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A General Control Variate Method for Lévy Models in Finance

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Abstract

This study proposes a new control variate method for Lévy models in finance. Our method generates a process of the control variate whose initial and terminal values coincide with those of the target Lévy model process, with both processes being driven by the same Brownian motion in the simulation. These features efficiently reduce the variance of the Monte Carlo simulation. As a typical application of this method, we provide the calculation scheme for pricing path-dependent exotic options.

We use numerical experiments to examine the validity of our method for both continuously and discretely monitored path-dependent options under variance gamma and normal inverse Gaussian models.

Keywords: Lévy processes; Control variate; Monte Carlo simulation; Path-dependent options

1 Introduction

While the Monte Carlo method is a powerful technique to estimate the expectations under various models, its calculation cost is a burden. In recent years, many methods have been developed to reduce variance. One of the most well-known methods is the control variate method, which uses a process of the control variate and estimates the difference between target values and control variate values. Identifying a good control variate is an important issue for reducing the variance. The control variate method is widely used in many areas, and a typical application is option pricing in finance.

In finance, the Black-Scholes model is the most common model, and many types of exotic options prices under the model are known. However, the Black-Scholes model cannot express the skew and smile structures of implied volatilities and practitioners use other models to evaluate exotic options. Lévy models are one of these models.

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Lévy models can express the skew and smile of the implied volatilities and include asset price jumps that are observed in real markets. Moreover, since Lévy processes have a closed form characteristic function, they are well developed mathematically.

However, the exotic options prices under Lévy models are not easily calculated and are examined in many articles. As examples of analytical calculations, Feng and Linetsky [9] used Fourier transform and Hilbert transform to price discretely monitored exotic options. Umezawa and Yamazaki [21] used a multivariate characteristic function to derive semi-analytical formulas for discretely monitored path-dependent options. Miyachi et al. [18] show approximation formulas for barrier and lookback options prices. While the approximation formula is useful to quote many options in a short time, the Monte Carlo method is required to obtain the accurate values that are used, for example, for competitive bidding or evaluating the closing values of traders' position.

There are mainly two types of problem regarding the Monte Carlo method under Lévy models. One is the number of time steps, which causes the discretization error bias. In particular, the discretization error of the barrier and lookback options is not small. Ribeiro and Webber [19] used Brownian bridge techniques to reduce the error for pricing these options. The other is the variance of the simulation. One of the variance reduction methods is the quasi-Monte Carlo (QMC) method. Avramidis and L'Ecuyer [2] examined the QMC method for the average, lookback, and barrier options under the variance gamma (VG) model with the gamma sampling method. Xie et al. [22] provided a method for discretely monitored barrier options with an importance sampling method.

Another variance reduction technique is the "control variate method." There are many studies of the control variate method for options whose underlying asset is expressed as a sum of asset prices (e.g., average, multi-asset options). For example, Caldana and Fusai [5], Caldana et al. [6], Dingeç et al. [8], Fusai and Meucci [10], and Fusai and Kyriakou [11] studied these options and obtained efficient results. Shiraya and Takahashi [20] studied local stochastic volatility models. On the other hand, reducing the variance for options related to the maximum or minimum value of the path (e.g., barrier options) is a challenge. Representative research of the control variate method for such path-dependent options is that of Dingeç and Hörmann [7]. They successfully reduced the variance for discretely monitored average, lookback, and barrier options using a control variate with the rapid numerical inversion of the cumulative distribution functions.

In this study, we propose a new control variate method under the processes expressed with subordinated Brownian motion, which are typical models in Lévy processes. The important factor that distinguishes our method is that it generates a process in which the initial and terminal values coincide with those of the original process, and these processes are driven by the same Brownian motion in the simulation. Our method works efficiently not only for discretely monitored but also continuously monitored path-dependent options even when the payoff depends on the maximum or minimum value of the path.

The rest of this paper is organized as follows. Section 2 explains the basic structure of the control variate method and Lévy models in finance. Section 3 presents the scheme of our new control variate method. Section 4 shows numerical experiments for the VG process (Madan and Seneta [15]) and the normal inverse Gaussian (NIG) process (Barndorff-Nielsen [3]).

2 Preliminary

2.1 Control variate method

First, we briefly explain the basic structure of the control variate method (e.g., see Glasserman [12] for the details).

Let Y and Z be random variates, and the n-th sample from Y and Z are set as Y^n and Z^n, respectively. To obtain $\mathbf{E}[Y]$, we use Z as a control variate and assume the expectation $\mathbf{E}[Z]$ is known. We define $\overline{Y}$ and $\overline{Z}$ as the sample means of Y and Z, that is, $\overline{Y} = \frac{1}{N}\sum_{n=1}^{N} Y^n$, $\overline{Z} = \frac{1}{N}\sum_{n=1}^{N} Z^n$ where N is the sampling number. Here, $\overline{Y}$ is an estimator of $\mathbf{E}[Y]$ and is simply calculated using the Monte Carlo method.

Next, we define δ as

$$\delta := \frac{\sum_{n=1}^{N}(Y^n - \overline{Y})(Z^n - \overline{Z})}{\sum_{n=1}^{N}(Z^n - \overline{Z})^2}, \tag{1}$$

and $\tilde{Y}$ is defined as an estimator of $\mathbf{E}[Y]$ using the control variate method calculated as

$$\tilde{Y} := \overline{Y} - \delta\{\overline{Z} - \mathbf{E}[Z]\}. \tag{2}$$

If the correlation between Y and Z is not 0, the variances of $\tilde{Y}$ and $\overline{Y}$ are estimated as

$$\mathrm{Var}[\tilde{Y}] < \mathrm{Var}[\overline{Y}]. \tag{3}$$

If Z is highly correlated with Y, the variance is reduced effectively. This variance reduction method is called the "control variate method."

2.2 Underlying asset price process

Let X be a Lévy process in a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{Q})$. We assume $\mathbf{Q}$ is a risk-neutral measure.

Following the Lévy-Khintchine formula, the characteristic function of X_t is written as

$$\phi_t(u) := \mathbf{E}\left[e^{iuX_t}\right] = e^{t\Psi_X(u)}, \quad u \in \mathbf{R}, \tag{4}$$

where Ψ_X is the characteristic exponent of X:

$$\Psi_X(u) := i\alpha u - \frac{1}{2}\beta^2 u^2 + \int_{\mathbf{R}} \left(e^{iux} - 1 - iux\mathbb{I}_{\{|x|\geq 1\}}(x)\right)\nu(dx), \quad \alpha \in \mathbf{R},\ \beta \geq 0. \tag{5}$$

$(\alpha, \beta^2, \nu(dx))$ is Lévy triplet, which characterizes X.

A class of Lévy processes is expressed by a time-changed Wiener process:

$$X_t = \mu\tau_t + \sigma W_{\tau_t}, \tag{6}$$

where $\mu \in \mathbf{R}$ and $\sigma > 0$ are constants, W is an $(\mathcal{F}_t)$-adapted standard Brownian motion, and τ is an $(\mathcal{F}_t)$-adapted non-decreasing Lévy process independent of W. X defined in

(6) is referred to as the subordinated Brownian motion. We can assume $\mathbf{E}[\tau_t] = t$ for any $t > 0$ without loss of generality.

Hereafter, we deal with subordinated Brownian motions. Under the risk-neutral measure, the underlying asset price process S is described by a geometric Lévy process

$$S_t = S_0 \exp((r - c)t + X_t), \tag{7}$$

where c is defined as

$$e^c = \mathbf{E}[\exp(X_1)], \tag{8}$$

and is obtained from the other parameters.

3 New control variate method

In this section, we explain the simulation scheme of our control variate method with the following steps.

1. Introduce a process of the control variate defined as U.
2. Generate paths of $\hat{S}$ and $\hat{U}$, which are discretized processes of S and U, respectively.
3. Calculate the estimator of the payoff functions with the control variate method.

The important factor that distinguishes our method is that it generates the process $\hat{U}$ in which initial and terminal values coincide with those of $\hat{S}$, and both processes are driven by the same Brownian motion.

3.1 Proxy process

To use the control variate method, we introduce a process of the control variate that follows a geometric Brownian motion model. Hereafter, we refer to the process of the control variate as the "proxy process."

Let S be a geometric Lévy model expressed with a subordinated Brownian motion, and T be the maturity of an exotic option. We express τ_T as the time-changed maturity. We also set U as the proxy process, which follows a Black-Scholes asset price process with the maturity τ_T. Then, S and U are written as

$$\ln \frac{S_t}{S_0} = (r - c)t + \mu\tau_t + \sigma W_{\tau_t} \quad t \in [0, T], \tag{9}$$

$$\ln \frac{U_t}{U_0} = \left(R(\tau_T) - \frac{\sigma^2}{2}\right)t + \sigma W_t \quad t \in [0, \tau_T], \tag{10}$$

where $S_0 = U_0$, and $R(\tau_T)$ is a pseudo risk-free rate defined as

$$R(\tau_T) = (r - c)\frac{T}{\tau_T} + \mu + \frac{\sigma^2}{2}. \tag{11}$$

By substituting (11) into (10), we obtain

$$\ln \frac{U_t}{U_0} = (r-c)\frac{T}{\tau_T} t + \mu t + \sigma W_t. \tag{12}$$

In our simulation, while the path of S is generated until the maturity T, the path of U is generated until τ_T.

3.2 Path generation

We provide the procedure of path generation for our control variate method in the following steps.

1. Divide the maturity T into M steps, and set t_m as

$$t_m = m\frac{T}{M} \quad (m = 0, 1, 2, \cdots, M). \tag{13}$$

2. Generate discretized time change process $\hat{\tau}^S$, which is used for generating $\hat{S}$. Set g_m as the m-th increment of $\hat{\tau}^S$. That is,

$$\hat{\tau}^S_{m+1} = \hat{\tau}^S_m + g_{m+1} \quad (m = 0, 1, 2, \cdots, M-1), \tag{14}$$

where $\hat{\tau}^S_m$ satisfies

$$\mathbf{E}[\hat{\tau}^S_m] = t_m. \tag{15}$$

We show g_m of a VG process and an NIG process as examples. Set κ as a constant parameter satisfying

$$\mathrm{Var}[\tau_t] = \kappa t. \tag{16}$$

- VG process

$$g_m \sim \mathrm{G}\left(\frac{t_{m+1} - t_m}{\kappa}, \kappa\right), \tag{17}$$

where $\mathrm{G}(a, b)$ denotes the gamma distribution with shape parameter a and rate parameter b, and the probability density function of the gamma distribution is expressed as

$$f(x) = \frac{\kappa^{\frac{t_{m+1}-t_m}{\kappa}}}{\Gamma\left(\frac{t_{m+1}-t_m}{\kappa}\right)} x^{\frac{t_{m+1}-t_m}{\kappa}-1} e^{-\kappa x}, \tag{18}$$

where $\Gamma(x)$ is the gamma function.

- NIG process

$$g_m \sim \mathrm{IG}\left(t_{m+1} - t_m, \frac{(t_{m+1} - t_m)^2}{\kappa}\right). \tag{19}$$

where $\mathrm{IG}(a, b)$ denotes the inverse Gaussian distribution with mean parameter a and shape parameter b. The probability density function of the inverse Gaussian distribution is expressed as

$$f(x) = \sqrt{\frac{(t_{m+1} - t_m)^2}{2\pi\kappa x^3}} \exp\left\{\frac{-(x - t_{m+1} + t_m)^2}{2\kappa x}\right\}. \tag{20}$$

3. Divide $\hat{\tau}^U_M$ into M steps and set $\hat{\tau}^U$, which is used for generating $\hat{U}$, as

$$\hat{\tau}^U_m = m\frac{\hat{\tau}^S_M}{M} \quad (m = 0, 1, 2, \cdots, M). \tag{21}$$

4. Sort the sequence

$$\{\hat{\tau}^S_0, \hat{\tau}^S_1, \cdots, \hat{\tau}^S_M, \hat{\tau}^U_1, \cdots, \hat{\tau}^U_{M-1}\}, \tag{22}$$

in ascending order. Here, we eliminate $\hat{\tau}^U_0, \hat{\tau}^U_M$ because

$$\hat{\tau}^S_0 = \hat{\tau}^U_0, \tag{23}$$
$$\hat{\tau}^S_M = \hat{\tau}^U_M. \tag{24}$$

We then relabel the elements of the sorted sequence of (22) as

$$\{\hat{\tau}_0, \hat{\tau}_1, \cdots, \hat{\tau}_{2M-1}\}. \tag{25}$$

5. Generate a discretized standard Brownian motion $\hat{W}$.

We generate $2M-1$ standard normal random numbers $z_0, z_1, \cdots, z_{2M-2}$ and generate discretized standard Brownian motion $\hat{W}$ as

$$\hat{W}_0 = 0, \tag{26}$$
$$\hat{W}_{\hat{\tau}_{l+1}} = \hat{W}_{\hat{\tau}_l} + z_l\sqrt{\hat{\tau}_{l+1} - \hat{\tau}_l} \quad (l = 0, 1, \cdots, 2M-2). \tag{27}$$

6. Generate $\hat{S}$ and $\hat{U}$.

For $m = 0, 1, 2, \cdots, M$, $\hat{S}$ and $\hat{U}$ are obtained as

$$\hat{S}_{t_m} = S_0 \exp((r-c)t_m + \mu\hat{\tau}^S_m + \sigma\hat{W}_{\hat{\tau}^S_m}), \tag{28}$$
$$\hat{U}_{\hat{\tau}^U_m} = S_0 \exp\left((r-c)\frac{T}{\tau_T}\hat{\tau}^U_m + \mu\hat{\tau}^U_m + \sigma\hat{W}_{\hat{\tau}^U_m}\right). \tag{29}$$

We note that $\hat{S}$ and $\hat{U}$ are driven by the same discretized Brownian motion $\hat{W}$. In addition, since $\hat{\tau}^S_M = \hat{\tau}^U_M$, $\hat{S}_T = \hat{U}_{\tau_T}$ holds.

If the payoff is expressed as a function of $(S_T - K)_+$, and $\hat{S}_T$ is out of the money, we can omit generating $\hat{U}$ because of $\hat{S}_T = \hat{U}_{\tau_T}$.

Regarding the crude Monte Carlo method, we obtain the procedure in a similar manner to the above steps. That is, we omit Steps 3 and 4 with setting $\hat{\tau}_m = \hat{\tau}^S_m$ $(m = 1, \cdots, M)$, and generate only $\hat{S}_{t_m}$ in Step 6.

Remark 1. *To generate $\hat{W}_{\hat{\tau}^S_m}$ and $\hat{W}_{\hat{\tau}^U_m}$, the typical method is*

$$\hat{W}_{\hat{\tau}^S_m} = \hat{W}_{\hat{\tau}^S_{m-1}} + z_{m-1}\sqrt{\hat{\tau}^S_m - \hat{\tau}^S_{m-1}}, \tag{30}$$
$$\hat{W}_{\hat{\tau}^U_m} = \hat{W}_{\hat{\tau}^U_{m-1}} + z_{m-1}\sqrt{\hat{\tau}^U_m - \hat{\tau}^U_{m-1}}, \tag{31}$$

, which uses the same z_{m-1} for generating the m-th values of Brownian motions for both processes. However, the terminal prices of $\hat{S}$ and $\hat{U}$ generated by this method do not coincide, and the control variate does not work efficiently in most cases.

To obtain highly correlated paths, Dingeç and Hörmann [7] use the same uniform random number with inverse functions and do not generate the time changed series $\{\hat{\tau}_0, \cdots, \hat{\tau}_M\}$. On the other hand, we generate the time changed series $\{\hat{\tau}_0, \cdots, \hat{\tau}_M\}$. While generating $\{\hat{\tau}_0, \cdots, \hat{\tau}_M\}$ creates additional computational cost, we can use the same Brownian motion for both S and U and obtain not only highly correlated paths but also the same terminal values. This feature produces another benefit. If the sequence $\{\hat{\tau}^S_0, \dots, \hat{\tau}^S_M\}$ is uniformly located in the interval $[0, \hat{\tau}^S_M]$ to a certain extent, a trajectory of $\hat{S}$ covaries with that of $\hat{U}$ due to the path generation. As a result, each value of the control variate process comes close to that of the original price process. This is the case that parameter κ is not so large.

In addition, since we generate the time changed series $\{\hat{\tau}_0, \cdots, \hat{\tau}_M\}$, we can easily apply our method to continuously monitored exotic options using the discretization error reduction technique introduced by Ribeiro and Webber [19] for the original process.

3.3 Estimator of payoff with control variate method

After generating N sample paths with the above scheme, we calculate the estimator of the payoff function. For the maturity t, let $F(S, t)$, $G(\hat{S}, t)$ and $H(\hat{U}, t)$ be payoff functions whose underlying asset price processes are the original process S and discretized processes of S and U, respectively. Then, the expectation of payoff $\mathbf{E}[F(S, T)]$ with the control variate method is estimated as

$$\mathbf{E}[F(S, T)] \approx \frac{1}{N}\sum_{n=1}^{N} G(\hat{S}^n, T) + \delta\left\{\frac{1}{N}\sum_{n=1}^{N} H(\hat{U}^n, \hat{\tau}^n_T) - \bar{V}_U(T)\right\}. \tag{32}$$

Here, δ is defined as

$$\delta = -\frac{\sum_{n=1}^{N}(G(\hat{S}^n, T) - \frac{1}{N}\sum_{n=1}^{N} G(\hat{S}^n, T))(H(\hat{U}^n, \hat{\tau}^n_T) - \frac{1}{N}\sum_{n=1}^{N} H(\hat{U}^n, \hat{\tau}^n_T))}{\sum_{n=1}^{N}(H(\hat{U}^n, \hat{\tau}^n_T) - \frac{1}{N}\sum_{n=1}^{N} H(\hat{U}^n, \hat{\tau}^n_T))^2}, \tag{33}$$

where $\hat{\tau}^n$, $\hat{S}^n$, and $\hat{U}^n$ are the n-th simulated values of $\hat{\tau}$, $\hat{S}$, and $\hat{U}$, respectively. $\bar{V}_U(T)$ is the option premium under the process U with the maturity T, and the value is obtained as follows: Firstly, we define $V_U(R(\tau), \tau_T)$ as

$$V_U(R(\tau), \tau_T) := \mathbf{E}[I(U, \tau_T) | \tau_T]. \tag{34}$$

where $I(U, \tau_T)$ is a payoff function with the underlying asset price process U and the maturity τ_T. The difference between $I(U, \tau_T)$ and $H(\hat{U}, \tau_T)$ is that while $I(U, \tau_T)$ is a continuously monitored payoff with the continuous process U, $H(\hat{U}, \tau_T)$ is a continuously monitored payoff with the discretized process $\hat{U}$. Then, for a fixed τ_T, V_U is obtained by a formula for the Black-Scholes model, and the analytical expression of exotic options prices is shown in Hull [13], for example.

Next, we assume $f_{\tau_T}(\cdot)$ as the probability density function of τ_T. Then, the expected value ($\bar{V}_U(T)$) of $V_U(R(\tau), \tau_T)$ is obtained as

$$\begin{aligned}\bar{V}_U(T) &= \mathbf{E}[V_U(R(\tau_T), \tau_T)] \\ &= \int_0^\infty V_U(R(y), y) f_{\tau_T}(y) dy,\end{aligned} \tag{35}$$

Here, $\bar{V}_U(T)$ is easily obtained through numerical integration. $\bar{V}_U(T)$ is partially considered in Miyachi et al. [18] for approximating exotic options prices.

4 Numerical example

In this section, we examine the validity of our method for up and out call, floating strike lookback put, and average call options under VG and NIG models. We calculate non-discounted option premiums using a personal computer Core i3 8100B, 8 GB RAM, Xcode C++.

To evaluate the efficiency of our method, we use the variance reduction factor (VRF) and efficiency factor (EF), which are defined as follows.

$$\begin{aligned}\text{VRF} &= \frac{\text{(Variance of the sample mean with crude Monte Carlo method)}}{\text{(Variance of the sample mean with the new control variate method)}} \\ \text{EF} &= \text{VRF} \times \frac{\text{(Calculation time of crude Monte Carlo method)}}{\text{(Calculation time of the new control variate method)}}.\end{aligned}$$

To generate random numbers of the gamma distribution and the inverse Gaussian distribution, we follow Ahrens and Dieter [1], Marsaglia and Tsang [16], and Michael et al. [17].

The parameters used in the numerical example are presented in Table 1.

	μ	σ	κ	r
VG	-0.1306	0.1594	0.0018	0.05
NIG	-0.1482	0.1597	0.0023	0.05

Table 1: The base parameters

These parameters are the same as those used in Dingeç and Hörmann [7] (obtained in Luethi and Breymann [14]). Since their parameterization of the models is different from ours, we convert them as $\kappa = \frac{1}{250\lambda}$, $\mu = \frac{500\lambda\beta}{\gamma^2}$, and $\sigma = \frac{\sqrt{500\lambda}}{\gamma^2}$ for the VG model, and $\kappa = \frac{1}{250\delta\gamma}$, $\mu = \frac{250\delta\beta}{\gamma}$, and $\sigma = \sqrt{\frac{250\delta}{\gamma}}$ for the NIG model. The constant c in the drift term is calculated from these parameters as $c = -\frac{1}{\kappa}\ln(1 - \mu\kappa - \frac{1}{2}\sigma^2\kappa)$ for the VG model, and $c = \frac{1-\sqrt{1-2\kappa\mu-\kappa\sigma^2}}{\kappa}$ for the NIG model.

The number of paths is 100,000, and the number of time steps is 250 per year (daily monitoring). We note that the referenced prices of the typical contract of these discretely monitored options are daily monitoring, and non-daily monitored options are almost never dealt with in practice.

To examine the efficiency of only the control variate method, we do not add other speed up techniques without the discretization error reduction method shown in each subsection.

4.1 Discretely monitored path-dependent options

This subsection presents the results of a discretely monitored average call option, floating strike lookback put option, and up and out call option. Since the payoff of our control variate for barrier and lookback options is assumed to be monitored continuously, we use the discretization error reduction technique (e.g., see Ribeiro and Webber [19]) for the proxy process.

First, we explain the payoff of each option.

- Average option

  The payoff of the average call option with the strike price K is expressed as $F(S,T) = (\frac{1}{M}\sum_{m=1}^{M} S_{t_m} - K)_+$, and for the discretized process, the payoff for $\hat{S}$ is $G(\hat{S},T) = (\frac{1}{M}\sum_{m=1}^{M} \hat{S}_{t_m} - K)_+$. Regarding the proxy process, we use a discretely monitored geometric average option as the control variate because the analytical formula of V_U for the discretely monitored version is well known. Thus, the payoff for the proxy process is expressed as

$$H(\hat{U},\tau_T) = \left(\exp\left(\frac{1}{M}\sum_{m=1}^{M}\ln \hat{U}_{\hat{\tau}_m^U}\right) - K\right)_+ . \tag{36}$$

- Lookback option

  The payoffs of the discretely monitored floating strike lookback put option for S and $\hat{S}$ are expressed as $F(S,T) = (\sup_{0\le m\le M} S_{t_m} - S_T)$, $G(\hat{S},T) = (\sup_{0\le m\le M}\hat{S}_{t_m} - \hat{S}_T)$, respectively. On the other hand, since $H(\hat{U},\tau_T)$ is assumed to be continuously monitored, we use the discretization error reduction method. Let $f_M(\cdot,\cdot,\cdot)$ be an approximated maximum function defined as

$$\begin{aligned} &f_M^U(\hat{U},m,I_m)\\ &:= \hat{U}_{\hat{\tau}_m^U}\exp\left(\frac{\ln(\hat{U}_{\hat{\tau}_m^U}\hat{U}_{\hat{\tau}_{m+1}^U}) + \sqrt{(\ln(\hat{U}_{\hat{\tau}_{m+1}^U}/\hat{U}_{\hat{\tau}_m^U}))^2 - 2\sigma^2(\hat{\tau}_{m+1}^U - \hat{\tau}_m^U)\ln(1-I_m)}}{2}\right), \end{aligned} \tag{37}$$

  where $I_m \sim \mathrm{U}(0,1)$ (uniform random number). Then, we use $H(\hat{U},\tau_T)$ set as

$$H(\hat{U},\tau_T) = \left(\sup_{0\le m\le M-1} f_M^U(\hat{U},m,I_m) - \hat{U}_{\tau_T}\right)_+ , \tag{38}$$

  as the payoff function for $\hat{U}$. See Ribeiro and Webber [19] for the details, for example.

- Barrier option

  The payoffs of the discretely monitored up and out call option for S and $\hat{S}$ with the barrier level B and the strike price K are expressed as $F(S,T) = 1_{\sup_{0\le m\le M} S_{t_m}<B}(S_T - K)_+$, $G(\hat{S},T) = 1_{\sup_{0\le m\le M} \hat{S}_{t_m}<B}(\hat{S}_T - K)_+$. Similar to the lookback option, we adjust the payoff of $\hat{U}$ as continuous monitoring.

  Firstly, let $M^U_{\hat{\tau}^U_m,\hat{\tau}^U_{m+1}} := \sup_{\hat{\tau}^U_m\le t\le\hat{\tau}^U_{m+1}} U_t$, and consider the case of $\hat{U}_{\hat{\tau}^U_m} < B$, $\hat{U}_{\hat{\tau}^U_{m+1}} < B$. Following Ribeiro and Webber [19], the conditional probability that $M^U_{\hat{\tau}^U_m,\hat{\tau}^U_{m+1}}$ reaches B is approximated as

$$\mathbf{P}[M^U_{\hat{\tau}^U_m,\hat{\tau}^U_{m+1}} \ge B|\hat{U}_{\hat{\tau}^U_m}, \hat{U}_{\hat{\tau}^U_{m+1}}] \approx \exp\left(-2\frac{(\ln B - \ln \hat{U}_{\hat{\tau}^U_m})(\ln B - \ln \hat{U}_{\hat{\tau}^U_{m+1}})}{\sigma^2(\hat{\tau}^U_{m+1} - \hat{\tau}^U_m)}\right). \quad (39)$$

  Here, we set

$$p_{\hat{U}}(B,m) := \exp\left(-2\frac{(\ln B - \ln \hat{U}_{\hat{\tau}^U_m})(\ln B - \ln \hat{U}_{\hat{\tau}^U_{m+1}})}{\sigma^2(\hat{\tau}^U_{m+1} - \hat{\tau}^U_m)}\right). \quad (40)$$

  Then, the payoff of the continuously monitored up and out call option for U is approximated as

$$(\hat{U}_{\tau_T} - K)_+ \mathbf{1}_{\sup_{0\le m\le M} \hat{U}_{\tau_m}<B} \prod_{m=0}^{M-1}(1 - p_{\hat{U}}(B,m)). \quad (41)$$

  In addition, we adjust the barrier level of the continuously monitored up and out call option to that of discretely monitored by a method proposed in Broadie et al. [4]. Let B be a barrier level for a discretely monitored barrier option, and Δt be a monitoring interval ($\Delta t = t_1 - t_0$). Then, the adjusted barrier level B_d for the continuous barrier option is written as

$$B_d := B\exp(0.5826\sigma\sqrt{\Delta t}). \quad (42)$$

  Using (41) and (42), the adjusted payoff for $\hat{U}$ is set as

$$H(\hat{U},\tau_T) = (\hat{U}_{\tau_T} - K)_+ \mathbf{1}_{\sup_{0\le m\le M} \hat{U}_{\tau_m}<B_d} \prod_{m=0}^{M-1}(1 - p_{\hat{U}}(B_d,m)). \quad (43)$$

With the above settings, we compare our results to those of Dingeç and Hörmann [7]. We examine $T = 1$, 2, and 3 with $K = 70$, 100, and 130 cases for the average option, $T = 1$, 2, and 3 for the lookback option, and $T = 1$ with $B = 150$, $T = 2$ with $B = 170$, and $T = 3$ with $B = 200$ for the barrier option. The strike price for barrier options is $K = 100$. The results are presented in Tables 2 and 3.

Option		T	Price	VRF		EF	Error		Time	
				CV	DH	CV	MC	CV	MC	CV
Average	$K = 70$	1	32.551	648	88	473	0.030	0.001	4.3	5.9
		2	35.183	609	82	442	0.044	0.002	8.4	11.6
		3	37.907	480	79	350	0.056	0.003	12.6	17.3
	$K = 100$	1	5.156	686	66	492	0.021	0.001	4.2	5.9
		2	8.354	613	68	445	0.033	0.001	8.4	11.6
		3	11.358	467	66	340	0.044	0.002	12.6	17.3
	$K = 130$	1	0.022	31	23	22	0.001	0.000	4.2	5.9
		2	0.387	79	31	57	0.007	0.001	8.4	11.6
		3	1.267	101	27	74	0.016	0.002	12.6	17.3
Lookback		1	10.636	182	52	112	0.026	0.002	3.7	6.0
		2	14.867	231	50	143	0.035	0.002	7.3	11.8
		3	18.129	247	50	154	0.043	0.003	10.9	17.5
Barrier	$B = 150$	1	8.479	90	15	57	0.035	0.004	3.7	5.8
	$B = 170$	2	12.906	74	9	47	0.051	0.006	7.3	11.3
	$B = 200$	3	18.511	98	13	63	0.071	0.007	10.9	16.8

Table 2: VG model results compared with Table 2 in Dingeç and Hörmann [7]

Option		T	Price	VRF		EF	Error		Time	
				CV	DH	CV	MC	CV	MC	CV
Average	$K = 70$	1	32.549	521	67	324	0.030	0.001	2.9	4.7
		2	35.179	520	66	327	0.044	0.002	5.7	9.1
		3	37.908	434	64	273	0.056	0.003	8.5	13.6
	$K = 100$	1	5.163	570	53	354	0.021	0.001	2.9	4.7
		2	8.364	533	51	334	0.033	0.001	5.7	9.1
		3	11.371	427	52	269	0.044	0.002	8.5	13.6
	$K = 130$	1	0.022	35	11	21	0.001	0.000	2.9	4.7
		2	0.392	70	18	44	0.007	0.001	5.7	9.1
		3	1.276	98	21	61	0.016	0.002	8.5	13.6
Lookback		1	10.657	158	41	77	0.026	0.002	2.3	4.7
		2	14.901	187	39	92	0.035	0.003	4.6	9.3
		3	18.179	193	39	96	0.043	0.003	6.8	13.7
Barrier	$B = 150$	1	8.481	66	13	34	0.035	0.004	2.3	4.5
	$B = 170$	2	12.916	68	8	35	0.051	0.006	4.5	8.8
	$B = 200$	3	18.494	89	8	46	0.070	0.007	6.8	13.1

Table 3: NIG model results compared with Table 4 in Dingeç and Hörmann [7]

The “Option” column shows the types of exotic options. “Average,” “Lookback,” and “Barrier” indicate average call, floating strike lookback put, and up and out barrier call options, respectively. The “T” column is the maturity of the option. “Price” shows the non-discounted option premium obtained by our new control variate method. “DH” shows the results taken from Tables 2 and 4 in Dingeç and Hörmann [7]. “CV” shows the results of our new control variate method. “Error” shows the 68.27% error bound of the simulation, and “Time” shows the calculation time.

Our new method reduces variances very efficiently. While the EFs of Dingeç and Hörmann [7] are not presented in their study, the EFs of our method are several times as large as those of Dingeç and Hörmann [7] considering the computational time ratio presented in their study. While EFs in $K = 130$ of the average option do not appear relatively large, the error itself is very small, and our method works efficiently enough.

For barrier option, if the proxy process hits the barrier and the original process does not, the difference in payoffs is $(S_T - K)_+$, which is much larger than that of other options. This is the reason the control variate method for the barrier option is harder than that for other options. However, our method successfully reduces the variance even in the case of up and out call options.

4.2 Continuously monitored path-dependent options

This subsection examines our method for a continuously monitored floating strike lookback put option and up and out call option.

Since the discretization bias of the Monte Carlo method is not negligible for pricing continuously monitored barrier and lookback options (please compare the values of “Price” in Tables 2 and 3 with those in Tables 4 and 5, respectively), we also use the discretization error reduction technique of Ribeiro and Webber [19] for the original process. We note that to use this technique, time series $\{\hat{\tau}_0^S, \cdots, \hat{\tau}_M^S\}$ is necessary.

Similar to Section 4.1, we explain the payoffs of each continuously monitored option.

- Lookback option

  The payoff of the continuously monitored floating strike lookback put option for S is defined as $F(S, T) = (\sup_{0 \le t \le T} S_t - S_T)$. Regarding $G(\hat{S}, T)$ and $H(\hat{U}, \tau_T)$, we use the same adjustment method in Section 4.1, and the payoffs are defined as

$$G(\hat{S}, T) = \left(\sup_{0 \le m \le M-1} f_M^S(\hat{S}, m, I_m) - \hat{S}_T \right)_+ , \tag{44}$$

$$H(\hat{U}, \tau_T) = \left(\sup_{0 \le m \le M-1} f_M^U(\hat{U}, m, I_m) - \hat{U}_{\tau_T} \right)_+ , \tag{45}$$

  where $f_M^U(\hat{U}, m, I_m)$ is the same as that in Section 4.1, and

$$f_M^S(\hat{S}, m, I_m) := \hat{S}_{t_m} \exp\left(\frac{\ln(\hat{S}_{t_m}\hat{S}_{t_{m+1}}) + \sqrt{(\ln(\hat{S}_{t_{m+1}}/\hat{S}_{t_m}))^2 - 2\sigma^2(\hat{\tau}_{t_{m+1}} - \hat{\tau}_{t_m})\ln(1 - I_m)}}{2} \right), \tag{46}$$

- Barrier option

The payoff of the continuously monitored up and out call option for S is defined as $F(S,T) = \mathbf{1}_{\sup_{0\le t\le T} S_t < B}(S_T - K)_+$. The payoffs for $\hat{S}$ and $\hat{U}$ are also set as

$$G(\hat{S}, \tau_T) = (\hat{S}_{\tau_T} - K)_+ \mathbf{1}_{\sup_{0\le m\le M} \hat{S}_{\tau_m} < B} \prod_{m=0}^{M-1} (1 - p_{\hat{S}}(B,m)), \tag{47}$$

$$H(\hat{U}, \tau_T) = (\hat{U}_{\tau_T} - K)_+ \mathbf{1}_{\sup_{0\le m\le M} \hat{U}_{\tau_m} < B} \prod_{m=0}^{M-1} (1 - p_{\hat{U}}(B,m)). \tag{48}$$

where $p_{\hat{U}}(B,m)$ is the same as that in Section 4.1, and

$$p_{\hat{S}}(B,m) := \exp\left(-2\frac{(\ln B - \ln \hat{S}_{t_m})(\ln B - \ln \hat{S}_{t_{m+1}})}{\sigma^2(\hat{\tau}^S_{t_{m+1}} - \hat{\tau}^S_{t_m})}\right). \tag{49}$$

The parameters are the same as those in Table 1, the number of time steps is 250, and the number of simulations is 100,000. The strike price for barrier options is K = 100.

The results of the VG and NIG processes are shown in Tables 4 and 5, respectively.

Option		T	Price	VRF	EF	Error		Time	
				CV	CV	MC	CV	MC	CV
Lookback		1	11.361	186	130	0.026	0.002	4.3	6.1
		2	15.657	232	163	0.035	0.002	8.5	12.0
		3	18.980	247	175	0.043	0.003	12.6	17.8
Barrier	B = 150	1	8.402	207	150	0.034	0.002	4.2	5.8
	B = 170	2	12.752	168	122	0.050	0.004	8.3	11.4
	B = 200	3	18.357	193	141	0.070	0.005	12.4	16.9

Table 4: VRFs and EFs for continuously monitored options (VG)

Option		T	Price	VRF	EF	Error		Time	
				CV	CV	MC	CV	MC	CV
Lookback		1	11.394	156	91	0.026	0.002	2.9	4.9
		2	15.706	184	109	0.035	0.003	5.7	9.7
		3	19.045	191	115	0.043	0.003	8.6	14.2
Barrier	B = 150	1	8.400	192	120	0.034	0.002	2.9	4.6
	B = 170	2	12.766	135	85	0.050	0.004	5.6	9.0
	B = 200	3	18.342	133	85	0.070	0.006	8.4	13.2

Table 5: VRFs and EFs for continuously monitored options (NIG)

Even in continuously monitored cases, our new control variate method works efficiently. Regarding the continuously monitored up and out call option, even if one process hits the barrier, and the other does not, the difference is small because the adjustment of barrier hitting probability (40) makes the payoff value small. Thus, EFs of the up and out call option are higher than those of the discretely monitored one.

5 Conclusion

We proposed a new control variate method for path dependent simulations under Lévy models and applied it not only to discretely monitored but also continuously monitored exotic options pricing. Since the initial and terminal values of the process of the control variate coincide with those of the original Lévy process, and both the processes are driven by the same Brownian motion in the simulation, our method works efficiently for reducing the variance. We also confirmed the validity of our method with the numerical examples for pricing path-dependent options.

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