CARF Working Paper

CARF-F-412

Pricing Average and Spread Options under Local-Stochastic Volatility Jump-Diffusion Models

Online Appendix

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July 6, 2017

CARF is presently supported by The Dai-ichi Life Insurance Company, Limited, Nomura Holdings, Inc., Sumitomo Mitsui Banking Corporation, Bank of Tokyo-Mitsubishi UFJ, Finatext Ltd., GCI Asset Management, Inc and The University of Tokyo Edge Capital Co.. This financial support enables us to issue CARF Working Papers.

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This online appendix provides results omitted in the paper with the same title. Appendix A explains all the definitions and equations necessary for practical computations of an option pricing formula in Theorem 4.3: Section A.1 gives a summary with Corollary A.1, which shows our pricing formula with complete expressions of constants $C_{i,k}$ $(i = 1, 2, 3)$, C_j $(j = 4, 5, 6)$ appearing in the theorem. Section A.2. provides the details of the derivation. Appendix B lists up the conditional expectation formulas used in the derivation of the theorem.

A Summary and Details of Calculations in Theorem 4.3

A.1 Summary of the calculations

Firstly, for a known parameter $\epsilon \in [0, 1]$ we consider the following stochastic integral equations: for $i = 1, \cdots, d$,

$$
\begin{aligned}
S_T^{i,(\epsilon)} &= s_0^i + \int_0^T \alpha_{t-}^i S_{t-}^{i,(\epsilon)} dt + \epsilon \int_0^T \phi_{S^i}\left(\sigma_{t-}^{(\epsilon)}, S_{t-}^{(\epsilon)}\right) dW_t \\
&\quad + \sum_{l=1}^{n} \left(\sum_{j=1}^{N_{l,T}} h_{S^i,l,j}^{(\epsilon)} S_{\tau_{j,l}-}^{i,(\epsilon)} - \int_0^T \Lambda_l S_{t-}^{i,(\epsilon)} \mathbf{E}[h_{S^i,l,1}^{(\epsilon)}] dt \right), \qquad (1) \\
\sigma_T^{i,(\epsilon)} &= \sigma_0^i + \int_0^T \mu_{\sigma^i}\left(\sigma_{t-}^{(\epsilon)}, t-\right) dt + \epsilon \int_0^T \phi_{\sigma^i}\left(\sigma_{t-}^{(\epsilon)}\right) dW_t \\
&\quad + \sum_{l=1}^{n} \left(\sum_{j=1}^{N_{l,T}} h_{\sigma^i,l,j}^{(\epsilon)} \sigma_{\tau_{j,l}-}^{i,(\epsilon)} - \int_0^T \Lambda_l \sigma_{t-}^{i,(\epsilon)} \mathbf{E}[h_{\sigma^i,l,1}^{(\epsilon)}] dt \right), \qquad (2)
\end{aligned}
$$

where s_0^i, σ_0^i are given constants, and α_t^i is a deterministic piecewise continuous function of t, and W is a $2d$-dimensional (independent) Brownian motion. Each N_l,

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$(l = 1, \cdots, n)$ is a Poisson process with a constant intensity Λ_l, which is independent of each other. N_l, $(l = 1, \cdots, n)$ are also independent of all W; $\tau_{j,l}$ stands for the j-th jump time of N_l; For each $l = 1, \cdots, n$ and $i = 1, \cdots, d$, both $\left(\sum_{j=1}^{N_{l,t}} h^{(\epsilon)}_{S^i,l,j}\right)_{t\geq 0}$ and $\left(\sum_{j=1}^{N_{l,t}} h^{(\epsilon)}_{\sigma^i,l,j}\right)_{t\geq 0}$ are compound Poisson processes. ($\sum_{j=1}^{N_{l,t}} \equiv 0$ when $N_{l,t} = 0$). $\mu_{\sigma^i} : \mathbf{R}^d \times \mathbf{R} \to \mathbf{R}$, and $\phi_{x^i} : \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}^{2d}$, ($x^i = S^i$ or σ^i), and we assume that $\mu^{(\epsilon)}_{x^i}$, $\phi^{(\epsilon)}_{x^i}$, $h^{(\epsilon)}_{x^i,l,j}$ satisfy some regularity conditions (see Section 2 and 3). We also suppose that $h^{(\epsilon)}_{x^i,l,j} = \epsilon m_{x^i,l}$ ($m_{x^i,l}$: a constant), or $h^{(\epsilon)}_{x^i,l,j} = e^{\epsilon Y_{x^i,l,j}} - 1$ where $Y_{x^i,l,j} \sim N(m_{x^i,l}, v^2_{x^i,l})$.

In order to reduce the computational costs for numerical experiments, we use the 3rd order (ϵ^3) corrections obtained by a no-jump model, which is defined as

$$S^{i,LSV(\epsilon)}_T = s^i_0 + \int_0^T \alpha^i_t S^{i,LSV(\epsilon)}_t dt + \epsilon \int_0^T \phi_{S^i}\left(\sigma^{i,LSV(\epsilon)}_t, S^{i,LSV(\epsilon)}_t\right) dW_t, \quad (3)$$

$$\sigma^{i,LSV(\epsilon)}_T = \sigma^i_0 + \int_0^T \mu_{\sigma^i}\left(\sigma^{i,LSV(\epsilon)}_t, t\right) dt + \epsilon \int_0^T \phi_{\sigma^i}\left(\sigma^{i,LSV(\epsilon)}_t\right) dW_t. \quad (4)$$

Next, let us define new processes $A^{i,(\epsilon)}_t$ based on the price processes (1) and (2) as well as $A^{i,LSV(\epsilon)}_t$ based on the price processes (3) and (4):

$$A^{i,(\epsilon)}_t = \sum_{j=1}^{m_i} w^{(i)}_j S^{i,(\epsilon)}_{t^{(i)}_j} 1_{\{t^{(i)}_j \leq t\}},$$

$$A^{i,LSV(\epsilon)}_t = \sum_{j=1}^{m_i} w^{(i)}_j S^{i,LSV(\epsilon)}_{t^{(i)}_j} 1_{\{t^{(i)}_j \leq t\}}, \quad (5)$$

where $0 \leq t^{(i)}_1 < \cdots < t^{(i)}_{m_i} \leq T$, each m_i denotes the number of the underlying asset price $S^{i,(\epsilon)}$ ($S^{i,LSV(\epsilon)}$) to which a discrete average option refers, and each $w^{(i)}_j$ stands for the weight of the asset price $S^{i,(\epsilon)}$ at date $t^{(i)}_j$. Then, we can describe the underlying asset price process $g(A^{(\epsilon)}_t)$ of an average option as

$$g(A^{(\epsilon)}_t) = \sum_{i=1}^d A^{i,(\epsilon)}_t, \quad g(A^{i,LSV(\epsilon)}_t) = \sum_{i=1}^d A^{i,LSV(\epsilon)}_t. \quad (6)$$

Remark A.1. *The underlying asset of spread option is the spread of futures prices with two different maturities ($T_1 < T_2$). More precisely, the spread consists of a long position in the first expiring futures (whose price is denoted by $S^{1,(\epsilon)}$) in the spread and a short position in the second expiring futures (whose price is denoted by $S^{2,(\epsilon)}$). Then, the underlying asset price process of an spread option is given as a special case of an average option:*

$$A^{i,(\epsilon)}_t = w^i S^{i,(\epsilon)}_{T_i} 1_{\{T_i \leq t\}}, \quad A^{i,LSV(\epsilon)}_t = w^i S^{i,LSV(\epsilon)}_{T_i} 1_{\{T_i \leq t\}}, \ (i = 1, 2, \ w^1 = 1, \ w^2 = -1),$$

$$g(A^{i,(\epsilon)}_t) = A^{1,(\epsilon)}_t - A^{2,(\epsilon)}_t, \quad g(A^{i,LSV(\epsilon)}_t) = A^{1,LSV(\epsilon)}_t - A^{2,LSV(\epsilon)}_t. \quad (7)$$

Then, the payoff function of an average/spread option with maturity T and strike K is defined as $(g(A_T^{(\epsilon)}) - K)^+ (:= \max\{g(A_T^{(\epsilon)}) - K, 0\})$ for a call option and $(K - g(A_T^{(\epsilon)}))^+ (:= \max\{K - g(A_T^{(\epsilon)}), 0\})$ for a put option. In this way, we are able to evaluate average and spread options in a unified manner.

By specifying the functions ϕ_{S^i}, μ_{σ^i} and ϕ_{σ^i} ($i = 1, \cdots, d$) in (1)–(4), we can express various types of local-stochastic volatility models.

In particular, an extended Bates model and an extended SABR model used in the numerical examples (that appear in Section 5 of the paper) are given as follows:

- (Extended Bates model) for $i = 1, 2, \cdots, d$,

$$\begin{aligned}
&\phi_{S^i}(\sigma, S) = (\rho_{i,1} v^i S^{i,\beta_i} \sqrt{\sigma^i}, \cdots, \rho_{i,2d} v^i S^{i,\beta_i} \sqrt{\sigma^i}),\ (v^i : \text{a constant}), \\
&\mu_{\sigma^i}(\sigma) = \lambda^i(\theta^i - \sigma^i),\ (\lambda^i, \theta^i : \text{constants}), \\
&\phi_{\sigma^i}(\sigma) = (\rho_{d+i,1} \nu^i \sqrt{\sigma^i}, \cdots, \rho_{d+i,2d} \nu^i \sqrt{\sigma^i}),\ (\nu^i : \text{a constant}).
\end{aligned}$$

- (Extended SABR model) for $i = 1, 2, \cdots, d$,

$$\begin{aligned}
&\phi_{S^i}(\sigma, S) = (\rho_{i,1} v^i S^{i,\beta_i} \sigma^i, \cdots, \rho_{i,2d} v^i S^{i,\beta_i} \sigma^i),\ (v^i : \text{a constant}), \\
&\mu_{\sigma^i}(\sigma^i) = 0,\ (\mu_{\sigma^i}(\sigma^i) = \lambda^i(\theta^i - \sigma^i) \text{ with } \lambda^i = 0), \\
&\phi_{\sigma^i}(\sigma) = (\rho_{i+d,1} \nu^i \sigma^i, \cdots, \rho_{i+d,1} \nu^i \sigma^i),\ (\nu^i : \text{a constant}).
\end{aligned}$$

  ρ_{ij} is the (i, j)-element of a $2d \times 2d$ matrix obtained by Cholesky decomposition of the correlation matrix of the correlated Brownian motions in Section 5.

Next, let us provide a concrete pricing formula that can be applied with $\epsilon = 1$ to numerical examples in Section 5 of the paper. More precisely, we show an approximation formula for option prices under the following setup:

- Only one type of the jump components, i.e. $n = 1$ in (1)–(2), and hence we use the notations independent of jumps types l (e.g. $\Lambda = \Lambda_l$).
- Specification of the jump size in (1)–(2):

  (For the log-normal jump case)

  $h_{x^i,j}^{(\epsilon)} = e^{\epsilon Y_{x^i,j}} - 1$ with $Y_{x^i,j} \sim N(m_{x^i}, \upsilon_{x^i}^2)$ (m_{x^i}, υ_{x^i}: constants),

  $x^i = S^i$ or σ^i,

  We also use the notation $\vartheta_{S^i,S^{i'}}$ that denotes the (i, i')-element of ϑ, $d \times d$ correlation matrix among the jump sizes of S^i and $S^{i'}$.

  (For the constant jump case)

  $h_{x^i,j}^{(\epsilon)} = \epsilon m_{x^i}$ (m_{x^i}: constant)

- The underlying asset price process is described as (6) with (5), which corresponds to the one for a discrete average option. The one for a spread option is given by (7) as its special case.

- The drift terms in volatility processes are set as the one for the extended Bates model, that is $\mu_{\sigma^i}(\sigma) = \lambda^i(\theta^i - \sigma^i)$, whence $\lambda^i = 0$ gives the one for the extended SABR model.

- We provide an explicit formula for the call option, since the call price $C(K,T)$ with maturity T and strike K gives the put price $P(K,T)$ based on the put-call parity as follows:

$$P(K,T) = K - \sum_{i=1}^{d}\sum_{j=1}^{m_i} w_j^{(i)} e^{\int_0^{t_j^{(i)}} \alpha_s^i ds} s_0^i + C(K,T).$$

Then, in Section 5 we apply the formula with:

- $\alpha^i \equiv 0$ for all i in (1)–(4)
- $\vartheta_{S^i,S^{i'}} = 1$ for all i, i'
- six specifications of the model (i)–(vi) stated in Section 5

Now, we state the next result as a corollary of Theorem 4.3 in the paper.

Corollary A.2. *An approximation formula for the initial value $C(K,T)$ of a discrete average call option with maturity T and strike price K with $K = \sum_{i=1}^{d}\sum_{j=1}^{m_i} w_j^{(i)} e^{\int_0^{t_j^{(i)}} \alpha_s^i ds} s_0^i - \epsilon\mathscr{Y}$ for an arbitrary $\mathscr{Y} \in \mathbf{R}$ is provided as follows:*

$$\begin{aligned} C(K,T) \approx & \sum_{k=0}^{\infty} p_{\{k\}} e^{-rT} \Bigg\{ \epsilon \Bigg\{ y_k N\left(\frac{y_k}{\sqrt{\Sigma_T^{\{k\}}}}\right) + \Sigma_T^{\{k\}} n(y_k; 0, \Sigma_T^{\{k\}}) \Bigg\} \\ & + \epsilon^2 \Bigg\{ C_{1,k} N\left(\frac{y_k}{\sqrt{\Sigma_T^{\{k\}}}}\right) + \left(C_{2,k}\Sigma_T^{\{k\}} + C_{3,k}\frac{H_1(y_k;\Sigma_T^{\{k\}})}{\Sigma_T^{\{k\}}}\right) n(y_k; 0, \Sigma_T^{\{k\}}) \Bigg\} \\ & + \epsilon^3 \Bigg\{ \left(C_4 \frac{H_2(\mathscr{Y};\Sigma_T)}{(\Sigma_T)^2} + C_5 \frac{H_4(\mathscr{Y};\Sigma_T)}{(\Sigma_T)^4} + C_6 \right) n(\mathscr{Y}; 0, \Sigma_T) \Bigg\} \Bigg\}, \end{aligned} \tag{8}$$

where r is a constant risk-free rate, $N(x)$ denotes the standard normal distribution function and $n(x;0,\Sigma) = \frac{1}{\sqrt{2\pi\Sigma}} \exp\left(\frac{-x^2}{2\Sigma}\right)$. Moreover, $H_k\left(x;\Sigma_T^{\{k\}}\right)$ denotes the k-th order Hermite polynomial: particularly, $H_1\left(x;\Sigma_T^{\{k\}}\right) = x$, $H_2\left(x;\Sigma_T^{\{k\}}\right) = x^2 - \Sigma_T^{\{k\}}$ and $H_4\left(x;\Sigma_T^{\{k\}}\right) = x^4 - 6\Sigma_T^{\{k\}}x^2 + 3\left(\Sigma_T^{\{k\}}\right)^2$. Also, $p_{\{k\}}$, $\xi^i_{\{k\}}$, $y_{\{k\}}$, $\Sigma_T^{\{k\}}$ and Σ_T are given as follows:

$$p_{\{k\}} = \frac{(\Lambda T)^k e^{-\Lambda T}}{k!}, \tag{9}$$

$$\xi^i_{\{k\}} = \tilde{w}_i \left(k - \Lambda T\right) m_{S^i} s^i_0, \tag{10}$$

$$y_{\{k\}} = g(\xi_{\{k\}}) + \mathscr{Y} = \sum_{i=1}^{d} \xi^i_{\{k\}} + \mathscr{Y}, \tag{11}$$

$$\Sigma^{\{k\}}_T = \sum_{i=1}^{d}\sum_{i'=1}^{d} \int_0^T \left(\bar{w}_i(t)\phi_{S^i}\left(\sigma^{(0)}_t, S^{(0)}_t\right)\right)\left(\bar{w}_{i'}(t)\phi_{S^{i'}}\left(\sigma^{(0)}_t, S^{(0)}_t\right)\right)' dt + \sum_{i=1}^{d}\sum_{i'=1}^{d} k\left(\tilde{w}_i \upsilon_{S^i} s^i_0\right)\vartheta_{S^i,S^{i'}}\left(\tilde{w}_{i'}\upsilon_{S^{i'}} s^{i'}_0\right), \tag{12}$$

$$\Sigma_T = \Sigma^{\{0\}}_T, \tag{13}$$

where x' means the transpose of x. We also introduce the notation ∂^i_x which stands for the i-th order partial differential operator with respect to x, and

$$\bar{w}_i(t) = \sum_{j=1}^{m_i} w^{(i)}_j 1_{\{t \le t^{(i)}_j\}} e^{\int_t^{t^{(i)}_j} \alpha^i_s ds}, \tag{14}$$

$$\tilde{w}_i = \frac{1}{T}\sum_{j=1}^{m_i} w^{(i)}_j t^{(i)}_j e^{\int_0^{t^{(i)}_j} \alpha^i_s ds}, \tag{15}$$

$$\hat{w}_i(t) = \sum_{j=1}^{m_i} w^{(i)}_j 1_{\{t \le t^{(i)}_j\}} e^{\int_0^{t^{(i)}_j} \alpha^i_s ds}. \tag{16}$$

Then, the coefficients $C_{1,k}, \cdots, C_{3,k}, C_4, \cdots, C_6$ are constants that are provided in the following:

(log-normal jump case)

$$C_{1_k} = \sum_{i=1}^{d} \tilde{w}_i k m^2_{S^i} s^i_0 + \sum_{i=1}^{d} \tilde{w}_i \Lambda_l (m^2_{S^i} + \upsilon^2_{S^i}) s^i_0 T + \sum_{i=1}^{d} \tilde{w}_i s^i_0 \sum_{j=1}^{k}\sum_{I=1}^{j-1} m_{S^i} m_{S^i} + \sum_{i=1}^{d} m_{S^i}\tilde{w}_i \Lambda m_{S^i} s^i_0 \frac{kT}{2} + \sum_{i=1}^{d} \tilde{w}_i \Lambda m_{S^i} s^i_0 m_{S^i} \frac{kT}{2}$$

$$+\sum_{i=1}^{d}\tilde{w}_i\int_0^T \Lambda m_{S^i}\Lambda m_{S^i} t s_0^i dt, \quad (17)$$

$$\begin{aligned}
C_{2_k} \quad = \quad & \sum_{i=1}^{d}\int_0^T \bar{w}_i(t)\partial_{S^i}\phi_{S^i}\frac{kt}{T}m_{S^i}s_0^i\sum_{I=1}^{d}\bar{w}_I(t)\phi'_{S^I}dt \\
& +\sum_{i=1}^{d}\int_0^T \bar{w}_i(t)\partial_{S^i}\phi_{S^i}\sum_{I=1}^{d}\bar{w}_I(t)\phi'_{S^I}\Lambda m_{S^i}e^{\int_0^t \alpha_s^i ds}s_0^i t dt \\
& +\sum_{i=1}^{d}\int_0^T \bar{w}_i(t)\partial_{\sigma^i}\phi_{S^i}\sum_{I=1}^{d}\bar{w}_I(t)\phi'_{S^I}\int_0^t \frac{k}{T}m_{\sigma^i}e^{-\lambda^i(t-u)}\sigma_u^{i,(0)}dudt \\
& +\sum_{i=1}^{d}\int_0^T \bar{w}_i(t)\partial_{\sigma^i}\phi_{S^i}\sum_{I=1}^{d}\bar{w}_I(t)\phi'_{S^I}\Lambda m_{\sigma^i}\int_0^t e^{-\lambda^i(t-u)}\sigma_u^{i,(0)}dudt \\
& +\sum_{i=1}^{d}\tilde{w}_i k 2 m_{S^i}\upsilon_{S^i}s_0^i\sum_{I=1}^{d}\vartheta_{S^i,S^I}\tilde{w}_I\upsilon_{S^I}s_0^I \\
& +\sum_{i=1}^{d}\int_0^T \tilde{w}_i m_{S^i}\int_0^t e^{-\int_0^u \alpha_s^i ds}\phi_{S^i}\sum_{I=1}^{d}\bar{w}_I(t)\phi'_{S^I}dudt\frac{k}{T} \\
& +2\sum_{i=1}^{d}\tilde{w}_i s_0^i\sum_{j=1}^{k}\sum_{I=1}^{j-1}m_{S^i}\upsilon_{S^i}\sum_{J=1}^{d}\vartheta_{S^i,S^J}\upsilon_{S^J}\tilde{w}_J s_0^J \\
& +\sum_{i=1}^{d}\upsilon_{S^i}\tilde{w}_i\sum_{I=1}^{d}\vartheta_{S^i,S^I}\tilde{w}_I\upsilon_{S^I}s_0^I\Lambda m_{S^i}s_0^i\frac{kT}{2} \\
& +\sum_{i=1}^{d}T\tilde{w}_i\Lambda m_{S^i}\int_0^T e^{-\int_0^t \alpha_s^i ds}\phi_{S^i}\sum_{I=1}^{d}\bar{w}_I(t)\phi'_{S^I}dt \\
& -\sum_{i=1}^{d}\tilde{w}_i\int_0^T t\Lambda m_{S^i}e^{\int_0^t \alpha_s^i ds}\phi_{S^i}\sum_{I=1}^{d}\bar{w}_I(t)\phi'_{S^I}dt \\
& +\sum_{i=1}^{d}\tilde{w}_i s_0^i\Lambda m_{S^i}\upsilon_{S^i}\sum_{J=1}^{d}\vartheta_{S^i,S^J}\upsilon_{S^J}\tilde{w}_J s_0^J\frac{kT}{2}, \quad (18)
\end{aligned}$$

$$\begin{aligned}
C_{3_k} \quad = \quad & \sum_{i=1}^{d}\int_0^T \bar{w}_i(t)\partial_{S^i}\phi_{S^i}\sum_{I=1}^{d}\bar{w}_I(t)\phi'_{S^I}\int_0^t e^{\int_u^t \alpha_s^i ds}\phi_{S^i}\sum_{J=1}^{d}\bar{w}_J(u)\phi'_{S^J}dudt \\
& +\sum_{i=1}^{d}\int_0^T \bar{w}_i(t)\partial_{S^i}\phi_{S^i}\sum_{I=1}^{d}\bar{w}_I(t)\phi'_{S^I}\frac{kt}{T}e^{\int_0^t \alpha_s^i ds}\upsilon_{S^i}s_0^i\sum_{J=1}^{d}\tilde{w}_J\vartheta_{S^i,S^J}\upsilon_{S^J}s_0^J dt, \\
& +\sum_{i=1}^{d}\int_0^T \bar{w}_i(t)\partial_{\sigma^i}\phi_{S^i}\sum_{I=1}^{d}\bar{w}_I(t)\phi'_{S^I}\int_0^t e^{-\lambda^i(t-u)}\phi_{\sigma^i}\sum_{J=1}^{d}\bar{w}_J(t)\phi'_{S^J}dudt
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{d}\int_0^T \bar{w}_i(t)\partial_{\sigma^i}\phi_{S^i}\sum_{I=1}^{d}\bar{w}_I(t)\phi'_{S^I}\\
&\times\int_0^t \frac{k}{T}\upsilon_{\sigma^i}e^{-\lambda^i(t-u)}\sigma_u^{i,(0)}\sum_{J=1}^{d}\vartheta_{S^i,\sigma^J}\upsilon_{S^J}\tilde{w}_J s_0^J dudt\\
&+\sum_{i=1}^{d}k(\upsilon_{S^i})^2(\tilde{w}_i)^2(s_0^i)^2+\sum_{i=1}^{d}k\left(\upsilon_{S^i}\tilde{w}_i s_0^i\sum_{I=1}^{d}\vartheta_{S^i,S^I}\upsilon_{S^I}\tilde{w}_I s_0^I\right)^2\\
&+\sum_{i=1}^{d}\int_0^T \tilde{w}_i\upsilon_{S^i}\sum_{I=1}^{d}T\tilde{w}_I\vartheta_{S^i,S^I}\upsilon_{S^I}s_0^I\sum_{J=1}^{d}\int_0^t e^{\int_0^u \alpha_s^i ds}\phi_{S^i}\bar{w}_J(t)\phi'_{S^J}dudt\frac{k}{T}\\
&+\sum_{i=1}^{d}\tilde{w}_i s_0^i\sum_{j=1}^{k}\sum_{I=1}^{j-1}\upsilon_{S^i}\sum_{\iota=1}^{d}\vartheta_{S^i,S^\iota}\upsilon_{S^\iota}s_0^\iota\upsilon_{S^i}\sum_{J=1}^{d}\vartheta_{S^i,S^J}\upsilon_{S^J}\tilde{w}_J s_0^J. \qquad (19)
\end{aligned}
$$

(constant jump case)

$$
\begin{aligned}
C_{1_k} &= \sum_{i=1}^{d}\tilde{w}_i s_0^i\sum_{j=1}^{k}\sum_{I=1}^{j-1}m_{S^i}m_{S^i}\\
&+\sum_{i=1}^{d}m_{S^i}\tilde{w}_i\Lambda m_{S^i}s_0^i\frac{kT}{2}\\
&+\sum_{i=1}^{d}\tilde{w}_i\Lambda m_{S^i}s_0^i m_{S^i}\frac{kT}{2}\\
&+\sum_{i=1}^{d}\tilde{w}_i\int_0^T \Lambda m_{S^i}\Lambda m_{S^i}ts_0^i dt, \qquad (20)\\
C_{2_k} &= \sum_{i=1}^{d}\int_0^T \bar{w}_i(t)\partial_{S^i}\phi_{S^i}\frac{kt}{T}m_{S^i}s_0^i\sum_{I=1}^{d}\bar{w}_I(t)\phi'_{S^I}dt\\
&+\sum_{i=1}^{d}\int_0^T \bar{w}_i(t)\partial_{S^i}\phi_{S^i}\sum_{I=1}^{d}\bar{w}_I(t)\phi'_{S^I}\Lambda m_{S^i}e^{\int_0^t \alpha_s^i ds}s_0^i tdt\\
&+\sum_{i=1}^{d}\int_0^T \bar{w}_i(t)\partial_{\sigma^i}\phi_{S^i}\sum_{I=1}^{d}\bar{w}_I(t)\phi'_{S^I}\int_0^t \frac{k}{T}m_{\sigma^i}e^{-\lambda^i(t-u)}\sigma_u^{i,(0)}dudt\\
&+\sum_{i=1}^{d}\int_0^T \bar{w}_i(t)\partial_{\sigma^i}\phi_{S^i}\sum_{I=1}^{d}\bar{w}_I(t)\phi'_{S^I}\Lambda m_{\sigma^i}\int_0^t e^{-\lambda^i(t-u)}\sigma_{u-}^{i,(0)}dudt\\
&+\sum_{i=1}^{d}\int_0^T \tilde{w}_i m_{S^i}\int_0^t e^{-\int_0^u \alpha_s^i ds}\phi_{S^i}\sum_{I=1}^{d}\bar{w}_I(t)\phi'_{S^I}dudt\frac{k}{T}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{d} T\tilde{w}_i \Lambda m_{S^i}\int_0^T e^{-\int_0^t \alpha_s^i ds}\phi_{S^i}\sum_{I=1}^{d}\bar{w}_I(t)\phi'_{S^I}dt \\
&-\sum_{i=1}^{d}\tilde{w}_i\int_0^T t\Lambda m_{S^i}e^{\int_0^t \alpha_s^i ds}\phi_{S^i}\sum_{I=1}^{d}\bar{w}_I(t)\phi'_{S^I}dt, \qquad (21)
\end{aligned}
$$

$$
\begin{aligned}
C_{3_k} &= \sum_{i=1}^{d}\int_0^T \bar{w}_i(t)\partial_{S^i}\phi_{S^i}\sum_{I=1}^{d}\bar{w}_I(t)\phi'_{S^I}\int_0^t e^{\int_u^t \alpha_s^i ds}\phi_{S^i}\sum_{J=1}^{d}\bar{w}_J(u)\phi'_{S^J}dudt \\
&+\sum_{i=1}^{d}\int_0^T \bar{w}_i(t)\partial_{\sigma^i}\phi_{S^i}\sum_{I=1}^{d}\bar{w}_I(t)\phi'_{S^I}\int_0^t e^{-\lambda^i(t-u)}\phi_{\sigma^i}\sum_{J=1}^{d}\bar{w}_J(t)\phi'_{S^J}dudt \\
&+\sum_{i=1}^{d}\int_0^T \bar{w}_i(t)\partial_{\sigma^i}\phi_{S^i}\sum_{I=1}^{d}\bar{w}_I(t)\phi'_{S^I}. \qquad (22)
\end{aligned}
$$

(log-normal and constant jump cases)

$$
\begin{aligned}
C_4 &= \frac{1}{2}I_2(134)+I_2(135)+\frac{1}{2}I_2(136) \\
&+\frac{1}{2}I_2(154)+I_2(155)+I_2(156)+\frac{1}{2}I_2(157)+I_2(158), \qquad (23)\\
C_5 &= \frac{1}{2}I_4(134)+I_4(135)+\frac{1}{2}I_4(136), \qquad (24)\\
C_6 &= \frac{1}{2}I_0(134)+I_0(135)+\frac{1}{2}I_0(136)+\frac{1}{2}I_2(154)+I_2(158), \qquad (25)
\end{aligned}
$$

where the parenthesized numbers correspond to the equation numbers in the next subsection A.2 (details of the calculations).

For $M=134,135,136$,

$$
\begin{aligned}
I_4(M) &= \sum_{i=1}^{d}\sum_{I=1}^{d}\left(\int_0^T q'_{M,3t,i}q_{M,1t,i}\int_0^t q'_{M,2s,i}q_{M,1s,i}dsdt\right)\times \\
&\quad \left(\int_0^T q'_{M,5r,I}q_{M,1r,I}\int_0^r q'_{M,4u,I}q_{M,1u,I}dudr\right), \\
I_2(M) &= \sum_{i=1}^{d}\sum_{I=1}^{d}\left\{\int_0^T q'_{M,3t,i}q_{M,1t,i}\int_0^t q'_{M,5r,I}q_{M,1r,I}\int_0^r q'_{M,2u,i}q_{M,4u,I}dudrdt\right. \\
&+\int_0^T q'_{M,5t,I}q_{M,1t,I}\int_0^t q'_{M,3r,i}q_{M,1r,i}\int_0^r q'_{M,2u,i}q_{M,4u,I}dudrdt \\
&+\int_0^T q'_{M,3t,i}q_{M,1t,i}\int_0^t q'_{M,2r,i}q_{M,5r,I}\int_0^r q'_{M,4u,I}q_{M,1u,I}dudrdt \\
&+\int_0^T q'_{M,3t,i}q_{M,5t,I}\left(\int_0^t q'_{M,2s,i}q_{1s}ds\right)\left(\int_0^t q'_{M,4u,I}q_{M,1u,I}du\right)dt \\
&\left.+\int_0^T q'_{M,5r,I}q_{M,1r,I}\int_0^r q'_{M,3u,i}q_{M,4u,I}\int_0^u q'_{M,2s,i}q_{M,1s,i}dsdudr\right\},
\end{aligned}
$$

$$I_0(M) = \sum_{i=1}^{d}\sum_{I=1}^{d}\int_0^T\int_0^t q'_{M,2u,i}q_{M,4u,I}du q'_{M,3t,i}q_{M,5t,I}dt,$$

where

$$q'_{134,1t,i} = \sum_{i=1}^{d}\bar{w}_i(t)\phi_{S^i}, \quad (26)$$
$$q'_{134,2t,i} = e^{-\int_0^t \alpha_s^i ds}\phi_{S^i}, \quad (27)$$
$$q'_{134,3t,i} = \hat{w}_i(t)\partial_{S^i}\phi_{S^i}, \quad (28)$$
$$q'_{134,4t,i} = e^{-\int_0^t \alpha_s^i ds}\phi_{S^i}, \quad (29)$$
$$q'_{134,5t,i} = \hat{w}_i(t)\partial_{S^i}\phi_{S^i}, \quad (30)$$
$$q'_{135,1t,i} = \sum_{i=1}^{d}\bar{w}_i(t)\phi_{S^i}, \quad (31)$$
$$q'_{135,2t,i} = e^{-\int_0^t \alpha_s^i ds}\phi_{S^i}, \quad (32)$$
$$q'_{135,3t,i} = \hat{w}_i(t)\partial_{S^i}\phi_{S^i}, \quad (33)$$
$$q'_{135,4t,i} = e^{\lambda^i t}\phi_{\sigma^i}, \quad (34)$$
$$q'_{135,5t,i} = \bar{w}_i(t)e^{-\lambda^i t}\partial_{\sigma^i}\phi_{S^i}, \quad (35)$$
$$q'_{136,1t,i} = \sum_{i=1}^{d}\hat{w}_i(t)\phi_{S^i}, \quad (36)$$
$$q'_{136,2t,i} = e^{\lambda^i t}\phi_{\sigma^i}, \quad (37)$$
$$q'_{136,3t,i} = \bar{w}_i(t)e^{-\lambda^i t}\partial_{\sigma^i}\phi_{S^i}, \quad (38)$$
$$q'_{136,4t,i} = e^{\lambda^i t}\phi_{\sigma}, \quad (39)$$
$$q'_{136,5t,i} = \bar{w}_i(t)e^{-\lambda^i t}\partial_{\sigma^i}\phi_{S^i}. \quad (40)$$

For $M = 155, 156, 158$*,*

$$I_2(M) = \sum_{i=1}^{d}\left(\int_0^T q'_{M,4t,i}q_{M,1t,i}\int_0^t q'_{M,3s,i}q_{M,1s,i}\int_0^s q'_{M,2u,i}q_{M,1u,i}dudsdt\right),$$

where

$$q'_{155,1t,i} = \sum_{i=1}^{d}\bar{w}_i(t)\phi_{S^i}, \quad (41)$$
$$q'_{155,2t,i} = e^{-\int_0^t \alpha_s^i ds}\phi_{S^i}, \quad (42)$$
$$q'_{155,3t,i} = \partial_{S^i}\phi_{S^i}, \quad (43)$$
$$q'_{155,4t,i} = \hat{w}_i(t)\partial_{S^i}\phi_{S^i}, \quad (44)$$

$$
\begin{aligned}
q'_{156,1t,i} &= \sum_{i=1}^{d} \bar{w}_i(t)\phi_{S^i}, & (45)\\
q'_{156,2t,i} &= e^{\lambda^i t}\phi_{\sigma^i}, & (46)\\
q'_{156,3t,i} &= e^{-\int_0^t \alpha_s^i ds - \lambda^i t}\partial_{\sigma^i}\phi_{S^i}, & (47)\\
q'_{156,4t,i} &= \hat{w}_i(t)\partial_{S^i}\phi_{S^i}, & (48)\\
q'_{158,1t,i} &= \sum_{i=1}^{d} \bar{w}_i(t)\phi_{S^i}, & (49)\\
q'_{158,2t,i} &= e^{\lambda^i t}\phi_{\sigma^i}, & (50)\\
q'_{158,3t,i} &= \partial_{\sigma^i}\phi_{\sigma^i}, & (51)\\
q'_{158,4t,i} &= \bar{w}_i(t)e^{-\lambda^i t}\partial_{\sigma^i}\phi_{S^i}. & (52)
\end{aligned}
$$

For $M = 154, 157$,

$$
\begin{aligned}
I_2(M) &= \sum_{i=1}^{d}\left\{\int_0^T \left(\int_0^t q'_{M,2u,i}q_{M,1u,i}du\right)\left(\int_0^t q'_{M,3s,i}q_{M,1s,i}ds\right) q'_{M,4t,i}q_{M,1t,i}dt\right\},\\
I_0(M) &= \sum_{i=1}^{d}\left(\int_0^T\int_0^t q'_{M,2u,i}q_{M,3u,i}du q'_{M,4t,i}q_{M,1t,i}dt\right),
\end{aligned}
$$

where

$$
\begin{aligned}
q'_{154,1t,i} &= \sum_{i=1}^{d} \bar{w}_i(t)\phi_{S^i}, & (53)\\
q'_{154,2t,i} &= q_{154,3t,i} = e^{-\int_0^t \alpha_s^i ds}\phi_{S^i}, & (54)\\
q'_{154,4t,i} &= \hat{w}_i(t)e^{\int_0^t \alpha_s^i ds}\partial^2_{S^i}\phi_{S^i}, & (55)\\
q'_{157,1t,i} &= \sum_{i=1}^{d} \bar{w}_i(t)\phi_{S^i}, & (56)\\
q'_{157,2t,i} &= q_{157,3t,i} = e^{\lambda^i t}\phi_{\sigma^i}, & (57)\\
q'_{157,4t,i} &= \bar{w}_i(t)e^{-2\lambda^i t}\partial^2_{\sigma^i}\phi_{S^i}. & (58)
\end{aligned}
$$

The derivation of the functions I_0, I_2, *and* I_4 *is provided in the next subsection.*

Remark A.3. *When we do not have closed-forms for the multiple integrals on the time parameter that appear in the calculation of the conditional expectation formulas, some numerical method is necessary.*

However, all the multiple integrals necessary for the evaluation of $C_{1,k}, \cdots, C_{3,k}, C_4, \cdots, C_6$ *are computed by the program code with only one loop against the time parameter. For instance, a multiple integral is approximated for the numerical integration as follows:*

$$
\int_0^T f(s)\int_0^t g(u)\int_0^s h(v)dvduds
$$

$$\approx \sum_{i=1}^{M} \Delta_{t_i} f(t_i) \sum_{j=1}^{i} \Delta_{t_j} g(t_j) \sum_{k=1}^{j} \Delta_{t_k} h(t_k)$$
$$= \sum_{i=1}^{M} \Delta_{t_i} f(t_i) \left(G(t_{i-1}) + \Delta_{t_i} g(t_i) \left(H(t_{i-1}) + \Delta_{t_i} h(t_i) \right) \right),$$

where $\Delta_{t_i} = (t_i - t_{i-1})$, $H(t_i) = H(t_{i-1}) + \Delta_{t_i} h(t_j)$ *and* $G(t_i) = G(t_{i-1}) + \Delta_{t_i} g(t_i) H(t_i)$.

Hence, the order of the computational effort is at most M*, where* M *is the number of time-steps for the discretization in the numerical integral. Note that we have no problems in terms of computational complexity and speed since various fast numerical integration methods are available such as the extrapolation method.*

A.2 Details of the calculations

For ease of the expressions we introduce the following notations:

- $X^{(n)} = \frac{\partial^n}{\partial \epsilon^n} X^{(\epsilon)}\big|_{\epsilon=0}$, where X corresponds to S, σ, h.
- $\Phi_S := (\phi_{S^1}(\sigma, S), \cdots, \phi_{S^d}(\sigma, S))'$ and $\Phi_\sigma := (\phi_{\sigma^1}(\sigma), \cdots, \phi_{\sigma^d}(\sigma))'$ are $d \times 2d$ matrices.
- We define an operator "$*$" as follows: When A and B are $d \times 2d$ matrices,

$$A * B := \begin{bmatrix} (A)_{1,1}(B)_{1,1} & \cdots & (A)_{1,2d}(B)_{1,2d} \\ \vdots & \ddots & \vdots \\ (A)_{d,1}(B)_{d,1} & \cdots & (A)_{d,2d}(B)_{d,2d} \end{bmatrix}. \tag{59}$$

When A is a $d \times 2d$ matrix and B is a d-dimensional vector,

$$A * B = B * A := \begin{bmatrix} (A)_{1,1}(B)_1 & \cdots & (A)_{1,2d}(B)_1 \\ \vdots & \ddots & \vdots \\ (A)_{d,1}(B)_d & \cdots & (A)_{d,2d}(B)_d \end{bmatrix}. \tag{60}$$

When A and B are d-dimensional vectors,

$$A * B := \begin{bmatrix} (A)_1(B)_1 \\ \vdots \\ (A)_d(B)_d \end{bmatrix}. \tag{61}$$

- We also define $\partial_x \Phi_{\hat{x}}$ ($x = S$ or σ, $\hat{x} = S$ or σ) as

$$\partial_x \Phi_{\hat{x}} := \begin{bmatrix} \frac{\partial}{\partial x^1}(\Phi_{\hat{x}})_{1,1} & \cdots & \frac{\partial}{\partial x^1}(\Phi_{\hat{x}})_{1,2d} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x^d}(\Phi_{\hat{x}})_{d,1} & \cdots & \frac{\partial}{\partial x^d}(\Phi_{\hat{x}})_{d,2d} \end{bmatrix}, \tag{62}$$

where $(\Phi_{\hat{x}})_{i,j}$ denotes the (i,j)-element of the $d \times 2d$ matrix $\Phi_{\hat{x}}$.

- Let us introduce the following notations:
  $S_t = (S_t^1, \cdots, S_t^d)$, $\sigma_t = (\sigma_t^1, \cdots, \sigma_t^d)$,
  $S_t^{(j)} = (S_t^{1,(j)}, \cdots, S_t^{d,(j)})$, $\sigma_t^{(j)} = (\sigma_t^{1,(j)}, \cdots, \sigma_t^{d,(j)})$,
  $h_{S,j}^{(i)} = (h_{S^1,j}^{(i)}, \cdots, h_{S^d,j}^{(i)})$, $h_{\sigma,j}^{(i)} = (h_{\sigma^1,j}^{(i)}, \cdots, h_{\sigma^d,j}^{(i)})$,
  $e^{\int_0^t \alpha_s ds} = (e^{\int_0^t \alpha_s^1 ds}, \cdots, e^{\int_0^t \alpha_s^d ds})$ and $e^{\lambda t} = (e^{\lambda^1 t}, \cdots, e^{\lambda^d t})$.

Based on these preparations, we obtain the next proposition.

Proposition A.4. *1. The coefficients, $S_T^{(i)}$, $h_{x,j}^{(i)}$ $(x =, S, \sigma)$, $i = 0, 1, 2$ and $\sigma_T^{(i)}$ $i = 0, 1$ are given as follows:*

$$S_T^{(0)} = e^{\int_0^T \alpha_t dt} * s_0, \tag{63}$$

$$\sigma_T^{(0)} = \theta + (\sigma_0 - \theta) * e^{-\lambda T}, \tag{64}$$

$$h_{x,j}^{(0)} = 0, \tag{65}$$

$$S_T^{(1)} = \int_0^T e^{\int_t^T \alpha_s ds} * \Phi_S\left(\sigma_{t-}^{(0)}, S_{t-}^{(0)}\right) dW_t + \left(\sum_{j=1}^{N_T} h_{S,j}^{(1)} - \Lambda T \mathbf{E}\left[h_{S,1}^{(1)}\right]\right) * S_T^{(0)}, \tag{66}$$

$$\sigma_T^{(1)} = \int_0^T e^{-\lambda(T-t)} * \Phi_\sigma\left(\sigma_{t-}^{(0)}\right) dW_t + \left(\sum_{j=1}^{N_T} h_{\sigma,j}^{(1)} * e^{-\lambda(T-\tau_j)} * \sigma_{\tau_j-}^{(0)} - \Lambda \mathbf{E}\left[h_{\sigma,1}^{(1)}\right] * e^{-\lambda T} * \int_0^T e^{\lambda t} * \sigma_{t-}^{(0)} dt\right), \tag{67}$$

$$h_{x,j}^{(1)} = m_x := (m_{x^1}, \cdots, m_{x^d}), \ \textit{(for all } j, \textit{ constant jump case)} \tag{68}$$

$$h_{x,j}^{(1)} = Y_{x,j} := (Y_{x^1,j}, \cdots, Y_{x^d,j}), \ \textit{(log-normal jump case)} \tag{69}$$

$$\begin{aligned} S_T^{(2)} = {} & 2\int_0^T e^{\int_t^T \alpha_s ds} * \partial_S \Phi_S\left(\sigma_{t-}^{(0)}, S_{t-}^{(0)}\right) * S_{t-}^{(1)} dW_t \\ & + 2\int_0^T e^{\int_t^T \alpha_s ds} * \partial_\sigma \Phi_S\left(\sigma_{t-}^{(0)}, S_{t-}^{(0)}\right) * \sigma_{t-}^{(1)} dW_t \\ & + \Bigg(\left\{\sum_{j=1}^{N_T} h_{S,j}^{(2)} - \Lambda T \mathbf{E}\left[h_{S,1}^{(2)}\right]\right\} * S_T^{(0)} \\ & + 2\sum_{j=1}^{N_T} h_{S,j}^{(1)} * e^{\int_{\tau_j}^T \alpha_s ds} * S_{\tau_j-}^{(1)} - 2\Lambda \mathbf{E}\left[h_{S,1}^{(1)}\right] * \int_0^T e^{\int_t^T \alpha_s ds} * S_{t-}^{(1)} dt\Bigg), \end{aligned} \tag{70}$$

$$h_{x,j}^{(2)} = 0 \in \mathbf{R^d}, \ \textit{(for all } j, \textit{ constant jump case)} \tag{71}$$

$$h_{x,j}^{(2)} = Y_{x,j} * Y_{x,j}. \text{ (log-normal jump case)} \tag{72}$$

2. *The coefficients, $S_T^{LSV(i)}$ $(i = 1, 2, 3)$ and $\sigma_T^{LSV(i)}$ $(i = 1, 2)$ in the asymptotic expansions are given as follows:*

$$S_T^{LSV(1)} = \int_0^T e^{\int_t^T \alpha_s ds} * \Phi_S\left(\sigma_t^{(0)}, S_t^{(0)}\right) dW_t, \tag{73}$$

$$\sigma_T^{LSV(1)} = \int_0^T e^{-\lambda(T-t)} * \Phi_\sigma\left(\sigma_t^{(0)}\right) dW_t, \tag{74}$$

$$\begin{aligned} S_T^{LSV(2)} &= 2\int_0^T e^{\int_t^T \alpha_s ds} * \partial_S \Phi_S\left(\sigma_t^{(0)}, S_t^{(0)}\right) * S_t^{LSV(1)} dW_t \\ &\quad +2\int_0^T e^{\int_t^T \alpha_s ds} * \partial_\sigma \Phi_S\left(\sigma_t^{(0)}, S_t^{(0)}\right) * \sigma_t^{LSV(1)} dW_t, \end{aligned} \tag{75}$$

$$\begin{aligned} \sigma_T^{LSV(2)} &= 2\int_0^T e^{-\lambda(T-t)} * \partial_\sigma \Phi_\sigma\left(\sigma_t^{(0)}\right) * \sigma_t^{LSV(1)} dW_t \\ &= 2\int_0^T e^{-\lambda(T-t)} * \partial_\sigma \Phi_\sigma\left(\sigma_t^{(0)}\right) * \int_0^t e^{-\lambda(t-u)} * \Phi_\sigma\left(\sigma_u^{(0)}\right) dW_u dW_t, \end{aligned} \tag{76}$$

$$\begin{aligned} S_T^{LSV(3)} &= 6\int_0^T e^{\int_t^T \alpha_s ds} * \partial_S^2 \Phi_S\left(\sigma_t^{(0)}, S_t^{(0)}\right) * (S_t^{LSV(1)}) * (S_t^{LSV(1)}) dW_t \\ &\quad +6\int_0^T e^{\int_t^T \alpha_s ds} * \partial_S \Phi_S\left(\sigma_t^{(0)}, S_t^{(0)}\right) * S_t^{LSV(2)} dW_t \\ &\quad +6\int_0^T e^{\int_t^T \alpha_s ds} * \partial_\sigma^2 \Phi_S\left(\sigma_t^{(0)}, S_t^{(0)}\right) * (\sigma^{LSV(1)}) * (\sigma^{LSV(1)}) dW_t \\ &\quad +6\int_0^T e^{\int_t^T \alpha_s ds} * \partial_\sigma \Phi_S\left(\sigma_t^{(0)}, S_t^{(0)}\right) * \sigma^{LSV(2)} dW_t. \end{aligned} \tag{77}$$

In order to obtain the approximate value (8), all we have to do is to calculate the constants $C_{1,k}, \cdots, C_{3,k}, C_4, \cdots, C_6$. Hereafter, we derive those concrete expressions for a discrete average option, whose payoffs is defined as $(g(A_T) - K)^+ (:= \max\{g(A_T) - K, 0\})$, where the strike price K is defined by $K = \sum_{i=1}^d \sum_{j=1}^{m_i} w_j^{(i)} e^{\int_0^{t_j^{(i)}}} s_0^i - \epsilon \mathscr{Y}$ for an arbitrary $\mathscr{Y} \in \mathbf{R}$. We remark that a spread option is regarded as its special case.

Firstly, let us define $A_t^{i,(\epsilon)}$ as

$$A_t^{i,(\epsilon)} = \sum_{j=1}^{m_i} w_j^{(i)} S_{t_j^{(i)}}^{i,(\epsilon)} 1_{\{t_j^{(i)} \le t\}}. \tag{78}$$

Then, a call payoff function f is expanded as:

$$f(g(A_T^{(\epsilon)})) = \left(g(A_T^{(\epsilon)}) - K\right)^+$$

$$
\begin{aligned}
&= \epsilon\left(\frac{g(A_T^{(\epsilon)})-g(A_T^{(0)})}{\epsilon}+\mathscr{Y}\right)^+ \\
&= \epsilon\left(g(A_T^{(1)})+\mathscr{Y}\right)^+ + \epsilon^2 1_{\left\{g(A_T^{(1)})>-\mathscr{Y}\right\}} g\left(\frac{1}{2!}A_T^{(2)}\right) \\
&\quad +\epsilon^3\left(1_{\left\{g(A_T^{(1)})>-\mathscr{Y}\right\}} g\left(\frac{1}{3!}A_T^{(3)}\right)+\frac{1}{2}\delta_{\left\{g(A_T^{(1)})=-\mathscr{Y}\right\}} g\left(\frac{1}{2!}A_T^{(2)}\right)^2\right)+o(\epsilon^3),
\end{aligned} \tag{79}
$$

where

$$
\begin{aligned}
g(A_T^{(n)}) &= g\left(\frac{\partial^n}{\partial\epsilon^n}A_T^{(\epsilon)}\Big|_{\epsilon=0}\right) \\
&= \sum_{i=1}^{d}\sum_{j=1}^{m_i} w_j^{(i)}\frac{\partial^n}{\partial\epsilon^n}S^{i,(\epsilon)}_{t_j^{(i)}}\Big|_{\epsilon=0}1_{\{t_j^{(i)}\le T\}} \\
&= \sum_{i=1}^{d}\sum_{j=1}^{m_i} w_j^{(i)} S^{i,(n)}_{t_j^{(i)}}1_{\{t_j^{(i)}\le T\}}.
\end{aligned} \tag{80}
$$

Here, the strike price is expressed as $K = g(A_T^{(0)}) - \epsilon\mathscr{Y}$ for some $\mathscr{Y} \in \mathbf{R}$, and g is given by (6).

We use a lemma in Appendix B to obtain values of each conditional expectation, and obtain an approximate pricing formula for average call options with $\epsilon = 1$.

We set $k_t := \sum_{p=1}^{\infty}\mathbf{1}_{\{\tau_p\le t\}}$, $k = k_T$, and $\check{S} := \left(\check{S}^1,\cdots,\check{S}^d\right)$ defined as

$$
\check{S}_t^i := \sum_{q=1}^{2d}\int_0^t e^{\int_u^t \alpha_s ds}\phi_{S^i,q}\left(\sigma_{u-}^{(0)}, S_{u-}^{(0)}\right)dW_u^q + \sum_{p=1}^{k_t}\upsilon_{S^i}\zeta_{S^i,p}e^{\int_0^t \alpha_s^i ds}s_0^i. \tag{81}
$$

In this setting, we have $g(A_T^{(1)}) = g(\check{A}_T) + g(\xi_{\{k\},T})(= \sum_{i=1}^{d}\check{A}_T^i + \sum_{i=1}^{d}\xi^i_{\{k\},T})$, where

$$
\begin{aligned}
\check{A}_T^i &:= \sum_{j=1}^{m_i} w_j^{(i)}\check{S}^i_{t_j^{(i)}}1_{\{t_j^{(i)}\le T\}}, \\
\xi^i_{\{k\},T} &:= \sum_{j=1}^{m_i} w_j^{(i)}\left(k_{t_j^{(i)}} - \Lambda t_j^{(i)}\right)m_{S^i}e^{\int_0^{t_j^{(i)}}\alpha_s^i ds}s_0^i.
\end{aligned} \tag{82}
$$

Then, for ease of numerical computation, that is to avoid the multiple time-integrals appearing in Corollary 3.3, we approximate $\xi^i_{\{k\},t}$ and the distribution of $g(\check{A}_T)$: With the notations $\bar{w}_i(t) := \sum_{j=1}^{m_i} w_j^{(i)} 1_{\{t\le t_j^{(i)}\}}e^{\int_t^{t_j^{(i)}}\alpha_s ds}$ and $\tilde{w}_i := \frac{1}{T}\sum_{j=1}^{m_i} w_j^{(i)} t_j^{(i)} e^{\int_0^{t_j^{(i)}}\alpha_s ds}$, let us approximate $\xi^i_{\{k\},t}$ by $\xi^i_{\{k\}}$, which is defined as

$$
\xi^i_{\{k\}} := \tilde{w}_i\left(k - \Lambda T\right)m_{S^i}s_0^i, \tag{83}
$$

where the number of jumps until $t_j^{(i)}$ (i.e. $k_{t_j^{(i)}}$) is replaced by the number of jumps until T (i.e. $k = k_T$). Moreover, the distribution of $g(\check{A}_T)$ is approximated as $N\left(0, \Sigma_T^{\{k\}}\right)$, that is the normal distribution with mean zero and variance $\Sigma_T^{\{k\}}$ defined as

$$\begin{aligned}\Sigma_T^{\{k\}} &:= \sum_{i=1}^{d}\sum_{q=1}^{2d}\sum_{i'=1}^{d}\sum_{q'=1}^{2d}\int_0^T \left(\bar{w}_i(t)\phi_{S^i,q}\left(\sigma_t^{(0)}, S_t^{(0)}\right)\right)\left(\bar{w}_{i'}(t)\phi_{S^{i'},q'}\left(\sigma_t^{(0)}, S_t^{(0)}\right)\right)dt \\ &\quad + \sum_{i=1}^{d}\sum_{i'=1}^{d} k\left(\tilde{w}_i \upsilon_{S^i} s_0^i\right)\vartheta_{S^i,S^{i'}}\left(\tilde{w}_{i'}\upsilon_{S^{i'}} s_0^{i'}\right). \end{aligned} \tag{84}$$

(In the case of $d = 1$ and $n = 1$, $\Sigma_T^{\{k\}}$ corresponds to $\Sigma_T^{\{k\}}(T)$ in Remark B.2 of Appendix B.) In sum, given $\{N_T = k\}$, we obtain an approximation for the distribution of $g(A_T^{(1)})$ as $N(g(\xi_{\{k\}}), \Sigma_T^{\{k\}})$ with $g(\xi_{\{k\}}) = \sum_{i=1}^d \xi_{\{k\}}^i$, i.e. in the subsequent analysis, we treat its distribution as if $g(A_T^{(1)}) \sim N(g(\xi_{\{k\}}), \Sigma_T^{\{k\}})$.

Next, let us evaluate the expectation of each term in the right hand side of (79): For the ϵ-term, with $y_k := \mathscr{Y} + g(\xi_{\{k\}})$,

$$\begin{aligned}&\epsilon\mathbf{E}\left[\left(g(A_T^{(1)}) + \mathscr{Y}\right)^+\right] \\ &= \epsilon\sum_{k=0}^{\infty} p_{\{k\}}\frac{k!}{T^k}\int_0^{\vec{T}}\int_{\mathbf{R}}(x+\mathscr{Y})^+ p^{A^{(1)}_{\{\tau=\vec{t}\},T}}(x)dx d\vec{t} \\ &\approx \epsilon\sum_{k=0}^{\infty} p_{\{k\}}\int_{\mathbf{R}}(x+\mathscr{Y})^+ n(x: g(\xi_k), \Sigma_T^{\{k\}})dx \\ &= \epsilon\sum_{k=0}^{\infty} p_{\{k\}}\int_{-y_k}^{\infty}(x+y_k)n(x:0,\Sigma_T^{\{k\}})dx \\ &= \epsilon\sum_{k=0}^{\infty} p_{\{k\}}\left\{y_k N\left(\frac{y_k}{\sqrt{\Sigma_T^{\{k\}}}}\right) + \Sigma_T^{\{k\}} n(y_k; 0, \Sigma_T^{\{k\}})\right\}, \end{aligned} \tag{85}$$

where $\int_0^{\vec{T}} = \int_0^T\int_0^{t_k}\cdots\int_0^{t_2}$, $d\vec{t} = dt_1\cdots dt_k$, $p_{\{k\}} = \frac{(\Lambda T)^k e^{-\Lambda T}}{k!}$, and $p^{A^{(1)}_{\{\tau=\vec{t}\},T}}(x)$ stands for the density function of $A_T^{(1)}$ conditioned on $\{\tau = \vec{t}\} = \{\tau_1 = t_1, \cdots, \tau_k = t_k\}$ $(0 < t_1 < \cdots < t_k \le T)$ with $N_T = k$.

For the ϵ^2-term and one of the ϵ^3-terms, with $A_T^{(m)} = \frac{\partial^m}{\partial\epsilon^m}A_T^{(\epsilon)}\big|_{\epsilon=0}$ $(m = 2, 3)$,

$$\begin{aligned}&\epsilon^m\mathbf{E}\left[1_{\left\{g(A_T^{(1)})>-\mathscr{Y}\right\}} g\left(\frac{1}{m!}A_T^{(m)}\right)\right] \\ &= \epsilon^m\sum_{k=0}^{\infty} p_{\{k\}}\frac{k!}{T^k}\int_0^{\vec{T}}\int_{\mathbf{R}} 1_{\{g(A_T^{(1)})>-\mathscr{Y}\}} g\left(\frac{1}{m!}A_T^{(m)}\right) p^{A^{(1)}_{\{\tau=\vec{t}\},T}}(x)dx d\vec{t}\end{aligned}$$

$$
\begin{aligned}
&= \epsilon^m \sum_{k=0}^{\infty} p_{\{k\}} \frac{k!}{T^k} \int_0^{\vec{T}} \int_{-\mathscr{Y}}^{\infty} \mathbf{E}\left[g\left(\frac{1}{m!} A_T^{(m)}\right) | g\left(A_T^{(1)}\right) = x, N_T = k, \{\tau = \vec{t}\}\right] p^{A^{(1)}_{\{\tau=\vec{t}\},T}}(x) dx d\vec{t} \\
&\approx \epsilon^m \sum_{k=0}^{\infty} p_{\{k\}} \frac{k!}{T^k} \int_0^{\vec{T}} \int_{-\mathscr{Y}}^{\infty} \mathbf{E}\left[g\left(\frac{1}{m!} A_T^{(m)}\right) | g\left(A_T^{(1)}\right) = x, N_T = k, \{\tau = \vec{t}\}\right] n(x : g(\xi_k), \Sigma_T^{\{k\}}) dx d\vec{t} \\
&= \epsilon^m \sum_{k=0}^{\infty} p_{\{k\}} \frac{k!}{T^k} \int_0^{\vec{T}} \int_{-y_k}^{\infty} \mathbf{E}\left[g\left(\frac{1}{m!} A_T^{(m)}\right) | g\left(\check{A}_T^{(1)}\right) = x, N_T = k, \{\tau = \vec{t}\}\right] n(x : 0, \Sigma_T^{\{k\}}) dx d\vec{t} \\
&= \epsilon^m \sum_{k=0}^{\infty} p_{\{k\}} \int_{-y_k}^{\infty} \frac{k!}{T^k} \int_0^{\vec{T}} \mathbf{E}\left[g\left(\frac{1}{m!} A_T^{(m)}\right) | g\left(\check{A}_T^{(1)}\right) = x, N_T = k, \{\tau = \vec{t}\}\right] d\vec{t}\ n(x : 0, \Sigma_T^{\{k\}}) dx.
\end{aligned}
\tag{86}
$$

For the remaining ϵ^3-term,

$$
\begin{aligned}
&\epsilon^3 \mathbf{E}\left[\delta_{\left\{g(A_T^{(1)}) = -\mathscr{Y}\right\}} g\left(\frac{1}{2!} A_T^{(2)}\right)^2 \right] \\
&= \epsilon^3 \sum_{k=0}^{\infty} p_{\{k\}} \frac{k!}{T^k} \int_0^{\vec{T}} \int_{\mathbf{R}} \delta_{\{g(A_T^{(1)}) = -\mathscr{Y}\}} g\left(\frac{1}{2!} A_T^{(2)}\right)^2 p^{A^{(1)}_{\{\tau=\vec{t}\},T}}(x) dx d\vec{t} \\
&= \epsilon^3 \sum_{k=0}^{\infty} p_{\{k\}} \frac{k!}{T^k} \int_0^{\vec{T}} \int_{\mathbf{R}} \delta_{\{g(A_T^{(1)}) = -\mathscr{Y}\}} \\
&\quad \times \mathbf{E}\left[g\left(\frac{1}{2!} A_T^{(2)}\right)^2 | g\left(A_T^{(1)}\right) = x, N_T = k, \{\tau = \vec{t}\}\right] p^{A^{(1)}_{\{\tau=\vec{t}\},T}}(x) dx d\vec{t} \\
&\approx \epsilon^3 \sum_{k=0}^{\infty} p_{\{k\}} \frac{k!}{T^k} \int_0^{\vec{T}} \int_{\mathbf{R}} \delta_{\{g(A_T^{(1)}) = -\mathscr{Y}\}} \\
&\quad \times \mathbf{E}\left[g\left(\frac{1}{2!} A_T^{(2)}\right)^2 | g\left(A_T^{(1)}\right) = x, N_T = k, \{\tau = \vec{t}\}\right] n(x : g(\xi_k), \Sigma_T^{\{k\}}) dx d\vec{t} \\
&= \epsilon^3 \sum_{k=0}^{\infty} p_{\{k\}} \frac{k!}{T^k} \int_0^{\vec{T}} \int_{\mathbf{R}} \delta_{\{g(A_T^{(1)}) = -y_k\}} \\
&\quad \times \mathbf{E}\left[g\left(\frac{1}{2!} A_T^{(2)}\right)^2 | g\left(\check{A}_T^{(1)}\right) = x, N_T = k, \{\tau = \vec{t}\}\right] n(x : 0, \Sigma_T^{\{k\}}) dx d\vec{t} \\
&= \epsilon^3 \sum_{k=0}^{\infty} p_{\{k\}} \frac{k!}{T^k} \int_0^{\vec{T}} \mathbf{E}\left[g\left(\frac{1}{2!} A_T^{(2)}\right)^2 | g\left(\check{A}_T^{(1)}\right) = -y_k, N_T = k, \{\tau = \vec{t}\}\right] d\vec{t} \\
&\quad \times n(-y_k : 0, \Sigma_T^{\{k\}}).
\end{aligned}
\tag{87}
$$

Moreover, since we use LSV model for the coefficient of ϵ^3 to reduce the computational costs, we evaluate the following equation for an approximation for average call prices: with the notation $\Sigma_T = \Sigma_T^{\{0\}}$,

$$\mathbf{E}\left[\left(g(A_T^{(\epsilon)}) - K\right)^+\right]$$

$$\begin{aligned}
&\approx \epsilon\sum_{k=0}^{\infty} p_{\{k\}}\left\{y_k N\left(\frac{y_k}{\sqrt{\Sigma_T^{\{k\}}}}\right) + \Sigma_T^{\{k\}} n(y_k; 0, \Sigma_T^{\{k\}})\right\} \\
&+\epsilon^2\sum_{k=0}^{\infty} p_{\{k\}}\int_{-y_k}^{\infty}\frac{k!}{T^k}\int_0^{\vec{T}}\mathbf{E}\left[g\left(\frac{1}{2!}A_T^{(2)}\right)|g\left(\check{A}_T^{(1)}\right) = x, N_T = k, \{\tau = \vec{t}\}\right]\vec{dt}\; n(x:0,\Sigma_T^{\{k\}})dx \\
&+\epsilon^3\int_{-y_k}^{\infty}\mathbf{E}\left[g\left(\frac{1}{3!}A_T^{LSV(3)}\right)|g\left(\check{A}_T^{LSV(1)}\right) = x\right]\; n(x:0,\Sigma_T)dx \\
&+\epsilon^3\mathbf{E}\left[g\left(\frac{1}{2!}A_T^{LSV(2)}\right)^2|g\left(\check{A}_T^{LSV(1)}\right) = -\mathscr{Y}\right] n(-\mathscr{Y}:0,\Sigma_T). \qquad (88)
\end{aligned}$$

In the rest of this subsection, we show the details of the calculation for the right hand side of the equation above. (In the following we omit some notations for simplicity.)

First, we define $\check{w}_i(t) := \sum_{j=1}^{m_i} w_j^{(i)} 1_{\{t\le t_j^{(i)}\}} e^{\int_0^t \alpha_s ds}$ and get an expression of $g\left(\frac{1}{2!}A_T^{(2)}\right)$ as follows:

$$\begin{aligned}
g\left(\frac{1}{2!}A_T^{(2)}\right) &= \int_0^T \bar{w}(t)*\partial_S\Phi_S*\int_0^t e^{\int_u^t \alpha_s ds}*\Phi_S dW_u dW_t && (89)\\
&+\int_0^T \bar{w}(t)*\partial_S\Phi_S*\sum_{j=1}^{N_t} h_{S,j}^{(1)}*e^{\int_{\tau_j}^t \alpha_s ds}*S_{\tau_j-}^{(0)} dW_t && (90)\\
&-\int_0^T \bar{w}(t)*\partial_S\Phi_S*\Lambda\mathbf{E}[h_{S,1}^{(1)}]*\int_0^t e^{\int_u^t \alpha_s ds}*S_{u-}^{(0)} du dW_t && (91)\\
&+\int_0^T \bar{w}(t)*\partial_\sigma\Phi_S*\int_0^t e^{-\lambda(t-u)}*\Phi_\sigma dW_u dW_t && (92)\\
&+\int_0^T \bar{w}(t)*\partial_\sigma\Phi_S*\sum_{j=1}^{N_t} h_{\sigma,j}^{(1)}*e^{-\lambda(t-\tau_j)}*\sigma_{\tau_j-}^{(0)} dW_t && (93)\\
&-\int_0^T \bar{w}(t)*\partial_\sigma\Phi_S*\Lambda\mathbf{E}[h_{\sigma,1}^{(1)}]*e^{-\lambda t}*\int_0^t e^{\lambda u-}*\sigma_{u-}^{(0)} du dW_t && (94)\\
&+\sum_{j=1}^{N_T}\check{w}(\tau_j)*h_{S,j}^{(2)}*s_0 && (95)\\
&-\tilde{w}*\Lambda\mathbf{E}[h_{S,1}^{(2)}]*s_0 T && (96)\\
&+\sum_{j=1}^{N_T}\check{w}(\tau_j)*h_{S,j}^{(1)}*\int_0^{\tau_j} e^{\int_0^u \alpha_s ds}*\Phi_S dW_u && (97)\\
&+\sum_{j=1}^{N_T}\check{w}(\tau_j)*h_{S,j}^{(1)}*\sum_{m=1}^{N_{\tau_j}} h_{S,m}^{(1)}*s_0 && (98)
\end{aligned}$$

$$-\sum_{j=1}^{N_T} \breve{w}(\tau_j) * h_{S,j}^{(1)} * \Lambda \mathbf{E}[h_{S,1}^{(1)}] * s_0 * \tau_j \tag{99}$$

$$+\int_0^T \tilde{w} \Lambda \mathbf{E}[h_{S,1}^{(1)}] * \int_0^t e^{\int_0^u \alpha_s ds} * \Phi_S dW_u dt \tag{100}$$

$$-\int_0^T \tilde{w} \Lambda \mathbf{E}[h_{S,1}^{(1)}] * \sum_{m=1}^{N_t} h_{S,m}^{(1)} * s_0 dt \tag{101}$$

$$+\int_0^T \tilde{w} \Lambda \mathbf{E}[h_{S,1}^{(1)}] * \Lambda \mathbf{E}[h_{S,1}^{(1)}] * s_0 dt, \tag{102}$$

where

$$\mathbf{E}\left[h_{x^i,1}^{(\epsilon)}\right] = \mathbf{E}\left[e^{\epsilon Y_{x^i,1}} - 1\right] = e^{\epsilon m_{x^i} + \frac{1}{2}\epsilon^2 \upsilon_{x^i}^2} - 1, \tag{103}$$

$$\mathbf{E}\left[h_{x^i,1}^{(0)}\right] = \mathbf{E}\left[h_{x^i,1}^{(\epsilon)}\right]\Big|_{\epsilon=0} = 1 - 1 = 0, \tag{104}$$

$$\mathbf{E}\left[h_{x^i,1}^{(1)}\right] = \partial_\epsilon \mathbf{E}\left[h_{x^i,1}^{(\epsilon)}\right]\Big|_{\epsilon=0} = (m_{x^i} + \epsilon \upsilon_{x^i}^2) e^{\epsilon m_{x^i} + \frac{1}{2}\epsilon^2 \upsilon_{x^i}^2}\Big|_{\epsilon=0} = m_{x^i}, \tag{105}$$

$$\begin{aligned}\mathbf{E}\left[h_{x^i,1}^{(2)}\right] &= \partial_\epsilon^2 \mathbf{E}\left[h_{x^i,1}^{(\epsilon)}\right]\Big|_{\epsilon=0} = \left(\upsilon_{x^i}^2 e^{\epsilon m_{x^i} + \frac{1}{2}\epsilon^2 \upsilon_{x^i}^2} + (m_{x^i} + \epsilon \upsilon_{x^i}^2)^2 e^{\epsilon m_{x^i} + \frac{1}{2}\epsilon^2 \upsilon_{x^i}^2}\right)\Big|_{\epsilon=0} \\ &= m_{x^i}^2 + \upsilon_{x^i}^2. \end{aligned} \tag{106}$$

Next, we define the expression $F(X)$ as

$$F(X) := \frac{k!}{T^k} \int_0^{\vec{T}} \mathbf{E}\left[X | g(\check{A}_T^{(1)}) = x, N_T = k, \{\tau = \vec{t}\}\right] d\vec{t}, \tag{107}$$

where X stands for the expression in the equation number (X). Then, we obtain the following approximations of $F(g(A_T^{(2)}))$ by using formulas in Lemma B.1 of Appendix B with the approximation of Remark B.2:

$$\begin{aligned} F(g(A_T^{(2)})) \approx\ & G(89) + G(90) + G(91) + G(92) + G(93) + G(94) + G(95) \\ & + G(96) + G(97) + G(98) + G(99) + G(100) + G(101) + G(102), \end{aligned} \tag{108}$$

where $G(89) \sim G(102)$ are provided with the k-th order Hermite polynomial $H_k\left(x; \Sigma_T^{\{k\}}\right)$ as follows:

$$G(89) = \sum_{i=1}^d \int_0^T \bar{w}_i(t) \partial_{S^i} \phi_{S^i} \sum_{I=1}^d \bar{w}_I(t) \phi'_{S^I} \int_0^t e^{\int_u^t \alpha_s^i ds} \phi_{S^i} \sum_{J=1}^d \bar{w}_J(u) \phi'_{S^J} du dt \frac{H_2(x, \Sigma_T^{\{k\}})}{(\Sigma_T^{\{k\}})^2}, \tag{109}$$

$$G(90) = \sum_{i=1}^d \int_0^T \bar{w}_i(t) \partial_{S^i} \phi_{S^i} \frac{kt}{T} m_{S^i} s_0^i \sum_{I=1}^d \bar{w}_I(t) \phi'_{S^I} dt \frac{H_1(x, \Sigma_T^{\{k\}})}{\Sigma_T^{\{k\}}}, \tag{110}$$

$$+\sum_{i=1}^d \int_0^T \bar{w}_i(t) \partial_{S^i} \phi_{S^i} \sum_{I=1}^d \bar{w}_I(t) \phi'_{S^I} \frac{kt}{T} e^{\int_0^t \alpha_s^i ds} \upsilon_{S^i} s_0^i \sum_{J=1}^d \tilde{w}_J \vartheta_{S^i,S^J} \upsilon_{S^J} s_0^J dt \frac{H_2(x, \Sigma_T^{\{k\}})}{(\Sigma_T^{\{k\}})^2}, \tag{111}$$

$$G(91) = \sum_{i=1}^d \int_0^T \bar{w}_i(t) \partial_{S^i} \phi_{S^i} \sum_{I=1}^d \bar{w}_I(t) \phi'_{S^I} \Lambda m_{S^i} e^{\int_0^t \alpha_s^i ds} s_0^i t dt \frac{H_1(x, \Sigma_T^{\{k\}})}{\Sigma_T^{\{k\}}}, \tag{112}$$

$$G(92) = \sum_{i=1}^{d}\int_0^T \bar{w}_i(t)\partial_{\sigma^i}\phi_{S^i}\sum_{I=1}^{d}\bar{w}_I(t)\phi'_{S^I}\int_0^t e^{-\lambda^i(t-u)}\phi_{\sigma^i}\sum_{J=1}^{d}\bar{w}_J(t)\phi'_{S^J}dudt\frac{H_2(x,\Sigma_T^{\{k\}})}{(\Sigma_T^{\{k\}})^2}, \quad (113)$$

$$G(93) = \sum_{i=1}^{d}\int_0^T \bar{w}_i(t)\partial_{\sigma^i}\phi_{S^i}\sum_{I=1}^{d}\bar{w}_(t)\phi'_{S^I}\int_0^t \frac{k}{T}m_{\sigma^i}e^{-\lambda^i(t-u)}\sigma_u^{i,(0)}dudt\frac{H_1(x,\Sigma_T^{\{k\}})}{\Sigma_T^{\{k\}}} \quad (114)$$

$$+\sum_{i=1}^{d}\int_0^T \bar{w}_i(t)\partial_{\sigma^i}\phi_{S^i}\sum_{I=1}^{d}\bar{w}_I(t)\phi'_{S^I}$$

$$\times\int_0^t \frac{k}{T}\upsilon_{\sigma^i}e^{-\lambda^i(t-u)}\sigma_u^{i,(0)}\sum_{J=1}^{d}\vartheta_{S^i,\sigma^J}\upsilon_{S^J}\tilde{w}_J s_0^J dudt\frac{H_2(x,\Sigma_T^{\{k\}})}{(\Sigma_T^{\{k\}})^2}, \quad (115)$$

$$G(94) = \sum_{i=1}^{d}\int_0^T \bar{w}_i(t)\partial_{\sigma^i}\phi_{S^i}\sum_{I=1}^{d}\bar{w}_I(t)\phi'_{S^I}\Lambda m_{\sigma^i}\int_0^t e^{-\lambda^i(t-u)}\sigma_{u-}^{i,(0)}dudt\frac{H_1(x,\Sigma_T^{\{k\}})}{\Sigma_T^{\{k\}}}, \quad (116)$$

$$G(95) = \sum_{i=1}^{d}\tilde{w}_i k m_{S^i}^2 s_0^i \quad (117)$$

$$+\sum_{i=1}^{d}\tilde{w}_i k 2 m_{S^i}\upsilon_{S^i}s_0^i\sum_{I=1}^{d}\vartheta_{S^i,S^I}\tilde{w}_I\upsilon_{S^I}s_0^I\frac{H_1(x,\Sigma_T^{\{k\}})}{\Sigma_T^{\{k\}}} \quad (118)$$

$$+\sum_{i=1}^{d}k(\upsilon_{S^i})^2(\tilde{w}_i)^2(s_0^i)^2+\sum_{i=1}^{d}k\left(\upsilon_{S^i}\tilde{w}_i s_0^i\sum_{I=1}^{d}\vartheta_{S^i,S^I}\upsilon_{S^I}\tilde{w}_I s_0^I\right)^2\frac{H_2(x,\Sigma_T^{\{k\}})}{(\Sigma_T^{\{k\}})^2}, \quad (119)$$

$$G(96) = \sum_{i=1}^{d}\tilde{w}_i\Lambda(m_{S^i}^2+\upsilon_{S^i}^2)s_0^i T, \quad (120)$$

$$G(97) = \sum_{i=1}^{d}\int_0^T \tilde{w}_i m_{S^i}\int_0^t e^{-\int_0^u \alpha_s ds}\phi_{S^i}\sum_{I=1}^{d}\bar{w}_I(t)\phi'_{S^I}dudt\frac{k}{T}\frac{H_1(x,\Sigma_T^{\{k\}})}{\Sigma_T^{\{k\}}} \quad (121)$$

$$+\sum_{i=1}^{d}\int_0^T \tilde{w}_i\upsilon_{S^i}\sum_{I=1}^{d}T\tilde{w}_I\vartheta_{S^i,S^I}\upsilon_{S^I}s_0^I\sum_{J=1}^{d}\int_0^t e^{\int_0^u \alpha_s^i ds}\phi_{S^i}\bar{w}_J(t)\phi'_{S^J}dudt\frac{k}{T}\frac{H_2(x,\Sigma_T^{\{k\}})}{(\Sigma_T^{\{k\}})^2}, \quad (122)$$

$$G(98) = \sum_{i=1}^{d}\tilde{w}_i s_0^i\sum_{j=1}^{k}\sum_{I=1}^{j-1}m_{S^i}m_{S^i} \quad (123)$$

$$+2\sum_{i=1}^{d}\tilde{w}_i s_0^i\sum_{j=1}^{k}\sum_{I=1}^{j-1}m_{S^i}\upsilon_{S^i}\sum_{J=1}^{d}\vartheta_{S^i,S^J}\upsilon_{S^J}\tilde{w}_J s_0^J\frac{H_1(x,\Sigma_T^{\{k\}})}{\Sigma_T^{\{k\}}} \quad (124)$$

$$+\sum_{i=1}^{d}\tilde{w}_i s_0^i\sum_{j=1}^{k}\sum_{I=1}^{j-1}\upsilon_{S^i}\sum_{\iota=1}^{d}\vartheta_{S^i,S^\iota}\upsilon_{S^\iota}s_0^\iota\upsilon_{S^i}\sum_{J=1}^{d}\vartheta_{S^i,S^J}\upsilon_{S^J}\tilde{w}_J s_0^J\frac{H_2(x,\Sigma_T^{\{k\}})}{(\Sigma_T^{\{k\}})^2}, \quad (125)$$

$$G(99) = \sum_{i=1}^{d}m_{S^i}\tilde{w}_i\Lambda m_{S^i}s_0^i\frac{kT}{2} \quad (126)$$

$$+\sum_{i=1}^{d}\upsilon_{S^i}\tilde{w}_i\sum_{I=1}^{d}\vartheta_{S^i,S^I}\tilde{w}_I\upsilon_{S^I}s_0^I\Lambda m_{S^i}s_0^i\frac{kT}{2}\frac{H_1(x,\Sigma_T^{\{k\}})}{\Sigma_T^{\{k\}}}, \quad (127)$$

$$G(100) = \sum_{i=1}^{d}T\tilde{w}_i\Lambda m_{S^i,j}\int_0^T e^{-\int_0^t \alpha_s^i ds}\phi_{S^i}\sum_{I=1}^{d}\bar{w}_I(t)\phi'_{S^I}dt\frac{H_1(x,\Sigma_T^{\{k\}})}{\Sigma_T^{\{k\}}} \quad (128)$$

$$-\sum_{i=1}^{d}\tilde{w}_i\int_0^T t\Lambda m_{S^i,j}e^{\int_0^t \alpha_s ds}\phi_{S^i}\sum_{I=1}^{d}\bar{w}_I(t)\phi'_{S^I}dt\frac{H_1(x,\Sigma_T^{\{k\}})}{\Sigma_T^{\{k\}}}, \quad (129)$$

$$
\begin{aligned}
G(101) &= \sum_{i=1}^{d} \tilde{w}_i \Lambda m_{S^i} s_0^i m_{S^i} \frac{kT}{2} && (130)\\
&\quad + \sum_{i=1}^{d} \tilde{w}_i s_0^i \Lambda m_{S^i} \upsilon_{S^i} \sum_{J=1}^{d} \vartheta_{S^i,S^J} \upsilon_{S^J} \tilde{w}_J s_0^J \frac{kT}{2} \frac{H_1(x, \Sigma_T^{\{k\}})}{\Sigma_T^{\{k\}}}, && (131)\\
G(102) &= \sum_{i=1}^{d} \tilde{w}_i \int_0^T \Lambda m_{S^i} \Lambda m_{S^i} t s_0^i dt. && (132)
\end{aligned}
$$

The results of the constant jump case can be derived in a similar way to that of the log normal jumps, and we omit them.

Next, let us show the expression of $F\left(g\left(\frac{1}{2!}A_T^{LSV(2)}\right)^2\right)$ by applying the formulas in Appendix B of Shiraya-Takahashi [1]. Hereafter, the expression $F(X)$ as

$$
F(X) := \mathbf{E}\left[X | g(\check{A}_T^{LSV,(1)}) = x\right]. \quad (133)
$$

Then,

$$
\begin{aligned}
F\left(g\left(\frac{1}{2!}A_T^{LSV(2)}\right)^2\right) &= F\left(g\left(\int_0^T \bar{w}(t) * \partial_S \Phi_S * \int_0^t e^{\int_u^t \alpha_s ds} * \Phi_S dW_u dW_t\right)^2\right) && (134)\\
&\quad +2F\left(\left(\int_0^T \bar{w}(t) * \partial_S \Phi_S * \int_0^t e^{\int_u^t \alpha_s ds} * \Phi_S dW_u dW_t\right)\right.\\
&\qquad \times \left.\left(\int_0^T \bar{w}(t) * \partial_\sigma \Phi_S * \int_0^t e^{-\lambda(t-u)} * \Phi_\sigma dW_u dW_t\right)\right) && (135)\\
&\quad +F\left(g\left(\int_0^T \bar{w}(t) * \partial_\sigma \Phi_S * \int_0^t e^{-\lambda(t-u)} * \Phi_\sigma dW_u dW_t\right)^2\right) && (136)\\
&= G(134) + 2G(135) + G(136). && (137)
\end{aligned}
$$

Here, we define $\hat{w}_i(t) := \sum_{j=1}^{m_i} w_j^{(i)} 1_{\{t \le t_j^{(i)}\}} e^{\int_0^{t_j^{(i)}} \alpha_s ds}$.

Then, we obtain the expressions of $G(M)$ for $M = 134, 135, 136$ as follows:

$$
\begin{aligned}
G(M) &= \sum_{i=1}^{d}\sum_{I=1}^{d} \left(\int_0^T q'_{M,3t,i} q_{M,1t,i} \int_0^t q'_{M,2s,i} q_{M,1s,i} ds dt\right)\\
&\quad \left(\int_0^T q'_{M,5r,I} q_{M,1r,I} \int_0^r q'_{M,4u,I} q_{M,1u,I} du dr\right) \frac{H_4(x; \Sigma_T)}{(\Sigma_T)^4}\\
&\quad + \sum_{i=1}^{d}\sum_{I=1}^{d} \left\{ \int_0^T q'_{M,3t,i} q_{M,1t,i} \int_0^t q'_{M,5r,I} q_{M,1r,I} \int_0^r q'_{M,2u,i} q_{M,4u,I} du dr dt \right.\\
&\quad + \int_0^T q'_{M,5t,I} q_{M,1t,I} \int_0^t q'_{M,3r,i} q_{M,1r,i} \int_0^r q'_{M,2u,i} q_{M,4u,I} du dr dt\\
&\quad + \int_0^T q'_{M,3t,i} q_{M,1t,i} \int_0^t q'_{M,2r,i} q_{M,5r,I} \int_0^r q'_{M,4u,I} q_{M,1u,I} du dr dt
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^T q'_{M,3t,i}q_{M,5t,I}\left(\int_0^t q'_{M,2s,i}q_{1s}ds\right)\left(\int_0^t q'_{M,4u,I}q_{M,1u,I}du\right)dt \\
&\left.+\int_0^T q'_{M,5r,I}q_{M,1r,I}\int_0^r q'_{M,3u,i}q_{M,4u,I}\int_0^u q'_{M,2s,i}q_{M,1s,i}dsdudr\right\}\frac{H_2(x;\Sigma_T)}{(\Sigma_T)^2} \\
&+\sum_{i=1}^d\sum_{I=1}^d\int_0^T\int_0^t q'_{M,2u,i}q_{M,4u,I}duq'_{M,3t,i}q_{M,5t,I}dt \\
= \;& I_4(M)\frac{H_4(x;\Sigma_T)}{(\Sigma_T)^4}+I_2(M)\frac{H_2(x;\Sigma_T)}{(\Sigma_T)^2}+I_0(M), \qquad (138)
\end{aligned}
$$

where

$$
\Sigma_T := \sum_{i=1}^d\sum_{q=1}^{2d}\sum_{i'=1}^d\sum_{q'=1}^{2d}\int_0^T\left(\bar{w}_i(t)\phi_{S^i,q}\left(\sigma_t^{(0)},S_t^{(0)}\right)\right)\left(\bar{w}_{i'}(t)\phi_{S^{i'},q'}\left(\sigma_t^{(0)},S_t^{(0)}\right)\right)dt.
$$

That is, for $M=134,135,136$,

$$
\begin{aligned}
I_4(M) &= \sum_{i=1}^d\sum_{I=1}^d\left(\int_0^T q'_{M,3t,i}q_{M,1t,i}\int_0^t q'_{M,2s,i}q_{M,1s,i}dsdt\right)\times \\
&\quad\left(\int_0^T q'_{M,5r,I}q_{M,1r,I}\int_0^r q'_{M,4u,I}q_{M,1u,I}dudr\right), \\
I_2(M) &= \sum_{i=1}^d\sum_{I=1}^d\left\{\int_0^T q'_{M,3t,i}q_{M,1t,i}\int_0^t q'_{M,5r,I}q_{M,1r,I}\int_0^r q'_{M,2u,i}q_{M,4u,I}dudrdt\right. \\
&\quad+\int_0^T q'_{M,5t,I}q_{M,1t,I}\int_0^t q'_{M,3r,i}q_{M,1r,i}\int_0^r q'_{M,2u,i}q_{M,4u,I}dudrdt \\
&\quad+\int_0^T q'_{M,3t,i}q_{M,1t,i}\int_0^t q'_{M,2r,i}q_{M,5r,I}\int_0^r q'_{M,4u,I}q_{M,1u,I}dudrdt \\
&\quad+\int_0^T q'_{M,3t,i}q_{M,5t,I}\left(\int_0^t q'_{M,2s,i}q_{1s}ds\right)\left(\int_0^t q'_{M,4u,I}q_{M,1u,I}du\right)dt \\
&\quad\left.+\int_0^T q'_{M,5r,I}q_{M,1r,I}\int_0^r q'_{M,3u,i}q_{M,4u,I}\int_0^u q'_{M,2s,i}q_{M,1s,i}dsdudr\right\}, \\
I_0(M) &= \sum_{i=1}^d\sum_{I=1}^d\int_0^T\int_0^t q'_{M,2u,i}q_{M,4u,I}duq'_{M,3t,i}q_{M,5t,I}dt,
\end{aligned}
$$

where

$$
\begin{aligned}
q'_{134,1t,i} &= \sum_{i=1}^d \bar{w}_i(t)\phi_{S^i}, \qquad (139) \\
q'_{134,2t,i} &= e^{-\int_0^t\alpha_s^i ds}\phi_{S^i}, \qquad (140) \\
q'_{134,3t,i} &= \hat{w}_i(t)\partial_{S^i}\phi_{S^i}, \qquad (141) \\
q'_{134,4t,i} &= e^{-\int_0^t\alpha_s^i ds}\phi_{S^i}, \qquad (142)
\end{aligned}
$$

$$q'_{134,5t,i} = \hat{w}_i(t)\partial_{S^i}\phi_{S^i}, \quad (143)$$

$$q'_{135,1t,i} = \sum_{i=1}^{d} \bar{w}_i(t)\phi_{S^i}, \quad (144)$$

$$q'_{135,2t,i} = e^{-\int_0^t \alpha_s^i ds}\phi_{S^i}, \quad (145)$$

$$q'_{135,3t,i} = \hat{w}_i(t)\partial_{S^i}\phi_{S^i}, \quad (146)$$

$$q'_{135,4t,i} = e^{\lambda^i t}\phi_{\sigma^i}, \quad (147)$$

$$q'_{135,5t,i} = \bar{w}_i(t)e^{-\lambda^i t}\partial_{\sigma^i}\phi_{S^i}, \quad (148)$$

$$q'_{136,1t,i} = \sum_{i=1}^{d} \hat{w}_i(t)\phi_{S^i}, \quad (149)$$

$$q'_{136,2t,i} = e^{\lambda^i t}\phi_{\sigma^i}, \quad (150)$$

$$q'_{136,3t,i} = \bar{w}_i(t)e^{-\lambda^i t}\partial_{\sigma^i}\phi_{S^i}, \quad (151)$$

$$q'_{136,4t,i} = e^{\lambda^i t}\phi_{\sigma}, \quad (152)$$

$$q'_{136,5t,i} = \bar{w}_i(t)e^{-\lambda^i t}\partial_{\sigma^i}\phi_{S^i}. \quad (153)$$

Next, we show the expressions of $g\left(\frac{1}{3!}A_T^{LSV(3)}\right)$ by applying the formulas in Appendix B of Shiraya-Takahashi [1].

$$F\left(g\left(\frac{1}{3!}A_T^{LSV(3)}\right)\right) = \frac{1}{2}F\left(\int_0^T \bar{w}(t) * \partial_S^2\Phi_S * \left(\int_0^t e^{\int_u^t \alpha_s ds} * \Phi_S dW_u\right)^2 dW_t\right) \quad (154)$$

$$+F\left(\int_0^T \bar{w}(t) * \partial_S\Phi_S * \int_0^t e^{\int_u^t \alpha_s ds} * \partial_S\Phi_S * \int_0^u e^{\int_v^u \alpha_s ds} * \Phi_S dW_v dW_u dW_t\right) \quad (155)$$

$$+F\left(\int_0^T \bar{w}(t) * \partial_S\Phi_S * \int_0^t e^{\int_u^t \alpha_s ds} * \partial_\sigma\Phi_S * \int_0^u e^{-\lambda(u-v)} * \Phi_\sigma dW_v dW_u dW_t\right) \quad (156)$$

$$+\frac{1}{2}F\left(\int_0^T \bar{w}(t) * \partial_\sigma^2\Phi_S * \left(\int_0^t e^{-\lambda(t-u)} * \Phi_\sigma dW_u\right)^2 dW_t\right) \quad (157)$$

$$+F\left(\int_0^T \bar{w}(t) * \partial_\sigma\Phi_S * \int_0^t e^{-\lambda(t-u)} * \partial_\sigma\Phi_\sigma * \int_0^u e^{-\lambda(u-v)} * \Phi_\sigma dW_v dW_u dW_t\right) \quad (158)$$

$$= \frac{1}{2}G(154) + G(155) + G(156) + \frac{1}{2}G(157) + G(158). \quad (159)$$

We obtain the expressions of $G(M)$ for $M = 155, 156, 158$:

$$G(M) = \sum_{i=1}^{d}\left(\int_0^T q'_{M,4t,i}q_{M,1t,i}\int_0^t q'_{M,3s,i}q_{M,1s,i}\int_0^s q'_{M,2u,i}q_{M,1u,i}dudsdt\right)\frac{H_3(x;\Sigma_T)}{(\Sigma_T)^3}$$

$$= \left(\frac{1}{2}I_2(154)+I_2(155)+I_2(156)+\frac{1}{2}I_2(157)+I_2(158)\right)\frac{H_3(\mathscr{Y};\Sigma_T)}{(\Sigma_T)^3}. \quad (160)$$

That is, for $M=155,156,158$:

$$I_2(M) = \sum_{i=1}^{d}\left(\int_0^T q'_{M,4t,i}q_{M,1t,i}\int_0^t q'_{M,3s,i}q_{M,1s,i}\int_0^s q'_{M,2u,i}q_{M,1u,i}dudsdt\right),$$

where

$$q'_{155,1t,i} = \sum_{i=1}^{d}\bar{w}_i(t)\phi_{S^i}, \quad (161)$$
$$q'_{155,2t,i} = e^{-\int_0^t\alpha_s^i ds}\phi_{S^i}, \quad (162)$$
$$q'_{155,3t,i} = \partial_{S^i}\phi_{S^i}, \quad (163)$$
$$q'_{155,4t,i} = \hat{w}_i(t)\partial_{S^i}\phi_{S^i}, \quad (164)$$
$$q'_{156,1t,i} = \sum_{i=1}^{d}\bar{w}_i(t)\phi_{S^i}, \quad (165)$$
$$q'_{156,2t,i} = e^{\lambda^i t}\phi_{\sigma^i}, \quad (166)$$
$$q'_{156,3t,i} = e^{-\int_0^t\alpha_s^i ds-\lambda^i t}\partial_{\sigma^i}\phi_{S^i}, \quad (167)$$
$$q'_{156,4t,i} = \hat{w}_i(t)\partial_{S^i}\phi_{S^i}, \quad (168)$$
$$q'_{158,1t,i} = \sum_{i=1}^{d}\bar{w}_i(t)\phi_{S^i}, \quad (169)$$
$$q'_{158,2t,i} = e^{\lambda^i t}\phi_{\sigma^i}, \quad (170)$$
$$q'_{158,3t,i} = \partial_{\sigma^i}\phi_{\sigma^i}, \quad (171)$$
$$q'_{158,4t,i} = \bar{w}_i(t)e^{-\lambda^i t}\partial_{\sigma^i}\phi_{S^i}. \quad (172)$$

We also have the expressions of $G(M)$ for $M=154,157$ as follows:

$$\begin{aligned} G(M) &= \sum_{i=1}^{d}\left\{\int_0^T\left(\int_0^t q'_{M,2u,i}q_{M,1u,i}du\right)\left(\int_0^t q'_{M,3s,i}q_{M,1s,i}ds\right)q'_{M,4t,i}q_{M,1t,i}dt\right\}\frac{H_3(x;\Sigma_T)}{(\Sigma_T)^3} \\ &\quad + \left(\int_0^T\int_0^t q'_{M,2u,i}q_{M,3u,i}du q'_{M,4t,i}q_{M,1t,i}dt\right)\frac{H_1(x;\Sigma_T)}{\Sigma_T} \\ &= I_2(M)\frac{H_3(x;\Sigma_T)}{(\Sigma_T)^3}+I_0(M)\frac{H_1(x;\Sigma_T)}{(\Sigma_T)}. \end{aligned} \quad (173)$$

That is, for $M=154,157$,

$$I_2(M) = \sum_{i=1}^{d}\left\{\int_0^T\left(\int_0^t q'_{M,2u,i}q_{M,1u,i}du\right)\left(\int_0^t q'_{M,3s,i}q_{M,1s,i}ds\right)q'_{M,4t,i}q_{M,1t,i}dt\right\},$$

$$I_0(M) = \sum_{i=1}^{d}\left(\int_0^T\int_0^t q'_{M,2u,i}q_{M,3u,i}du q'_{M,4t,i}q_{M,1t,i}dt\right),$$

where

$$q'_{154,1t,i} = \sum_{i=1}^{d} \bar{w}_i(t)\phi_{S^i}, \tag{174}$$

$$q'_{154,2t,i} = q_{154,3t,i} = e^{-\int_0^t \alpha_s^i ds}\phi_{S^i}, \tag{175}$$

$$q'_{154,4t,i} = \hat{w}_i(t)e^{\int_0^t \alpha_s^i ds}\partial^2_{S^i}\phi_{S^i}, \tag{176}$$

$$q'_{157,1t,i} = \sum_{i=1}^{d} \bar{w}_i(t)\phi_{S^i}, \tag{177}$$

$$q'_{157,2t,i} = q_{157,3t,i} = e^{\lambda^i t}\phi_{\sigma^i}, \tag{178}$$

$$q'_{157,4t,i} = \bar{w}_i(t)e^{-2\lambda^i t}\partial^2_{\sigma^i}\phi_{S^i}. \tag{179}$$

Finally, if necessary, we use the next integral formula:

$$\int_{-z}^{\infty} H_k(x,\Sigma)n(x;0,\Sigma)dx = \Sigma H_{k-1}(-z,\Sigma)n(z;0,\Sigma), \tag{180}$$

and collecting the terms with the same order of x in Helmite polynomials, we obtain the coefficients $C_{1,k}, \cdots, C_{3,k}, C_4, \cdots, C_6$ in (8) as follows: with $\Sigma_T^{\{k\}}$ given by (84),

$$C_{1_k} = (117) + (120) + (123) + (126) + (130) + (132), \tag{181}$$

$$C_{2_k} = ((110) + (112) + (114) + (116) + (118) + (121) + (124) + (127) + (128) + (129) + (131))\frac{\Sigma_T^{\{k\}}}{H_1(x;\Sigma_T^{\{k\}})}, \tag{182}$$

$$C_{3_k} = ((109) + (111) + (113) + (115) + (119) + (122) + (125))\frac{(\Sigma_T^{\{k\}})^2}{H_2(x;\Sigma_T^{\{k\}})}, \tag{183}$$

$$C_4 = \frac{1}{2}I_2(134) + I_2(135) + \frac{1}{2}I_2(136) + \frac{1}{2}I_2(154) + I_2(155) + I_2(156) + \frac{1}{2}I_2(157) + I_2(158), \tag{184}$$

$$C_5 = \frac{1}{2}I_4(134) + I_4(135) + \frac{1}{2}I_4(136), \tag{185}$$

$$C_6 = \frac{1}{2}I_0(134) + I_0(135) + \frac{1}{2}I_0(136) + \frac{1}{2}I_2(154) + I_2(158). \tag{186}$$

B Conditional Expectation Formulas

Appendix B lists up the conditional expectation formulas used in the derivation of Theorem 4.3. While we show the one-dimensional case for ease of exposition, one can obtain the formulas for the multi-dimensional case in a similar way.

Lemma B.1. *We assume $f_{1,t}, f_{2,t}, g_{1,t}, g_{2,t}, q_{i,t}$, $i = 1, 2, ..., 5$ are $\mathbf{R}_+ \to \mathbf{R}$ deterministic piecewise continuous functions on $[0, T]$. Each N is a Poisson process with intensity Λ, which is independent of each other. τ_j stands for the time of the j-th jump in N. W is a one-dimensional Brownian motion, that is independent of N. X_j follows a one-dimensional standard normal distribution $N(0, 1)$. X_j and $X_{j'}$ are independent for $j \neq j'$. X_j is also independent of W and N. $H_n(x; \Sigma)$ denotes the Hermite polynomial of degree n.*

Firstly, let us define $\hat{Y}_T(\mathbf{s})$ and $\Sigma_T^{\{k\}}(\mathbf{s})$ as follows:

$$\hat{Y}_T(\mathbf{s}) := \int_0^T f_{2,t} dW_t + \sum_{j=1}^{k} f_{1,x_j-} X_j, \; (0 < s_1 < s_2 < \cdots < s_k \leq T), \tag{187}$$

$$\Sigma_T^{\{k\}}(\mathbf{s}) := \int_0^T |f_{2,s}|^2 ds + \sum_{j=1}^{k} |f_{1,s_j-}|^2 \; (0 < s_1 < s_2 < \cdots < s_k \leq T). \tag{188}$$

Moreover, we define $\{\tau = \vec{s}\}$ as $\{\tau = \vec{s}\} := \{\tau_1 = s_1, \cdots, \tau_k = s_k\}$.

Then, we have the following formulas.

1. $$\mathbf{E}\left[\int_0^T q_{2,t} dW_t | \hat{Y}_T(\mathbf{s}) = y, N_T = k, \{\tau = \vec{s}\}\right] = \left(\int_0^T q_{2,t} f_{2,t} dt\right) \frac{H_1(y; \Sigma_T^{\{k\}}(\mathbf{s}))}{\Sigma_T^{\{k\}}(\mathbf{s})}. \tag{189}$$

2. $$\mathbf{E}\left[\int_0^T \int_0^t q_{2,u} dW_u q_{3,t} dW_t | \hat{Y}_T(\mathbf{s}) = y, N_T = k, \{\tau = \vec{s}\}\right] = \left(\int_0^T \int_0^t q_{2,u} f_{2,u} du q_{3,t} f_{2,t} dt\right) \frac{H_2(y; \Sigma_T^{\{k\}}(\mathbf{s}))}{\Sigma_T^{\{k\}}(\mathbf{s})^2}. \tag{190}$$

3. $$\mathbf{E}\left[\left(\int_0^T q_{2,u} dW_u\right)\left(\int_0^T q_{3,s} dW_s\right) | \hat{Y}_T(\mathbf{s}) = y, N_T = k, \{\tau = \vec{s}\}\right] = \left(\int_0^T q_{2,u} f_{2,u} du\right)\left(\int_0^T q_{3,s} f_{2,s} ds\right) \frac{H_2(y; \Sigma_T^{\{k\}}(\mathbf{s}))}{\Sigma_T^{\{k\}}(\mathbf{s})^2} + \int_0^T q_{2,t} q_{3,t} dt. \tag{191}$$

4. $$\mathbf{E}\left[\int_0^T \int_0^t \int_0^s q_{2,u} dW_u q_{3,s} dW_s q_{4,t} dW_t | \hat{Y}_T(\mathbf{s}) = y, N_T = k, \{\tau = \vec{s}\}\right] = \left(\int_0^T q_{4,t} f_{2,t} \int_0^t q_{3,s} f_{2,s} \int_0^s q_{2,u} f_{2,u} du ds dt\right) \frac{H_3(y; \Sigma_T^{\{k\}}(\mathbf{s}))}{\Sigma_T^{\{k\}}(\mathbf{s})^3}. \tag{192}$$

5. $$\mathbf{E}\left[\int_0^T \left(\int_0^t q_{2,u} dW_u\right)\left(\int_0^t q_{3,s} dW_s\right) q_{4,t} dW_t | \hat{Y}_T(\mathbf{s}) = y, N_T = k, \{\tau = \vec{s}\}\right] = \left\{\int_0^T \left(\int_0^t q_{2,u} f_{2,u} du\right)\left(\int_0^t q_{3,s} f_{2,s} ds\right) q_{4,t} f_{2,t} dt\right\} \frac{H_3(y; \Sigma_T^{\{k\}}(\mathbf{s}))}{\Sigma_T^{\{k\}}(\mathbf{s})^3} + \left(\int_0^T \int_0^t q_{2,u} q_{3,u} du q_{4,t} f_{2,t} dt\right) \frac{H_1(y; \Sigma_T^{\{k\}}(\mathbf{s}))}{\Sigma_T^{\{k\}}(\mathbf{s})}. \tag{193}$$

6. $$\mathbf{E}\left[\left(\int_0^T\int_0^t q_{2,s}dW_s q_{3,t}dW_t\right)\left(\int_0^T\int_0^r q_{4,u}dW_u q_{5,r}dW_r\right) \mid \hat{Y}_T(\mathbf{s})=y, N_T=k, \{\tau=\vec{s}\}\right]=$$
$$\left(\int_0^T q_{3,t}f_{2,t}\int_0^t q_{2,s}f_{2,s}dsdt\right)\left(\int_0^T q_{5,r}f_{2,r}\int_0^r q_{4,u}f_{2,u}dudr\right)\frac{H_4(y;\Sigma_T^{\{k\}}(\mathbf{s}))}{\Sigma_T^{\{k\}}(\mathbf{s})^4}$$
$$+\left\{\int_0^T q_{3,t}f_{2,t}\int_0^t q_{5,r}f_{2,r}\int_0^r q_{2,u}q_{4,u}dudrdt+\int_0^T q_{5,t}f_{2,t}\int_0^t q_{3,r}f_{2,r}\int_0^r q_{2,u}q_{4,u}dudrdt\right.$$
$$+\int_0^T q_{3,t}f_{2,t}\int_0^t q_{2,r}q_{5,r}\int_0^r q_{4,u}f_{2,u}dudrdt+\int_0^T q_{3,t}q_{5,t}\left(\int_0^t q_{2,s}f_{2,s}ds\right)\left(\int_0^t q_{4,u}f_{2,u}du\right)dt$$
$$\left.+\int_0^T q_{5,r}f_{2,r}\int_0^r q_{3,u}q_{4,u}\int_0^u q_{2,s}f_{2,s}dsdudr\right\}\frac{H_2(y;\Sigma_T^{\{k\}}(\mathbf{s}))}{\Sigma_T^{\{k\}}(\mathbf{s})^2}$$
$$+\int_0^T\int_0^t q_{2,u}q_{4,u}duq_{3,t}q_{5,t}dt. \tag{194}$$

7. $$\mathbf{E}\left[\sum_{j=1}^{N_T} g_{1,\tau_j-}\middle|\hat{Y}_T(\mathbf{s})=y, N_T=k, \{\tau=\vec{s}\}\right]$$
$$=\sum_{j=1}^{k} g_{1,s_j-}. \tag{195}$$

8. $$\mathbf{E}\left[\sum_{j=1}^{N_T} g_{1,\tau_j-}X_j\middle|\hat{Y}_T(\mathbf{s})=y, N_T=k, \{\tau=\vec{s}\}\right]$$
$$=\sum_{j=1}^{k} g_{1,s_j-}f_{1,s_j-}\frac{H_1(y,\Sigma_T^{\{k\}}(\mathbf{s}))}{\Sigma_T^{\{k\}}(\mathbf{s})}. \tag{196}$$

9. $$\mathbf{E}\left[\int_0^T g_{2,t}\sum_{j=1}^{N_t} g_{1,\tau_j-}dW_t\middle|\hat{Y}_T(\mathbf{s})=y, N_T=k, \{\tau=\vec{s}\}\right]$$
$$=\sum_{j=1}^{k}\left(\int_{s_j}^T g_{2,t}f_{2,t}dt\right)g_{1,s_j-}\frac{H_1(y,\Sigma_T^{\{k\}}(\mathbf{s}))}{\Sigma_T^{\{k\}}(\mathbf{s})}. \tag{197}$$

10. $$\mathbf{E}\left[\int_0^T g_{2,t}\sum_{j=1}^{N_t} g_{1,\tau_j-}X_j dW_t\middle|\hat{Y}_T(\mathbf{s})=y, N_T=k, \{\tau=\vec{s}\}\right]$$
$$=\sum_{j=1}^{k}\left(\int_{s_j}^T g_{2,t}f_{2,t}dt\right)\left(g_{1,s_j-}f_{1,s_j-}\right)\frac{H_2(y,\Sigma_T^{\{k\}}(\mathbf{s}))}{(\Sigma_T^{\{k\}}(\mathbf{s}))^2}. \tag{198}$$

11. $$\mathbf{E}\left[\int_0^T g_{2,t}\sum_{j=1}^{N_t} g_{1,\tau_j-}dt\middle|\hat{Y}_T(\mathbf{s})=y, N_T=k, \{\tau=\vec{s}\}\right]$$
$$=\sum_{j=1}^{k}\left(\int_0^T g_{2,t}dt\right)g_{1,s_j-}. \tag{199}$$

12. $\mathbf{E}\left[\int_0^T g_{2,t}\sum_{j=1}^{N_t} g_{1,\tau_j-}X_j dt \middle| \hat{Y}_T(\mathbf{s}) = y, N_T = k, \{\tau = \vec{s}\}\right]$

$$= y\sum_{j=1}^{k}\left(\int_{s_j}^{T} g_{2,t}dt\right)\left(g_{1,s_j-}f_{1,s_j-}\right)\frac{1}{\Sigma_T^{\{k\}}(\mathbf{s})}. \tag{200}$$

13. $\mathbf{E}\left[\sum_{j=1}^{N_T} g_{1,\tau_j-}\int_0^{\tau_j-} g_{2,t}dW_t \middle| \hat{Y}_T(\mathbf{s}) = y, N_T = k, \{\tau = \vec{s}\}\right]$

$$= y\sum_{j=1}^{k} g_{1,s_j-}\left(\int_0^{s_j} g_{2,t}f_{2,t}dt\right)\frac{1}{\Sigma_T^{\{k\}}(\mathbf{s})}. \tag{201}$$

14. $\mathbf{E}\left[\sum_{j=1}^{N_T} g_{1,\tau_j-}\int_0^{\tau_j-} g_{2,t}dW_t X_j \middle| \hat{Y}_T(\mathbf{s}) = y, N_T = k, \{\tau = \vec{s}\}\right]$

$$= \sum_{j=1}^{k}\left(g_{1,s_j-}f_{1,s_j-}\right)\left(\int_0^{s_j} g_{2,t}f_{2,t}dt\right)\frac{H_2(y, \Sigma_T^{\{k\}}(\mathbf{s}))}{(\Sigma_T^{\{k\}}(\mathbf{s}))^2}. \tag{202}$$

15. $\mathbf{E}\left[\sum_{j=1}^{N_T}(g_{1,\tau_j-}X_j)(g_{2,\tau_j-}X_j) \middle| \hat{Y}_T(\mathbf{s}) = y, N_T = k, \{\tau = \vec{s}\}\right]$

$$= \sum_{j=1}^{k}\left(g_{1,s_j-}f_{1,s_j-}g_{2,s_j-}f_{1,s_j-}\frac{H_2(y, \Sigma_T^{\{k\}}(\mathbf{s}))}{(\Sigma_T^{\{k\}}(\mathbf{s}))^2} + g_{1,s_j-}g_{2,s_j-}\right). \tag{203}$$

16. $\mathbf{E}\left[\sum_{j=2}^{N_T} g_{1,\tau_j-}\sum_{J=1}^{N_{\tau_j-}} g_{2,\tau_J-} \middle| \hat{Y}_T(\mathbf{s}) = y, N_T = k, \{\tau = \vec{s}\}\right]$

$$= \sum_{j=2}^{k} g_{1,s_j-}\sum_{J=1}^{j-1} g_{2,s_j-}. \tag{204}$$

17. $\mathbf{E}\left[\sum_{j=2}^{N_T} g_{1,\tau_j-}\sum_{J=1}^{N_{\tau_j-}} g_{2,\tau_J-}X_J \middle| \hat{Y}_T(\mathbf{s}) = y, N_T = k, \{\tau = \vec{s}\}\right]$

$$= y\sum_{j=2}^{k} g_{1,s_j-}\sum_{J=1}^{j-1} g_{2,s_J-}f_{1,s_J-}\frac{1}{\Sigma_T^{\{k\}}(\mathbf{s})}. \tag{205}$$

18. $\mathbf{E}\left[\sum_{j=2}^{N_T} g_{1,\tau_j-}X_j\sum_{J=1}^{N_{\tau_j-}} g_{2,\tau_J-} \middle| \hat{Y}_T(\mathbf{s}) = y, N_T = k, \{\tau = \vec{s}\}\right]$

$$= \sum_{j=2}^{k} g_{1,s_j-}f_{1,s_j-}\sum_{J=1}^{j-1} g_{2,s_J-}\frac{1}{\Sigma_T^{\{k\}}(\mathbf{s})}. \tag{206}$$

19. $\mathbf{E}\left[\sum_{j=2}^{N_T} g_{1,\tau_j-}\sum_{J=1}^{N_{\tau_j-}} g_{2,\tau_J-}X_J X_j \middle| \hat{Y}_T(\mathbf{s}) = y, N_T = k, \{\tau = \vec{s}\}\right]$

$$= \sum_{j=2}^{k} g_{1,s_j-} f_{1,s_j-} \sum_{J=1}^{j-1} g_{2,s_J-} f_{1,s_J-} \frac{H_2(y, \Sigma_T^{\{k\}}(\mathbf{s}))}{(\Sigma_T^{\{k\}}(\mathbf{s})^2}. \tag{207}$$

Remark B.2. *Due to the time dependence of average options, multiple integrals appear in (107). To reduce computational burdens in practice, we apply the following approximation:*

$$\frac{k!}{T^k} \sum_{j=1}^{k} \int_0^T \int_0^{s_k} \cdots \int_0^{s_2} h_{s_j} g(\Sigma_T^{\{k\}}(\mathbf{s})) ds_1 \cdots ds_{k-1} ds_k \approx \frac{k}{T} g(\Sigma_T^{\{k\}}(T)) \int_0^T h_s ds, \tag{208}$$

$$\int_0^T \int_0^{s_k} \cdots \int_0^{s_2} f(\Sigma_T^{\{k\}}(\mathbf{s})) ds_1 \cdots ds_{k-1} ds_k \approx f(\Sigma_T^{\{k\}}(T)), \tag{209}$$

where $\Sigma_T^{\{k\}}(\mathbf{s}) := \int_0^T |f_{2,s}|^2 ds + \sum_{j=1}^{k} |f_{1,s_j}|^2$ *and* $\Sigma_T^{\{k\}}(T) := \int_0^T |f_{2,s}|^2 ds + \frac{k}{T} \int_0^T |f_{1,s}|^2 ds.$

References

[1] Shiraya, K. and Takahashi, A., *An approximation formula for basket option prices under local stochastic volatility with jumps: an application to commodity markets*, Journal of Computational and Applied Mathematics, 292(15), 2016, 230-256.