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Yuan Li
Graduate School of Economics, University of Tokyo

Kaimon Miyachi
Graduate School of Economics, University of Tokyo

Kenichiro Shiraya
Graduate School of Economics, University of Tokyo

Akira Yamazaki
Graduate School of Business Administration, Hosei University

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# Approximation Method Using Black-Scholes Formula for Barrier Option Pricing under Lévy Models*


Yuan Li[†] Kaimon Miyachi[‡] Kenichiro Shiraya[§] Akira Yamazaki[¶]


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**Abstract**

This study proposes an approximation method for pricing continuously monitored barrier options. We employ a class of Lévy processes as the driving factor of an underlying stock price and consider a mimicking process for approximation. Randomizing the Black-Scholes formula associated with the mimicking process leads to a primary approximation formula. We then develop a probability matching adjustment for improving the accuracy of the primary approximation formula. This method is straightforward and easily implementable. Nevertheless, the approximation prices are reasonably accurate, and the calculation speed is remarkably fast, regardless of time to maturity.



**Keywords:** barrier option; Lévy process; Black-Scholes formula; randomized maturity; Brownian bridge


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[†]Graduate School of Economics, University of Tokyo.

[‡]Graduate School of Economics, University of Tokyo.

[§]Graduate School of Economics, University of Tokyo.

[¶]Graduate School of Business Administration, Hosei University. His work is also supported by the Research Institute for Innovation Management at Hosei University.

# 1 Introduction

In derivatives markets, pricing exotic options based on calibration to liquid plain vanilla option prices is a de facto standard. For that reason, a large number of studies have addressed the development of non-Gaussian pricing models, almost all of which are regarded as a modified version of the Black-Scholes (BS) model (Black and Scholes [4]), to capture the implied volatility smiles observed in option markets. Furthermore, various kinds of methods for pricing exotic options based on such pricing models have been actively explored. Even now, practitioners engaged in derivatives business desire more effective pricing methods for some frequently traded exotic options such as barrier options.

The objective of this study is to propose an approximation method for pricing continuously monitored barrier options. In contrast to recent advance in highly complicated option pricing methods, we aim to develop a *simple* and *practical* pricing method that might not be necessarily novel as an aspect of mathematical finance. We present the desired attributes for our method as follows:

- **Volatility smiles:** According to the de facto standard in derivatives markets, models describing the dynamics of an underlying asset price have to capture implied volatility smiles. Under such pricing models, we should develop a pricing method for exotic options.
- **Easy understanding:** It would be desirable that the principle of a pricing method is easily understandable for quantitative analysts in financial institutions. To achieve this, we should circumvent highly complicated mathematical discussion.
- **Simple implementation:** It would be preferable for someone to be able to apply the pricing method easily and to be provided with stable pricing results at any time. Therefore, we should avoid highly complicated and unpredictable computations.
- **Fast computation:** Calculation time of option pricing is crucial in the derivatives business. It would be desirable that the calculation time is within a fraction of a second, regardless of time to maturity. This level of computational speed is our target.
- **Reasonably accurate pricing:** Barrier options are less liquid and have wider offer-bid spreads than plain vanilla options in general. Thus, we aim to obtain reasonably accurate prices, whereas we allow for some degree of pricing errors caused by an approximation method.

For this purpose, we explore an approximation method for pricing barrier options with *continuous* monitoring as follows: We begin by employing a class of Lévy processes called *subordinated Brownian motions* as a driving factor of an underlying stock price. A stock price process modeled as a subordinated Brownian motion, which can be regarded as a time-changed BS model subordinated by a non-decreasing Lévy process, has the ability to generate negatively skewed and fat-tailed return distributions. We then consider a mimicking process, a version of the BS model with a parameterized interest rate, to approximate the true stock price process. Next, we apply the BS formula to evaluate an exotic option written on the mimicking stock. Randomizing the option expiry date and the parameter of the pseudo interest rate and taking the expectation of the randomized BS formula lead to a primary approximation formula. To improve the accuracy, we add a probability matching adjustment based on the Brownian bridge probability to the primary approximation formula. Finally, numerical examples confirm the effectiveness of our approximation method.

In this paragraph, we state the results of this study in line with the desired method attributes mentioned above. First, in the numerical examples, we employ the normal inverse

Gaussian process (Barndorff-Nielsen [1]) with plausible parameter sets as the driving factor of the underlying stock price. This process belongs to subordinated Brownian motions. The numerical examples illustrate that the pricing model based on the normal inverse Gaussian process depicts implied volatility smiles. Second, we derive a primary approximation formula for pricing continuously monitored barrier options. We then develop a probability matching adjustment for improving the accuracy of the primary approximation formula. This method is straightforward because it is based on the BS formula and the Brownian bridge probability. An additional procedure is numerical integration. Only a suitable quadrature rule is needed for computation. Third, as will be shown by the numerical examples, the calculation time of our method is only a few milliseconds. The computational speed does not depend on time to maturity. The approximation prices in the numerical examples are reasonably accurate. However, we acknowledge that it is necessary to selectively use various patterns of the approximation method. In conclusion, our approximation method achieves all the desired attributes.

There are a number of related studies. A list of the prominent literature addressing path-dependent option pricing under Lévy processes or their subspecies is as follows: Boyarchenko et al. [5], Kudryavtsev and Levendorskiĭ [19], Boyarchenko and Levendorskiĭ [6] [7], and Jeannin and Pistorius [18]. These five use the Wiener-Hopf factorization for pricing continuously monitored barrier options, whereas Kudryavtsev and Levendorskiĭ [20] also apply the Wiener-Hopf factorization to evaluate continuously monitored lookback options. However, applying the Wiener-Hopf factorization requires the double inverse Laplace transform and the inverse Fourier transform. Accordingly, the computational implementation involves some complexity. Feng and Linetsky [14] develops a pricing method for discretely monitored barrier options by using the Hilbert transform combined with the Fourier transform. Zhen and Kwok [30] extend the pricing method of Feng and Linetsky [14] to the time-changed Lévy processes introduced by Carr and Wu [11]. Fusai and Meucci [15] provide semi-analytical solutions for both discretely and continuously monitored geometric average option prices. Umezawa and Yamazaki [27] derive semi-analytical solutions for some exotic option prices with discrete monitoring under time-changed Lévy processes from multivariate characteristic functions. Yamazaki [29] extends the pricing formulas of Umezawa and Yamazaki [27] to Barndorff-Nielsen and Shephard's [2] stochastic volatility model. Fusai and Meucci [15] apply a Fourier transform to derive semi-analytical solutions for geometric average option prices, whereas Yamazaki [28] uses the Gram-Charlier expansion for pricing arithmetic average options under time-changed Lévy processes. Carr and Crosby [9] calibrate exponential Lévy models to barrier options and plain vanilla options simultaneously. Broadie et al. [8] examine the relation between discretely and continuously monitored path-dependent option prices in the BS model, and Dia and Lamberton [13] extend the result of Broadie et al. [8] to jump-diffusion models.

The remainder of this paper is organized as follows: Section 2 describes the model. Section 3 provides the general formula of the primary approximation for pricing path-dependent options. Section 4 shows a modified approximation formula with the probability matching adjustment for pricing barrier options. Section 5 presents numerical examples and Section 6 concludes.

# 2 Model

Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a probability space carrying a one-dimensional Lévy process $X$ with the associated filtration $(\mathcal{F}_t)_{t\geq 0}$. A stochastic process $X$ on $(\Omega, \mathcal{F}, \mathbb{Q})$ with values in $\mathbf{R}$ such that $X_0 = 0$ is a Lévy process if it has the following properties: (1) $X$ is $(\mathcal{F}_t)_{t\geq 0}$-adapted. (2) The sample paths of $X$ are right continuous with left limits. (3) $X_u - X_t$ is independent of $\mathcal{F}_t$ for $0 \leq t \leq u$. (4) $X_u - X_t$ has the same distribution as $X_{u-t}$ for $0 \leq t \leq u$. We assume frictionless markets and absence of arbitrage opportunities and assume that the risk-neutral measure $\mathbb{Q}$ is given.

This study focuses on a class of Lévy processes called *subordinated Brownian motions*. A Lévy process in this class is defined as

$$X_t = \mu\tau_t + \sigma W(\tau_t), \tag{2.1}$$

for each $t \geq 0$. Here, $\mu \in \mathbf{R}$ and $\sigma > 0$ are some constants, $W$ denotes a one-dimensional standard Brownian motion, and $\tau$ denotes a non-decreasing Lévy process independent of $W$. That is, the process $X$ in (2.1) is a time-changed Brownian motion equipped with a stochastic time change $\tau$. The time change process $\tau$ known as the *subordinator* in the field of stochastic calculus is normalized so that $\mathbb{E}[\tau_t] = t$ for any $t > 0$ without loss of generality. Notice that drifted Brownian motions are a special case of the class, and a subordinated Brownian motion deviates from the underlying Brownian motion due to the stochastic time change. As will be shown in the next section, this property is used in our approximation method.

Using the independence of $\tau$ from $W$, the characteristic function of $X_t$ in (2.1) can be written as

$$\phi_{X_t}(u) := \mathbb{E}\left[e^{iuX_t}\right] = \exp\left\{t\Psi_X(u)\right\}, \quad u \in \mathbf{R}, \tag{2.2}$$

with

$$\Psi_X(u) = \mathcal{L}\left(i\mu u - \frac{1}{2}\sigma^2 u^2\right), \tag{2.3}$$

where $\mathbb{E}[\,\cdot\,]$ is the risk-neutral expectation operator, $\Psi_X(u) := \ln \mathbb{E}\left[e^{iuX_1}\right]$ is the characteristic exponent of $X$, and $\mathcal{L}(u) := \ln \mathbb{E}\left[e^{u\tau_1}\right]$ is the Laplace exponent of $\tau$. Thus, these two exponents define the stochastic properties of $X$ and $\tau$ over the unit time interval. Taking an arbitrary time change process $\tau$ leads to a subordinated Brownian motion $X$, and its probabilistic features are derived from Eqs.(2.2) and (2.3). Additionally, we assume that the probability density function of $\tau_t$ denoted by $f_{\tau_t}(u)$ is known even if the probability density function of $X_t$ is unknown or does not exist. The following examples provide two Lévy processes used in the existing literature belonging to the class of subordinated Brownian motions, employed in further numerical examples.

**Example 1 (Normal Inverse Gaussian Process)** Taking the inverse Gaussian process as the time change process in (2.1) leads to the normal inverse Gaussian (NIG) process proposed by Barndorff-Nielsen [1]. Increments of the inverse Gaussian process have the inverse Gaussian distribution with only one parameter, $\kappa > 0$, because of the normalization. Thus, the Laplace exponent and the probability density function of the inverse Gaussian process are expressed as

$$\mathcal{L}(u) = -\sqrt{\frac{2}{\kappa}}\left(\sqrt{\frac{1}{2\kappa} - u} - \sqrt{\frac{1}{2\kappa}}\right), \tag{2.4}$$

and

$$f_{\tau_t}(u) = \frac{t}{u^{\frac{3}{2}}\sqrt{2\kappa\pi}} \exp\left\{\frac{1}{2\kappa}\left(2t - u - \frac{t^2}{u}\right)\right\}, \tag{2.5}$$

respectively. (2.4) yields $\mathbf{Var}[\tau_1] = \kappa$. Applying (2.4) to (2.3) leads to the characteristic exponent of the NIG process

$$\Psi_X(u) = \frac{1}{\kappa} - \frac{1}{\kappa}\sqrt{1 - 2i\mu\kappa u + \sigma^2\kappa u^2}. \tag{2.6}$$

The NIG process is a Lévy process without a diffusion component and has infinite variation at any time, with high activity of small jumps. Parameter $\kappa$ generates the excess kurtosis of the distribution of $X_t$ and parameter $\mu$ controls its skewness, whereas parameter $\sigma$ determines the degree of its standard deviation. The NIG process converges to a Brownian motion as $\kappa$ approaches zero.

**Example 2 (Variance Gamma Process)** Taking the gamma process as the time change process in (2.1) leads to the variance gamma (VG) process introduced by Madan and Seneta [24]. Increments of the gamma process have the gamma distribution with only one parameter, $\kappa > 0$, for the same reason as that in the NIG process. Thus, the Laplace exponent and the probability density function of the gamma process can be written as

$$\mathcal{L}(u) = -\frac{1}{\kappa}\ln(1-\kappa u), \tag{2.7}$$

and

$$f_{\tau_t}(u) = \frac{1}{\kappa\Gamma(\frac{t}{\kappa})}\left(\frac{u}{\kappa}\right)^{\frac{t}{\kappa}-1} e^{-\frac{u}{\kappa}}, \tag{2.8}$$

respectively, where $\Gamma(x)$ is the gamma function. (2.7) also yields $\mathbf{Var}[\tau_1] = \kappa$. Applying (2.7) to (2.3), we obtain the characteristic exponent of the VG process

$$\Psi_X(u) = -\frac{1}{\kappa}\ln\left(1 - i\mu\kappa u + \frac{1}{2}\sigma^2\kappa u^2\right). \tag{2.9}$$

Although the VG process is also a pure jump Lévy process, it has finite variation at any time, with infinite, but relatively low activity of small jumps unlike the NIG process. The parameters $\sigma$, $\mu$, and $\kappa$ of the VG process play a similar role to those of the NIG process.

The NIG process, as well as the VG process, can capture the fat tails and negative skewness of the risk-neutral return distributions observed in option markets, whereas it produces a parsimonious pricing model with fewer parameters than other non-Gaussian models such as Heston's stochastic volatility model (Heston [16]) and Merton's jump-diffusion model (Merton [25]). For the detailed features of the latter two important processes, see Cont and Tankov [12] and references therein.

Suppose that the risk-neutral price process of an underlying stock is given by

$$\ln\frac{S_t}{S_0} := (r-\phi)t + X_t, \quad \text{for every} \quad t \geq 0, \tag{2.10}$$

where $r$ is a positive constant denoting a risk-free interest rate and $S_0 > 0$ is the initial stock price. Here, $\phi := \Psi_X(-i)$ is a constant such that the discounted price process is a $\mathbb{Q}$-martingale. According to (2.1), the price process in (2.10) can be rewritten as

$$\ln\frac{S_t}{S_0} := (r-\phi)t + \mu\tau_t + \sigma W(\tau_t). \tag{2.11}$$

Let $H(S,T)$ denote the payoff function of a continuously monitored path-dependent option maturing at time $T$ and $V(T)$ denote the unconditional expected value of the payoff function, that is,

$$V(T) := \mathbb{E}\left[H(S,T)\right]. \tag{2.12}$$

The objective of this study is to develop an analytical approximation method for (2.12) associated with payoff functions of barrier options and lookback options.

# 3 Primary Approximation

The basic idea of our approximation method is straightforward and based on the BS formulas for exotic options. First, we consider a pseudo stock price process $\tilde{S}^{x,T}$ governed by the BS model

$$\ln \frac{\tilde{S}_t^{x,T}}{\tilde{S}_0^{x,T}} := \left( R(x,T) - \frac{\sigma^2}{2} \right) t + \sigma W(t), \quad \text{for} \quad t \in [0,T], \tag{3.1}$$

where the initial price is set to be $\tilde{S}_0^{x,T} = S_0$ and $R(x,T)$ denotes the pseudo risk-free interest rate parameterized by $x > 0$, whose explicit representation will be shown below. Let $V_{\mathrm{BS}}(R(x,T), y)$ denote the unconditional expected value of payoff $H(\tilde{S}^{x,T}, y)$ under BS model (3.1) with maturity $y$, defined as

$$V_{\mathrm{BS}}(R(x,T), y) := \mathbb{E}\left[ H(\tilde{S}^{x,T}, y) \right]. \tag{3.2}$$

Note that the closed-form solutions to (3.2) for some exotic derivatives, including barrier options and lookback options, are the well-known BS formulas. Replacing the maturity $T$ and parameter $x$ in (3.2) with the random time $\tau_T$ and taking the expectation of (3.2) with respect to the random time lead to the following equation:

$$\begin{aligned} U(T) &:= \mathbb{E}\left[ \mathbb{E}\left[ H(\tilde{S}^{\tau_T,T}, \tau_T) \middle| \tau_T \right] \right] \\ &= \mathbb{E}\left[ V_{\mathrm{BS}}(R(\tau_T, T), \tau_T) \right] \\ &= \int_0^\infty V_{\mathrm{BS}}(R(u,T), u) f_{\tau_T}(u) du. \end{aligned} \tag{3.3}$$

If a closed-form solution to (3.2) exists and the probability density function of $\tau_T$ is given, the value of $U(T)$, which is called the *primary approximation* hereafter, can be easily obtained by a suitable quadrature rule.

Next, we define the pseudo interest rate as

$$R(x,T) := (r - \phi)\frac{T}{x} + \mu + \frac{\sigma^2}{2}. \tag{3.4}$$

Then, substituting (3.4) with $x = \tau_T$ into (3.1), we obtain

$$\ln \frac{\tilde{S}_t^{\tau_T,T}}{\tilde{S}_0^{\tau_T,T}} = (r - \phi)\left( \frac{T}{\tau_T} \right) t + \mu t + \sigma W(t). \tag{3.5}$$

Consider a randomized time interval $[0, \tau_T]$ for the pseudo price process described by (3.5) and the corresponding fixed time interval $[0,T]$ for the true price process of (2.11). Notice that the terminal price of the pseudo stock equals the terminal price of the true stock, that is, $\tilde{S}_{\tau_T}^{\tau_T,T} = S_T$. Because $\tilde{S}_0^{\tau_T,T} = S_0$, the pseudo interest rate defined in (3.4) with $x = \tau_T$ induces both ends of any sample paths of the pseudo price process over the time interval $[0, \tau_T]$ to coincide with those of the true price process over the time interval $[0,T]$. Further, recall that $\mathbb{E}[\tau_t] = t$ for every $t \geq 0$. Accordingly, if the variance of the time change process $\tau$ is not very large, each sample path of the pseudo price process over $[0, \tau_T]$ is expected to be close to that of the true price process over $[0,T]$. This is why we regard $U(T)$ defined in (3.3) as the approximation of $V(T)$ of (2.12). Since a direct application of $U(T)$ might deliver a crude approximation, in the following subsections, we explore *bias correction* to improve the accuracy of the primary approximation $U(T)$. Before doing so, we present special cases as examples. Notice that when a payoff function $H(S,T)$ depends only on the terminal stock price $S_T$, approximation formula (3.3) yields the exact value of $V(T)$.

**Example 3 (Plain Vanilla Option)** Because the payoff function of a European call option with strike price $K$ and maturity $T$ is

$$H(S,T) = (S_T - K)^+, \tag{3.6}$$

the expected payoff is given by

$$V(T) = \mathbb{E}\left[(S_T - K)^+\right] = U(T). \tag{3.7}$$

Madan et al. [23] began with the integral form of (3.3) to evaluate a European call option under the VG process and obtained a closed-form solution.

**Example 4 (Digital Option)** The payoff function of a digital put option with strike price $K$ and maturity $T$ is

$$H(S,T) = \mathbb{I}(S_T \leq K), \tag{3.8}$$

where $\mathbb{I}(A)$ denotes the indicator function that outputs the value 1 if event $A$ is true and 0 otherwise. The expected value of the payoff is given by

$$V(T) = \mathbb{E}\left[\mathbb{I}(S_T \leq K)\right] = \mathbb{Q}(S_T \leq K) = U(T). \tag{3.9}$$

Since some effective pricing methods for European options under Lévy process have been proposed (see Carr and Madan [10], Lewis [22], and Lee [21], among others), applying our method to evaluate European options as shown in the above examples seems less attractive. However, such examples can be supplementary to our approximation method. In the next section, we focus on down-and-out call barrier options, which are one of the most frequently traded path-dependent options.

# 4 Barrier Option

The payoff function of a down-and-out call (DOC) option with maturity $T$ can be written as

$$\begin{aligned} H(S,T) &= (S_T - K)^+ \mathbb{I}\left(B < \inf_{t\in[0,T]} S_t\right) \\ &= (S_T - K)^+ - (S_T - K)^+ \mathbb{I}\left(B \geq \inf_{t\in[0,T]} S_t\right), \end{aligned} \tag{4.1}$$

where $K$ is the strike price and $B$ is the level of a lower barrier below the initial stock price $S_0$. Since the second term on the right side of (4.1) is the payoff function of a down-and-in call (DIC) option, in the following we consider the expected payoff of the DIC option

$$V(T) = \mathbb{E}\left[(S_T - K)^+ \mathbb{I}\left(B \geq \inf_{t\in[0,T]} S_t\right)\right]. \tag{4.2}$$

The primary approximation of the expected payoff (4.2) is

$$U(T) = \int_0^\infty \mathbb{E}\left[(\tilde{S}_u^{u,T} - K)^+ \mathbb{I}\left(B \geq \inf_{t\in[0,u]} \tilde{S}_t^{u,T}\right)\right] f_{\tau_T}(u)du. \tag{4.3}$$

Unfortunately, our numerical example in the next section shows that the primary approximation (4.3) dose not have enough accurate if the barrier price is close to the initial stock price. Thus, we must develop the modification of the primary approximation to improve the accuracy.

## 4.1 Adjusted Approximation

Let us consider the hitting probability of a geometric Brownian motion $Y$:

$$dY_t = aY_t dt + bY_t dW(t), \tag{4.4}$$

where $Y_0 = S_0$ and $a, b$ are some constants. For given $Y_m$, the probability that $Y$ hits a constant $L$ in the period $[0, m]$, which is denoted by $p\left(m, a, b, L|Y_m\right)$, is expressed as

$$\begin{aligned} & p\left(m, a, b, L|Y_m\right) \\ = \quad & \exp\left\{ \frac{-2}{b^2 m} \ln\left( \frac{S_0 \exp\left\{ \left(a - \frac{b^2}{2}\right) m + b\sqrt{m} y \right\}}{L} \right) \ln\left(\frac{S_0}{L}\right) \right\}, \end{aligned} \tag{4.5}$$

where $L \leq S_0, Y_m$ and $y := \frac{\ln(Y_m/S_0) - (a - b^2/2)m}{b\sqrt{m}}$. This conditional probability (4.5) is shown in e.g., Section 3 of Beghin and Orsingher [3].

Instead of the primary approximation (4.3), we consider

$$\Lambda(T) = \int_0^\infty \mathbb{E}\left[ \left( \tilde{S}_u^{u,T} - K \right)^+ p\left(m, a, b, L|Y_m\right) \right] f_{\tau_T}(u) du, \tag{4.6}$$

which is called the *adjusted approximation* hereafter. Note that if $m = u$, $a = R(u, T)$, $b = \sigma$, and $L = B$, the adjusted approximation (4.6) coincides with the primary approximation (4.3).

On the payoff function, the path-independent parts $(S_T - K)^+$ and $(S_{\tau_T}^{\tau_T, T} - K)^+$ are equal almost surely, but the path-dependent parts $\mathbb{I}\left(B \geq \inf_{t\in[0,T]} S_t\right)$ and $\mathbb{I}\left(B \geq \inf_{t\in[0,\tau_T]} \tilde{S}_t^{\tau_T, T}\right)$ are not so. This disparity causes considerable approximation errors in the primary approximation. In the adjusted approximation, we replace the path-dependent part of the primary approximation with the hitting probability of a Brownian bridge on $Y$. We attempt to adjust the hitting probability of a Brownian bridge by choice of its parameters to reduce the errors. This adjustment is not theoretical, but experimental. A similar adjustment is used in Shiraya [26] for local stochastic volatility models.

In our numerical example, we employ nine patterns of parameter setting for the hitting probability of a Brownian bridge, which are exhibited in Table 1. Some parameters are given in advance, while others are free parameters. In the next subsection, we develop a probability matching adjustment to determine free parameters.

At the end of this subsection, we provide a closed-form of the expected value in (4.6):

$$\begin{aligned} & \mathbb{E}\left[ \left( \tilde{S}_u^{u,T} - K \right)^+ p\left(m, a, b, L|Y_m\right) \right] = \\ & \left(\frac{S_0}{L}\right)^A S_0 N\left(d + F\right) \exp\left\{ \left( R(u,T) - \frac{\sigma^2}{2} \right) u + D + \frac{F^2}{2} \right\} - \left(\frac{S_0}{L}\right)^A K N\left(d + E\right) \exp\left\{ D + \frac{E^2}{2} \right\}, \end{aligned} \tag{4.7}$$

where

$$\begin{cases} d = \frac{\ln(S_0/K) + \left(R(u,T) - \sigma^2/2\right)u}{\sigma\sqrt{u}}, \\ A = \frac{-2\ln(S_0/L)}{b^2 m}, \\ D = Am\left(a - \frac{b^2}{2}\right), \\ E = Ab\sqrt{m}, \\ F = \sigma\sqrt{u} + E. \end{cases} \tag{4.8}$$

Here, $N(x)$ denotes the cumulative distribution function of the standard normal distribution. Thus, the value of $\Lambda(T)$ can be also easily obtained by a suitable quadrature rule. The derivation of the formula (4.7) can be found in Appendix.

Table 1: Parameter patterns

| | $m$ | $a$ | $b$ | $L$ | free parameters |
|---|---|---|---|---|---|
| Pattern 1 | $u$ | | | $B$ | $a, b$ |
| Pattern 2 | $T$ | | | $B$ | $a, b$ |
| Pattern 3 | $u$ | $\nu \frac{T}{u}$ | | $B$ | $\nu, b$ |
| Pattern 4 | $u$ | $\nu \frac{T}{u} + \mu$ | | $B$ | $\nu, b$ |
| Pattern 5 | $u$ | $(r - \phi)\frac{T}{u} + \nu$ | | $B$ | $\nu, b$ |
| Pattern 6 | $T$ | $\nu + \mu \frac{u}{T}$ | | $B$ | $\nu, b$ |
| Pattern 7 | $u$ | $r - \phi + \mu + \frac{b^2}{2}$ | | $B$ | $b$ |
| Pattern 8 | $u$ | $r - \phi + \mu + \frac{\sigma^2}{2}$ | $\sigma$ | | $L$ |
| Pattern 9 | $u$ | $\nu$ | $\sigma$ | | $\nu, L$ |

## 4.2 Determination of Free Parameters

To determine free parameters in Table 1, we propose the probability matching adjustment stated below. The probability of the event $\{Y_{T_n} \leq L\}$ is written as

$$\mathbb{Q}\left(Y_{T_n} \leq L\right) = \mathbb{E}\left[\mathbb{I}\left(Y_{T_n} \leq L\right)\right]. \tag{4.9}$$

Note that (4.9) is the expected payoff of a digital put option with strike price $L$ and maturity $T_n$ written on $Y$ and its closed-form is given by the Black-Scholes formula. The probability of the event $\{S_{T_n} \leq B\}$ is written as

$$Q_n = \mathbb{Q}\left(S_{T_n} \leq B\right) = \mathbb{E}\left[\mathbb{I}\left(S_{T_n} \leq B\right)\right]. \tag{4.10}$$

$Q_n$ is the expected payoff of a digital put option with strike price $B$ and maturity $T_n$ written on $S$. As shown in Example 4, the value of $Q_n$ can be easily obtained.

We solve the following system of nonlinear equations to determine the free parameters.

$$\mathbb{Q}\left(Y_{T_n} \leq L\right) = Q_n, \quad \text{for} \quad n = 1, 2. \tag{4.11}$$

Here, the probability matching point $T_n \in (0, T]$ is arbitrarily given. The number of equations in the system corresponds to the degree of the freedom of free parameters. The system of the equations (4.11) means that the free parameters are calibrated such that the probability of $\{Y_{T_n} \leq L\}$ equals the probability of $\{S_{T_n} \leq B\}$. In the following, we will provide the explicit solutions to (4.11) for each pattern in Table 1.

### 4.2.1 Free Parameters of Patterns 1-6

Free parameters of Patterns 1-6 in Table 1 are $a$ (or $\nu$) and $b$. In this case, the system of the equations (4.11) has the form

$$\mathbb{Q}\left(Y_{T_1} \leq B\right) = Q_1, \tag{4.12}$$

$$\mathbb{Q}\left(Y_{T_2} \leq B\right) = Q_2. \tag{4.13}$$

The solutions to (4.12)-(4.13) are

$$a = \frac{b^2}{2} + \frac{\ln\left(\frac{B}{S_0}\right) - b\sqrt{T_1}N^{-1}(Q_1)}{T_1}, \tag{4.14}$$

$$b = \frac{(T_1 - T_2)\ln\left(\frac{B}{S_0}\right)}{T_2\sqrt{T_1}N^{-1}(Q_1) - T_1\sqrt{T_2}N^{-1}(Q_2)}, \tag{4.15}$$

where $N^{-1}(x)$ denotes the inverse function of the cumulative distribution function of the standard normal distribution.

### 4.2.2 Free Parameters of Pattern 7

A free parameter of Pattern 7 is $b$. In this case, the system of the equations (4.11) has the form

$$\mathbb{Q}\left(Y_{T_1} \leq B\right) = Q_1, \tag{4.16}$$

The solution to (4.16) is

$$b = \frac{\ln\left(\frac{B}{S_0}\right) - (r - \phi + \mu)T_1}{\sqrt{T_1}N^{-1}(Q_1)}. \tag{4.17}$$

#### 4.2.3 Free Parameters of Pattern 8

A free parameter of Pattern 8 is $L$. In this case, the system of the equations (4.11) is

$$\mathbb{Q}\left(Y_{T_1} \le L\right) = Q_1, \tag{4.18}$$

The solution to (4.18) is

$$L = S_0 \exp\left\{(r - \phi + \mu)T_1 + \sigma\sqrt{T_1}N^{-1}(Q_1)\right\}. \tag{4.19}$$

#### 4.2.4 Free Parameters of Pattern 9

Free parameters of Pattern 9 are $\nu$ and $L$. In this case, the system of the equations (4.11) has the form

$$\mathbb{Q}\left(Y_{T_1} \le L\right) = Q_1, \tag{4.20}$$

$$\mathbb{Q}\left(Y_{T_2} \le L\right) = Q_2. \tag{4.21}$$

The solutions to (4.20)-(4.21) are

$$\nu = \frac{\sigma\left[\sqrt{T_2}N^{-1}(Q_2) - \sqrt{T_1}N^{-1}(Q_1)\right]}{T_1 - T_2} - \mu, \tag{4.22}$$

$$L = S_0 \exp\left\{(\nu + \mu)T_1 + \sigma\sqrt{T_1}N^{-1}(Q_1)\right\}. \tag{4.23}$$

## 5 Numerical Example

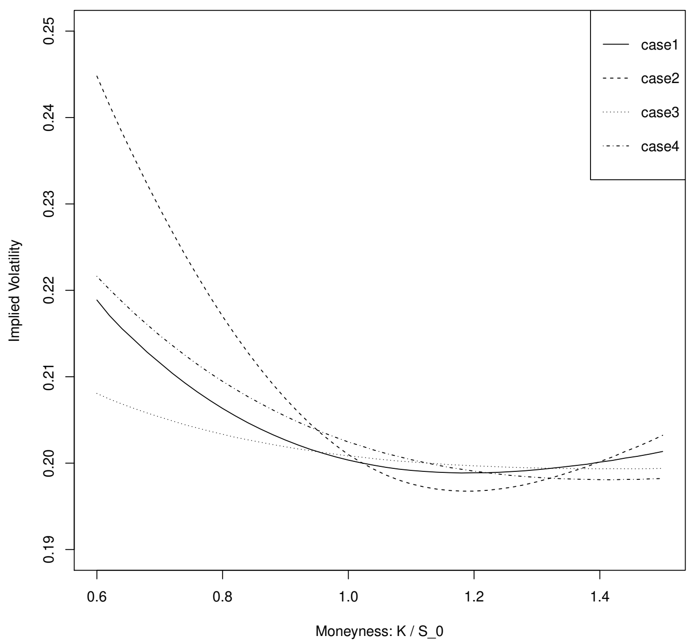
This section provides numerical examples to confirm how accurately our approximation method evaluates continuously monitored barrier option. In the numerical examples, we compute the prices of DOC options expiring at $T = 0.5$ or $1.0$. Suppose that the underlying stock price follows the NIG model. We consider four cases with different parameter and expiry sets. The values of model parameters and expires are listed in Table 2. The negative value of parameter $\mu$ indicates that the return distribution on the stock is negatively skewed, which is consistent with actual option markets. Parameter $\kappa$ takes the value 0.02 or 0.06. The NIG model with $\kappa = 0.02$ is relatively closer to the BS model than that with $\kappa = 0.06$. Figure 1 depicts the implied volatilities generated by the NIG model. As expected, the slopes of the implied volatilities with $\kappa = 0.06$ are steeper than those with $\kappa = 0.02$. As is well-known, the figure shows that the implied volatility smile becomes flat as expiry goes away.

For approximating the improper integral in (3.3), we truncate the infinite interval of the integration to a finite interval. The upper bound of the finite interval is set to $T + 4\sqrt{\kappa T}$ and the lower bound is set to almost zero. That is, the intervals of the truncated integral are $[0.001, 0.9]$ for Case 1, $[0.001, 1.19]$ for Cases 2, $[0.001, 1.57]$ for Case 3, and $[0.001, 1.98]$ for Case 4. In the computations, the trapezoidal rule with 128 partitions is applied to the truncated integral. The closed-form solution to (3.2) for the primary approximation can be found in Hull [17], for instance. Monte Carlo estimates of the option prices are regarded as the exact values and are simulated with 10 million sample paths and 512 monitoring points per year. The calculation time for each option depends on the difference of barrier level and spot price. The larger the difference is, the longer the time is. The calculation time for each option ranges from about 1000 to 2000 seconds. Note that the maturity does not have significant effect on the time because we fix the number of steps. On the other hand, the calculation time of our method is about a few milliseconds for all options. We would like to stress that the calculation time does not depend on time to maturity unlike Monte Carlo pricing. We use a PC with Intel(R) Core(TM) i5-7200U CPU, 8GB RAM and the C++ programing language for the implementation of our method.

Table 2: Model parameters and expiries

| | $r$ | $S_0$ | $\sigma$ | $\mu$ | $\kappa$ | $T$ |
|---|---|---|---|---|---|---|
| Case 1 | 0.03 | 100 | 0.2 | -0.18 | 0.02 | 0.5 |
| Case 2 | 0.03 | 100 | 0.2 | -0.18 | 0.06 | 0.5 |
| Case 3 | 0.03 | 100 | 0.2 | -0.18 | 0.02 | 1.0 |
| Case 4 | 0.03 | 100 | 0.2 | -0.18 | 0.06 | 1.0 |

Figure 1: Implied volatility in NIG model




We set $T_1 = T/3$ and $T_2 = 2T/3$ in Patterns 1-6 and 9 for maturity $T$, while we set $T_1 = T/2$ in Patterns 7 and 8.

Tables 3-6 exhibit the DOC option prices in Cases 1-4, respectively. We set the barrier level as $B = 80$, 90, or 95 and strike price as $K = 90$, 100, or 110. In the tables, "MC" denotes the Monte Carlo estimates and the values in parentheses are the absolute errors between the approximation prices and the Monte Carlo estimates.

We first focus on Table 3, which shows the results of Case 1. In the case that the barrier level is far away from the initial stock price, the primary approximation prices are very close to the MC prices. However, the primary approximation is not satisfactory when the barrier level is near the initial stock price. In such a case, the adjusted approximation improves the accuracy of the primary approximation prices. For example, the absolute error of the primary approximation with $K = 100$ and $B = 95$ is 0.515, while that of the adjusted approximation in Pattern 4 is 0.364. Unfortunately, the accuracy of the adjusted approximation gets worse than that of the primary approximation when the barrier level departs from the initial stock price.

We turn our attention to Table 4, which exhibits the results of Case 2. We find that almost all the absolute errors in this table are larger than those in Table 3. It is worthwhile recalling that the sample paths of the true stock price process converge to those of the BS model as $\kappa \to 0$. Thus, the NIG model in Case 2 ($\kappa$ =0.06) is more divergent from the BS model than the NIG model in Case 1 ($\kappa = 0.02$). Since the primary approximation is based on the BS formula, the absolute errors are increasing in $\kappa$. Similar to Case 1, the adjusted approximation in Case 2 is more accurate than the primary approximation if the barrier level is close to the initial stock price, whereas the adjusted approximation deteriorates the accuracy otherwise.

Next, we examine the impact of time to maturity on the approximation prices. Tables 5 and 6 exhibit option prices with longer maturity ($T$ =1.0) than Tables 3 and 4 ($T$ =0.5). The results of Tables 5 and 6 are comparable with those of Tables 3 and 4, respectively, in terms of time to maturity. As seen in Table 5, the absolute errors of the primary approximation prices with maturity $T$ =1.0 are mostly larger than those with $T$ =0.5 as shown in Table 3. There could be at least two possible reasons for this result. One is the levels of the option prices: The levels of the prices in Table 5 are larger than those in Table 3 due to the difference in time values. A large price level of a barrier option causes a large absolute error. Another possibility is the magnitude of the approximation errors in the knock-out values. Since the terminal price of the pseudo stock coincides with the terminal true stock price, the expected value of the terminal payoff is always exact in our method. On the other hand, we do approximate the knock-out value of a DOC option. A sample path of the pseudo price process deviates from that of the true price process although both ends of the pseudo sample path coincide with those of the true sample path. The deviation is increasing in time to maturity. For this reason, it is difficult to obtain an accurate approximation of a knock-out value with longer maturity. The adjusted approximation is expected to improve the accuracy of the knock-out value of the primary approximation. In fact, the absolute errors of the adjusted approximation in Pattern 5 are smaller than those of the primary approximation in not only the case of $B = 95$, but also the case of $B = 90$.

# 6 Conclusion

This study suggested a simple approximation method for pricing barrier option with continuous monitoring under a class of Levy processes. We then performed a numerical experiment to examine the accuracy of our approximation method. The method is straightforward, and the computational procedure can be easily implemented without complicated calculations. This feature might be preferable for practitioners in financial institutions. Nevertheless, the method generates reasonably accurate approximation prices and the computational speed is remarkably fast, regardless of time to maturity. However, we acknowledge that it is necessary to selectively use various patterns of the approximation method. We believe that the method developed in this study satisfies practical requirements to a great degree.

Table 3: Barrier option prices in Case 1

| $\{K, B\}$ | {90, 80} | {100, 80} | {100, 90} | {100, 95} | {110, 80} | {110, 90} | {110, 95} |
|---|---|---|---|---|---|---|---|
| MC | 13.006 | 6.462 | 6.145 | 4.885 | 2.618 | 2.561 | 2.189 |
| Primary approximation | | | | | | | |
| | 13.002 | 6.473 | 6.021 | 4.371 | 2.630 | 2.535 | 2.007 |
| | (0.004) | (0.010) | (0.123) | (0.515) | (0.012) | (0.026) | (0.183) |
| Adjusted approximation | | | | | | | |
| Pattern 1 | 12.974 | 6.465 | 6.025 | 4.492 | 2.627 | 2.537 | 2.061 |
| | (0.031) | (0.002) | (0.119) | (0.393) | (0.010) | (0.024) | (0.128) |
| Pattern 2 | 12.965 | 6.464 | 6.002 | 4.447 | 2.627 | 2.536 | 2.052 |
| | (0.040) | (0.002) | (0.142) | (0.438) | (0.010) | (0.025) | (0.137) |
| Pattern 3 | 12.973 | 6.464 | 6.026 | 4.497 | 2.627 | 2.537 | 2.061 |
| | (0.033) | (0.002) | (0.119) | (0.388) | (0.009) | (0.024) | (0.128) |
| Pattern 4 | 12.966 | 6.463 | 6.020 | 4.521 | 2.627 | 2.532 | 2.061 |
| | (0.040) | (0.000) | (0.124) | (0.364) | (0.009) | (0.028) | (0.128) |
| Pattern 5 | 12.970 | 6.464 | 6.022 | 4.520 | 2.627 | 2.533 | 2.062 |
| | (0.035) | (0.002) | (0.134) | (0.423) | (0.010) | (0.025) | (0.138) |
| Pattern 6 | 12.971 | 6.465 | 6.011 | 4.462 | 2.627 | 2.536 | 2.051 |
| | (0.035) | (0.002) | (0.134) | (0.423) | (0.010) | (0.025) | (0.138) |
| Pattern 7 | 12.952 | 6.462 | 6.022 | 4.509 | 2.627 | 2.534 | 2.056 |
| | (0.054) | (0.001) | (0.123) | (0.376) | (0.009) | (0.027) | (0.133) |
| Pattern 8 | 12.951 | 6.462 | 6.022 | 4.510 | 2.627 | 2.534 | 2.057 |
| | (0.055) | (0.001) | (0.123) | (0.375) | (0.009) | (0.027) | (0.132) |
| Pattern 9 | 12.965 | 6.463 | 6.022 | 4.519 | 2.627 | 2.533 | 2.062 |
| | (0.040) | (0.000) | (0.123) | (0.367) | (0.009) | (0.028) | (0.128) |

Table 4: Barrier option prices in Case 2

| $\{K, B\}$ | {90, 80} | {100, 80} | {100, 90} | {100, 95} | {110, 80} | {110, 90} | {110, 95} |
|---|---|---|---|---|---|---|---|
| MC | 13.099 | 6.474 | 6.223 | 5.192 | 2.580 | 2.530 | 2.245 |
| Primary approximation | | | | | | | |
| | 13.082 | 6.488 | 6.050 | 4.473 | 2.590 | 2.490 | 1.992 |
| | (0.017) | (0.014) | (0.174) | (0.718) | (0.010) | (0.039) | (0.252) |
| Adjusted approximation | | | | | | | |
| Pattern 1 | 13.018 | 6.470 | 6.068 | 4.764 | 2.583 | 2.499 | 2.119 |
| | (0.081) | (0.004) | (0.156) | (0.428) | (0.003) | (0.031) | (0.126) |
| Pattern 2 | 12.978 | 6.467 | 6.003 | 4.636 | 2.583 | 2.496 | 2.096 |
| | (0.121) | (0.007) | (0.221) | (0.556) | (0.003) | (0.034) | (0.149) |
| Pattern 3 | 13.011 | 6.466 | 6.069 | 4.774 | 2.582 | 2.497 | 2.119 |
| | (0.088) | (0.008) | (0.154) | (0.417) | (0.002) | (0.033) | (0.125) |
| Pattern 4 | 12.992 | 6.461 | 6.054 | 4.833 | 2.581 | 2.487 | 2.115 |
| | (0.107) | (0.013) | (0.169) | (0.358) | (0.001) | (0.043) | (0.129) |
| Pattern 5 | 13.011 | 6.466 | 6.060 | 4.832 | 2.582 | 2.489 | 2.116 |
| | (0.087) | (0.007) | (0.164) | (0.359) | (0.002) | (0.040) | (0.129) |
| Pattern 6 | 12.998 | 6.469 | 6.027 | 4.680 | 2.583 | 2.495 | 2.093 |
| | (0.100) | (0.005) | (0.197) | (0.512) | (0.003) | (0.034) | (0.151) |
| Pattern 7 | 12.964 | 6.460 | 6.058 | 4.815 | 2.581 | 2.490 | 2.109 |
| | (0.135) | (0.014) | (0.165) | (0.376) | (0.001) | (0.040) | (0.136) |
| Pattern 8 | 12.958 | 6.458 | 6.059 | 4.814 | 2.581 | 2.490 | 2.111 |
| | (0.141) | (0.015) | (0.165) | (0.377) | (0.001) | (0.040) | (0.133) |
| Pattern 9 | 12.989 | 6.460 | 6.057 | 4.827 | 2.581 | 2.488 | 2.118 |
| | (0.110) | (0.014) | (0.166) | (0.365) | (0.001) | (0.041) | (0.127) |

Table 5: Barrier option prices in Case 3

| $\{K, B\}$ | {90, 80} | {100, 80} | {100, 90} | {100, 95} | {110, 80} | {110, 90} | {110, 95} |
|---|---|---|---|---|---|---|---|
| MC | 15.663 | 9.639 | 8.367 | 5.984 | 5.425 | 4.923 | 3.726 |
| Primary approximation | | | | | | | |
| | 15.600 | 9.633 | 8.041 | 5.216 | 5.433 | 4.779 | 3.288 |
| | (0.063) | (0.006) | (0.326) | (0.768) | (0.008) | (0.145) | (0.439) |
| Adjusted approximation | | | | | | | |
| Pattern 1 | 15.551 | 9.615 | 8.056 | 5.339 | 5.427 | 4.792 | 3.369 |
| | (0.112) | (0.024) | (0.311) | (0.645) | (0.002) | (0.131) | (0.357) |
| Pattern 2 | 15.520 | 9.606 | 8.013 | 5.297 | 5.425 | 4.775 | 3.346 |
| | (0.143) | (0.032) | (0.354) | (0.687) | (0.001) | (0.148) | (0.380) |
| Pattern 3 | 15.553 | 9.615 | 8.062 | 5.348 | 5.427 | 4.794 | 3.373 |
| | (0.110) | (0.024) | (0.305) | (0.635) | (0.002) | (0.129) | (0.354) |
| Pattern 4 | 15.545 | 9.609 | 8.081 | 5.399 | 5.424 | 4.796 | 3.392 |
| | (0.118) | (0.029) | (0.286) | (0.584) | (0.001) | (0.127) | (0.334) |
| Pattern 5 | 15.548 | 9.611 | 8.080 | 5.396 | 5.425 | 4.796 | 3.391 |
| | (0.115) | (0.028) | (0.287) | (0.588) | (0.000) | (0.127) | (0.336) |
| Pattern 6 | 15.540 | 9.611 | 8.039 | 5.326 | 5.426 | 4.783 | 3.357 |
| | (0.123) | (0.028) | (0.328) | (0.657) | (0.001) | (0.141) | (0.369) |
| Pattern 7 | 15.533 | 9.609 | 8.080 | 5.376 | 5.425 | 4.796 | 3.378 |
| | (0.130) | (0.030) | (0.287) | (0.607) | (0.000) | (0.127) | (0.348) |
| Pattern 8 | 15.533 | 9.608 | 8.080 | 5.380 | 5.425 | 4.796 | 3.381 |
| | (0.130) | (0.031) | (0.287) | (0.604) | (0.000) | (0.127) | (0.345) |
| Pattern 9 | 15.546 | 9.610 | 8.080 | 5.393 | 5.425 | 4.796 | 3.390 |
| | (0.117) | (0.029) | (0.287) | (0.591) | (0.000) | (0.127) | (0.336) |

Table 6: Barrier option prices in Case 4

| $\{K, B\}$ | {90, 80} | {100, 80} | {100, 90} | {100, 95} | {110, 80} | {110, 90} | {110, 95} |
|---|---|---|---|---|---|---|---|
| MC | 15.789 | 9.714 | 8.615 | 6.496 | 5.437 | 5.021 | 3.982 |
| Primary approximation | | | | | | | |
| | 15.707 | 9.692 | 8.163 | 5.373 | 5.440 | 4.811 | 3.355 |
| | (0.082) | (0.022) | (0.452) | (1.123) | (0.003) | (0.209) | (0.627) |
| Adjusted approximation | | | | | | | |
| Pattern 1 | 15.586 | 9.650 | 8.207 | 5.697 | 5.426 | 4.850 | 3.569 |
| | (0.203) | (0.065) | (0.408) | (0.799) | (0.011) | (0.171) | (0.413) |
| Pattern 2 | 15.488 | 9.621 | 8.082 | 5.569 | 5.420 | 4.802 | 3.502 |
| | (0.301) | (0.093) | (0.533) | (0.927) | (0.017) | (0.219) | (0.480) |
| Pattern 3 | 15.595 | 9.650 | 8.224 | 5.722 | 5.425 | 4.854 | 3.578 |
| | (0.194) | (0.065) | (0.391) | (0.774) | (0.012) | (0.167) | (0.403) |
| Pattern 4 | 15.576 | 9.637 | 8.271 | 5.875 | 5.419 | 4.858 | 3.633 |
| | (0.212) | (0.077) | (0.344) | (0.622) | (0.018) | (0.163) | (0.349) |
| Pattern 5 | 15.589 | 9.643 | 8.270 | 5.867 | 5.421 | 4.859 | 3.630 |
| | (0.200) | (0.071) | (0.345) | (0.629) | (0.015) | (0.161) | (0.351) |
| Pattern 6 | 15.550 | 9.638 | 8.156 | 5.662 | 5.423 | 4.823 | 3.536 |
| | (0.239) | (0.077) | (0.459) | (0.834) | (0.014) | (0.198) | (0.446) |
| Pattern 7 | 15.552 | 9.636 | 8.270 | 5.823 | 5.420 | 4.859 | 3.603 |
| | (0.236) | (0.078) | (0.344) | (0.673) | (0.016) | (0.161) | (0.379) |
| Pattern 8 | 15.550 | 9.634 | 8.271 | 5.823 | 5.420 | 4.860 | 3.606 |
| | (0.239) | (0.080) | (0.344) | (0.674) | (0.017) | (0.161) | (0.375) |
| Pattern 9 | 15.577 | 9.637 | 8.270 | 5.851 | 5.419 | 4.859 | 3.625 |
| | (0.212) | (0.077) | (0.345) | (0.646) | (0.018) | (0.161) | (0.357) |

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## Appendix

This appendix provides the detail of how we obtain the result in (4.7). First we have

$$\begin{aligned}&\mathbf{E}\left[(\tilde{S}_u^{u,T}-K)^+p(m,a,b,L|Y_m)\right]\\ =\ &\mathbf{E}\Bigg[\left(S_0\exp\left\{R(u,T)-\sigma^2/2)u+\sigma\sqrt{u}Z\right\}-K\right)^+\\ &\times\exp\left\{\frac{-2}{b^2m}\left(\ln\left(S_0\exp\left\{(a-b^2/2)m+b\sqrt{m}Z\right\}\right)-\ln L\right)\ln\left(\frac{S_0}{L}\right)\right\}\Bigg] \qquad (6.1)\end{aligned}$$

where $Z$ is a standard normal variable. Since the expectation is an integral, let us first specify the integral range. From

$$S_0\exp\left\{R(u,T)-\sigma^2/2)u+\sigma\sqrt{u}Z\right\}-K\geq 0 \qquad (6.2)$$

we have

$$Z\geq -d \qquad (6.3)$$

where $d:=\frac{ln(\frac{S_0}{K})+(R(u,T)-\frac{\sigma^2}{2})u}{\sigma\sqrt{u}}$. Let us also define

$$S_u \quad := \quad S_0\exp\left\{(R(u,T)-\sigma^2/2)u+\sigma\sqrt{u}Z\right\} \qquad (6.4)$$

then (6.1) can be written as the difference of

$$\int_{-d}^{\infty}S_u\exp\left\{\frac{-2}{b^2m}\ln\left(\frac{S_0\exp\left\{(a-b^2/2)m+b\sqrt{m}z\right\}}{L}\right)\ln\left(\frac{S_0}{L}\right)\right\}f_Z(z)dz \qquad (6.5)$$

and

$$\int_{-d}^{\infty}K\exp\left\{\frac{-2}{b^2m}\ln\left(\frac{S_0\exp\left\{(a-b^2/2)m+b\sqrt{m}z\right\}}{L}\right)\ln\left(\frac{S_0}{L}\right)\right\}f_Z(z)dz \qquad (6.6)$$

For simplicity, let us define $A:=\frac{-2\ln(S_0/L)}{b^2m}$, then (6.6) can be transformed to

$$K\int_{-d}^{\infty}(S_0/L)^A\exp\left\{A(a-b^2/2)m+Ab\sqrt{m}z\right\}f_Z(z)dz \qquad (6.7)$$

Let us define $D:=A(a-b^2/2)m$ and $E:=Ab\sqrt{m}$, then the above formula becomes

$$\begin{aligned}&K(S_0/L)^Ae^D\textstyle\int_{-d}^{\infty}e^{Ez}f_Z(z)dz &(6.8)\\ &=K(\tfrac{S_0}{L})^A\exp\left\{D+\tfrac{E^2}{2}\right\}N(d+E) &(6.9)\end{aligned}$$

Similarly, (6.5) can be written as

$$S_0(S_0/L)^A e^{(R(u,T)-\sigma^2/2)u} \int_{-d}^{\infty} e^{\sigma\sqrt{u}z} e^{D+Ez} f_Z(z)dz \tag{6.10}$$

$$= (S_0/L)^A S_0 \exp\left\{\left(R(u,T)-\sigma^2/2\right)u + D + F^2/2\right\} N(d+F) \tag{6.11}$$

where $F := \sigma\sqrt{u} + E$. Therefore, (6.1) can finally be expressed as

$$\mathbb{E}\left[\left(\tilde{S}_u^{u,T} - K\right)^+ p\left(m, a, b, L|Y_m\right)\right] =$$

$$\left(\frac{S_0}{L}\right)^A S_0 N\left(d+F\right) \exp\left\{\left(R(u,T) - \frac{\sigma^2}{2}\right)u + D + \frac{F^2}{2}\right\} - \left(\frac{S_0}{L}\right)^A KN\left(d+E\right)\exp\left\{D + \frac{E^2}{2}\right\}, \tag{6.12}$$

where

$$\begin{cases} d = \frac{\ln(S_0/K)+\left(R(u,T)-\sigma^2/2\right)u}{\sigma\sqrt{u}}, \\ A = \frac{-2\ln(S_0/L)}{b^2 m}, \\ D = Am\left(a - \frac{b^2}{2}\right), \\ E = Ab\sqrt{m}, \\ F = \sigma\sqrt{u} + E. \end{cases} \tag{6.13}$$

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