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Abstract

This study proposes an approximation method for pricing barrier options and lookback options with continuous monitoring. We employ a class of Lévy processes as the driving factor of an underlying stock price and consider a mimicking process for approximation. Randomizing the Black-Scholes formula associated with the mimicking process leads to an approximation formula. Two features of this method are that it is straightforward and easily implementable. Nevertheless, the approximation prices generated by the method are quite accurate and the calculation speed is remarkably fast, regardless of the type of option and time to maturity.

Keywords: barrier options; lookback options; continuous monitoring; normal inverse Gaussian process; variance gamma process; Black-Scholes formula; randomized maturity

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1 Introduction

In derivatives markets, pricing exotic options based on calibration to liquid plain vanilla option prices is a de facto standard. For that reason, a large number of studies have addressed the development of non-Gaussian pricing models, almost all of which are regarded as a modified version of the Black-Scholes (BS) model (Black and Scholes [3]), to capture the implied volatility smiles observed in option markets. Furthermore, various kinds of methods for pricing exotic options based on such pricing models have been actively explored. Even now, practitioners engaged in derivatives business desire more effective pricing methods for some frequently traded exotic options, including barrier options and lookback options.

The objective of this study is to propose an approximation method for pricing path-dependent exotic options. In contrast to recent advance in highly complicated option pricing methods, we aim to develop a *simple* and *practical* pricing method that might not be necessarily novel as an aspect of mathematical finance. We present the desired attributes for our method as follows:

- **Volatility smiles:** According to the de facto standard in derivatives markets, models describing the dynamics of an underlying asset price have to be able to capture implied volatility smiles. Under such pricing models, we should develop a pricing method for exotic options.
- **Derivative products:** The derivative products applicable to the pricing method should be frequently traded exotic options. Barrier options and lookback options with daily or more frequent monitoring are probably the most accepted path-dependent derivatives among practitioners.
- **Easy understanding:** It would be desirable that the principle of a pricing method is easily understandable for quantitative analysts in financial institutions. To achieve this, we should circumvent highly complicated mathematical discussion.
- **Simple implementation:** It would be preferable for someone to be able to apply the pricing method easily and to be provided with stable pricing results at any time. Therefore, we should avoid highly complicated and unpredictable computations.
- **Fast computation:** Calculation time of option pricing is crucial in the derivatives business. It would be desirable that the calculation time is within a fraction of a second, regardless of the type of option and time to maturity. This level of computational speed is our target.
- **Accurate pricing:** Exotic options are less liquid and have wider offer-bid spreads than plain vanilla options in general. We assume that the offer-bid spreads of exotic options are larger than or equal to 0.3% of the underlying asset price. Accordingly, it would be acceptable for the absolute errors of the approximation prices to be within 0.15% of the underlying price. This accuracy level is our target.

For this purpose, we explore an approximation method for pricing barrier options and lookback options with *continuous* monitoring as a proxy for more than daily monitoring as follows: We begin by employing a class of Lévy processes called *subordinated Brownian motions* as a driving factor of an underlying stock price. A stock price process modeled as a subordinated Brownian motion, which can be regarded as a time-changed BS model subordinated by a non-decreasing Lévy process, has the ability to generate negatively skewed and fat-tailed return distributions. We then consider a mimicking process, a version of the BS model with a parameterized interest rate, to approximate the true stock price process. Next, we apply the BS formula to evaluate an exotic option written on the mimicking stock. Randomizing the option expiry date and the parameter of the pseudo interest rate and taking the expectation of the randomized BS formula lead to a primary approximation formula for pricing the exotic option written on the true stock. To improve accuracy, we add a bias correction to the primary approximation formula. Finally, numerical examples confirm the effectiveness of our approximation method.

In this paragraph, we state the results of this study in line with the desired method attributes mentioned above. First, in the numerical examples, we employ the variance gamma process (Madan and Seneta [23]) and the normal inverse Gaussian process (Barndorff-Nielsen [1]) with plausible parameter sets as the driving factor of the underlying stock price. Both processes belong to subordinated Brownian motions. The numerical examples illustrate that the pricing models based on these processes exhibit implied volatility smiles. Second, we derive approximation formulas for pricing continuously monitored barrier options as well as continuously monitored lookback options. Each approximation formula is straightforward because each is based on the BS formula. An additional procedure is the integral of the BS formula and bias correction. Only a suitable quadrature rule applied for well-known functions, including the cumulative normal distribution function and the gamma function, is needed for computation. Third, as will be shown by the numerical examples, the calculation time of our method is only a few milliseconds. The computational speed does not depend on the type of option and time to maturity. All the absolute errors of the approximation prices in the numerical examples remain within 0.15% of the underlying stock price. In fact, they are at most 0.11% of the initial stock price, although the magnitude of the approximation errors is uneven and depends on the type of option and the levels of strike price and barrier. In conclusion, our approximation method achieves all the desired attributes.

We acknowledge that there are a number of related studies. A list of the prominent literature addressing path-dependent option pricing under Lévy processes or their subspecies is as follows: Boyarchenko et al. [4], Kudryavtsev and Levendorskiĭ [18], Boyarchenko and Levendorskiĭ [5] [6], and Jeannin and Pistorius [17]. These five use the Wiener-Hopf factorization for pricing continuously monitored barrier options, whereas Kudryavtsev and Levendorskiĭ [19] also apply the Wiener-Hopf factorization to evaluate continuously monitored lookback options. However, applying the Wiener-Hopf factorization requires the double inverse Laplace transform and the inverse Fourier transform. Accordingly, the computational implementation involves some complexity. Feng and Linetsky [13] develops a pricing method for discretely monitored barrier options by using the Hilbert transform combined with the Fourier transform. Zhen and Kwok [29] extend the pricing method of Feng and Linetsky [13] to the time-changed Lévy processes introduced by Carr and Wu [10]. Fusai and Meucci [14] provide semi-analytical solutions for both discretely and continuously monitored geometric average option prices. Umezawa and Yamazaki [26] derive semi-analytical solutions for some exotic option prices with discrete monitoring under time-changed Lévy processes from multivariate characteristic functions. Yamazaki [28] extends the pricing formulas of Umezawa and Yamazaki [26] to Barndorff-Nielsen and Shephard's [2] stochastic volatility model. Fusai and Meucci [14] apply a Fourier transform to derive semi-analytical solutions for geometric average option prices, whereas Yamazaki [27] uses the Gram-Charlier expansion for pricing arithmetic average options under time-changed Lévy processes. Carr and Crosby [8] calibrate exponential Lévy models to barrier options and plain vanilla options simultaneously. Broadie et al. [7] examine the relation between discretely and continuously monitored path-dependent option prices in the BS model, and Dia and Lamberton [12] extend the result of Broadie et al. [7] to jump-diffusion models.

The remainder of this paper is organized as follows: Section 2 describes the model. Section 3 provides approximation formulas for pricing barrier options and lookback options. Section 4 presents numerical examples and Section 5 concludes.

2 Model

Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a probability space carrying a one-dimensional Lévy process X with the associated filtration $(\mathcal{F}_t)_{t\geq 0}$. A stochastic process X on $(\Omega, \mathcal{F}, \mathbb{Q})$ with values in $\mathbf{R}$ such that $X_0 = 0$ is a Lévy process if it has the following properties: (1) X is $(\mathcal{F}_t)_{t\geq 0}$-adapted. (2) The sample paths of X are right continuous with left limits. (3) $X_u - X_t$ is independent of $\mathcal{F}_t$ for $0 \leq t \leq u$. (4) $X_u - X_t$ has the same distribution as X_{u-t} for $0 \leq t \leq u$. We assume frictionless markets and absence of arbitrage opportunities and assume that the risk-neutral measure $\mathbb{Q}$ is given.

This study focuses on a class of Lévy processes called *subordinated Brownian motions.* A Lévy process in this class is defined as

$$X_t = \mu\tau_t + \sigma W(\tau_t), \tag{2.1}$$

for each $t \geq 0$. Here, $\mu \in \mathbf{R}$ and $\sigma > 0$ are some constants, W denotes a one-dimensional standard Brownian motion, and τ denotes a non-decreasing Lévy process independent of W. That is, the process X in Eq.(2.1) is a time-changed Brownian motion equipped with a stochastic time change τ. The time change process τ known as the *subordinator* in the field of stochastic calculus is normalized so that $\mathbb{E}[\tau_t] = t$ for any $t > 0$ without loss of generality. Notice that drifted Brownian motions are a special case of the class, and a subordinated Brownian motion deviates from the underlying Brownian motion due to the stochastic time change. As will be shown in the next section, this property is used in our approximation method.

Using the independence of τ from W, the characteristic function of X_t in Eq.(2.1) can be written as

$$\phi_{X_t}(u) := \mathbb{E}\left[e^{iuX_t}\right] = \exp\left\{t\Psi_X(u)\right\}, \quad u \in \mathbf{R}, \tag{2.2}$$

with

$$\Psi_X(u) = \mathcal{L}\left(i\mu u - \frac{1}{2}\sigma^2 u^2\right), \tag{2.3}$$

where $\mathbb{E}[\,\cdot\,]$ is the risk-neutral expectation operator, $\Psi_X(u) := \ln \mathbb{E}\left[e^{iuX_1}\right]$ is the characteristic exponent of X, and $\mathcal{L}(u) := \ln \mathbb{E}\left[e^{u\tau_1}\right]$ is the Laplace exponent of τ. Thus, these two exponents define the stochastic properties of X and τ over the unit time interval. Taking an arbitrary time change process τ leads to a subordinated Brownian motion X, and its probabilistic features are derived from Eqs.(2.2) and (2.3). Additionally, we assume that the probability density function of τ_t denoted by $f_{\tau_t}(u)$ is known even if the probability density function of X_t is unknown or does not exist. The following examples provide two Lévy processes used in the existing literature belonging to the class of subordinated Brownian motions, employed in further numerical examples.

Example 1 (Normal Inverse Gaussian Process) Taking the inverse Gaussian process as the time change process in Eq.(2.1) leads to the normal inverse Gaussian (NIG) process proposed by Barndorff-Nielsen [1]. Increments of the inverse Gaussian process have the inverse Gaussian distribution with only one parameter, $\kappa > 0$, because of the normalization. Thus, the Laplace exponent and the probability density function of the inverse Gaussian process are expressed as

$$\mathcal{L}(u) = -\sqrt{\frac{2}{\kappa}}\left(\sqrt{\frac{1}{2\kappa} - u} - \sqrt{\frac{1}{2\kappa}}\right), \tag{2.4}$$

and

$$f_{\tau_t}(u) = \frac{t}{u^{\frac{3}{2}}\sqrt{2\kappa\pi}} \exp\left\{\frac{1}{2\kappa}\left(2t - u - \frac{t^2}{u}\right)\right\}, \tag{2.5}$$

respectively. Eq.(2.4) yields $\mathbf{Var}[\tau_1] = \kappa$. Applying Eq.(2.4) to Eq.(2.3) leads to the characteristic exponent of the NIG process

$$\Psi_X(u) = \frac{1}{\kappa} - \frac{1}{\kappa}\sqrt{1 - 2i\mu\kappa u + \sigma^2\kappa u^2}. \tag{2.6}$$

The NIG process is a Lévy process without a diffusion component and has infinite variation at any time, with high activity of small jumps. Parameter κ generates the excess kurtosis of the distribution of X_t and parameter μ controls its skewness, whereas parameter σ determines the degree of its standard deviation. The NIG process converges to a Brownian motion as κ approaches zero.

Example 2 (Variance Gamma Process) Taking the gamma process as the time change process in Eq.(2.1) leads to the variance gamma (VG) process introduced by Madan and Seneta [23]. Increments of the gamma process have the gamma distribution with only one parameter, $\kappa > 0$, for the same reason as that in the NIG process. Thus, the Laplace exponent and the probability density function of the gamma process can be written as

$$\mathcal{L}(u) = -\frac{1}{\kappa}\ln(1 - \kappa u), \tag{2.7}$$

and

$$f_{\tau_t}(u) = \frac{1}{\kappa\Gamma(\frac{t}{\kappa})}\left(\frac{u}{\kappa}\right)^{\frac{t}{\kappa}-1} e^{-\frac{u}{\kappa}}, \tag{2.8}$$

respectively, where $\Gamma(x)$ is the gamma function. Eq.(2.7) also yields $\mathbf{Var}[\tau_1] = \kappa$. Applying Eq.(2.7) to Eq.(2.3), we obtain the characteristic exponent of the VG process

$$\Psi_X(u) = -\frac{1}{\kappa}\ln\left(1 - i\mu\kappa u + \frac{1}{2}\sigma^2\kappa u^2\right). \tag{2.9}$$

Although the VG process is also a pure jump Lévy process, it has finite variation at any time, with infinite, but relatively low activity of small jumps unlike the NIG process. The parameters σ, μ, and κ of the VG process play a similar role to those of the NIG process.

The NIG process, as well as the VG process, can capture the fat tails and negative skewness of the risk-neutral return distributions observed in option markets, whereas it produces a parsimonious pricing model with fewer parameters than other non-Gaussian models such as Heston's stochastic volatility model (Heston [15]) and Merton's jump-diffusion model (Merton [24]). For the detailed features of the latter two important processes, see Cont and Tankov [11] and references therein.

Suppose that the risk-neutral price process of an underlying stock is given by

$$\ln\frac{S_t}{S_0} := (r - \varphi)t + X_t, \quad \text{for every} \quad t \geq 0, \tag{2.10}$$

where r is a positive constant denoting a risk-free interest rate and $S_0 > 0$ is the initial stock price. Here, $\varphi := \Psi_X(-i)$ is a constant such that the discounted price process is a $\mathbb{Q}$-martingale. According to Eq.(2.1), the price process in Eq.(2.10) can be rewritten as

$$\ln\frac{S_t}{S_0} := (r - \varphi)t + \mu\tau_t + \sigma W(\tau_t). \tag{2.11}$$

Let $H(S,T)$ denote the payoff function of a continuously monitored path-dependent option maturing at time T and $V(T)$ denote the unconditional expected value of the payoff function, that is,

$$V(T) := \mathbb{E}\left[H(S,T)\right]. \tag{2.12}$$

The objective of this study is to develop an analytical approximation method for Eq.(2.12) associated with payoff functions of barrier options and lookback options.

3 Approximation Method

The basic idea of our approximation method is straightforward and based on the BS formulas for exotic options. First, we consider a pseudo stock price process $\tilde{S}^x$ governed by the BS model

$$\ln \frac{\tilde{S}^x_t}{\tilde{S}^x_0} := \left(R(x) - \frac{\sigma^2}{2}\right) t + \sigma W(t), \quad \text{for every} \quad t \geq 0, \tag{3.1}$$

where the initial price is set to be $\tilde{S}^x_0 = S_0$ and $R(x)$ denotes the pseudo risk-free interest rate parameterized by $x > 0$, whose explicit representation will be shown below. Let $V_{BS}(R(x), y)$ denote the unconditional expected value of payoff $H(\tilde{S}^x, y)$ under BS model (3.1) with maturity y, defined as

$$V_{BS}(R(x), y) := \mathbb{E}\left[H(\tilde{S}^x, y)\right]. \tag{3.2}$$

Note that the closed-form solutions to Eq.(3.2) for some exotic derivatives, including barrier options and lookback options, are the well-known BS formulas. Replacing the maturity T and parameter x in Eq.(3.2) with the random time τ_T and taking the expectation of Eq.(3.2) with respect to the random time lead to the following equation:

$$\begin{aligned} U(T) &:= \mathbb{E}\left[\mathbb{E}\left[H(\tilde{S}^{\tau_T}, \tau_T)\middle|\tau_T\right]\right] \\ &= \mathbb{E}\left[V_{BS}(R(\tau_T), \tau_T)\right] \\ &= \int_0^\infty V_{BS}(R(u), u) f_{\tau_T}(u) du. \end{aligned} \tag{3.3}$$

If a closed-form solution to Eq.(3.2) exists and the probability density function of τ_T is given, the value of $U(T)$, which is called the *primary approximation* hereafter, can be easily obtained by a suitable quadrature rule.

Next, we define the pseudo interest rate as

$$R(x) := (r - \varphi)\frac{T}{x} + \mu + \frac{\sigma^2}{2}. \tag{3.4}$$

Then, substituting Eq.(3.4) with $x = \tau_T$ into Eq.(3.1), we obtain

$$\ln \frac{\tilde{S}^{\tau_T}_t}{\tilde{S}^{\tau_T}_0} = (r - \varphi)\left(\frac{T}{\tau_T}\right) t + \mu t + \sigma W(t). \tag{3.5}$$

Consider a randomized time interval $[0, \tau_T]$ for the pseudo price process described by Eq.(3.5) and the corresponding fixed time interval $[0, T]$ for the true price process of Eq.(2.11). Notice that the terminal price of the pseudo stock equals the terminal price of the true stock, that is, $\tilde{S}^{\tau_T}_{\tau_T} = S_T$. Because $\tilde{S}^{\tau_T}_0 = S_0$, the pseudo interest rate defined in Eq.(3.4) with $x = \tau_T$ induces both ends of any sample paths of the pseudo price process over the time interval $[0, \tau_T]$

to coincide with those of the true price process over the time interval $[0, T]$. Further, recall that $\mathbb{E}[\tau_t] = t$ for every $t \geq 0$. Accordingly, if the variance of the time change process τ is not very large, each sample path of the pseudo price process over $[0, \tau_T]$ is expected to be close to that of the true price process over $[0, T]$. This is why we regard $U(T)$ defined in Eq.(3.3) as the approximation of $V(T)$ of Eq.(2.12). Since a direct application of $U(T)$ might deliver a crude approximation, in the following subsections, we explore *bias correction* to improve the accuracy of the primary approximation $U(T)$. Before doing so, we present special cases as examples. Notice that when a payoff function $H(S, T)$ depends only on the terminal stock price S_T, approximation formula (3.3) yields the exact value of $V(T)$.

Example 3 (Plain Vanilla Option) Because the payoff function of a European call option with strike price K and maturity T is

$$H(S, T) = (S_T - K)^+, \tag{3.6}$$

the expected payoff is given by

$$V(T) = \mathbb{E}\left[(S_T - K)^+\right] = U(T). \tag{3.7}$$

Madan et al. [22] began with the integral form of Eq.(3.3) to evaluate a European call option under the VG process and obtained a closed-form solution.

Example 4 (Digital Option) The payoff function of a digital call option with strike price K and maturity T is

$$H(S, T) = \mathbb{I}(S_T > K), \tag{3.8}$$

where $\mathbb{I}(A)$ denotes the indicator function that outputs the value 1 if event A is true and 0 otherwise. The expected value of the payoff is given by

$$V(T) = \mathbb{E}\left[\mathbb{I}(S_T > K)\right] = \mathbb{Q}(S_T > K) = U(T). \tag{3.9}$$

Since some effective pricing methods for European options under Lévy process have been proposed (see Carr and Madan [9], Lewis [21], and Lee [20], among others), applying our method to evaluate European options as shown in the above examples seems less attractive. However, such examples can be supplementary to our approximation method. In the following subsections, we focus on barrier options and lookback options.

3.1 Barrier Option

3.1.1 Up-and-Out Call Option

The payoff function of an up-and-out call (UOC) option maturing at time T can be written as

$$H(S, T) = (S_T - K)^+ \mathbb{I}\left(B > \sup_{t \in [0,T]} S_t\right), \tag{3.10}$$

where K is the strike price and B is the level of an upper barrier above the initial stock price S_0. Then, the expected value of the payoff

$$V(T) = \mathbb{E}\left[(S_T - K)^+ \mathbb{I}\left(B > \sup_{t \in [0,T]} S_t\right)\right], \tag{3.11}$$

can be decomposed into the primary approximation term defined in Eq.(3.3) and a residual term as follows:

$$V(T) = U(T) + RT_{\mathrm{uoc}}. \tag{3.12}$$

In this case,

$$U(T) = \mathbb{E}\left[(\tilde{S}^{\tau_T}_{\tau_T} - K)^+ \mathbb{I}\left(B > \sup_{t\in[0,\tau_T]} \tilde{S}^{\tau_T}_t\right)\right], \tag{3.13}$$

and we define

$$RT_{\mathrm{uoc}} := \mathbb{E}\left[(S_T - K)^+ \left(\mathbb{I}\left(B > \sup_{t\in[0,T]} S_t\right) - \mathbb{I}\left(B > \sup_{t\in[0,\tau_T]} \tilde{S}^{\tau_T}_t\right)\right)\right]. \tag{3.14}$$

In what follows, we attempt to approximate the residual term RT_{uoc} heuristically rather than theoretically. For the heuristic approximation, we postulate the following: The difference between the probability of the event that the true price does not reach the barrier until maturity T and that of the event that the pseudo price does not reach the barrier until random maturity τ_T is proportional to the difference between the probability of the event that the true price at time $T/2$ remains below the barrier and that of the event that the pseudo price at time $\tau_{T/2}$ remains below the barrier. In addition, the terminal stock price is independent of the event that the true price at time T remains below the barrier and that the pseudo price at time $\tau_{T/2}$ remains below the barrier.

Thus, we propose the approximation

$$\begin{aligned} RT_{\mathrm{uoc}} &\approx \epsilon\mathbb{E}\left[(S_T - K)^+ \left(\mathbb{I}\left(B > S_{\frac{T}{2}}\right) - \mathbb{I}\left(B > \tilde{S}^{\tau_T}_{\tau_{\frac{T}{2}}}\right)\right)\right] \\ &\approx \epsilon\mathbb{E}\left[(S_T - K)^+\right] \left(\mathbb{Q}\left(B > S_{\frac{T}{2}}\right) - \mathbb{Q}\left(B > \tilde{S}^{\tau_T}_{\tau_{\frac{T}{2}}}\right)\right), \end{aligned} \tag{3.15}$$

where we define the proportionality constant ϵ as

$$\epsilon := \frac{U(T)}{\mathbb{E}\left[(S_T - K)^+\right] \mathbb{Q}\left(B > \tilde{S}^{\tau_T}_{\tau_{\frac{T}{2}}}\right)}. \tag{3.16}$$

Consequently, we obtain the approximation formula

$$V(T) \approx U(T) \frac{\mathbb{Q}\left(B > S_{\frac{T}{2}}\right)}{\mathbb{Q}\left(B > \tilde{S}^{\tau_T}_{\tau_{\frac{T}{2}}}\right)}. \tag{3.17}$$

Approximation formula (3.17) can be interpreted as follows: The ratio of $V(T)$ to $U(T)$ is approximately equal to the ratio of the probability of the event that the true price at time $T/2$ remains below the barrier to that of the event that the pseudo price at time $\tau_{T/2}$ remains below the barrier. When the pseudo stock price is less likely to reach the barrier than the true stock price, the primary approximation $U(T)$ might be an overestimate. On the other hand, the probability ratio in Eq.(3.17) might be less than 1. Accordingly, it is expected to adjust the level of $U(T)$ to the true value of $V(T)$.

Since the increments of the stochastic time change process τ are independent and stationary, the denominator on the right side of approximation formula (3.17) can be expressed as

$$\mathbb{Q}\left(B > \tilde{S}^{\tau_T}_{\tau_{\frac{T}{2}}}\right) = \int_0^\infty \int_0^\infty V^{\mathrm{dp}}_{BS}(R(u_1 + u_2), u_1) f_{\tau_{\frac{T}{2}}}(u_1) du_1 f_{\tau_{\frac{T}{2}}}(u_2) du_2, \tag{3.18}$$

where $V_{BS}^{\mathrm{dp}}(R(x), T)$ denotes the expected payoff of the digital put option with strike price S_0 and maturity T under the BS model (3.1) with the pseudo interest rate $R(x)$ of Eq.(3.4). An application of Example 4 yields the value of the numerator.

3.1.2 Down-and-Out Call Option

The payoff function of a down-and-out call (DOC) option maturing at time T can be written as

$$H(S,T) = (S_T - K)^+ \mathbb{I}\left(B < \inf_{t\in[0,T]} S_t\right), \tag{3.19}$$

where K is the strike price and B is the level of a lower barrier below the initial stock price S_0. Then, the expected value of the payoff

$$V(T) = \mathbb{E}\left[(S_T - K)^+ \mathbb{I}\left(B < \inf_{t\in[0,T]} S_t\right)\right], \tag{3.20}$$

can also be decomposed into the primary approximation term and the residual term as follows:

$$V(T) = U(T) + RT_{\mathrm{doc}}. \tag{3.21}$$

In this case,

$$U(T) = \mathbb{E}\left[(\tilde{S}_{\tau_T}^{\tau_T} - K)^+ \mathbb{I}\left(B < \inf_{t\in[0,\tau_T]} \tilde{S}_t^{\tau_T}\right)\right], \tag{3.22}$$

and we define

$$RT_{\mathrm{doc}} := \mathbb{E}\left[(S_T - K)^+ \left(\mathbb{I}\left(B < \inf_{t\in[0,T]} S_t\right) - \mathbb{I}\left(B < \inf_{t\in[0,\tau_T]} \tilde{S}_t^{\tau_T}\right)\right)\right]. \tag{3.23}$$

Similar to the UOC option, the approximation pricing formula for the DOC option is

$$V(T) \approx U(T) \frac{\mathbb{Q}\left(B < S_{\frac{T}{2}}\right)}{\mathbb{Q}\left(B < \tilde{S}_{\tau_{\frac{T}{2}}}^{\tau_T}\right)}. \tag{3.24}$$

Approximation formula (3.24) has a similar interpretation to that of the formula in the UOC option case stated above. The probability ratio on the right side of Eq.(3.24) is computed similar to the ratio of Eq.(3.17).

3.2 Lookback Option

3.2.1 Floating Strike Call Option

The payoff function of a floating strike lookback call option maturing at time T can be expressed as

$$H(S,T) = S_T - \inf_{t\in[0,T]} S_t. \tag{3.25}$$

Then, the expected value of the payoff

$$V(T) = \mathbb{E}\left[S_T - \inf_{t\in[0,T]} S_t\right], \tag{3.26}$$

can be decomposed into the primary approximation term and the residual term as follows.

$$V(T) = U(T) + RT_{\mathrm{floating}}. \tag{3.27}$$

In this case,

$$U(T) = \mathbb{E}\left[\tilde{S}^{\tau_T}_{\tau_T} - \inf_{t\in[0,\tau_T]} \tilde{S}^{\tau_T}_t\right], \tag{3.28}$$

and we define

$$\begin{aligned} RT_{\text{floating}} &:= \mathbb{E}\left[\inf_{t\in[0,\tau_T]} \tilde{S}^{\tau_T}_t - \inf_{t\in[0,T]} S_t\right] \\ &= \mathbb{E}\left[\left(S_0 - \inf_{t\in[0,T]} S_t\right)^+\right] - \mathbb{E}\left[\left(S_0 - \inf_{t\in[0,\tau_T]} \tilde{S}^{\tau_T}_t\right)^+\right]. \end{aligned} \tag{3.29}$$

Likewise, we attempt to approximate the residual term RT_{floating} heuristically. For the heuristic approximation, we postulate what follows. Consider: (1) The difference between the expected payoff of a fixed strike lookback put option written on a true stock with strike price S_0 maturing at time T and the expected payoff of such an option written on the pseudo stock with strike price S_0 maturing at time τ_T. (2) The difference between the expected payoff of a plain vanilla put option written on the true stock with strike price S_0 maturing at time $T/2$ and the expected payoff of such an option written on the pseudo stock with strike price S_0 maturing at time $\tau_{T/2}$. We assume that the difference in (1) is proportional to the difference in (2).

Thus, we have the approximation

$$RT_{\text{floating}} \approx \epsilon\left(\mathbb{E}\left[\left(S_0 - S_{\frac{T}{2}}\right)^+\right] - \mathbb{E}\left[\left(S_0 - \tilde{S}^{\tau_T}_{\tau_{\frac{T}{2}}}\right)^+\right]\right), \tag{3.30}$$

where we define the proportionality constant ϵ as

$$\epsilon := \frac{U(T)}{\mathbb{E}\left[\left(S_0 - \tilde{S}^{\tau_T}_{\tau_{\frac{T}{2}}}\right)^+\right]}. \tag{3.31}$$

Consequently, we obtain the approximation formula

$$V(T) \approx U(T)\frac{\mathbb{E}\left[\left(S_0 - S_{\frac{T}{2}}\right)^+\right]}{\mathbb{E}\left[\left(S_0 - \tilde{S}^{\tau_T}_{\tau_{\frac{T}{2}}}\right)^+\right]}. \tag{3.32}$$

Approximation formula (3.32) can be interpreted as follows: The ratio of $V(T)$ to $U(T)$ is approximately equal to the ratio of the expected payoff of a plain vanilla put option written on the true stock with strike price S_0 expiring at time $T/2$ to the expected payoff of such an option written on the pseudo stock with strike price S_0 expiring at time $\tau_{T/2}$. If the pseudo stock price tends to remain below the true stock price much of the time, the primary approximation $U(T)$ might be an underestimate. On the other hand, the expected payoff ratio in Eq.(3.32) might be greater than 1. Therefore, it is expected to adjust the level of $U(T)$ to the true value of $V(T)$.

The denominator on the right side of approximation formula (3.32) is computed similarly to Eq.(3.18). An application of Example 3 yields the value of the numerator.

3.2.2 Fixed Strike Call Option

The payoff function of a fixed strike lookback call option maturing at time T is written as

$$H(S,T) = \left(\sup_{t\in[0,T]} S_t - K\right)^+, \tag{3.33}$$

where K is the strike price. Unlike the previous heuristic approximations, we decompose the expected value of the payoff as follows:

$$V(T) = \mathbb{E}\left[\left(\sup_{t\in[0,T]} S_t - \sup_{t\in[0,\tau_T]} \tilde{S}_t^{\tau_T} + \sup_{t\in[0,\tau_T]} \tilde{S}_t - K\right)^+\right]. \tag{3.34}$$

We approximate Eq.(3.34) as follows:

$$\begin{aligned} V(T) &\approx \mathbb{E}\left[\left(\sup_{t\in[0,\tau_T]} \tilde{S}_t^{\tau_T} - (K+\delta)\right)^+\right] \\ &= U(T; K+\delta). \end{aligned} \tag{3.35}$$

Here, $U(T;K)$ denotes the primary approximation of the expected payoff of a fixed strike lookback call option with strike price K and we define the strike price adjustment δ as

$$\begin{aligned} \delta &:= \mathbb{E}\left[\sup_{t\in[0,\tau_T]} \tilde{S}_t^{\tau_T} - \sup_{t\in[0,T]} S_t\right] \\ &= \mathbb{E}\left[\left(\sup_{t\in[0,\tau_T]} \tilde{S}_t^{\tau_T} - S_0\right)^+ - \left(\sup_{t\in[0,T]} S_t - S_0\right)^+\right]. \end{aligned} \tag{3.36}$$

In Eq.(3.36), we postulate that the difference between the supremum of the true price process and the supremum of the pseudo price process is always close to its expected value. That is, the difference is postulated to be nearly constant. Furthermore, we attempt to approximate the strike price adjustment as follows:

$$\begin{aligned} \delta &\approx \frac{\mathbb{E}\left[\left(\sup_{t\in[0,\tau_T]} \tilde{S}_t^{\tau_T} - S_0\right)^+\right]}{\mathbb{E}\left[\left(\tilde{S}_{\tau_{\frac{T}{2}}}^{\tau_T} - S_0\right)^+\right]} \left(\mathbb{E}\left[\left(\tilde{S}_{\tau_{\frac{T}{2}}}^{\tau_T} - S_0\right)^+\right] - \mathbb{E}\left[\left(S_{\frac{T}{2}} - S_0\right)^+\right]\right) \\ &= U(T;S_0)\left(1 - \frac{\mathbb{E}\left[\left(S_{\frac{T}{2}} - S_0\right)^+\right]}{\mathbb{E}\left[\left(\tilde{S}_{\tau_{\frac{T}{2}}}^{\tau_T} - S_0\right)^+\right]}\right). \end{aligned} \tag{3.37}$$

The approximation in (3.37) indicates that the ratio of $V(T)$ to $U(T;S_0)$ is nearly equal to the ratio of the expected payoff of a plain vanilla call option written on the true stock with strike price S_0 expiring at time $T/2$ to the expected payoff of such an option written on the pseudo stock with strike price S_0 expiring at time $\tau_{T/2}$. For this reason, a similar adjustment to the previous heuristic approximations is expected to work the approximation in (3.37). The expected payoff ratio is computed similarly to the floating strike call option case.

4 Numerical Example

This section provides numerical examples to confirm how accurately our approximation method evaluates continuously monitored exotic options. In the numerical examples, we compute the prices of DOC and UOC options as well as floating and fixed strike call options, all of which expire at $T = 0.5$ or 1.0. Suppose that the underlying stock price follows the NIG model or the VG model. We consider four cases with different parameter and expiry sets. Commonly used values of model parameters for both models are listed in Table 1. The negative value of parameter μ indicates that the return distribution on the stock is negatively skewed, which is consistent with actual option markets. Parameter κ takes the value 0.02 or 0.06. Recall that $\mathbf{Var}[\tau_t] = \kappa t$ in the two pricing models. The model with $\kappa = 0.02$ is relatively closer to the BS model than that with $\kappa = 0.06$. Figures 1 and 2 depict the implied volatilities generated by the NIG and VG models, respectively. As expected, the slopes of the implied volatilities with $\kappa = 0.06$ are steeper than those with $\kappa = 0.02$. As is well-known, the figures show that the implied volatility smile becomes flat as expiry goes away.

For approximating the improper integral in Eq.(3.3), we truncate the infinite interval of the integration to the finite interval $[0.001, 1.6]$ for Case 1, $[0.001, 2.5]$ for Cases 2 and 3, and $[0.001, 3.0]$ for Case 4. In the computations, the trapezoidal rule with 64 partitions is applied to the truncated integral. Closed-form solutions to Eq.(3.2) for barrier and lookback options can be found in Hull [16], for instance. Monte Carlo estimates of the option prices are regarded as the exact values and are simulated by the method of Ribeiro and Webber [25] with 10 million sample paths and 512 monitoring points per year. The calculation time for the Monte Carlo simulation is about 800 seconds for options with maturity $T = 0.5$ and about 1,600 seconds for options with maturity $T = 1.0$. On the other hand, the calculation time of our method is about a few milliseconds for all options. We would like to stress that the calculation time does not depend on the type of option and time to maturity. We use a PC with Core i5-5250U, 8GB RAM and the C++ programing language for the implementation of our method.

Table 1: Model parameters and expiries

	r	S_0	σ	μ	κ	T
Case 1	0.03	100	0.2	-0.18	0.02	0.5
Case 2	0.03	100	0.2	-0.18	0.06	0.5
Case 3	0.03	100	0.2	-0.18	0.02	1.0
Case 4	0.03	100	0.2	-0.18	0.06	1.0

4.1 Barrier Option

Tables 2 through 5 exhibit the DOC and UOC option prices in Cases 1 through 4, respectively. We set the barrier level for the DOC options as $B = 80$, 90, or 105, and that for the UOC options as $B = 105$, 110, and 120. Both types of options have a strike price of $K = 90$, 100, or 110. Notice that the UOC options for which the barrier level is less than or equal to the strike price are worthless. In the tables, the approximation prices by our method and the Monte Carlo estimates are displayed in parallel and labeled as "Approx" and "MC", respectively. The tables also show the errors between the approximation prices and the corresponding Monte Carlo estimates, which are labeled as "Error".

As shown by the tables, our approximation method yields adequately accurate option prices overall. The largest absolute error in all the tables, arising in the DOC option with barrier level $B = 95$ and strike price $K = 90$ under the NIG model in Table 5, is 0.110 against the initial stock price of 100. The degree of the approximation errors might be acceptable in actual derivatives markets.

We first focus on Table 2, which shows the results of Case 1. The approximation errors range from -0.021 to 0.012. In the table, the DOC option with barrier level $B = 95$ and strike price $K = 90$ under the NIG model has the largest absolute error, 0.021, whereas the UOC options

Figure 1: Implied volatility in NIG model

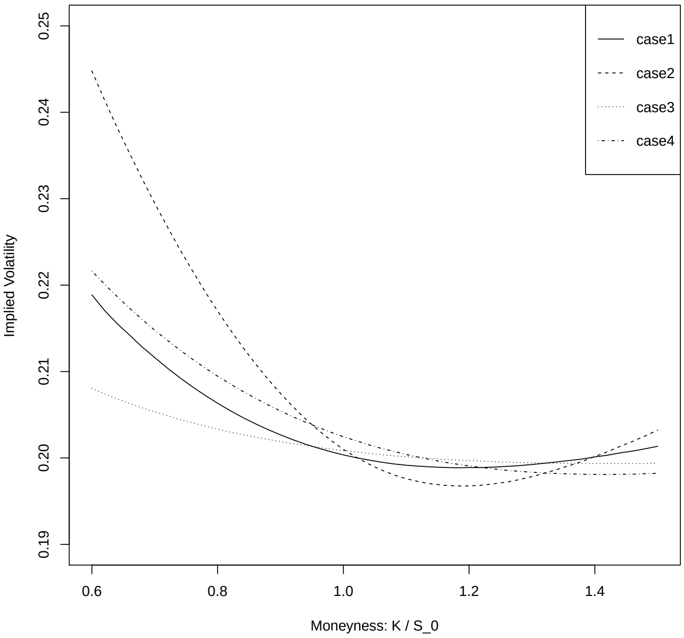
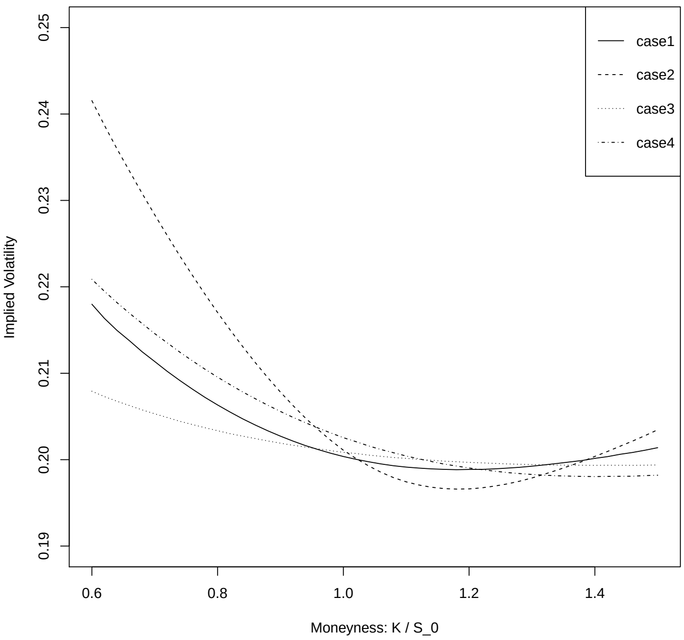


Figure 2: Implied volatility in VG model



with barrier level $B = 105$ and strike price $K = 100$ under the NIG and VG models have the smallest absolute error, 0.001. As regards the DOC option prices with the same barrier level, the higher the strike price, the greater the absolute error is. That is, the degree of the absolute errors appears to depend on the level of the DOC option prices. Recall that the options with strike price $K = 100$ are at the money. For example, in the case of the DOC options with barrier level $B = 80$ in the NIG model, the relative error with strike price $K = 90$ (in the money) is -0.115%, whereas the relative error with strike price $K = 110$ (out of the money) is -0.190%. The error magnitude pattern in the absolute values is reversed for the relative values. On the other hand, such a feature is not observed in the UOC option prices, and the absolute errors are relatively small.

We turn our attention to Table 3, which exhibits the result of Case 2. We find that almost all the absolute errors in this table are larger than those in Table 2. It is worthwhile recalling that the sample paths of the true stock price process converge to those of the BS model as $\kappa \to 0$. Thus, the models in Case 2 (κ =0.06) are more divergent from the BS model than the models in Case 1 ($\kappa = 0.02$). Because the approximation method is based on the BS formulas, approximation errors are expected to be increasing in κ. Besides, the pattern of the approximation errors in Table 3 is similar to that of the errors in Table 2. Specifically, the higher the strike price of a DOC option with the same barrier level, the larger the absolute error is, except for a few instances. The approximation errors in the UOC options seem to have no pattern, although they are small on average because of the relatively small price levels.

Next, we examine the impact of time to maturity on the approximation option prices. Tables 4 and 5 exhibit option prices with longer maturity (T =1.0) than those of Tables 2 and 3 (T =0.5). The results of Tables 4 and 5 are comparable with those of Tables 2 and 3, respectively, in terms of time to maturity. As seen in Table 4, the absolute errors of the DOC option prices with maturity T =1.0 are mostly larger than those with T =0.5 as shown in Table 2. There are at least two possible reasons for this result. One is the levels of the option prices: The levels of the prices in Table 4 are larger than those in Table 2 due to the difference in time values. As mentioned above, a large price level for a DOC option causes a large absolute error. Another possibility is the magnitude of the approximation errors in the knock-out values. Since the terminal price of the pseudo stock coincides with the terminal true stock price, the expected value of the terminal payoff is always exact in our method. On the other hand, we do approximate the knock-out value of a DOC option. A sample path of the pseudo price process deviates from that of the true price process although both ends of the pseudo sample path coincide with those of the true sample path. The deviation is increasing in time to maturity. For this reason, it is difficult to obtain an accurate approximation of a knock-out value with longer maturity. In contrast to the DOC option prices, the approximation prices of the UOC options in Table 4 are sufficiently accurate. Note that the levels of these prices are relatively small. Comparing Table 5, which is the case of κ =0.06 and T =1.0, to Table 3 leads to remarks similar to those above.

We then reveal the effect of bias correction on the approximation prices. As an example, we shed light on the approximation price for the DOC option with strike price $K = 90$ and barrier level $B = 90$ under the NIG model in Table 3. Notice that the approximation error is much smaller than those of adjacent options in the table. The discounted value of $U(T)$ is 11.434 and its error with respect to the MC estimate is 0.044. Thus, the primary approximation is an overestimate. On the other hand, the probability ratio in Eq.(3.24) is 0.9961, which is the correction coefficient. As a result, the approximation price equals $11.434 \times 0.9961 = 11.389$, that is, very close to the exact price. Other examples are available upon request. We acknowledge that some approximation prices become worse after the bias correction. However, the bias correction improves approximation errors on average.

Table 2: Barrier option prices in Case 1

			DOC			UOC		
			$B = 80$	$B = 90$	$B = 95$	$B = 105$	$B = 110$	$B = 120$
NIG	$K = 90$	Approx	12.985	11.248	7.375	0.646	2.230	6.801
		MC	13.000	11.241	7.396	0.644	2.233	6.810
		Error	-0.015	0.007	-0.021	0.002	-0.003	-0.009
	$K = 100$	Approx	6.464	6.012	4.379	0.027	0.321	2.248
		MC	6.474	6.018	4.396	0.028	0.326	2.258
		Error	-0.010	-0.006	-0.017	-0.001	-0.005	-0.010
	$K = 110$	Approx	2.626	2.531	2.010	-	-	0.293
		MC	2.631	2.536	2.019	-	-	0.299
		Error	-0.005	-0.005	-0.009	-	-	-0.006
VG	$K = 90$	Approx	12.985	11.246	7.379	0.647	2.232	6.806
		MC	12.999	11.234	7.394	0.643	2.236	6.813
		Error	-0.014	0.012	-0.015	0.004	-0.004	-0.007
	$K = 100$	Approx	6.464	6.011	4.380	0.027	0.321	2.250
		MC	6.471	6.014	4.395	0.028	0.328	2.261
		Error	-0.007	-0.003	-0.015	-0.001	-0.007	-0.011
	$K = 110$	Approx	2.626	2.530	2.010	-	-	0.293
		MC	2.629	2.533	2.017	-	-	0.300
		Error	-0.003	-0.003	-0.007	-	-	-0.007

Table 3: Barrier option prices in Case 2

			DOC			UOC		
			$B = 80$	$B = 90$	$B = 95$	$B = 105$	$B = 110$	$B = 120$
NIG	$K = 90$	Approx	13.042	11.389	7.671	0.681	2.355	7.077
		MC	13.077	11.390	7.727	0.675	2.359	7.093
		Error	-0.035	-0.001	-0.056	0.006	-0.004	-0.016
	$K = 100$	Approx	6.468	6.026	4.491	0.030	0.356	2.389
		MC	6.490	6.047	4.537	0.059	0.367	2.413
		Error	-0.022	-0.021	-0.046	-0.029	-0.011	-0.024
	$K = 110$	Approx	2.582	2.481	2.000	-	-	0.319
		MC	2.591	2.493	2.024	-	-	0.332
		Error	-0.009	-0.012	-0.024	-	-	-0.013
VG	$K = 90$	Approx	13.046	11.381	7.697	0.682	2.370	7.108
		MC	13.081	11.372	7.740	0.670	2.375	7.131
		Error	-0.035	0.009	-0.043	0.012	-0.005	-0.023
	$K = 100$	Approx	6.470	6.018	4.497	0.030	0.362	2.408
		MC	6.490	6.038	4.541	0.034	0.379	2.438
		Error	-0.020	-0.020	-0.044	-0.004	-0.017	-0.030
	$K = 110$	Approx	2.577	2.472	1.994	-	-	0.322
		MC	2.585	2.485	2.018	-	-	0.340
		Error	-0.008	-0.013	-0.024	-	-	-0.018

Table 4: Barrier option prices in Case 3

			DOC			UOC		
			$B = 80$	$B = 90$	$B = 95$	$B = 105$	$B = 110$	$B = 120$
NIG	$K = 90$	Approx	15.572	12.108	7.401	0.249	0.961	3.855
		MC	15.586	12.088	7.448	0.247	0.962	3.855
		Error	-0.014	0.020	-0.047	0.002	-0.001	0.000
	$K = 100$	Approx	9.615	8.039	5.227	0.009	0.126	1.222
		MC	9.631	8.035	5.264	0.010	0.129	1.229
		Error	-0.016	0.004	-0.037	-0.001	-0.003	-0.007
	$K = 110$	Approx	5.423	4.777	3.294	-	-	0.156
		MC	5.435	4.780	3.319	-	-	0.160
		Error	-0.012	-0.003	-0.025	-	-	-0.004
VG	$K = 90$	Approx	15.571	12.108	7.404	0.249	0.961	3.856
		MC	15.591	12.088	7.450	0.246	0.960	3.855
		Error	-0.020	0.020	-0.046	0.003	0.001	0.001
	$K = 100$	Approx	9.615	8.039	5.229	0.009	0.126	1.223
		MC	9.635	8.036	5.266	0.010	0.129	1.230
		Error	-0.020	0.003	-0.037	-0.001	-0.003	-0.007
	$K = 110$	Approx	5.423	4.777	3.295	-	-	0.156
		MC	5.438	4.781	3.321	-	-	0.160
		Error	-0.015	-0.004	-0.026	-	-	-0.004

Table 5: Barrier option prices in Case 4

			DOC			UOC		
			$B = 80$	$B = 90$	$B = 95$	$B = 105$	$B = 110$	$B = 120$
NIG	$K = 90$	Approx	15.634	12.333	7.686	0.255	0.987	3.970
		MC	15.670	12.298	7.796	0.249	0.986	3.971
		Error	-0.036	0.035	-0.110	0.006	0.001	-0.001
	$K = 100$	Approx	9.647	8.155	5.403	0.010	0.133	1.283
		MC	9.682	8.151	5.488	0.011	0.139	1.299
		Error	-0.035	0.004	-0.085	-0.001	-0.006	-0.016
	$K = 110$	Approx	5.415	4.807	3.373	-	-	0.167
		MC	5.438	4.811	3.428	-	-	0.176
		Error	-0.023	-0.004	-0.055	-	-	-0.009
VG	$K = 90$	Approx	15.632	12.335	7.705	0.254	0.987	3.976
		MC	15.666	12.274	7.807	0.247	0.984	3.978
		Error	-0.034	0.061	-0.102	0.007	0.003	-0.002
	$K = 100$	Approx	9.648	8.155	5.414	0.010	0.134	1.288
		MC	9.680	8.137	5.497	0.011	0.141	1.307
		Error	-0.032	0.018	-0.083	-0.001	-0.007	-0.019
	$K = 110$	Approx	5.414	4.804	3.377	-	-	0.169
		MC	5.436	4.802	3.432	-	-	0.181
		Error	-0.022	0.002	-0.055	-	-	-0.012

4.2 Lookback Option

Table 6 displays lookback call option prices. We set the strike price for fixed strike call options to $K = 90$, 100, or 110. The labels in the table have the same meanings as those in the previous subsection. Table 6 shows that the approximation prices of the lookback options are accurate enough to be considered acceptable by practitioners. The largest absolute error in the table, in the fixed strike call option with strike price $K = 110$ under the VG model in Case 2, is 0.063 for the initial stock price of 100.

Observing the floating strike call options in Table 6, we find that the absolute errors of the approximation option prices with short maturity are very small, whereas the approximation prices with long maturity are also moderately good. Different values of the parameter κ seems not to affect the approximation errors very much. In particular, the approximation formula under the NIG model works well, whereas the price levels of the options are somewhat high.

Next, we turn our attention to fixed strike call options. In contrast to the DOC options shown in the previous subsection, the absolute error of a fixed strike call option tends to be decreasing in strike price. This seemingly contradictory result is caused by bias correction. To understand the result, we compare the fixed strike call option with strike price $K = 90$ to that with $K = 110$ under the NIG model in Case 1. The discounted value of the primary approximation $U(T)$ for the option with $K = 90$ is 22.622 and its error with respect to the MC estimate is -0.044, whereas that for the option with $K = 110$ is 5.170 and its error is 0.001. The primary approximation of the option with $K = 90$ is an underestimate, whereas that of the latter option is definitely accurate. However, as shown by Table 6, the bias correction improves the approximation error of the former option, while it makes the error of the latter option worse. Recall that the bias correction for a fixed strike call option, in which the strike price adjustment δ is applied, is different from that in the other cases. The bias correction is a double-edged sword, but, in general, it contributes to improving the accuracy of the approximation prices.

5 Conclusion

This study suggests a simple approximation method for pricing barrier options and lookback options with continuous monitoring under a class of Lévy processes. The approximation method is straightforward, and the computational procedure can be comparatively easily implemented without complicated calculations. This feature might be preferable for practitioners in financial institutions. Nevertheless, the method generates sufficiently accurate approximation prices and the computational speed is remarkably fast, regardless of the type of option and time to maturity. We believe that the method developed in this study satisfies practical requirements to a great degree.

References

[1] Barndorff-Nielsen, O. "Processes of normal inverse Gaussian type." *Finance and Stochastics* 2.1 (1997): 41-68.

[2] Barndorff-Nielsen, O., and N. Shephard. "Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics." *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 63.2 (2001): 167-241.

[3] Black, F., and M. Scholes. "The pricing of options and corporate liabilities." *Journal of Political Economy* 81.3 (1973): 637-654.

Table 6: Lookback option prices

				Case 1	Case 2	Case 3	Case 4
NIG	Floating		Approx	11.647	11.633	16.827	16.881
			MC	11.650	11.633	16.833	16.893
			Error	-0.003	-0.000	-0.006	-0.012
	Fixed	$K = 90$	Approx	22.663	22.664	28.874	28.960
			MC	22.667	22.662	28.884	28.979
			Error	-0.004	0.001	-0.010	-0.019
		$K = 100$	Approx	12.663	12.664	18.874	18.960
			MC	12.667	12.662	18.884	18.979
			Error	-0.004	0.001	-0.010	-0.019
		$K = 110$	Approx	5.191	5.150	10.651	10.685
			MC	5.170	5.101	10.639	10.661
			Error	0.022	0.048	0.012	0.024
VG	Floating		Approx	11.649	11.647	16.828	16.894
			MC	11.658	11.650	16.850	16.918
			Error	-0.009	-0.004	-0.022	-0.024
	Fixed	$K = 90$	Approx	22.665	22.678	28.876	28.974
			MC	22.679	22.687	28.901	29.016
			Error	-0.015	-0.009	-0.025	-0.042
		$K = 100$	Approx	12.665	12.678	18.876	18.974
			MC	12.679	12.687	18.901	19.016
			Error	-0.015	-0.009	-0.025	-0.042
		$K = 110$	Approx	5.192	5.153	10.652	10.693
			MC	5.174	5.090	10.650	10.671
			Error	0.017	0.063	0.002	0.022

[4] Boyarchenko, M., M. Innocentis, and S. Levendorskiĭ. "Prices of barrier and first-touch digital options in Lévy -driven models, near barrier." *International Journal of Theoretical and Applied Finance* 14.07 (2011): 1045-1090.

[5] Boyarchenko, S. and S. Levendorskiĭ. "Barrier options and touch-and-out options under regular Lévy processes of exponential type." *Annals of Applied Probability* 12.4 (2002): 1261-1298.

[6] Boyarchenko, M., and S. Levendorskiĭ. "Prices and sensitivities of barrier and first-touch digital options in Lévy -driven models." *International Journal of Theoretical and Applied Finance* 12.08 (2009): 1125-1170.

[7] Broadie, M., P. Glasserman, and S. Kou. "Connecting discrete and continuous path-dependent options." *Finance and Stochastics* 3.1 (1999): 55-82.

[8] Carr, P., and J. Crosby. "A class of Lévy process models with almost exact calibration to both barrier and vanilla FX options." *Quantitative Finance* 10.10 (2010): 1115-1136.

[9] Carr, P., and D. Madan. "Option valuation using the fast Fourier transform." *Journal of Computational Finance* 2.4 (1999): 61-73.

[10] Carr, P., and L. Wu. "Time-changed Lévy processes and option pricing." *Journal of Financial Economics* 71.1 (2004): 113-141.

[11] Cont, R., and Tankov, P. *Financial Modelling with Jump Processes* (2003): CRC press.

[12] Dia, E. H. A., and D. Lamberton. "Connecting discrete and continuous lookback or hindsight options in exponential Lévy models." *Advances in Applied Probability* 43.4 (2011): 1136-1165.

[13] Feng, L., and V. Linetsky. "Pricing discretely monitored barrier options and defaultable bonds in Lévy process models: a fast Hilbert transform approach." *Mathematical Finance* 18.3 (2008): 337-384.

[14] Fusai, G., and A. Meucci. "Pricing discretely monitored Asian options under Lévy processes." *Journal of Banking & Finance* 32.10 (2008): 2076-2088.

[15] Heston, S. "A closed-form solution for options with stochastic volatility with applications to bond and currency options." *Review of Financial Studies* 6.2 (1993): 327-343.

[16] Hull, J. *Options Futures and Other Derivatives.* Pearson Education India, 2003.

[17] Jeannin, M., and M. Pistorius. "A transform approach to compute prices and Greeks of barrier options driven by a class of Lévy processes." *Quantitative Finance* 10.6 (2010): 629-644.

[18] Kudryavtsev, O., and S. Levendorskiĭ. "Fast and accurate pricing of barrier options under Lévy processes." *Finance and Stochastics* 13.4 (2009): 531-562.

[19] Kudryavtsev, O., and S. Levendorskiĭ. "Efficient pricing options with barrier and lookback features under Lévy processes." Working Paper, Social Science Research Network (2011).

[20] Lee, R. "Option pricing by transform methods: extensions, unification and error control." *Journal of Computational Finance* 7.3 (2004): 51-86.

[21] Lewis, A. "A simple option formula for general jump-diffusion and other exponential Lévy processes." Working Paper, Social Science Research Network (2001).

[22] Madan, D., P. Carr, and E. Chang. "The variance gamma process and option pricing." *Review of Finance* 2.1 (1998): 79-105.

[23] Madan, D., and E. Seneta. "The variance gamma (VG) model for share market returns." *Journal of Business* 64.4 (1990): 511-524.

[24] Merton, R. "Option pricing when underlying stock returns are discontinuous." *Journal of Financial Economics* 3.1-2 (1976): 125-144.

[25] Ribeiro, C., and N. Webber. "Correcting for simulation bias in Monte Carlo methods to value exotic options in models driven by Lévy processes." *Applied Mathematical Finance* 13.4 (2006): 333-352.

[26] Umezawa, Y., and A. Yamazaki. "Pricing path-dependent options with discrete monitoring under time-changed Lévy processes." *Applied Mathematical Finance* 22.2 (2015): 133-161.

[27] Yamazaki, A. "Pricing average options under time-changed Lévy processes." *Review of Derivatives Research* 17.1 (2014): 79-111.

[28] Yamazaki, A. "Generalized Barndorff-Nielsen and Shephard model and discretely monitored option pricing." *International Journal of Theoretical and Applied Finance* 19.04 (2016): 1650024.

[29] Zeng, P., and Y. K. Kwok. "Pricing barrier and Bermudan style options under time-changed Lévy processes: fast Hilbert transform approach." *SIAM Journal on Scientific Computing* 36.3 (2014): B450-B485.