$

$$+ \frac{C}{|z|^p}\mathbb{E}\left[\left(\int_0^T \big[|\delta Y_r^m| + \|\delta\psi_r^m\|_{\mathbb{L}^2(\nu)} + \mathcal{H}_{s,z}^m(r)|\delta Z_r^m| \big] dr \right)^{p\bar{q}^2} \right]^{\frac{1}{\bar{q}^2}}, \tag{5.13}$$

where the positive constants C and $\bar{q} > 1$ are m-independent as before.

Due to the result of the *Second step*, the order of limit operations can be changed,

$$\lim_{m\to\infty}\lim_{\epsilon\downarrow0}\int_0^T\int_{|z|>\epsilon}\left|\left|(\Delta^{s,z}Y^m,\Delta^{s,z}Z^m,\Delta^{s,z}\psi^m)\right|\right|_{\mathcal{K}^p[0,T]}^p m(dz)ds$$

$$=\lim_{\epsilon\downarrow0}\lim_{m\to\infty}\int_0^T\int_{|z|>\epsilon}\left|\left|(\Delta^{s,z}Y^m,\Delta^{s,z}Z^m,\Delta^{s,z}\psi^m)\right|\right|_{\mathcal{K}^p[0,T]}^p m(dz)ds$$

$$=\lim_{\epsilon\downarrow0}\int_0^T\int_{|z|>\epsilon}\lim_{m\to\infty}\left|\left|(\Delta^{s,z}Y^m,\Delta^{s,z}Z^m,\Delta^{s,z}\psi^m)\right|\right|_{\mathcal{K}^p[0,T]}^p m(dz)ds,$$

where (5.9) is used to obtain the second equality. Therefore, in order to prove the convergence $\Delta^{s,z}\Theta^m\to\Theta^{s,z}$ in $\mathbb{L}^{1,2}$, it suffices to show, for $m(dz)ds$-a.e. $(s,z)\in[0,T]\times\mathbb{R}_0$,

$$\lim_{m\to\infty}\left|\left|(\Delta^{s,z}Y^m,\Delta^{s,z}Z^m,\Delta^{s,z}\psi^m)\right|\right|_{\mathcal{K}^p[0,T]}^p=0.$$

This can be easily confirmed from (5.13) by noticing the fact that $(\Theta^m,\varphi_m(\Theta^m))\to\Theta$ and $(\check{\Xi}_{s,z}^m,\varphi_m(\check{\Xi}_{s,z}^m))\to\Xi^{s,z}$ converge in $\mathbb{S}^\infty\times\mathbb{H}_{BMO}^2\times\mathbb{J}_{BMO}^2$ and hence also in $\mathcal{K}^{p'}$ for $\forall p'\geq2$.

Finally, the closability of the Malliavin derivatives in $\mathbb{L}^{1,2}$ (See Theorem 12.6 in [14].), one concludes $(Y,Z,\psi)\in\mathbb{L}^{1,2}$ and that $(Y^{s,z},Z^{s,z},\psi^{s,z})$ is a version of $(D_{s,z}Y,D_{s,z}Z,D_{s,z}\psi)$. $\square$

Corollary 5.1. *Under the assumptions of Theorem 5.1, we have*
(i) $\left(\left(D_{t,0}^iY_t\right)^{\mathcal{P}},t\in[0,T]\right)$ *is a version of* $\left(Z_t^i,t\in[0,T]\right)$ *for* $i\in\{1,\cdots,d\}$,
(ii) $\left(\left(zD_{t,z}^iY_t\right)^{\mathcal{P}},(t,z)\in[0,T]\times\mathbb{R}_0\right)$ *is a version of* $\left(\psi_t^i(z),(t,z)\in[0,T]\times\mathbb{R}_0\right)$ *for* $i\in\{1,\cdots,k\}$,
where $(\cdot)^{\mathcal{P}}$ *denotes the predictable projection of a process.*

Proof. See Corollory 4.1 in [13]. $\square$

6 An application: Markovian forward-backward system

6.1 Forward SDE

As an important application, we consider a $Q_{\exp}$-growth BSDE driven by an n-dimensional Markovian process $\left(X_s^{t,x},s\in[0,T]\right)$ defined by the next SDE:

$$X_s^{t,x}=x+\int_t^s b(r,X_r^{t,x})dr+\int_t^s\sigma(r,X_r^{t,x})dW_r+\int_t^s\int_E\gamma(r,X_{r-}^{t,x},e)\widetilde{\mu}(dr,de)\qquad(6.1)$$

for $s\in[t,T]$ and put $X_s^{t,x}\equiv x$ for $s<t$. Here, $x\in\mathbb{R}^n$, $b:[0,T]\times\mathbb{R}^n\to\mathbb{R}^n$, $\sigma:[0,T]\times\mathbb{R}^n\to\mathbb{R}^{n\times d}$ and $\gamma:[0,T]\times\mathbb{R}^n\times E\to\mathbb{R}^{n\times k}$. Let us introduce $\eta:\mathbb{R}\to\mathbb{R}_+$ by $\eta(e)=1\wedge|e|$.

Assumption 6.1. *The functions $b(t,x)$, $\sigma(t,x)$ and $\gamma(t,x,e)$ are continuous in all their arguments and one-time continuously differentiable with respect to x with continuous derivatives. Furthermore, there exists some positive constant K such that*
(i) $|b(t,0)|+|\sigma(t,0)|\leq K$ *uniformly in* $t\in[0,T]$.
(ii) $|\partial_x b(t,x)|+|\partial_x\sigma(t,x)|\leq K$ *uniformly in* $t\in[0,T]$.
(iii) For each column vector $i\in\{1,\cdots,k\}$, $|\gamma^i(t,0,e)|\leq K\eta(e)$ *uniformly in* $(t,e)\in[0,T]\times\mathbb{R}_0$.

(iv) For each column vector $i \in \{1, \cdots, k\}$, $|\partial_x \gamma^i(t, x, e)| \leq K\eta(e)$ uniformly in $(t, x, e) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}_0$.

We have the following result:

Proposition 6.1. *Under Assumption 6.1, there exists a unique solution $X^{t,x} \in \mathbb{S}^p[0, T]$ with $\forall p \geq 2$ for every initial data $(t, x) \in [0, T] \times \mathbb{R}^n$. Furthermore, the process $X^{t,x}$ is Malliavin differentiable $X^{t,x} \in \mathbb{L}^{1,2}$ and satisfies, for $\forall p \geq 2$,*

$$\int_{\widetilde{E}} \mathbb{E}\Big[||D_{u,z}X^{t,x}||_T^p\Big] q(du, dz) \leq C(1 + |x|^p)$$

with some positive constant C depending only on (p, T, K).

Proof. The fact that $X^{t,x} \in \mathbb{S}^p[0, T]$ with $\forall p \geq 2$ is rather standard. See, for example, Lemma A.3 in [16]. The existence of Malliavin derivative follows from Theorem 3 of Petrou (2008) [31]. This implies, for $u \in [t, s]$ and $i \in \{1, \cdots, d\}$,

$$D_{u,0}^i X_s^{t,x} = \sigma^i(u, X_u^{t,x}) + \int_u^s \partial_x b(r, X_r^{t,x}) D_{u,0}^i X_r^{t,x} + \int_u^s \partial_x \sigma(r, X_r^{t,x}) D_{u,0}^i X_r^{t,x} dW_r$$

$$+ \int_u^s \int_E \partial_x \gamma(r, X_{r-}^{t,x}, e) D_{u,0}^i X_r^{t,x} \widetilde{\mu}(dr, de) \ ,$$

and for $(u, z) \in [t, s] \times \mathbb{R}_0$ and $i \in \{1, \cdots, k\}$,

$$D_{u,z}^i X_s^{t,x} = \frac{\gamma^i(u, X_{u-}^{t,x}, z)}{z} + \int_u^s D_{u,z}^i b(r, X_r^{t,x}) dr + \int_u^s D_{u,z}^i \sigma(r, X_r^{t,x}) dW_r$$

$$+ \int_u^s \int_E D_{u,z}^i \gamma(r, X_{r-}^{t,x}, e) \widetilde{\mu}(dr, de) \ ,$$

where both σ^i and γ^i denote the i-th column vectors, and for $\varphi = b, \sigma, \gamma$,

$$D_{u,z}^i \varphi(r, X_r^{t,x}) := \frac{\varphi(r, X_r^{t,x} + zD_{u,z}^i X_r^{t,x}) - \varphi(r, X_r^{t,x})}{z}.$$

By Lemma A.3 [16], the above SDEs satisfy the a priori estimates

$$\mathbb{E}\Big[||D_{u,0}X^{t,x}||_T^p\Big] \leq C_{p,T,K}\mathbb{E}\Big[|\sigma(u, X_u^{t,x})|^p\Big]$$

$$\leq C_{p,T,K}\mathbb{E}\Big[|\sigma(u, 0)|^p + ||X^{t,x}||_T^p\Big] \leq C_{p,T,K}(1 + |x|^p)$$

and

$$\mathbb{E}\Big[||D_{u,z}X^{t,x}||_T^p\Big] \leq C_{p,T,K}\mathbb{E}\left[\Big|\frac{\gamma(u, X_{u-}^{t,x}, z)}{z}\Big|^p\right]$$

$$\leq C_{p,T,K}\mathbb{E}\left[\Big|\frac{\gamma(u, 0, z)}{z}\Big|^p + ||X^{t,x}||_T^p\right] \leq C_{p,T,K}(1 + |x|^p) \ .$$

Since $q(du, dz)$ on $\widetilde{E}$ is a finite measure, the claim is proved. $\qquad \square$

6.2 $Q_{\exp}$-growth BSDE driven by $X^{t,x}$

In many applications, there appears a BSDE driven by a Markovian forward process. Let us consider a $Q_{\exp}$-BSDE driven by the process $\big(X_s^{t,x}, s \in [0, T]\big)$ introduced in the last

section;

$$Y_s^{t,x} = \xi(X_T^{t,x}) + \int_s^T f\Big(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}, \int_{\mathbb{R}_0} \rho(e)G(r, \psi_r(e))\nu(de)\Big) dr$$

$$- \int_s^t Z_r^{t,x} dW_r - \int_s^T \int_E \psi_r^{t,x}(e)\widetilde{\mu}(dr, de) \tag{6.2}$$

for $s \in [t, T]$ and put $(Y_s^{t,x}, Z_s^{t,x}, \psi_s^{t,x}) \equiv (Y_t^{t,x}, 0, 0)$ for $s < t$. Here, $\xi : \mathbb{R}^n \to \mathbb{R}$, $f : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}$ are measurable functions. We treat Z and ψ as row vectors for notational simplicity. In this setup, the driver f is deterministic without explicit dependence on ω, which is now provided by the dependence on $X^{t,x}$.

Assumption 6.2. *For every* $(x, y, z, \psi) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^2(E, \nu; \mathbb{R}^k)$, *there exist two positive constants* $\beta \geq 0$, $\gamma > 0$ *and the non-negative measurable function* $l : [0, T] \to \mathbb{R}_+$ *such that the measurable function* f *satisfies*

$$-l_t - \beta|y| - \frac{\gamma}{2}|z|^2 - \int_E j_\gamma\big(-\psi(e)\big)\nu(de) \leq f\Big(t, x, y, z, \int_{\mathbb{R}_0} \rho(e)G(t, \psi(e))\nu(de)\Big)$$

$$\leq l_t + \beta|y| + \frac{\gamma}{2}|z|^2 + \int_E j_\gamma\big(\psi(e)\big)\nu(de)$$

dt-a.e. $t \in [0, T]$, *where* $j_\gamma(u) := \frac{1}{\gamma}\Big(e^{\gamma u} - 1 - \gamma u\Big)$.

Assumption 6.3. *(i)* $|\xi(x)| + l_t$ *is bounded uniformly in* $(t, x) \in [0, T] \times \mathbb{R}^n$.
(ii) For each $M > 0$, *for every* $x \in \mathbb{R}^n$ *and* $(y, z, \psi), (y', z', \psi') \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^2(E, \nu; \mathbb{R}^k)$ *satisfying*

$$|y|, |y'|, ||\psi||_{\mathbb{L}^\infty(\nu)}, ||\psi'||_{\mathbb{L}^\infty(\nu)} \leq M,$$

there exists some positive constant K_M *(possibly dependent on* M*) such that*

$$\big|f(t, x, y, z, u_t) - f(t, x, y', z', u'_t)\big|$$
$$\leq K_M\Big(|y - y'| + |u_t - u'_t|\Big) + K_M\Big(1 + |z| + |z'| + |u_t| + |u'_t|\Big)|z - z'|$$

where we have used the short-hand notation $u_t := \int_{\mathbb{R}_0} \rho(e)G(t, \psi(e))\nu(de)$ *and* $u'_t := \int_{\mathbb{R}_0} \rho(e)G(t, \psi'(e))\nu(de)$.

The following result is obvious:

Lemma 6.1. *Under Assumptions 5.1, 6.1, 6.2 and 6.3, there exists a unique solution* $(Y^{t,x}, Z^{t,x}, \psi^{t,x}) \in \mathbb{S}^\infty[0, T] \times \mathbb{H}^2_{BMO}[0, T] \times \mathbb{J}^2_{BMO}[0, T]$ *to the BSDE (6.2) for every* $(t, x) \in [0, T] \times \mathbb{R}^n$.

We denote $\Theta_r^{t,x} := \Big(Y^{t,x}, Z^{t,x}, \int_{\mathbb{R}_0} \rho(e)G(r, \psi_r^{t,x}(e))\nu(de)\Big)$ as a collective arguments for the solution indexed by the initial data (t, x).

Assumption 6.4. *(i)* ξ *and the driver* f *are one-time continuously differentiable with respect to the spacial variables with continuous derivatives.*
(ii) There exists some positive constant K *such that* $|\partial_x \xi(x)| \leq K$ *as well as* $|\partial_x f(t, x, 0, 0, 0)| \leq K$ *uniformly in* $(t, x) \in [0, T] \times \mathbb{R}^n$.
(iii) For each $M > 0$, *for every* $x \in \mathbb{R}^n$ *and* $(y, z, \psi), (y', z', \psi') \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^2(E, \nu; \mathbb{R}^k)$ *satisfying*

$$|y|, |y'|, ||\psi||_{\mathbb{L}^\infty(\nu)}, ||\psi'||_{\mathbb{L}^\infty(\nu)} \leq M,$$

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there exists some positive constant K_M (possibly dependent on M) such that

$$\left|\partial_x f(t, x, y, z, u_t) - \partial_x f(t, x, y', z', u'_t)\right|$$
$$\leq K_M\Big(|y - y'| + |u_t - u'_t|\Big) + K_M\Big(1 + |z| + |z'| + |u_t| + |u'_t|\Big)|z - z'|$$

with the short-hand notation $u_t := \int_{\mathbb{R}_0} \rho(e)G(t, \psi(e))\nu(de)$ and $u'_t := \int_{\mathbb{R}_0} \rho(e)G(t, \psi'(e))\nu(de)$.

One sees that Assumption 6.4, together with Assumption 6.3, implies

$$|\partial_x f(t, x, y, z, u_t)| \leq CK_M\big(1 + |y| + |z|^2 + |u_t|^2\big), \quad |\partial_y f(t, x, y, z, u_t)| \leq K_M,$$
$$|\partial_z f(t, x, y, z, u_t)| \leq K_M\big(1 + 2|z| + 2|u_t|\big), \quad |\partial_u f(t, x, y, z, u_t)| \leq K_M ,$$

where C is some positive constant.

Theorem 6.1. *Under Assumptions 5.1, 6.1, 6.2, 6.3 and 6.4, the solution of the BSDE (6.2) is Malliavin differentiable $(Y^{t,x}, Z^{t,x}, \psi^{t,x}) \in \mathbb{L}^{1,2} \times \mathbb{L}^{1,2} \times \mathbb{L}^{1,2}$ for every initial data $(t, x) \in [0, T] \times \mathbb{R}^n$. A version of $\Big(\big(D^i_{s,0}Y^{t,x}_r, D^i_{s,0}Z^{t,x}_r, D^i_{s,0}\psi^{t,x}_r(e)\big), 0 \leq s, r \leq T, e \in \mathbb{R}_0\Big)_{i \in \{1,\cdots,d\}}$ is the unique solution of the BSDE*

$$D^i_{s,0}Y^{t,x}_u = D^i_{s,0}Z^{t,x}_u = D^i_{s,0}\psi^{t,x}_u(\cdot) = 0, \qquad 0 \leq u < s \leq T,$$
$$D^i_{s,0}Y^{t,x}_u = \partial_x\xi(X^{t,x}_T)D^i_{s,0}X^{t,x}_T + \int_u^T f^{s,0,i}(r)dr - \int_u^T D^i_{s,0}Z^{t,x}_r dW_r$$
$$- \int_u^T \int_E D^i_{s,0}\psi^{t,x}_r \widetilde{\mu}(dr, de), \quad u \in [s, T]$$

where

$$f^{s,0,i}(r) := \partial_x f(r, X^{t,x}_r, \Theta^{t,x}_r)D_{s,0}X^{t,x}_r + \partial_\Theta f(r, X^{t,x}_r, \Theta^{t,x}_r)D_{s,0}\Theta^{t,x}_r .$$

Moreover, for a given ds-a.e. $s \in [0, T]$, $(D^i_{s,0}Y^{t,x}, D^i_{s,0}Z^{t,x}, D^i_{s,0}\psi^{t,x}) \in \mathcal{K}^p[0, T]$ with $\forall p \geq 2$.
A version of $\Big(\big(D^i_{s,z}Y^{t,x}_r, D^i_{s,z}Z^{t,x}_r, D^i_{s,z}\psi^{t,x}_r(e)\big), 0 \leq s, r \leq T, e, z \in \mathbb{R}_0\Big)_{i \in \{1,\cdots,k\}}$ is the unique solution of the BSDE

$$D^i_{s,z}Y^{t,x}_u = D^i_{s,z}Z^{t,x}_u = D^i_{s,z}\psi^{t,x}_u(\cdot) = 0, \qquad 0 \leq u < s \leq T,$$
$$D^i_{s,z}Y^{t,x}_u = \xi^{s,z,i} + \int_u^T f^{s,z,i}(r)dr - \int_u^T D^i_{s,z}Z^{t,x}_r dW_r - \int_u^T \int_E D^i_{s,z}\psi^{t,x}_r(e)\widetilde{\mu}(dr, de) ,$$

for $u \in [s, T]$ where

$$\xi^{s,z,i} := \frac{\xi(X^{t,x}_T + zD^i_{s,z}X^{t,x}_T) - \xi(X^{t,x}_T)}{z},$$
$$f^{s,z,i}(r) := \frac{1}{z}\Big\{f\Big(r, X^{t,x}_r + zD^i_{s,z}X^{t,x}_r, \ Y^{t,x}_r + zD^i_{s,z}Y^{t,x}_r, \ Z^{t,x}_r + zD^i_{s,z}Z^{t,x}_r$$
$$, \int_{\mathbb{R}_0} \rho(e)G(r, \psi^{t,x}_r(e) + zD^i_{s,z}\psi^{t,x}_r(e))\nu(e)de\Big) - f(r, X^{t,x}_r, \Theta^{t,x}_r)\Big\} .$$

Moreover, for a given $m^i(dz)ds$-a.e. $(s, z) \in [0, T] \times \mathbb{R}_0$, $(D^i_{s,z}Y^{t,x}, D^i_{s,z}Z^{t,x}, D^i_{s,z}\psi^{t,x}) \in \mathbb{S}^\infty[0, T] \times \mathbb{H}^2_{BMO}[0, T] \times \mathbb{J}^2_{BMO}[0, T]$.

Proof. It suffices to check Assumption 5.4 to hold so that Theorem 5.1 can be applied.

$(i), (ii)$ are obviously satisfied due to the Malliavin's differential rule (Theorem 3.5 and Theorem 12.8 in [14]). The local Lipschitz condition (iii) is satisfied if we replace $K_{s,z}^M(r)$ by $K_M|D_{s,z}X_r^{t,x}|$. This is easy to see for a Wiener direction $(z = 0)$. For a jump direction $(z \neq 0)$, notice that

$$
\begin{aligned}
(D_{s,z}f)(r,y,z,u_r) &= \frac{1}{z}\big[f(r,X_r^{t,x}+zD_{s,z}X_r^{t,x},y,z,u_r) - f(r,X_r^{t,x},y,z,u_r)\big] \\
&= \left(\int_0^1 \partial_x f\Big(r,X_r^{t,x}+\theta z D_{s,z}X_r^{t,x},y,z,u_r\Big)d\theta\right)D_{s,z}X_r^{t,x} \,,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\big|(D_{s,z}f)(r,y,z,u_r) - (D_{s,z}f)(r,y',z',u_r')\big| \\
&\leq |D_{s,z}X_r^{t,x}|\int_0^1\Big|\partial_x f(r,X_r^{t,x}+\theta z D_{s,z}X_r^{t,x},y,z,u_r) - \partial_x f(r,X_r^{t,x}+\theta z D_{s,z}X_r^{t,x},y',z',u_r')\Big|d\theta \\
&\leq K_M|D_{s,z}X_r^{t,x}|\Big(|y-y'| + |u_r - u_r'| + \big(1+|z|+|z'|+|u_r|+|u_r'|\big)|z-z'|\Big) \,.
\end{aligned}
$$

Since $|D_{s,z}\xi| \leq K|D_{s,z}X_T^{t,x}|$ and $|(D_{s,z}f)(r,0,0,0)| \leq K|D_{s,z}X_r^{t,x}|$, one can confirm that $(iv), (v)$ are satisfied from an inequality

$$
\begin{aligned}
&\mathbb{E}\left[|D_{s,z}\xi|^p + \left(\int_0^T |(D_{s,z}f)(r,0,0,0)|dr\right)^p + K_M^{2p}||D_{s,z}X^{t,x}||_T^{2p}\right] \\
&\leq C_{p,K,K_M,T}\mathbb{E}\Big[1 + ||D_{s,z}X^{t,x}||_T^{2p}\Big] \leq C_{p,K,K_M,T}(1+|x|^{2p})
\end{aligned}
$$

uniformly in $(s,z) \in [0,T] \times \mathbb{R}$ for $\forall p \geq 2$ (See, proof of Proposition 6.1.). $\qquad\square$

Corollary 6.1. *Under the assumptions of Theorem 6.1, let us define the deterministic function $u : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ by $u(t,x) := Y_t^{t,x}$. Then, $u(t,x)$ is continuous in (t,x), one-time continuously differentiable with respect to x with continuous derivative. Moreover,*

$$
\begin{aligned}
\left(Z^{t,x}(s)\right)^i &= \partial_x u(s,X_{s-}^{t,x})\sigma^i(s,X_{s-}^{t,x}), \quad t \leq s \leq T \,, i \in \{1,\cdots,d\} \\
\left(\psi_s^{t,x}(z)\right)^i &= u(s,X_{s-}^{t,x}+\gamma^i(s,X_{s-}^{t,x},z)) - u(s,X_{s-}^{t,x}), \quad t \leq s \leq T \,, i \in \{1,\cdots,k\}
\end{aligned}
$$

where σ^i and γ^i denotes the i-th column vectors.

Proof. By replacing *a priori* estimates for the Lipschitz BSDEs of Lemma 5.1 in [16] with the local Lipschitz ones given in Theorem A.1 and Lemma A.2, one can follow the same arguments in Theorem 3.1 in [25] to show that the function $u(t,x)$ is continuous in the both arguments and one-time continuously differentiable with respect to x with continuous derivatives. Then the fact that

$$
D_{s,0}^i X_s^{t,x} = \sigma^i(s,X_s^{t,x}), \quad z D_{s,z}^i X_s^{t,x} = \gamma^i(s,X_s^{t,x},z) \,,
$$

Corollary 5.1, and the Malliavin differential rule for a continuously differentiable function give the desired result. $\qquad\square$

A An a priori estimate and BMO-Lipschitz BSDEs

A.1 An a priori estimate

Firstly, we establish a priori estimate which plays a crucial role throughout the paper. Although it is very similar to that of BMO-Lipschitz BSDEs, which will be discussed in the next section, it has a much wider range of application. See discussion in Section 3 of Ankirchner et.al. [1] for diffusion setup. Let us consider the BSDE, for $t \in [0, T]$,

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, \psi_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_E \psi_s(x)\widetilde{\mu}(ds, dx) , \qquad (A.1)$$

where $\xi : \Omega \to \mathbb{R}$, $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^2(E, \nu; \mathbb{R}^k) \to \mathbb{R}$. We treat Z, ψ are row vectors for simplicity. We introduce another driver $\widetilde{f} : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^2(E, \nu; \mathbb{R}^k) \to \mathbb{R}$. The crucial point of the next assumption is that the process $(H_t)_{t \in [0,T]}$ is not forbidden to be a function of $(Y_t, Z_t, \psi_t)_{t \in [0,T]}$.

Assumption A.1. *(i) The maps $(\omega, t) \mapsto f(\omega, t, \cdot), \widetilde{f}(\omega, t, \cdot)$ are $\mathbb{F}$-progressively measurable. ξ is an $\mathcal{F}_T$-measurable random variable.*
(ii) There exists a solution (Y, Z, ψ) to the BSDE (A.1) satisfying $Y \in \mathbb{S}^p$ for $\forall p \geq 2$.
(iii) For every $(y, z, \psi) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^2(E, \nu; \mathbb{R}^k)$, the driver $\widetilde{f}$ satisfies with some positive constant K such that [3]

$$|\widetilde{f}(\omega, t, y, z, \psi)| \leq g_t + K\big(|y| + |z| + ||\psi||_{\mathbb{L}^2(\nu)}\big)$$

$d\mathbb{P} \otimes dt$-a.e. $(\omega, t) \in \Omega \times [0, T]$, where $(g_t, t \in [0, T])$ is an $\mathbb{F}$-progressively measurable positive process. Moreover, ξ and g satisfy, for $\forall p \geq 2$, $\mathbb{E}\left[|\xi|^p + \left(\int_0^T g_s ds\right)^p\right] < \infty$.
(iv) With the solution (Y, Z, ψ) to the BSDE (A.1), there exists an $\mathbb{F}$-progressively measurable positive process $(H_t, t \in [0, T])$, $H \in \mathbb{H}^2_{BMO}$ such that

$$|f(s, Y_s, Z_s, \psi_s) - \widetilde{f}(s, Y_s, Z_s, \psi_s)| \leq H_s|Z_s|$$

for $d\mathbb{P} \otimes ds$-a.e. $(\omega, s) \in \Omega \times [0, T]$.

Lemma A.1. *Suppose Assumption A.1 hold true. Then the solution (Y, Z, ψ) to the BSDE (A.1) satisfies, for $\forall p \geq 2$,*

$$\big|\big|(Y, Z, \psi)\big|\big|_{\mathcal{K}^p[0,T]}^p \leq C \left(\mathbb{E}\left[|\xi|^{p\bar{q}^2} + \left(\int_0^T g_s ds\right)^{p\bar{q}^2} \right] \right)^{\frac{1}{\bar{q}^2}}$$

with a positive constant $\bar{q}$ satisfying $q_* \leq \bar{q} < \infty$ whose lower bound $q_* > 1$ is controlled only by $||H||_{\mathbb{H}^2_{BMO}}$, and some positive constant C depending only on $(p, \bar{q}, T, K, ||H||_{\mathbb{H}^2_{BMO}})$.

Proof. Define a d-dimensional progressively measurable process $(b_s, s \in [0, T])$ by

$$b_s := \frac{f(s, Y_s, Z_s, \psi_s) - \widetilde{f}(s, Y_s, Z_s, \psi_s)}{|Z_s|^2}\mathbf{1}_{Z_s \neq 0}Z_s,$$

which satisfies $|b_s| \leq H_s$ and hence $b \in \mathbb{H}^2_{BMO}$ whose norm is dominated by $||H||_{\mathbb{H}^2_{BMO}}$. Using the process b, (A.1) can be written as

$$Y_t = \xi + \int_t^T \left(\widetilde{f}(s, Y_s, Z_s, \psi_s) + b_s \cdot Z_s\right)ds - \int_t^T Z_s dW_s - \int_t^T \int_E \psi_s(x)\widetilde{\mu}(ds, dx)$$

---

[3]This can be generalized to a monotone condition.

and hence under the new measure $\mathbb{Q}$ defined by $d\mathbb{Q}/d\mathbb{P} = \mathcal{E}_T(b * W)$, one obtains

$$Y_t = \xi + \int_t^T \widetilde{f}(s, Y_s, Z_s, \psi_s)ds - \int_t^T Z_s dW_s^{\mathbb{Q}} - \int_t^T \int_E \psi_s(x)\widetilde{\mu}^{\mathbb{Q}}(ds, dx) \qquad (A.2)$$

where $W^{\mathbb{Q}} := W - \int_0^\cdot b_s ds$ and $\widetilde{\mu}^{\mathbb{Q}} = \widetilde{\mu}$ due to the independence of $(W, \widetilde{\mu})$. By the linear growth property of $\widetilde{f}$, one has

$$Y_s \widetilde{f}(s, Y_s, Z_s, \psi) \leq |Y_s| \Big(g_s + K(|Y_s| + |Z_s| + ||\psi_s||_{\mathbb{L}^2(\nu)}) \Big) ,$$

and hence for $\forall \lambda > 0$

$$Y_s \widetilde{f}(s, Y_s, Z_s, \psi) \leq |Y_s|^2 \Big(K + \frac{K^2}{2\lambda} \Big) + |Y_s|g_s + \lambda(|Z_s|^2 + ||\psi_s||^2_{\mathbb{L}^2(\nu)}) .$$

Thus by choosing $V_t^\lambda := \Big(K + \frac{K^2}{2\lambda} \Big)t$ and $N_t^\lambda = \int_0^t g_s ds$, the BSDE (A.2) satisfies Assumption B.1 in [16]. Then Lemma B.1 in [16] of an a prior estimate for the BSDEs with a monotone driver implies, for $\forall p \geq 2$,

$$||(Y, Z, \psi)||^p_{\mathcal{K}^p(\mathbb{Q})[0,T]} \leq C\mathbb{E}^{\mathbb{Q}} \left[|\xi|^p + \Big(\int_0^T g_s ds \Big)^p \right]$$

with some positive constant $C = C_{p,K,T}$ depending only on (p, K, T).

By the properties of the BMO martingales, one can choose $\bar{r} > 1$ with which both of $\mathcal{E}(b*W)$ and $\mathcal{E}(-b*W^{\mathbb{Q}})$ satisfy the reverse Hölder inequality (See Lemma 2.4 and the following remark.). Define $\bar{q} = \frac{\bar{r}}{\bar{r}-1}$ as its dual. Let us put $D := \max\big(||\mathcal{E}(b*W)||_{\mathbb{L}^{\bar{r}}(\mathbb{P})}, ||\mathcal{E}(-b*W^{\mathbb{Q}})||_{\mathbb{L}^{\bar{r}}(\mathbb{Q})}\big)$, which is dominated by some constant depending only on $||H||_{\mathbb{H}^2_{BMO}(\mathbb{P})}$. Then one obtains

$$
\begin{aligned}
||(Y, Z, \psi)||^p_{\mathcal{K}^p(\mathbb{P})[0,T]} &= \mathbb{E}^{\mathbb{Q}} \left[\mathcal{E}_T(-b*W^{\mathbb{Q}}) \Big(||Y||_T^p + \Big(\int_0^T |Z_s|^2 ds \Big)^{\frac{p}{2}} + \Big(\int_0^T ||\psi_s||^2_{\mathbb{L}^2(\nu)} ds \Big)^{\frac{p}{2}} \Big) \right] \\
&\leq D||(Y, Z, \psi)||^p_{\mathcal{K}^{p\bar{q}}(\mathbb{Q})[0,T]} \\
&\leq C_{p,\bar{q},K,T} D \left(\mathbb{E}^{\mathbb{Q}} \left[|\xi|^{p\bar{q}} + \Big(\int_0^T g_s ds \Big)^{p\bar{q}} \right] \right)^{\frac{1}{\bar{q}}} \\
&\leq C_{p,\bar{q},K,T} D^{1+\frac{1}{\bar{q}}} \left(\mathbb{E} \left[|\xi|^{p\bar{q}^2} + \Big(\int_0^T g_s ds \Big)^{p\bar{q}^2} \right] \right)^{\frac{1}{\bar{q}^2}} ,
\end{aligned}
$$

which proves the desired result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

A.2 BMO-Lipschitz BSDE

In this subsection, we study the properties of the BSDE with a locally Lipschitz driver where the Lipschitz coefficient for the control variable belongs to $\mathbb{H}^2_{BMO}$. In the diffusion setup, the details have been discussed by Briand & Confortola (2008) [6]. As we have announced before, we keep the reverse Hölder property only to the continuous part and assume only the standard Lipschitz continuity for the jump coefficient.

Assumption A.2. *The map* $(\omega, t) \mapsto f(\omega, t, \cdot)$ *is* $\mathbb{F}$-*progressively measurable.*
(i) There exist a positive constant K and a positive $\mathbb{F}$-progressively measurable process $(H_t, t \in [0,T]) \in \mathbb{H}^2_{BMO}$ *such that, for every* $(y, z, \psi), (y', z', \psi') \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{L}^2(E, \nu; \mathbb{R}^k)$,

$$|f(\omega, t, y, z, \psi) - f(\omega, t, y', z', \psi')| \leq K\Big(|y - y'| + ||\psi - \psi'||_{\mathbb{L}^2(\nu)} \Big) + H_t(\omega)|z - z'|$$

$d\mathbb{P} \otimes dt$-a.e. $(\omega, t) \in \Omega \times [0, T]$.

(ii) ξ is $\mathcal{F}_T$-measurable and, for $\forall p \geq 2$,

$$\mathbb{E}\left[|\xi|^p + \left(\int_0^T |f(s, 0, 0, 0)|ds\right)^p\right] < \infty .$$

Theorem A.1. *Under Assumption A.2, there exists a unique solution (Y, Z, ψ) to the BSDE (A.1) and it satisfies, for $\forall p \geq 2$,*

$$||(Y, Z, \psi)||_{\mathcal{K}^p[0,T]}^p \leq C \left(\mathbb{E}\left[|\xi|^{p\bar{q}^2} + \left(\int_0^T |f(s, 0, 0, 0)|ds\right)^{p\bar{q}^2}\right]\right)^{\frac{1}{\bar{q}^2}}$$

with a positive constant $\bar{q}$ satisfying $q_* \leq \bar{q} < \infty$ whose lower bound $q_* > 1$ is controlled only by $||H||_{\mathbb{H}^2_{BMO}}$, and some positive constant C depending only on $(p, \bar{q}, T, K, ||H||_{\mathbb{H}^2_{BMO}})$.

Proof. Define a progressively measurable process $(b_s, s \in [0, T])$ taking values in $\mathbb{R}^d$ by

$$b_s := \frac{f(s, Y_s, Z_s, \psi_s) - f(s, Y_s, 0, \psi_s)}{|Z_s|^2}\mathbf{1}_{Z_s \neq 0}Z_s$$

then $|b_s| \leq H_s$ and hence $b \in \mathbb{H}^2_{BMO}$ and its norm is dominated by $||H||_{\mathbb{H}^2_{BMO}}$. Under the measure $\mathbb{Q}$ defined by $d\mathbb{Q}/d\mathbb{P} = \mathcal{E}_T(b * W)$,

$$Y_t = \xi + \int_t^T f(s, Y_s, 0, \psi_s)ds - \int_t^T Z_s dW_s^{\mathbb{Q}} - \int_t^T \psi_s(x)\widetilde{\mu}^{\mathbb{Q}}(ds, dx) \tag{A.3}$$

where $W^{\mathbb{Q}} = W - \int_0^{\cdot} b_s ds$ and $\widetilde{\mu}^{\mathbb{Q}} = \widetilde{\mu}$. As discussed in Lemma A.1, one can choose $\bar{r} > 1$ with which both of $\mathcal{E}(b * W)$ and $\mathcal{E}(-b * W^{\mathbb{Q}})$ satisfy the reverse Hölder inequality and $\bar{q} = \frac{\bar{r}}{\bar{r}-1}$ as its dual. Let us put $D := \max\left(||\mathcal{E}(b * W)||_{\mathbb{L}^{\bar{r}}(\mathbb{P})}, ||\mathcal{E}(-b * W^{\mathbb{Q}})||_{\mathbb{L}^{\bar{r}}(\mathbb{Q})}\right)$, which is dominated by some constant depending only on $||H||_{\mathbb{H}^2_{BMO}(\mathbb{P})}$.

It is clear that the BSDE satisfies the global Lipschitz properties under the measure $\mathbb{Q}$. Furthermore, the following inequality is satisfied due to (reverse) Hölder inequalities:

$$\mathbb{E}^{\mathbb{Q}}\left[|\xi|^p + \left(\int_0^T |f(s, 0)|ds\right)^p\right] = \mathbb{E}\left[\mathcal{E}(b * W)\left(|\xi|^p + \left(\int_0^T |f(s, 0)|ds\right)^p\right)\right]$$

$$\leq \left(\mathbb{E}\left[\mathcal{E}(b * W)^{\bar{r}}\right]\right)^{\frac{1}{\bar{r}}}\left(\mathbb{E}\left[\left(|\xi|^p + \left(\int_0^T |f(s, 0)|ds\right)^p\right)^{\bar{q}}\right]\right)^{\frac{1}{\bar{q}}}$$

$$\leq C_{\bar{q}}D\mathbb{E}\left[|\xi|^{p\bar{q}} + \left(\int_0^T |f(s, 0)|ds\right)^{p\bar{q}}\right]^{\frac{1}{\bar{q}}} < \infty .$$

Thus, by Lemma B.2 in [16], one concludes that there exists a unique solution (Y, Z, ψ) to (A.3) in $\mathbb{Q}$ and hence also to (A.1) in $\mathbb{P}$. Furthermore by the same Lemma, it also satisfies,

$$||(Y, Z, \psi)||_{\mathcal{K}^p(\mathbb{Q})}^p \leq C_{p,K,T}\mathbb{E}^{\mathbb{Q}}\left[|\xi|^p + \left(\int_0^T |f(s, 0)|ds\right)^p\right] .$$

We thus have

$$||(Y, Z, \psi)||_{\mathcal{K}^p(\mathbb{P})}^p \leq C_{\bar{q}}D \, ||(Y, Z, \psi)||_{\mathcal{K}^{p\bar{q}}(\mathbb{Q})}^p$$

$$\leq C_{p,\bar{q},K,T}D^{1+\frac{1}{\bar{q}}}\left(\mathbb{E}\left[|\xi|^{p\bar{q}^2} + \left(\int_0^T |f(s, 0)|ds\right)^{p\bar{q}^2}\right]\right)^{\frac{1}{\bar{q}^2}} ,$$

34

which proves the second part of the claim. $\qquad\square$

Now, we gives the stability result which is required to show the strong convergence of the quadratic-exponential growth BSDE. Consider the two BSDEs with $i \in \{1,2\}$ satisfying Assumption A.2;

$$Y_t^i = \xi^i + \int_t^T f^i(s, Y_s^i, Z_s^i, \psi_s^i)ds - \int_t^T Z_s^i dW_s - \int_t^T \int_E \psi_s^i(x)\widetilde{\mu}(ds, dx) \qquad \text{(A.4)}$$

and put

$$\delta Y := Y^1 - Y^2, \quad \delta Z := Z^1 - Z^2, \quad \delta\psi := \psi^1 - \psi^2,$$
$$\delta f(s) := (f^1 - f^2)(s, Y_s^1, Z_s^1, \psi_s^1).$$

Lemma A.2. *The unique solutions* $(Y^i, Z^i, \psi^i), i \in \{1,2\}$ *to the BSDEs (A.4) under Assumption A.2 satisfy*

$$||(\delta Y, \delta Z, \delta\psi)||_{\mathcal{K}^p[0,T]}^p \leq C \left(\mathbb{E}\left[|\delta\xi|^{p\bar{q}^2} + \left(\int_0^T |\delta f(s)|ds \right)^{p\bar{q}^2} \right] \right)^{\frac{1}{\bar{q}^2}}$$

with a positive constant $q_* \leq \bar{q} < \infty$ *whose lower bound* $q_* > 1$ *is controlled only by* $||H||_{\mathbb{H}_{BMO}^2}$, *and some positive constant* C *depending only on* $(p, \bar{q}, T, K, ||H||_{\mathbb{H}_{BMO}^2})$.

Proof. Let a process $(b_s, s \in [0, T])$ be defined by

$$b_s = \frac{f^2(s, Y_s^1, Z_s^1, \psi_s^1) - f^2(s, Y_s^1, Z_s^2, \psi_s^1)}{|\delta Z_s|^2} \mathbf{1}_{\delta Z_s \neq 0} \delta Z_s$$

which satisfies $|b_s| \leq H_s$ by the assumption and hence $b \in \mathbb{H}_{BMO}^2$. By defining the measure $\mathbb{Q}$ by $d\mathbb{Q}/d\mathbb{P} = \mathcal{E}_T(b * W)$, one obtains

$$\delta Y_t = \delta\xi + \int_t^T \left(\delta f(s) + f^2(s, Y_s^1, Z_s^2, \psi_s^1) - f^2(s, Y_s^2, Z_s^2, \psi_s^2) \right) ds$$
$$- \int_t^T \delta Z_s dW_s^{\mathbb{Q}} - \int_t^T \int_E \delta\psi_s(x)\widetilde{\mu}^{\mathbb{Q}}(ds, dx). \qquad \text{(A.5)}$$

As in the previous theorem, we can choose $\bar{r} > 1$ with which both of $\mathcal{E}(b * W)$ and $\mathcal{E}(-b * W^{\mathbb{Q}})$ satisfy the reverse Hölder inequality and $\bar{q}$ as its dual. We also put $D := \max\left(||\mathcal{E}(b * W)||_{\mathbb{L}^{\bar{r}}(\mathbb{P})}, ||\mathcal{E}(-b * W^{\mathbb{Q}})||_{\mathbb{L}^{\bar{r}}(\mathbb{Q})} \right)$ as before.

Since $|f^2(t, y, z, \psi) - f^2(t, y', z, \psi')| \leq K(|\delta y| + ||\delta\psi||_{\mathbb{L}^2(\nu)})$, the driver of the BSDE (A.5) satisfies the linear growth. Thus, the same technique used to derive an a priori estimate for the linear-growth BSDEs yields

$$||(\delta Y, \delta Z, \delta\psi)||_{\mathcal{K}^p(\mathbb{Q})}^p \leq C_{p,K,T} \mathbb{E}^{\mathbb{Q}}\left[|\delta\xi|^p + \left(\int_0^T |\delta f(s)|ds \right)^p \right].$$

(See, for example, Lemma B.1 of [16] and its proof.) Thus, one obtains from the same procedures used in the previous theorem

$$||(\delta Y, \delta Z, \delta\psi)||_{\mathcal{K}^p(\mathbb{P})}^p \leq C_{p,\bar{q},K,T} D^{1+\frac{1}{\bar{q}}} \left(\mathbb{E}\left[|\delta\xi|^{p\bar{q}^2} + \left(\int_0^T |\delta f(s)|ds \right)^{p\bar{q}^2} \right] \right)^{\frac{1}{\bar{q}^2}},$$

which proves the claim. $\qquad\square$

B Malliavin differentiability for Lipschitz BSDEs with jumps

In order to show Malliavin's differentiability of $Q_{\exp}$-growth BSDEs, we have to establish the differentiability for Lipschitz BSDEs with slightly more general setup than what was proved in [13] and [12]. For convenience of the readers, we give the detailed proof in this section. We closely follow the arguments used in El Karoui et.al. (1997) [15]. The complication relative to a diffusion case is the treatment of small jumps. The difference from the work [13] is a local Lipschitz condition instead of the global Lipschitz condition for the Malliavin derivative of the driver [4].

We consider a BSDE defined by

$$Y_t = \xi + \int_t^T f\Big(s, Y_s, Z_s, \int_{\mathbb{R}_0} \rho(x) G(s, \psi_s(x)) \nu(dx)\Big) ds$$
$$- \int_t^T Z_s dW_s - \int_t^T \int_E \psi_s(x) \widetilde{\mu}(ds, dx) , \tag{B.1}$$

where $\xi : \Omega \to \mathbb{R}^m$, $f : \Omega \times [0,T] \times \mathbb{R}^{m \times d} \times \mathbb{R}^{m \times k} \to \mathbb{R}^m$. Here, $\int_{\mathbb{R}_0} \rho(x) G(s, \psi_s(x)) \nu(dx)$ denotes a k-dimensional vector whose i-th element is given by $\int_{\mathbb{R}_0} \rho^i(x) G^i(s, \psi_s^i(x)) \nu^i(dx)$ where $\rho^i : \mathbb{R} \to \mathbb{R}$, $\mathbf{G}^i : [0,T] \times \mathbb{R} \to \mathbb{R}$. With slight abuse of notation, we use

$$\Theta_r := \Big(Y_r, Z_r, \int_{\mathbb{R}_0} \rho(x) G(r, \psi_r(x)) \nu(dx)\Big)$$

as a collective argument in this section.

Assumption B.1. *(i) For every $i \in \{1, \cdots, k\}$, $\rho^i(s)$ and $G^i(s,v)$ are continuous functions in $s \in [0,T]$ and $(s,v) \in [0,T] \times \mathbb{R}$, respectively. We set without loss of generality that $G^i(\cdot, 0) = 0$. In addition $\int_{\mathbb{R}_0} |\rho^i(x)|^2 \nu^i(dx) < \infty$, and with some positive constant K, G^i satisfies*

$$|G^i(s,v) - G^i(s,v')| \le K|v - v'|, \quad \text{for every } s \in [0,T] \text{ and } v, v' \in \mathbb{R}.$$

(ii) The map $(\omega, t) \mapsto f(\omega, t, \cdot)$ is $\mathbb{F}$-progressively measurable, and for every $(y,z,u), (y',z',u') \in \mathbb{R}^m \times \mathbb{R}^{m \times d} \times \mathbb{R}^{m \times k}$, there exists some positive constant K such that

$$|f(\omega, t, y, z, u) - f(\omega, t, y', z', u')| \le K(|y - y'| + |z - z'| + |u - u'|)$$

$d\mathbb{P} \otimes dt$-a.e. $(\omega, t) \in \Omega \times [0,T]$.
(iii) $\xi \in \mathbb{L}^4(\Omega, \mathcal{F}_T, \mathbb{P})$ and $\big(f(t,0), t \in [0,T]\big) \in \mathbb{H}^4[0,T]$.

Remark

Due to the property of G and ρ, it is easy to see that

$$\Big|\int_{\mathbb{R}_0} \rho(x) G(s, \psi_s(x)) \nu(dx) - \int_{\mathbb{R}_0} \rho(x) G(s, \psi_s'(x)) \nu(dx)\Big| \le K'||\psi_s - \psi_s'||_{\mathbb{L}^2(\nu)}$$

---

[4]In addition, we think that the result of [13] misses the one condition for the driver

$$\lim_{\epsilon \downarrow 0} \int_0^T \int_{|z| \le \epsilon} \mathbb{E}\left[\left(\int_0^T |(D_{s,z}f)(r,0)| dr\right)^2\right] z^2 \nu(dz) ds = 0 .$$

This stems from an error in the estimate (4.15) and (4.16) (and hence (4.21),(4.22)) in the proof of Theorem 4.1 of [13]. Note that if one choose $\widetilde{f} = f$, then $(\widetilde{Y}^{s,z}, \widetilde{Z}^{s,z}, \widetilde{U}^{s,z})$ cannot be chosen as zero. The left hand side of (4.16), for example, should still be $||Y^{s,z} - \widetilde{Y}^{s,z}||_{\mathbb{S}^2} + \cdots$. Adding the contributions from $\widetilde{Y}^{s,z}$ etc would yield the consistent result to the analysis given here.

with some constant $K' > 0$. Thus, Assumption B.1 yields the standard global Lipschitz conditions. By Lemma B.2 in [16], the BSDE (B.1) has a unique solution $(Y, Z, \psi) \in \mathcal{K}^4[0,T]$. In order to show the Malliavin differentiability, we need additional assumptions.

Assumption B.2. *(i) For every $i \in \{1, \cdots, k\}$, G^i is one-time continuously differentiable with respect to its spacial variable v with a uniformly bounded and continuous derivative.*
(ii) The terminal value is Malliavin differentiable $\xi \in \mathbb{D}^{1,2}$ and satisfies

$$\mathbb{E}\left[\int_{\widetilde{E}} |D_{s,z}\xi|^2 q(ds, dz)\right] < \infty.$$

(iii) The driver $f(\cdot, y, z, u)$ is one-time continuously differentiable with respect to (y, z, u) with uniformly bounded and continuous derivatives. For every $(y, z, u) \in \mathbb{R}^m \times \mathbb{R}^{m \times d} \times \mathbb{R}^{m \times k}$, the driver $\big(f(t, y, z, u), t \in [0,T]\big)$ belongs to $\mathbb{L}^{1,2}(\mathbb{R}^m)$ and its Malliavin derivative is denoted by $(D_{s,z}f)(t, y, z, u)$.
(iv) For every Wiener as well as jump direction, and for every $(y, z, u), (y', z', u') \in \mathbb{R}^m \times \mathbb{R}^{m \times d} \times \mathbb{R}^{m \times k}$ and $d\mathbb{P} \otimes dt$-a.e. $(\omega, t) \in \Omega \times [0,T]$, the Malliavin derivative of the driver satisfies the following local Lipschitz conditions [5];

$$|(D^i_{s,0}f)(t, y, z, u) - (D^i_{s,0}f)(t, y', z', u')| \leq K^i_{s,0}(t)\Big(|y - y'| + |z - z'| + |u - u'|\Big),$$

for ds-a.e. $s \in [0,T]$ with $i \in \{1, \cdots, d\}$, and

$$|(D^i_{s,z}f)(t, y, z, u) - (D^i_{s,z}f)(t, y', z', u')| \leq K^i_{s,z}(t)\Big(|y - y'| + |z - z'| + |u - u'|\Big),$$

for $m^i(dz)ds$-a.e. $(s, z) \in [0,T] \times \mathbb{R}_0$ with $i \in \{1, \cdots, k\}$. Here, $\Big(K^i_{s,0}(t), t \in [0,T]\Big)_{i \in \{1, \cdots, d\}}$ and $\Big(K^i_{s,z}(t), t \in [0,T]\Big)_{i \in \{1, \cdots, k\}}$ are $\mathbb{R}_+$-valued $\mathbb{F}$-progressively measurable processes satisfying $\int_{\widetilde{E}} \|K_{s,z}(\cdot)\|^4_{\mathbb{S}^4[0,T]} q(ds, dz) < \infty$.
(v) The following equality is supposed to hold;

$$\lim_{\epsilon \downarrow 0} \sum_{i=1}^k \int_0^T \int_{|z| \leq \epsilon} \mathbb{E}\left[|D^i_{s,z}\xi|^2 + \Big(\int_0^T |(D^i_{s,z}f)(r, 0)|dr\Big)^2 + \|K^i_{s,z}\|^4_T\right] z^2 \nu^i(dz)ds = 0.$$

Theorem B.1. *Suppose that Assumptions B.1 and B.2 hold true and denote the solution to the BSDE (B.1) as $(Y, Z, \psi) \in \mathcal{K}^4[0,T]$. Then, the following statements hold:*
(a) For each Wiener direction $i \in \{1, \cdots, d\}$ and ds-a.e. $s \in [0,T]$, there exists a unique solution $(Y^{s,0,i}, Z^{s,0,i}, \psi^{s,0,i}) \in \mathcal{K}^2[0,T]$ to the BSDE

$$Y^{s,0,i}_t = D^i_{s,0}\xi + \int_t^T f^{s,0,i}(r)dr - \int_t^T Z^{s,0,i}_r dW_r - \int_t^T \int_E \psi^{s,0,i}_r(x)\widetilde{\mu}(dr, dx) \qquad \text{(B.2)}$$

for $0 \leq s \leq t \leq T$, where

$$
\begin{aligned}
f^{s,0,i}(r) &:= (D^i_{s,0}f)(r, \Theta_r) + \partial_\Theta f(r, \Theta_r)\Theta^{s,0,i}_r \\
&= (D^i_{s,0}f)(r, \Theta_r) + \partial_y f(r, \Theta_r)Y^{s,0,i}_r + \partial_z f(r, \Theta_r)Z^{s,0,i}_r \\
&\quad + \partial_u f(r, \Theta_r)\int_{\mathbb{R}_0} \rho(x)\partial_v G(r, \psi_r(x))\psi^{s,0,i}_r(x)\nu(dx).
\end{aligned}
$$

---

[5]Delong & Imkeller (2010) [13] has treated a special case where $(K_{s,0}, K_{s,z})$ are positive constants. The current generalization is necessary when one introduces a Markovian process X driven by a FSDE to create a forward-backward SDE system, which is the subject of interests in many applications.

(b) For each jump direction $i \in \{1, \cdots, k\}$ and $m^i(dz)ds$-a.e. $(s, z) \in [0, T] \times \mathbb{R}_0$, there exists a unique solution $(Y^{s,z,i}, Z^{s,z,i}, \psi^{s,z,i}) \in \mathcal{K}^2[0, T]$ to the BSDE

$$Y_t^{s,z,i} = D_{s,z}^i \xi + \int_t^T f^{s,z,i}(r)dr - \int_t^T Z_r^{s,z,i} dW_r - \int_t^T \int_E \psi_r^{s,z,i}(x)\widetilde{\mu}(dr, dx) \qquad (B.3)$$

for $0 \leq s \leq t \leq T$ and $z \neq 0$, where

$$
\begin{aligned}
f^{s,z,i}(r) \quad &:= \quad \frac{1}{z}\Big(f(\omega^{s,z}, r, \Theta_r + z\Theta_r^{s,z,i}) - f(\omega, r, \Theta_r)\Big) \\
&= \quad \frac{1}{z}\Big\{f\Big(\omega^{s,z}, r, Y_r + zY_r^{s,z,i}, Z_r + zZ_r^{s,z,i} \\
&\qquad \qquad , \int_{\mathbb{R}_0} \rho(x) G\big(r, \psi_r(x) + z\psi_r^{s,z,i}(x)\big)\nu(dx)\Big) - f(\omega, r, \Theta_r)\Big)\Big\}.
\end{aligned}
$$

(c) Solution of the BSDE (B.1) is Malliavin differentiable $(Y, Z, \psi) \in \mathbb{L}^{1,2} \times \mathbb{L}^{1,2} \times \mathbb{L}^{1,2}$. Put, for every i, $Y_t^{s,\cdot,i} = Z_t^{s,\cdot,i} = \psi_t^{s,\cdot,i}(\cdot) \equiv 0$ for $t < s \leq T$, then $\Big((Y_t^{s,z,i}, Z_t^{s,z,i}, \psi_t^{s,z,i}(x)), 0 \leq s, t \leq T, x \in \mathbb{R}_0, z \in \mathbb{R}\Big)$ is a version of the Malliavin derivative $\Big((D_{s,z}^i Y_t, D_{s,z}^i Z_t, D_{s,z}^i \psi_t(x)), 0 \leq s, t \leq T, x \in \mathbb{R}_0, z \in \mathbb{R}\Big)$ for every Wiener and jump direction.

Proof. For notational simplicity, we omit i denoting the direction of derivative by assuming that we consider each direction separately (and summing them up whenever necessary, such as when considering integration on $\widetilde{E}$).

Proof for (a) and (b)
It is easy to see that both of the BSDEs (B.2) and (B.3) satisfy the standard global Lipschitz conditions. We have

$$|f^{s,0}(r)| \leq |(D_{s,0}f)(r, 0)| + K_{s,0}(r)|\Theta_r| + K|\Theta_r^{s,0}| \ .$$

Since

$$
\begin{aligned}
f^{s,z}(r) \quad &= \quad \frac{f(\omega^{s,z}, r, \Theta_r) - f(\omega, r, \Theta_r)}{z} + \frac{f(\omega^{s,z}, r, \Theta_r + z\Theta_r^{s,z}) - f(\omega^{s,z}, r, \Theta_r)}{z} \\
&= \quad (D^{s,z}f)(r, \Theta_r) + \frac{f(\omega^{s,z}, r, \Theta_r + z\Theta_r^{s,z}) - f(\omega^{s,z}, r, \Theta_r)}{z}.
\end{aligned}
$$

we also have $|f^{s,z}(r)| \leq |(D_{s,z}f)(r, 0)| + K_{s,z}(r)|\Theta_r| + K|\Theta_r^{s,z}| \ .$ Thus, Lemma B.2 in [16] tells us that for all $(s, z) \in [0, T] \times \mathbb{R}$ (thus including $\Theta^{s,0}$) there exists a unique solution $\Theta^{s,z} \in \mathcal{K}^2[0, T]$ satisfying

$$
\begin{aligned}
||(Y^{s,z}, Z^{s,z}, \psi^{s,z})||_{\mathcal{K}^2[0,T]}^2 &\leq C_{K,T}\mathbb{E}\left[|D_{s,z}\xi|^2 + \Big(\int_0^T \Big[|(D_{s,z}f)(r, 0)| + K_{s,z}(r)|\Theta_r|\Big]dr\Big)^2\right] \\
&\leq C_{K,T}\mathbb{E}\left[|D_{s,z}\xi|^2 + \Big(\int_0^T |(D_{s,z}f)(r, 0)|dr\Big)^2 + ||K_{s,z}||_T^4 + \Big(\int_0^T |\Theta_r|^2 dr\Big)^2\right] < \infty.
\end{aligned}
$$

Note that $\Theta \in \mathcal{K}^4[0, T]$. Assumption B.2 also yields

$$\int_{\widetilde{E}} ||(Y^{s,z}, Z^{s,z}, \psi^{s,z})||_{\mathcal{K}^2[0,T]}^2 q(ds, dz) < \infty \ .$$

Proof for (c)

We consider an approximating sequence of BSDEs;

$$Y_t^{n+1} = \xi + \int_t^T f^n(r) - \int_t^T Z_r^{n+1} dW_r - \int_t^T \int_E \psi_r^{n+1}(x)\widetilde{\mu}(dr, dx), \tag{B.4}$$

for $t \in [0, T]$ and $n \in \mathbb{N}$, where

$$f^n(r) := f\left(r, Y_r^n, Z_r^n, \int_{\mathbb{R}_0} \rho(x) G(r, \psi_r^n(x)) \nu(dx)\right).$$

It converges to (Y, Z, ψ) of (B.1) in $\mathcal{K}^4[0, T]$ by the standard arguments of contraction mapping for the Lipschitz BSDEs. See, for example, Lemma B.2 in [16] and its proof.

First step: confirm $(Y^{n+1}, Z^{n+1}, \psi^{n+1}) \in (\mathbb{L}^{1,2})^{\otimes 3}$
We now suppose that $(Y^n, Z^n, \psi^n) \in \mathcal{K}^4[0, T] \cap (\mathbb{L}^{1,2})^{\otimes 3}$ and are going to prove that $(Y^{n+1}, Z^{n+1}, \psi^{n+1}) \in \mathcal{K}^4[0, T] \cap (\mathbb{L}^{1,2})^{\otimes 3}$. Firstly, the *chain* rules (Theorem 3.5 and Theorem 12.8 in [14] with the division by the jump size in the current convention) and Lemma 3.2 in [13] shows that

$$\int_{\mathbb{R}_0} \rho(x) G(r, \psi_r^n(x)) \nu(dx) dr \in \mathbb{D}^{1,2}. \tag{B.5}$$

In particular, this is because

$$\int_{\widetilde{E}} \left|\left| D_{t,z} G(\cdot, \psi_\cdot^n) \right|\right|_{\mathbb{J}^2[0,T]}^2 q(dt, dz) \le K^2 \int_{\widetilde{E}} \left|\left| D_{t,z} \psi_\cdot^n \right|\right|_{\mathbb{J}^2[0,T]}^2 q(dt, dz) < \infty,$$

where we have used the bounded derivative and the Lipschitz condition for G and the assumption that $\psi^n \in \mathbb{L}^{1,2}$. This also shows that $G(\cdot, \psi_\cdot^n) \in \mathbb{L}^{1,2}$. By (B.5) and again by the *chain* rule, we see $f^n(r) = f(r, \Theta_r^n) \in \mathbb{D}^{1,2}$ for every $r \in [0, T]$.

It is obvious to see $||f^n(\cdot)||_{\mathbb{H}^2[0,T]}^2 < \infty$. Moreover, with some positive constant C_K depending only on the Lipschitz constant,

$$\int_{\widetilde{E}} ||D_{t,z} f^n(\cdot)||_{\mathbb{H}^2[0,T]}^2 q(dt, dz)$$

$$\le C_K \int_{\widetilde{E}} \mathbb{E}\left[\int_0^T \left(|(D_{t,z}f)(r, 0)|^2 + |K_{t,z}(r)|^2 |\Theta_r^n|^2 + |D_{t,z}\Theta_r^n|^2\right) dr\right] q(dt, dz)$$

$$\le C_K \int_{\widetilde{E}} \mathbb{E}\left[\int_0^T \left(|(D_{t,z}f)(r, 0)|^2 + |D_{t,z}\Theta_r^n|^2\right) dr + ||K_{t,z}||_T^4 + \left(\int_0^T |\Theta_r^n|^2 dr\right)^2\right] q(dt, dz)$$

$$< \infty$$

due to Assumption B.2 and the fact that $\Theta^n \in \mathcal{K}^4[0, T] \cap (\mathbb{L}^{1,2})^{\otimes 3}$. Thus, Lemma 3.2 [13] shows that $\int_t^T f^n(r) dr \in \mathbb{D}^{1,2}$ for every $t \in [0, T]$.

As a result, we have for $t \in [0, T]$, $\xi + \int_t^T f^n(r) \in \mathbb{D}^{1,2}$. Thus, by Lemma 3.1 [13], we conclude that $Y_t^{n+1} = \mathbb{E}\left[\xi + \int_t^T f^n(r)\Big|\mathcal{F}_t\right] \in \mathbb{D}^{1,2}$, which then implies

$$\int_t^T Z_r^{n+1} dW_r + \int_t^T \int_E \psi_r^{n+1}(x)\widetilde{\mu}(dr, dx)$$

$$= -Y_t^{n+1} + \xi + \int_t^T f^n(r) dr \in \mathbb{D}^{1,2},$$

which, together with Lemma 3.3 [13], shows $Z^{n+1}, \psi^{n+1} \in \mathbb{L}^{1,2}$.

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We are now going to prove $Y^{n+1} \in \mathbb{L}^{1,2}$. For a Wiener ($z = 0$) as well as a jump ($z \neq 0$) direction, we have, by Lemma 3.3 [13], that

$$D_{s,z}Y_t^{n+1} = D_{s,z}\xi + \int_t^T D_{s,z}f^n(r)dr - \int_t^T D_{s,z}Z_r^{n+1}dW_r - \int_t^T \int_E D_{s,z}\psi_r^{n+1}(x)\widetilde{\mu}(dr, dx)$$

for $q(ds, dz)$-a.e. $(s, z) \in [0, T] \times \mathbb{R}^k$. One can check that

$$|D_{s,z}f^n(r)| \leq |(D_{s,z}f)(r, 0)| + K_{s,z}(r)|\Theta_r^n| + K|D_{s,z}\Theta_r^n| . \tag{B.6}$$

Thus, by Lemmas B.1 in [16], one can show that

$$\int_{\widetilde{E}} \big|\big|(D_{s,z}Y^{n+1}, D_{s,z}Z^{n+1}, D_{s,z}\psi^{n+1})\big|\big|_{\mathcal{K}^2[0,T]}^2 q(ds, dz)$$

$$\leq C_{K,T} \int_{\widetilde{E}} \mathbb{E}\left[|D_{s,z}\xi|^2 + \left(\int_0^T \big[|(D_{s,z}f)(r, 0)| + K_{s,z}|\Theta_r^n| + |D_{s,z}\Theta_r^n|\big]dr\right)^2\right] q(ds, dz)$$

$$\leq C_{K,T} \int_{\widetilde{E}} \mathbb{E}\left[|D_{s,z}\xi|^2 + \int_0^T \Big(|(D_{s,z}f)(r, 0)|^2 + |D_{s,z}\Theta_r^n|^2\Big)dr\right.$$

$$\left. + ||K_{s,z}||_T^4 + \left(\int_0^T |\Theta_r^n|^2 dr\right)^2\right] q(ds, dz) < \infty . \tag{B.7}$$

This proves $Y^{n+1} \in \mathbb{L}^{1,2}$. As a result, we have shown $\Theta^{n+1} \in \mathbb{L}^{1,2}$.

Second step: convergence of $D_{s,0}\Theta^n \to \Theta^{s,0}$ as $n \to \infty$ in $\mathbb{L}^{1,2}$
Let us set the difference process as follows:

$$\Delta^{s,0}Y^n := Y^{s,0} - D_{s,0}Y^n, \quad \Delta^{s,0}Z^n := Z^{s,0} - D_{s,0}Z^n, \quad \Delta^{s,0}\psi^n := \psi^{s,0} - D_{s,0}\psi^n.$$

and denote $\Delta^{s,0}\Theta^n := (\Delta^{s,0}Y^n, \Delta^{s,0}Z^n, \Delta^{s,0}\psi^n)$ for every $n \in \mathbb{N}$. Since the derivative of the driver is uniformly bounded, one has

$$|f^{s,0}(r) - D_{s,0}f^n(r)| \leq K_{s,0}(r)|\Theta_r - \Theta_r^n| + |\partial_\Theta f(r, \Theta_r) - \partial_\Theta f(r, \Theta_r^n)||\Theta_r^{s,0}| + K|\Delta^{s,0}\Theta_r^n|$$

then, the a priori estimate given in Lemma B.1 [16] gives

$$\int_0^T ||(\Delta^{s,0}Y^{n+1}, \Delta^{s,0}Z^{n+1}, \Delta^{s,0}\psi^{n+1})||_{\mathcal{K}^2[0,T]}^2 ds \leq C_T \int_0^T \mathbb{E}\left[\left(\int_0^T |f^{s,0}(r) - D_{s,0}f^n(r)|dr\right)^2\right]ds$$

$$\leq C_T \int_0^T \mathbb{E}\left[\left(\int_0^T \Big[K_{s,0}(r)|\Theta_r - \Theta_r^n| + |\partial_\Theta f(r, \Theta_r) - \partial_\Theta f(r, \Theta_r^n)||\Theta_r^{s,0}|\Big]dr\right)^2\right]ds$$

$$+ C_{T,K} \int_0^T \mathbb{E}\left[\left(\int_0^T |\Delta^{s,0}\Theta_r^n|dr\right)^2\right]ds .$$

One sees that the first line converges to zero when passing to the limit $n \to \infty$. Thus, by a sequence of small positive constants $\epsilon_n, n \in \mathbb{N}$ which converges to zero, one can write

$$\int_0^T ||(\Delta^{s,0}Y^{n+1}, \Delta^{s,0}Z^{n+1}, \Delta^{s,0}\psi^{n+1})||_{\mathcal{K}^2[0,T]}^2 ds \leq \epsilon_n + C_{T,K} \int_0^T \mathbb{E}\left[\left(\int_0^T |\Delta^{s,0}\Theta_r^n|dr\right)^2\right]ds$$

$$\leq \epsilon_n + C_{T,K}' \max(T^2, T) \int_0^T ||(\Delta^{s,0}Y^n, \Delta^{s,0}Z^n, \Delta^{s,0}\psi^n)||_{\mathcal{K}^2[0,T]}^2 ds. \tag{B.8}$$

We first consider a short maturity case. Choosing the terminal time T small so that

$$\alpha := C'_{T,K} \max(T^2, T) < 1 \tag{B.9}$$

then one obtains

$$\int_0^T ||(\Delta^{s,0}\Theta^{n+1})||^2_{\mathcal{K}^2[0,T]} ds \le \epsilon_n + \alpha \int_0^T ||(\Delta^{s,0}\Theta^n)||^2_{\mathcal{K}^2[0,T]} ds \ .$$

Then, fixing some $n_0 \in \mathbb{N}$, it yields

$$\int_0^T ||(\Delta^{s,0}\Theta^{n+n_0})||^2_{\mathcal{K}^2[0,T]} ds \le \frac{\epsilon_{n_0}}{1-\alpha} + \alpha^n \int_0^T ||(\Delta^{s,0}\Theta^{n_0})||^2_{\mathcal{K}^2[0,T]} ds.$$

Thus, passing n and then n_0 to ∞, one can conclude that

$$\lim_{n\to\infty} \int_0^T ||(\Delta^{s,0}\Theta^n)||^2_{\mathcal{K}^2[0,T]} ds = 0.$$

For general maturity $T > 0$, one can use a time partition $0 = T_0 < T_1 < \cdots < T_N = T$ fine enough to make $\alpha < 1$ in every time interval. Due to the uniqueness of the solution, by setting $Y^{s,0}_{T_i}$ as the common terminal condition for the interval $[T_{i-1}, T_i]$, one can repeat the same arguments to obtain the convergence in $\mathbb{L}^{1,2}$ for the whole range.

Third step: convergence of $D_{s,z}\Theta^n \to \Theta^{s,z}(z \neq 0)$ as $n \to \infty$ in $\mathbb{L}^{1,2}$
Choosing one direction of jump (omit i for simplicity) and put

$$\Delta^{s,z}Y^n := Y^{s,z} - D_{s,z}Y^n, \quad \Delta^{s,z}Z^n := Z^{s,z} - D_{s,z}Z^n, \quad \Delta^{s,z}\psi^n := \psi^{s,z} - D_{s,z}\psi^n.$$

and denote $\Delta^{s,z}\Theta^n := (\Delta^{s,z}Y^n, \Delta^{s,z}Z^n, \Delta^{s,z}\psi^n)$ for every $n \in \mathbb{N}$. By Lemma B.1 [16], one can show that

$$\int_0^T \int_{\mathbb{R}_0} ||(\Delta^{s,z}\Theta^{n+1})||^2_{\mathcal{K}^2[0,T]} m(dz) ds$$
$$\le C_T \int_0^T \int_{\mathbb{R}_0} \mathbb{E}\left[\left(\int_0^T |f^{s,z}(r) - D_{s,z}f^n(r)| dr\right)^2\right] m(dz) ds \ .$$

Before discussing $\lim_{n\to\infty}$, we have to prove that the convergence

$$\lim_{\epsilon\downarrow 0} \int_0^T \int_{|z|>\epsilon} ||(\Delta^{s,z}\Theta^{n+1})||^2_{\mathcal{K}^2[0,T]} m(dz) ds = \int_0^T \int_{\mathbb{R}_0} ||(\Delta^{s,z}\Theta^{n+1})||^2_{\mathcal{K}^2[0,T]} m(dz) ds \tag{B.10}$$

occurs uniformly in n.

By Assumption B.2 (v), for an arbitrary small $\epsilon > 0$, there exists $\bar{\epsilon} > 0$ such that

- $\int_0^T \int_{|z|\le\bar{\epsilon}} \mathbb{E}\left[|D_{s,z}\xi|^2 + \left(\int_0^T |(D_{s,z}f)(r,0)| dr\right)^2 + ||K_{s,z}||^4_T\right] m(dz) ds < \epsilon$ (B.11)

- $\int_0^T \int_{|z|\le\bar{\epsilon}} m(dz) ds < \epsilon.$ (B.12)

For proving the claim, it suffices to show that there exists a positive constant independent

of n such that

$$\int_0^T \int_{|z|\leq\bar\epsilon} ||(\Delta^{s,z}\Theta^{n+1})||^2_{\mathcal{K}^2[0,T]} m(dz)ds < C\epsilon$$

for the above defined $\bar\epsilon$.

Using the (local) Lipschitz properties, it is easy to show that

$$|f^{s,z}(r) - D_{s,z}f^n(r)| \leq K_{s,z}(r)|\Theta_r - \Theta^n_r| + K|\Theta^{s,z}_r| + K|D_{s,z}\Theta^n_r|$$

and hence

$$\int_0^T \int_{|z|\leq\bar\epsilon} ||(\Delta^{s,z}\Theta^{n+1})||^2_{\mathcal{K}^2[0,T]} m(dz)ds \leq C_{T,K} \int_0^T \int_{|z|\leq\bar\epsilon} \mathbb{E}\left[\left(\int_0^T K_{s,z}(r)|\Theta_r - \Theta^n_r|dr\right)^2\right.$$
$$+ \left(\int_0^T |\Theta^{s,z}_r|dr\right)^2 + \left.\left(\int_0^T |D_{s,z}\Theta^n_r|dr\right)^2\right] m(dz)ds .$$

We are now going to discuss each term. The first term can be evaluated as

$$C_{T,K} \int_0^T \int_{|z|\leq\bar\epsilon} \mathbb{E}\left[\left(\int_0^T K_{s,z}(r)|\Theta_r - \Theta^n_r|dr\right)^2\right]$$
$$\leq C_{T,K} \int_0^T \int_{|z|\leq\bar\epsilon} \mathbb{E}\left[||K_{s,z}||^4_T + \left(\int_0^T |\Theta_r - \Theta^n_r|^2 dr\right)^2\right] m(dz)ds < C\epsilon$$

where the last inequality follows from (B.11), (B.12) and the fact that $\Theta^n \to \Theta$ in $\mathcal{K}^4[0,T]$. For the second term, one can show

$$C_{T,K} \int_0^T \int_{|z|\leq\bar\epsilon} \mathbb{E}\left[\left(\int_0^T |\Theta^{s,z}_r|dr\right)^2\right] m(dz)ds$$
$$\leq C_{T,K} \int_0^T \int_{|z|\leq\bar\epsilon} ||(\Theta^{s,z})||^2_{\mathcal{K}^2[0,T]} m(dz)ds$$
$$\leq C_{T,K} \int_0^T \int_{|z|\leq\bar\epsilon} \mathbb{E}\left[|D_{s,z}\xi|^2 + \left(\int_0^T |(D_{s,z}f)(r,0)|dr\right)^2 + ||K_{s,z}||^4_T\right.$$
$$+ \left.\left(\int_0^T |\Theta_r|^2 dr\right)^2\right] m(dz)ds < C\epsilon \tag{B.13}$$

where the last equality follows from (B.11), (B.12) and the fact that $\Theta \in \mathcal{K}^4[0,T]$.

Finally, the third term can be evaluated as

$$C_{T,K} \int_0^T \int_{|z|\leq\bar\epsilon} \mathbb{E}\left[\left(\int_0^T |D_{s,z}\Theta^n_r|dr\right)^2\right] m(dz)ds$$
$$\leq C_{T,K} \int_0^T \int_{|z|\leq\bar\epsilon} ||(D_{s,z}\Theta^n)||^2_{\mathcal{K}^2[0,T]} m(dz)ds .$$

Here, by the same a priori estimate used in (B.7),

$$C_{T,K}||(D_{s,z}\Theta^n)||^2_{\mathcal{K}^2[0,T]} \leq C_{K,T}\mathbb{E}\left[|D_{s,z}\xi|^2 + \left(\int_0^T |(D_{s,z}f)(r,0)|dr\right)^2 + ||K_{s,z}||^4_T\right.$$

$$\left.+\left(\int_0^T |\Theta_r^{n-1}|^2 dr\right)^2\right] + C_{K,T}\mathbb{E}\left[\left(\int_0^T |D_{s,z}\Theta_r^{n-1}|dr\right)^2\right]$$

$$\leq C_{K,T}\mathbb{E}\left[\epsilon_{n-1} + |D_{s,z}\xi|^2 + \left(\int_0^T |(D_{s,z}f)(r,0)|dr\right)^2 + ||K_{s,z}||^4_T + \left(\int_0^T |\Theta_r|^2 dr\right)^2\right]$$

$$+C_{K,T}\max(T^2,T)||(D_{s,z}\Theta^{n-1})||^2_{\mathcal{K}^2[0,T]}\ . \tag{B.14}$$

By the convergence of $\Theta^n \to \Theta$ in $\mathcal{K}^4[0,T]$, $(\epsilon_n)_{n\in\mathbb{N}}$ with $\epsilon_n \to 0$ is a sequence positive constants bounded uniformly in n.

Choosing the terminal time T small enough so that $\alpha := C_{K,T}\max(T^2,T) < 1$, (B.14) yields

$$C_{T,K}\int_0^T \int_{|z|\leq\bar{\epsilon}} ||(D_{s,z}\Theta^n)||^2_{\mathcal{K}^2[0,T]}m(dz)ds$$

$$\leq \frac{C_{K,T}}{1-\alpha}\int_0^T \int_{|z|\leq\bar{\epsilon}} \mathbb{E}\left[\epsilon_1 + |D_{s,z}\xi|^2 + \left(\int_0^T |(D_{s,z}f)(r,0)|dr\right)^2 + ||K_{s,z}||^4_T\right.$$

$$\left.+\left(\int_0^T |\Theta_r|^2 dr\right)^2\right]m(dz)ds + \alpha^n\int_0^T \int_{|z|\leq\bar{\epsilon}} ||(D_{s,z}\Theta^1)||^2_{\mathcal{K}^2[0,T]}m(dz)ds.$$

It is free to choose $\Theta^1 \equiv 0$ in our fixed point iteration (B.4). Thus, the right hand side is dominated by $C\epsilon$ with n independent constant C due to (B.11) and (B.12).

Small terminal time T

By the previous arguments, we have shown that the convergence of (B.10) is uniform in n, at least for small terminal time T. Firstly, let us consider this case, where one can exchange the order of limit operations and obtain

$$\lim_{n\to\infty}\lim_{\epsilon\downarrow 0}\int_0^T \int_{|z|>\epsilon} ||(\Delta^{s,z}\Theta^{n+1})||^2_{\mathcal{K}^2[0,T]}m(dz)ds$$

$$= \lim_{\epsilon\downarrow 0}\lim_{n\to\infty}\int_0^T \int_{|z|>\epsilon} ||(\Delta^{s,z}\Theta^{n+1})||^2_{\mathcal{K}^2[0,T]}m(dz)ds\ . \tag{B.15}$$

Therefore, in order to show the convergence of $D_{s,z}\Theta^n \to \Theta^{s,z}$ in $\mathbb{L}^{1,2}$, it is enough to prove

$$\lim_{n\to\infty}\int_0^T \int_{|z|>\epsilon} ||(\Delta^{s,z}\Theta^{n+1})||^2_{\mathcal{K}^2[0,T]}m(dz)ds = 0$$

for a given $\forall\epsilon > 0$. An inequality from the Lipschitz property of the driver

$$|f^{s,z}(r) - D_{s,z}f^n(r)| \leq \frac{1}{|z|}|f(\omega^{s,z},r,\Theta_r + z\Theta_r^{s,z}) - f(\omega^{s,z},r,\Theta_r^n + zD_{s,z}\Theta_r^n)|$$

$$+\frac{1}{|z|}|f(\omega,r,\Theta_r) - f(\omega,r,\Theta_r^n)|$$

$$\leq \frac{2K}{|z|}|\Theta_r - \Theta_r^n| + K|\Delta^{s,z}\Theta_r^n|$$

implies

$$\int_0^T \int_{|z|>\epsilon} ||(\Delta^{s,z}\Theta^{n+1})||^2_{\mathcal{K}^2[0,T]} m(dz)ds$$

$$\leq C_{T,K} \int_0^T \int_{|z|>\epsilon} \mathbb{E}\left[\frac{1}{|z|^2}\left(\int_0^T |\Theta_r - \Theta_r^n| dr\right)^2 + \left(\int_0^T |\Delta^{s,z}\Theta_r^n| dr\right)^2\right] m(dz)ds$$

$$\leq \epsilon_n + C_{T,K} \max(T^2, T) \int_0^T \int_{|z|\leq\bar{\epsilon}} ||(\Delta^{s,z}\Theta^n)||^2_{\mathcal{K}^2[0,T]} m(dz)ds$$

where $\epsilon_n \to 0$ as $n \to 0$ due to the convergence of $\Theta^n \to \Theta$. If necessary by re-choosing T small enough so that $\alpha := C_{T,K} \max(T^2, T) < 1$, one gets

$$\int_0^T \int_{|z|>\epsilon} ||(\Delta^{s,z}\Theta^{n+n_0})||^2_{\mathcal{K}^2[0,T]} m(dz)ds$$

$$\leq \frac{\epsilon_{n_0}}{1-\alpha} + \alpha^n \int_0^T \int_{|z|>\epsilon} ||(\Delta^{s,z}\Theta^{n_0})||^2_{\mathcal{K}^2[0,T]} m(dz)ds.$$

Passing to the limit $n, n_0 \to \infty$, one obtains the desired result.

General terminal time T

For general $T > 0$, one can construct a partition $0 = T_0 < T_1 < \cdots < T_N = T$ fine enough so that one can conclude by the previous arguments

$$\lim_{n\to 0} \int_{T_{N-1}}^T \int_{|z|>\epsilon} ||(\Delta^{s,z}\Theta^n)||^2_{\mathcal{K}^2[0,T]} m(dz)ds = 0 ,$$

and hence $D_{s,z}\Theta^n$ converges to $\Theta^{s,z}$ in $\mathbb{L}^{1,2}$ in $[T_{N-1}, T_N]$. Note that (B.13) holds for arbitrary $T > 0$, which implies in particular

$$\lim_{\epsilon\downarrow 0} \int_0^T \int_{|z|<\bar{\epsilon}} \mathbb{E}\left[|Y_{T_{N-1}}^{s,z}|^2\right] m(dz)ds = 0.$$

Therefore, by the same procedures with a new terminal value $Y_{T_{N-1}}^{s,z}$ instead of $D_{s,z}\xi$, one can prove the convergence in $[T_{N-2}, T_{N-1}]$. Now, repeat the same arguments, one can conclude $D_{s,z}Y^n \to Y^{s,z}$ in $\mathbb{L}^{1,2}$ for the whole region.

Finally, thanks to the closability of the Malliavin derivatives in $\mathbb{L}^{1,2}$ (See Theorem 12.6 in [14].), one can conclude $(Y, Z, \psi) \in \mathbb{L}^{1,2}$ and that $(Y^{s,z}, Z^{s,z}, \psi^{s,z})$ is a version of $(D_{s,z}Y, D_{s,z}Z, D_{s,z}\psi)$. $\square$

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References

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