A New-Keynesian Model with Estimated Shadow Rate for Japan Economy

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Abstract

In this paper, we use Japanese government bond yield curve data to estimate the shadow rate from a shadow rate term structure model. We firstly confirm the traceability of the shadow rate as a consistent and compatible measure of monetary policy which can be used to gauge the stance of policy implemented by the Bank of Japan, then we introduce the shadow rate in a standard NK-DSGE model and estimate the model using the shadow rate estimated previously. Finally, we check the empirical results from the NK-DSGE model with shadow rate. The results show a good compatibility of the shadow rate used as a proxy of monetary policy in the estimation of NK-DSGE model, providing credible policy implications as the benchmark NK-DSGE model does in a standard context without the restriction of the zero lower bound. The empirical results show that the monetary easing policy conducted by the Bank of Japan has positive effects on the improvement of the output gap and deflation. Using the shadow rate in the estimation of DSGE models also avoids the technical difficulties incurred by the nonlinearity of the zero lower bound.

Keywords: shadow rate, NK-DSGE, unconventional monetary policy, zero lower bound

1 Introduction

When the short-term nominal interest rate is at or near zero, central banks have to face the problems incurred by the Zero Lower Bound (ZLB) because the ZLB invalidates the implementation of conventional monetary policy, the adjustment of short policy rate. Facing the constraint of ZLB, central banks conduct unconventional monetary policy to stabilize and stimulate the economy. This is what Japan economy has experienced and Bank of Japan (BoJ) has done since 1999 when the call rate decreased to a very low level near zero. The Great Recession incurred by the Global Financial Crisis 2007-2009 brought the same

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problem to US, UK and Euro area. Besides the practical policy issues faced by the monetary authorities in advanced economies, the ZLB and the related unconventional monetary policy also pose academic issues and new challenges for macroeconomic research.

New Keynesian Dynamic Stochastic General Equilibrium (NK-DSGE) model and Gaussian Affine Term Structure (GATS) model are two main workhorses in monetary economics and macro-finance. But when the ZLB on the short-term nominal interest rate binds, unfortunately, these models have deficient performance and undesired economic implications, leading to implausible and weird policy paradoxes and unsatisfied fitness of data.

As a standard methodology for monetary policy analysis, a NK-DSGE model in the ZLB environment predicts that positive temporary supply shocks have contractionary effects and vice versa, negative supply shocks have expansionary effects. Also, fiscal and forward guidance multipliers can be implausibly larger than one. All these conclusions from the standard NK-DSGE models are inconsistent with economic intuition and empirical facts. Besides the misleading policy implications, the ZLB also brings many technical problems in DSGE methodology. The explicit introduction of the ZLB constraint into DSGE models accompanies with structural break or nonlinear kink. Such kind of nonlinearity invalidates the linear approximation and the Kalman filter. Some researchers use global projection method and the particle filter to deal with the nonlinearity in solution and estimation of DSGE models, but these methods are technically difficult and demand for numerous computation.

As another main methodology in macro-finance research, GATS models which are widely used to fit the term structure of interest rate can provide good description of the dynamics of short interest rate in many macro-financial applications in the non-ZLB environment, but in the ZLB environment, the GATS models provide barely satisfactory fitness of data and can not accommodate the "sticky" property of short interest rate.

In recent research, the Shadow Rate Term Structure (SRTS) model which was firstly proposed by Black (1995) has become an alternative to overcome the poor performance incurred by the ZLB in general GATS model. Kim and Singleton (2012) and Bauer and Rudebusch (2013) have used the shadow rate estimated from a SRTS model to capture the behavior of interest rates and unconventional monetary policy in the ZLB environment. Krippner (2013) proposed a continuous-time formulation of SRTS model, known as Krippner Affine Gaussian model (K-AGM), where he added a call option feature to derive the closed-form solution. Wu and Xia (2016) took an analogous discrete-time ap-

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1Wieland (2015) and Garín et al. (2016) showed similar impulse responses of output to a supply shock in both ZLB and non-ZLB environment.

2The "sticky" property means that the short interest rate tends to keep approximately static around the ZLB with lower volatilities for a long period of time.
approach which can be directly applied to discrete-time yield curve data without any numerical error associated with simulation methods and numerical integration in other models. All of these literatures advocate the effectiveness of the shadow rate as a kind of measure for describing the monetary policy stance.

Wu and Zhang (2016) established the equivalence between shadow rate and unconventional monetary policy in a standard NK-DSGE model. The equivalence between shadow rate and unconventional monetary policy is established on the empirical findings that have shown the highly correlation between the quantity of government bond purchase and the estimated shadow rate. The shadow rate can take both positive and negative values and show consistent response to monetary policy in both non-ZLB and ZLB environment. Introducing the shadow rate into a DSGE model can provide more insights for the propagation and amplification mechanism of unconventional monetary policy without introducing the complications incurred by the ZLB constraint.

We take a combination of these two methods, using the shadow rate estimated from a SR TS model as the data for the estimation of a NK-DSGE model where the general policy rate is replaced by the shadow rate. Then we use the NK-DSGE model with shadow rate to do some monetary policy analysis for Japan economy. This may be a circuitous route, but the logic is valid and consistent from the beginning to the end. Also, this approach salvages the DSGE models from the nonlinearity incurred by the ZLB. Standard procedures such as the linear approximation and the Kalman filter can be used instead of complicated nonlinear solution and estimation techniques.

The remaining of this paper is organized as follows. Section 2 presents the estimation of shadow rate from a SRTS model of Krippner (2013). We also check the shadow rate’s traceability as a summary for monetary policy. In Section 3, we estimate a New-Keynesian DSGE model with shadow rate. Section 4 checks the empirical results such as historical decomposition and impulse response from the estimated DSGE model. Section 5 concludes this paper and gives the prospect for further research.

2 Shadow rate

Bauer and Rudebusch (2013) and Christensen and Rudebusch (2014) pointed out that the estimated shadow rates vary across different model specifications and estimation methods. Kim and Singleton (2012) and Bauer and Rudebusch (2013) use simulation-based estimation method to estimate the shadow rate. Krippner (2013) and Wu and Xia (2016) estimated an analogous SRTS model, but with different time specifications, the former is in continuous-time and the latter is in discrete-time. Ichiu and Ueno (2013)’s estimation is based on the approximation of bond prices by ignoring Jensen’s inequality. Imakubo and Nakajima (2015) also estimated a shadow rate model to extract inflation risk
premium from nominal and real term structures. By surveying related literature and replicating the estimation results, we find that although the shadow rates estimated from different model specifications and estimation methods do have different values, they show common trend and dynamics and lead to same economic implications in the analysis of monetary policy, at least for the major advanced economies, US, UK, Japan and Euro area.

Also, most existing literatures advocate the potential usefulness of shadow rate as a measure for the stance of monetary policy. Bullard (2012), Krippner (2012), Lombardi and Zhu (2014) and Wu and Xia (2016) all support the view that the shadow rate is a powerful tool to summarize useful information from yield curve data and describe the conventional monetary policy stance in the non-ZLB environment and unconventional monetary policy stance in the ZLB environment in a consistent manner.

2.1 Shadow rate estimated from Krippner SRTSM

Krippner (2012, 2013, 2014, 2015) provide a detailed introduction of his methodology of term structure modeling of shadow rate. For derivation of the model, please refer to Appendix A. We use yield curve data of Japanese government bond to estimate the shadow rate. See Table 1 for the summary statistics of yield curve data.

The Figure 1 shows the plot of data from which we can find that since 1999, the yield curve has shifted down to very low level. The 3-months bond yield has begun to take negative values since 2014M10. The yield curve data of Japan is Japanese government bond data from 1992-Jul-10 to 2016-Nov-24, daily frequency of 5-business days week with 6360 observations, obtained from the Bloomberg database. The maturity of yield curve is from 3-months to 30-years and 3-months interest rate is adapted to be the short interest rate \( r_t \). Interest rates with other maturities are equivalent to be \( R_{t,\tau} \) in term structure models.

The Figure 2 gives the estimation result of shadow rate. The blue area band is the 95% of the confidence interval. During non-ZLB period, the shadow rate and call rate have almost same dynamics. The shadow rate shows a good approximation to the call rate. But during the ZLB period, the call rate is approximately near to zero. We can’t read too much information about monetary policy from the call rate. But the shadow rate has keeping decreasing. In next section, we check the relation between the shadow rate and the balance sheet of BoJ. Besides other unconventional operations of the central bank, such as

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3Krippner (2013) and Wu and Xia (2016) pose their estimation codes on Internet. By replicating the estimation of shadow rates for US, UK, Japan and EU, the results don’t show much difference in two methods.

forward guidance and direct facility lending, the enlargement of balance sheet, widely known as quantitative easing (QE), is the essence of unconventional monetary policy, especially for the monetary policy of BoJ.

Figure 1: yield curve data of Japanese government bond

Figure 2: estimated shadow rate and call rate and lending rate
2.2 Empirical evidence of shadow rate

We establish some empirical relations between the shadow rate and the balance sheet.

![Figure 3: Balance sheet of BoJ](image)

BoJ is the first central bank which introduced unconventional monetary policy among major advanced economies. The essence of the policy programs conducted by BoJ is the large scale purchase of Japanese government bond. From the Figure 3, we can find since 2010M10, the balance sheet of BoJ has increased aggressively. What is the relation between the shadow rate and the size of balance sheet? The Figure 4 shows the time series plot of shadow rate, minus log of bond holdings and minus log of monetary base. The yellow shaded area in Figure 1 represents the zero interest rate policy from 1999M3 to 2000M18. The green shaded area shows the first round of QE from 2001M3 to 2006M3. The blue shaded area represents the QE since 2010M10.

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5 The full series of daily estimation for shadow rate from 1992-Jul-10 is available upon request. We did the daily-frequency estimation of the shadow rate and aggregated the daily-frequency data to average monthly-frequency data.
The correlation between shadow rate and bond holdings is -0.81 and the correlation between shadow rate and monetary base is -0.82. The X-Y scatter plots also show obvious negative correlation of the shadow rate and the variables of balance sheet. Note that the shadow rate can’t be controlled directly by the central bank in the ZLB environment. What the central bank can manipulate is its balance sheet. The shadow rate just summarizes the stance of policy. According to the empirical relation between the shadow rate and the balance sheet, we can map the manipulation of the balance sheet into the change of the shadow rate.

3 Shadow rate NK-DSGE model

In this section, we first derive a standard NK-DSGE model, then we incorporate the shadow rate into this standard framework. We show that the shadow rate can be used as an equivalence for QE so we can use the NK-DSGE model with

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*In the non-ZLB environment, the shadow rate takes positive value which is same to the general short-term policy rate. The short-term policy rate can be controlled by the central bank through open market operations.*
shadow rate to analyze the unconventional monetary policy without considering the technical complexity incurred by the ZLB.

3.1 Standard NK-DSGE model

A representative infinitely-living household maximizes lifetime utility

\[ E_t \sum_{t=0}^{\infty} \beta^t \left[ \frac{C_t^{1-\sigma} - \chi L_t^{1+\eta}}{1-\sigma} \right] \]

subject to the budget constraint

\[ C_t + \frac{B_t}{P_t} \leq \frac{R^B_{t-1} B_{t-1}}{P_t} + W_t L_t + T_t \]

where \( C_t \) and \( L_t \) denote household’s consumption and labor supply. \( P_t \) is the price level. The nominal gross bond return paid for bonds \( B_t \) is \( R^B_{t-1} \). \( W_t \) is real wage and \( T_t \) is the real transfer. Two first-order conditions decide the optimal decision of consumption and labor supply

\[ C_t^{-\sigma} = \beta E_t R^B_t \left( \frac{C_{t+1}^{-\sigma}}{\Pi_{t+1}} \right) \]

\[ W_t = \frac{\chi L_t^{\eta}}{C_t^{-\sigma}} \]

where \( \Pi_{t+1} = \frac{P_{t+1}}{P_t} \) is the gross inflation from \( t \) period to \( t + 1 \) period. The specification of intermediate-good firms and final-good firms is same as the standard NK-DSGE model. A continuum of intermediate-good firms exist, producing heterogenous intermediate-goods and selling them into final-good firms. Let \( Y_t \) be the output of the final-good which is produced using inputs of the intermediate-goods according to a bundle production function

\[ Y_t = \left[ \int_0^1 Y_{j,t}^{\frac{\varepsilon-1}{\varepsilon}} dj \right]^\frac{1}{\varepsilon} \]

where \( \varepsilon > 1 \) is the elasticity of substitution among differentiated intermediate-goods and \( Y_{j,t} \) is the input of intermediate-good \( j \in [0,1] \). Final-good firms in completely competitive market maximize profits

\[ \max_{Y_{j,t}} P_t Y_t - \int_0^1 P_j Y_{j,t} dj \]

subject to the bundle production function. The optimal input for intermediate-good \( Y_{j,t} \) is

\[ Y_{j,t} = \left( \frac{P_{j,t}}{P_t} \right)^{-\varepsilon} Y_t \]
and the zero profit condition of final-good market leads to the general price level index \( P_t \):
\[
P_t = \left( \int_0^1 p_{j,t}^{1-\varepsilon} dj \right)^{\frac{1}{\varepsilon}}
\]

Intermediate-good firms produce and sell differentiated products to final-good firms in monopolistically competitive markets. A intermediate-good firm \( j \) minimizes the total cost \( W_t L_{j,t} \) subject to a Cobb-Douglas production function given by
\[
Y_{j,t} = A_t L_{j,t}^{1-a}
\]
where \( A_t \) is aggregate productivity shock and \( L_{j,t} \) is labor input. Cost minimization leads to the real marginal cost.
\[
MC_{j,t} = \frac{W_t}{A_t(1-a)(Y_{j,t}/A_t)^{-a/(1-a)}}
\]
Define the economy-wide average real marginal cost as
\[
MC_t = \frac{W_t}{A_t(1-a)(Y_t/A_t)^{-a/(1-a)}}
\]
and the relation between intermediate-good firm \( j \)'s marginal cost and the economy-wide average real marginal cost is
\[
MC_{j,t} = MC_t \left( \frac{Y_t}{Y_{j,t}} \right)^{\frac{1}{1-a}} = MC_t \left( \frac{P_{j,t}}{P_t} \right)^{\frac{a}{1-a}}
\]
which can be derived by the demand curve of intermediate-good \( j \). Intermediate-good firm’s objective is to maximize the discounted present value of real profits according to Calvo (1983) price setting mechanism
\[
\max_{P_t} \mathbb{E}_t \sum_{k=0}^{\infty} (\beta \theta)^k \frac{\Lambda_{t+k}}{\Lambda_t} \left( \frac{P_{t+k}^{*} Y_{j,t+k|t}}{P_{t+k}} - MC_{j,t+k|t} Y_{j,t+k|t} \right)
\]
subject to demand curve of \( Y_{j,t+k|t} \),
\[
Y_{j,t+k|t} = \left( \frac{P_{t+k}^{*}}{P_{t+k}} \right)^{-\varepsilon} Y_{t+k}
\]
where \( \theta \) is the probability that the intermediate-good firm can’t adjust its price and \( \Lambda_t \) is the Lagrange multiplier in the optimization of household which represents the marginal utility of consumption. The first-order condition leads to the optimal price setting for intermediate-firm
\[
P_{t+k}^{*} = \frac{\varepsilon}{\varepsilon - 1} \frac{\mathbb{E}_t \sum_{k=0}^{\infty} (\beta \theta)^k \frac{\Lambda_{t+k}}{\Lambda_t} P_{t+k}^{*} Y_{t+k|t} MC_{j,t+k|t}^{*}}{\mathbb{E}_t \sum_{k=0}^{\infty} (\beta \theta)^k \Lambda_{t+k} P_{t+k}^{* -1} Y_{t+k}}
\]
where $MC_{j,t+k|I} = MC_{t+k} \left( \frac{P_{t+k}}{P_{t+k}} \right)^{\frac{\varepsilon}{\varepsilon - 1}}$. By a law of large number, a fraction $\theta$ of intermediate-good firms can’t adjust prices and keep prices at the previous period price level $P_{t-1}$ and the remaining fraction $1 - \theta$ of intermediate-good firms adjust to the new level $P_t$, so the price level $P_t = \left( \int_0^1 P_{j,t}^1 \, dj \right)^{\frac{1}{1+\alpha}}$ can be rewritten as a weighted sum of all intermediate-good firms’ prices.

$$P_t^{1-\varepsilon} = \theta P_{t-1}^{1-\varepsilon} + \theta (P_t^1)^{1-\varepsilon}$$

Under flexible price equilibrium, $\theta = 0$ and price rigidity disappears. The optimal price setting of intermediate-good firm is the standard result in microeconomics

$$P_t^* = P_t \frac{\varepsilon}{\varepsilon - 1} MC_{j,t}$$

where $\frac{\varepsilon}{\varepsilon - 1}$ can be explained as a markup charged by the intermediate-good firm. When prices are flexible, all intermediate-good firms are symmetric and charge the same price such that $P_t^* = P_t$, $MC_{j,t} = \frac{\varepsilon-1}{\varepsilon}$, $Y_{j,t} = Y_t$ and $L_{j,t} = L_t$ for all $j$.

Real wage is equal to marginal productivity of labor.

$$W_t = \frac{\varepsilon - 1}{\varepsilon} (1 - \alpha) A_t L_t^{-\alpha}$$

Combining this equation to the first-order condition of household’s labor supply leads to

$$\frac{\chi L_t^\eta}{C_t^{-\theta}} = \frac{\varepsilon - 1}{\varepsilon} (1 - \alpha) A_t L_t^{-\alpha}$$

from which we can solve the output $Y_t^f$ with the resource constraint $Y_t = C_t = A_t L_t^{1-\alpha}$ under flexible price equilibrium.

$$Y_t^f = \left[ \frac{1}{\varepsilon} (\varepsilon - 1)(1 - \alpha) \right] \left( \frac{1-\alpha}{\varepsilon \chi} \right) A_t^{\frac{1+\eta}{\eta(1-\alpha)^{1+\alpha}}}$$

Log-linearization of all equilibrium conditions of final-good and intermediate-good firms leads to the standard New-Keynesian Phillips Curve

$$\pi_t = \beta E_t \pi_{t+1} + \frac{(1 - \beta)(1 - \beta \theta)(1 - \alpha)}{\theta(1 - \alpha + \varepsilon \alpha)} MC_t$$

where

$$MC_t = \frac{\eta + \alpha + \sigma(1 - \alpha)}{1 - \alpha} \dot{Y}_t - \frac{1 + \eta}{1 - \alpha} \dot{A}_t$$

is the percentage deviation of economy-wide average real marginal cost. Using the output under flexible price equilibrium, this can be written as

$$MC_t = \frac{\eta + \alpha + \sigma(1 - \alpha)}{1 - \alpha} \left( \dot{Y}_t - \dot{Y}_t^f \right) = \frac{\eta + \alpha + \sigma(1 - \alpha)}{1 - \alpha} x_t$$
where $x_t = \hat{Y}_t - \hat{Y}_{f_t}$ can be explained as the output gap. So relation between the inflation and the output gap can be rewritten as

$$\pi_t = \beta \mathbb{E}_t \tau_{t+1} + \kappa x_t$$

where $\kappa = \frac{(1-\beta)(1-\beta\theta)[c+\alpha+\sigma(1-\alpha)]}{\sigma(1-\alpha+\alpha\eta)}$. Similarly, log-linearization of the household’s Euler equation leads to

$$\hat{Y}_t = \mathbb{E}_t \hat{Y}_{t+1} - \frac{1}{\sigma} \left( r_{t}^{B} + \ln \beta - \mathbb{E}_t \tau_{t+1} \right)$$

which also holds at the flexible price equilibrium as

$$\hat{Y}_{f_t} = \mathbb{E}_t \hat{Y}_{f_{t+1}} - \frac{1}{\sigma} \left( r_{t}^{N} + \ln \beta \right)$$

where $r_{t}^{N} = -\ln \beta + \sigma \left( \mathbb{E}_t \hat{Y}_{f_{t+1}} - \hat{Y}_{f_t} \right)$ is the natural interest rate. Using the definition of output gap, the New Keynesian IS curve is

$$x_t = \mathbb{E}_t \tau_{t+1} - \frac{1}{\sigma} \left( r_{t}^{B} - \mathbb{E}_t \pi_{t+1} - r_{t}^{N} \right)$$

where $r_{t}^{N} = -\ln \beta + \frac{\sigma(1+\eta)}{\sigma(1-\alpha+\alpha\eta)} \left( \mathbb{E}_t \hat{A}_{t+1} - \hat{A}_t \right)$ only depends on exogenous productivity shock.

### 3.2 Shadow rate in NK-DSGE model

According to the empirical evidence of shadow rate presented in Section 2, we introduce shadow rate into standard NK-DSGE model. For general NK-DSGE model, the economic agents face risk-free short rate $r_t$ and hold risk-free bond. $r_t$ is generally recognized as the short policy rate which can be controlled by the central bank. In actual, the relevant interest rates affecting economic agents’ decisions are private interest rates $r_t^{B}$, through which both conventional and unconventional monetary policies transmit into the economy.

Generally, the private interest rates $r_t^{B}$ can be represented as the sum of risk-free short rate $r_t$ plus a time-varying risk premium $r_t^{P}$

$$r_t^{B} = r_t + r_t^{P}$$

where $r_t$ is assumed that can be adjusted by the conventional monetary policy of the central bank. Empirical works such as Gagnon et al. (2011), Krishnamurthy and Vissing-Jorgensen (2012) and Hamilton and Wu (2012) advocate that the large-scale asset purchase by the central banks can reduce the risk premium which means

$$\frac{\partial r_t^{P}}{\partial B_t^{C}} < 0$$
where $b^G_t$ is the log of bond holdings of the central bank. This is known as the risk premium channel of QE.

Figure 5: credit spread and balance sheet

The Figure 5 shows the relation between credit spread and the balance sheet. The credit spread used here is defined as the difference between 10-years and 2-years Japanese government bond returns. The correlation of credit spread and log of bond holdings is -0.80 and the correlation of credit spread and log of monetary base is -0.81.

According to the regression lines in Figure 5, we assume that the response of risk premium $r^P_t$ to bond holdings $b^G_t$ follows a simple linear form

$$r^P_t = r^P - \gamma \left( b^G_t - b^G \right) + \epsilon^P_t$$

where $-\gamma = \frac{\partial r^P}{\partial b^G} < 0$, $r^P$ is the constant component of risk premium and $\epsilon^P_t$ is the exogenous time-varying component of risk premium which is interpreted as the liquidity preference shock in Campbell et al. (2016). In the non-ZLB
environment, \( b_t^G = b^G, r_t^P = r^P + \epsilon_t^P \) such that
\[
r_t^B = r_t + r_t^P = r_t + r^P + \epsilon_t^P
\]
which means that the private interest rate is the short rate controlled by the central bank plus risk premium. When \( r_t \) is restricted by the ZLB, approximately \( r_t = 0 \) and
\[
r_t^B = r^P - \gamma \left( b_t^G - b^G \right) + \epsilon_t^P
\]
through which the unconventional monetary policy affects risk premium to reduce private interest rate and stimulate the economy. According to the empirical evidence of the shadow rate, we also assume that the shadow rate has a same response to the log of bond holdings in a linear form like
\[
s_t = -\gamma \left( b_t^G - b^G \right)
\]
then
\[
r_t^B = s_t + r^P + \epsilon_t^P
\]
can capture the both conventional and unconventional monetary policies.

In the non-ZLB environment, \( s_t = r_t > 0, b_t^G = b^G \) and \( r_t^B = r_t + r^P + \epsilon_t^P \), the New Keynesian IS curve is
\[
x_t = E_t x_{t+1} - \frac{1}{\sigma} \left( r_t^B - E_t \pi_{t+1} - r_t^N \right)
\]
\[
= E_t x_{t+1} - \frac{1}{\sigma} \left( r_t + r^P + \epsilon_t^P - E_t \pi_{t+1} - r_t^N \right)
\]
\[
= E_t x_{t+1} - \frac{1}{\sigma} \left( s_t - E_t \pi_{t+1} + \epsilon_t^x \right)
\]
\[
= E_t x_{t+1} - \frac{1}{\sigma} \left( s_t - E_t \pi_{t+1} + \epsilon_t^x \right)
\]
where \( \epsilon_t^x = -\frac{1}{\sigma} \left( r^P + \epsilon_t^P - r_t^N \right) \) is a compound of exogenous shocks. The risk premium shock \( \epsilon_t^P \) and \( r_t^N = -\ln \beta + \frac{\sigma(1+\eta)}{\eta(1-\beta)} (E_t \hat{A}_{t+1} - \hat{A}_t) \) can’t be identified separately, so we denote the compound of these exogenous shocks as a demand shock \( \epsilon_t^X \). In the ZLB environment, the New Keynesian IS curve is
\[
x_t = E_t x_{t+1} - \frac{1}{\sigma} \left( r_t^B - E_t \pi_{t+1} - r_t^N \right)
\]
\[
= E_t x_{t+1} - \frac{1}{\sigma} \left( s_t + r^P + \epsilon_t^P - E_t \pi_{t+1} - r_t^N \right)
\]
\[
= E_t x_{t+1} - \frac{1}{\sigma} \left( s_t - E_t \pi_{t+1} + \epsilon_t^x \right)
\]
which is same as its counterpart in the non-ZLB environment.
Finally, we define a Taylor rule of shadow rate

\[ s_t = \phi_s s_{t-1} + (1 - \phi_s) (\phi_x x_t + \phi_\pi \pi_t) + \epsilon^s_t \]

where \( \epsilon^s_t \) is the monetary policy shock and \( \phi_\pi > 1 \) guarantees a unique, non-explosive equilibrium.

### 3.3 Estimation of shadow rate NK-DSGE model

From the analysis in Section 3.2, we can find the NK-DSGE model with shadow rate has the same formulation in both ZLB and non-ZLB environment. Because we have three observable variables, output gap, inflation rate and shadow rate, we add a shock term to the New Keynesian Phillips Curve to avoid the stochastic singularity in estimation.

\[
\begin{align*}
    x_t &= \mathbb{E}_t x_{t+1} - \frac{1}{\sigma} (s_t - \mathbb{E}_t \pi_{t+1}) + \epsilon^x_t \\
    \pi_t &= \beta \mathbb{E}_t \pi_{t+1} + k x_t + \epsilon^\pi_t \\
    s_t &= \phi_s s_{t-1} + (1 - \phi_s) (\phi_x x_t + \phi_\pi \pi_t) + \epsilon^s_t
\end{align*}
\]

All shocks follow autoregressive processes

\[ \epsilon^\text{shock}_t = \rho^x \epsilon^\text{shock}_{t-1} + \mu^\text{shock}_t, \quad \text{shock} \in (x, \pi, s) \]

where \( \mu^\text{shock}_t \sim N(0, \sigma^2 \text{shock}) \) is exogenous innovation term.

![Figure 6: data for estimation](image)

The data used for output gap \( x_t \) which is estimated by BoJ is from 1983Q1 to 2016Q3. The data series of inflation \( \pi_t \) is CPI inflation rate from 1980Q3.

\footnote{According to Gali (2014, chapter 5), the shock term \( \epsilon^x_t \) can be explained as a cost-push shock which may come from the exogenous variations in desired price markups or exogenous variations in wage markups.}
to 2016Q3. The data series of shadow rate is from 1992Q3 to 2017Q1. For non-ZLB period, the shadow rate is approximately equal to the call rate, we use call rate from 1980Q1 to 1992Q2 to complete the data series. Other parameters are generally calibrated as $\alpha = 0, \beta = 0.99, \varepsilon = 6$. The prior and posterior distributions of parameters and standard deviations are given in Table 2. The multivariate convergence diagnostic also shows good convergence for all estimates.

4 Shadow rate NK-DSGE in monetary policy analysis

By introducing the concept of shadow rate into a standard NK-DSGE model and estimating it with the data of shadow rate, the monetary policy analysis in the ZLB environment can be done in a same way as the analysis in the non-ZLB environment. We run some standard procedures of DSGE methodology. Firstly, we show the historical decomposition of the variables to check the contributions of each shock.

4.1 Historical decomposition

From the Figure 7, we can confirm the effects of monetary policy represented by shadow rate, especially from 2014. The positive contributions of the shadow rate monetary policy shock (green area in Figure 7) from 2014 confirm the effects of Quantitative Qualitative Monetary Easing (QQE) of BoJ.

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8 The yield curve data that can be obtained from the Bloomberg database is from 1992M7, so the estimated series of the shadow rate is from 1992M7. During non-ZLB period, the shadow rate can be replaced by the call rate that has positive values.

9 The elasticity of substitution $\varepsilon$ is calibrated to be 6 which means an average 20% markup charged by intermediate-good firms in steady state.
The Figure 8 and Figure 9 show the historical decompositions of inflation and shadow rate. The most of fluctuations of inflation rate come from the supply shock and shadow rate monetary policy shock. Also, shadow rate monetary policy shock has positive contribution to positive inflation since 2014. We can conclude from the historical decompositions of output gap and inflation, the enlargement of balance sheet of BoJ does have its effect to improve the output gap and deflation. The fluctuations of shadow rate are due to all three exogenous shocks because we still assume the shadow rate follows a Taylor-type monetary policy rule that responses to both output gap and inflation.

Figure 8: historical decomposition of inflation

Figure 9: historical decomposition of shadow rate
4.2 Impulse response function

Bayesian impulse response functions (IRFs) are calculated based on the posterior distribution of structural parameters and exogenous shock standard deviations. The Figure 10 shows a positive demand shock can improve the output gap. The Figure 11 gives the supply shock (cost-push shock) can trigger the increasing of inflation. These results are same to those in the standard NK-DSGE models although here we used the shadow rate to estimate the model.

Figure 10: IRF of demand shock $\epsilon^x_t$

Figure 11: IRF of supply shock $\epsilon^p_t$

The Figure 12 shows that a positive shadow rate monetary policy shock can deteriorate the output gap. Note that even the shadow rate can take negative values, the change of shadow rate, a rise or a reduction, still has the same effect as the positive policy rate has in the non-ZLB environment. The shadow rate NK-DSGE model shows consistent IRFs no matter the policy rate is restricted by the ZLB or not.
Figure 12: IRF of shadow rate monetary policy shock $\epsilon^s_t$

5 Conclusion

In this paper, we estimated the shadow rate of Japan economy from a shadow rate term structure model. We adopt the shadow rate as a measure of monetary policy of BoJ because since 1995, the general policy rate of BoJ, call rate, has kept been near zero and already lost its function as a operating target for the conduct of monetary policy. The concept of shadow rate is not new and for a long time, it is not widely used in macroeconomics. Since the ZLB has become a common issue for monetary authorities in the advanced economies, using the shadow rate to observe and analyze the monetary policy has been applied in many empirical works. We also confirmed that the shadow rate does have credible traceability of the BoJ’s policy in the ZLB environment, providing us a new perspective to check the unconventional monetary policy of BoJ.

Most of existing literatures use the shadow rate in reduced-form time series econometric models such as FAVAR or TVP-VAR to find the empirical evidences of unconventional monetary policy, but these models are not structural. The introduction of the shadow rate into the DSGE framework is a new attempt. Using the shadow rate does relieve us from the technical difficulties incurred by the ZLB, but whether this method is robust or not is still unclear. As far as we know, there doesn’t exist other similar works that use the shadow rate in the estimation of DSGE model. However, the historical decomposition of output gap and inflation advocates the effectiveness of the monetary easing conducted by BoJ. The transmission channels of the monetary easing policy which are assumed in this paper are based on the empirical findings and these empirical evidences have been mapped into the model. Than we find that the impulse response functions of shadow rate NK-DSGE model are similar to the benchmark NK-DSGE model. The mechanism of the general policy rate and shadow rate is same as the standard NK-DSGE framework.

For further research, we want to introduce the shadow rate to a Smets and Wouters (2003, 2007) type middle-scale DSGE model, which is the prototype of the DSGE models used in major central banks. Also we shouldn’t forget that the shadow rate only exists as a economic concept and the central bank can’t
control the shadow rate as a operating target, but we can use the information from it as a guidance for monetary policy operation.
References


[19] Leo Krippner. Modifying gaussian term structure models when interest rates are near the zero lower bound. 2012.


[22] Leo Krippner. A comment on wu and xia (2015), and the case for two-factor shadow short rates. 2015.


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Appendix

Appendix A: Krippner shadow rate term structure model

We present a general GA TS model with two factors (state variables), level component $L_t$ and slope component $S_t$ with the same factor loading in the style of Arbitrage-free Nelson and Siegel (1987) model which is widely used for interest rate modeling, hereafter abbreviated to ANSM(2). Then we extend ANSM(2) to adapt to the ZLB environment by incorporating explicit zero lower bound restriction for short-term interest rate in a mathematically consistent fashion.

Short-term interest rate $r_t$ can be represented as a linear combination of unobservable state variables, $L_t$ and $S_t$.

$$ r_t = c^\top x_t = [1 \ 1] \begin{bmatrix} L_t \\ S_t \end{bmatrix} = L_t + S_t $$

$c^\top = [1 \ 1]$ is the factor loading vector. The state variables vector $x_t^\top = [L_t \ S_t]$ under the objective physical $\mathbb{P}$ measure evolves as a correlated vector Ornstein-Uhlenbeck stochastic process

$$ dx_t = k^\mathbb{P} (\bar{q}^\mathbb{P} x_t) dt + \sigma dW_t^\mathbb{P} $$

where $dW_t^\mathbb{P}$ is a $2 \times 1$ vector of independent Wiener increments with each component of $dW_t^\mathbb{P}$ follows $\mathcal{N}(0, 1)\sqrt{dt}$. $\mathcal{N}(0, 1)$ is the standard normal distribution. $\theta^\mathbb{P}$ is a $2 \times 1$ constant vector which represents the mean level of $x_t$ in long-run. $k^\mathbb{P}$ is a $2 \times 2$ constant parameter matrix that controls the deterministic mean reversion of $x_t$ to its long-run mean level $\bar{q}^\mathbb{P}$. $\sigma$ is a $2 \times 2$ constant variance-covariance matrix $\begin{bmatrix} \sigma_1^2 & \rho_{12} \sigma_1 \sigma_2 \\ \rho_{12} \sigma_1 \sigma_2 & \sigma_2^2 \end{bmatrix}$ of innovations to $x_t$. Note that $\sigma x_t^\top = \begin{bmatrix} \sigma_1 \\ \rho_{12} \sigma_1 \sigma_2 \sigma_2^2 \end{bmatrix}$ which means that the two disturbance terms in vector stochastic process $x_t$ have covariance $\rho_{12} \sigma_1 \sigma_2$. The dynamics of $x_t$ under objective $\mathbb{P}$ measure can be solved using Ito’s lemma.

$$ x_{t+\tau} = \theta^\mathbb{P} + e^{-k^\mathbb{P} \tau} (x_t - \theta^\mathbb{P}) + \int_t^{t+\tau} e^{-k^\mathbb{P} (\tau-u)} \sigma dW_u^\mathbb{P} $$

In general GATSMs, the market prices of risk $\Pi_t$ are typically specified as a linear function of the state variables $x_t$, which allows the market prices of risk to vary over time,

$$ \Pi_t = \sigma^{-1} (\gamma + \Gamma x_t) $$

where $\gamma$ is the constant component of the market prices of risks and $\Gamma$ determines the variations in market prices of risks with respect to the state variables $x_t$. Since the bonds or securities in financial markets are priced under the risk-adjusted $\mathbb{Q}$ measure, the process of state variables $x_t$ must be adjusted to
represent the observed term structure of interest rates by modification of parameters. Under the risk-adjusted Q measure, $x_t$ is analogous to its counterpart under P measure,

$$dx_t = \kappa^Q (\theta^Q - x_t) \, dt + \sigma^Q \, dW^Q_t$$

where $\kappa^Q = \kappa^P + \Gamma$, $\theta^Q = (\kappa^Q)^{-1} (\kappa^P \theta^P - \gamma)$ and $dW^Q_t = dW^P_t + \Pi_t dt$. Solving $x_t$ under the risk-adjusted Q measure leads to the expression of $x_{t+\tau}$.

$$x_{t+\tau} = \theta^Q + e^{-\kappa^Q \tau} (x_t - \theta^Q) + \int_t^{t+\tau} e^{-\kappa^Q (t-u) \sigma} dW^Q_u$$

Then the expectation of $x_{t+\tau}$ is $E_t^Q[x_{t+\tau}|x_t] = \theta^Q + e^{-\kappa^Q \tau} (x_t - \theta^Q)$. With ANSM(2) specification, $\kappa^Q = \begin{bmatrix} 0 & 0 \\ 0 & \varphi \end{bmatrix}$ and $\theta^Q = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The expectation of short interest rate $r_t$ at time $t + \tau$ is

$$E_t^Q[r_{t+\tau}|x_t] = c^T E_t^Q[x_{t+\tau}|x_t] = \begin{bmatrix} 1 & 1 \end{bmatrix} \exp \left( - \begin{bmatrix} 0 & 0 \\ 0 & \varphi \end{bmatrix} \right) \begin{bmatrix} L_t \\ S_t \end{bmatrix} = L_t + e^{-\varphi \tau} S_t$$

The variance of short interest rate is calculated as follows.

$$\text{VAR}^Q_t [r_{t+\tau}|x_t] = \omega^2 = \int_0^T \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \left[ \begin{array}{c} e^{-\kappa^Q u} \sigma^Q \, e^{-\kappa^Q u} \sigma^Q \end{array} \right] \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] du$$

$$= \sigma_1^2 + \sigma_2^2 - \frac{1 - e^{-2\varphi \tau}}{2\varphi} + 2\rho_{12}\sigma_1\sigma_2 \frac{1 - e^{-\varphi \tau}}{\varphi}$$

Volatility effect captures the influence that volatility in the short interest rate has on expected returns due to Jensen’s Inequality which can be calculated using the Heath, Jarrow and Morton (1992) double integral.

$$\nu_t = \int_0^T \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \left[ \begin{array}{c} e^{-\kappa^Q (t-s) \sigma} \right] \left[ \begin{array}{c} \sigma^Q \int_s^T e^{-\kappa^Q (u-s) \sigma} du \\ e^{-\kappa^Q (t-s) \sigma} \int_s^T e^{-\kappa^Q (u-s) \sigma} du \end{array} \right] ds$$

$$= \int_0^T \left\{ \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \left[ \begin{array}{c} e^{-\varphi (t-s)} \right] \left[ \begin{array}{c} \sigma_1^2 \\ \rho_{12}\sigma_1\sigma_2 \end{array} \right] \left[ \begin{array}{c} e^{-\varphi (t-s)} \sigma_2^2 \right] \right] \\ \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \left[ \begin{array}{c} e^{-\varphi (t-s)} \sigma_2^2 \right] \right] \right\} ds$$

$$= \frac{\sigma_1^2 \tau^2}{2} + \rho_{12}\sigma_1\sigma_2 \frac{1 - e^{-\varphi \tau}}{\varphi} + \frac{\sigma_2^2}{2\varphi^2} \left( 1 - 2e^{-\varphi \tau} + e^{-2\varphi \tau} \right)$$

Forward interest rate is given by the expectation of short interest rate minus volatility effect.

$$f_{t,\tau} = E_t^Q[r_{t+\tau}|x_t] - \nu_t = L_t + e^{-\varphi \tau} S_t - \nu_t$$

Interest rate $R_{t,\tau}$ at time $t$ with maturity $\tau$ can be obtained using the standard
term structure relation in continuous time.

\[ R_{t, \tau} = \frac{1}{\tau} \int_0^\tau E_t^Q [r_{t+\tau} | x_t] \, du - \frac{1}{\tau} \int_0^\tau V_t \, d\tau = \frac{1}{\tau} \int_0^\tau \left[ 1 - e^{-\rho \tau} \right] \left[ L_t \right] \, du \]

\[ \quad - \frac{1}{\tau} \int_0^\tau \left[ \frac{\sigma_1^2 \tau^2}{2} + \frac{\rho_1 \sigma_1 \sigma_2 \tau^2}{\phi} \left( 1 - e^{-\phi \tau} \right) + \frac{\sigma_2^2}{2 \phi^2} \left( 1 - 2e^{-\phi \tau} + e^{-2\phi \tau} \right) \right] \, d\tau \]

\[ = a_\tau + b_\tau^\top x_t \]

\[ a_\tau = -\frac{\sigma_1^2 \tau^2}{6} - \frac{\sigma_2^2}{2 \phi^2} \left[ 1 - \frac{1}{2 \phi^2} - \frac{3}{2 \phi^2} \right] \]

\[ b_\tau = \left[ \frac{1}{\phi^2} \left( 1 - e^{-\phi \tau} \right) \right] \]

From the previous analysis, we can find that short interest rate can be written as a linear function of state variables like \( r_1 = e^\top x_t \) and the interest rates with other maturities \( \tau \) can also be written as a linear function of state variables \( x_t \), maturity \( \tau \) and parameters like \( R_{t, \tau} = a_\tau + b_\tau^\top x_t \).

We then extend general GATSM by an intuitive modification to adapt GATSM to the ZLB environment. In Krippner’s framework, imposing ZLB restriction can be represented by a \textbf{max} operator.

\[ \underline{r}_t = \max \{0, r_t\} = r_t + \max \{-r_t, 0\} \]

Here \( \underline{r}_t \) means the actual short interest rate and the \( r_t \) is the shadow short interest rate. In general situations, \( r_t \geq 0 \), the economic agent invests at the instantaneous interest rate \( r_t \) and \( \underline{r}_t = r_t \). But in the ZLB environment, \( r_t < 0 \), the economic agent will choose to hold physical currency and obtain zero return actually with \( \underline{r}_t = 0 \). The short interest rate under forward \( t + \tau \) risk-adjusted \( Q \) measure follows the normal distribution

\[ r_{t+\tau} | x_t \sim N \left( f_{t+\tau}, \omega_\tau^2 \right) \]

where \( f_{t+\tau} \) and \( \omega_\tau^2 \) represent the forward interest rate and its variance, which means that

\[ f_{t+\tau} = E_t^Q (r_{t+\tau} | x_t) \]

where \( E_t^Q \) represents the expectation under forward \( t + \tau \) risk-adjusted \( Q \) measure. The conditional variance of \( r_{t+\tau} | x_t \) is time-invariant.

\[ \text{VAR}_t^Q [r_{t+\tau} | x_t] = \text{VAR}_t^Q [r_{t+\tau} | x_t] = \omega_\tau^2 \]

Given the distribution of \( r_{t+\tau} | x_t \), its probability density function PDF\((r_{t+\tau} | x_t)\) is given as follows.

\[ \text{PDF}(r_{t+\tau} | x_t) = \frac{1}{\omega_\tau \sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{r_{t+\tau} - f_{t+\tau}}{\omega_\tau} \right)^2 \right] \]

\[ ^{10}\text{In this section, the notation with underbar } \_ \text{ means the restriction of the ZLB.} \]
In practice, we allow a non-zero lower bound \( r_L \) to represent the ZLB which may be a very small number, approximately equal to zero but not zero actually. In our two-factor model, this can be written as follows.

\[
\Xi = \max \{r_L, r_t\} = \max \{r_L, L_t + S_t\}
\]

\[
\Xi_t + T = \max \{r_L, r_{t+T}\} = r_{t+T} + \max \{r_L - r_{t+T}, 0\}
\]

The forward interest rate as the expectation of \( \Xi_t + T \) can be modified as

\[
f_{t,T} = E^{Q}_{t+T} [\Xi_{t+T}|x_t] = E^{Q}_{t+T} [r_{t+T}|x_t] + E^{Q}_{t+T} [\max \{r_L - r_{t+T}, 0\}|x_t] = f_{t,T} + z_{t,T}
\]

and if we set \( r_L = 0 \),

\[
f_{t,T} = E^{Q}_{t+T} [\Xi_{t+T}|x_t] = E^{Q}_{t+T} [r_{t+T}|x_t] + E^{Q}_{t+T} [\max \{-r_{t+T}, 0\}|x_t] = f_{t,T} + z_{t,T}
\]

where \( E^{Q}_{t+T} [r_{t+T}|x_t] = f_{t,T} \) has been obtained in previous analysis. We now evaluate another part \( z_{t,T} \) in forward interest rate \( f_{t,T} \).

\[
z_{t,T} = E^{Q}_{t+T} [\max \{r_L - r_{t+T}, 0\}|x_t] = \int_{-\infty}^{\infty} \max \{r_L - r_{t+T}, 0\} \cdot PDF(r_{t+T}|x_t) dr_{t+T}
\]

\[
= \int_{-\infty}^{f_{t,T}} (r_L - r_{t+T}) \cdot PDF(r_{t+T}|x_t) dr_{t+T} + \int_{f_{t,T}}^{\infty} 0 \cdot PDF(r_{t+T}|x_t) dr_{t+T}
\]

\[
= \int_{-\infty}^{f_{t,T}} (r_L - f_{t,T} + f_{t,T} - r_{t+T}) \frac{1}{\omega_T \sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{r_{t+T} - f_{t,T}}{\omega_T} \right)^2 \right] dr_{t+T}
\]

\[
= \int_{-\infty}^{f_{t,T}} (r_L - f_{t,T}) \frac{1}{\omega_T \sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{r_{t+T} - f_{t,T}}{\omega_T} \right)^2 \right] dr_{t+T}
\]

Define \( \frac{r_{t+T} - f_{t,T}}{\omega_T} = t \) and we have \( r_{t+T} = t \omega_T + f_{t,T} \). Change the range of integral to adapt the new variable \( t \) and rewrite \( z_{t,T} \). The first part of RHS of \( z_{t,T} \) is

\[
(r_L - f_{t,T}) \int_{-\infty}^{\frac{f_{t,T} - r_L}{\omega_T}} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} t^2 \right) dt = (r_L - f_{t,T}) \Phi \left( \frac{r_L - f_{t,T}}{\omega_T} \right)
\]

and the second part is

\[
\int_{-\infty}^{\frac{f_{t,T} - r_L}{\omega_T}} (-t \omega_T) \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} t^2 \right) dt = -\omega_T \int_{-\infty}^{\frac{f_{t,T} - r_L}{\omega_T}} t \exp \left( -\frac{1}{2} t^2 \right) dt
\]
where $\Phi \left( \frac{r_L - f_{L,T}}{\omega_T} \right)$ is the cumulative density function of standard normal distribution. Using the property of standard normal distribution, we can rewrite it as follows.

$$
\Phi \left( \frac{r_L - f_{L,T}}{\omega_T} \right) = 1 - \Phi \left( -\frac{r_L - f_{L,T}}{\omega_T} \right) = 1 - \Phi \left( \frac{f_{L,T} - r_L}{\omega_T} \right)
$$

Then calculate the second part of RHS of $z_{t,T}$.

$$
-w_t \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{r_L-f_{L,T}} \exp \left( -\frac{1}{2} t^2 \right) dt = \omega_T \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{r_L - f_{L,T}}{\omega_T} \right)^2 \right]
$$

Substituting these results leads to the $z_{t,T}$.

$$
z_{t,T} = (r_L - f_{L,T}) \left[ 1 - \Phi \left( \frac{f_{L,T} - r_L}{\omega_T} \right) \right] + \omega_T \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{r_L - f_{L,T}}{\omega_T} \right)^2 \right]
$$

Substituting $z_{t,T}$ into $f_{L,T} = f_{L,T} + z_{t,T}$ leads to the expression of $f_{L,T}$.

$$
f_{L,T} = f_{L,T} + z_{t,T} = r_L + (f_{L,T} - r_L) \Phi \left( \frac{f_{L,T} - r_L}{\omega_T} \right) + \omega_T \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{f_{L,T} - r_L}{\omega_T} \right)^2 \right]
$$

Given all results derived above, the interest rate $R_{t,T}$ in SRTSM still has the analogous expression of the counterpart in general GATSM framework.

$$
R_{t,T} = \int_0^T f_{L,u} du
$$

where $f_{L,u}$ has been derived in previous analysis.

GATS model is generally estimated in state-space form with Kalman filter and maximum likelihood method. Compared to general GATS model, the SRTS model proposed by Krippner and its specified implementation have nonlinear functions of state variables $x_t$ via the CDF or PDF of normal distribution. Iterated Extended Kalman Filter (IEKF) can handle the nonlinearity. For technical details of IEKF, please refer to Krippner (2015, pp.117-126) which provides the instruction of IEKF in the estimation of shadow rate term structure model. Wu and Xia (2016) also provides a short introduction of the estimation of shadow rate term structure model using IEKF.

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11 This expression does not have analytic solution in closed-form and need to be evaluated numerically.
Appendix B: Figures and Tables

prior distribution and posterior distribution

multivariate convergence diagnostic
### Table 1: Summary Statistics of Yield Curve Data (M=Month and Y=Year)

<table>
<thead>
<tr>
<th>Maturity τ</th>
<th>3M</th>
<th>6M</th>
<th>1Y</th>
<th>2Y</th>
<th>3Y</th>
<th>4Y</th>
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<tbody>
<tr>
<td>Mean</td>
<td>0.465</td>
<td>0.468</td>
<td>0.513</td>
<td>0.626</td>
<td>0.766</td>
<td>0.937</td>
</tr>
<tr>
<td>Median</td>
<td>0.123</td>
<td>0.129</td>
<td>0.149</td>
<td>0.249</td>
<td>0.404</td>
<td>0.571</td>
</tr>
<tr>
<td>Minimum</td>
<td>-0.421</td>
<td>-0.434</td>
<td>-0.371</td>
<td>-0.360</td>
<td>-0.360</td>
<td>-0.361</td>
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<tr>
<td>Std. Dev.</td>
<td>0.849</td>
<td>0.833</td>
<td>0.850</td>
<td>0.900</td>
<td>0.968</td>
<td>1.060</td>
</tr>
<tr>
<td>Observations</td>
<td>6360</td>
<td>6360</td>
<td>6360</td>
<td>6360</td>
<td>6360</td>
<td>6360</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Maturity τ</th>
<th>5Y</th>
<th>7Y</th>
<th>10Y</th>
<th>15Y</th>
<th>20Y</th>
<th>30Y</th>
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</thead>
<tbody>
<tr>
<td>Mean</td>
<td>1.104</td>
<td>1.402</td>
<td>1.805</td>
<td>2.135</td>
<td>2.532</td>
<td>2.695</td>
</tr>
<tr>
<td>Median</td>
<td>0.735</td>
<td>1.031</td>
<td>1.487</td>
<td>1.833</td>
<td>2.197</td>
<td>2.478</td>
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<tr>
<td>Maximum</td>
<td>5.252</td>
<td>5.713</td>
<td>5.640</td>
<td>6.159</td>
<td>6.435</td>
<td>6.239</td>
</tr>
<tr>
<td>Minimum</td>
<td>-0.370</td>
<td>-0.393</td>
<td>-0.282</td>
<td>-0.125</td>
<td>0.025</td>
<td>0.053</td>
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<tr>
<td>Std. Dev.</td>
<td>1.129</td>
<td>1.219</td>
<td>1.256</td>
<td>1.291</td>
<td>1.264</td>
<td>1.162</td>
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<tr>
<td>Observations</td>
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<td>6360</td>
<td>6360</td>
<td>6360</td>
<td>6360</td>
<td>6360</td>
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</table>

### Table 2: Prior and Posterior Distribution of Structural Parameters

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Mean</th>
<th>St.Dev.</th>
<th>Prior</th>
<th>Mean</th>
<th>Mode</th>
<th>St.Dev.</th>
<th>90% HPD Interval</th>
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</thead>
<tbody>
<tr>
<td>σ</td>
<td>1.000</td>
<td>0.375</td>
<td>Γ</td>
<td>3.5126</td>
<td>3.4864</td>
<td>0.5779</td>
<td>[2.6515, 4.3759]</td>
</tr>
<tr>
<td>η</td>
<td>2.000</td>
<td>0.750</td>
<td>Γ</td>
<td>1.9945</td>
<td>1.7187</td>
<td>0.7397</td>
<td>[0.7778, 3.1092]</td>
</tr>
<tr>
<td>θ</td>
<td>0.375</td>
<td>0.100</td>
<td>β</td>
<td>0.3758</td>
<td>0.3628</td>
<td>0.1002</td>
<td>[0.2099, 0.5384]</td>
</tr>
<tr>
<td>φ_π</td>
<td>1.500</td>
<td>0.100</td>
<td>Γ</td>
<td>1.5385</td>
<td>1.5250</td>
<td>0.1060</td>
<td>[1.3613, 1.7091]</td>
</tr>
<tr>
<td>φ_x</td>
<td>0.125</td>
<td>0.050</td>
<td>Γ</td>
<td>0.2955</td>
<td>0.2866</td>
<td>0.0915</td>
<td>[0.1482, 0.4468]</td>
</tr>
<tr>
<td>φ_s</td>
<td>0.800</td>
<td>0.100</td>
<td>β</td>
<td>0.7989</td>
<td>0.8000</td>
<td>0.0211</td>
<td>[0.7675, 0.8322]</td>
</tr>
<tr>
<td>ρ_π</td>
<td>0.800</td>
<td>0.100</td>
<td>β</td>
<td>0.8432</td>
<td>0.8452</td>
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<tr>
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<td>0.100</td>
<td>β</td>
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<td>[0.7940, 0.8401]</td>
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<tr>
<td>ρ_s</td>
<td>0.800</td>
<td>0.100</td>
<td>β</td>
<td>0.8017</td>
<td>0.8462</td>
<td>0.1003</td>
<td>[0.6513, 0.9609]</td>
</tr>
<tr>
<td>σ_π</td>
<td>0.500</td>
<td>0.500</td>
<td>Inv-Γ</td>
<td>0.1604</td>
<td>0.1036</td>
<td>0.0122</td>
<td>[0.0863, 0.1260]</td>
</tr>
<tr>
<td>σ_x</td>
<td>0.500</td>
<td>0.500</td>
<td>Inv-Γ</td>
<td>0.2249</td>
<td>0.2209</td>
<td>0.0257</td>
<td>[0.1847, 0.2671]</td>
</tr>
<tr>
<td>σ_s</td>
<td>0.500</td>
<td>0.500</td>
<td>Inv-Γ</td>
<td>0.2040</td>
<td>0.1996</td>
<td>0.0176</td>
<td>[0.1754, 0.2313]</td>
</tr>
</tbody>
</table>

Table 2: Prior and Posterior Distribution of Structural Parameters