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# A General Framework for the Benchmark pricing in a Fully Collateralized Market

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# A General Framework for the Benchmark pricing in a Fully Collateralized Market *

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## Abstract

Collateralization with daily margining has become a new standard in the post-crisis market. Although there appeared vast literature on a so-called multi-curve framework, a complete picture of a multi-currency setup with cross-currency basis can be rarely found since our initial attempts. This work gives its extension regarding a general framework of interest rates in a fully collateralized market. It gives a new formulation of the currency funding spread which is better suited for the general dependence. In the last half, it develops a discretization of the HJM framework with a fixed *tenor structure*, which makes it implementable as a traditional Market Model.

**Keywords :** swap, collateral, derivatives, Libor, currency, OIS, basis, HJM, CSA

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# 1  Introduction

The landscape of derivative modeling has experienced a radical change since the financial crisis in 2008. On the one hand, evaluation of the counterparty credit risk has become an unavoidable element for all the types of financial contracts. This is a rather natural consequence of significant number of credit events, well exemplified by a collapse of Lehman brothers, which was one of the most prestigious investment banks at that time. On the other hand, clean or benchmark pricing framework has also changed significantly. Here, explosion of various basis spreads, which were mostly negligible before the crisis, and the recognition of collateral cost have played the central role. After some initial attempts to explain the effects of collateralization, such as Fujii, Shimada and Takahashi (2010a) [8], Bianchetti (2010) [1] and Piterbarg (2010) [18], there appeared vast literature to build a dynamic multi-curve framework such as Mercurio (2009) [17], Fujii, Shimada and Takahashi (2011) [10], Crépey, Grbac and Nguyen (2012) [5], Filipović and Trolle (2013) [7], Grbac, Papapantoleon, Schoenmakers and Skovmond (2014) [14], and articles in a recent book Bianchetti and Morini (editors) (2013) [2] just name a few. See Brigo, Morini and Pllavicini (2013) [4], Crépey and Bielecki (2014) [6] and references therein for closely related topics, such as CVA and FVA.

A term structure model in a multi-currency setup with non-zero cross currency basis was firstly developed in [10], where LIBOR-OIS spreads were stochastic but currency funding spreads were left deterministic. The funding spreads were made stochastic firstly in Fujii, Shimada and Takahashi (2010b) [9] and Fujii and Takahashi (2011) [11] in a continuous Heath-Jarrow-Morton (Heath, Jarrow and Morton, 1992 [15]) framework.

This article is a revised and extended version of our previous works and presents a general framework for the benchmark pricing in a fully collateralized market. The formulation of the dynamics of the currency funding spread is changed from the original one [9, 11], which will be more useful to implement in the presence of non-zero correlation among the collateral rates and funding spreads. In particular, in the last half of the paper, we provide a complete picture of a discretized HJM framework for stochastic collateral rates, LIBORs, foreign exchange rates currency funding spreads, and equities with a fixed *tenor structure*, which is readily implementable as a traditional Market Model (Brace, Gatarek and Musiela, 1997 [3]). We do not repeat the details of the curve bootstrapping procedures. In fact the methodology given in the previous papers [8, 11, 13] can be applied without significant change.

# 2  Pricing in a fully collateralized market

Let us start from the review of [8, 13]. We make the following assumptions that has become popular for the benchmark pricing:

1.  Full collateralization (zero threshold) by cash.
2.  The collateral is adjusted continuously with zero minimum transfer amount.

This means that the party who has negative mark-to-market posts the equal amount of cash collateral to the counterparty, and this is done continuously until the contract expires. Actually, daily margin call is becoming popular, and that is a market standard at least among major broker-dealers and central counterparties. This observation allows us to see

the above assumptions a reasonable proxy for the reality. One might consider there is no counterparty risk remains under the above assumption. In fact, however, this is not always the case, if there exists a sudden jump of the underlying asset and/or the collateral values at the time of counterparty default. This is the so-called "gap risk". If this is the case, the remaining risk should be taken into account as a part of credit risk valuation adjustment (CVA). In this article, we assume that no counterparty risk remains and focus on the clean benchmark pricing. For the interested readers, we refer to [4, 6] as well as Fujii and Takahashi (2013a) [12] to handle more generic situations with non-zero credit as well as funding risks.

Let us mention the standing assumption of the paper:

**Assumption**

*Throughout the paper, we assume that the appropriate regularity conditions are satisfied whenever they are required. In particular every local martingale is assumed to be a true martingale.*

We consider a derivative whose payoff at time $T$ is given by $h^{(i)}(T)$ in terms of currency $(i)$. We suppose that currency $(j)$ is used as the collateral for the contract. Let us introduce an important spread process:

$$y^{(j)}(t) = r^{(j)}(t) - c^{(j)}(t) \ , \tag{2.1}$$

where $r^{(j)}$ and $c^{(j)}$ denote the risk-free interest rate and the collateral rate of the currency $(j)$, respectively. A common practice in the market is to set $c^{(j)}$ as the overnight (ON) rate of currency $(j)$. Economically, the spread $y^{(j)}$ can be interpreted as the dividend yield from the collateral account of currency $(j)$ from the view point of a collateral receiver. On the other hand, from the view point of a collateral payer, it can be considered as a collateral funding cost. Of course, the return from risky investments, or the borrowing cost from the external market can be quite different from the risk-free rate. However, if one wants to treat these effects directly, an explicit modeling of the associated risks is required. Here, we use the risk-free rate as net return/cost after hedging these risks, which can be justified in a simple credit model (see the arguments given in Section 9.4.1 of [4].). As we shall see, the final formula does not require any knowledge of the risk-free rate, and hence there is no need of its estimation, which is crucial for the practical implementation.

Now, we explain the derivation of the pricing formula. If we denote the present value of the derivative at time $t$ by $h^{(i)}(t)$ in terms of currency $(i)$, collateral amount of currency $(j)$ posted from the counterparty is given by

$$\frac{h^{(i)}(t)}{f_x^{(i,j)}(t)}, \tag{2.2}$$

where $f_x^{(i,j)}(t)$ is the foreign exchange rate at time $t$ representing the price of the unit amount of currency $(j)$ in terms of currency $(i)$. It should be interpreted that the investor posts the collateral to the counterparty when $h^{(i)}(t)$ is negative.

These considerations lead to the following calculation for the collateralized derivative price,

$$
\begin{aligned}
h^{(i)}(t) \ = \ & \mathbb{E}_t^{\mathbb{Q}^{(i)}} \left[ e^{-\int_t^T r^{(i)}(s)ds} h^{(i)}(T) \right] \\
& + f_x^{(i,j)}(t) \mathbb{E}_t^{\mathbb{Q}^{(j)}} \left[ \int_t^T e^{-\int_t^s r^{(j)}(u)du} y^{(j)}(s) \left( \frac{h^{(i)}(s)}{f_x^{(i,j)}(s)} \right) ds \right]
\end{aligned}
\tag{2.3}
$$

where $\mathbb{E}_t^{\mathbb{Q}^{(i)}}[\cdot]$ is the time $t$ conditional expectation under the risk-neutral measure of currency $(i)$, where the money-market account of currency $(i)$

$$\beta^{(i)} = \exp\left(\int_0^{\cdot} r_s^{(i)} ds\right) \tag{2.4}$$

is used as the numeraire. Here, the second line of (2.3) represents the effective dividend yield from the collateral account, or the cost of posting collateral to the counterparty. One can see the second term changes its sign properly according to the value of the contract. An economic discussion on the *risk-free* rate will be given also in Section 7.

By changing the measure using the Radon-Nikodym density

$$\left.\frac{d\mathbb{Q}^{(i)}}{d\mathbb{Q}^{(j)}}\right|_t = \frac{\beta_t^{(i)} f_x^{(i,j)}(0)}{\beta_t^{(j)} f_x^{(i,j)}(t)} \tag{2.5}$$

one can show

$$h^{(i)}(t) = \mathbb{E}_t^{\mathbb{Q}^{(i)}}\left[e^{-\int_t^T r^{(i)}(s)ds} h^{(i)}(T) + \int_t^T e^{-\int_t^s r^{(i)}(u)du} y^{(j)}(s) h^{(i)}(s) ds\right] . \tag{2.6}$$

Thus, it is easy to check that the process $X = \{X_t; \ t \geq 0\}$

$$X(t) := e^{-\int_0^t r^{(i)}(s)ds} h^{(i)}(t) + \int_0^t e^{-\int_0^s r^{(i)}(u)du} y^{(j)}(s) h^{(i)}(s) ds \tag{2.7}$$

is a $\mathbb{Q}^{(i)}$-martingale under the appropriate integrability conditions. This tells us that the process of the option price can be written as

$$dh^{(i)}(t) = \left(r^{(i)}(t) - y^{(j)}(t)\right) h^{(i)}(t) dt + dM(t) \tag{2.8}$$

with some $\mathbb{Q}^{(i)}$-martingale $M$. It can be expressed as a linear backward stochastic differential equation:

$$h^{(i)}(t) = h^{(i)}(T) - \int_t^T \left(r^{(i)}(s) - y^{(j)}(s)\right) h^{(i)}(s) ds - \int_t^T dM(s) . \tag{2.9}$$

As a result, we have the following theorem:

**Theorem 2.1.** *Suppose that $h^{(i)}(T)$ is a derivative's payoff at time $T$ in terms of currency $(i)$ and that currency $(j)$ is used as the collateral for the contract. Then, the value of the derivative at time $t$, $h^{(i)}(t)$ is given by*

$$\begin{aligned}
h^{(i)}(t) &= \mathbb{E}_t^{\mathbb{Q}^{(i)}}\left[e^{-\int_t^T r^{(i)}(s)ds}\left(e^{\int_t^T y^{(j)}(s)ds}\right) h^{(i)}(T)\right] \tag{2.10}\\
&= \mathbb{E}_t^{\mathbb{Q}^{(i)}}\left[e^{-\int_t^T c^{(i)}(s)ds - \int_t^T y^{(i,j)}(s)ds} h^{(i)}(T)\right]\\
&= D^{(i)}(t,T)\mathbb{E}_t^{\mathbb{T}^{(i)}}\left[e^{-\int_t^T y^{(i,j)}(s)ds} h^{(i)}(T)\right] , \tag{2.11}
\end{aligned}$$

*where*

$$y^{(i,j)}(s) = y^{(i)}(s) - y^{(j)}(s) \tag{2.12}$$

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with $y^{(i)}(s) = r^{(i)}(s) - c^{(i)}(s)$ and $y^{(j)}(s) = r^{(j)}(s) - c^{(j)}(s)$. *Here, we have defined the collateralized zero-coupon bond of currency $(i)$ as*

$$D^{(i)}(t,T) = \mathbb{E}_t^{\mathbb{Q}^{(i)}} \left[ e^{-\int_t^T c^{(i)}(s)ds} \right] . \qquad (2.13)$$

*We have also defined the "collateralized forward measure" $\mathbb{T}^{(i)}$ of currency $(i)$, for which $\mathbb{E}_t^{\mathbb{T}^{(i)}}[\cdot]$ denotes the time $t$ conditional expectation. Here, $D^{(i)}(t,T)$ is used as its numeraire, and the associated Radon-Nikodym density is defined by*

$$\left. \frac{d\mathbb{T}^{(i)}}{d\mathbb{Q}^{(i)}} \right|_t = \frac{D^{(i)}(t,T)}{\beta_c^{(i)}(t)D^{(i)}(0,T)} \qquad (2.14)$$

*where*

$$\beta_c^{(i)}(t) := \exp \left( \int_0^t c_s^{(i)}ds \right) . \qquad (2.15)$$

Notice that the collateralized zero-coupon bond is actually a "dividend yielding" asset due to the cash flow arising from the collateral account. This makes $D^{(i)}(t,T)/\beta^{(i)}(t)$ inappropriate for the definition of the forward measure. In fact it is not a $\mathbb{Q}^{(i)}$-martingale unless one has to compensate the dividend yield $y^{(i)}$. This adjustment has changed $\beta^{(i)}$ to $\beta_c^{(i)}$ in the expression of the Radon-Nikodym density. See Section 6 for more discussions on the arbitrage-free conditions in a collateralized market where every contract has a common dividend yield $y$.

Since $y^{(i)}$ and $y^{(j)}$ denote the collateral funding cost of currency $(i)$ and $(j)$ respectively, $y^{(i,j)}$ represents the difference of the funding cost between the two currencies. As a corollary of the theorem, we have

$$h^{(i)}(t) = \mathbb{E}_t^{\mathbb{Q}^{(i)}} \left[ e^{-\int_t^T c^{(i)}(s)ds} h^{(i)}(T) \right] = D^{(i)}(t,T)\mathbb{E}_t^{\mathbb{T}^{(i)}}[h^{(i)}(T)] \qquad (2.16)$$

when both of the payment and collateralization are done in a common currency $(i)$.

Theorem 2.1 provides the generic pricing formula under the full collateralization including the case of foreign collateral currency. The result will be used repeatedly throughout the article as the foundation of theoretical discussions. We would like to emphasize the importance of the correct understanding for the reason why the collateral rate appears in the discounting. It is the well-known fact that the "effective" discounting rate of an asset with dividend yield $y$ is given by $(r - y)$, or the difference of the risk-free rate and the dividend yield. Under the full collateralization, we can interpret the return (which can be negative) from the collateral account $y = r - c$ as the dividend yield, which leads to $r - (r - c) = c$ as the discounting rate. The arguments can be easily extended to the case of foreign collateral currency. Now, it is clear that the so called "OIS-discounting" can be justified only when the collateral rate "$c$" is given by the overnight rate of the domestic currency. For example, suppose that there is a trade where the collateral rate or "fee" on the posted collateral, specified in the contractual agreement, is given by the LIBOR, then one should use the LIBOR as the discounting rate of the contract, no matter how different it is from the risk-free rate.

# 3 Heath-Jarrow-Morton framework under collateralization

In this section, we discuss the way to give no-arbitrage dynamics to the relevant quantities using Heath-Jarrow-Morton framework. This is a generalized version of Fujii, Shimada and Takahashi (2011) [10] and Fujii and Takahashi (2011) [11] and also provides more rigorous arguments.

## 3.1 Dynamics of the forward collateral rate

We work in the filtered probability space $(\Omega, \mathcal{F}_t, \mathbb{F}(:= (\mathcal{F}_t)_{t \geq 0}), \mathbb{P})$ where $\mathbb{F}$ is an augmented filtration generated by a $d$-dimensional Brownian motion $W$. We assume that the market is complete and that the measure $\mathbb{Q}^{(i)}$ is equivalent to $\mathbb{P}$. We use a simple notation $\mathbb{E}_t[\cdot]$ instead of $\mathbb{E}_t[\cdot | \mathcal{F}_t]$ since $\mathbb{F}$ is the only relevant filtration.

Firstly, we consider the dynamics of the forward collateral rate, which is defined by

$$c^{(i)}(t, T) = -\frac{\partial}{\partial T} \ln D^{(i)}(t, T) \tag{3.1}$$

or equivalently

$$D^{(i)}(t, T) = \exp\left(-\int_t^T c^{(i)}(t, s) ds\right) . \tag{3.2}$$

Since it can be written as

$$c^{(i)}(t, T) = \frac{1}{D^{(i)}(t, T)} \mathbb{E}_t^{\mathbb{Q}^{(i)}}\left[ e^{-\int_t^T c^{(i)}(s) ds} c^{(i)}(T) \right] \tag{3.3}$$

one can see $c^{(i)}(t, t) = c^{(i)}(t)$ by passing the limit $T \downarrow t$.

Suppose that the dynamics of the forward collateral rate under the measure $Q^{(i)}$ is given by

$$dc^{(i)}(t, s) = \alpha^{(i)}(t, s) dt + \sigma_c^{(i)}(t, s) \cdot dW_t^{\mathbb{Q}^{(i)}} , \tag{3.4}$$

where $W^{\mathbb{Q}^{(i)}}$ is a $d$-dimensional $\mathbb{Q}^{(i)}$-Brownian motion, $\alpha^{(i)} : \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ and and $\sigma_c^{(i)} : \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^d$ and $\alpha(\cdot, s), \sigma_c^{(i)}(\cdot, s)$ are both $\mathbb{F}$-adapted processes for every $s \geq 0$. Here, we have used the abbreviation

$$\sigma_c^{(i)}(t, s) \cdot dW_t^{\mathbb{Q}^{(i)}} := \sum_{k=1}^d \left(\sigma_c^{(i)}(t, s)\right)_k dW_k^{\mathbb{Q}^{(i)}}(t) \tag{3.5}$$

to lighten the notation.

**Proposition 3.1.** *Under the setup given in Section 3.1, the no-arbitrage dynamics of the collateral forward rate $\{c^{(i)}(t, s), t \in [0, s]\}$ for every $s \geq 0$ is given by*

$$dc^{(i)}(t, s) = \sigma_c^{(i)}(t, s) \cdot \left(\int_t^s \sigma_c^{(i)}(t, u) du\right) dt + \sigma_c^{(i)}(t, s) \cdot dW_t^{\mathbb{Q}^{(i)}} \tag{3.6}$$

$$c^{(i)}(t, t) = c^{(i)}(t) . \tag{3.7}$$

*Proof.* Simple application of Itô's formula yields

$$
dD^{(i)}(t,T)/D^{(i)}(t,T) = \left\{ c^{(i)}(t) - \int_t^T \alpha^{(i)}(t,s)ds + \frac{1}{2} \left\| \int_t^T \sigma_c^{(i)}(t,s)ds \right\|^2 \right\} dt
$$
$$
- \left( \int_t^T \sigma_c^{(i)}(t,s)ds \right) \cdot dW_t^{\mathbb{Q}^{(i)}} . \tag{3.8}
$$

From the definition of the zero coupon bond $D^{(i)}$ in Theorem 2.1, $\{D^{(i)}(t,T)/\beta_c^{(i)}(t), t \in [0,T]\}$ must be a $\mathbb{Q}^{(i)}$-martingale. This requires $D^{(i)}$'s drift to be equal to $c^{(i)}$, which yields

$$
\alpha^{(i)}(t,s) = \sigma_c^{(i)}(t,s) \cdot \left( \int_t^s \sigma_c^{(i)}(t,u)du \right) . \tag{3.9}
$$

This gives the desired result. $\qquad \square$

## 3.2 Dynamics of the forward LIBOR-OIS spread

We denote the LIBOR of currency $(i)$ fixed at $T_{n-1}$ and maturing at $T_n$ as $L^{(i)}(T_{n-1},T_n)$. Instead of modeling $L^{(i)}$ directly, we consider the dynamics of LIBOR-OIS spread, which is defined by

$$
B^{(i)}(T_{n-1},T_n) = L^{(i)}(T_{n-1},T_n) - \frac{1}{\delta_n^{(i)}} \left( \frac{1}{D^{(i)}(T_{n-1},T_n)} - 1 \right) \tag{3.10}
$$

where $\delta_n^{(i)}$ denotes the day-count fraction of $L^{(i)}$ for the period of $[T_{n-1},T_n]$. It is clear that both $L^{(i)}(T_{n-1},T_n)$ and $B^{(i)}(T_{n-1},T_n)$ are $\mathcal{F}_{T_{n-1}}$ measurable.

**Definition 3.1.** *The forward LIBOR-OIS spread for the period $[T_{n-1},T_n]$ is defined by*

$$
B^{(i)}(t;T_{n-1},T_n) := \mathbb{E}_t^{\mathbb{T}_n^{(i)}} \left[ B^{(i)}(T_{n-1},T_n) \right]
$$
$$
= \mathbb{E}_t^{\mathbb{T}_n^{(i)}} \left[ L^{(i)}(T_{n-1},T_n) \right] - \frac{1}{\delta_n^{(i)}} \left( \frac{D^{(i)}(t,T_{n-1})}{D^{(i)}(t,T_n)} - 1 \right) . \tag{3.11}
$$

Here, the Radon-Nikodym density for the forward measure is given, as before, by

$$
\left. \frac{d\mathbb{T}_n^{(i)}}{d\mathbb{Q}^{(i)}} \right|_t = \frac{D^{(i)}(t,T_n)}{\beta_c^{(i)}(t)D^{(i)}(0,T_n)} . \tag{3.12}
$$

Let us denote the volatility process for $B^{(i)}(\cdot;T_{n-1},T_n)$ by some appropriate adapted process $\sigma_B^{(i)}(\cdot;T_{n-1},T_n) : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$.

**Proposition 3.2.** *Under the setup given in Sections 3.1 and 3.2, the no-arbitrage dynamics of the LIBOR-OIS spread $\{B^{(i)}(t;T_{n-1},T_n), t \in [0,T_{n-1}]\}$ for every $0 \leq T_{n-1} < T_n$ is given by*

$$
\frac{dB^{(i)}(t;T_{n-1},T_n)}{B^{(i)}(t;T_{n-1},T_n)} = \sigma_B^{(i)}(t;T_{n-1},T_n) \cdot \left( \int_t^{T_n} \sigma_c^{(i)}(t,s)ds \right) dt + \sigma_B^{(i)}(t;T_{n-1},T_n) \cdot dW_t^{\mathbb{Q}^{(i)}}
$$

$$
\tag{3.13}
$$

*with*

$$B^{(i)}(0; T_{n-1}, T_n) = \mathbb{E}^{\mathbb{T}_n^{(i)}}\left[L^{(i)}(T_{n-1}, T_n)\right] - \frac{1}{\delta_n^{(i)}}\left(\frac{D^{(i)}(0, T_{n-1})}{D^{(i)}(0, T_n)} - 1\right) . \tag{3.14}$$

*Proof.* By construction, $B^{(i)}(\cdot; T_{n-1}, T_n)$ is a martingale under the forward measure $\mathbb{T}_n^{(i)}$. Thus, in general, one can write its dynamics as

$$dB^{(i)}(t; T_{n-1}, T_n)/B^{(i)}(t; T_{n-1}, T_n) = \sigma_B^{(i)}(t; T_{n-1}, T_n) \cdot dW_t^{\mathbb{T}_n^{(i)}} \tag{3.15}$$

where $W_t^{\mathbb{T}_n^{(i)}}$ is a $d$-dimensional $\mathbb{T}_n^{(i)}$-Brownian motion. By Maruyama-Girsanov theorem, one can check that the relation

$$dW_t^{\mathbb{T}_n^{(i)}} = dW_t^{\mathbb{Q}^{(i)}} + \left(\int_t^{T_n} \sigma_c^{(i)}(t, s)ds\right)dt \tag{3.16}$$

holds, which gives the desired result. $\square$

## 3.3   Dynamics of the forward currency funding spread

The remaining important ingredient for the term structure modeling is the dynamics of the currency funding spread, $y^{(i,j)}$. To understand the direct relationship between the observed cross currency basis and this currency funding spread, see [11]. It has shown that the funding spread $y^{(i,j)}$ is the main cause of the currency basis by analyzing the actual market data.

**Definition 3.2.** *The instantaneous forward funding spread at time t with maturity $T \geq t$ of currency (j) with respect to currency (i) is defined by*

$$y^{(i,j)}(t, T) := -\frac{\partial}{\partial T}\ln Y^{(i,j)}(t, T) \tag{3.17}$$

*where*

$$Y^{(i,j)}(t, T) := \mathbb{E}_t^{\mathbb{T}^{(i)}}\left[e^{-\int_t^T y^{(i,j)}(s)ds}\right] . \tag{3.18}$$

Let us denote the volatility process of the forward funding spread $y^{(i,j)}$ by $\sigma^{(i,j)} : \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^d$ where $\{\sigma_y^{(i,j)}(t, s), t \in [0, s]\}$ is $\mathbb{F}$-adapted for every $s \geq 0$.

**Proposition 3.3.** [1] *Under the setup given in Sections 3.1 and 3.3, the no-arbitrage dynamics of the forward funding spread $\{y^{(i,j)}(t, s), t \in [0, s]\}$ for every $s \geq 0$ is given by*

$$dy^{(i,j)}(t, s) = \left\{\sigma_y^{(i,j)}(t, s) \cdot \int_t^s \sigma_y^{(i,j)}(t, u)du + \sigma_y^{(i,j)}(t, s) \cdot \int_t^s \sigma_c^{(i)}(t, u)du\right.$$
$$\left. + \sigma_c^{(i)}(t, s) \cdot \int_t^s \sigma_y^{(i,j)}(t, u)du\right\}dt + \sigma_y^{(i,j)}(t, s) \cdot dW_t^{\mathbb{Q}^{(i)}} \tag{3.19}$$

*with $y^{(i,j)}(t, t) = y^{(i,j)}(t)$ .*

---

[1]Notice that in Eq (6.31) and hence also in (6.39) of [13], the third component $\sigma_c^{(i)}(t, s) \cdot \int_t^s \sigma_y^{(i,j)}(t, u)du$ of the drift term is missing, which should be corrected as in this proposition.

*Proof.* Define

$$\widetilde{Y}^{(i,j)}(t,T) := \mathbb{E}_t^{\mathbb{Q}^{(i)}}\left[e^{-\int_t^T (c^{(i)}(s)+y^{(i,j)}(s))ds}\right] \tag{3.20}$$

and the corresponding forward rate $\{\widetilde{y}^{(i,j)}(t,T), t \in [0,T]\}$ as

$$\widetilde{y}^{(i,j)}(t,T) = -\frac{\partial}{\partial T}\ln \widetilde{Y}^{(i,j)}(t,T) . \tag{3.21}$$

Notice that

$$\begin{aligned}
\widetilde{y}^{(i,j)}(t,T) &= -\frac{\partial}{\partial T}\ln \widetilde{Y}^{(i,j)}(t,T) \\
&= \frac{1}{\widetilde{Y}^{(i,j)}(t,T)}\mathbb{E}_t^{\mathbb{Q}^{(i)}}\left[e^{-\int_t^T (c^{(i)}(s)+y^{(i,j)}(s))ds}(c^{(i)}(T)+y^{(i,j)}(T))\right]
\end{aligned} \tag{3.22}$$

and hence using the fact that $\widetilde{Y}^{(i,j)}(t,t) = 1$, one obtains by passing to the limit $T \downarrow t$,

$$\widetilde{y}^{(i,j)}(t,t) = c^{(i)}(t) + y^{(i,j)}(t) . \tag{3.23}$$

Using the above result and the fact that

$$\left\{e^{-\int_0^t (c^{(i)}(s)+y^{(i,j)}(s))ds}\widetilde{Y}^{(i,j)}(t,T),\ t \in [0,T]\right\} \tag{3.24}$$

is a $\mathbb{Q}^{(i)}$-martingale for every $T \geq 0$, one obtains, for $t \in [0,s]$ for every $s \geq 0$, that

$$d\widetilde{y}^{(i,j)}(t,s) = \widetilde{\sigma}^{(i,j)}(t,s) \cdot \left(\int_t^s \widetilde{\sigma}^{(i,j)}(t,u)du\right)dt + \widetilde{\sigma}^{(i,j)}(t,s) \cdot dW_t^{\mathbb{Q}^{(i)}} \tag{3.25}$$

$$\widetilde{y}^{(i,j)}(t,t) = c^{(i)}(t) + y^{(i,j)}(t) \tag{3.26}$$

from exactly the same arguments in Proposition 3.1. Here, $\widetilde{\sigma}^{(i,j)} : \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^d$ corresponds to some volatility process, and $\{\widetilde{\sigma}^{(i,j)}(t,s), t \in [0,s]\}$ is $\mathbb{F}$-adapted.

Now, let us decompose $\widetilde{y}^{(i,j)}$ into two parts as

$$\widetilde{y}^{(i,j)}(t,s) = c^{(i)}(t,s) + y^{(i,j)}(t,s) . \tag{3.27}$$

From the definition of the forward collateral rate, one sees the second component satisfies

$$\exp\left(-\int_t^T y^{(i,j)}(t,s)ds\right) = Y^{(i,j)}(T) \tag{3.28}$$

which is consistent with Definition 3.2.

One can write a general dynamics of the funding spread as

$$dy^{(i,j)}(t,s) = \mu^{(i,j)}(t,s)dt + \sigma_y^{(i,j)}(t,s) \cdot dW_t^{\mathbb{Q}^{(i)}} \tag{3.29}$$

using an appropriate adapted drift process $\{\mu^{(i,j)}(t,s), t \in [0,s]\}$. Then, by construction, one must have

$$\widetilde{\sigma}^{(i,j)}(t,s) = \sigma_c^{(i)}(t,s) + \sigma_y^{(i,j)}(t,s) . \tag{3.30}$$

By taking the difference $dy^{(i,j)}(t,s) = d\widetilde{y}^{(i,j)}(t,s) - dc^{(i)}(t,s)$, one obtains

$$\mu^{(i,j)}(t,s) = \sigma_y^{(i,j)}(t,s) \cdot \int_t^s \left[\sigma_y^{(i,j)}(t,u) + \sigma_c^{(i)}(t,u)\right]du + \sigma_c^{(i)}(t,s) \cdot \int_t^s \sigma_y^{(i,j)}(t,u)du \tag{3.31}$$

for every $s \geq 0$. This gives the desired result. $\qquad\square$

## 3.4 Dynamics of the spot foreign exchange rate

Now, the last piece for the term structure modeling is the dynamics of the spot foreign exchange rate $f_x^{(i,j)}$. Since we have the relation

$$r_t^{(i)} - r_t^{(j)} = c_t^{(i)} - c_t^{(j)} + y_t^{(i,j)} \tag{3.32}$$

it is easy to see that the relevant dynamics is given by

$$df_x^{(i,j)}(t)/f_x^{(i,j)}(t) = \left( c_t^{(i)} - c_t^{(j)} + y_t^{(i,j)} \right) dt + \sigma_X^{(i,j)}(t) \cdot dW_t^{\mathbb{Q}^{(i)}} \,. \tag{3.33}$$

where $\sigma_X^{(i,j)} : \Omega \times \mathbb{R} \to \mathbb{R}^d$ is an appropriate adapted process for the spot FX volatility.

The Radon-Nikodym density between two money-market measures with different currencies are given by

$$\left. \frac{d\mathbb{Q}^{(j)}}{d\mathbb{Q}^{(i)}} \right|_t = \frac{\beta^{(j)}(t)f_x^{(i,j)}(t)}{\beta^{(i)}(t)f_x^{(i,j)}(0)} = \frac{\beta_c^{(j)}(t)f_x^{(i,j)}(t)}{\beta_c^{(i)}(t)\beta_y^{(i,j)}(t)f_x^{(i,j)}(0)} \tag{3.34}$$

where $\beta_y^{(i,j)}(t) = e^{\int_0^t y_s^{(i,j)} ds}$. Since we have the relation

$$dW_t^{\mathbb{Q}^{(j)}} = dW_t^{\mathbb{Q}^{(i)}} - \sigma_X^{(i,j)}(t)dt \,, \tag{3.35}$$

it is easy to change the currency measure. For example, one can check that the dynamics of the forward collateral rate of currency $(j)$ becomes

$$dc^{(j)}(t,s) = \sigma_c^{(j)}(t,s) \cdot \left[ \left( \int_t^s \sigma_c^{(j)}(t,u)du \right) - \sigma_X^{(i,j)}(t) \right] dt + \sigma_c^{(j)}(t,s) \cdot dW_t^{\mathbb{Q}^{(i)}} \tag{3.36}$$

under the money-market measure of currency $(i)$.

## 3.5 Summary of the dynamics

From Sections 3.1 to 3.4, we have derived the dynamics of all the relevant processes in the HJM framework. For the convenience of readers, let us summarize the resultant system of stochastic differential equations (SDEs). Here, we set currency $(i)$ as the base currency.

**Rates for the base currency**

$$dc^{(i)}(t,s) = \sigma_c^{(i)}(t,s) \cdot \left( \int_t^s \sigma_c^{(i)}(t,u)du \right) dt + \sigma_c^{(i)}(t,s) \cdot dW_t^{\mathbb{Q}^{(i)}} \tag{3.37}$$

$$\frac{dB^{(i)}(t;T_{n-1},T_n)}{B^{(i)}(t;T_{n-1},T_n)} = \sigma_B^{(i)}(t;T_{n-1},T_n) \cdot \left( \int_t^{T_n} \sigma_c^{(i)}(t,s)ds \right) dt + \sigma_B^{(i)}(t;T_{n-1},T_n) \cdot dW_t^{\mathbb{Q}^{(i)}} \tag{3.38}$$

**Funding spreads with respect to the base currency**

$$dy^{(i,j)}(t,s) = \left\{ \sigma_y^{(i,j)}(t,s) \cdot \left( \int_t^s [\sigma_y^{(i,j)}(t,u) + \sigma_c^{(i)}(t,u)]du \right) \right.$$
$$\left. + \sigma_c^{(i)}(t,s) \cdot \int_t^s \sigma_y^{(i,j)}(t,u)du \right\} dt + \sigma_y^{(i,j)}(t,s) \cdot dW_t^{\mathbb{Q}^{(i)}} \tag{3.39}$$

**Foreign exchange rate**

$$df_x^{(i,j)}(t)/f_x^{(i,j)}(t) = \left(c_t^{(i)} - c_t^{(j)} + y_t^{(i,j)}\right)dt + \sigma_X^{(i,j)}(t) \cdot dW_t^{\mathbb{Q}^{(i)}} \tag{3.40}$$

**Rates for a foreign currency**

$$dc^{(j)}(t,s) = \sigma_c^{(j)}(t,s) \cdot \left[\left(\int_t^s \sigma_c^{(j)}(t,u)du\right) - \sigma_X^{(i,j)}(t)\right]dt + \sigma_c^{(j)}(t,s) \cdot dW_t^{\mathbb{Q}^{(i)}} \tag{3.41}$$

$$\frac{dB^{(j)}(t;T_{n-1},T_n)}{B^{(j)}(t;T_{n-1},T_n)} = \sigma_B^{(j)}(t;T_{n-1},T_n) \cdot \left[\left(\int_t^{T_n} \sigma_c^{(i)}(t,s)ds\right) - \sigma_X^{(i,j)}(t)\right]dt$$
$$+ \sigma_B^{(i)}(t;T_{n-1},T_n) \cdot dW_t^{\mathbb{Q}^{(i)}} \tag{3.42}$$

**Funding spreads with respect to a foreign currency**

$$dy^{(j,k)}(t,s) = \left\{\sigma_y^{(j,k)}(t,s) \cdot \left(\int_t^s [\sigma_y^{(j,k)}(t,u) + \sigma_c^{(j)}(t,u)]du - \sigma_X^{(i,j)}(t)\right)\right.$$
$$\left. + \sigma_c^{(j)}(t,s) \cdot \int_t^s \sigma_y^{(j,k)}(t,u)du\right\}dt + \sigma_y^{(j,k)}(t,s) \cdot dW_t^{\mathbb{Q}^{(i)}} \tag{3.43}$$

# 4   Some remarks on the formulation in [9, 11]

In our previous works, we have defined the instantaneous forward rate of the funding spread as

$$e^{\int_t^T y^{(i,j)}(t,s)ds} = \mathbb{E}_t^{\mathbb{Q}^{(i)}}\left[e^{-\int_t^T y^{(i,j)}(s)ds}\right] \tag{4.1}$$

or equivalently,

$$y^{(i,j)}(t,T) = -\frac{\partial}{\partial T}\ln \mathbb{E}_t^{\mathbb{Q}^{(i)}}\left[e^{-\int_t^T y^{(i,j)}(s)ds}\right]. \tag{4.2}$$

Notice the difference in the associated measure in Definition 3.2. In the above definition, the dynamics of the funding spread is actually simpler:

$$dy^{(i,j)}(t,s) = \sigma_y^{(i,j)}(t,s) \cdot \left(\int_t^s \sigma_y^{(i,j)}(t,u)du\right)dt + \sigma_y^{(i,j)}(t,s) \cdot dW_t^{\mathbb{Q}^{(i)}}. \tag{4.3}$$

However, the above formulation makes the calibration procedures rather complicated in the presence of non-zero correlation between the collateral rate and the funding spread. Suppose that there is a currency-$(j)$ collateralized zero coupon bond denominated by currency $(i)$. This is an important building block for the pricing of the collateralized contracts with a foreign currency. The present value of this bond is

$$\mathbb{E}_t^{\mathbb{Q}^{(i)}}\left[e^{-\int_t^T (c^{(i)}(s)+y^{(i,j)}(s))ds}\right]. \tag{4.4}$$

Due to the non-zero correlation, the above quantity depends on the dynamics of both of the forward rates, and is impossible to separate the two effects. On the other hand, in the formulation proposed in Section 3.3, the above quantity is given by $D^{(i)}(t,T)Y^{(i,j)}(t,T)$ without any independence assumption (See Eqs.(2.11) and (3.18).). In addition, if the market quotes are available for this type of products, then one can easily get the clean information for $Y^{(i,j)}$ which directly gives the initial condition for the funding spread by differentiation.

# 5 Collateralized FX

## 5.1 Collateralized FX forward

Let first consider a collateralized FX forward contract between currency $(i)$ and $(j)$, in which a unit amount of currency $(j)$ is going to be exchanged by $K$ units amount of currency $(i)$ at the maturity $T$. The amount of $K$ is fixed at $t$, the trade inception time. Assume the contract is fully collateralized by currency $(k)$. Here, the amount of $K$ that makes this exchange have the present value 0 at time $t$ is called the forward FX rate.

The break-even condition for the amount of $K$ is given by

$$K\mathbb{E}_t^{\mathbb{Q}^{(i)}}\left[e^{-\int_t^T (c_s^{(i)}+y_s^{(i,k)})ds}\right] = f_x^{(i,j)}(t)\mathbb{E}_t^{\mathbb{Q}^{(j)}}\left[e^{-\int_t^T (c_s^{(j)}+y_s^{(j,k)})ds}\right] . \tag{5.1}$$

Here, the left-hand side represents the value of $K$ units amount of currency $(i)$ paid at $T$ with collateralization in currency $(k)$. In the right-hand side the value of a unit amount of currency $(j)$ is represented in terms of currency $(i)$ by multiplying the spot FX rate. By solving the equation for $K$, we obtain the forward FX as (See definition (3.20).)

$$f_x^{(i,j)}(t;T,(k)) = f_x^{(i,j)}(t)\frac{\widetilde{Y}^{(j,k)}(t,T)}{\widetilde{Y}^{(i,k)}(t,T)} \tag{5.2}$$

where the last argument $(k)$ denotes the currency used as the collateral. In general, the currency triangle relation among forward FXs only holds with the common collateral currency:

$$f_x^{(i,j)}(t,T;(k)) \times f_x^{(j,l)}(t,T;(k)) = f_x^{(i,l)}(t,T;(k)) . \tag{5.3}$$

Suppose the same contract is made with a collateral currency either $(i)$ or $(j)$, which seems more natural in the market. In this case, we have one-to-one relation between the forward FX value and the forward funding spread. For example, if currency $(i)$ is used as the collateral, we have

$$f_x^{(i,j)}(t;T,(i)) = f_x^{(i,j)}(t)\frac{D^{(j)}(t,T)}{D^{(i)}(t,T)}Y^{(j,i)}(t,T). \tag{5.4}$$

Therefore, if we can observe the forward FX with various maturities in the market, we can bootstrap the forward curve of $\{y^{(j,i)}(t,\cdot)\}$ since we already know $D^{(i)}$ and $D^{(j)}$ from each OIS market. When collateralization is done by currency $(j)$, we can extract $\{y^{(i,j)}(t,\cdot)\}$ by the same arguments.

When the collateral currency is $(i)$, one can easily show that

$$f_x^{(i,j)}(t)\mathbb{E}_t^{\mathbb{Q}^{(j)}}\left[e^{-\int_t^T (c_s^{(j)}+y_s^{(j,i)})ds}\right] = \mathbb{E}_t^{\mathbb{Q}^{(i)}}\left[e^{-\int_t^T c_s^{(i)}ds}f_x^{(i,j)}(T)\right] \tag{5.5}$$

by using the Radon-Nikodym density (3.34). Thus, we see

$$f_x^{(i,j)}(t;T,(i)) = \mathbb{E}_t^{\mathbb{T}^{(i)}}\left[f_x^{(i,j)}(T)\right] \tag{5.6}$$

from Eq. (5.4). Hence, the forward FX collateralized by its domestic currency $(i)$ is a martingale under the associated forward measure $\mathbb{T}^{(i)}$. In more general situation, we have the expression

$$f_x^{(i,j)}(t;T,(k)) = \mathbb{E}_t^{\mathbb{T}^{(i,k)}}\left[f_x^{(i,j)}(T)\right] \tag{5.7}$$

where the new forward measure is defined by

$$
\left. \frac{d\mathbb{T}^{(i,k)}}{d\mathbb{Q}^{(i)}} \right|_t = \frac{\widetilde{Y}^{(i,k)}(t,T)}{\beta_c^{(i)}(t)\beta_y^{(i,k)}(t)\widetilde{Y}^{(i,k)}(0,T)} \ , \tag{5.8}
$$

and $\{f_x^{(i,j)}(t;T,(k)), t \in [0,T]\}$ is a $\mathbb{T}^{(i,k)}$-martingale. One can see, $\widetilde{Y}^{(i,k)}(t,T)$, the price of a zero-coupon bond of currency $(i)$ collateralized by currency $(k)$ is associated as a numeraire for the forward measure $\mathbb{T}^{(i,k)}$.

## 5.2   FX option

Just for completeness, we give a brief remark on collateralized FX European options. Let us consider a $T$-maturing European call option on $f_x^{(i,j)}$ with strike $K$ collateralized by a currency $(k)$. The present value of the option is

$$
PV_t = \mathbb{E}_t^{\mathbb{Q}^{(i)}}\left[ e^{-\int_t^T (c_s^{(i)} + y_s^{(i,k)})ds} \left( f_x^{(i,j)}(T) - K \right)^+ \right] \ , \tag{5.9}
$$

where $X^+$ denotes $\max(X,0)$. Since we can rewrite it as

$$
PV_t = D^{(i)}(t,T)Y^{(i,k)}(t,T)\mathbb{E}_t^{\mathbb{T}^{(i,k)}}\left[ \left( f_x^{(i,j)}(T;T,(k)) - K \right)^+ \right] \tag{5.10}
$$

it is enough to know the dynamics of $f_x(t;T,(k))$ under the measure $\mathbb{T}^{(i,k)}$. By applying Itô formula to (5.2), we obtain

$$
df_x^{(i,j)}(t;T,(k))/f_x^{(i,j)}(t;T,(k)) = \left\{ \sigma_X^{(i,j)}(t) + \Gamma^{(i,k)}(t,T) - \Gamma^{(j,k)}(t,T) \right\} \cdot dW_t^{\mathbb{T}^{(i,k)}} \tag{5.11}
$$

where $W^{\mathbb{T}^{(i,k)}}$ is a $d$-dimensional $\mathbb{T}^{(i,k)}$-Brownian motion. Here, for $m = \{i,j\}$, we have defined

$$
\Gamma^{(m,k)}(t,T) = \int_t^T \left( \sigma_c^{(m)}(t,s) + \sigma_y^{(m,k)}(t,s) \right)ds \ . \tag{5.12}
$$

Thus, when both of the collateral and funding spread volatilities are deterministic (hence follow Gaussian HJM), one can use Black's formula. Although there is no closed-form solution except this special case, asymptotic expansion technique can be applied to derive an analytical approximate solution. See, for example, Takahashi (2015) [22] and references therein.

# 6   Implementation with a fixed tenor structure

Due to the infinite dimensionality, a direct numerical implementation of the proposed HJM framework is impossible. An obvious solution is to reduce its dimension by discretization following the idea of BGM (or LIBOR market) model [3]. One can see, for example, Mercurio [17] (2009) as an early attempt. In recent years, many researchers have followed the same route or adopted an Affine model for tractability. However, a complete picture including multiple currencies and currency funding spreads, which are necessary to explain

the cross currency basis, have only been found in continuous HJM framework given in our previous works [10, 9, 11].

In the remainder of the paper, we shall give a directly implementable discretization for the complete HJM framework including multiple currencies and the associated funding spreads. Due to the limited capacity for numerical simulation, using a daily span for the overnight rates looks difficult. When using a more sparse time partition, considering a daily compounding nature of the OIS, it seems more useful to define a continuously compounded forward rate with a finite accrual period. It also looks suitable for the modeling of the funding spread since it is the dividend yield being paid everyday to the collateral holder. In addition, it possesses a much stronger similarity to the HJM framework than a traditional BGM-like discretization. Thanks to this similarity, one can use a continuous HJM framework, which allows more concise treatments, for the analytical study of the implemented model with a fixed tenor structure.

## 6.1   Setup

We work in a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfying the usual conditions, where $\mathbb{F}$ is assumed to be an augmented filtration generated by the $d$-dimensional Brownian motion $W$. It is assumed to support every stochastic variable to appear in the model. We first introduce a partition of the time interval called a *tenor structure* as

$$0 = T_0 < T_1 < T_2 < \cdots < T_{N^*} \tag{6.1}$$

and $\delta_i := T_{i+1} - T_i$, where $T_{N^*}$ is some fixed time horizon. We also define

$$q(t) := \min\left\{n; T_n \geq t\right\} \tag{6.2}$$

and hence we always have $T_{q(t)-1} < t \leq T_{q(t)}$. We assume, for each currency, that the traditional risk-free zero coupon bonds and the collateralized zero coupon bonds are continuously tradable at any time $t \in [0, T_{N^*}]$. Their price processes are assumed to be strictly positive and denoted by

$$P^{(i)}(t, T_{q(t)}), \cdots, P^{(i)}(t, T_{N^*}) \tag{6.3}$$

and

$$D^{(i)}(t, T_{q(t)}), \cdots, D^{(i)}(t, T_{N^*}) \tag{6.4}$$

for each currency.

Following the idea of Schlögl [20], for every currency $(i)$, we assume that the prices of the shortest bonds $P^{(i)}(t, T_{q(t)})$ $D^{(i)}(t, T_{q(t)})$ are $\mathcal{F}_{T_{q(t)-1}}$-measurable and absolutely continuous with respect to the Lebesgue measure. Then, we can define the associated short rate processes $(r^{(i)}, c^{(i)})$ in such a way that

$$\exp\left(-\int_t^{T_{q(t)}} r^{(i)}(s)ds\right) = P^{(i)}(t, T_{q(t)}) \tag{6.5}$$

$$\exp\left(-\int_t^{T_{q(t)}} c^{(i)}(s)ds\right) = D^{(i)}(t, T_{q(t)}) \tag{6.6}$$

for every $t \in [0, T_{N^*}]$, and $(r^{(i)}, c^{(i)})$ are $\mathcal{F}_{T_{q(t)-1}}$-measurable processes. We denote

$$y^{(i)}(t) := r^{(i)}(t) - c^{(i)}(t), \quad t \in [0, T_{N^*}] \tag{6.7}$$

which corresponds to the dividend yield associated to the currency-$(i)$ collateralized products.

Let us first define a discrete bank account for a given currency $(i)$:

$$B^{(i)}(t) := \exp\left( \sum_{m=0}^{q(t)-1} \delta_m f_m^{(i)}(T_m) \right) P^{(i)}(t, T_{q(t)}) . \tag{6.8}$$

Here,

$$f_m^{(i)}(t) := -\frac{1}{\delta_m} \ln\left( \frac{P^{(i)}(t, T_{m+1})}{P^{(i)}(t, T_m)} \right), t \in [0, T_m] \tag{6.9}$$

is a forward rate at time $t$ for the period $[T_m, T_{m+1}]$ with a continuous compounding [2]. It is easy to show that the market is arbitrage free if there exists a measure $\mathbb{Q}_B^{(i)}$ equivalent to $\mathbb{P}$ such that every (non-dividend paying) asset price denominated in currency $(i)$ discounted by $B^{(i)}$ is a $\mathbb{Q}_B^{(i)}$-martingale.

**Remark**

Suppose we create the continuous money market account $\beta^{(i)}(t) = \exp\left( \int_0^t r^{(i)}(s)ds \right)$ by using the rate process defined in (6.5). Then the new measure $\mathbb{Q}^{(i)}$ associated with this $\beta^{(i)}$ as the numeraire, is actually *equal* to $\mathbb{Q}_B^{(i)}$ since both numeraires have finite variation.

Although the above numeraire asset is conceptually useful, it remains completely implicit in a collateralized market. In the presence of collateralization, the fundamental assets are dividend yielding assets. Let us define a discrete collateral account by

$$C^{(i)}(t) := \exp\left( \sum_{m=0}^{q(t)-1} \delta_m c_m^{(i)}(T_m) \right) D^{(i)}(t, T_{q(t)}) , \tag{6.10}$$

where

$$c_m^{(i)}(t) := -\frac{1}{\delta_m} \ln\left( \frac{D^{(i)}(t, T_{m+1})}{D^{(i)}(t, T_m)} \right), t \in [0, T_m] \tag{6.11}$$

is the corresponding collateral forward rate.

Now, let us choose a certain base currency and omit its specification $(i)$. Consider the market that contains $m$ non-dividend paying assets and $n$ $y$-yielding collateralized assets. Let us denote their price processes by $\{S_i(t), t \in [0, T_{N^*}]\}$ and $\{U_j(t), t \in [0, T_{N^*}]\}$ for $i \in \{1, \cdots, m\}$ and $j \in \{1, \cdots, n\}$.

**Proposition 6.1.** *The market is arbitrage free if* $\{S_i(t)/B(t), t \in [0, T_{N^*}]\}_{1 \le i \le m}$ *and* $\{U_j(t)/C(t), t \in [0, T_{N^*}]\}_{1 \le j \le n}$ *are all* $\mathbb{Q}_B$*-martingales.*

---

[2]If we adopt a simple rate convention, we obtain a traditional BGM model.

*Proof.* Let us denote the self-financing trading strategy by $\{(\pi_i(t), \eta_i(t)), t \in [0, T_{N^*}]\}$, $i \in \{1, \cdots, m\}$, $j \in \{1, \cdots, n\}$. The associated wealth process $\{V(t), t \in [0, T_{N^*}]\}$ is given by

$$
\begin{aligned}
dV(t) &= \pi(t) \cdot dS(t) + \eta(t) \cdot dU(t) + y(t)(\eta(t) \cdot U(t))dt \\
&+ \left[ V_t - \pi(t) \cdot S(t) - \eta(t) \cdot U(t) \right] \frac{dB(t)}{B(t)} .
\end{aligned}
\tag{6.12}
$$

Note that the investments into the collateralized zero coupon bonds require continuous reinvestments of their dividends, which eventually leads to the same expression.

For the absence of arbitrage, it is enough to show that $\{V(t)/B(t), t \in [0, T_{N^*}]\}$ is a $\mathbb{Q}_B$-martingale. It is easy to see

$$
\begin{aligned}
d\left( \frac{V(t)}{B(t)} \right) &= \frac{dV(t)}{B(t)} - \frac{V(t)}{B(t)^2} dB(t) \\
&= \pi_t \cdot \left( \frac{dS(t)}{B(t)} - \frac{S(t)}{B(t)^2} dB(t) \right) + \frac{C(t)}{B(t)} \eta(t) \cdot \left( \frac{dU(t)}{C(t)} - \frac{U(t)}{C(t)^2} dC(t) \right) \\
&+ \frac{\eta(t) \cdot U(t)}{B(t)} \left( \frac{dC(t)}{C(t)} + y(t)dt - \frac{dB(t)}{B(t)} \right) .
\end{aligned}
\tag{6.13}
$$

Since the last term vanishes, one obtains

$$
d\left( \frac{V(t)}{B(t)} \right) = \pi(t) \cdot d\left( \frac{S(t)}{B(t)} \right) + \frac{C(t)}{B(t)} \eta(t) \cdot d\left( \frac{U(t)}{C(t)} \right)
\tag{6.14}
$$

which proves the desired result. $\qquad\square$

The above result implies that one can guarantee the absence of arbitrage in a market collateralized by the domestic currency $(i)$ by imposing that every $C^{(i)}$-discounted price process is a $\mathbb{Q}_B^{(i)}$-martingale. The arguments is easily extendable to the situation where there exist collateralized products by a foreign currency $(j)$. In this market, one can additionally trade the set of domestic zero-coupon bonds with the same tenor structure but collateralized by the foreign currency $(j)$:

$$
\widetilde{Y}^{(i,j)}(t, T_{q(t)}), \cdots, \widetilde{Y}^{(i,j)}(t, T_{N^*}) ,
\tag{6.15}
$$

which are assumed to have a common dividend yield $y^{(j)} = r^{(j)} - c^{(j)}$. The shortest bond satisfies

$$
\begin{aligned}
\widetilde{Y}^{(i,j)}(t, T_{q(t)}) &= \exp\left( -\int_t^{T_{q(t)}} (r^{(i)}(s) - y^{(j)}(s))ds \right) \\
&= \exp\left( -\int_t^{T_{q(t)}} (c^{(i)}(s) + y^{(i,j)}(s))ds \right)
\end{aligned}
\tag{6.16}
$$

where $y^{(i,j)} = y^{(i)} - y^{(j)}$ and are $\mathcal{F}_{T_{q(t)-1}}$-measurable. Let us define a new collateralized account by the currency $(j)$ as

$$
C^{(i,j)}(t) = \exp\left( \sum_{m=0}^{q(t)-1} \delta_m (c_m^{(i)}(T_m) + y_m^{(i,j)}(T_m)) \right) \widetilde{Y}^{(i,j)}(t, T_{q(t)}) .
\tag{6.17}
$$

with

$$y_m^{(i,j)}(t) = -\frac{1}{\delta_m} \ln\left(\frac{Y^{(i,j)}(t, T_{m+1})}{Y^{(i,j)}(t, T_m)}\right) \tag{6.18}$$

$$Y^{(i,j)}(t, T_m) := \widetilde{Y}^{(i,j)}(t, T_m)/D^{(i)}(t, T_m) . \tag{6.19}$$

It is clear that $Y^{(i,j)}(t, T_{q(t)})$ is $\mathcal{F}_{T_{q(t)-1}}$-measurable. By exactly the same arguments in Proposition 6.1, the absence of arbitrage is guaranteed by requiring that every price process of $(j)$-collateralized $(i)$-denominated asset divided by $C^{(i,j)}$ is a $\mathbb{Q}_B^{(i)}$-martingale.

## 6.2 Dynamics of discretized collateral rates

We want to derive the dynamics of $\{c_m^{(i)}\}_{m \in \{0, \cdots, N^*-1\}}$ under $\mathbb{Q}_B^{(i)}$. Its dynamics in our Brownian setup can generally be represented by

$$dc_m^{(i)}(t) = \alpha_m^{(i)}(t)dt + \sigma_m^{(i)}(t) \cdot dW_t^{\mathbb{Q}_B^{(i)}} \tag{6.20}$$

for $m \in \{0, \cdots, N^*-1\}$. Here, $\alpha_m^{(i)} : \Omega \times [0, T_m] \to \mathbb{R}$ and $\sigma_m^{(i)} : \Omega \times [0, T_m] \to \mathbb{R}$ are $\mathbb{F}$-adapted processes. We define $c_m^{(i)}(t) = c_m^{(i)}(T_m)$ for $t \geq T_m$.

**Proposition 6.2.** *The no-arbitrage dynamics of the collateral forward rate is given by*

$$dc_n^{(i)}(t) = \left\{ \sigma_n^{(i)}(t) \cdot \left( \sum_{m=q(t)}^{n-1} \delta_m \sigma_m^{(i)}(t) \right) + \frac{1}{2}\delta_n |\sigma_n^{(i)}(t)|^2 \right\} dt + \sigma_n^{(i)}(t) \cdot dW_t^{\mathbb{Q}_B^{(i)}} \tag{6.21}$$

*for $t \in [0, T_n]$ with every $n \in \{0, \cdots, N^*-1\}$.*

*Proof.* For the absence of arbitrage,

$$\frac{D^{(i)}(t, T_n)}{C^{(i)}(t)}, \quad t \in [0, T_n] \tag{6.22}$$

must be a $\mathbb{Q}_B^{(i)}$-martingale for every $n \in \{0, \cdots, N^*\}$. It is not difficult to confirm that the ratio (6.22) is continuous at every $T_i, i \in \{0, \cdots, n\}$ by using the expression

$$D^{(i)}(t, T_n) = D^{(i)}(t, T_{q(t)}) \prod_{m=q(t)}^{n-1} \frac{D^{(i)}(t, T_{m+1})}{D^{(i)}(t, T_m)}$$

$$= D^{(i)}(t, T_{q(t)}) \exp\left(- \sum_{m=q(t)}^{n-1} \delta_m c_m^{(i)}(t)\right), \tag{6.23}$$

and (6.10). Thus, it is enough to require

$$X_n^{(i)}(t) := \exp\left(- \sum_{m=q(t)}^{n-1} \delta_m c_m^{(i)}(t)\right) \tag{6.24}$$

to be a $\mathbb{Q}_B^{(i)}$-martingale within each interval $(T_{q(t)-1}, T_{q(t)})$ for every $n$.

An application of Itô-formula in a given interval yields that

$$
\begin{aligned}
dX_n^{(i)}(t) &= X_n^{(i)}(t) \left\{ -\sum_{m=q(t)}^{n-1} \delta_m dc_m^{(i)}(t) + \frac{1}{2} \sum_{m,m'=q(t)}^{n-1} \delta_m \delta_{m'} d\langle c_m^{(i)}, c_{m'}^{(i)} \rangle_t \right\} \\
&= X_n^{(i)}(t) \left\{ -\sum_{m=q(t)}^{n-1} \delta_m \alpha_m^{(i)}(t) + \frac{1}{2} \Big| \sum_{m=q(t)}^{n-1} \delta_m \sigma_m^{(i)}(t) \Big|^2 \right\} dt \\
&\quad - X_n^{(i)}(t) \sum_{m=q(t)}^{n-1} \delta_m \sigma_m^{(i)}(t) \cdot dW_t^{\mathbb{Q}_B^{(i)}} .
\end{aligned}
\tag{6.25}
$$

This implies

$$
\sum_{m=q(t)}^{n-1} \delta_m \alpha_m^{(i)}(t) = \frac{1}{2} \Big| \sum_{m=q(t)}^{n-1} \delta_m \sigma_m^{(i)}(t) \Big|^2 .
\tag{6.26}
$$

Since this relation must hold for every $n \in \{0, \cdots, N^* - 1\}$, one obtains, by taking the difference, that

$$
\begin{aligned}
\delta_n \alpha_n^{(i)}(t) &= \frac{1}{2} \Big| \sum_{m=q(t)}^{n} \delta_m \sigma_m^{(i)}(t) \Big|^2 - \frac{1}{2} \Big| \sum_{m=q(t)}^{n-1} \delta_m \sigma_m^{(i)}(t) \Big|^2 \\
&= \frac{1}{2} \delta_n^2 |\sigma_n^{(i)}(t)|^2 + \delta_n \sigma_n^{(i)}(t) \cdot \Big( \sum_{m=q(t)}^{n-1} \delta_m \sigma_m^{(i)}(t) \Big)
\end{aligned}
\tag{6.27}
$$

and hence

$$
\alpha_n^{(i)}(t) = \frac{1}{2} \delta_n |\sigma_n^{(i)}(t)|^2 + \sigma_n^{(i)}(t) \cdot \Big( \sum_{m=q(t)}^{n-1} \delta_m \sigma_m^{(i)}(t) \Big) .
\tag{6.28}
$$

Once we fix $\alpha_n^{(i)}$ as above, one can recursively verify that the relation (6.26) is satisfied for every $n$. This proves the proposition. $\qquad\square$

## 6.3 Dynamics of collateralized LIBOR-OIS spreads

The LIBOR-OIS spread $B^{(i)}(T_{n-1}, T_n)$ itself is merely an index and not a tradable asset. However, a collateralized forward contract is of course tradable and hence

$$
\mathbb{E}_t^{\mathbb{Q}_B^{(i)}} \left[ e^{-\int_t^{T_n} c^{(i)}(s) ds} B^{(i)}(T_{n-1}, T_n) \right] = D^{(i)}(t, T_n) B^{(i)}(t; T_{n-1}, T_n), \; t \in [0, T_{n-1}]
\tag{6.29}
$$

is a price process of a "dividend $y^{(i)}$" yielding asset.

Let us write the general dynamics as

$$
dB^{(i)}(t; T_{n-1}, T_n) := B^{(i)}(t; T_{n-1}, T_n) \left( b_n^{(i)}(t) dt + \sigma_{B,n}^{(i)}(t) \cdot dW_t^{\mathbb{Q}_B^{(i)}} \right) ,
\tag{6.30}
$$

with some appropriate $\mathbb{F}$-adapted processes $b_n^{(i)} : \Omega \times [0, T_{n-1}] \to \mathbb{R}$ and $\sigma_{B,n}^{(i)} : \Omega \times [0, T_{n-1}] \to \mathbb{R}^d$.

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**Proposition 6.3.** *The no-arbitrage dynamics of the LIBOR-OIS forward spread is given by*

$$dB^{(i)}(t; T_{n-1}, T_n) = B^{(i)}(t; T_{n-1}, T_n) \left( \sigma_{B,n}^{(i)}(t) \cdot \Big( \sum_{m=q(t)}^{n-1} \delta_m \sigma_m^{(i)}(t) \Big) dt + \sigma_{B,n}^{(i)}(t) \cdot dW_t^{\mathbb{Q}_B^{(i)}} \right)$$
(6.31)

*for $t \in [0, T_{n-1}]$ with every $n \in \{0, \cdots, N^*\}$ .*

*Proof.* For the absence of arbitrage,

$$\frac{D^{(i)}(t, T_n) B^{(i)}(t; T_{n-1}, T_n)}{C^{(i)}(t)}, \quad t \in [0, T_{n-1}]$$
(6.32)

must be a $\mathbb{Q}_B^{(i)}$-arbitrage. This is equivalent to require the process

$$X_{B,n}^{(i)}(t) := \exp\Big( - \sum_{m=q(t)}^{n-1} \delta_m c_m^{(i)}(t) \Big) B^{(i)}(t; T_{n-1}, T_n)$$
(6.33)

to be a $\mathbb{Q}_B^{(i)}$-martingale within an each interval $(T_{q(t)-1}, T_{q(t)})$, since (6.32) is continuous at every node $T_i, i \in \{0, \cdots, n-1\}$.

An application of Itô-formula in a given interval and the result of Proposition 6.2 yield

$$\begin{aligned}
dX_{B,n}^{(i)}(t) &= X_{B,n}^{(i)}(t) \left( b_n^{(i)}(t) - \sigma_{B,n}^{(i)}(t) \cdot \Big( \sum_{m=q(t)}^{n-1} \delta_m \sigma_m^{(i)}(t) \Big) \right) dt \\
&\quad + X_{B,n}^{(i)}(t) \Big( \sigma_{B,n}^{(i)}(t) - \sum_{m=q(t)}^{n-1} \delta_m \sigma_m^{(i)}(t) \Big) \cdot dW_t^{\mathbb{Q}_B^{(i)}} .
\end{aligned}$$
(6.34)

This implies

$$b_n^{(i)}(t) = \sigma_{B,n}^{(i)}(t) \cdot \Big( \sum_{m=q(t)}^{n-1} \delta_m \sigma_m^{(i)}(t) \Big),$$
(6.35)

which proves the proposition [3]. $\qquad\square$

## 6.4   Dynamics of currency funding spreads

As in the previous section, the funding spread itself is not directly tradable but a zero coupon bond collateralized by a foreign currency is a tradable asset. Hence

$$\begin{aligned}
\widetilde{Y}^{(i,j)}(t, T_n) &= \mathbb{E}^{\mathbb{Q}_B^{(i)}} \left[ e^{-\int_t^{T_n} (c^{(i)}(s) + y^{(i,j)}(s)) ds} \right] \\
&= D^{(i)}(t, T_n) Y^{(i,j)}(t, T_n), \quad t \in [0, T_n]
\end{aligned}$$
(6.36)

---

[3]The same result can be obtained from the measure-change technique and the fact that $\{B^{(i)}(t; T_{n-1}, T_n), t \in [0, T_{n-1}]\}$ is a martingale under the forward measure $\mathbb{T}_n^{(i)}$.

is a price process of a tradable asset. An important fact is that this asset has a stream of dividend yield

$$y^{(j)} = y^{(i)} - y^{(i,j)} \ . \tag{6.37}$$

According to the discussion in Section 6.1, we have to discount it by $C^{(i,j)}(t)$ of (6.17) to impose a martingale condition.

Let us write the dynamics of $y_m^{(i,j)}$ as follows:

$$dy_m^{(i,j)}(t) = \alpha_m^{(i,j)}(t)dt + \sigma_{y,m}^{(i,j)}(t) \cdot dW_t^{\mathbb{Q}_B^{(i)}} \tag{6.38}$$

where $\alpha_m^{(i,j)} : \Omega \times [0, T_m] \to \mathbb{R}$ and $\sigma_m^{(i,j)} : \Omega \times [0, T_m] \to \mathbb{R}^d$ are $\mathbb{F}$-adapted processes.

**Proposition 6.4.** *The no-arbitrage dynamics of the forward funding spread is given by*

$$dy_n^{(i,j)}(t) = \left\{ \sigma_{y,n}^{(i,j)}(t) \cdot \left( \sum_{m=q(t)}^{n-1} \delta_m[\sigma_{y,m}^{(i,j)}(t) + \sigma_m^{(i)}(t)] \right) + \sigma_n^{(i)}(t) \cdot \left( \sum_{m=q(t)}^{n-1} \delta_m \sigma_{y,m}^{(i,j)}(t) \right) \right.$$
$$\left. + \frac{1}{2}\delta_n|\sigma_{y,n}^{(i,j)}(t)|^2 + \delta_n(\sigma_{y,n}^{(i,j)}(t) \cdot \sigma_n^{(i)}(t)) \right\} dt + \sigma_{y,n}^{(i,j)}(t) \cdot dW_t^{\mathbb{Q}_B^{(i)}} \tag{6.39}$$

*for $t \in [0, T_n]$ with every $n \in \{0, \cdots, N^* - 1\}$.*

*Proof.* The absence of arbitrage requires

$$\frac{\widetilde{Y}^{(i,j)}(t, T_n)}{C^{(i,j)}(t)}, \quad t \in [0, T_n] \tag{6.40}$$

to be a $\mathbb{Q}_B^{(i)}$-martingale for every $n \in \{0, \cdots, N^*\}$. As before, one can check that the ratio (6.40) is continuous at every $T_i, i \in \{0, \cdots, n\}$. Thus, this condition is equivalent to require, for every $n \in \{0, \cdots, N^*\}$,

$$X_n^{(i,j)}(t) := \exp\left( - \sum_{m=q(t)}^{n-1} \delta_m \widetilde{y}_m^{(i,j)}(t) \right) \tag{6.41}$$

to be a $\mathbb{Q}_B^{(i)}$-martingale within an each interval $(T_{q(t)-1}, T_{q(t)})$. Here, we have put

$$\widetilde{y}_m^{(i,j)}(t) := c_m^{(i)}(t) + y_m^{(i,j)}(t) \tag{6.42}$$

for notational simplicity.

By following the same arguments given in Proposition 6.2, the no-arbitrage dynamics of $\widetilde{y}_n^{(i,j)}$ for $n \in \{0, \cdots, N^* - 1\}$ is give by

$$d\widetilde{y}_n^{(i,j)}(t) = \left\{ \widetilde{\sigma}_n^{(i,j)}(t) \cdot \left( \sum_{m=q(t)}^{n-1} \delta_m \widetilde{\sigma}_m^{(i,j)}(t) \right) + \frac{1}{2}\delta_n|\widetilde{\sigma}_n^{(i,j)}|^2 \right\} dt + \widetilde{\sigma}_n^{(i,j)}(t) \cdot dW_t^{\mathbb{Q}_B^{(i)}} \tag{6.43}$$

with some appropriate $\mathbb{F}$-adapted processes $\widetilde{\sigma}_m^{(i,j)} : \Omega \times [0, T_m] \to \mathbb{R}^d$, $m \in \{0, \cdots, N^* - 1\}$. Thus, by the relation $dy_n^{(i,j)}(t) = d\widetilde{y}_n^{(i,j)}(t) - dc_n^{(i)}(t)$, the drift term is obtained by

$$
\begin{aligned}
\alpha_n^{(i,j)}(t) &= \widetilde{\sigma}_n^{(i,j)}(t) \cdot \Big( \sum_{m=q(t)}^{n-1} \delta_m \widetilde{\sigma}_m^{(i,j)}(t) \Big) + \frac{1}{2} \delta_n |\widetilde{\sigma}_n^{(i,j)}|^2 \\
&\quad - \sigma_n^{(i)}(t) \cdot \Big( \sum_{m=q(t)}^{n-1} \delta_m \sigma_m^{(i)}(t) \Big) - \frac{1}{2} \delta_n |\sigma_n^{(i)}(t)|^2 \ .
\end{aligned}
\tag{6.44}
$$

By the definition of $\widetilde{y}^{(i,j)}$, we have for every $m \in \{0, \cdots, N^* - 1\}$,

$$
\widetilde{\sigma}_m^{(i,j)}(t) = \sigma_m^{(i)}(t) + \sigma_{y,m}^{(i,j)}(t) \ ,
\tag{6.45}
$$

and hence

$$
\begin{aligned}
\alpha_n^{(i,j)}(t) &= \sigma_{y,n}^{(i,j)}(t) \cdot \Big( \sum_{m=q(t)}^{n-1} \delta_m [\sigma_{y,m}^{(i,j)}(t) + \sigma_m^{(i)}(t)] \Big) + \sigma_n^{(i)}(t) \cdot \Big( \sum_{m=q(t)}^{n-1} \delta_m \sigma_{y,m}^{(i,j)}(t) \Big) \\
&\quad + \frac{1}{2} \delta_n |\sigma_{y,n}^{(i,j)}(t)|^2 + \delta_n (\sigma_{y,n}^{(i,j)}(t) \cdot \sigma_n^{(i)}(t))
\end{aligned}
\tag{6.46}
$$

which proves the claim. $\qquad\square$

**Remark**

Although the dynamics of the funding spread is rather complicated, it has exactly the same structure as the forward *hazard* rate model [21]. Thus, if a term structure model of a default intensity has already been implemented, then modifying it for the stochastic funding spread should not be difficult for practitioners.

## 6.5   Dynamics of foreign exchange rates

Finally, let us consider the dynamics of the foreign exchange rate and the associated measure change. The spot foreign exchange rate $\{f_x^{(i,j)}(t)\}_{t \geq 0}$ itself is not tradable but the foreign currency is.

We can define the measure change between $\mathbb{Q}_B^{(i)}$ and $\mathbb{Q}_B^{(j)}$ by

$$
\frac{d\mathbb{Q}_B^{(j)}}{d\mathbb{Q}_B^{(i)}} \Big|_{\mathcal{F}_t} = \frac{B^{(j)}(t) f_x^{(i,j)}(t)}{B^{(i)}(t) f_x^{(i,j)}(0)}
\tag{6.47}
$$

since the right hand side should be a positive $\mathbb{Q}_B^{(i)}$-martingale for the absence of arbitrage. This fact immediately tells us that

**Proposition 6.5.** *The no-arbitrage dynamics of the foreign exchange rate of a foreign currency (j) with respect to a domestic currency (i) is given by*

$$
\begin{aligned}
df_x^{(i,j)}(t) / f_x^{(i,j)}(t) &= \big( r^{(i)}(t) - r^{(j)}(t) \big) dt + \sigma_X^{(i,j)}(t) \cdot dW_t^{\mathbb{Q}_B^{(i)}} \\
&= \big( c^{(i)}(t) - c^{(j)}(t) + y^{(i,j)}(t) \big) dt + \sigma_X^{(i,j)}(t) \cdot dW_t^{\mathbb{Q}_B^{(i)}}
\end{aligned}
\tag{6.48}
$$

*with some appropriate volatility process $\sigma_X^{(i,j)} : \Omega \times [0, T_{N^*}] \to \mathbb{R}^d$, which is $\mathbb{F}$-adapted.*

By the non-randomness assumption on the shortest bonds, one can easily check that the relation

$$\frac{d\mathbb{Q}_B^{(j)}}{d\mathbb{Q}_B^{(i)}}\bigg|_{\mathcal{F}_t} = \frac{C^{(j,k)}(t)f_x^{(i,j)}(t)}{C^{(i,k)}(t)f_x^{(i,j)}(0)} \tag{6.49}$$

holds for an arbitrary currency $(k)$. It is interesting to observe that one must use the common collateral currency to define the measure change. This is due to the existence of the funding spreads $\{y^{(i,j)}\}$, which is a direct consequence of the non-zero cross currency basis being observed in the market [11].

By arranging the right hand side and using the definition (5.2), one sees that

$$f_x^{(i,j)}(t, T_{q(t)}; (k)), t \in [T_{q(t)-1}, T_q(t)] \tag{6.50}$$

is a $\mathbb{Q}_B^{(i)}$-martingale regardless of the collateral currency $(k)$. In addition, its volatility must be identical to that of the spot exchange rate so that its gives the same Radon-Nikodym density. This gives the following result:

**Proposition 6.6.** *The no-arbitrage dynamics of the rolling forward exchange rate of a foreign currency $(j)$ with respect to a domestic currency $(i)$ collateralized by a currency $(k)$ is given by*

$$df_x^{(i,j)}(t, T_{q(t)}; (k)) = f_x^{(i,j)}(t, T_{q(t)}; (k))\sigma_X^{(i,j)}(t) \cdot dW_t^{\mathbb{Q}_B^{(i)}} , \tag{6.51}$$

*with a condition at each rollover date* $f_x^{(i,j)}(T_n, T_{n+1}; (k)) = f_x^{(i,j)}(T_n, T_n; (k))\dfrac{\widetilde{Y}^{(j,k)}(T_n, T_{n+1})}{\widetilde{Y}^{(i,k)}(T_n, T_{n+1})}$

*for every* $n \in \{0, \cdots, N^* - 1\}$. $\sigma_X^{(i,j)} : \Omega \times [0, T_{N^*}] \to \mathbb{R}^d$ *is an $\mathbb{F}$-adapted volatility process equal to that of the spot exchange rate (6.48).*

Note that $f_x^{(i,j)}(t, t; (k)) = f_x^{(i,j)}(t)$. The following corollary is obvious by the standard application of Girsanov-Maruyama theorem.

**Corollary 6.1.** *The Brownian motion under the measure* $\mathbb{Q}_B^{(j)}$ *is related to that of* $\mathbb{Q}_B^{(i)}$ *by*

$$W_t^{\mathbb{Q}_B^{(j)}} = W_t^{\mathbb{Q}_B^{(i)}} - \int_0^t \sigma_X^{(i,j)}(s)ds . \tag{6.52}$$

## 6.6 Dynamics of Equities

If one tries to model a spot equity dynamics directly, it requires to simulate the risk-free rate $r$ or the dividend yield implied by the collateral account $y$, neither of them is easy to observe. The best way to avoid this problem is to use the equity forward price. Let us consider a price process of a equity $\{S_t^{(i)}\}_{t \geq 0}$ denominated by currency $(i)$. We do not ask whether the equity has dividend payments or not. Let us denote the price of a forward contract on this equity with maturity $T_n$ collateralized by the same currency $(i)$ as $S^{(i)}(t, T_n)$:

$$S^{(i)}(t, T_n) = \frac{\mathbb{E}_t^{\mathbb{Q}_B^{(i)}}\left[e^{-\int_t^{T_n} c^{(i)}(s)ds}S^{(i)}(T_n)\right]}{D^{(i)}(t, T_n)}, \quad t \in [0, T_n]. \tag{6.53}$$

Then, by exactly the same reasoning of Proposition 6.3, one obtains:

**Proposition 6.7.** *The no-arbitrage dynamics of the equity forward price is given by*

$$dS^{(i)}(t, T_n) = S^{(i)}(t, T_n) \left( \sigma_{S^{(i)},n}(t) \cdot \Big( \sum_{m=q(t)}^{n-1} \delta_m \sigma_m^{(i)}(t) \Big) dt + \sigma_{S^{(i)},n}(t) \cdot dW_t^{\mathbb{Q}_B^{(i)}} \right) \quad (6.54)$$

*for* $t \in [0, T_n]$, $n \in \{0, \cdots, N^* - 1\}$ *with some appropriate* $\mathbb{F}$-*adapted volatility process* $\sigma_{S^{(i)},n} : \Omega \times [0, T_n] \to \mathbb{R}^d$.

The same technique can be applied to an arbitrary asset or index, as long as there exists a corresponding forward market.

## 6.7 Summary of the dynamics

For easy reference, we summarize the dynamics of various rates in the multi-currency setup with a base currency ($i$) below. For actual implementation, one has to decide the size of dimension "$d$" and reproduce the observed correlation among the underlyings as accurately as possible. For these points, we recommend Rebonato (2004) [19]), which contains many valuable insights for implementation, in particular Chapter 19 and 20.

**Rates for the base currency**

$$dc_n^{(i)}(t) = \left\{ \sigma_n^{(i)}(t) \cdot \Big( \sum_{m=q(t)}^{n-1} \delta_m \sigma_m^{(i)}(t) \Big) + \frac{1}{2} \delta_n |\sigma_n^{(i)}(t)|^2 \right\} dt + \sigma_n^{(i)}(t) \cdot dW_t^{\mathbb{Q}_B^{(i)}} \quad (6.55)$$

$$\frac{dB^{(i)}(t; T_{n-1}, T_n)}{B^{(i)}(t; T_{n-1}, T_n)} = \sigma_{B,n}^{(i)}(t) \cdot \Big( \sum_{m=q(t)}^{n-1} \delta_m \sigma_m^{(i)}(t) \Big) dt + \sigma_{B,n}^{(i)}(t) \cdot dW_t^{\mathbb{Q}_B^{(i)}} \quad (6.56)$$

**Funding spreads with respect to the base currency**

$$dy_n^{(i,j)}(t) = \left\{ \sigma_{y,n}^{(i,j)}(t) \cdot \Big( \sum_{m=q(t)}^{n-1} \delta_m [\sigma_{y,m}^{(i,j)}(t) + \sigma_m^{(i)}(t)] \Big) + \sigma_n^{(i)}(t) \cdot \Big( \sum_{m=q(t)}^{n-1} \delta_m \sigma_{y,m}^{(i,j)}(t) \Big) \right.$$

$$\left. + \frac{1}{2} \delta_n |\sigma_{y,n}^{(i,j)}(t)|^2 + \delta_n (\sigma_{y,n}^{(i,j)}(t) \cdot \sigma_n^{(i)}(t)) \right\} dt + \sigma_{y,n}^{(i,j)}(t) \cdot dW_t^{\mathbb{Q}_B^{(i)}} \quad (6.57)$$

**Foreign exchange rate**

$$df_x^{(i,j)}(t)/f_x^{(i,j)}(t) = \Big( c_t^{(i)} - c_t^{(j)} + y_t^{(i,j)} \Big) dt + \sigma_X^{(i,j)}(t) \cdot dW_t^{\mathbb{Q}^{(i)}} \quad (6.58)$$

**Rates for a foreign currency**

$$dc_n^{(j)}(t) = \left\{ \sigma_n^{(j)}(t) \cdot \Big( \sum_{m=q(t)}^{n-1} \delta_m \sigma_m^{(j)}(t) - \sigma_X^{(i,j)}(t) \Big) + \frac{1}{2} \delta_n |\sigma_n^{(j)}(t)|^2 \right\} dt + \sigma_n^{(j)}(t) \cdot dW_t^{\mathbb{Q}_B^{(i)}}$$

$$\frac{dB^{(j)}(t; T_{n-1}, T_n)}{B^{(j)}(t; T_{n-1}, T_n)} = \sigma_{B,n}^{(j)}(t) \cdot \Big( \sum_{m=q(t)}^{n-1} \delta_m \sigma_m^{(j)}(t) - \sigma_X^{(i,j)}(t) \Big) dt + \sigma_{B,n}^{(j)}(t) \cdot dW_t^{\mathbb{Q}_B^{(i)}} \quad (6.59)$$

**Funding spreads with respect to a foreign currency**

$$dy_n^{(j,k)}(t) = \left\{ \sigma_{y,n}^{(j,k)}(t) \cdot \left( \sum_{m=q(t)}^{n-1} \delta_m [\sigma_{y,m}^{(j,k)}(t) + \sigma_m^{(j)}(t)] - \sigma_X^{(i,j)}(t) \right) + \sigma_n^{(j)}(t) \cdot \left( \sum_{m=q(t)}^{n-1} \delta_m \sigma_{y,m}^{(j,k)}(t) \right) \right.$$

$$\left. + \frac{1}{2} \delta_n |\sigma_{y,n}^{(j,k)}(t)|^2 + \delta_n (\sigma_{y,n}^{(j,k)}(t) \cdot \sigma_n^{(j)}(t)) \right\} dt + \sigma_{y,n}^{(j,k)}(t) \cdot dW_t^{\mathbb{Q}_B^{(i)}} \qquad (6.60)$$

# 7 Remarks on a risk-free money market account

It is clear from the results in Sections 3 and 6, one need not refer to *risk-free interest rates* and the associated money-market accounts in a fully collateralized market. In fact, it is matter of preference for the user to choose a certain currency $(i)$ as a base currency and treat its collateral account $C^{(i)}$ as a unique risk (default)-free bank account. It makes $y^{(i)} = 0$ but does not change dynamics given in the previous sections at all.

However, due to the presence of non-zero cross currency basis, this requires asymmetric treatments for the other collateral rates of foreign currencies, which cannot be treated as traditional risk-free rates. Furthermore, it has become clear that the authority can force the collateral rate to be negative as observed in recent EONIA market, for example. Considering the fact that there exist firms which are outside the banking regulation and do not have to collateralize their contracts, it is not always rational to treat the overnight rate as risk-free. Even for those who are forced to collateralize the contracts, it is more natural to consider the collateral agreement has some dividend yield which makes overnight rate effectively negative.

In our setup, there is no problem to make a collateral rate negative since there is no a priori restriction to the dividend yield process $\{y^{(i)}\}$. Due to these observations, we have chosen to use a risk-free money market account separately from a collateral account. The biggest assumption for our framework is the existence of a common "risk-free" money market account for every market participant. It is completely rational that each financial firm wants to reflect its own funding/investment conditions instead of a common "risk-free" rate, but then it will inevitably produce the company dependent price of derivatives [6]. Therefore, we think that the company-specific effects should be treated in FVA, separately from the benchmark pricing we have described in this article.

# 8 Conclusion

This paper is an extension of previous works [10, 9, 11] and provided more detailed explanation for the general framework for the interest rate modeling in a fully collateralized market. In particular, we gave a new formulation for the funding spread dynamics which is more suitable in the presence of non-zero correlation to the collateral rates. We also presented a complete picture of a discretized HJM model with a fixed tenor structure under a multi-currency setup, which is arbitrage free, readily implementable and capable of taking the stochastic cross currency basis into account.

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