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A Market Model of Interest Rates with Dynamic Basis Spreads in the presence of Collateral and Multiple Currencies*

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Abstract

The recent financial crisis caused dramatic widening and elevated volatilities among basis spreads in cross currency as well as domestic interest rate markets. Furthermore, the widespread use of cash collateral, especially in fixed income contracts, has made the effective funding cost of financial institutions for the trades significantly different from the Libor of the corresponding payment currency. Because of these market developments, the text-book style application of a market model of interest rates has now become inappropriate for financial firms; It cannot even reflect the exposures to these basis spreads in pricing, to say nothing of proper delta and vega (or kappa) hedges against their movements. This paper presents a new framework of the market model to address all these issues.

Keywords : Market Model, HJM model, Libor, tenor, swap, curve, OIS, cross currency, basis spread, interest rate model, derivatives, multi-currency

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1 Introduction

The recent financial crisis and the following liquidity and credit squeeze have caused significant widening and elevated volatilities among various types of basis spreads[1]. In particular, we have witnessed dramatic moves of cross currency swap (CCS), Libor-OIS, and tenor swap[2] (TS) basis spreads. In some occasions, the size of spreads has exceeded several tens of basis points, which is far wider than the general size of bid/offer spreads. Furthermore, there has been a dramatic increase of collateralization in financial contracts recent years, and it has become almost a market standard at least in the fixed income world [11]. As seen later, the existence of collateral agreement reduces the discounting rate significantly relative to the Libor of a given currency through frequent mark-to-market and collateral postings that follow. Although the Libor Market Model has been widely used among market participants since its invention, its text-book style application does not provide an appropriate tool to handle these new realities; It can only treat one type of Libor, and is unable to reflect the movement of spreads among Libors with different tenors. The discounting of a future cash flow is done by the same Libor, which does not reflect the existence of collaterals and the funding cost differentials among multiple currencies in CCS markets[3].

As a response to these market developments, the invention of a more sophisticated financial model which is able to reflect all the relevant swap prices and their behavior has risen as an urgent task among academics and market participants. Surprisingly, it is not at all a trivial task even constructing a set of yield curves explaining the various swap prices in the market consistently while keeping no-arbitrage conditions intact. Ametrano and Bianchetti (2009) [1] proposed a simple scheme that is able to recover the level of each swap rate in the market, but gives rise to arbitrage possibilities due to the existence of multiple discounting rates within a single currency. The model proposed by Bianchetti (2008) [3] using a multi-currency analogy does not seem to be a practical solution although it is at least free from arbitrage. The main problem of the model is that the curve calibration can not be separated from the option calibration due to the entanglement of volatility specifications, since it treats the usual Libor payment as a quanto of different currencies with a pegged FX rate. It also makes the daily hedge against the move of basis spreads quite complicated. In addition, neither of Bianchetti (2008) and Ametrano and Bianchetti (2009) has discussed how to make the model consistent with the collateralization and cross currency swap markets.

Our recent work, "A Note on Construction of Multiple Swap Curves with and without Collateral" [6], have developed a method of swap-curve construction which allows us to treat overnight index swap (OIS), interest rate swaps (IRS), tenor swaps (TS), and cross currency swaps (CCS) consistently with explicit considerations of the effects from collateralization. The current paper presents a framework of stochastic interest rate models with dynamic basis spreads addressing all the above mentioned issues, where the output

[1] A basis spread generally means the interest rate differentials between two different floating rates.

[2] It is a floating-vs-floating swap that exchanges Libors with two different tenors with a fixed spread in one side.

[3] As for the cross currency basis spread, it has been an important issue for global financial institutions for many years. However, there exists no literature that directly takes its dynamics into account consistently in a multi-currency setup of an interest rate model.

of curve calibrations in the work [6] can be directly used as a starting point of simulation. In the most generic setup in Ref.[6], there remained a difficulty to calibrate all the parameters due to the lack of separate quotes of foreign-currency collateralized swaps in the current market. This new work presents a simplified but practical way of implementation which allows exact fits to the domestic-currency collateralized OIS, IRS and TS, together with FX forward and mark-to-market CCS (MtMCCS) without referring to the quotes of foreign collateralized products. Also, this paper adopts an HJM(Heath-Jarrow-Morton)-type framework just for clarity of presentation: Of course, it is quite straightforward to write the model using a discretized interest rates, which becomes an extension of the Libor and Swap market models([4],[12]). Since our motivation is to explain the generic modeling framework, the details of volatility processes are not specified. Such as analytic expressions of vanilla options and implications to the risk management for various types of exotics will be presented somewhere else in the future adopting a fully specified model.

The organization of the paper is as follows: The next section firstly reminds readers of the pricing formula under the collateral agreement. Then, after reviewing the fundamental interest rate products, it presents the modeling framework with stochastic basis spreads in a single currency environment, which enables us to explain these instruments consistently. Section 3 extends the model into the multi-currency environment and explains how to make the model consistent with the FX forward and MtMCCS. Finally, after Section 4 briefly comments on inflation modeling, Section 5 concludes.

2 Single Currency Market

This section develops a HJM-type framework of an interest rate model in a single currency market. Our goal is to construct a framework which is able to explain all the OIS, IRS and TS markets consistently in an unified way. Here, it is assumed that every trade has a collateral agreement using a domestic currency as collateral [4].

2.1 Collateralization

Firstly, let us briefly explain the effects of collateralization. Under the collateral agreement, the firm receives the collateral from the counter party when the present value of the net position is positive and needs to pay the margin called "collateral rate" on the outstanding collateral in exchange. On the other hand, if the present value of the net position is negative, the firm is asked to post the collateral to the counter party and receives the collateral rate in return. Although the details can possibly differ trade by trade due to the OTC nature of the fixed income market, the most commonly used collateral is a currency of developed countries, such as USD, EUR and JPY [11]. In this case, the collateral rate is usually fixed by the overnight rate of the collateral currency: for example, Fed-Fund rate, EONIA, and Mutan for USD, EUR and JPY, respectively.

In general setup, pricing of collateralized products is very hard due to the non-linearity arising from the residual credit risk. Due to the netting procedures, the pricing of each product becomes dependent on the whole contracts with the counter party, which makes the use of model unpractical for the daily pricing and hedging. In order to make the

[4] It is easy to apply the similar methodology to the unsecured (or uncollateralized) trade by approximately taking into account the credit risk by using Libor as the effective discounting rate.

problem tractable, we will assume the perfect and continuous collateralization with zero threshold by cash, which means that the mark-to-market and collateral posting is to be made continuously, and the posted amount of cash is 100% of the contract's present value. Actually, the daily mark-to-market and adjustment of collateral amount is the market best practice, and the approximation should not be too far from the reality. Under the above simplification, we can think that there remains no counter party default risk and recover the linearity among different payments. This means that a generic derivative is treated as a portfolio of the independently collateralized strips of payments.

We would like to ask readers to consult Sec.3 of Ref. [6] for details, but the present value of a collateralized derivative with payment $h(T)$ at time T is given by [5]

$$h(t) = E_t^Q\left[e^{-\int_t^T c(s)ds}h(T)\right] , \tag{2.1}$$

where $E_t^Q[\cdot]$ denotes the expectation under the Money-Market (MM) measure Q conditioned on the time-t filtration, and $c(s)$ is the time-s value of the collateral rate. Note that $c(s)$ is not necessarily equal to the risk-free interest rate $r(s)$ of a given currency.

For the later purpose, let us define the collateralized zero-coupon bond D as

$$D(t,T) = E_t^Q\left[e^{-\int_t^T c(s)ds}\right] , \tag{2.2}$$

which is the present value of the unit amount of payment under the contract of continuous collateralization with the same currency. In later sections, we will frequently use the expectation $E^{\mathcal{T}^c}[\cdot]$ under the collateralized-forward measure $\mathcal{T}^c$ defined as

$$E_t^Q\left[e^{-\int_t^T c(s)ds}h(T)\right] = D(t,T)E_t^{\mathcal{T}^c}\left[h(T)\right] , \tag{2.3}$$

where the collateralized zero-coupon bond $D(\cdot,T)$ is used as a numeraire.

2.2 Market Instruments

Before going to discuss the modeling framework, this subsection briefly summarizes the important swaps in a domestic market as well as the conditions that par swap rates have to satisfy. They are the most important calibration instruments to fix the starting points of simulation.

2.2.1 Overnight index swap

An overnight index swap (OIS) is a fixed-vs-floating swap whose floating rate is given by the daily compounded overnight rate. Since the overnight rate is same as the collateral rate of the corresponding currency, the following relation holds[6]:

$$\mathrm{OIS}_N(t)\sum_{n=1}^{N}\Delta_n E_t^Q\left[e^{-\int_t^{T_n} c(s)ds}\right] = \sum_{n=1}^{N} E_t^Q\left[e^{-\int_t^{T_n} c(s)ds}\left(e^{\int_{T_{n-1}}^{T_n} c(s)ds} - 1\right)\right] , \tag{2.4}$$

[5] In this section, the collateral currency is the same as the payment currency.

[6] Typically, there is only one payment at the very end for the swap with short maturity ($< 1yr$) case, and otherwise there are periodical payments, quarterly for example.

or equivalently,

$$\mathrm{OIS}_N(t)\sum_{n=1}^{N}\Delta_n D(t,T_n) = D(t,T_0) - D(t,T_N)\ , \tag{2.5}$$

where $\mathrm{OIS}_N(t) = \mathrm{OIS}(t,T_0,T_N)$ is the market quote at time t of the T_0-start T_N-maturing OIS rate, and T_0 is the effective date in the case of spot-start OIS. Also Δ_n denotes the fixed leg day count fraction for the period of (T_{n-1},T_n).

2.2.2 Interest rate swap

In an interest rate swap (IRS), two parties exchange a fixed coupon and Libor for a certain period with a given frequency. The tenor of Libor "τ" is determined by the frequency of floating payments, i.e., 6m-tenor for semi-annual payments, for example. For a T_0-start T_M-maturing IRS with the Libor of tenor τ, we have

$$\mathrm{IRS}_M(t)\sum_{m=1}^{M}\Delta_m D(t,T_m) = \sum_{m=1}^{M}\delta_m D(t,T_m)E_t^{\mathcal{T}_m^c}[L(T_{m-1},T_m;\tau)] \tag{2.6}$$

as a consistency condition. Here, $\mathrm{IRS}_M(t) = \mathrm{IRS}(t,T_0,T_M;\tau)$ is the time-t value of the corresponding IRS quote, $L(T_{m-1},T_m;\tau)$ is the Libor rate with tenor τ for a period of (T_{m-1},T_m), and δ_m is its day count fraction. In the remainder of the paper, we distinguish the difference of day count conventions between the fixed and floating legs by Δ and δ, respectively.

Here, it is assumed that the frequencies of both legs are equal just for simplicity, and it does not affect our later arguments even if this is not the case. Usually, IRS with a specific choice of τ has dominant liquidity in a given currency market, such as 6m for JPY IRS and 3m for USD IRS. Information of forward Libors with other tenors is provided by tenor swaps, which will be explained next.

2.2.3 Tenor swap

A tenor swap is a floating-vs-floating swap where the parties exchange Libors with different tenors with a fixed spread on one side, which we call TS basis spread in this paper. Usually, the spread is added on top of the Libor with shorter tenor. For example, in a 3m/6m tenor swap, quarterly payments with 3m Libor plus spread are exchanged by semi-annual payments of 6m Libor flat. The condition that the tenor spread should satisfy is given by

$$\sum_{n=1}^{N}\delta_n D(t,T_n)\left(E_t^{\mathcal{T}_n^c}\left[L(T_{n-1},T_n;\tau_S)\right] + TS(t)\right) = \sum_{m=1}^{M}\delta_m D(t,T_m)E_t^{\mathcal{T}_m^c}\left[L(T_{m-1},T_m;\tau_L)\right]\ , \tag{2.7}$$

where $T_N = T_M$, "m" and "n" distinguish the difference of payment frequency. $TS(t) = TS(t,T_0,T_N;\tau_S,\tau_L)$ denotes the time-t value of TS basis spread for the T_0-start T_N-maturing tenor swap. The spread is added on the Libor with the shorter tenor τ_S in exchange for the Libor with longer tenor τ_L.

Here, we have explained using slightly simplified terms of contract. In the actual market, the terms of contract in which coupons of the Leg with the short tenor are compounded by Libor flat and paid with the same frequency of the other Leg is more popular. However, the size of correction from the above simplified result can be shown to be negligibly small. Please see Appendix for details.

2.2.4 Underlying factors in the Model

Using the above instruments and the method explained in Ref. [6], we can extract

$$\{D(t,T)\}, \quad \{E_t^{\mathcal{T}^c}[L(T-\tau,T;\tau)]\} \tag{2.8}$$

for continuous time $T \in [0, T_H]$ where T_H is the time horizon of relevant pricing[7], and each relevant tenor τ, such as 1m, 3m, 6m, 12m, for example[8]. The next section will explain how to make these underlying factors consistently with no-arbitrage conditions in an HJM-type framework.

2.3 Model with Dynamic basis spreads in a Single Currency

As seen in Sec.2.1, the collateral rate plays a critical role as the effective discounting rate, which leads us to consider its dynamics first. Let us define the continuous forward collateral rate as

$$c(t,T) = -\frac{\partial}{\partial T}\ln D(t,T) \tag{2.9}$$

or, equivalently

$$D(t,T) = e^{-\int_t^T c(t,s)ds} \ , \tag{2.10}$$

where it is related to the spot rate as $c(t,t) = c(t)$. Then, assume that the dynamics of the forward collateral rate under the MM measure Q is given by

$$dc(t,s) = \alpha(t,s)dt + \sigma_c(t,s) \cdot dW^Q(t) \ , \tag{2.11}$$

where $\alpha(t,s)$ is a scalar function for its drift, and $W^Q(t)$ is a d-dimensional Brownian motion under the Q-measure. $\sigma_c(t,s)$ is a d-dimensional vector and the following abbreviation have been used:

$$\sigma_c(t,s) \cdot dW^Q(t) = \sum_{j=1}^{d} [\sigma_c(t,s)]_j \, dW_j^Q(t). \tag{2.12}$$

As mentioned in the introduction, the details of volatility process will not be specified: It can depend on the collateral rate itself, or any other state variables.

Applying Itô's formula to Eq.(2.10), we have

$$\frac{dD(t,T)}{D(t,T)} = \left\{ c(t) - \int_t^T \alpha(t,s)ds + \frac{1}{2}\left\| \int_t^T \sigma_c(t,s)ds \right\|^2 \right\} dt - \left(\int_t^T \sigma_c(t,s)ds \right) \cdot dW_t^Q. \tag{2.13}$$

[7] Basically, OIS quotes allow us to fix the collateralized zero coupon bond values, and then the combinations of IRS and TS will give us the Libor forward expectations.

[8] We need to use proper spline technique to get smooth continuous result. See Hagan and West (2006) [10], for example.

On the other hand, from the definition of (2.2), the drift rate of $D(t,T)$ should be $c(t)$. Therefore, it is necessary that

$$\begin{aligned}
\alpha(t,s) &= \sum_{j=1}^{d}[\sigma_c(t,s)]_j\left(\int_t^s \sigma_c(t,u)du\right)_j && (2.14)\\
&= \sigma_c(t,s)\cdot\left(\int_t^s \sigma_c(t,u)du\right)\ , && (2.15)
\end{aligned}$$

and as a result, the process of $c(t,s)$ under the Q-measure is obtained by

$$dc(t,s) = \sigma_c(t,s)\cdot\left(\int_t^s \sigma_c(t,u)du\right)dt + \sigma_c(t,s)\cdot dW^Q(t)\ . \tag{2.16}$$

Now, let us consider the dynamics of Libors with various tenors. Mercurio (2008) [14] has proposed an interesting simulation scheme[9]. He follows the original idea of Libor Market Model, and has modeled the market observables or forward expectations of Libors directly, instead of considering the corresponding spot process as Ref.[3]. We will adopt the Mercurio's scheme, but separating the spread processes explicitly.

Firstly, define the collateralized forward Libor, and OIS forward as

$$\begin{aligned}
L^c(t,T_{k-1},T_k;\tau) &= E_t^{\mathcal{T}_k^c}\left[L(T_{k-1},T_k;\tau)\right]\ , && (2.17)\\
L^{\rm OIS}(t,T_{k-1},T_k) &= E_t^{\mathcal{T}_k^c}\left[\frac{1}{\delta_k}\left(\frac{1}{D(T_{k-1},T_k)}-1\right)\right] && (2.18)\\
&= \frac{1}{\delta_k}\left(\frac{D(t,T_{k-1})}{D(t,T_k)}-1\right)\ , && (2.19)
\end{aligned}$$

and also define the Libor-OIS spread process:

$$B(t,T_k;\tau) = L^c(t,T_{k-1},T_k;\tau) - L^{\rm OIS}(t,T_{k-1},T_k)\ . \tag{2.20}$$

By construction, $B(t,T;\tau)$ is a martingale under the collateralized forward measure $\mathcal{T}^c$, and its stochastic differential equation can be written as

$$dB(t,T;\tau) = B(t,T;\tau)\sigma_B(t,T;\tau)\cdot dW^{\mathcal{T}^c}(t)\ , \tag{2.21}$$

where d-dimensional volatility function σ_B can depend on B or other state variables as before. Using Maruyama-Girsanov's theorem, one can see that the Brownian motion under the $\mathcal{T}^c$-measure, $W^{\mathcal{T}^c}(t)$, is related to $W^Q(t)$ by the following relation:

$$dW^{\mathcal{T}^c}(t) = \left(\int_t^T \sigma_c(t,s)ds\right)dt + dW^Q(t)\ . \tag{2.22}$$

As a result, the process of $B(t,T;\tau)$ under the Q-measure is obtained by

$$\frac{dB(t,T;\tau)}{B(t,T;\tau)} = \sigma_B(t,T;\tau)\cdot\left(\int_t^T \sigma_c(t,s)ds\right)dt + \sigma_B(t,T;\tau)\cdot dW^Q(t)\ . \tag{2.23}$$

[9] Exactly the same idea has been also adopted in inflation modeling [2] as will be seen later.

We need to specify B-processes for all the relevant tenors in the market, such as 1m, 3m, 6m, and 12m, for example. If one wants to guarantee the positivity for $B(\cdot,T;\tau_L) - B(\cdot,T;\tau_S)$ where $\tau_L > \tau_S$, it is possible to model this spread as Eq. (2.23) directly.

The list of what we need only consists of these two types of underlyings. As one can see, there is no explicit need to simulate the risk-free interest rate in a single currency environment if all the interested trades are collateralized with the same domestic currency. Let us summarize the relevant equations:

$$dc(t,s) = \sigma_c(t,s)\cdot\left(\int_t^s \sigma_c(t,u)du\right)dt + \sigma_c(t,s)\cdot dW^Q(t) \ , \tag{2.24}$$

$$\frac{dB(t,T;\tau)}{B(t,T;\tau)} = \sigma_B(t,T;\tau)\cdot\left(\int_t^T \sigma_c(t,s)ds\right)dt + \sigma_B(t,T;\tau)\cdot dW^Q(t) \ . \tag{2.25}$$

Since we already have $\{c(t,s)\}_{s\geq t}$, and $\{B(t,T;\tau)\}_{T\geq t}$ each for the relevant tenor, after curve construction explained in Ref. [6], we can directly use them as starting points of simulation. If one needs an equity process $S(t)$ with an effective dividend yield given by $q(t)$ with the same collateral agreement, we can model it as

$$dS(t)/S(t) = (c(t) - q(t))\,dt + \sigma_S(t)\cdot dW^Q(t) \ , \tag{2.26}$$

and σ_S and q can be state dependent. Note that the effective dividend yield q is not equal to the dividend yield in the non-collateralized trade but should be adjusted by the difference between the collateral rate and the risk-free rate [10]. In practice, it is likely not a big problem to use the same value or process of the usual definition of dividend yield. Here, we are not trying to reflect the details of repo cost for an individual stock, but rather try to model a stock index, such as S&P500, for IR-Equity hybrid trades.

2.4 Simple options in a single currency

This subsection explains the procedures for simple option pricing in a single currency environment. In the following, suppose that all the forward and option contracts themselves are collateralized with the same domestic currency.

2.4.1 Collateralized overnight index swaption

As was seen from Sec.2.2.1, a T_0-start T_N-maturing forward OIS rate at time t is given by

$$\mathrm{OIS}(t,T_0,T_N) = \frac{D(t,T_0) - D(t,T_N)}{\sum_{n=1}^N \Delta_n D(t,T_n)} \ . \tag{2.27}$$

When the length of OIS is very short and there is only one final payment, one can get the correct expression by simply replacing the annuity in the denominator by $\Delta_N D(t,T_N)$, a collateralized zero coupon bond times a day count fraction for the fixed payment.

[10]The effective dividend yield is given by $q(t) = q_{\mathrm{org}}(t) - (r(t) - c(t))$ with the original dividend yield q_{org}. In later sections, we will use a simplified assumption that $(r(t) - c(t))$ is a deterministic function of time.

Under the annuity measure $\mathcal{A}$, where the annuity $A(t,T_0,T_N) = \sum_{n=1}^{N} \Delta_n D(t,T_n)$ is being used as a numeraire, the above OIS rate becomes a martingale. Therefore, the present value of a collateralized payer option on the OIS with strike K is given by

$$PV(t) = A(t,T_0,T_N) E_t^{\mathcal{A}} \left[(\mathrm{OIS}(T_0,T_0,T_N) - K)^+ \right] \ , \tag{2.28}$$

where one can show that the stochastic differential equation for the forward OIS is given as follows under the $\mathcal{A}$-measure:

$$\begin{aligned} d\mathrm{OIS}(t,T_0,T_N) &= \mathrm{OIS}(t,T_0,T_N) \left\{ \frac{D(t,T_N)}{D(t,T_0) - D(t,T_N)} \left(\int_{T_0}^{T_N} \sigma_c(t,s) ds \right) \right. \\ &\left. + \frac{1}{A(t,T_0,T_N)} \sum_{n=1}^{N} \Delta_n D(t,T_n) \left(\int_{T_0}^{T_n} \sigma_c(t,s) ds \right) \right\} \cdot dW^{\mathcal{A}}(t) \ , \end{aligned} \tag{2.29}$$

where $W^{\mathcal{A}}(t)$ is the Brownian motion under the $\mathcal{A}$-measure, and is related to $W^Q(t)$ as

$$dW^{\mathcal{A}}(t) = dW^Q(t) + \frac{1}{A(t,T_0,T_N)} \sum_{n=1}^{N} \Delta_n D(t,T_n) \left(\int_{t}^{T_n} \sigma_c(t,s) ds \right) dt \ . \tag{2.30}$$

We can derive an accurate approximation of Eq.(2.28) by applying asymptotic expansion technique [16, 17, 18], or ad hoc but simpler methods given, for example, in Brigo and Mercurio (2006) [5].

2.4.2 Collateralized interest rate swaption

Next, let us consider the usual swaption with the collateral agreement. As we have seen in Sec.2.2.2, a T_0-start T_N-maturing collateralized forward swap rate is given by

$$\begin{aligned} \mathrm{IRS}(t,T_0,T_N;\tau) &= \frac{\sum_{n=1}^{N} \delta_n D(t,T_n) L^c(t,T_{n-1},T_n;\tau)}{\sum_{n=1}^{N} \Delta_n D(t,T_n)} && (2.31) \\ &= \frac{D(t,T_0) - D(t,T_N)}{\sum_{n=1}^{N} \Delta_n D(t,T_n)} + \frac{\sum_{n=1}^{N} \delta_n D(t,T_n) B(t,T_n;\tau)}{\sum_{n=1}^{N} \Delta_n D(t,T_n)} && (2.32) \\ &= \mathrm{OIS}(t,T_0,T_N) + Sp^{\mathrm{OIS}}(t,T_0,T_N;\tau), && (2.33) \end{aligned}$$

where we have defined IRS-OIS spread Sp^{OIS} as

$$Sp^{\mathrm{OIS}}(t,T_0,T_N;\tau) = \frac{\sum_{n=1}^{N} \delta_n D(t,T_n) B(t,T_n;\tau)}{\sum_{n=1}^{N} \Delta_n D(t,T_n)} \ . \tag{2.34}$$

Note that we have slightly abused the notation of $\mathrm{OIS}(t)$. In reality, there is no guarantee that the day count conventions and frequencies are the same between IRS and OIS, which may require appropriate adjustments.

Sp^{OIS} is a martingale under the $\mathcal{A}$-measure, and one can show that its stochastic differential equation is given by

$$dSp^{\mathrm{OIS}}(t,T_0,T_N;\tau) = Sp^{\mathrm{OIS}}(t)\left\{\frac{1}{A(t,T_0,T_N)}\sum_{j=1}^{N}\Delta_j D(t,T_j)\left(\int_{T_0}^{T_j}\sigma_c(t,s)ds\right)\right.$$
$$\left. + \frac{1}{A_{sp}(t,T_0,T_N;\tau)}\sum_{n=1}^{N}\delta_n D(t,T_n)B(t,T_n;\tau)\left(\sigma_B(t,T_n;\tau) - \int_{T_0}^{T_n}\sigma_c(t,s)ds\right)\right\}\cdot dW^{\mathcal{A}}(t)\ , \tag{2.35}$$

where we have defined

$$A_{sp}(t,T_0,T_N;\tau) = \sum_{n=1}^{N}\delta_n D(t,T_n)B(t,T_n;\tau)\ . \tag{2.36}$$

Since IRS forward rate is a martingale under the annuity measure $\mathcal{A}$, the present value of a T_0 into T_N collateralized payer swaption is expressed as

$$PV(t) = A(t,T_0,T_N)E_t^{\mathcal{A}}\left[\left(\mathrm{OIS}(T_0,T_0,T_N) + Sp^{\mathrm{OIS}}(T_0,T_0,T_N;\tau) - K\right)^+\right]\ . \tag{2.37}$$

As in the previous OISwaption case, we can use asymptotic expansion technique or other methods to derive analytic approximation for this option.

2.4.3 Collateralized tenor swaption

Finally, consider an option on tenor swap. From Sec.2.2.3, the forward TS spread for a collateralized T_0-start T_N $(= T_M)$-maturing swap which exchanges Libors with tenor τ_S and τ_L is given by

$$\begin{aligned} &TS(t,T_0,T_N;\tau_S,\tau_L) \\ &= \frac{\sum_{m=1}^{M}\delta_m D(t,T_m)L^c(t,T_{m-1},T_m;\tau_L) - \sum_{n=1}^{N}\delta_n D(t,T_n)L^c(t,T_{n-1},T_n;\tau_S)}{\sum_{n=1}^{N}\delta_n D(t,T_n)}, \\ &= \frac{\sum_{m=1}^{M}\delta_m D(t,T_m)B(t,T_m;\tau_L)}{\sum_{n=1}^{N}\delta_n D(t,T_n)} - \frac{\sum_{n=1}^{N}\delta_n D(t,T_n)B(t,T_n;\tau_L)}{\sum_{n=1}^{N}\delta_n D(t,T_n)}, \end{aligned} \tag{2.38}$$

where we have distinguished the different payment frequencies by "n" and "m". In the case of a 3m/6m tenor swap, for example, $N = 2M$, $\tau_S = 3\mathrm{m}$ and $\tau_L = 6\mathrm{m}$. Since the two terms in Eq.(2.38) are equal to Sp^{OIS} except the difference in day count conventions, the tenor swaption is basically equivalent to a spread option between two different Sp^{OIS}s. The present value of collateralized payer tenor swaption with strike K can be expressed as

$$PV(t) = \left(\sum_{n=1}^{N}\delta_n D(t,T_n)\right)E_t^{\tilde{\mathcal{A}}}\left[(TS(T_0,T_0,T_N;\tau_S,\tau_L) - K)^+\right]\ . \tag{2.39}$$

Here, $E_t^{\tilde{\mathcal{A}}}[\cdot]$ denote the expectation under the annuity measure with day count fraction specified by that of floating leg, δ.

These options explained in Secs. 2.4.1, 2.4.2, and 2.4.3, can allow us to extract volatility information for our model. Considering the current situation where there is no liquid market of options on the relevant basis spreads, we probably need to combine some historical estimation for the volatility calibration.

3 Multiple Currency Market

This section extends the framework developed in the previous section into multi-currency environment. For later purpose, let us define several variables first. The T-maturing risk-free zero coupon bond of currency "k" is denoted by $P^{(k)}(\cdot, T)$, and is calculated from the equation

$$P^{(k)}(t,T) = E_t^{Q_k}\left[e^{-\int_t^T r^{(k)}(s)ds}\right] , \tag{3.1}$$

where Q_k and $r^{(k)}$ denote the MM measure and risk-free interest rate for the k-currency. Also define the instantaneous risk-free forward rate by

$$f^{(k)}(t,T) = -\frac{\partial}{\partial T}\ln P^{(k)}(t,T) \tag{3.2}$$

as usual, and $r^{(k)}(t) = f^{(k)}(t,t)$.

As is well known, its stochastic differential equation under the domestic MM measure Q_k is given by

$$df^{(k)}(t,s) = \sigma^{(k)}(t,s)\cdot\left(\int_t^s \sigma^{(k)}(t,u)du\right)dt + \sigma^{(k)}(t,s)\cdot dW^{Q_k}(t) , \tag{3.3}$$

where $W^{Q_k}(t)$ is the d-dimensional Brownian motion under the Q_k-measure. The volatility term $\sigma^{(k)}$ is d-dimensional vector and possibly depends on $f^{(k)}$ or any other state variables. Here, we have shown the risk-free interest rate to make the structure of the model easy to understand though our scheme does not directly simulate it as will be seen later.

Let us also define the spot foreign exchange rate between currency "i" and "j":

$$f_x^{(i,j)}(t) . \tag{3.4}$$

It denotes the time-t value of unit amount of currency "j" in terms of currency "i". Then, define its dynamics under the Q_i-measure as

$$df_x^{(i,j)}(t)/f_x^{(i,j)}(t) = \left(r^{(i)}(t) - r^{(j)}(t)\right)dt + \sigma_X^{(i,j)}(t)\cdot dW^{Q_i}(t) . \tag{3.5}$$

The volatility term can depend on $f_x^{(i,j)}$ or any other state variables. The Brownian motions of two different MM measures are connected each other by the relation

$$dW^{Q_i}(t) = \sigma_X^{(i,j)}(t)dt + dW^{Q_j}(t) , \tag{3.6}$$

as indicated by Maruyama-Girsanov's theorem.

3.1 Collateralization with foreign currencies

Until this point, the collateral currency have been assumed to be the same as the payment currency of the contract. However, this assumption cannot be maintained in multi-currency environment, since multi-currency trades contain different currencies in their payments in general. In fact, this currency mismatch is inevitable in a CCS trade whose payments contain two different currencies, but only one collateral currency.

Our previous work [6] have provided a pricing formula for a generic financial product whose collateral currency "j" is different from its payment currency "k":

$$\begin{aligned} h^{(k)}(t) &= E_t^{Q_k}\left[e^{-\int_t^T r^{(k)}(s)ds}\left(e^{\int_t^T (r^{(j)}(s)-c^{(j)}(s))ds}\right)h^{(k)}(T)\right] & (3.7)\\ &= P^{(k)}(t,T)E_t^{\mathcal{T}_{(k)}}\left[\left(e^{\int_t^T (r^{(j)}(s)-c^{(j)}(s))ds}\right)h^{(k)}(T)\right] . & (3.8)\end{aligned}$$

Here, $h^{(k)}(t)$ is the present value of a financial derivative whose payment $h^{(k)}(T)$ is to be made at time T in k-currency. The collateralization is assumed to be made continuously by cash of j-currency with zero threshold, and $c^{(j)}$ is the corresponding collateral rate. $E_t^{\mathcal{T}_{(k)}}[\cdot]$ denotes the expectation under the risk-free forward measure of currency k, $\mathcal{T}_{(k)}$, where the risk-free zero coupon bond $P^{(k)}(\cdot,T)$ is used as a numeraire.

As is clear from these arguments, the price of a financial product depends on the choice of collateral currency. Let us check this impact for the most fundamental instruments, i.e., FX forward contracts and Libor payments in the next sections.

3.1.1 FX forward and Currency triangle

As is well known, the currency triangle relation should be satisfied among arbitrary combinations of currencies (j,k,l),

$$f_x^{(j,k)}(t) = f_x^{(j,l)}(t) \times f_x^{(l,k)}(t) \tag{3.9}$$

otherwise, the difference will soon be arbitraged away in the current liquid foreign exchange market. In the default-free market without collateral agreement, this relation should hold also in FX forward market. However, it is not a trivial issue in the presence of collateral as will be seen below[11].

Let us consider a k-currency collateralized FX forward contract between the currencies (i,j). The FX forward rate $f_x^{(i,j)}(t,T)$ is given by the amount of i-currency to be exchanged by the unit amount of j-currency at time T with zero present value:

$$\begin{aligned} & f_x^{(i,j)}(t,T)P^{(i)}(t,T)E_t^{\mathcal{T}_{(i)}}\left[e^{\int_t^T (r^{(k)}(s)-c^{(k)}(s))ds}\right] \\ &\qquad = f_x^{(i,j)}(t)P^{(j)}(t,T)E_t^{\mathcal{T}_{(j)}}\left[e^{\int_t^T (r^{(k)}(s)-c^{(k)}(s))ds}\right] , \qquad (3.10)\end{aligned}$$

and hence

$$f_x^{(i,j)}(t,T) = f_x^{(i,j)}(t)\frac{P^{(j)}(t,T)}{P^{(i)}(t,T)}\left(\frac{E_t^{\mathcal{T}_{(j)}}\left[e^{\int_t^T (r^{(k)}(s)-c^{(k)}(s))ds}\right]}{E_t^{\mathcal{T}_{(i)}}\left[e^{\int_t^T (r^{(k)}(s)-c^{(k)}(s))ds}\right]}\right) . \tag{3.11}$$

[11] FX forward contract is usually included in the list of trades for which netting and collateral postings are to be made.

From the above equation, it is clear that the currency triangle relation only holds among the trades with the common collateral currency, in general.

3.1.2 Libor payment collateralized with a foreign currency

Next, let us consider the implications to a foreign-currency collateralized Libor payment. Using the result of Sec.3.1, the present value of a k-currency Libor payment with cash collateral of j-currency is given by

$$PV(t) = \delta_n P^{(k)}(t,T_n) E_t^{\mathcal{T}_{n,(k)}}\left[e^{\int_t^{T_n}(r^{(j)}(s)-c^{(j)}(s))ds} L^{(k)}(T_{n-1},T_n;\tau) \right] . \quad (3.12)$$

Remind that if the Libor is collateralized by the same domestic currency k, the present value of the same payment is given by

$$\begin{aligned} PV(t) &= \delta_n D^{(k)}(t,T_n) E_t^{\mathcal{T}^c_{n,(k)}}\left[L^{(k)}(T_{n-1},T_n;\tau) \right] & (3.13) \\ &= \delta_n P^{(k)}(t,T_n) E_t^{\mathcal{T}_{n,(k)}}\left[e^{\int_t^{T_n}(r^{(k)}(s)-c^{(k)}(s))ds} L^{(k)}(T_{n-1},T_n;\tau) \right] . & (3.14) \end{aligned}$$

Here, the superscript "c" in $\mathcal{T}^c_{n,(k)}$ of $E_t^{\mathcal{T}^c_{n,(k)}}[\cdot]$ denotes that the expectation is taken under the collateralized forward measure instead of the risk-free forward measure. The above results suggest that the price of an interest rate product, such as IRS, does depend on the choice of its collateral currency.

3.1.3 Simplification for practical implementation

The findings of Secs.3.1.1 and 3.1.2 give rise to a big difficulty for practical implementation. If all the relevant vanilla products have separate quotes as well as sufficient liquidity for each collateral currency, it is possible to set up a separate multi-currency model for each choice of a collateral currency. However, separate quotes for different collateral currencies are unobservable in the actual market. Furthermore, closing the hedges within each collateral currency is unrealistic. This is because one would like to use JPY domestic IR swaps to hedge the JPY Libor exposure in a complicated multi-currency derivatives collateralized by EUR, for example. The setup of a separate model for each collateral currency will make these hedges too complicated.

In order to avoid these difficulties, let us adopt a very simple assumption that

$$\sigma^{(k)}(t,s) = \sigma_c^{(k)}(t,s) . \quad (3.15)$$

or

$$y^{(k)}(t,s) = f^{(k)}(t,s) - c^{(k)}(t,s) \quad (3.16)$$

is a deterministic function of t for each s and for every currency k. Here, $\sigma_c^{(k)}$ is the volatility term defined for the forward collateral rate of the k-currency as in Eq.(2.16). Under this assumption, one can show that

$$r^{(k)}(t) - c^{(k)}(t) = f^{(k)}(s,t) - c^{(k)}(s,t) \quad (3.17)$$

for any $s \leq t$. Hence, it follows that

$$y^{(k)}(t) = r^{(k)}(t) - c^{(k)}(t) \quad (3.18)$$

as a deterministic function of time.

Under this assumption, one can see that the FX forward rate in Eq.(3.11) becomes

$$f_x^{(i,j)}(t,T) = f_x^{(i,j)}(t)\frac{P^{(j)}(t,T)}{P^{(i)}(t,T)} \tag{3.19}$$

and it is independent from the choice of collateral currency. Therefore, the relation of cross currency triangle holds among FX forwards even when they contain multiple collateral currencies.

In addition, the collateralized forward expectation and the risk-free forward expectation are equal for each currency k,

$$E_t^{\mathcal{T}^c_{(k)}}[\cdot] = E_t^{\mathcal{T}_{(k)}}[\cdot] \tag{3.20}$$

since the corresponding Radon-Nikodym derivative becomes constant:

$$e^{-\int_0^{\cdot}(r^{(k)}(s)-c^{(k)}(s))ds}\frac{P^{(k)}(\cdot,T)}{D^{(k)}(\cdot,T)}\frac{D^{(k)}(0,T)}{P^{(k)}(0,T)} \equiv 1 \ . \tag{3.21}$$

Now, Eq.(3.12) turns out to be

$$\begin{aligned} PV(t) &= \delta_n P^{(k)}(t,T_n)e^{\int_t^{T_n} y^{(j)}(s)ds}E_t^{\mathcal{T}_{n,(k)}}\left[L^{(k)}(T_{n-1},T_n;\tau)\right] && (3.22)\\ &= \delta_n D^{(k)}(t,T_n)e^{\int_t^{T_n}(y^{(j)}(s)-y^{(k)}(s))ds}E_t^{\mathcal{T}_{n,(k)}}\left[L^{(k)}(T_{n-1},T_n;\tau)\right] \ . && (3.23)\end{aligned}$$

Since it holds that $E_t^{\mathcal{T}^c_{(k)}}[\cdot] = E_t^{\mathcal{T}_{(k)}}[\cdot]$ under the current assumption, even if the Libor payment is collateralized by a foreign j-currency, it is straight forward to calculate the exposure in terms of the standard IRS collateralized by the domestic currency.

One can see that all the corrections from our simplifying assumption arise from either the convexity correction in $E\left[e^{\int_t^T y^{(k)}(s)ds}\right]$ or from the covariance between $e^{\int_t^T y^{(k)}(s)ds}$ and other stochastic variable such as Libor and FX rates. Considering the absolute size of the spread y and its volatility, one can reasonably expect that the corrections are quite small. Actually, the fact that separate quotes of these instruments for each collateral currency are unobservable indicates that the corrections induced from the assumptions are well within the current market bid/offer spreads. As will be seen in the following sections, the above assumption will allow a flexible enough framework to address the issues described in the introduction without causing unnecessary complications.

3.2 Model with Dynamic basis spreads in Multiple Currencies

Now, let us finally preset the modeling framework in the multi-currency environment under the simplified assumption given in Sec. 3.1.3. We have already set up the dynamics for the forward collateral rate, Libor-OIS spread for each tenor, and an equity with an effective dividend yield q for each currency as in Seq. 2.3:

$$dc^{(i)}(t,s) = \sigma_c^{(i)}(t,s)\cdot\left(\int_t^s \sigma_c^{(i)}(t,u)du\right)dt + \sigma_c^{(i)}(t,s)\cdot dW^{Q_i}(t) \ , \tag{3.24}$$

$$\frac{dB^{(i)}(t,T;\tau)}{B^{(i)}(t,T;\tau)} = \sigma_B^{(i)}(t,T;\tau)\cdot\left(\int_t^T \sigma_c^{(i)}(t,s)ds\right)dt + \sigma_B^{(i)}(t,T;\tau)\cdot dW^{Q_i}(t) \ , \tag{3.25}$$

$$dS^{(i)}(t)/S^{(i)}(t) = \left(c^{(i)}(t) - q^{(i)}(t)\right)dt + \sigma_S^{(i)}(t)\cdot dW^{Q_i}(t). \tag{3.26}$$

We have the above set of stochastic differential equations for each currency i. The foreign exchange dynamics between currency i and j is given by

$$df_x^{(i,j)}(t)/f_x^{(i,j)}(t) = \left(c^{(i)}(t) - c^{(j)}(t) + y^{(i,j)}(t)\right)dt + \sigma_X^{(i,j)}(t)\cdot dW^{Q_i}(t) \ , \tag{3.27}$$

where $y^{(i,j)}(t)$ is defined as

$$y^{(i,j)}(t) = y^{(i)}(t) - y^{(j)}(t) \tag{3.28}$$

$$= \left(r^{(i)}(t) - r^{(j)}(t)\right) - \left(c^{(i)}(t) - c^{(j)}(t)\right), \tag{3.29}$$

which is a deterministic function of time.

If a specific currency i is chosen to be a home currency for simulation, the stochastic differential equations for other currencies $j \neq i$ are given by

$$dc^{(j)}(t,s) = \sigma_c^{(j)}(t,s)\cdot\left[\left(\int_t^s \sigma_c^{(j)}(t,u)du\right) - \sigma_X^{(i,j)}(t)\right]dt + \sigma_c^{(j)}(t,s)\cdot dW^{Q_i}(t) \ , \tag{3.30}$$

$$\frac{dB^{(j)}(t,T;\tau)}{B^{(j)}(t,T;\tau)} = \sigma_B^{(j)}(t,T;\tau)\cdot\left[\left(\int_t^T \sigma_c^{(j)}(t,s)ds\right) - \sigma_X^{(i,j)}(t)\right]dt + \sigma_B^{(j)}(t,T;\tau)\cdot dW^{Q_i}(t) \ , \tag{3.31}$$

$$dS^{(j)}(t)/S^{(j)}(t) = \left[\left(c^{(j)}(t) - q^{(j)}(t)\right) - \sigma_S^{(j)}(t)\cdot\sigma_X^{(i,j)}(t)\right]dt + \sigma_S^{(j)}(t)\cdot dW^{Q_i}(t) \ , \tag{3.32}$$

where the relation (3.6) has been used. These are the relevant underlying factors for multi-currency environment.

3.3 Curve calibration

This section explains how to set up the initial conditions for the modeling framework explained in the previous section. As will see, the spread curves $\{y(t)^{(i,j)}\}$ for the relevant currency pairs can be bootstrapped by fitting to the term structure of CCS basis spread, or equivalently to the FX forwards.

3.3.1 Single currency instruments

Let us first remind the setup of single currency sector of the model. As explained in Sec. 2.3, the collateralized zero coupon bonds $D(t,T)$ and Libor expectations $E_t^{\mathcal{T}_k^c}[L(T_{k-1},T_k;\tau)]$ can be extracted from the following set of equations:

$$\mathrm{OIS}_N^{(i)}(t)\sum_{n=1}^{N}\Delta_n^{(i)}D^{(i)}(t,T_n)=D^{(i)}(t,T_0)-D^{(i)}(t,T_N) \tag{3.33}$$

$$\mathrm{IRS}_M^{(i)}(t)\sum_{m=1}^{M}\Delta_m^{(i)}D^{(i)}(t,T_m)=\sum_{m=1}^{M}\delta_m^{(i)}D^{(i)}(t,T_m)E_t^{\mathcal{T}_{m,(i)}^c}[L^{(i)}(T_{m-1},T_m;\tau)] \tag{3.34}$$

$$\sum_{n=1}^{N}\delta_n^{(i)}D^{(i)}(t,T_n)\left(E_t^{\mathcal{T}_{n,(i)}^c}\left[L^{(i)}(T_{n-1},T_n;\tau_S)\right]+TS^{(i)}(t)\right)$$
$$=\sum_{m=1}^{M}\delta_m^{(i)}D^{(i)}(t,T_m)E_t^{\mathcal{T}_{m,(i)}^c}\left[L^{(i)}(T_{m-1},T_m;\tau_L)\right]\ , \tag{3.35}$$

from OIS, IRS, and TS contracts respectively. Using the relations

$$c^{(i)}(t,s)=-\frac{\partial}{\partial s}\ln D^{(i)}(t,s) \tag{3.36}$$

and

$$B^{(i)}(t,T_n;\tau)=E_t^{\mathcal{T}_{n,(i)}^c}\left[L^{(i)}(T_{n-1},T_n;\tau)\right]-\frac{1}{\delta_n^{(i)}}\left(\frac{D^{(i)}(t,T_{n-1})}{D^{(i)}(t,T_n)}-1\right)\ , \tag{3.37}$$

one can get the initial conditions for the collateral rate $c(t,s)$, and the Libor-OIS spreads $B(t,T;\tau)$ for each currency.

3.3.2 FX forward

Next, let us consider FX forward contracts. In the current setup, a FX forward contract maturing at time T between currency (i,j) becomes

$$f_x^{(i,j)}(t,T) = f_x^{(i,j)}(t)\frac{P^{(j)}(t,T)}{P^{(i)}(t,T)} \tag{3.38}$$

$$= f_x^{(i,j)}(t)\frac{D^{(j)}(t,T)}{D^{(i)}(t,T)}e^{\int_t^T y^{(i,j)}(s)ds}\ . \tag{3.39}$$

By the quotes of spot and forward FX rates, and the $\{D(t,T)\}$ derived in the previous section, the value of $\int_t^T y^{(i,j)}(s)ds$ can be found. Based on the quotes for various maturities T and proper spline technique, $y^{(i,j)}(s)$ will be obtained as a continuous function of time s. This can be done for all the relevant pairs of currencies. This will give another important input of the model required in Eq.(3.27). If one needs to assume that the collateral rate of a given currency i is actually the risk-free rate, the set of functions $\{y^{(j)}(s)\}_{j\neq i}$ can be obtained by combination of the information of FX forwards with $y^{(i)}(s) \equiv 0$. Note that one cannot assume the several collateral rates are equal to the risk-free rates simultaneously since the model should be made consistent with FX forwards (and CCS).

As mentioned before, the current setup does not recognize the differences among FX forwards from their choice of collateral currencies. It arises from our simplified assumption that the spread between the risk-free and collateral rates of a given currency is a deterministic function of time. This seems consistent with the reality, at least in the current market [12].

3.4 Other Vanilla Instruments

The instruments explained in the previous sections 3.3.1 and 3.3.2 are sufficient to fix the initial conditions of the curves used in the model. Next, let us check other fundamental instruments and the implications of the model.

3.4.1 European FX option

Calculation of European FX option is quite simple. Let us consider the T-maturing FX call option for $f_x^{(i,j)}$ collateralized by k-currency. The present value can be written as

$$
\begin{aligned}
PV(t) &= E_t^{Q_i}\left[e^{-\int_t^T r^{(i)}(s)ds} e^{\int_t^T y^{(k)}(s)ds}\left(f_x^{(i,j)}(T) - K\right)^+\right] && (3.40)\\
&= D^{(i)}(t,T) e^{\int_t^T y^{(k,i)}(s)ds} E_t^{\mathcal{T}^c_{(i)}}\left[\left(f_x^{(i,j)}(T,T) - K\right)^+\right] . && (3.41)
\end{aligned}
$$

The FX forward $f_x^{(i,j)}(\cdot, T)$ is a martingale under the forward measure $\mathcal{T}_{(i)}$ (or equivalently $\mathcal{T}^c_{(i)}$ in our assumption), and its stochastic differential equation is given by

$$
\begin{aligned}
\frac{df_x^{(i,j)}(t,T)}{f_x^{(i,j)}(t,T)} &= \sigma_{FX}^{(i,j)}(t,T) \cdot dW^{\mathcal{T}^c_{(i)}}(t) && (3.42)\\
&= \left\{\sigma_X^{(i,j)}(t) + \int_t^T \sigma_c^{(i)}(t,s)ds - \int_t^T \sigma_c^{(j)}(t,s)ds\right\} \cdot dW^{\mathcal{T}^c_{(i)}}(t) \ , && (3.43)
\end{aligned}
$$

under the same forward measure. It is straightforward to obtain an analytical approximation of Eq.(3.41).

3.4.2 Constant notional cross currency swap

A constant notional CCS (CNCCS) of a currency pair (i, j) is a floating-vs-floating swap where the two parties exchange the i-Libor flat vs j-Libor plus fixed spread periodically for a certain period. There are both the initial and final notional exchanges, and the notional for each leg is kept constant throughout the contract. The currency i, in which

[12] Note however that the choice of collateral currency does affect the present value of a trade. As can be seen from Eq. (3.23), the present value of a payment at time T in j-currency collateralized with i-currency is proportional to $D^{(j)}(t,T)e^{\int_t^T y^{(i,j)}(s)ds}$, and hence the payer of collateral may want to choose the collateral currency "i" for each period in such a way that it minimizes $\left(\int_t^T y^{(i,j)}(s)ds\right)$.

Libor is paid in flat is dominated by USD in the market. CNCCS has been used to convert a loan denominated in a given currency to that of another currency to reduce its funding cost. Due to its significant FX exposure, mark-to-market CCS (MtMCCS), which will be explained in the next section, has now become quite popular. The information in CNCCS is equivalent to the one extracted from FX forwards, since CNCCS combined with IRS and TS with the same collateral currency can replicate a FX forward contract.

Here, we will provide the formula for the CNCCS of a currency pair (i,j), just for completeness. Assume that the collateral is posted in i-currency. Then, the present value of i-Leg for unit notional is given by

$$\begin{aligned} PV_i(t) &= \sum_{n=1}^{N} \delta_n^{(i)} D^{(i)}(t,T_n) E_t^{\mathcal{T}^c_{n,(i)}} \left[L^{(i)}(T_{n-1},T_n;\tau) \right] - D^{(i)}(t,T_0) + D^{(i)}(t,T_N) \\ &= \sum_{n=1}^{N} \delta_n^{(i)} D^{(i)}(t,T_n) B^{(i)}(t,T_n;\tau), \end{aligned} \tag{3.44}$$

where T_0 is the effective date of the contract. On the other hand, the present value of j-Leg with a spread $B_N^{\mathrm{CCS}}(t) = B_N^{\mathrm{CCS}}(t,T_0,T_N;\tau)$ for the unit notional is

$$\begin{aligned} PV_j(t) &= -E_t^{Q_j} \left[e^{-\int_t^{T_0} \left(r^{(j)}(s) - y^{(i)}(s) \right) ds} \right] + E_t^{Q_j} \left[e^{-\int_t^{T_N} \left(r^{(j)}(s) - y^{(i)}(s) \right) ds} \right] \\ &\quad + \sum_{n=1}^{N} \delta_n^{(j)} E_t^{Q_j} \left[e^{-\int_t^{T_n} \left(r^{(j)}(s) - y^{(i)}(s) \right) ds} \left(L^{(j)}(T_{n-1},T_n;\tau) + B_N^{\mathrm{CCS}}(t) \right) \right] , \end{aligned} \tag{3.45}$$

and using the assumption of the deterministic spread y leads to

$$\begin{aligned} PV_j(t) &= \sum_{n=1}^{N} \delta_n^{(j)} D^{(j)}(t,T_n) e^{\int_t^{T_n} y^{(i,j)}(s) ds} \left(B^{(j)}(t,T_n;\tau) + B_N^{\mathrm{CCS}}(t) \right) \\ &\quad + \sum_{n=1}^{N} D^{(j)}(t,T_{n-1}) e^{\int_t^{T_{n-1}} y^{(i,j)}(s) ds} \left(e^{\int_{T_{n-1}}^{T_n} y^{(i,j)}(s) ds} - 1 \right) . \end{aligned} \tag{3.46}$$

Let us denote the notional of i-Leg per unit amount of j-notional as $N^{(i)}$. Usually, it is fixed by the forward FX at the time of inception of the contract as $N^{(i)} = f_x^{(i,j)}(t,T_0)$, and then the total present value of i-Leg in terms of currency j is given by

$$\frac{N^{(i)}}{f_x^{(i,j)}(t)} PV_i(t) = \sum_{n=1}^{N} \delta_n^{(i)} \frac{N^{(i)}}{f_x^{(i,j)}(t)} D^{(i)}(t,T_n) B^{(i)}(t,T_n;\tau) \tag{3.47}$$

$$= \sum_{n=1}^{N} \delta_n^{(i)} \frac{N^{(i)}}{f_x^{(i,j)}(t,T_n)} D^{(j)}(t,T_n) e^{\int_t^{T_n} y^{(i,j)}(s) ds} B^{(i)}(t,T_n;\tau) . \tag{3.48}$$

Hence, the following expression of the T_0-start T_N-maturing CNCCS basis spread is obtained:

$$
\begin{aligned}
&B_N^{\mathrm{CCS}}(t,T_0,T_N;\tau)\\
&=\left[\sum_{n=1}^{N}\delta_n^{(j)}D^{(j)}(t,T_n)e^{\int_t^{T_n}y^{(i,j)}(s)ds}\left\{\frac{\delta_n^{(i)}}{\delta_n^{(j)}}\frac{N^{(i)}}{f_x^{(i,j)}(t,T_n)}B^{(i)}(t,T_n;\tau)-B^{(j)}(t,T_n;\tau)\right\}\right.\\
&\left.-\sum_{n=1}^{N}D^{(j)}(t,T_{n-1})e^{\int_t^{T_{n-1}}y^{(i,j)}(s)ds}\left(e^{\int_{T_{n-1}}^{T_n}y^{(i,j)}(s)ds}-1\right)\right]\Big/\sum_{n=1}^{N}\delta_n^{(j)}D^{(j)}(t,T_n)e^{\int_t^{T_n}y^{(i,j)}(s)ds}.
\end{aligned}
\tag{3.49}
$$

One can also get a formula for different collateral currency by repeating similar calculation.

Note that the $B_N^{\mathrm{CCS}}(t,T_0,T_N;\tau)$ in Eq.(3.49) is a martingale under the annuity measure $\hat{\mathcal{A}}$ where the i-collateralized j-annuity $\sum_{n=1}^{N}\delta_n^{(j)}D^{(j)}(t,T_n)e^{\int_t^{T_n}y^{(i,j)}(s)ds}$ is used as the numeraire. Therefore, the present value of a T_0-start T_N-maturing constant-notional cross currency payer swaption with strike spread K is given as

$$
PV(t)=\sum_{n=1}^{N}\delta_n^{(j)}D^{(j)}(t,T_n)e^{\int_t^{T_n}y^{(i,j)}(s)ds}E_t^{\hat{\mathcal{A}}}\left[\left(B_N^{\mathrm{CCS}}(T_0,T_0,T_N;\tau)-K\right)^+\right]\ , \tag{3.50}
$$

where the notional of j-Leg is assumed to be the unit amount of a corresponding currency. Once every volatility process is specified, it will be tedious but possible to derive an analytic approximation by, for example, applying asymptotic expansion technique.

3.4.3 Mark-to-Market cross currency swap

Mark-to-Market cross currency swap (MtMCCS) is a similar contract to the aforementioned CNCCS except that the notional of the Leg which pays Libor flat is refreshed at the every start of the Libor calculation period based on the spot FX at that time. The notional for the other leg is kept constant throughout the contract. More specifically, let us consider a MtMCCS for (i,j) currency pair where j-Libor plus spread is exchanged for i-Libor flat. In this case, the notional of the i-Leg is going to be set at $f_x^{(i,j)}(t)$ times the notional of j-Leg at beginning of every period and the amount of notional change is exchanged at the same time. Due to the notional refreshment, a (i,j)-MtMCCS can be considered as a portfolio of one-period (i,j)-CNCCS, where the notional of j-Leg of every contract is the same. Here, the net effect from the final notional exchange of the (n)-th CNCCS and the initial exchange of the $(n+1)$-th CNCCS is equivalent to the notional adjustment at the star of the $(n+1)$-th period of the MtMCCS.

Let us assume the collateral currency is i as before. The present value of j-Leg can be calculated exactly in the same way as CNCCS, and is given by

$$
\begin{aligned}
PV_j(t)\ &=\ \sum_{n=1}^{N}\delta_n^{(j)}D^{(j)}(t,T_n)e^{\int_t^{T_n}y^{(i,j)}(s)ds}\left(B^{(j)}(t,T_n;\tau)+B_N^{\mathrm{MtM}}(t)\right)\\
&\quad+\sum_{n=1}^{N}D^{(j)}(t,T_{n-1})e^{\int_t^{T_{n-1}}y^{(i,j)}(s)ds}\left(e^{\int_{T_{n-1}}^{T_n}y^{(i,j)}(s)ds}-1\right)\ ,
\end{aligned}
\tag{3.51}
$$

where $B_N^{\mathrm{MtM}}(t) = B_N^{\mathrm{MtM}}(t, T_0, T_N; \tau)$ is the time-t value of the MtMCCS basis spread for this contract. On the other hand, the present value of i-Leg can be calculated as

$$
\begin{aligned}
PV_i(t) &= -\sum_{n=1}^{N} E_t^{Q_i}\left[e^{-\int_t^{T_{n-1}} c^{(i)}(s)ds} f_x^{(i,j)}(T_{n-1})\right] \\
&\quad + \sum_{n=1}^{N} E_t^{Q_i}\left[e^{-\int_t^{T_n} c^{(i)}(s)ds} f_x^{(i,j)}(T_{n-1})\left(1 + \delta_n^{(i)} L^{(i)}(T_{n-1}, T_n; \tau)\right)\right] \\
&= \sum_{n=1}^{N} \delta_n^{(i)} D^{(i)}(t, T_n) E_t^{\mathcal{T}_{n,(i)}^c}\left[f_x^{(i,j)}(T_{n-1}) B^{(i)}(T_{n-1}, T_n; \tau)\right] . \qquad (3.52)
\end{aligned}
$$

As a result, the MtMCCS basis spread is given by

$$
\begin{aligned}
&B_N^{\mathrm{MtM}}(t, T_0, T_N; \tau) = \\
&\left[\sum_{n=1}^{N} \delta_n^{(j)} D^{(j)}(t, T_n) e^{\int_t^{T_n} y^{(i,j)}(s)ds}\left\{\frac{\delta_n^{(i)}}{\delta_n^{(j)}} E_t^{\mathcal{T}_{n,(i)}^c}\left[\frac{f_x^{(i,j)}(T_{n-1})}{f_x^{(i,j)}(t, T_n)} B^{(i)}(T_{n-1}, T_n; \tau)\right] - B^{(j)}(t, T_n; \tau)\right\}\right. \\
&\left. - \sum_{n=1}^{N} D^{(j)}(t, T_{n-1}) e^{\int_t^{T_{n-1}} y^{(i,j)}(s)ds}\left(e^{\int_{T_{n-1}}^{T_n} y^{(i,j)}(s)ds} - 1\right)\right] \Big/ \sum_{n=1}^{N} \delta_n^{(j)} D^{(j)}(t, T_n) e^{\int_t^{T_n} y^{(i,j)}(s)ds} ,
\end{aligned}
\qquad (3.53)
$$

and, after some calculation, we get

$$
\begin{aligned}
&B_N^{\mathrm{MtM}}(t, T_0, T_N; \tau) = \\
&\left[\sum_{n=1}^{N} \delta_n^{(j)} D^{(j)}(t, T_n) e^{\int_t^{T_n} y^{(i,j)}(s)ds}\left\{\frac{\delta_n^{(i)}}{\delta_n^{(j)}} \frac{f_x^{(i,j)}(t, T_{n-1})}{f_x^{(i,j)}(t, T_n)} B^{(i)}(t, T_n; \tau) Y_n^{(i,j)}(t) - B^{(j)}(t, T_n; \tau)\right\}\right. \\
&\left. - \sum_{n=1}^{N} D^{(j)}(t, T_{n-1}) e^{\int_t^{T_{n-1}} y^{(i,j)}(s)ds}\left(e^{\int_{T_{n-1}}^{T_n} y^{(i,j)}(s)ds} - 1\right)\right] \Big/ \sum_{n=1}^{N} \delta_n^{(j)} D^{(j)}(t, T_n) e^{\int_t^{T_n} y^{(i,j)}(s)ds} .
\end{aligned}
\qquad (3.54)
$$

Here, $Y_n^{(i,j)}(t)$ is defined by

$$
\begin{aligned}
Y_n^{(i,j)}(t) &= E_t^{\mathcal{T}_{n,(i)}^c}\left[\exp\left\{\int_t^{T_{n-1}} \sigma_{FX}^{(i,j)}(s, T_{n-1}) \cdot \left(\sigma_B^{(i)}(s, T_n; \tau) - \int_{T_{n-1}}^{T_n} \sigma_c^{(i)}(s, u)du\right) ds\right.\right. \\
&\quad \left.\left. + \int_t^{T_{n-1}} \left(\sigma_{X_n}(s) \cdot dW^{\mathcal{T}_{n,(i)}^c}(s) - \frac{1}{2}\sigma_{X_n}(s)^2 ds\right)\right\}\right] , \qquad (3.55)
\end{aligned}
$$

where

$$
\sigma_{X_n}(t) = \sigma_{FX}^{(i,j)}(t, T_{n-1}) + \sigma_B^{(i)}(t, T_n; \tau) . \qquad (3.56)
$$

If we have liquid markets for FX forward and CNCCS, volatility and correlation parameters involved in the expression of $Y_n^{(i,j)}$ needs to be adjusted to make the model consistent with the MtMCCS. However, considering the popularity of MtMCCS and limited liquidity of FX forwards with long maturities, it may be more practical to calibrate

$\{y^{(i,j)}(t)\}$ using MtMCCS directly. One can see easily that approximating $Y_n^{(i,j)} \simeq 1$ allows us straightforward bootstrapping of $\{y^{(i,j)}(t)\}$.

As is the case in CNCCS, the forward MtMCCS basis spread given in Eq.(3.53) is a martingale under the annuity measure $\hat{\mathcal{A}}$, where i-collateralized j-annuity, $\sum_{n=1}^N \delta_n^{(j)} D^{(j)}(t,T_n)e^{\int_t^{T_n} y^{(i,j)}(s)ds}$ is used as the numeraire. Therefore, a T_0-start T_N-maturing mark-to-market cross currency payer swaption with strike spread K is calculated as

$$PV(t) = \sum_{n=1}^N \delta_n^{(j)} D^{(j)}(t,T_n)e^{\int_t^{T_n} y^{(i,j)}(s)ds} E_t^{\hat{\mathcal{A}}}\left[\left(B_N^{\mathrm{MtM}}(T_0,T_0,T_N;\tau) - K\right)^+\right] , \qquad (3.57)$$

where we have used the unit amount of j-Leg notional. A similar formula for a different collateral currency case can be also derived. One can see that forward MtMCCS basis spread has much smaller volatility than that of CNCCS due to the cancellation of FX exposure thanks to its notional refreshments.

By comparing the expression in Eq. (3.49), we can also derive the difference of i-collateralized CNCCS and MtMCCS basis spread as follows:

$$\begin{aligned} & B_N^{\mathrm{MtM}}(t,T_0,T_n;\tau) - B_N^{\mathrm{CCS}}(t,T_0,T_n;\tau) \\ &= \frac{\sum_{n=1}^N \delta_n^{(i)} D^{(j)}(t,T_n)e^{\int_t^{T_n} y^{(i,j)}(s)ds}\left\{\frac{f_x^{(i,j)}(t,T_{n-1})}{f_x^{(i,j)}(t,T_n)}B^{(i)}(t,T_n;\tau)Y_n^{(i,j)}(t) - \frac{N^{(i)}}{f_x^{(i,j)}(t,T_n)}B^{(i)}(t,T_n;\tau)\right\}}{\sum_{n=1}^N \delta_n^{(j)} D^{(j)}(t,T_n)e^{\int_t^{T_n} y^{(i,j)}(s)ds}} . \end{aligned} \qquad (3.58)$$

One can check that the difference of FX exposure and the correction term $Y_n^{(i,j)}$ give rise to the gap between the two CCS's.

4 Comments on Inflation Modeling

Before closing the paper, let us briefly comment on the inflation modeling in the presence of collateral. Although it is straightforward to use the multi-currency framework as was proposed in the work of Jarrow and Yildirim [13], it requires the simulation of unobservable real interest rates. It is quite difficult to estimate the real rate volatilities and its correlations to the other underlying factors. Here, let us present the method by which the collateralized forward CPI is directly simulated in the same way as for the Libor-OIS spreads. This is a simple extension of the model proposed by Belgrade and Benhamou [2] for collateralized contracts.

First, define the forward CPI as the fixed amount of payment which is exchanged for $I(T)$ units of the corresponding currency at time T. Here, $I(T)$ is the time-T CPI index. Let us consider CPI of i-currency continuously collateralized by j-currency. Then, the forward CPI $I^{(i)}(t,T)$ should satisfy

$$I^{(i)}(t,T)E_t^{Q_i}\left[e^{-\int_t^T r^{(i)}(s)ds}e^{\int_t^T y^{(j)}(s)ds}\right] = E_t^{Q_i}\left[e^{-\int_t^T r^{(i)}(s)ds}e^{\int_t^T y^{(j)}(s)ds}I(T)\right] . (4.1)$$

Under the assumption of deterministic spread $y^{(j)}$, it becomes

$$I^{(i)}(t,T) = E_t^{\mathcal{T}_{(i)}}[I(T)] = E_t^{\mathcal{T}^c_{(i)}}[I(T)], \qquad (4.2)$$

and is independent from the collateralized currency as for the multi-currency example in the previous section. The present value of a future CPI payment of the currency i collateralized by the foreign currency j is expressed by using the forward CPI as

$$PV_i(t) = D^{(i)}(t,T)e^{\int_t^T y^{(j,i)}(s)ds} I^{(i)}(t,T) \ , \tag{4.3}$$

where $y^{(j,i)}(s)$ is available after the multi-currency curve calibration.

The forward CPI can be easily extracted from a set of zero coupon inflation swap (ZCIS), which is the most liquid inflation product in the current market. The break-even rate K_N of the N-year zero coupon inflation swap satisfies

$$\left[(1+K_N)^N - 1\right] D(t,T_N) = \left(\frac{E_t^{\mathcal{T}^c}[I(T_N)]}{I(t)} - 1\right) D(t,T_N) \ , \tag{4.4}$$

and hence

$$I(t,T_N) = I(t)(1+K_N(t))^N \ . \tag{4.5}$$

Here, the collateral currency is assumed to be the same as the payment currency. It is straightforward to construct a smooth forward CPI curve using appropriate spline technique. Although we are not going into details, it is also quite important to estimate month-on-month (MoM) seasonality factors using historical data. As is clear from its property, it should not be treated as a diffusion process, and hence it should be added on top of the simulated forward CPI based on the smooth YoY trend process.

Since $I(t,T)$ is a martingale under the $\mathcal{T}^c$ measure, its stochastic differential equation under the MM measure Q can be specified as follows:

$$dI(t,T) = \sigma_I(t,T) \cdot \left(\int_t^T \sigma_c(t,s)ds\right) dt + \sigma_I(t,T) \cdot dW^Q(t) \ . \tag{4.6}$$

This should be understood as the trend forward CPI process, and needs to be adjusted properly by the use of seasonality factors to derive a forward CPI with odd period. As a summary, necessary stochastic differential equations for IR-Inflation Hybrids are given by

$$\begin{aligned}
dc(t,s) &= \sigma_c(t,s) \cdot \left(\int_t^s \sigma_c(t,u)du\right) dt + \sigma_c(t,s) \cdot dW^Q(t) \ , && (4.7)\\
\frac{dB(t,T;\tau)}{B(t,T;\tau)} &= \sigma_B(t,T;\tau) \cdot \left(\int_t^T \sigma_c(t,s)ds\right) dt + \sigma_B(t,T;\tau) \cdot dW^Q(t) \ , && (4.8)\\
dI(t,T) &= \sigma_I(t,T) \cdot \left(\int_t^T \sigma_c(t,s)ds\right) dt + \sigma_I(t,T) \cdot dW^Q(t) \ . && (4.9)
\end{aligned}$$

5 Conclusions

This paper has presented a new framework of interest rate models which reflects the existence as well as dynamics of various basis spreads in the market. It has also explicitly taken the impacts from the collateralization into account, and provided its extension for multi-currency environment consistently with FX forwards and MtMCCS in the first time. It has also commented on the inflation modeling in the presence of collateral.

Finally, let us provide a possible order of calibration in this framework.
1, Calibrate domestic swap curves and extract $\{D(t,T)\}$ and $\{B(t,T;\tau)\}$ following the method in Ref. [6] for each currency.
2, Calibrate domestic interest rate options, such as swaptions and caps/floors, and determine the volatility curves (or surface) of IR sector for each currency. For the setup of correlation structure, option implied information or historical data can be used. If one has a set of calibrated swap curves for a certain period of history, it is straight forward to carry out the principal component analysis and extract the several dominant factors. See the explanation given, for example, in the work of Rebonato [15].
3, Calibrate FX forwards (or CNCCS) and extract the set of $\{y^{(i,j)}(s)\}$ for all the relevant currency pairs.
4, Calibrate the vanilla FX options and determine the spot FX volatility for all the relevant currency pairs. The resultant spot FX volatility does depend on the correlation structure between the spot FX and collateral rates of the two currencies. It should be estimated using quanto products and/or historical data.
5, Calibrate MtMCCS and determine the correlation curve between spot FX and Libor-OIS spread. Considering the size of correction, one will have quite a good fit after the calibration of FX forwards, though.

There remain various interesting topics for the practical implementation of this new framework; Analytic approximation for vanilla options will be necessary for fast calibration and for the use as regressors for Bermudan/American type of exotics. Because of the separation of discounting curve and Libor-OIS spread, there will be some important implications to the price of convexity products, such as constant-maturity swap (CMS). It is also an important problem to consider the method to obtain stable attribution of vega (kappa) exposure to each vanilla options for generic exotics [13].

A Compounding in Tenor Swap

As we have mentioned in Sec.2.2.3, there is a slight complication in TS due to the compounding in the Leg with the short tenor. For example, in a USD 3m/6m-tenor swap, coupon payments from the 3m-Leg occur semiannually where the previous coupon (3m-Libor plus tenor spread) is compounded by 3m-Libor flat. As a result, the present value

[13] After completion of the original version of this paper, we have published several new works for the related issues: Fujii and Takahashi (2010,2011) [7, 8, 9], which include improvements and further extensions as well as some numerical examples.

of the 3m-Leg is calculated as

$$
\begin{aligned}
&PV_{\tau_S}(t) \\
&= \sum_{m=1}^{M} E_t^Q \left[e^{-\int_t^{T_{2m}} c(s)ds} \{ \delta_{2m-1} \left(L(T_{2m-2}, T_{2m-1}; \tau_S) + TS(t) \right) \left(1 + \delta_{2m} L(T_{2m-1}, T_{2m}; \tau_S) \right) \right. \\
&\quad \left. + \ \delta_{2m} \left(L(T_{2m-1}, T_{2m}; \tau_S) + TS(t) \right) \} \right] \\
&= \sum_{n=1}^{2M} D(t, T_n) \delta_n \left(E_t^{\mathcal{T}_n^c} \left[L(T_{n-1}, T_n; \tau_S) \right] + TS(t) \right) \\
&\quad + \sum_{m=1}^{M} \delta_{2m-1} \delta_{2m} D(t, T_{2m}) TS(t) B(t, T_{2m}; \tau_S) \\
&\quad + \sum_{m=1}^{M} \delta_{2m-1} \delta_{2m} D(t, T_{2m}) E_t^{\mathcal{T}_{2m}^c} \left[L(T_{2m-2}, T_{2m-1}; \tau_S) B(T_{2m-1}, T_{2m}; \tau_S) \right] \ , \qquad \text{(A.1)}
\end{aligned}
$$

where $\tau_S = 3$m. Note that the second and third terms are correction to the left-hand side of Eq.(2.7). Since the size of Libor-OIS and tenor spreads have similar sizes, the correction term can not affect the calibration meaningfully. Considering the bid/offer spread, one can safely neglect the compounding effects in most situations.

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