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Mechanism Design in Hidden Action and Hidden Information: Richness and Pure-VCG[1]

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Abstract

We investigate general mechanism design problems in which agents can take hidden actions that influence state distribution. Their action choices exert significant externality effects on their valuation functions through this influence. We characterize all mechanisms that resolve the hidden action problem (i.e., that *induce* a targeted action profile). A variety of action choices shrinks the set of mechanisms that induce the targeted action profile, leading to the equivalence properties in the ex-post term with respect to payoffs, payments, and revenues. When the agents can take unilateral deviations to change the state distribution in various directions (i.e., when the action profile satisfies *richness*), *pure-VCG mechanisms*—the simplest form of canonical VCG mechanism, which is implemented via open-bid descending procedures that determine the losers' compensation—are the only mechanisms that induce an efficient action profile. Contrariwise, the popular pivot mechanism, implemented by ascending auctions that determine the winner's payment, generally fails to induce any efficient action profile.

Keywords: hidden action, hidden information, externality, richness, pure-VCG, equivalence
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1. Introduction

This study investigates a general class of mechanism design problems that include hidden action and hidden information issues with the assumptions of quasi-linearity and risk-neutrality, such as principal–agent relationships, partnerships, and general resource allocation (including auctions and public good provision). Multiple agents make action choices independently in early stages before the realization of the state. Their action choices influence the state distribution. According to a prespecified mechanism, the central planner determines an allocation in a state-contingent manner, which affects the payoffs of all agents and that of the central planner. The central planner cannot observe the agents' action choices; thus, we face a *hidden action* problem. The central planner therefore designs a state-contingent mechanism consisting of an allocation rule and a payment rule in advance to incentivize the agents to select the action profile the central planner desires.

This study assumes that their action choices exert significant externality effects on their valuation functions through the action-contingent stochastic state determination. This study seeks to clarify whether and how the central planner can construct a mechanism that resolves the hidden action problem and investigates whether a mechanism resolves both the hidden action problem and the hidden information problem simultaneously.

We first propose a benchmark model that addresses only the hidden action problem. We then incorporate *hidden information* by assuming that the central planner can observe neither the state nor the agents' action choices and therefore requires agents to report their private information regarding the state (i.e., their type).

This study differs substantially from previous research on mechanism design and contract theory in that it assumes that each agent can have various activity aspects such as information acquisition, R&D investment, patent control, standardization, M&A, rent-seeking, positive/negative campaigns, environmental concern, product differentiation, entry/exit decisions, preparation of infrastructure, and headhunting. The central planner lacks information about the breadth of these potential aspects due to, for example, the separation between ownership and control. Accordingly, the central planner may be unable to grasp which aspects of agents' activities are actually relevant to the current problem. A conservative central planner must account for all of these aspects in mechanism design.

The hidden action problem is particularly severe when each agent's action has significant externality effects on the other agents' valuation functions (i.e., when each agent can change the distribution of the state that shapes the other agents' valuation functions). This externality effect severely restricts the range of mechanisms that can incentivize agents to make the desired action choices (i.e., that can *induce* the desired action profile).

The *pivot mechanism*, which aligns each agent's payoff with the agent's marginal contribution, has generally been considered the most desirable mechanism in private-value environments because it implements an efficient allocation rule in hidden information and collects a moderate amount of revenue. Importantly, without externality effects, the pivot mechanism can induce an efficient action profile (e.g., Bergemann and Välimäki, 2002; Hatfield, Kojima, and Kominers, 2015). However, the pivot mechanism generally fails to induce an efficient action profile once externality effects are taken into account.

Example 1 (R&D Investment): The government allocates a resource to one of two firms, and the firms can make ex-ante R&D investments to increase the profitability of this resource. Assume that the R&D investment has a positive externality (i.e., an increase in the level of one firm's R&D investment strengthens not only that firm but also its rival). In such a case, a firm's expected payoff in the pivot mechanism (i.e., the second-price auction or the ascending auction) is decreasing in the strength of the rival firm because a stronger rival makes a higher price-bid. Hence, the pivot mechanism discourages a firm from making such an R&D investment, even if a higher level of R&D investment is more socially desirable because of its positive externality.

Example 2 (Preemptive Behavior): An incumbent and a potential entrant (rival firm) participate in a procurement auction. The incumbent can choose the level of its preemptive action, which disturbs the entry of the rival firm. In such a case, the pivot mechanism strongly encourages the incumbent to take the preemptive action because the incumbent seeks to avoid price competition. However, paying such a positive cost to reduce the rival's profitability is socially wasteful.

In brief, when agents can change the other agents' type distributions in the pivot mechanism, they have an incentive to weaken the other agents' types to exaggerate their contributions. The abovementioned examples illustrate that we need to look at mechanisms other than the pivot mechanism to achieve full efficiency (i.e., to induce an efficient action profile and implement an efficient allocation rule simultaneously).

The main contribution of this study is that it characterizes all mechanisms that induce the desired action profile. In particular, by assuming that each agent can unilaterally change the state distribution for all directions by taking a mixed action, (i.e., when the targeted action profile satisfies *richness*), we demonstrate strong equivalence properties in the ex-post term; the ex-post payments, the ex-post revenue, and the ex-post payments are unique up to constants.

We further consider the possibility that the central planner achieves both an efficient action profile and efficient allocations. We introduce *pure-VCG mechanisms* as the simplest form of canonical VCG (Vickrey–Clarke–Groves) mechanisms,[4] in which the central planner gives each agent the welfare of the other agents and then imposes on the agents a fixed monetary fee. A pure-VCG mechanism is implemented via an open-bid descending procedure that determines the compensation for losers. This is in contrast to the pivot mechanism, which is implemented through an open-bid ascending auction that determines the winner's payment. We show that pure-VCG mechanisms are the only efficient mechanisms that resolve the hidden action problem when the targeted action profile is rich. Hence, a mechanism induces an efficient action profile if and only if it is pure-VCG.

This theoretical finding has important implications not only in hidden action but also in hidden information. Suppose the central planner cannot observe the state and therefore requires agents to report their private information regarding the state (i.e., their respective types). Under the assumption of private values, once the central planner designs a mechanism that induces the efficient action profile (i.e., resolves the hidden

[4] See Vickrey (1961), Clarke (1971), and Groves (1973).

action problem), this mechanism, which is equivalent to a pure-VCG mechanism, automatically resolves the hidden information problem due to its internalization feature (whereas, without private values, pure-VCG mechanisms generally fail to resolve the hidden information problem).

With the assumption of private values, the generally accepted view in mechanism design is that VCG mechanisms are (with some regularity conditions) the only efficient mechanisms that resolve the *hidden information* problem. However, this study shows that pure-VCG mechanisms are the only efficient mechanisms that resolve the hidden action problem, and, since pure-VCG mechanisms are special cases of VCG mechanisms, pure-VCG mechanisms also resolve the hidden information problem.

Since the class of pure-VCG mechanisms is a proper subclass of VCG mechanisms, it is more difficult for the central planner to earn non-negative revenues with hidden action than it is to earn them without hidden action. In fact, the pivot mechanism generally guarantees non-negative revenues, but it is not pure-VCG and thus generally fails to induce an efficient action profile. We show an impossibility result in which, under the assumption of richness, no pure-VCG mechanism (i.e., no well-behaved mechanism in hidden action) satisfies non-negative revenues and ex-post individual rationality simultaneously, while the pivot mechanism satisfies both.

The results of this study crucially depend on the presence of externality effects. Previous works such as Bergemann and Välimäki (2002) and Hatfield, Kojima, and Kominers (2015) showed that, when such externality effects are absent, the pivot mechanism can resolve the hidden action problem. However, whenever the externality effects are non-negligible, the pivot mechanism generally fails to resolve the hidden action problem.

Without externality effects, well-behaved mechanisms are not very limited, even if we permit a sufficiently high availability of unilateral deviations. We introduce a class of mechanisms, *expectation-VCG mechanisms*, which are more general than VCG and require each agent to pay the same amount as VCG in expectation. The AGV mechanism (Arrow, 1979; d'Aspremont and Gerard-Varet; 1979), which satisfies both budget-balance and Bayesian incentive compatibility, is also expectation-VCG. We show that a mechanism resolves the hidden action problem if and only if it is expectation-VCG.[5]

The remainder of this paper is organized as follows. Section 2 reviews the relevant literature. Section 3 presents the benchmark model, where we account for only hidden actions. Section 4 incorporates hidden information into the model. Section 5 focuses on fully efficient mechanisms. Section 6 discusses the implementation of the pure-VCG mechanism through an open-bid procedure. Section 7 examines the central planner's revenues. Section 8 considers the case without externality, and Section 9 concludes the paper.

2. Literature Review

When an agent has a wide variety of action choices, a complicated contract design might motivate the agent to deviate from desired behavior. In this case, a simply designed contract could function better than a complex one. For example, Holmström and Milgrom (1987) study principal–agent relationships in a dynamic context, where randomly determined outputs are accumulated through time, and the agent flexibly adjusts the effort level depending on output histories. They show that the optimal incentive contract that maximizes the principal's revenue must be linear with respect to the output accumulated at the ending time. Carroll (2014) investigates optimal contract design in a static principal–agent relationship in which the principal experiences ambiguity about the range of activities the agent can undertake. Carroll shows that the optimal contract must be linear with respect to the resultant output, given that the principal follows the maximin expected utility hypothesis.

This study examines the hidden action problem by introducing multiple agents, general state spaces, general allocation rules, and various criteria such as efficiency

[5] Matthews (1984), Hausch and Li (1991), and Tan (1992) show that the first- and second-price auctions provide each agent with the same incentive in hidden action in symmetric environments, in which the first-price auction is an indirect implementation of an expectation-VCG mechanism and the second-price auction is equivalent to the pivot mechanism. Tan (1992), Stegeman (1996), and Arozamena and Cantillon (2004) also show that, with private values, the second-price auction (the pivot mechanism) induces an efficient hidden action. This study provides a comprehensive understanding of all such previous works.

instead of revenue optimization. We account deliberately for a wide range of externality effects of each agent's action choice on other agents' valuation functions. We then demonstrate a characterization result for well-behaved incentive mechanisms, which imply that only a simple form of mechanism design (i.e., pure-VCG) functions, and we further show the general equivalence properties in the ex-post term.

Any well-behaved efficient mechanism must be pure-VCG. Athey and Segal (2013) show that, under the assumption of private values, pure-VCG mechanisms induce efficient action profiles. We extend their study to show that, under the assumption of richness, pure-VCG mechanisms are the only mechanisms that can induce efficiency in hidden action.

The literature regarding the hidden action problem has shown that, without richness (i.e., when the scope of action spaces is sufficiently limited), we can design efficient incentive mechanisms by tailoring the payment rule to detailed specifications. We can even achieve either full surplus extraction or budget balance (e.g., Matsushima, 1989; Legros and Matsushima, 1991; Williams and Radner, 1995; Obara, 2008). With richness, however, a mechanism's dependence on the detail even encourages each agent to deviate.

This study also makes important contributions to the literature on the hidden information problem. Green and Laffont (1977, 1979) and Holmström (1979) show that, in hidden information environments with differentiable valuation functions, differentiable path-connectedness, and private values, VCG mechanisms are the only efficient mechanisms that satisfy incentive compatibility in dominant strategies. This study reconsiders VCG mechanisms from the viewpoint of hidden action and shows that only pure-VCG mechanisms, which are special cases of VCG, resolve the hidden action problem. While the resolution of the hidden action problem automatically resolves the hidden information problem under the assumption of private values, the reverse is not true.[6]

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[6] Hausch and Li (1993) and Persico (2000) are relevant to this point. They demonstrate that first- and second-price auctions provide different incentives for information acquisition that make the other agents' valuation more accurate. Taking the mechanism design approach rather than comparing between special auction formats, we explain that the difference in the induced action profile originates from the difference in ex-post payoffs between these auction formats.

It is worth noting that our equivalence theorems do not rely on the assumptions of the continuum state spaces and the differentiability of valuation functions. In this sense, we can interpret our characterization theorem as a new version of VCG-necessity theorems, whereby this study articulates the desirability of pure-VCG mechanisms and expectation-VCG mechanisms even if the assumption of Green–Laffont–Holmström fails to hold (e.g., even if type spaces are finite).

3. Hidden Action

Consider a setting with one central planner and n agents indexed by $i \in N = \{1, 2, ..., n\}$. This section investigates an allocation problem consisting of the four stages below and focuses on the incentive issue in hidden action.

Stage 1: The central planner commits to a *mechanism* defined as (g, x), where Ω denotes the set of states, A denotes the set of allocations, $g : \Omega \to A$, and $x \equiv (x_i)_{i \in N} : \Omega \to R^n$. We assume that Ω and A are finite. We call g and x the *allocation rule* and *payment rule*, respectively.[7]

For each $i \in N$, we call a pair of an allocation rule and a payment rule for agent i (i.e., (g, x_i)) *a mechanism for agent* i. We denote $\Omega = \{1, 2, ..., |\Omega|\}$ and regard x_i as the $|\Omega|$-dimensional vector (i.e., $x_i = (x_i(1), ..., x_i(|\Omega|)) \in R^{|\Omega|}$). We also denote $x = (x_1, ..., x_n) = (x_i(\omega))_{\substack{i \in N \\ \omega \in \{1, ..., |\Omega|\}}} \in R^{n|\Omega|}$.

[7] This study assumes that the central planner commits to the mechanism *before* agents take action. Without the assumption of commitment, the desired outcomes, such as efficiency, may be unachievable due to time inconsistency. Consider a central planner who commits to a mechanism *after* the agents' action choices. In this case, the central planner prefers the pivot mechanism because it yields greater revenue than does the mechanism this study discusses. The pivot mechanism typically fails to induce any efficient action profile, however (see Proposition 3). Anticipating that the central planner will use a pivot mechanism, agents are willing to select inefficient actions.

Stage 2: Each agent $i \in N$ selects a hidden action $b_i \in B_i$, where B_i denotes the set of all actions for agent i. The cost function of agent $i's$ action choice is given by $c_i : B_i \to R_+$. We assume that there is a no-effort option $b_i^0 \in B_i$ such that $c_i(b_i^0) = 0$. Let $B \equiv \underset{i \in N}{\times} B_i$ and $b \equiv (b_1, ..., b_n) \in B$.

Stage 3: The state $\omega \in \Omega$ is randomly drawn from a conditional probability function $f(\cdot | b) \in \Delta(\Omega)$, where $b \in B$ is the action profile selected at stage 2, $\Delta(\Omega)$ denotes the set of all distributions (i.e., lotteries) over states, and $f(\omega | b)$ denotes the probability that state ω occurs provided that the agents selected action profile b.

Stage 4: The central planner determines the allocation $g(\omega) \in A$ and the side payment vector paid to the central planner $x(\omega) = (x_i(\omega))_{i \in N} \in R^n$, where ω is the state that occurred at stage 3. The resultant payoff of each agent $i \in N$ is given by

(1) $$v_i(g(\omega), \omega) - x_i(\omega) - c_i(b_i),$$

where we assume that each agent's payoff function is quasi-linear and risk-neutral, and the cost of the agent's action choice is additively separable.

Figure 1 describes the timeline of the benchmark model. This section concentrates on the incentives in hidden action at stage 2, assuming that the realization of state ω is publicly observable.

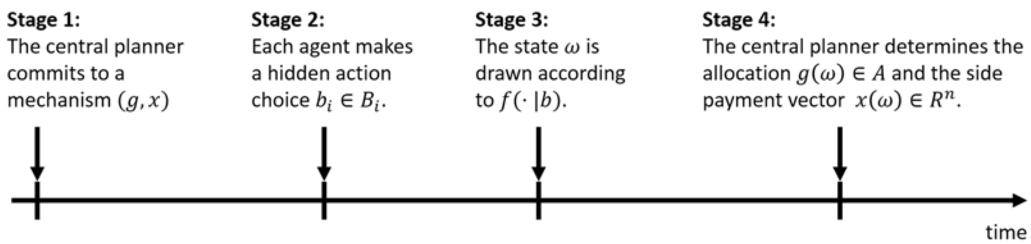

Figure 1: Timeline with Hidden Action

Definition 1 (Inducibility): A mechanism for agent i, (g, x_i), is said to *induce* an action profile $b \in B$ *for agent* i if b_i is a best response to b_{-i}, i.e.,

(2) $$E[v_i(g(\omega),\omega) - x_i(\omega) \mid b] - c_i(b_i)$$
$$\geq E[v_i(g(\omega),\omega) - x_i(\omega) \mid b_i', b_{-i}] - c_i(b_i') \text{ for all } b_i' \in B_i,$$

where $E[\cdot \mid b]$ denotes the expectation operator conditional on b, that is, for every function $\xi : \Omega \to R$:

$$E[\xi(\omega) \mid b] \equiv \sum_{\omega \in \Omega} \xi(\omega) f(\omega \mid b).$$

A mechanism (g, x) is said to *induce* an action profile b if (g, x_i) induces b for every $i \in N$, i.e., b is a Nash equilibrium in the game implied by the mechanism (g, x). We denote the set of all payment rules for agent i that induces targeted action profile b, together with allocation rule g, by

$$X_i(b, g) \equiv \left\{ \tilde{x}_i \in R^{|\Omega|} : (g, \tilde{x}_i) \text{ induces } b \text{ for } i \right\}.$$

We assume that $X_i(b, g)$ is non-empty. Note that $X_i(b, g)$ is non-empty if and only if there exists a function $u_i : \Omega \to R$ such that

$$b_i \in \arg\max_{b_i' \in B_i} E[u_i(\omega) \mid b_i', b_{-i}] \text{ for all } i \in N.$$

Hence, the payment rule for agent i specified as $\tilde{x}_i(\omega) = v_i(g(\omega), \omega) - u_i(\omega)$ induces the targeted action profile b, together with an allocation rule g (i.e., $\tilde{x}_i \in X_i(b, g)$). This implies that, if the planner has an objective function to maximize and b is a solution of the maximization problem, $X_i(b, g)$ is non-empty.

We define the dimensionality of a subset $L \subset R^h$ as the maximal number of linearly independent h-dimensional vectors that span L, denoted by $\dim L$. Since

$$[\tilde{x}_i \in X_i(b, g)] \Leftrightarrow [\tilde{x}_i + \overline{z}_i \in X_i(b, g)] \text{ for all } \overline{z}_i \in R$$

and $X_i(b, g)$ is non-empty,

$$\dim X_i(b, g) \geq 1.$$

We regard the dimensionality of $X_i(b,g)$, $\dim X_i(b,g)$, as a measure expressing the degree of the availability of payment rules x such that the associated mechanisms (g,x) induce the targeted action profile b.

We introduce a concept concerning the availability of unilateral deviations, termed "differentiable path," as follows.

Definition 2 (Differentiable Path): A mapping $\beta_i : [-1,1] \to B_i$ is said to be a *differentiable path of* $(i,b) \in N \times B$ if

$$\beta_i(0) = b_i,$$

$$[\beta_i(\alpha) = \beta_i(\alpha')] \Rightarrow [\alpha = \alpha'],$$

$c_i(\beta_i(\alpha))$ is differentiable in α at $\alpha = 0$,

and there exists a *tangent* of β_i at b, which is denoted by $t(\beta_i) \in R^{|\Omega|}$, where

(3) $$t(\beta_i) \equiv \lim_{\alpha \to 0} \frac{f(\cdot \mid \beta_i(\alpha), b_{-i}) - f(\cdot \mid b)}{\alpha}.$$

Figure 2a illustrates a case of differentiable path β_i, where we assume $|\Omega| = 3$, and the triangle represents the probability simplex $\Delta(\Omega)$. The blue point represents $f(\cdot \mid b)$. Each point on the orange curve corresponds to a probability distribution that can be generated by agent i's unilateral deviation along the differentiable path β_i. The tangent of β_i at b, $t(\beta_i)$, is depicted as the green-dotted arrow. Note that the local change of $f(\cdot \mid \beta_i(\alpha), b_{-i})$ around $\beta_i(0) = b_i$ is approximated by the tangent of the differentiable path β_i, $t(\beta_i)$.

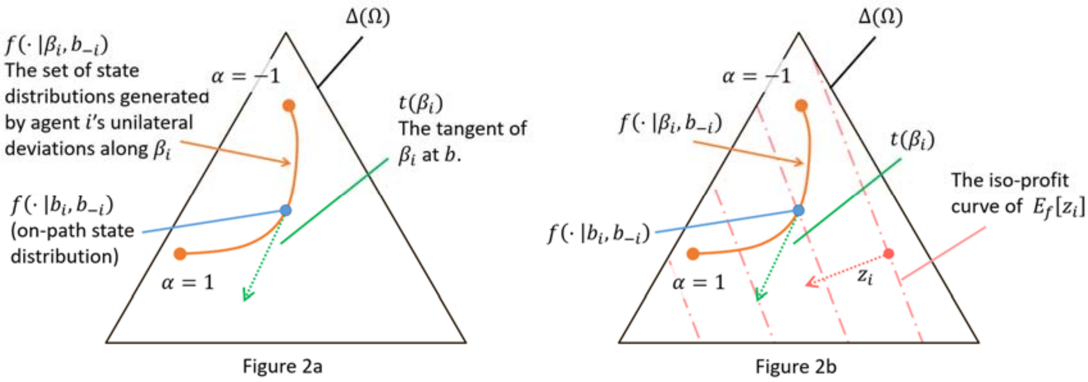

Figure 2: Differentiable path. If z_i is not perpendicular to $t(\beta_i)$, agent i has an incentive to make a unilateral deviation (i.e., increase α slightly).

Fix an arbitrary payment rule for agent i, $x_i \in X_i(b,g)$. We define

$$Z_i(b,g,x_i) \equiv \left\{ z_i \in R^{|\Omega|} : x_i + z_i \in X_i(b,g) \right\}.$$

Clearly,

$$\dim X_i(b,g) = \dim Z_i(b,g,x_i).$$

Proposition 1: *Suppose that* (g,x_i) *induces* b *for agent* i *and there exists a differentiable path* β_i *of* $(i,b) \in N \times B$. *Then,*

(4) $$[z_i \in Z_i(b,g,x_i)] \Rightarrow [z_i \cdot t(\beta_i) = 0].$$

Proof: Consider an arbitrary $z_i \in Z_i(b,g,x_i)$. Since (g,x_i) induces b for agent i, the following first-order necessary condition must hold to prevent agent i from locally deviating along β_i:

(5) $$\frac{\partial}{\partial \alpha} \left\{ E\left[v_i(g(\omega),\omega) - x_i(\omega) \mid \beta_i(\alpha), b_{-i} \right] - c_i(\beta_i(\alpha)) \right\} \Big|_{\alpha=0} = 0.$$

Similarly, $(g, x_i + z_i)$ must satisfy the following first-order condition:

(6) $$\frac{\partial}{\partial \alpha} \left\{ E\left[v_i(g(\omega),\omega) - x_i(\omega) - z_i(\omega) \mid \beta_i(\alpha), b_{-i} \right] - c_i(\beta_i(\alpha)) \right\} \Big|_{\alpha=0} = 0.$$

Subtracting (6) from (5), we obtain

$$\frac{\partial}{\partial \alpha} E\left[z_i(\omega) \mid \beta_i(\alpha), b_{-i}\right]\Big|_{\alpha=0} = 0,$$

or equivalently, $z_i \cdot t(\beta_i) = 0$.

Q.E.D.

Proposition 1 implies that a payment rule $\tilde{x}_i$ for agent i is never included in $X_i(b,g)$ if the difference $z_i \equiv \tilde{x}_i - x_i$ is not perpendicular to the tangent of the differentiable path β_i (i.e., $z_i \cdot t(\beta_i) \neq 0$). Since (g, x_i) induces b, agent i is (approximately) indifferent for local deviations along the differentiable path β_i in (g, x_i). If $(g, x_i + z_i)$ induces b, agent i is also indifferent for local deviations along β_i in $(g, x_i + z_i)$. This implies that the expected value of z_i is (approximately) unchanged even if agent i locally deviates. More formally, the expected payoff of z_i when agent i takes $\beta_i(\alpha)$ is

$$z_i \cdot f(\cdot \mid b) + \alpha \cdot z_i \cdot t(\beta_i) + o(\alpha).$$

Whenever $z_i \cdot t(\beta_i) > 0$ ($z_i \cdot t(\beta_i) < 0$), then agent i can increase this expected payoff by slightly increasing (decreasing) α from zero. Hence, in order for $(g, x_i + z_i)$ to induce b for agent i, z_i must be perpendicular to the tangent of the differentiable path at b, i.e., $z_i \cdot t(\beta_i) = 0$.

Figure 2b shows a case of $z_i \cdot t(\beta_i) > 0$, where agent i has an incentive to take α slightly larger than zero, implying the failure of $(g, x_i + z_i)$ to induce b.

In general, there may exist multiple differentiable paths whose tangents are linearly independent. We define the dimensionality of the availability of unilateral deviations as the maximal number of differentiable paths whose tangents are linearly independent, which is denoted by a positive integer, K. The following theorem shows that the dimensionality of mechanisms, $\dim X_i(b,g)$, is less than or equal to $|\Omega| - K$:

Theorem 1: *Suppose there exist* K *differentiable paths of* (i,b)*, denoted by* β_i^1*, …, and* β_i^K *, such that the respective tangents* $t(\beta_i^1)$ *, …, and* $t(\beta_i^K)$ *are linearly independent. Then,*

$$\dim X_i(b,g) \le |\Omega| - K .$$

Proof: Proposition 1 implies

$$Z_i(b,g,x_i) \subset \left\{ z_i \in R^{|\Omega|} : z_i \cdot t(\beta_i^k) = 0 \text{ for } k = 1,2,\cdots,K \right\}.$$

Hence,

$$\dim Z_i(b,g,x_i) \le \dim \left\{ z_i \in R^{|\Omega|} : z_i \cdot t(\beta_i^k) = 0 \text{ for } k = 1,2,\cdots,K \right\}.$$

Since $t(\beta_i^1)$, …, and $t(\beta_i^K)$ are linearly independent,

$$\dim \left\{ z_i \in R^{|\Omega|} : z_i \cdot t(\beta_i^k) = 0 \text{ for } k = 1,2,\cdots,K \right\} \le |\Omega| - K .$$

Therefore,

$$\dim Z_i(b,g,x_i) \le |\Omega| - K .$$

This, along with $\dim Z_i(b,g,x_i) = \dim X_i(b,g)$, implies $\dim X_i(b,g) \le |\Omega| - K$.

Q.E.D.

Hence, the dimensionality of payment rules (i.e., $\dim X_i(b,g)$) is limited below $|\Omega| - K$, which is decreasing in the dimensionality of unilateral deviations (i.e., in K). Let

$$T(\Omega) \equiv \{ t \in R^{|\Omega|} : \sum_{\omega\in\Omega} t(\omega) = 0 \} .$$

Note that $\dim T(\Omega) = |\Omega| - 1$. For every $\alpha \in [-1,1] \setminus \{0\}$ and every differentiable path β_i,

$$\frac{f(\cdot \mid \beta_i(\alpha), b_{-i}) - f(\cdot \mid b)}{\alpha} \in T(\Omega) .$$

Since $T(\Omega)$ is a closed set, its limit at $\alpha \to 0$ also belongs to $T(\Omega)$ (i.e., $t(\beta_i) \in T(\Omega)$). Hence, there are at most $|\Omega| - 1$ differentiable paths whose tangents are linearly independent. We call the targeted action profile "b rich" for agent i if there indeed exist $|\Omega|-1$ differentiable paths whose tangents are linearly independent.

Definition 3: An action profile $b \in B$ is said to be *rich for agent* i if there exist $|\Omega|-1$ differentiable paths of (i,b) such that these tangents are linearly independent. An action profile $b \in B$ is said to be *rich* if b is rich for all $i \in N$.

Figure 3 illustrates a case of rich action profile, where we assume $|\Omega|=3$. If we can take two (i.e., $|\Omega|-1$) different differentiable paths whose tangents are linearly independent, then b is rich for i by definition. Since the tangents span $\Delta(\Omega)$, by taking mixtures of unilateral deviations along the first differentiable path β_i^1 and the second differentiable path β_i^2, agent i can locally change the state distribution for all directions.[8]

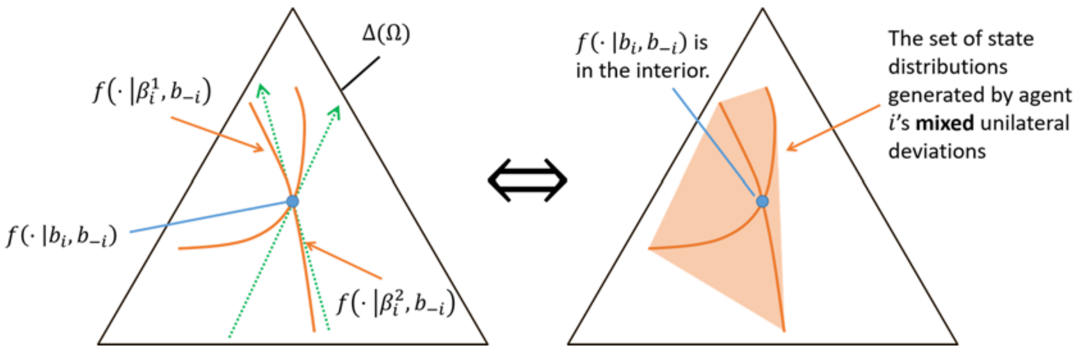

Figure 3: Richness. The case of $|\Omega|=3$. b is said to be rich for i if there are two $(=|\Omega|-1)$ different differentiable paths.

When b is rich for i, it follows from Theorem 1 and $\dim X_i(b,g) \geq 1$ that

$$\dim X_i(b,g) = 1.$$

---

[8] This indicates that $f(\cdot\,|b)$ locates in the interior of the set of state distributions.

This implies that, whenever x_i and $\tilde{x}_i$ induce b for agent i, then x_i and $\tilde{x}_i$ are the same up to constants. Hence, we have proved the following equivalence theorem:

Theorem 2: *Suppose that* b *is rich for agent* i *and* (g, x_i) *induces* b *for agent* i. *Then,*

$$[\, x_i \in X_i(b,g) \text{ and } \tilde{x}_i \in X_i(b,g)\,] \Leftrightarrow [\, \tilde{x}_i = x_i + \overline{z}_i \text{ for some } \overline{z}_i \in R\,].$$

Theorem 2 implies *general equivalence properties in the ex-post term* as follows. Consider an arbitrary combination of an action profile and an allocation rule (b, g). Consider two arbitrary payment rules x and $\tilde{x}$ such that both (g, x) and $(g, \tilde{x})$ induce b. Let $U_i \in R$ and $\tilde{U}_i \in R$ denote the respective ex-ante expected payoff for each agent $i \in N$:

$$U_i \equiv E[v_i(g(\omega),\omega) - x_i(\omega) \,|\, b] - c_i(b_i),$$

and

$$\tilde{U}_i \equiv E[v_i(g(\omega),\omega) - \tilde{x}_i(\omega) \,|\, b] - c_i(b_i).$$

Theorem 2 exhibits that

(i) the ex-post payment for each agent i is unique up to constants in that

$$\tilde{x}_i(\omega) = x_i(\omega) - U_i + \tilde{U}_i \text{ for all } \omega \in \Omega,$$

(ii) the ex-post revenue for the central planner is unique up to constants in that

$$\sum_{i \in N} \tilde{x}_i(\omega) = \sum_{i \in N} x_i(\omega) - \sum_{i \in N} (U_i - \tilde{U}_i) \text{ for all } \omega \in \Omega,$$

(iii) the ex-post payoff for each agent i is unique up to constants in that

$$v_i(g(\omega),\omega) - \tilde{x}_i(\omega) - c_i(b_i) = \{v_i(g(\omega),\omega) - x_i(\omega) - c_i(b_i)\} + U_i - \tilde{U}_i$$

for all $\omega \in \Omega$.

4. Hidden Information

We specify the hidden information structure as follows. The state $\omega \in \Omega$ is decomposed as

$$\omega = (\omega_0, \omega_1, ..., \omega_n) .$$

We call ω_0 a "public signal" and ω_i a "type" for each agent $i \in N$. Let Ω_0 denote the set of all public signals and Ω_i denote the set of all types for each agent $i \in N$. Let $\Omega \equiv \underset{i \in N \cup \{0\}}{\times} \Omega_i$.

We assume that the public signal $\omega_0 \in \Omega_0$ becomes observable to all agents as well as the central planner just before the central planner determines an allocation and side payments; it is therefore contractible. However, the central planner cannot observe the profiles of all agents' types, which is denoted by $\omega_{-0} = (\omega_i)_{i \in N} \in \Omega_{-0} \equiv \underset{i \in N}{\times} \Omega_i$. Agents $i \in N$ can observe their own type $\omega_i \in \Omega_i$ but cannot observe the profiles of the other agents' types, which is denoted by $\omega_{-i} = (\omega_j)_{j \in N \cup \{0\} \setminus \{i\}} \in \Omega_{-i} \equiv \underset{j \in N \cup \{0\} \setminus \{i\}}{\times} \Omega_j$.

Because of the abovementioned hidden information structure, we replace stages 3 and 4 in Section 3 with the following stages 3' and 4', respectively. Each agent observes her own type at stage 3'. The central planner requires each agent to reveal her type at stage 4'. Without loss of generality, we can focus on revelation mechanisms in which each agent reports only her type, because the central planner attempts to induce a pure action profile (see Proposition 1 in Obara [2008] for the revelation principle with ex-ante actions[9]).

Stage 3': The state $\omega = (\omega_0, \omega_1, \cdots, \omega_n) \in \Omega$ is randomly drawn from the conditional probability function $f(\cdot | b) \in \Delta(\Omega)$, where $b \in B$ is the action profile selected at stage 2. Each agent $i \in N$ observes her type $\omega_i \in \Omega_i$ but cannot observe $\omega_{-i} \in \Omega_{-i}$ at this stage.

[9] Obara (2008) also argued that, if the central planner attempts to induce a mixed action profile, we need to consider mechanisms in which each agent reports not only her type but also her selection of pure action. In a model with finite action spaces, by carefully mixing agents' actions, Obara (2008) generated a correlation between (b_i, ω_i) and (b_{-i}, ω_{-i}) and then applies the technique of Crémer and McLean (1985, 1988) to the incentives in revelations. However, the argument by Obara needs to make extremely large side payments and utilize very detailed knowledge about specifications.

To include contract design issues, we permit the presence of publicly observable signal at stage 4’ as follows:

Stage 4’: Each agent $i \in N$ announces $\tilde{\omega}_i \in \Omega_i$ about her type. Afterward, all agents, as well as the central planner, observe the public signal $\omega_0 \in \Omega_0$. According to the profile of the agents’ announcements $\tilde{\omega}_{-0} = (\tilde{\omega}_i)_{i \in N} \in \Omega_{-0}$ and the observed public signal $\omega_0 \in \Omega_0$, the central planner determines the allocation $g(\omega_0, \tilde{\omega}_{-0}) \in A$ and the side payment vector $x(\omega_0, \tilde{\omega}_{-0}) \in R^n$. The resultant payoff of each agent i is given by

$$v_i(g(\omega_0, \tilde{\omega}_{-0}), \omega) - x_i(\omega_0, \tilde{\omega}_{-0}) - c_i(b_i).$$

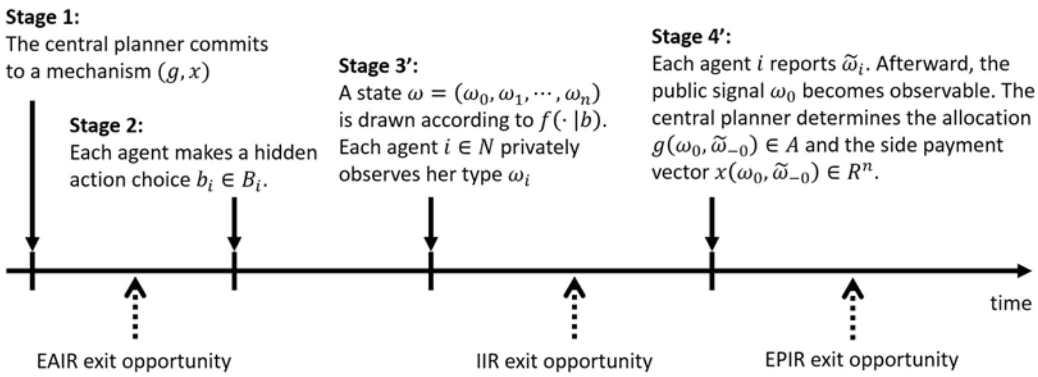

Figure 4: Timeline with Hidden Action and Hidden Information

Figure 4 describes the timeline of the model with hidden action and hidden information. We introduce notions of incentive compatibility as follows.

Definition 4 (Ex-post Incentive Compatibility): A mechanism (g, x) is said to be *ex-post incentive compatible* (EPIC) if truth-telling is an ex-post equilibrium; for every $i \in N$ and $\omega \in \Omega$,

$$v_i(g(\omega), \omega) - x_i(\omega) \geq v_i(g(\tilde{\omega}_i, \omega_{-i}), \omega) - x_i(\tilde{\omega}_i, \omega_{-i}) \quad \text{for all } \tilde{\omega}_i \in \Omega_i.$$

EPIC is independent of the action profile due to additive separability. We further introduce a notion weaker than EPIC or inducibility: *Bayesian Inducibility and Incentive Compatibility*.

Definition 5 (Bayesian Inducibility and Incentive Compatibility): A combination of an action profile and a mechanism $(b,(g,x))$ is said to be *Bayesian inducible and incentive compatible* (BIIC) if the selection of the action profile b at stage 2 and the truthful revelation at stage 4' result in a perfect Bayesian equilibrium; for every $i \in N$, $b_i' \in B_i$, and $\sigma_i : \Omega_i \to \Omega_i$,

$$E[v_i(g(\omega),\omega) - x_i(\omega) \mid b] - c_i(b_i)$$
$$\geq E[v_i(g(\sigma_i(\omega_i),\omega_{-i}),\omega_{-i}) - x_i(\sigma_i(\omega_i),\omega_{-i}) \mid b_i', b_{-i}] - c_i(b_i').$$

BIIC requires $(b,(g,x))$ to exclude the possibility that each agent i benefits by deviating from both the action choice b_i at stage 2 and the truthful revelation at stage 4'.

The following theorem states that, if there is a mechanism that induces an action profile b but fails to be incentive compatible, it is generally impossible to discover a mechanism that satisfies both inducibility and incentive compatibility:

Theorem 3: *Consider an arbitrary combination of an allocation rule and an action profile* (b,g) *such that* b *is rich.*

(i) *Suppose that there exists a payment rule* x *such that* (g,x) *induces* b *and satisfies EPIC. Then, for every payment rule* $\tilde{x}$*, whenever* $(g,\tilde{x})$ *induces* b*, it satisfies EPIC.*

(ii) *Suppose that there exists a payment rule* x *such that* $(b,(g,x))$ *satisfies BIIC. Then, for every payment rule* $\tilde{x}$*, whenever* $(g,\tilde{x})$ *induces* b*,* $(b,(g,\tilde{x}))$ *satisfies BIIC.*

Proof: Suppose that both (g,x) and $(g,\tilde{x})$ induce b. By Theorem 2, there exists $\bar{z}\in R^n$ such that

$$x(\omega)-\tilde{x}(\omega)=\bar{z} \quad \text{for all} \quad \omega\in\Omega,$$

which implies that (g,x) satisfies EPIC if and only if $(g,\tilde{x})$ satisfies EPIC. We can similarly prove that $(b,(g,x))$ satisfies BIIC if and only if $(b,(g,\tilde{x}))$ satisfies BIIC.

Q.E.D.

Suppose that there exists at least one mechanism that satisfies both inducibility and incentive compatibility. From Theorem 3, it then follows that every mechanism that satisfies inducibility automatically satisfies incentive compatibility (i.e., every mechanism that resolves the hidden action problem automatically resolves the hidden information problem).

5. Efficiency

We denote by $v_0 : A\times\Omega\to R$ the valuation function of the central planner. This section and the next intensively study allocation rules and action profiles that are *efficient* (i.e., that maximize the total welfare of all agents and the central planner). An allocation rule g is said to be *allocatively efficient* if

$$\sum_{i\in N\cup\{0\}} v_i(g(\omega),\omega)\geq \sum_{i\in N\cup\{0\}} v_i(a,\omega) \quad \text{for all} \quad a\in A \quad \text{and} \quad \omega\in\Omega.$$

A combination of an action profile and an allocation rule (b,g) is said to be *fully efficient* if g is allocatively efficient and the selection of b maximizes the total welfare in expectation—i.e., it satisfies

$$E[\sum_{i\in N\cup\{0\}} v_i(g(\omega),\omega)\,|\,b]-\sum_{i\in N\cup\{0\}} c_i(b_i)$$

$$\geq E[\sum_{i\in N\cup\{0\}} v_i(g(\omega),\omega)\,|\,\tilde{b}]-\sum_{i\in N\cup\{0\}} c_i(\tilde{b}_i) \quad \text{for all} \quad \tilde{b}\in B.$$

A mechanism (g,x) is said to be *fully efficient with inducibility* if there exists an action profile $b\in B$ such that (g,x) induces b and (b,g) is fully efficient.

A payment rule x is said to be *VCG* if there exists $y_i : \Omega_{-i} \to R$ for each $i \in N$ such that

$$x_i(\omega) = -\sum_{j \in N \cup \{0\} \setminus \{i\}} v_j(g(\omega), \omega) + y_i(\omega_{-i}) \text{ for all } i \in N \text{ and } \omega \in \Omega .$$

According to a VCG payment rule, the central planner gives each agent $i \in N$ the monetary amount equivalent to the other agents' welfare plus the central planner's welfare (i.e., $\sum_{j \in N \cup \{0\} \setminus \{i\}} v_j(g(\omega), \omega)$) and retrieves the monetary payment $y_i(\omega_{-i})$ that is independent of ω_i.

We now introduce a subclass of VCG mechanisms, the "pure-VCG mechanism," as follows. A payment rule x is said to be *pure-VCG* if it is VCG and $y_i(\cdot)$ is constant for each $i \in N$; there exists a vector $\bar{y} = (\bar{y}_i)_{i \in N} \in R^n$ such that

$$x_i(\omega) = -\sum_{j \in N \cup \{0\} \setminus \{i\}} v_j(g(\omega), \omega) + \bar{y}_i \text{ for all } i \in N \text{ and } \omega \in \Omega .$$

Pure-VCG payment rules are special cases of VCG payment rules, where the central planner imposes on each agent i a fixed amount $\bar{y}_i$ as a non-incentive term. We call a combination of efficient allocation rule and a VCG payment rule (pure-VCG payment rule) a "VCG mechanism" (or "pure-VCG mechanism").[10]

The following theorem states that pure-VCG mechanisms are fully efficient with inducibility and that, under the assumption of richness, only pure-VCG mechanisms can achieve full efficiency with inducibility:

Theorem 4:

(i) *Pure-VCG mechanisms are fully efficient with inducibility.*

(ii) *Suppose that, for every fully efficient* (b, g), b *is rich. Then, a mechanism is fully efficient with inducibility if and only if it is pure-VCG.*

Proof: If (b, g) is fully efficient and x is pure-VCG, then, for every $i \in N$ and $b_i' \in B_i$,

[10] This study slightly extends the canonical definition of a VCG mechanism because we account for the valuations of the central planner.

$$E[v_i(g(\omega),\omega)-x_i(\omega)\,|\,b]-c_i(b_i)-E[v_i(g(\omega),\omega)-x_i(\omega)\,|\,b_i',b_{-i}]+c_i(b_i')$$
$$=E[\sum_{j\in N\cup\{0\}} v_j(g(\omega),\omega)\,|\,b]-c_i(b_i)-E[\sum_{j\in N\cup\{0\}} v_j(g(\omega),\omega)\,|\,b_i',b_{-i}]+c_i(b_i')$$
$$\geq 0,$$

which implies that (g,x) induces b. Furthermore, if b is rich, it is clear from Theorem 2 and the definition of the pure-VCG payment rule that only pure-VCG mechanisms induce b.

Q.E.D.

We can construct any pure-VCG mechanism without utilizing any detailed knowledge about (f,B,c). Even if such detailed knowledge is available, the pure-VCG mechanism is the best choice for the central planner who is attempting to maximize total welfare.

Theorem 4 provides us a necessary and sufficient condition for the existence of a mechanism that satisfies both full efficiency with inducibility and incentive compatibility (i.e., satisfies either EPIC or BIIC), provided the action profile is rich. Such a mechanism exists only when pure-VCG mechanisms are incentive compatible. However, when each agent's payoff function has interdependent values, VCG mechanisms rarely satisfy either EPIC or BIIC, indicating that there is no fully efficient mechanism with inducibility in the interdependent value case.[11]

Based on this observation, the remainder of this study will focus on the case of *private values*, where we assume that, for every $i\in N$, the valuation $v_i(a,\omega)$ is independent of the other agents' type profile $\omega_{-0-i}\equiv(\omega_j)_{j\in N\setminus\{i\}}\in\Omega_{-0-i}\equiv\times_{j\in N\setminus\{i\}}\Omega_j$,

[11] Maskin (1992), Dasgupta and Maskin (2000), Jehiel and Moldovanu (2001), and Bergemann and Välimäki (2002) propose the generalized VCG mechanism and show that it satisfies EPIC for some environments, even with interdependent values. The generalized VCG mechanism fails to induce an efficient action profile, however, because it is not pure-VCG. On the other hand, Mezzetti (2004) shows that, if the realized valuation $v_i(g(\omega),\omega)$ is observable as an ex-post public signal and is contractible, even with interdependent values, we can implement ex-post efficient mechanism providing any VCG expected payoffs to each agent (see Noda [2016] for more general ex-post signals). If we can use these schemes, we may achieve full efficiency with interdependent values.

and $v_0(a,\omega)$ is independent of ω_{-0}. We write $v_i(a,\omega_0,\omega_i)$ instead of $v_i(a,\omega)$ for each $i \in N$, and $v_0(a,\omega_0)$ instead of $v_0(a,\omega)$.[12]

With the assumption of private values, EPIC is equivalent to *incentive compatibility in dominant strategies* (DIC); for every $i \in N$, $\omega \in \Omega$, and $\tilde{\omega}_i \in \Omega_i$,

$$v_i(g(\omega),\omega_0,\omega_i) - x_i(\omega) \geq v_i(g(\tilde{\omega}_i,\omega_{-i}),\omega_0,\omega_i) - x_i(\tilde{\omega}_i,\omega_{-i}).$$

With the assumption of private values, any VCG mechanism satisfies DIC; thus, we have proved the following proposition:

Proposition 2: *With the assumption of private values, pure-VCG mechanisms are fully efficient with inducibility and satisfy DIC.*

Although VCG mechanisms generally satisfy DIC and allocative efficiency in the private value case, they are generally not fully efficient with inducibility. For example, consider the *pivot mechanism* (g,x), which is a VCG mechanism specified by

$$x_i(\omega) = -\sum_{j \in N \cup \{0\} \setminus \{i\}} v_i(g(\omega),\omega_0,\omega_j) + \min_{\omega_i' \in \Omega_i} \sum_{j \in N \cup \{0\}} v_i(g(\omega_i',\omega_{-i}),\omega_0,\omega_j)$$

for all $i \in N$ and $\omega \in \Omega$.

Clearly, the pivot mechanism is generally not pure-VCG because the non-incentive term is not constant; thus, it is not always fully efficient with inducibility. The following proposition demonstrates a sufficient condition under which the pivot mechanism fails to be fully efficient with inducibility:

Proposition 3: *Let* g *be an allocatively efficient allocation rule. Define* $z_i : \Omega \to R$ *by*

(7) $$z_i(\omega) = \min_{\omega_i' \in \Omega_i} \sum_{j \in N \cup \{0\}} v_i(g(\omega_i',\omega_{-i}),\omega_0,\omega_j) \quad \textit{for all } i \in N.$$

[12] We permit the valuations to depend on the public signal ω_0. We write $v_0(a,\omega_0,\omega_0)$ instead of $v_0(a,\omega_0)$.

Then, a pivot mechanism (g,x) *is not fully efficient with inducibility if, for every* $b \in B$ *such that* (b,g) *is fully efficient, there exists* $i \in N$ *and a differentiable path* β_i *of* (i,b) *such that*

(8) $$z_i \cdot t(\beta_i) \neq 0 .$$

Proof: Note that z_i, defined by (7), is the difference between a pure-VCG payment rule and the pivot payment rule. Since pure-VCG mechanisms induce b, it follows from Proposition 1 that the pivot mechanism induces b for i only if $z_i \cdot t(\beta_i) = 0$.

Q.E.D.

From Proposition 3, it is clear that, if the targeted efficient action profile is rich, then the pivot mechanism is not fully efficient with inducibility. Importantly, Proposition 3 indicates that the pivot mechanism fails to be fully efficient with inducibility under a much weaker condition than richness. Indeed, even if there exists at least one differentiable path around the efficient action profile, the pivot mechanism generically fails to be fully efficient with inducibility. We have no reasons to expect that every tangent of differentiable paths is perpendicular to z_i, defined by (7). Accordingly, we cannot expect the pivot mechanism to induce an efficient action profile. For example, we have observed in Examples 1 and 2 that the pivot mechanism encourages agents to select inefficient actions.

6. Open-Bid Descending Procedure

So far, we have considered static revelation mechanisms at stage 4' and have shown that only pure VCG mechanisms induce efficient action profiles. In a single-unit auction with private values, we can implement a pure-VCG mechanism with an open-bid procedure, just as we can implement a pivot mechanism with an ascending auction. The allocation $a \in A \equiv \{1,2,\cdots,n\}$ specifies the agent who is the winner (i.e., obtains the object). Each agent's type $\omega_i \in \Omega_i \subset R_+$ specifies the payoff of the agent when she obtains the object, i.e.,

$$v_i(a,\omega_i)=\begin{cases}\omega_i & \text{if } a=i\\ 0 & \text{otherwise}\end{cases}.$$

Fix an arbitrary real number $\bar{y}_i \in R$ for each agent $i \in N$. A pure-VCG mechanism (g,x) is specified in the manner that the allocation rule g is allocatively efficient; thus,

$$g(\omega) \in \arg\max_{i\in N} \omega_i \quad \text{for all} \quad \omega \in \Omega,$$

and the payment rule is given by

$$x_i(\omega)=\begin{cases}\bar{y}_i & \text{if } g(\omega)=i\\ -\omega_{g(\omega)}+\bar{y}_i & \text{otherwise}\end{cases}.$$

We consider the following open-bid descending procedure:

1. Each agent pays a fixed participation fee $\bar{y}_i$ to the central planner at the beginning.
2. The central planner initially sets the price sufficiently high and then gradually descends the price.
3. When an agent declares to take the object, this descending procedure immediately terminates. The agent who declares to take the object becomes the winner and obtains this object.
4. The central planner does *not* require the winner to pay any additional fee. Instead, the central planner gives any other agent (i.e., any loser) the price at the ending time as a compensation.

Given this procedure, it is a weakly dominant strategy[13] for each agent to "declare to take the object" exactly when the current price is equal to the agents' value. If all the agents take this dominant strategy, then (i) the agent with the highest value wins, (ii) the winner's payment is $\bar{y}_i$, and (iii) the loser's payment is $-\omega_{g(\omega)}+\bar{y}_i$. Accordingly,

[13] It is worth noting that this open-bid descending procedure satisfies *obvious strategy-proofness* as defined in Li (2016), as in ascending auctions.

this open-bid descending procedure generates the same outcome as a pure-VCG mechanism.

The pivot mechanism (the second-price auction) is implemented by the popular ascending auction that determines the winner's payment. By contrast, our procedure descends the price and determines the compensation for losers, implementing a pure-VCG mechanism.

7. Revenues and Deficits

Assuming private values, this section studies whether the central planner can achieve efficiency *without deficits*. We define the central planner's ex-post revenue by

(9) $$v_0(g(\omega),\omega_0)+\sum_{i\in N}x_i(\omega)\,,$$ [14]

and the expected revenue in the ex-ante term by

$$E[v_0(g(\omega),\omega_0)+\sum_{i\in N}x_i(\omega)\,|\,b].$$

We introduce three notions of individual rationality below. The timings of the exit opportunities are depicted in Figure 2.

Definition 6 (Ex-ante Individual Rationality): A combination of an action profile and a mechanism $(b,(g,x))$ is said to satisfy *ex-ante individual rationality (hereafter EAIR)* if

$$E[v_i(g(\omega),\omega_0,\omega_i)-x_i(\omega)\,|\,b]-c_i(b_i)\geq 0 \quad \text{for all } i\in N.$$

Definition 7 (Interim Individual Rationality): A combination of an action profile and a mechanism $(b,(g,x))$ is said to satisfy *interim individual rationality* (IIR) if

$$E_{\omega_{-i}}[v_i(g(\omega_i,\omega_{-i}),\omega_0,\omega_i)-x_i(\omega_i,\omega_{-i})\,|\,b,\omega_i]\geq 0$$

for all $i\in N$ and $\omega_i\in\Omega_i$,

where

[14] Note that the central planner's revenue includes not only payments from agents but also the valuation of the central planner. In the single agent case, where the allocation space is degenerate, the value of (10) corresponds to the principal's payoff in a standard principal–agent model with a risk-neutral principal and agent.

$$E_{\omega_{-i}}[\xi(\tilde{\omega}_i, \omega_{-i}) \mid b, \omega_i] \equiv \sum_{\omega_{-i} \in \Omega_{-i}} \xi(\tilde{\omega}_i, \omega_{-i}) f_{-i}(\omega_{-i} \mid b, \omega_i)$$

denotes the expectation of a function $\xi(\tilde{\omega}_i, \cdot) : \Omega_{-i} \to R$ conditional on (b, ω_i).

Definition 8 (Ex-post Individual Rationality): A mechanism (g, x) is said to satisfy *ex-post individual rationality* (EPIR) if

$$v_i(g(\omega), \omega_0, \omega_i) - x_i(\omega) \geq 0 \text{ for all } i \in N \text{ and } \omega \in \Omega.$$

EPIR implies IIR; however, IIR does not necessarily imply EAIR because the cost for the action choice at stage 2 is sunk. The following proposition shows that EPIR implies EAIR:

Proposition 4: *Suppose that* (g, x) *induces* b. *Whenever* (g, x) *satisfies EPIR,* $(b, (g, x))$ *satisfies EAIR.*

Proof: Because $c_i(b_i^0) = 0$, it follows from EPIR and inducibility that

$$E[v_i(g(\omega), \omega_0, \omega_i) - x_i(\omega) \mid b] - c_i(b_i)$$
$$\geq E[v_i(g(\omega), \omega_0, \omega_i) - x_i(\omega) \mid b_i^0, b_{-i}] - c_i(b_i^0) \geq 0,$$

which implies EAIR.

Q.E.D.

EPIR is the strongest requirement for voluntary participation among the above three concepts. This section shows that EPIR is not compatible with the non-negativity of revenue in expectation.

The following proposition calculates the maximal expected revenues (i.e., the least upper bounds of the central planner's expected revenues):

Proposition 5: *Suppose that* (b, g) *is fully efficient,* b *is rich, and the private value assumption is satisfied. Then, the maximal expected revenue from* $(b, (g, x))$ *in terms of* $x \in \times_{i \in N} X_i(b, g)$ *that satisfies EPIR, IIR, EAIR, and IIR and EAIR are given by*

(10) $$R^{EPIR} \equiv n \min_{\omega \in \Omega} \sum_{j \in N \cup \{0\}} v_j(g(\omega), \omega_0, \omega_j) - (n-1) E\left[\sum_{j \in N \cup \{0\}} v_j(g(\omega), \omega_0, \omega_j) \middle| b \right],$$

(11) $$R^{IIR} \equiv \sum_{i \in N} \min_{\omega_i \in \Omega_i} E_{\omega_{-i}} \left[\sum_{j \in N \cup \{0\}} v_j(g(\omega_i, \omega_{-i}), \omega_0, \omega_j) \middle| b, \omega_i \right]$$

$$-(n-1) E\left[\sum_{j \in N \cup \{0\}} v_j(g(\omega), \omega_0, \omega_j) \middle| b \right],$$

(12) $$R^{EAIR} \equiv E\left[\sum_{j \in N \cup \{0\}} v_j(g(\omega), \omega_0, \omega_j) \middle| b \right] - \sum_{j \in N} c_j(b_j), \textit{ and}$$

(13) $$R^{EAIR,IIR} \equiv E\left[v_0(g(\omega), \omega_0) \middle| b \right] + \sum_{i \in N} \min \Big\{ E\left[v_i(g(\omega), \omega_0, \omega_i) \middle| b \right] - c_i(b_i),$$

$$\min_{\omega_i \in \Omega_i} E_{\omega_{-i}} \left[\sum_{j \in N \cup \{0\}} v_j(g(\omega_i, \omega_{-i}), \omega_0, \omega_j) \middle| b, \omega_i \right] - E\left[\sum_{j \in N \cup \{0\} \setminus \{i\}} v_j(g(\omega), \omega_0, \omega_j) \middle| b \right] \Big\},$$

respectively.

Proof: See the Appendix.

Clearly, R^{EAIR} is equal to the *maximized expected social welfare*. From the relative strength of the incentive compatibility constraints, it is also clear that

$$R^{EPIR} \le R^{IIR,EAIR} \le R^{IIR} \text{ and } R^{IIR,EAIR} \le R^{EAIR}.$$

However, which is greater between R^{IIR} and R^{EAIR} depends on the specifications.

The following proposition indicates that, with the constraints of EPIR, it is generally difficult for the central planner to achieve full efficiency without deficits:

Proposition 6: *Suppose that* (b, g) *is fully efficient,* b *is rich, and the private value assumption is satisfied. Suppose also that* $\sum_{i \in N} c_i(b_i) > 0$ *and there exists a null state* $\underline{\omega} = (\underline{\omega}_0, ..., \underline{\omega}_n) \in \Omega$ *in the sense that*

$$v_0(a, \underline{\omega}_0) = 0 \textit{ for all } a \in A,$$

and for every $i \in N$,

$$v_i(a, \omega_0, \underline{\omega}_i) = 0 \textit{ for all } a \in A \textit{ and } \omega_0 \in \Omega_0.$$

With EPIR, the central planner has a deficit in expectation: $R^{EPIR} < 0$.

Proof: See the Appendix.

If $R^{EAIR} < 0$, the conclusion is immediate from $R^{EPIR} \leq R^{EAIR}$. Suppose $R^{EAIR} \geq 0$. It follows from $\sum_{i \in N} c_i(b_i) > 0$ that the second term of R^{EPIR} in (10) is negative. Due to the presence of the null state, the first term of R^{EPIR} in (10) is non-positive. Accordingly, R^{EPIR} is negative.

By replacing EPIR with weaker participation constraints, such as IIR and EAIR, and adding some restrictions, it becomes much easier for the central planner to achieve full efficiency without deficits. The following proposition states that the central planner can earn the same expected revenue as in the case of observable actions:

Proposition 7: *Suppose that* (b, g) *is fully efficient,* b *is rich, and the private value assumption is satisfied. Suppose also that we have:*

Conditionally Independent Types: *For every* $\tilde{b} \in B$ *and* $\omega \in \Omega$,

$$f(\omega \mid \tilde{b}) = \prod_{i \in N \cup \{0\}} f_i(\omega_i \mid \tilde{b}),$$

where $f_i(\omega_i \mid \tilde{b})$ *denotes the probability of* ω_i *occurring when the agents select the action profile* $\tilde{b}$. *Then, the expected revenue achieved by any VCG mechanism that satisfies IIR and EAIR is less than or equal to* $R^{IIR,EAIR}$. *Furthermore, there exists a pure-VCG mechanism that satisfies IIR and EAIR and achieves* $R^{IIR,EAIR}$.

Proof: See the Appendix.

One may expect that the central planner can receive greater expected revenue than $R^{IIR,EAIR}$ once we remove the requirement of inducibility. However, Proposition 7 indicates that, with conditionally independent types, no VCG mechanism is able to make the expected revenue greater than $R^{IIR,EAIR}$. This implies that $R^{IIR,EAIR}$ is the upper bound of expected revenue regardless of whether we require inducibility.[15]

Proposition 7 is related to the observation from a classical principal–agent model. When both the principal and agent are risk-neutral, one of the revenue-maximizing contracts (which achieves the first-best of the principal) is to sell the company to the agent—that is, to give the entire outcome (externalities to the principal) in exchange for a fixed constant fee. By regarding pure-VCG mechanisms as an extension of such selling-out contracts, Proposition 7 indicates that selling the company is the *unique* revenue-maximizing contract when the efficient action is rich and the agent has unlimited liability (which corresponds to EAIR and IIR).[16]

It is widely accepted that efficiency is achievable through a VCG mechanism without running expected deficits if we do not require inducibility. With conditionally independent types and some moderate restrictions, we can guarantee the non-negativity of $R^{IIR,EAIR}$ as follows:

Proposition 8: *Assume the suppositions in Proposition 5, conditionally independent types, and the following conditions:*

Non-Negative Valuation: *For every* $i \in N \cup \{0\}$ *and* $\omega \in \Omega$,

$$v_i(g(\omega), \omega_0, \omega_i) \geq 0.$$

Non-Negative Expected Payoff: *For every* $i \in N$,

$$E[v_i(g(\omega), \omega_0, \omega_j) \mid b] - c_i(b_i) \geq 0. \tag{14}$$

With IIR and EAIR, the central planner has non-negative expected revenue: $R^{IIR,EAIR} \geq 0$.

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[15] In the main body of this paper, we assume finite state spaces. Accordingly, some non-VCG mechanisms may be allocatively efficient and satisfy DIC. However, if we take a fine grid of the states to make the state space closer to the continuum (with which the regularity condition of the Green–Laffont–Holmström theorem is satisfied), only the mechanisms that are close to VCG can satisfy DIC. In such a case, the maximum possible revenue is also close to $R^{IIR,EAIR}$.

[16] Harris and Raviv (1979) showed that the principal can extract full surplus if the agent is risk-neutral and has unlimited liability. Holmström and Milgrom (1987) established that, if the agent has a CRRA utility function and can take various deviations, there exists a unique contract that induces an efficient action. We generalize their insights to multi-agent mechanism design problems to derive the impossibility results.

Proof: See the Appendix.

Non-negative valuation excludes the case of bilateral bargaining addressed by Myerson and Satterthwaite (1983), where it is impossible for the central planner to achieve allocative efficiency without deficits. Non-negative expected payoff excludes the case of opportunism in the hold-up problem, where the sunk cost $c_i(b_i)$ is so large that it violates inequality (14). By eliminating these cases and replacing EPIR with IIR and EAIR, we can derive the possibility result in liability implied by Proposition 8.

8. Absence of Externality

So far, we have permitted each agent's action choice to influence the other agents' valuation functions through the action-dependence of the state distribution (i.e., we have assumed the presence of externality). We have seen that, even if the dimensionality of unilateral deviations is low, the pivot mechanism generally fails to satisfy the full efficiency with inducibility (Proposition 3). If the dimensionality of unilateral deviations is sufficient (i.e., if efficient action profiles are rich), only a very special VCG mechanism, a pure-VCG mechanism, satisfies full efficiency with inducibility (Theorem 4). Consequently, it is difficult for the central planner to earn a moderate amount of revenue in a manner compatible with EPIR and full efficiency with inducibility (Proposition 6).

In this section, we articulate that the externality effects of actions are crucial for the abovementioned results. To demonstrate the importance of the externalities, this section assumes that each agent's action does not have externality effects but that each agent has a very large set of actions. We then show a characterization of all mechanisms that satisfy full efficiency with inducibility. This characterization implies that it is generally possible for the central planner to earn a positive revenue and even make the payment rule budget-balancing.

As in Bergemann and Välimäki (2002) and Hatfield, Kojima, and Kominers (2015), this section assumes *independent types* of the information structure, which expresses the absence of externality. Independent types requires that, in addition to conditionally independent types, each agent i 's action choice $b_i \in B_i$ influences only the marginal distribution of the agent's type ω_i; for every $\omega \in \Omega$ and $b \in B$,

$$f(\omega \mid b) = f_0(\omega_0)\prod_{i\in N} f_i(\omega_i \mid b_i).$$

Here, $f_i(\cdot \mid b_i)$ denotes the marginal distribution of each agent i 's type ω_i that is assumed to depend only on b_i, and $f_0(\cdot)$ denotes the distribution of the public signal ω_0 that is assumed to be independent of the action profile $b \in B$. When we have independent types, agent i 's action space is equivalent to the set of available marginal distributions on agent i 's type. Accordingly, this restriction expresses no externality. With independent types, for every $\xi : \Omega \to R$ and $\xi_i : \Omega_i \to R$, we can simply write $E_{\omega_{-i}}[\xi(\tilde{\omega}_i, \omega_{-i}) \mid b_{-i}]$ and $E[\xi_i(\omega_i) \mid b_i]$ instead of $E_{\omega_{-i}}[\xi(\tilde{\omega}_i, \omega_{-i}) \mid b, \omega_i]$ and $E[\xi_i(\omega_i) \mid b]$, respectively.

Since the choice of b_i affects only agent i 's marginal type distribution, we can define the differentiable path of (i,b) independent of the other agents' action profile b_{-i}. With independent types, if $\beta_i : [-1,1] \to \beta_i$ is a differentiable path of $(i,b) \in N \times B$, then, for every $b_{-i} \in B_{-i}$, β_i is also a differentiable path of (i, b_i, b'_{-i}). Slightly abusing the notation, we write "β_i is a differentiable path of (i, b_i)" instead of "β_i is a differentiable path of (i, b_i, b_{-i})."

With independent types, the choice of b_i is isomorphic to the choice of $f_i \in \Delta(\Omega_i)$, where $\dim \Delta(\Omega_i) = |\Omega_i| - 1$. Accordingly, there are at most $|\Omega_i| - 1$ differentiable paths of (i, b_i) whose tangents are linearly independent. We say that a targeted action b_i is privately rich if such differentiable paths actually exist. If the targeted action is privately rich, an agent has an extremely large action set, while the assumption of independent types excludes externality effects.

Definition 9 (Private Richness): An action $b_i \in B_i$ is said to be *privately rich for agent* $i \in N$ if there exist $|\Omega_i| - 1$ differentiable paths of (i, b_i) whose tangents are linearly independent. An action profile $b \in B$ is said to be privately *rich* if b_i is privately rich for all $i \in N$.

A payment rule x is said to be *expectation-VCG* if, for each $i \in N$, there exist $r_i : \Omega \to R$ such that for every $i \in N$,

$$x_i(\omega) = -\sum_{j \in N \cup \{0\} \setminus \{i\}} v_j(g(\omega), \omega) + r_i(\omega) \quad \text{for all} \quad \omega \in \Omega,$$

and $E_{\omega_{-i}}[r_i(\omega_i, \omega_{-i}) \mid b_{-i}]$ is independent of $\omega_i \in \Omega_i$. We call the combination of an efficient allocation rule and expectation-VCG payment rule an "expectation-VCG mechanism."

Recall that the distribution of ω_{-i} is determined solely by the choice of b_{-i}, which agent i cannot manipulate. Accordingly, agent i's unilateral deviation does not change the value of $E_{\omega_{-i}}[r_i(\omega_i, \omega_{-i}) \mid b_{-i}]$ whenever it is independent of ω_i. Now, we obtain the following characterization result:

Proposition 9: *Under the assumption of independent types:*

(i) *Any expectation-VCG mechanism is fully efficient with inducibility.*

(ii) *Suppose that, for every fully efficient* (b, g), b *is privately rich. Then, a mechanism is fully efficient with inducibility if and only if it is expectation-VCG.*

Proof: See the Appendix.

Eliminating externality by assuming independent types, we can obtain a very permissive result in that a large class of mechanisms, expectation-VCG, satisfies full efficiency with inducibility regardless of the dimensionality of unilateral deviations. The class of expectation-VCG includes all VCG mechanisms, which guarantees the compatibility of full efficiency with inducibility with EPIR and positive revenues. Bergemann and Välimäki (2002) studied information acquisition for each agent's own hidden state with independent types and private values and showed that VCG mechanisms are fully efficient with inducibility. We extend their result to more general action spaces and mechanisms and derive the full characterization.

Hatfield, Kojima, and Kominers (2015) studied detail-free mechanisms and showed that a fully efficient and detail-free mechanism must be VCG. Proposition 10 clarifies that there are very important mechanisms that are fully efficient with inducibility but not detail-free, as follows. Arrow (1979) and D'Aspremont and Gérard-Varet (1979) specified an efficient mechanism (g,x), the "AGV mechanism," in the manner that

$$r_i(\omega) = \sum_{j\in N\cup\{0\}\setminus\{i\}} v_j(g(\omega),\omega_j) - E_{\tilde{\omega}_{-i}}[\sum_{j\in N\cup\{0\}\setminus\{i\}} v_j(g(\omega_i,\tilde{\omega}_{-i}),\tilde{\omega}_j)\,|\,b_{-i}]$$

$$+\frac{1}{n-1}\sum_{j\in N\setminus\{i\}} E_{\tilde{\omega}_{-j}}[\sum_{h\in N\cup\{0\}\setminus\{j\}} v_h(g(\omega_j,\tilde{\omega}_{-j}),\tilde{\omega}_h)\,|\,b_{-j}].$$

By definition, the AGV mechanism satisfies *budget balance*:

$$\sum_{i\in N} x_i(\omega) = 0 \quad \text{for all } \omega\in\Omega.$$

It is important to note that the AGV mechanism is expectation-VCG. Accordingly, with independent types, the central planner can achieve full efficiency with inducibility in a manner consistent with BIIC and budget balance.

9. Conclusion

We studied a general mechanism design problem in which agents take hidden actions that determine the state distribution; their action choices produce significant externality effects on their valuations. We have shown that the class of mechanisms that successfully induce a desired action profile is substantially restrictive. Indeed, the popular pivot mechanism generally fails to induce an efficient action profile. Hence, it is difficult for a mechanism to satisfy no-deficits and incentive compatibility.

If the availability of unilateral deviations is sufficient (i.e., if the targeted action profile is rich), we obtained quite strong equivalence properties; ex-post payoffs, ex-post payments, and ex-post revenues are all unique up to constants. This implies that the class of mechanisms that satisfy inducibility is severely restricted. In particular, focusing on the achievement of full efficiency, any mechanism that satisfies inducibility must be pure-VCG (i.e., the simplest form of VCG that excludes the pivot mechanism). Accordingly, it is difficult to satisfy both inducibility and incentive compatibility in the interdependent value case, while it is generally possible in the private value case. However, we have severe difficulty achieving non-deficits on the constraints of inducibility.

We also discussed the implementation of the pure-VCG mechanism via an open-bid procedure. We emphasized the importance of descending formats and a determination of the losers' compensation instead of the winner's payment, to overcome the hidden action problem. However, this study considered only the single-object allocation problem with private values. Future research should investigate open-bid mechanisms in hidden action in more general environments.

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Appendix

Proof of Proposition 5: By Theorem 4, we can focus on pure-VCG payment rules, where the constraint for EPIR is equivalent to

(A1) $$\min_{\omega\in\Omega} \sum_{j\in N\cup\{0\}} v_j(g(\omega),\omega_0,\omega_j) \geq z_i \quad \text{for all } i\in N.$$

We can maximize the expected revenue by letting z_i satisfy (A1) with equality for each $i\in N$. Accordingly, the central planner can receive from each agent i the expected value given by

$$\min_{\omega\in\Omega} \sum_{j\in N\cup\{0\}} v_j(g(\omega),\omega_0,\omega_j) - E\left[\sum_{j\in N\cup\{0\}\setminus\{i\}} v_j(g(\omega),\omega_0,\omega_j)\middle| b\right],$$

which implies (10), where we add $E[v_0(g(\omega),\omega_0)\,|\,b]$.

Similarly, the constraint for IIR is equivalent to

(A2) $$\min_{\omega_i\in\Omega_i} E_{\omega_{-i}}\left[\sum_{j\in N\cup\{0\}} v_j(g(\omega_i,\omega_{-i}),\omega_0,\omega_j)\middle|\omega_i,b\right]\geq z_i \text{ for all } i\in N \text{ and } \omega_i\in\Omega_i.$$

We can maximize the expected revenue by letting z_i satisfy (A2) with equality for each $i\in N$. Accordingly, the central planner can receive from each agent i the expected value given by

$$\min_{\omega_i\in\Omega_i} E_{\omega_{-i}}\left[\sum_{j\in N\cup\{0\}} v_j(g(\omega_i,\omega_{-i}),\omega_0,\omega_j)\middle|\omega_i,b\right]-E\left[\sum_{j\in N\cup\{0\}\setminus\{i\}} v_j(g(\omega),\omega_0,\omega_j)\middle|b\right],$$

which implies (11), where we add $E[v_0(g(\omega),\omega_0)\,|\,b]$.

The constraint for EAIR is equivalent to

(A3) $$E\left[\sum_{j\in N\cup\{0\}} v_j(g(\omega),\omega_0,\omega_j)\middle|b\right]-c_i(b_i)\geq z_i \text{ for all } i\in N.$$

We can maximize the expected revenue by letting z_i satisfy (A3) with equality for each $i\in N$. Accordingly, the central planner can receive from each agent i the expected value given by

$$E\left[\sum_{j\in N\cup\{0\}} v_j(g(\omega),\omega_0,\omega_j)\middle|b\right]-c_i(b_i)-E\left[\sum_{j\in N\cup\{0\}\setminus\{i\}} v_j(g(\omega),\omega_0,\omega_j)\middle|b\right]$$

$$=E\left[v_i(g(\omega),\omega_0,\omega_i)\middle|b\right]-c_i(b_i),$$

which implies (12), where we add $E[v_0(g(\omega),\omega_0)\,|\,b]$.

The constraint for IIR and EAIR is equivalent to

(A4) $$\min\left\{E\left[\sum_{j\in N\cup\{0\}} v_j(g(\omega),\omega_0,\omega_j)\middle|b\right]-c_i(b_i),\right.$$

$$\left.\min_{\omega_i\in\Omega_i} E_{\omega_{-i}}\left[\sum_{j\in N\cup\{0\}} v_j(g(\omega_i,\omega_{-i}),\omega_0,\omega_j)\middle|b,\omega_i\right]\right\}\geq z_i.$$

We can maximize the expected revenue by letting z_i satisfy (A4) with equality for each $i\in N$. Accordingly, the central planner can receive from each agent i the expected value given by

$$\min\Big\{E\big[v_i(g(\omega),\omega_0,\omega_i)\big|b\big]-c_i(b_i),$$

$$\min_{\omega_i\in\Omega_i}E_{\omega_{-i}}\left[\sum_{j\in N\cup\{0\}}v_j(g(\omega_i,\omega_{-i}),\omega_0,\omega_j)\middle|b,\omega_i\right]-E\left[\sum_{j\in N\cup\{0\}\setminus\{i\}}v_j(g(\omega),\omega_0,\omega_j)\middle|b\right]\Bigg\}\,,$$

which implies (13), where we add $E[v_0(g(\omega),\omega_0)\,|\,b]$.

Q.E.D.

Proof of Proposition 6: If $R^{EAIR}<0$, the conclusion would be immediate from the fact that $R^{EPIR}\leq R^{EAIR}$. Suppose $R^{EAIR}\geq 0$. Because for $(\underline{\omega}_0,\underline{\omega}_1,\cdots,\underline{\omega}_n)$, $\sum_{i\in N\cup\{0\}}v_i(a,\underline{\omega}_0,\underline{\omega}_i)=0$ holds for all $a\in A$, we have

$$\min_{\omega\in\Omega}\sum_{i\in N\cup\{0\}}v_i(g(\omega),\omega_0,\omega_i)\leq 0\,.$$

It follows from $R^{EAIR}\geq 0$ and $\sum_{i\in N}c_i(b_i)>0$ that we have

$$E\left[\sum_{i\in N\cup\{0\}}v_i(g(\omega),\omega_0,\omega_i)\middle|b\right]\geq\sum_{i\in N}c_i(b_i)>0\,.$$

From these observations,

$$R^{EPIR}=n\min_{\omega\in\Omega}\sum_{j\in N\cup\{0\}}v_j(g(\omega),\omega_j)-(n-1)E\left[\sum_{j\in N\cup\{0\}}v_j(g(\omega),\omega_0,\omega_j)\middle|b\right]<0\,.$$

Q.E.D.

Proof of Proposition 7: We have already shown that there is a pure-VCG mechanism that achieves $R^{IIR,EAIR}$. Assuming conditionally independent types and private values, we consider an arbitrary VCG mechanism (g,x) such that for each $i\in N$, there exists $y_i:\Omega_{-i}\to R$ such that

$$x_i(\omega)=-\sum_{j\in N\cup\{0\}\setminus\{i\}}v_j(g(\omega),\omega_0,\omega_j)+y_i(\omega_{-i})\quad\text{for all }\ \omega\in\Omega\,.$$

EAIR implies

$$E\big[y_i(\omega_{-i})\big|b\big]\leq E\left[\sum_{j\in N\cup\{0\}}v_j(g(\omega),\omega_0,\omega_j)\middle|b\right]-c_i(b_i)\,,$$

while IIR requires

$$E\left[y_i(\omega_{-i})\middle|b,\omega_i\right] \le E_{\omega_{-i}}\left[\sum_{j\in N\cup\{0\}} v_j(g(\omega_i,\omega_{-i}),\omega_0,\omega_j)\middle|b,\omega_i\right]$$

for all $\omega_i \in \Omega_i$,

or, equivalently,

$$E\left[y_i(\omega_{-i})\middle|b\right] \le \min_{\omega_i\in\Omega_i} E_{\omega_{-i}}\left[\sum_{j\in N\cup\{0\}} v_j(g(\omega_i,\omega_{-i}),\omega_0,\omega_j)\middle|b,\omega_i\right].$$

Here, we used the fact that $E\left[y_i(\omega_{-i})\middle|b\right] = E\left[y_i(\omega_{-i})\middle|b,\omega_i\right]$ because of conditional independence. Accordingly, we have

$$E\left[y_i(\omega_{-i})\middle|b\right] \le \min\left\{E\left[\sum_{j\in N\cup\{0\}} v_j(g(\omega),\omega_0,\omega_j)\middle|b\right] - c_i(b_i),\right.$$

$$\left.\min_{\omega_i\in\Omega_i} E_{\omega_{-i}}\left[\sum_{j\in N\cup\{0\}} v_j(g(\omega_i,\omega_{-i}),\omega_0,\omega_j)\middle|b,\omega_i\right]\right\},$$

which implies

$$E[v_0(g(\omega),\omega_0) + \sum_{i\in N} x_i(\omega) \,|\, b] \le R^{IIR,EAIR}.$$

Q.E.D.

Proof of Proposition 8: From conditionally independent types, non-negative valuations, and null state, it follows that, for every $i \in N$ and $\omega_i \in \Omega_i$,

$$E_{\omega_{-i}}\left[\sum_{j\in N\cup\{0\}} v_j(g(\omega_i,\omega_{-i}),\omega_0,\omega_j)\middle|b\right] = E_{\omega_{-i}}\left[\max_{a\in N}\sum_{j\in N\cup\{0\}} v_j(a,\omega_0,\omega_j)\middle|b\right]$$

$$\ge E_{\omega_{-i}}\left[\max_{a\in N}\sum_{j\in N\cup\{0\}\setminus\{i\}} v_j(a,\omega_0,\omega_j)\middle|b\right]$$

$$= E_{\omega_{-i}}\left[v_i(g(\underline{\omega}_i,\omega_{-i}),\omega_0,\underline{\omega}_i) + \sum_{j\in N\cup\{0\}\setminus\{i\}} v_j(g(\underline{\omega}_i,\omega_{-i}),\omega_0,\omega_j)\middle|b\right],$$

which implies

$$\min_{\omega_i\in\Omega_i} E_{\omega_{-i}}\left[\sum_{j\in N\cup\{0\}} v_j(g(\omega_i,\omega_{-i}),\omega_0,\omega_j)\middle|b\right] = E_{\omega_{-i}}\left[\max_{a\in A}\sum_{j\in N\cup\{0\}\setminus\{i\}} v_j(a,\omega_0,\omega_j)\middle|b\right],$$

that is,

$$\min_{\omega_i \in \Omega_i} E_{\omega_{-i}}\left[\sum_{j \in N \cup \{0\}} v_j(g(\omega_i, \omega_{-i}), \omega_0, \omega_j) \middle| b\right] - E[\sum_{j \in N \cup \{0\} \setminus \{i\}} v_j(g(\omega), \omega_0, \omega_j) \,|\, b] \geq 0 \ .$$

From the assumption of non-negative expected payoffs, we have

$$E\left[v_i(g(\omega), \omega_0, \omega_i) \middle| b\right] - c_i(b_i) \geq 0 \, .$$

From these observations, for every $i \in N$

$$\min\Big\{E\left[v_i(g(\omega), \omega_0, \omega_i) \middle| b\right] - c_i(b_i),$$

$$\min_{\omega_i \in \Omega_i} E_{\omega_{-i}}\left[\sum_{j \in N \cup \{0\}} v_j(g(\omega_i, \omega_{-i}), \omega_0, \omega_j) \middle| b\right] - E[\sum_{j \in N \cup \{0\} \setminus \{i\}} v_j(g(\omega), \omega_0, \omega_j) \,|\, b]\Big\}$$

$$\geq 0 \, ,$$

which, along with non-negative valuations, implies $R^{IIR,EAIR} \geq 0$.

Q.E.D.

Proof of Proposition 9: Sufficiency is straightforward from the fact that $E_{\omega_{-i}}[r_i(\omega_i, \omega_{-i}) \,|\, b_{-i}]$ is independent of ω_i and b_i.

For necessity, first, we prove the following lemma.

Lemma 1: With the assumption of independent types, if β_i is a differentiable path of b_i and $t(\beta_i) \in R^{|\Omega|}$ is its tangent, then there exists $t_i(\beta_i) \in R^{|\Omega_i|}$ such that

$$\lim_{\alpha \to 0} \frac{f_i(\cdot \,|\, \beta_i(\alpha)) - f_i(\cdot \,|\, b_i)}{\alpha} \equiv t_i(\beta_i) \, .$$

Furthermore, for all differentiable paths $\beta_i^1, \cdots, \beta_i^K$, their respective tangents $t(\beta_i^1)$, …, $t(\beta_i^K)$ are linearly independent if and only if $t_i(\beta_i^1)$, …, $t_i(\beta_i^K)$ are linearly independent.

Proof of Lemma 1: By independent types,

$$f(\omega \,|\, \beta_i(\alpha), b_{-i}) - f(\omega \,|\, b_i, b_{-i})$$

$$= \prod_{j \neq i} f_j(\omega_j \,|\, b_j) \cdot \big(f_i(\omega_i \,|\, \beta_i(\alpha)) - f_i(\omega_i \,|\, b_i) \big) \, .$$

Accordingly,

$$t(\omega, \beta_i)$$

$$= \prod_{j \neq i} f_j(\omega_j \mid b_j) \cdot \lim_{\alpha \to 0} \frac{f_i(\omega_i \mid \beta_i(\alpha)) - f_i(\omega_i \mid b_i)}{\alpha}$$

(A5) $$= \prod_{j \neq i} f_j(\omega_j \mid b_j) \cdot t_i(\omega_i, \beta_i).$$

This indicates that whenever $t(\beta_i)$ exists, $t_i(\beta_i)$ actually exists; thus, $t_i(\beta_i)$ is well-defined.

Suppose that $t(\beta_i^1)$, …, $t(\beta_i^K)$ are not linearly independent; thus, there exists $(\lambda_k)_{k=0}^K$ such that $\lambda_k \neq 0$ for some $k \in \{1, \cdots, K\}$, and

$$\sum_{k=1}^{K} \lambda_k \cdot t(\omega, \beta_i^k) = 0 \quad \text{for all} \quad \omega \in \Omega.$$

Fix some $\omega_{-i} \in \Omega_{-i}$ arbitrarily. It follows from (A5) that

$$\sum_{k=1}^{K} \lambda_k \cdot t_i(\omega_i, \beta_i^k) = \frac{1}{\prod_{j \neq i} f_j(\omega_j \mid b_j)} \sum_{k=1}^{K} \lambda_k \cdot t(\omega, \beta_i^k) = 0 \quad \text{for all} \quad \omega_i \in \Omega_i.$$

Hence, $t_i(\beta_i^1)$, …, $t_i(\beta_i^K)$ are not linearly independent either. If $t_i(\beta_i^1)$, …, $t_i(\beta_i^K)$ are not linearly independent, we can show that $t(\beta_i^1)$, …, $t(\beta_i^K)$ are not independent in a similar manner.

Q.E.D.

Using Lemma 1, we show the following equivalence result.

Lemma 2: Suppose that b is privately rich and (g, x) induces b. Then, for every payment rule $\tilde{x}$, the associated mechanism $(g, \tilde{x})$ induces b if and only if $E_{\omega_{-i}}[x_i(\omega_i, \omega_{-i}) - \tilde{x}_i(\omega_i, \omega_{-i}) \mid b_{-i}]$ is independent of ω_i, i.e., there exists $\bar{z} \in R^n$ such that

(A6) $$E_{\omega_{-i}}[x_i(\omega_i, \omega_{-i}) - \tilde{x}_i(\omega_i, \omega_{-i}) \mid b_{-i}] = \bar{z}_i \quad \text{for all} \quad i \in N \quad \text{and} \quad \omega_i \in \Omega_i.$$

Proof of Lemma 2: The proof of the sufficiency part is straightforward. Let us show the proof of the necessity part as follows. Suppose that both (g, x) and $(g, \tilde{x})$

induces b. Then, for all $i \in N$, the following first-order condition of necessary condition must be satisfied:

$$\frac{\partial}{\partial \alpha}\left\{E\left[v_i(g(\omega),\omega) - x_i(\omega) \mid \beta_i^k(\alpha), b_{-i}\right] - c_i(\beta_i^k(\alpha))\right\}\Big|_{\alpha=0} = 0$$

for $k = 1,\cdots,|\Omega_i| - 1$,

$$\frac{\partial}{\partial \alpha}\left\{E\left[v_i(g(\omega),\omega) - \tilde{x}_i(\omega) \mid \beta_i^k(\alpha), b_{-i}\right] - c_i(\beta_i^k(\alpha))\right\}\Big|_{\alpha=0} = 0$$

for $k = 1,\cdots,|\Omega_i| - 1$.

Comparing these equations, we obtain

(A7) $$\frac{\partial}{\partial \alpha} E\left[x_i(\omega) - \tilde{x}_i(\omega) \mid \beta_i^k(\alpha), b_{-i}\right]\Big|_{\alpha=0} = 0 \text{ for } k = 1,\cdots,|\Omega_i| - 1.$$

Using the assumption of independent types, (A7) can be rewritten as

(A8) $$\frac{\partial}{\partial \alpha} E_{\omega_i}\left[E_{\omega_{-i}}[x_i(\omega_i,\omega_{-i}) - \tilde{x}_i(\omega_i,\omega_{-i}) \mid b_{-i}] \mid \beta_i^k(\alpha)\right]\Big|_{\alpha=0} = 0$$

for $k = 1,\cdots,|\Omega_i| - 1$.

Define

$$w_i(\omega_i) \equiv E_{\omega_{-i}}[x_i(\omega_i,\omega_{-i}) - \tilde{x}_i(\omega_i,\omega_{-i}) \mid b_{-i}].$$

Then, (A8) is equivalent to

(A9) $$w_i \cdot t_i(\beta_i^k) = 0 \text{ for } k = 1,\cdots,|\Omega_i| - 1,$$

where $t_i(\beta_i^1)$, …, $t_i(\beta_i^{|\Omega_i|-1})$ are obtained by Lemma 1. Since $\left\{t_i(\beta_i^k)\right\}_{k=1}^{|\Omega_i|-1}$ are linearly independent,

$$\dim\left\{w_i \in R^{|\Omega_i|} : w_i \cdot t_i(\beta_i^k) = 0 \text{ for } k = 1,\cdots,|\Omega_i| - 1\right\} = 1.$$

This and (A9) implies that there exists $\overline{z}_i \in R$ such that

$$w_i(\omega_i) = E_{\omega_{-i}}[x_i(\omega_i,\omega_{-i}) - \tilde{x}_i(\omega_i,\omega_{-i}) \mid b_{-i}] = \overline{z}_i \text{ for all } \omega_i \in \Omega_i$$

as desired. Hence, we have proved Lemma 2.

Q.E.D.

Proof of Proposition 9, continued: Recall that, whenever (b,g) is fully efficient and x is pure-VCG, then (g,x) induces b. Furthermore, by the definition of expectation-VCG payment rules, if x is a pure-VCG payment rule and $\tilde{x}$ satisfies (A6), $\tilde{x}$ is an expectation-VCG payment rule. Accordingly, by Lemma 2, $(g,\tilde{x})$ induces b if and only if $\tilde{x}$ is an expectation-VCG payment rule.

Q.E.D.