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Probabilistic Approach to Mean Field Games and Mean Field Type Control Problems with Multiple Populations *

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Abstract

In this work, we systematically investigate mean field games and mean field type control problems with multiple populations using a coupled system of forward-backward stochastic differential equations of McKean-Vlasov type stemming from Pontryagin's stochastic maximum principle. Although the same cost functions as well as the coefficient functions of the state dynamics are shared among the agents within each population, they can be different population by population. We study the mean field limit for the three different situations; (i) every agent is non-cooperative; (ii) the agents within each population are cooperative; and (iii) the agents in some populations are cooperative but those in the other populations are not. We provide several sets of sufficient conditions for the existence of a mean field equilibrium for each of these cases. Furthermore, under appropriate conditions, we show that the mean field solution to each of these problems actually provides an approximate Nash equilibrium for the corresponding game with a large but finite number of agents.

Keywords : mean field game, mean field type control, FBSDE of McKean-Vlasov type

1 Introduction

In pioneering works of Lasry & Lions [26, 27, 28] and Huang, Malhame & Caines [22], the two groups of researchers independently proposed a powerful technique to produce an approximate Nash equilibrium for stochastic differential games among a large number of agents with symmetric interactions. Importantly, each agent is assumed to be affected by the states of the other agents only through their empirical distribution. In the large population limit, the problem is shown to result in two highly coupled nonlinear partial differential equations (PDEs), the one is of the Hamilton-Jacobi-Bellman type, which takes care of the optimization problem, while the other is of the Kolmogorov type guaranteeing the consistent time evolution of the distribution of the individual states of the agents. The greatest benefit of the mean-field game approach is to render notoriously intractable problems of stochastic differential games among many agents into simpler stochastic optimal control problems. For details of the analytical approach and its various applications, one may consult the monographs by Bensoussan, Frehse & Yam [2], Gomes, Nurbekyan & Pimentel [20], Gomes, Pimentel & Voskanyan [21] and also Kolokoltsov & Malafeyev [24].

*All the contents expressed in this research are solely those of the author and do not represent any views or opinions of any institutions. The author is not responsible or liable in any manner for any losses and/or damages caused by the use of any contents in this research.

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In a series of works [7, 8, 9], Carmona & Delarue developed a probabilistic approach to these problems, where forward-backward stochastic differential equations (FBSDEs) of McKean-Vlasov type instead of PDEs were shown to be the relevant objects for investigation. In particular, they provided the sufficient conditions for the existence of an equilibrium for mean field games with the cost functions of quadratic growth in [8]. In the case of cooperative agents who adopt the common feedback control function, they showed in [9] that the large population limit results in the optimization problem with respect to a controlled McKean-Vlasov SDE. Using the notion of so-called L-derivative, which is a type of differential for functions defined on the space of probability measures, they solved the problem by a new class of FBSDEs of McKean-Vlasov type. A probabilistic but weak formulation of the mean-field games was studied in Carmona & Lacker [12] and, in particular, in Carmona, Delarue & Lacker [13] in the presence of common noise. The details of probabilistic approaches, concrete examples, and many references for various applications are available in the recent two volumes of monograph [10, 11].

In this work, we are interested in mean field games and mean field type control problems in the presence of multiple populations. Here, the same cost functions as well as the coefficient functions of the state dynamics are shared among the agents *within* each population, but they can be different population by population. Mean field games with multiple populations arise naturally in most of the practical applications, and were already considered in the first original work of [22]. Lachapelle & Wolfram [25] modeled a congestion problem of pedestrian crowds, and Achdou, Bardi & Cirant [1] studied the issue of urban settlements and residential choice using the mean-field game representation. Feleqi [17] and Cirant [14] dealt with ergodic mean field games of multiple populations under different boundary conditions. Recently, Bensoussan, Huang & Lauriere have considered a new type of problem in [3], where the agents within each population are cooperative but compete with those in the other populations. They gave necessary conditions for equilibrium in terms of master equations. Note that, in all of these existing works, the analytic approach based on coupled nonlinear PDEs has been adopted.

In the current paper, differently from the existing works, we have adopted the probabilistic approach and closely followed the procedures developed in [8, 9]. In addition to the mean field games of multiple populations, we have studied the situation where the agents in each population are cooperative as in [3], and yet another situation which is a mixture of the first two cases: the agents in some populations are cooperative within their own but those in the other populations are not. The presence of multiple populations induces a system of FBSDEs of McKean-Vlasov type. Although it is a *coupled* system of FBSDEs due to the interactions among different populations, the couplings appear only through the mean field interactions i.e., the distribution of the state of the representative agent of each population. This feature allows us to solve a matching problem corresponding to the state of equilibrium by Schauder's fixed point theorem in a quite similar manner to [8]. In each of the three cases mentioned above, we have found several sets of sufficient conditions for the existence of an equilibrium, in particular the one which allows the cost functions of quadratic growth both in the state variable as well as in its distribution so that it is applicable to some of the popular linear quadratic problems. Moreover, we have investigated the quantitative relationships between the mean field problems discussed above and those with finite agents. In particular, under additional assumptions, we have proved that each mean field solution actually provides an approximate Nash equilibrium for the corresponding game with a large but finite number of agents. It highlights some interesting differences between the game where all the agents are non-cooperative and the one where the agents are cooperative in some populations.

The organization of the paper is as follows: after explaining notations in Section 2, we study the mean field problems in the first half of the paper; in Section 3 (i) the case of non-cooperative agents, in Section 4 (ii) the case where the agents are cooperative within each population, and in Section 5 (iii) the agents in some populations are cooperative but those in the other populations are not. In the second half of the paper, we investigate the corresponding problem with finite number of agents; we treats in Section 6 the case (i), in Section 7 the case (ii), and finally in Section 8 we treats the case (iii). Although we set the number of populations to two in the main analysis, this is just for notational convenience. We shall see that the analysis can be easily generalized to any finite number of populations. Finally, we conclude in Section 9.

2 Notations

Throughout the paper, we work on some complete probability space $(\Omega,\mathcal{F},\mathbb{P})$ equipped with a right-continuous and complete filtration $\mathbb{F}=(\mathcal{F}_t)_{t\in[0,T]}$ supporting two independent d-dimensional standard Brownian motions $\big(\boldsymbol{W}^1=(W_t^1)_{t\in[0,T]},\boldsymbol{W}^2=(W_t^2)_{t\in[0,T]}\big)$ as well as two independent random variables $\xi^1,\xi^2\in\mathbb{L}^2(\Omega,\mathcal{F}_0,\mathbb{P};\mathbb{R}^d)$. For each $i\in\{1,2\}$, $\mathbb{F}^i:=(\mathcal{F}_t^i)_{t\in[0,T]}\subset\mathbb{F}$ is a complete and right-continuous filtration generated by $(\xi^i,\boldsymbol{W}^i)$. Here, $T>0$ is a given terminal time. To lighten the notation, unless otherwise stated, we use indices i and j specifically to represent an element in $\{1,2\}$, and we always suppose that $j\neq i$ when they appear in the same expression. Moreover, we use the symbol C to represent a general nonnegative constant which may change line by line. When we want to emphasize that C depends only on some specific variables, say a and b, we use the symbol $C(a,b)$. We let $||\cdot||_2$ denote the $\mathbb{L}^2(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^d)$-norm. We use the following notations for frequently encountered spaces:

- $\mathbb{S}^2$ is the set of $\mathbb{R}^d$-valued continuous processes $\boldsymbol{X}$ satisfying

$$||X||_{\mathbb{S}^2}:=\mathbb{E}\big[\sup_{t\in[0,T]}|X_t|^2\big]^{\frac{1}{2}}<\infty\ .$$

- $\mathbb{S}^\infty$ is the set of $\mathbb{R}^d$-valued essentially bounded continuous processes $\boldsymbol{X}$ satisfying

$$||X||_{\mathbb{S}^\infty}:=\big|\big|\sup_{t\in[0,T]}|X_t|\big|\big|_\infty<\infty\ .$$

- $\mathbb{H}^2$ is the set of $\mathbb{R}^{d\times d}$-valued progressively measurable processes Z satisfying

$$||Z||_{\mathbb{H}^2}:=\mathbb{E}\Big[\Big(\int_0^T|Z_t|^2dt\Big)\Big]^{\frac{1}{2}}<\infty\ .$$

- $\mathcal{L}(X)$ denotes the law of a random variable X.
- $\mathcal{M}_f^1(\mathbb{R}^d)$ is the set of finite signed measures μ on $(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d))$ such that $\int_{\mathbb{R}^d}|x|d|\mu|(x)<\infty$.
- $\mathcal{P}(\mathbb{R}^d)$ is the set of probability measures on $(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d))$.
- $\mathcal{P}_p(\mathbb{R}^d)$ with $p\geq 1$ is the subset of $\mathcal{P}(\mathbb{R}^d)$ with finite p-th moment; i.e., the set of $\mu\in\mathcal{P}(\mathbb{R}^d)$ satisfying

$$M_p(\mu):=\Big(\int_{\mathbb{R}^d}|x|^p\mu(dx)\Big)^{\frac{1}{p}}<\infty\ .$$

We always assign $\mathcal{P}_p(\mathbb{R}^d)$ with $(p\geq 1)$ the p-Wasserstein distance W_p, which makes the space $\mathcal{P}_p(\mathbb{R}^d)$ a complete separable metric space. As an important property, for any $\mu,\nu\in\mathcal{P}_p(\mathbb{R}^d)$, we have

$$W_p(\mu,\nu)=\inf\Big\{\mathbb{E}[|X-Y|^p]^{\frac{1}{p}};\mathcal{L}(X)=\mu,\mathcal{L}(Y)=\nu\Big\}\ .$$

For more details, see Chapter 5 in [10] or Chapter 3 in [5].

3 Mean Field Games with Multiple Populations

In this section, we consider a mean-field limit of a game among a large number of non-cooperative agents in the presence of two populations. Here, each agent competes with all the other agents but shares the common cost functions as well as coefficient functions of the state dynamics within each population. As we shall see, extending to the general situation with finite number of populations is straightforward.

3.1 Definition of the Mean Field Problem

Before specifying detailed assumptions, let us formulate the problem of finding an equilibrium in the limiting framework. It proceeds in the following three steps.
(i) Fix any two deterministic flows of probability measures $(\boldsymbol{\mu}^i = (\mu_t^i)_{t\in[0,T]})_{i\in\{1,2\}}$ given on $\mathbb{R}^d$.
(ii) Solve the two optimal control problems

$$\inf_{\boldsymbol{\alpha}^1\in\mathbb{A}_1} J_1^{\boldsymbol{\mu}^1,\boldsymbol{\mu}^2}(\boldsymbol{\alpha}^1), \qquad \inf_{\boldsymbol{\alpha}^2\in\mathbb{A}_2} J_2^{\boldsymbol{\mu}^2,\boldsymbol{\mu}^1}(\boldsymbol{\alpha}^2) \tag{3.1}$$

over some admissible strategies $\mathbb{A}_i$ ($i \in \{1,2\}$), where

$$\begin{aligned} J_1^{\boldsymbol{\mu}^1,\boldsymbol{\mu}^2}(\boldsymbol{\alpha}^1) &:= \mathbb{E}\Big[\int_0^T f_1(t, X_t^1, \mu_t^1, \mu_t^2, \alpha_t^1)dt + g_1(X_T^1, \mu_T^1, \mu_T^2)\Big] ~, \\ J_2^{\boldsymbol{\mu}^2,\boldsymbol{\mu}^1}(\boldsymbol{\alpha}^2) &:= \mathbb{E}\Big[\int_0^T f_2(t, X_t^2, \mu_t^2, \mu_t^1, \alpha_t^2)dt + g_2(X_T^2, \mu_T^2, \mu_T^1)\Big] ~, \end{aligned}$$

subject to the d-dimensional diffusion dynamics:

$$\begin{aligned} dX_t^1 &= b_1(t, X_t^1, \mu_t^1, \mu_t^2, \alpha_t^1)dt + \sigma_1(t, X_t^1, \mu_t^1, \mu_t^2)dW_t^1 ~, \\ dX_t^2 &= b_2(t, X_t^2, \mu_t^2, \mu_t^1, \alpha_t^2)dt + \sigma_2(t, X_t^2, \mu_t^2, \mu_t^1)dW_t^2 ~, \end{aligned}$$

for $t \in [0,T]$ with $\big(X_0^i = \xi^i \in \mathbb{L}^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d)\big)_{1\le i\le 2}$. For each population $i \in \{1,2\}$, we suppose that $\mathbb{A}_i$ is the set of A_i-valued $\mathbb{F}^i$-progressively measurable processes $\boldsymbol{\alpha}^i$ satisfying $\mathbb{E}\int_0^T |\alpha_t^i|^2 dt < \infty$ where $A_i \subset \mathbb{R}^k$ is closed and convex.
(iii) Find a pair of probability flows $(\boldsymbol{\mu}^1, \boldsymbol{\mu}^2)$ as a solution to the matching problem:

$$\forall t \in [0,T], \qquad \mu_t^1 = \mathcal{L}(\hat{X}_t^{1,\boldsymbol{\mu}^1,\boldsymbol{\mu}^2}), \quad \mu_t^2 = \mathcal{L}(\hat{X}_t^{2,\boldsymbol{\mu}^2,\boldsymbol{\mu}^1}) ~, \tag{3.2}$$

where $(\hat{X}^{i,\boldsymbol{\mu}^i,\boldsymbol{\mu}^j})_{i\in\{1,2\},j\neq i}$ are the solutions to the optimal control problems in (ii).

Remark 3.1. *It is just for convenience to use the common dimension d (as well as k for A_i) for both populations.*

3.2 Optimization for given flows of probability measures

The main assumptions in this section are as follows:

Assumption 3.1. (MFG-a) *$L, K \ge 0$ and $\lambda > 0$ are some constants. For $1 \le i \le 2$, the measurable functions $b_i : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)^2 \times A_i \to \mathbb{R}^d$, $\sigma_i : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)^2 \to \mathbb{R}^{d\times d}$, $f_i : [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)^2 \times A_i \to \mathbb{R}$, and $g_i : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)^2 \to \mathbb{R}$ satisfy the following conditions:*
(A1) *The functions b_i and σ_i are affine in (x, α) in the sense that, for any $(t, x, \mu, \nu, \alpha) \in [0,T] \times$*

$\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)^2 \times A_i$,

$$b_i(t,x,\mu,\nu,\alpha) := b_{i,0}(t,\mu,\nu) + b_{i,1}(t,\mu,\nu)x + b_{i,2}(t,\mu,\nu)\alpha \ ,$$
$$\sigma_i(t,x,\mu,\nu) := \sigma_{i,0}(t,\mu,\nu) + \sigma_{i,1}(t,\mu,\nu)x \ ,$$

where $b_{i,0}, b_{i,1}, b_{i,2}, \sigma_{i,0}$ and $\sigma_{i,1}$ defined on $[0,T]\times\mathcal{P}_2(\mathbb{R}^d)^2$ are $\mathbb{R}^d$, $\mathbb{R}^{d\times d}, \mathbb{R}^{d\times k}, \mathbb{R}^{d\times d}$ and $\mathbb{R}^{d\times d\times d}$-valued measurable functions, respectively.
(A2) *For any $t\in[0,T]$, the functions $\mathcal{P}_2(\mathbb{R}^d)^2 \ni (\mu,\nu) \mapsto (b_{i,0}, b_{i,1}, b_{i,2}, \sigma_{i,0}, \sigma_{i,1})(t,\mu,\nu)$ are continuous in W_2-distance. Moreover for any $(t,\mu,\nu)\in[0,T]\times\mathcal{P}_2(\mathbb{R}^d)^2$,*

$$|b_{i,0}(t,\mu,\nu)|, |\sigma_{i,0}(t,\mu,\nu)| \le K + L\big(M_2(\mu)+M_2(\nu)\big) \ ,$$
$$|b_{i,1}(t,\mu,\nu)|, |b_{i,2}(t,\mu,\nu)|, |\sigma_{i,1}(t,\mu,\nu)| \le L \ .$$

(A3) *The function $\mathbb{R}^d\times A_i \ni (x,\alpha)\mapsto f_i(t,x,\mu,\nu,\alpha)\in\mathbb{R}$ is once continuously differentiable with L-Lipschitz derivatives, i.e. for any $t\in[0,T], \mu,\nu\in\mathcal{P}_2(\mathbb{R}^d)$, $x,x'\in\mathbb{R}^d$, $\alpha,\alpha'\in A_i$,*

$$|\partial_{(x,\alpha)} f_i(t,x',\mu,\nu,\alpha') - \partial_{(x,\alpha)} f_i(t,x,\mu,\nu,\alpha)| \le L\big(|x'-x|+|\alpha'-\alpha|\big) \ ,$$

where $\partial_{(x,\alpha)} f_i$ denotes the gradient in the joint variables (x,α). f_i also satisfies the λ-convexity:

$$f_i(t,x',\mu,\nu,\alpha') - f_i(t,x,\mu,\nu,\alpha) - \langle (x'-x,\alpha'-\alpha), \partial_{(x,\alpha)} f_i(t,x,\mu,\nu,\alpha)\rangle \ge \lambda|\alpha'-\alpha|^2 \ .$$

(A4) *For any $(t,x,\mu,\nu,\alpha)\in[0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)^2\times A_i$,*

$$|\partial_{(x,\alpha)} f_i(t,x,\mu,\nu,\alpha)| \le K + L\big(|x|+|\alpha|+M_2(\mu)+M_2(\nu)\big) \ .$$
$$|f_i(t,x,\mu,\nu,\alpha)| \le K + L\big(|x|^2+|\alpha|^2+M_2(\mu)^2+M_2(\nu)^2\big) \ .$$

Moreover, for any $(t,x,\alpha)\in[0,T]\times\mathbb{R}^d\times A_i$, the functions $\mathcal{P}_2(\mathbb{R}^d)^2\ni(\mu,\nu)\mapsto f_i(t,x,\mu,\nu,\alpha)$ and $\mathcal{P}_2(\mathbb{R}^d)^2\ni(\mu,\nu)\mapsto \partial_{(x,\alpha)} f_i(t,x,\mu,\nu,\alpha)$ are continuous in W_2-distance.
(A5) *For any $\mu,\nu\in\mathcal{P}_2(\mathbb{R}^d)$, the function $\mathbb{R}^d\ni x\mapsto g_i(x,\mu,\nu)\in\mathbb{R}$ is convex. It is also once continuously differentiable with L-Lipschitz derivatives, i.e. $\forall x,x'\in\mathbb{R}^d, \mu,\nu\in\mathcal{P}_2(\mathbb{R}^d)$, it holds*

$$|\partial_x g_i(x',\mu,\nu) - \partial_x g_i(x,\mu,\nu)| \le L|x'-x| \ .$$

For any $x\in\mathbb{R}^d$, the functions $\mathcal{P}_2(\mathbb{R}^d)^2\ni(\mu,\nu)\mapsto g_i(x,\mu,\nu)$ and $\mathcal{P}_2(\mathbb{R}^d)^2\ni(\mu,\nu)\mapsto\partial_x g_i(x,\mu,\nu)$ are continuous in W_2-distance. Moreover, the growth conditions

$$|\partial_x g_i(x,\mu,\nu)| \le K + L\big(|x|+M_2(\mu)+M_2(\nu)\big) \ ,$$
$$|g_i(x,\mu,\nu)| \le K + L\big(|x|^2+M_2(\mu)^2+M_2(\nu)^2\big) \ ,$$

are satisfied.

We first consider the optimal control problem (3.1) for given deterministic flows of probability measures. The corresponding Hamiltonian for each population $H_i : [0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)^2\times\mathbb{R}^d\times\mathbb{R}^{d\times d}\times A_i\to\mathbb{R}$ is defined by:

$$H_i(t,x,\mu,\nu,y,z,\alpha) := \langle b_i(t,x,\mu,\nu,\alpha), y\rangle + \mathrm{tr}[\sigma_i(t,x,\mu,\nu)^\top z] + f_i(t,x,\mu,\nu,\alpha) \ . \tag{3.3}$$

Since σ_i is independent of the control parameter, the minimizer $\hat{\alpha}_i(t,x,\mu,\nu,y)$ of the Hamiltonian

H_i can also be defined by a reduced Hamiltonian $H_i^{(r)}$:

$$\hat{\alpha}_i(t,x,\mu,\nu,y) := \operatorname{argmin}_{\alpha\in A_i} H_i^{(r)}(t,x,\mu,\nu,y,\alpha) \tag{3.4}$$

where

$$H_i^{(r)}(t,x,\mu,\nu,y,\alpha) := \langle b_i(t,x,\mu,\nu,\alpha), y\rangle + f_i(t,x,\mu,\nu,\alpha).$$

The following result regarding the regularity of $\hat{\alpha}_i$ is a straightforward extension of Lemma 2.1 [8].

Lemma 3.1. *Under Assumption* **(MFG-a)**, *for all* $(t,x,\mu,\nu,y) \in [0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)^2\times\mathbb{R}^d$, *there exists a unique minimizer* $\hat{\alpha}_i(t,x,\mu,\nu,y)$ *of* $H_i^{(r)}$, *where the map* $[0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)^2\times\mathbb{R}^d \ni (t,x,\mu,\nu,y)\mapsto \hat{\alpha}_i(t,x,\mu,\nu,y)\in A_i$ *is measurable. There exist constants* C *depending only on* (L,λ) *and* C' *depending additionally on* K *such that, for any* $t\in[0,T], x,x',y,y'\in\mathbb{R}^d, \mu,\nu\in\mathcal{P}_2(\mathbb{R}^d)$,

$$\begin{aligned}
&|\hat{\alpha}_i(t,x,\mu,\nu,y)| \le C' + C\big(|x|+|y|+M_2(\mu)+M_2(\nu)\big)\\
&|\hat{\alpha}_i(t,x,\mu,\nu,y) - \hat{\alpha}_i(t,x',\mu,\nu,y')| \le C\big(|x-x'|+|y-y'|\big)\ .
\end{aligned}$$

Moreover, for any $(t,x,y)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^d$, *the map* $\mathcal{P}_2(\mathbb{R}^d)^2 \ni (\mu,\nu)\mapsto\hat{\alpha}_i(t,x,\mu,\nu,y)$ *is continuous with respect to* W_2*-distance:*

$$\begin{aligned}
&|\hat{\alpha}_i(t,x,\mu,\nu,y) - \hat{\alpha}_i(t,x,\mu',\nu',y)|\\
&\le (2\lambda)^{-1}\Big(|b_{i,2}(t,\mu,\nu) - b_{i,2}(t,\mu',\nu')||y| + |\partial_\alpha f_i(t,x,\mu,\nu,\hat{\alpha}_i) - \partial_\alpha f_i(t,x,\mu',\nu',\hat{\alpha}_i)|\Big)
\end{aligned}$$

where $\hat{\alpha}_i := \hat{\alpha}_i(t,x,\mu,\nu,y)$.

Proof. To lighten the notation, let us write $\rho=(\mu,\nu)\in\mathcal{P}_2(\mathbb{R}^d)^2$. Since the function $A_i\ni\alpha\mapsto H_i^{(r)}(t,x,\rho,y,\alpha)$ is strictly convex and once continuously differentiable, $\hat{\alpha}_i(t,x,\rho,y)$ is given by the unique solution to the variational inequality:

$$\forall\beta\in A_i,\quad \langle \beta-\hat{\alpha}_i(t,x,\rho,y), \partial_\alpha H_i^{(r)}(t,x,\rho,y,\hat{\alpha}_i(t,x,\rho,y))\rangle \ge 0\ . \tag{3.5}$$

By strict convexity, the measurability is a consequence of the gradient descent algorithm (Lemma 3.3 [10]).

With an arbitrary point $\beta_i\in A_i$, the λ-convexity implies that

$$\begin{aligned}
H_i^{(r)}(t,x,\rho,y,\beta_i) &\ge H_i^{(r)}(t,x,\rho,y,\hat{\alpha}_i)\\
&\ge H_i^{(r)}(t,x,\rho,y,\beta_i) + \langle\hat{\alpha}_i-\beta_i, \partial_\alpha H_i^{(r)}(t,x,\rho,y,\beta_i)\rangle + \lambda|\hat{\alpha}_i-\beta_i|^2\ ,
\end{aligned}$$

where $\hat{\alpha}_i := \hat{\alpha}_i(t,x,\rho,y)$. Hence we have

$$|\hat{\alpha}_i-\beta_i| \le \lambda^{-1}\big(|b_{i,2}(t,\rho)||y| + |\partial_\alpha f_i(t,x,\rho,\beta_i)|\big)\ . \tag{3.6}$$

This gives the first growth condition. Next, with $\hat{\alpha}_i := \hat{\alpha}(t,x,\rho,y)$ and $\hat{\alpha}_i' := \hat{\alpha}_t(t,x',\rho,y')$, the optimality condition implies

$$\langle\hat{\alpha}_i'-\hat{\alpha}_i, \partial_\alpha H_i^{(r)}(t,x,\rho,y,\hat{\alpha}_i) - \partial_\alpha H_i^{(r)}(t,x',\rho,y',\hat{\alpha}_i')\rangle \ge 0\ .$$

This inequality, together with the λ-convexity, gives

$$\langle\hat{\alpha}_i'-\hat{\alpha}_i,\ b_{i,2}(t,\rho)\cdot(y-y') + \partial_\alpha f_i(t,x,\rho,\hat{\alpha}_i) - \partial_\alpha f_i(t,x',\rho,\hat{\alpha}_i)\rangle \ge 2\lambda|\hat{\alpha}_i-\hat{\alpha}_i'|^2,$$

and thus $|\hat{\alpha}_i - \hat{\alpha}'_i| \le (2\lambda)^{-1}\big(|b_{i,2}(t,\rho)||y-y'| + |\partial_\alpha f_i(t,x,\rho,\hat{\alpha}_i) - \partial_\alpha f_i(t,x',\rho,\hat{\alpha}_i)|\big)$. This proves the Lipschitz continuity in (x,y). The continuity with respect to the measure arguments follows exactly in the same way. □

For given flows $\boldsymbol{\mu}^1, \boldsymbol{\mu}^2 \in \mathcal{C}([0,T];\mathcal{P}_2(\mathbb{R}^d))$, the adjoint equation of the optimal control problem (3.1) for each population $1 \le i \le 2$ is given by

$$
\begin{aligned}
dX_t^i &= b_i(t, X_t^i, \mu_t^i, \mu_t^j, \hat{\alpha}_i(t, X_t^i, \mu_t^i, \mu_t^j, Y_t^i))dt + \sigma_i(t, X_t^i, \mu_t^i, \mu_t^j)dW_t^i \ , \\
dY_t^i &= -\partial_x H_i(t, X_t^i, \mu_t^i, \mu_t^j, Y_t^i, Z_t^i, \hat{\alpha}_i(t, X_t^i, \mu_t^i, \mu_t^j, Y_t^i))dt + Z_t^i dW_t^i \ ,
\end{aligned} \tag{3.7}
$$

with $j \neq i$, $X_0^i = \xi^i \in \mathbb{L}^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d)$ and $Y_T^i = \partial_x g_i(X_T^i, \mu_T^i, \mu_T^j)$, which is a $C(L,\lambda)$-Lipschitz FBSDE. Notice that H_i must be the full Hamiltonian instead of reduced one due to the state dependence in σ_i. Here, $\partial_x H_i$ has the form:

$$
\partial_x H_i(t,x,\mu,\nu,y,z,\alpha) := b_{i,1}(t,\mu,\nu)^\top y + \mathrm{tr}[\sigma_{i,1}(t,\mu,\nu)^\top z] + \partial_x f_i(t,x,\mu,\nu,\alpha) \ .
$$

Theorem 3.1. *Under Assumption* **(MFG-a)***, for any flows $\boldsymbol{\mu}^1, \boldsymbol{\mu}^2 \in \mathcal{C}([0,T];\mathcal{P}_2(\mathbb{R}^d))$, the adjoint FBSDE* (3.7) *for each $i \in \{1,2\}$ has a unique solution $(\hat{X}_t^i, \hat{Y}_t^i, \hat{Z}_t^i)_{t\in[0,T]} \in \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{H}^2$. Moreover, there exits a measurable function $u_i^{\boldsymbol{\mu}^i,\boldsymbol{\mu}^j} : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ such that, with some constant C depending only on (L,λ),*

$$
\forall t \in [0,T], \forall x, x' \in \mathbb{R}^d, \quad |u_i^{\boldsymbol{\mu}^i,\boldsymbol{\mu}^j}(t,x) - u_i^{\boldsymbol{\mu}^i,\boldsymbol{\mu}^j}(t,x')| \le C|x-x'| \tag{3.8}
$$

and also that $\forall t \in [0,T]$, $\hat{Y}_t^i = u_i^{\boldsymbol{\mu}^i,\boldsymbol{\mu}^j}(t, \hat{X}_t^i)$, $\mathbb{P}$-a.s.

If we set $\hat{\boldsymbol{\alpha}}^i = (\hat{\alpha}_t^i = \hat{\alpha}_i(t, \hat{X}_t^i, \mu_t^i, \mu_t^j, \hat{Y}_t^i))_{t\in[0,T]}$, then for any $\boldsymbol{\beta}^{\boldsymbol{i}} = (\beta_t^i)_{t\in[0,T]} \in \mathbb{A}_i$, it holds:

$$
J_i^{\boldsymbol{\mu}^i,\boldsymbol{\mu}^j}(\hat{\boldsymbol{\alpha}}^i) + \lambda\mathbb{E}\int_0^T |\beta_t^i - \hat{\alpha}_t^i|^2 dt \le J_i^{\boldsymbol{\mu}^i,\boldsymbol{\mu}^j}(\boldsymbol{\beta}^i) \ . \tag{3.9}
$$

Proof. The last claim regarding the sufficiency of the stochastic maximal principle is well known. Indeed, if a solution $(\hat{X}_t^i, \hat{Y}_t^i, \hat{Z}_t^i)_{t\in[0,T]}$ exists, then using the convexity $g_i(X_T^i, \mu_T^i, \mu_T^j) - g_i(\hat{X}_T^i, \mu_T^i, \mu_T^j) \ge \langle (X_T^i - \hat{X}_T^i), \partial_x g_i(\hat{X}_T^i, \mu_T^i, \mu_T^j)\rangle$, evaluating the expectation $\mathbb{E}[\langle X_T^i - \hat{X}_T^i, \hat{Y}_T^i\rangle]$ by the Ito formula, making use of the λ-convexity of the Hamiltonian, we get the desired result. Here, $(X_t^i)_{t\in[0,T]}$ with $X_0^i = \xi^i$ denotes the solution of the SDE (3.7) with $\boldsymbol{\beta}_i$ instead of $\hat{\boldsymbol{\alpha}}_i$ as its control. See, for example, Theorem 6.4.6 in [31].

The existence of a unique solution to the adjoint FBSDE as wall as the Lipschitz continuous decoupling field follows from a straightforward extension of Lemma 3.5 in [8]. First, since the adjoint FBSDE (3.7) is $C(L,\lambda)$-Lipschitz continuous in (x,y,z) and σ_i is independent of Z^i, Theorem 1.1 in [15] guarantees the existence of a unique solution for small time $T \le c$, where $c = c(L,\lambda)$ is a constant depending only on (L,λ). Thus, for a general T, we still have the unique solvability on $[T-\delta, T]$ with $0 < \delta \le c$ and any initial condition $\xi^i \in \mathbb{L}^2(\Omega, \mathcal{F}_{t_0}, \mathbb{P}; \mathbb{R}^d)$ at $t_0 \in [T-\delta, T]$. We let $(X_t^{i,t_0,\xi^i}, Y_t^{i,t_0,\xi^i}, Z_t^{i,t_0,\xi^i})_{t\in[t_0,T]}$ denote this solution. Following the proof of Theorem 2.6 in [15] (see also Proposition 4.8 in [10]), we can establish the existence and uniqueness on the whole $[0,T]$ by connecting the short-term solutions provided we have

$$
\forall x, y \in \mathbb{R}^d, \quad |Y_{t_0}^{i,t_0,x} - Y_{t_0}^{i,t_0,y}| \le C|x-y| \ , \tag{3.10}
$$

for some C independent of t_0 and δ. Here, by Blumenthal's zero-one law, $Y_{t_0}^{i,t_0,x}$ and $Y_{t_0}^{i,t_0,y}$ are deterministic. We are now going to prove (3.10). Let us put

$$\hat{J}_i^{t_0,x} := \mathbb{E}\Big[\int_{t_0}^T f_i(t, X_t^{i,t_0,x}, \mu_t^i, \mu_t^j, \hat{\alpha}_t^{i,t_0,x})dt + g_i(X_T^{i,t_0,x}, \mu_T^i, \mu_T^j)\Big] ,$$

with $\hat{\alpha}_t^{i,t_0,x} := \hat{\alpha}_i(t, X_t^{i,t_0,x}, \mu_t^i, \mu_t^j, Y_t^{i,t_0,x})$. Then similar arguments deriving the relation (3.9) gives

$$\langle y - x, Y_{t_0}^{t_0,x}\rangle + \hat{J}_i^{t_0,x} + \lambda \mathbb{E}\int_{t_0}^T |\hat{\alpha}_t^{i,t_0,y} - \hat{\alpha}_t^{i,t_0,x}|^2 dt \le J_i^{t_0,y} . \tag{3.11}$$

Exchanging the role of x and y in (3.11) and adding the two inequalities, we obtain that

$$2\lambda \mathbb{E}\int_{t_0}^T |\hat{\alpha}_t^{i,t_0,x} - \hat{\alpha}_t^{i,t_0,y}|^2 dt \le \langle x - y, Y_{t_0}^{i,t_0,x} - Y_{t_0}^{i,t_0,y}\rangle . \tag{3.12}$$

Now, treating the controls $\hat{\alpha}^{i,t_0,x}, \hat{\alpha}^{i,t_0,y}$ as well as the forward varibles $X^{i,t_0,x}, X^{i,t_0,y}$ as external inputs, we apply the standard stability result of Lipschitz BSDEs (e.g., see Theorem 4.2.3 in [36]) to obtain the estimate for $\mathbb{E}[\sup_{t\in[t_0,T]} |Y_t^{i,t_0,x} - Y_t^{i,t_0,y}|^2]$. Then, applying the standard stability result of Lipschitz SDEs (e.g., see Theorem 3.24 in [36]) to this estimate gives

$$\begin{aligned}
&\mathbb{E}\big[\sup_{t\in[t_0,T]} |X_t^{i,t_0,x} - X_t^{i,t_0,y}|^2 + \sup_{t\in[t_0,T]} |Y_t^{i,t_0,x} - Y_t^{i,t_0,y}|^2\big] \\
&\le C(L)\Big(|x-y|^2 + \mathbb{E}\int_{t_0}^T |\hat{\alpha}_t^{i,t_0,x} - \hat{\alpha}_t^{i,t_0,y}|^2 dt\Big) .
\end{aligned}$$

Now the inequality (3.12) proves the relation (3.10) with C depending only on (L, λ), and hence also the existence of a unique solution for general T. The decoupling field is defined by $u_i^{\boldsymbol{\mu}^i,\boldsymbol{\mu}^j} : [0,T]\times\mathbb{R}^d \ni (t,x) \mapsto Y_t^{i,t,x}$, and the representation $\mathbb{P}\big(\forall t\in[0,T],\ \hat{Y}_t^i = u_i^{\boldsymbol{\mu}^i,\boldsymbol{\mu}^j}(t,\hat{X}_t^i)\big) = 1$ follows from the uniqueness of the solution as well as its continuity (Corollary 1.5 in [15]). Its Lipschitz continuity is a direct result of (3.10). □

Remark 3.2. *In the remainder, we often use the simpler notation u_i for the decoupling field without the superscripts $(\boldsymbol{\mu}^i, \boldsymbol{\mu}^j)$.*

Lemma 3.2. *Suppose that two set of functions $(b_i, \sigma_i, f_i, g_i)$ and $(b_i', \sigma_i', f_i', g_i')$ satisfy Assumption* **(MFG-a)**. *For given inputs $\xi^i, \xi'^{,i} \in \mathbb{L}^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d)$ and $(\boldsymbol{\mu}^i, \boldsymbol{\mu}^j), (\boldsymbol{\mu}'^{,i}, \boldsymbol{\mu}'^{,j}) \in \mathcal{C}([0,T]; \mathcal{P}_2(\mathbb{R}^d))^2$, let us denote the corresponding solution to (3.7) by $(X_t^i, Y_t^i, Z_t^i)_{t\in[0,T]}$ and $(X_t'^{,i}, Y_t'^{,i}, Z_t'^{,i})_{t\in[0,T]}$, respectively. Then, there exists a constant C depending only on (L,λ) such that*

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T]} |X_t^i - X_t'^{,i}|^2 + \sup_{t\in[0,T]} |Y_t^i - Y_t'^{,i}|^2 + \int_0^T |Z_t^i - Z_t'^{,i}|^2 dt\Big] \\
&\le C\mathbb{E}\Big[|\xi^i - \xi'^{,i}|^2 + |\partial_x g_i(X_T^i, \mu_T^i, \mu_T^j) - \partial_x g_i'(X_T^i, \mu_T'^{,i}, \mu_T'^{,j})|^2 \\
&\quad + \int_0^T |b_i(t, X_t^i, \mu_t^i, \mu_t^j, \hat{\alpha}_i(t, X_t^i, \mu_t^i, \mu_t^j, Y_t^i)) - b_i'(t, X_t^i, \mu_t'^{,i}, \mu_t'^{,j}, \hat{\alpha}_i'(t, X_t^i, \mu_t'^{,i}, \mu_t'^{,j}, Y_t^i))|^2 dt \\
&\quad + \int_0^T |\sigma_i(t, X_t^i, \mu_t^i, \mu_t^j) - \sigma_i'(t, X_t^i, \mu_t'^{,i}, \mu_t'^{,j})|^2 dt + \int_0^T \Big(|\partial_x H_i(t, X_t^i, \mu_t^i, \mu_t^j, Y_t^i, Z_t^i, \hat{\alpha}_i(t, X_t^i, \mu_t^i, \mu_t^j, Y_t^i)) \\
&\qquad -\partial_x H_i'(t, X_t^i, \mu_t'^{,i}, \mu_t'^{,j}, Y_t^i, Z_t^i, \hat{\alpha}_i'(t, X_t^i, \mu_t'^{,i}, \mu_t'^{,j}, Y_t^i))|^2\Big) dt\Big],
\end{aligned} \tag{3.13}$$

where the functions $H_i, H_i' : [0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)^2\times\mathbb{R}^d\times\mathbb{R}^{d\times d}\times A_i \to \mathbb{R}$ are the Hamiltonians

(3.3) *associated with the coefficients* (b_i, σ_i, f_i) *and* (b_i', σ_i', f_i')*, respectively, and* $\hat{\alpha}_i, \hat{\alpha}_i'$ *are their minimizers. In particular, there is another constant* C' *depending additionally on* K *such that*

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|X_t^i|^2+\sup_{t\in[0,T]}|Y_t^i|^2+\int_0^T|Z_t^i|^2dt\Big]\le C\Big(||\xi^i||_2^2+\sup_{t\in[0,T]}\sum_{j=1}^2 M_2(\mu_t^j)^2\Big)+C', \qquad (3.14)$$

and, for any $t\in[0,T]$,

$$|Y_t^i|\le C\Big(|X_t^i|+\sup_{t\in[0,T]}\sum_{j=1}^2 M_2(\mu_t^j)\Big)+C',\ \mathbb{P}\text{-}a.s. \qquad (3.15)$$

Proof. Since the FBSDE (3.7) has Lipschitz continuous coefficients, it is standard to show that there exists some constant c depending only on (L,λ) such that the estimate (3.13) holds for small $T\le c$. In particular, by applying Ito formula to $|Y_t^i-Y_t^{\prime,i}|^2$, we see that $\mathbb{E}[\sup_{t\in[0,T]}|Y_t^i-Y_t^{\prime,i}|^2+\int_0^T|Z_t^i-Z_t^{\prime,i}|^2]$ is bounded by the terms related to the backward equation in (3.13) plus the term $C\mathbb{E}[\sup_{t\in[0,T]}|X_t^i-X_t^{\prime,i}|^2]$. On the other hand, the similar calculation shows that $\mathbb{E}[\sup_{t\in[0,T]}|X_t^i-X_t^{\prime,i}|^2]$ is bounded by the remaining terms in (3.13) plus $CT\mathbb{E}[\sup_{t\in[0,T]}|Y_t^i-Y_t^{i,\prime}|^2]$ where C depends only on the Lipschitz constants of the system. Hence, for small $T\le c$, we obtain the desired estimate.

For general T, the estimate is a result of connecting the short-term estimates (Lemma 4.9 [10]). Since the same technique will be used also in Lemma 4.3, let us explain it here in details. We first divide the interval $[0,T]$ into finite number of subintervals $([T_{k-1},T_k])_{1\le k\le N}$ with $T_0=0, T_N=T$ and $T_k-T_{k-1}\le c$ for each k. The estimate for $\Theta(T_{k-1},T_k):=\mathbb{E}\big[\sup_{t\in[T_{k-1},T_k]}|X_t^i-X_t^{\prime,i}|^2+\sup_{t\in[T_{k-1},T_k]}|Y_t^i-Y_t^{\prime,i}|^2+\int_{T_{k-1}}^{T_k}|Z_t^i-Z_t^{\prime,i}|^2dt\big]$ on each interval can be written in the from

$$\begin{aligned}\Theta(T_{k-1},T_k) \;\le\; & C\mathbb{E}\Big[|X_{T_{k-1}}^i-X_{T_{k-1}}^{\prime,i}|^2+|(u_i-u_i')(T_k,X_{T_k}^i)|^2\\ &+\int_{T_{k-1}}^{T_k}|(b_i-b_i',\sigma_i-\sigma_i',\partial_xH_i-\partial_xH_i')(s,X_s^i,Y_s^i,Z_s^i)|^2ds\Big]\end{aligned}$$

where we have used the notation $u_i:=u_i^{\boldsymbol{\mu}^i,\boldsymbol{\mu}^j}, u_i':=u_i^{\prime,\boldsymbol{\mu}^{\prime,i},\boldsymbol{\mu}^{\prime,j}}$ and omitted the arguments regarding $(\boldsymbol{\mu}^j,\boldsymbol{\mu}^{\prime,j})_{1\le j\le 2}$ to lighten the expression. The Lipschitz continuity in (3.8) is crucial to derive this expression. For $k=N$, it gives $\Theta(T_{N-1},T_N)\le C\mathbb{E}\Big[|X_{T_{N-1}}^i-X_{T_{N-1}}^{\prime,i}|^2+|\delta_T|^2+\int_{T_{N-1}}^T|\delta h_s|^2ds\Big]$ where $\delta_T:=(\partial_xg_i-\partial_xg_i')(T,X_T^i)$ and $\delta h_s:=(b_i-b_i',\sigma_i-\sigma_i',\partial_xH_i-\partial_xH_i')(s,X_s^i,Y_s^i,Z_s^i)$. This means, in particular, $\mathbb{E}\big[|(u_i-u_i')(T_{N-1},X_{T_{N-1}}^i)|^2\big]\le C\mathbb{E}\Big[|X_{T_{N-1}}^i-X_{T_{N-1}}^{\prime,i}|^2+|\delta_T|^2+\int_{T_{N-1}}^T|\delta h_s|^2ds\Big]$. Since it holds for any *initial value* $X_{T_{N-1}}^{\prime,i}$, we obtain $\mathbb{E}\big[|(u_i-u_i')(T_{N-1},X_{T_{N-1}}^i)|^2\big]\le C\mathbb{E}\Big[|\delta_T|^2+\int_{T_{N-1}}^{T_N}|\delta h_s|^2ds\Big]$ by choosing $X_{T_{N-1}}^{\prime,i}=X_{T_{N-1}}^i$. This estimate then implies $\Theta(T_{N-2},T_{N-1})\le C\mathbb{E}\Big[|X_{T_{N-2}}^i-X_{T_{N-2}}^{\prime,i}|^2+|\delta_T|^2+\int_{T_{N-2}}^T|\delta h_s|^2ds\Big]$. By iteration, we get for any k,

$$\Theta(T_k,T_{k+1})\le C\mathbb{E}\Big[|X_{T_k}^i-X_{T_k}^{\prime,i}|^2+|\delta_T|^2+\int_{T_k}^T|\delta h_s|^2ds\Big]\ . \qquad (3.16)$$

Moreover, by iterating the relation $\mathbb{E}\big[|X_{T_k}^i-X_{T_k}^{\prime,i}|^2\big]\le C\mathbb{E}\Big[|X_{T_{k-1}}^i-X_{T_{k-1}}^{\prime,i}|^2+|\delta_T|^2+\int_{T_{k-1}}^T|\delta h_s|^2ds\Big]$, we get

$$\mathbb{E}\big[|X^i_{T_k}-X'^{,i}_{T_k}|^2\big]\le C\mathbb{E}\Big[|\xi^i-\xi'^{,i}|^2+|\delta_T|^2+\int_0^T|\delta h_s|^2ds\Big]~. \tag{3.17}$$

Inserting the estimate (3.17) into (3.16) and summing over k, we obtain the desired estimate.

In order to obtain the growth estimate, we put, for any $(t,x,\mu,\nu,\alpha)\in[0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)^2\times A_i$,

$$b_i(t,x,\mu,\nu,\alpha)=\sigma_i(t,x,\mu,\nu)=g_i(x,\mu,\nu)=0~,\quad f_i(t,x,\mu,\nu,\alpha)=\lambda|\alpha|^2~,$$

and $\xi^i=0$, which then satisfies Assumption **(MFG-a)** and makes (X^i,Y^i,Z^i) identically zero. Plugging them into (3.13), we obtain the estimate:

$$\begin{aligned}&\mathbb{E}\Big[\sup_{t\in[0,T]}|X'^{,i}_t|^2+\sup_{t\in[0,T]}|Y'^{,i}_t|^2+\int_0^T|Z'^{,i}_t|^2dt\Big]\\&\le C\Big(||\xi'^{,i}||_2^2+|\partial_x g'_i(0,\mu'^{,i}_T,\mu'^{,j}_T)|^2+\int_0^T|\partial_x f'_i(t,0,\mu'^{,i}_t,\mu'^{,j}_t,\hat\alpha'_i(t,0,\mu'^{,i}_t,\mu'^{,j}_t,0))|^2dt\\&+\int_0^T\big[|b'_i(t,0,\mu'^{,i}_t,\mu'^{,j}_t,\hat\alpha'_i(t,0,\mu'^{,i}_t,\mu'^{,j}_t,0))|^2+|\sigma'_i(t,0,\mu'^{,i}_t,\mu'^{,j}_t)|^2\big]dt\Big)~.\end{aligned} \tag{3.18}$$

Now, by symmetry, the desired estimate (3.14) holds for (X^i,Y^i,Z^i). Finally, using the initial condition $X^i_t=0$ at time t yields

$$|Y^{i,t,0}_t|\le\mathbb{E}\Big[\sup_{s\in[t,T]}|Y^{i,t,0}_s|^2\Big]^{\frac12}\le C\sup_{s\in[t,T]}\sum_{j=1}^2 M_2(\mu^j_s)+C'.$$

Now, by the Lipschitz continuity (3.8)(or equivalently (3.10)), we have $|Y^i_t-Y^{i,t,0}_t|=|Y^{i,t,X^i_t}_t-Y^{i,t,0}_t|\le C|X^i_t|$. This proves the growth estimate (3.15). □

3.3 MFG equilibrium under boundedness assumptions

In preceding subsections, we have seen that, for given deterministic flows of probability measures $\boldsymbol{\mu}^1,\boldsymbol{\mu}^2\in\mathcal{C}([0,T];\mathcal{P}_2(\mathbb{R}^d))$, the solution to each optimal control problem of (3.1) is characterized by the uniquely solvable FBSDE (3.7). Hence finding an equilibrium condition (3.2) results in finding a solution to the following system of FBSDEs of McKean-Vlasov (MKV) type: for $i,j\in\{1,2\}, j\neq i$;

$$\begin{aligned}dX^i_t&=b_i\big(t,X^i_t,\mathcal{L}(X^i_t),\mathcal{L}(X^j_t),\hat\alpha_i(t,X^i_t,\mathcal{L}(X^i_t),\mathcal{L}(X^j_t),Y^i_t)\big)dt+\sigma_i\big(t,X^i_t,\mathcal{L}(X^i_t),\mathcal{L}(X^j_t)\big)dW^i_t,\\dY^i_t&=-\partial_x H_i\big(t,X^i_t,\mathcal{L}(X^i_t),\mathcal{L}(X^j_t),Y^i_t,Z^i_t,\hat\alpha_i(t,X^i_t,\mathcal{L}(X^i_t),\mathcal{L}(X^j_t),Y^i_t)\big)dt+Z^i_tdW^i_t,\end{aligned} \tag{3.19}$$

with $X^i_0=\xi^i\in\mathbb{L}^2(\Omega,\mathcal{F}_0,\mathbb{P};\mathbb{R}^d)$ and $Y^i_T=\partial_x g_i(X^i_T,\mathcal{L}(X^i_T),\mathcal{L}(X^j_T))$. Although two MKV-type FBSDEs are now coupled, their interactions appear only through the laws of the two populations. Thanks to this property, we can still apply a similar strategy developed by Carmona & Delarue [8, 7]. A crucial tool to prove the existence of an equilibrium is the Schauder's fixed point theorem [32] generalized by Tychonoff [35][1]. The following form is taken from Theorem 4.32 in [10].

Theorem 3.2. *(Schauder FPT) Let $(V,||\cdot||)$ be a normed linear vector space and E be a nonempty closed convex subset of V. Then, any continuous mapping from E into itself which has a relatively compact range has a fixed point.*

[1] See, for example Shapiro [33], for pedagogical introduction of fixed-point theorems and relevant references.

In this subsection, we prove the existence of a solution to the system of FBSDEs (3.19) under additional assumptions.

Assumption 3.2. (MFG-b) *For $1 \le i \le 2$, there exist some element $0_{A_i} \in A_i$ and a constant Λ such that, for any $(t,\mu,\nu) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d)^2$,*

$$\begin{aligned}&|b_{i,0}(t,\mu,\nu)|, |\sigma_{i,0}(t,\mu,\nu)| \le \Lambda\ ,\\ &|\partial_x g_i(0,\mu,\nu)|, |\partial_{(x,\alpha)} f_i(t,0,\mu,\nu,0_{A_i})| \le \Lambda\ .\end{aligned}$$

Here is the main result of this subsection.

Theorem 3.3. *Under Assumptions* **(MFG-a,b)***, the system of FBSDEs* (3.19) *(and hence the matching problem* (3.2)*) is solvable for any $\xi^1, \xi^2 \in \mathbb{L}^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d)$.*

Proof. With slight abuse of notation, we let $(X_t^{i,\boldsymbol{\rho}}, Y_t^{i,\boldsymbol{\rho}}, Z_t^{i,\boldsymbol{\rho}})_{t\in[0,T]}$ denote the solution to the FBSDE (3.7) for a given flows $\boldsymbol{\rho} := (\boldsymbol{\mu}^1, \boldsymbol{\mu}^2) \in \mathcal{C}([0,T]; \mathcal{P}_2(\mathbb{R}^d))^2$ and the initial condition $X_0^{i,\boldsymbol{\rho}} = \xi^i$. By Theorem 3.1, we can define a map:

$$\Phi : \mathcal{C}([0,T];\mathcal{P}_2(\mathbb{R}^d))^2 \ni (\boldsymbol{\mu}^1, \boldsymbol{\mu}^2) \mapsto (\mathcal{L}(X_t^{1,\boldsymbol{\rho}})_{t\in[0,T]}, \mathcal{L}(X_t^{2,\boldsymbol{\rho}})_{t\in[0,T]}) \in \mathcal{C}([0,T];\mathcal{P}_2(\mathbb{R}^d))^2\ .$$

In the following, we are going to check the conditions necessary for the application of Schauder FPT to this map. As a linear vector space V in the FPT, we use the product space $\mathcal{C}([0,T];\mathcal{M}_f^1(\mathbb{R}^d))^2$ equipped with the supremum of the Kantorovich-Rubinstein norm:

$$\begin{aligned}&||(\boldsymbol{\mu}^1, \boldsymbol{\mu}^2)|| := \sum_{i=1}^2 \sup_{t\in[0,T]} ||\mu_t^i||_{\mathrm{KR}_*}, && (3.20)\\ &\text{with :}\quad ||\mu||_{\mathrm{KR}_*} := |\mu(\mathbb{R}^d)| + \sup\Big\{\int_{\mathbb{R}^d} l(x)\mu(dx);\ l \in \mathrm{Lip}_1(\mathbb{R}^d), l(0)=0\Big\}\ ,\end{aligned}$$

where $\mathrm{Lip}_1(\mathbb{R}^d)$ is the set of 1-Lipschitz continuous functions on $\mathbb{R}^d$. Importantly, the norm $||\cdot||_{\mathrm{KR}_*}$ is known to coincide with the 1-Wasserstein distance W_1 on $\mathcal{P}_1(\mathbb{R}^d)$ (Corollary 5.4 in [10]). Of course, the reason to use a space of signed measures is to make it linear.

From (3.6) with $(\beta_i = 0_{A_i})$, it is immediate to see that $|\hat{\alpha}_i(t,0,\rho,0)| \le C(\lambda,\Lambda)$. Hence, by using the estimate (3.18) in Lemma 3.2, we get $\mathbb{E}\big[\sup_{t\in[0,T]} |Y_t^{i,\boldsymbol{\rho}}|^2\big] \le C(L,\lambda,\Lambda)(1+||\xi^i||_2^2)$. Then the Lipschitz continuity of the decoupling field implies that $|Y_t^{i,\boldsymbol{\rho}}| \le C\big(1+|X_t^{i,\boldsymbol{\rho}}|\big)$ with C independent of $\boldsymbol{\rho}$. Therefore, again by (3.6), we have $|\hat{\alpha}_i(t, X_t^{i,\boldsymbol{\rho}}, \rho_t, Y_t^{i,\boldsymbol{\rho}})| \le C(1+|X_t^{i,\boldsymbol{\rho}}|)$. Now, it is standard to check that $\mathbb{E}\big[|X_t^{i,\boldsymbol{\rho}} - X_s^{i,\boldsymbol{\rho}}|^2\big] \le C|t-s|$ and hence

$$W_2(\mathcal{L}(X_t^{i,\boldsymbol{\rho}}), \mathcal{L}(X_s^{i,\boldsymbol{\rho}})) \le C|t-s|^{\frac{1}{2}} \tag{3.21}$$

holds uniformly in $\boldsymbol{\rho}$. Since $\hat{\alpha}_i$ is of linear growth in $X_t^{i,\boldsymbol{\rho}}$, it is also straightforward to obtain $\mathbb{E}\Big[\sup_{t\in[0,T]} |X_t^{i,\boldsymbol{\rho}}|^4 | \mathcal{F}_0\Big]^{\frac{1}{2}} \le C\big(1+|\xi^i|^2\big)$ uniformly in $\boldsymbol{\rho}$. This inequality guarantees the uniform square integrability. In fact, the following estimate holds uniformly in $\boldsymbol{\rho}$ with any $a \ge 1$;[2]

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |X_t^{i,\boldsymbol{\rho}}|^2 \mathbf{1}_{\{\sup_{t\in[0,T]}|X_t^{i,\boldsymbol{\rho}}| \ge a\}}\Big] \le C\Big(a^{-1} + \mathbb{E}\big[|\xi^i|^2 \mathbf{1}_{\{|\xi^i| \ge \sqrt{a}\}}\big]\Big)^{\frac{1}{2}}. \tag{3.22}$$

Since the relation will be used repeatedly in the following, let us explain it here. For any $D \in \mathcal{F}$

[2]See p. 259 in [10].

and $\epsilon > 0$, we have, by Cauchy-Schwarz inequality,

$$\begin{aligned}\mathbb{E}\big[\sup_{t\in[0,T]}|X_t^{i,\boldsymbol{\rho}}|^2\mathbf{1}_D\big] &\le C\mathbb{E}\big[(1+|\xi^i|^2)\mathbb{P}(D|\mathcal{F}_0)^{\frac{1}{2}}\big]\\ &\le C\Big(\epsilon+\epsilon^{-1}\mathbb{E}\big[(1+|\xi^i|^2)\mathbf{1}_D\big]\Big) \le C\mathbb{E}\big[(1+|\xi^i|^2)\mathbf{1}_D\big]^{\frac{1}{2}}.\end{aligned}$$

In the last inequality, we have maximized in ϵ. Here, C depends on $||\xi^i||_2$ but not on $\boldsymbol{\rho}$. We also have

$$\begin{aligned}\sup_{D\in\mathcal{F};\mathbb{P}(D)\le Ca^{-2}}\mathbb{E}\big[(1+|\xi^i|^2)\mathbf{1}_D\big] &\le Ca^{-2}+\mathbb{E}\big[|\xi^i|^2(\mathbf{1}_{\{|\xi^i|\le\sqrt{a}\}}+\mathbf{1}_{\{|\xi^i|\ge\sqrt{a}\}})\big]\\ &\le 2Ca^{-1}+\mathbb{E}\big[|\xi^i|^2\mathbf{1}_{\{|\xi^i|\ge\sqrt{a}\}}\big]\ .\end{aligned}$$

Since $\mathbb{P}\Big(\sup_{t\in[0,T]}|X_t^{i,\boldsymbol{\rho}}|\ge a\Big)\le Ca^{-2}$ by Chevyshev's inequality, the estimate (3.22) is now established.

The above estimate suggests us to restrict the map Φ to the following domain:

$$\begin{aligned}\mathcal{E} := \Big\{&(\boldsymbol{\mu}^1,\boldsymbol{\mu}^2)\in\mathcal{C}([0,T];\mathcal{P}_2(\mathbb{R}^d))^2;\\ &\forall a\ge 1, 1\le i\le 2, \sup_{t\in[0,T]}\int_{|x|\ge a}|x|^2\mu_t^i(dx)\le C\Big(a^{-1}+\mathbb{E}\big[|\xi^i|^2\mathbf{1}_{\{|\xi^i|\ge\sqrt{a}\}}\big]\Big)^{\frac{1}{2}}\Big\},\end{aligned}$$

which is a closed and convex subset of $C([0,T];\mathcal{M}_f^1(\mathbb{R}^d))^2$. Choosing C sufficiently large, we can make Φ a self-map on $\mathcal{E}$. By the estimate (3.22) and Corollary 5.6 in [10], there exists a compact subset $\mathcal{K}\subset\mathcal{P}_2(\mathbb{R}^d)^2$ such that $\forall t\in[0,T]$, $[\Phi(\boldsymbol{\rho})]_t\in\mathcal{K}$ for any $\boldsymbol{\rho}\in\mathcal{E}$. Combined with the equicontinuity (3.21), Arzela-Ascoli theorem implies that the image $\Phi(\mathcal{E})$ is a relatively compact subset of $\mathcal{C}([0,T];\mathcal{P}_2(\mathbb{R}^d))^2$, and in particular of $\mathcal{C}([0,T];\mathcal{P}_1(\mathbb{R}^d))^2$.

Finally, by Lemma 3.2 and also by the continuity of coefficients in the measure arguments in W_2-distance, the dominated convergence theorem implies that $\mathbb{E}\big[\sup_{t\in[0,T]}|X_t^{i,\boldsymbol{\rho}}-X_t^{i,\boldsymbol{\rho}'}|^2\big]\to 0$ when $1\le i\le 2, \forall t\in[0,T], W_2(\mu_t^i,\mu_t^{\prime,i})\to 0$. Note that, by Theorem 5.5 in [10], when $\boldsymbol{\rho}$ converge with respect to the norm $||\cdot||$ in (3.20) under the restriction to the domain $\mathcal{E}$, $\boldsymbol{\rho}$ actually converges in W_2-distance. This proves the continuity of the map Φ. Now the existence of a fixed point (not necessarily unique) of the map Φ is guaranteed by Schauder FPT, which provides a solution to the system of FBSDEs (3.19). □

3.4 MFG equilibrium for small T or small coupling

In order to allow the quadratic cost functions relevant for popular Linear-Quadratic problems, we want to relax Assumption **(MFG-b)**. This is exactly what Carmona & Delarue have done in [8] for single population. Although we can follow the same route, it requires much stronger assumptions than **(MFG-a)**. Unfortunately, the conditions required in [8] preclude most of the interesting interactions among different populations through their state dynamics. In this work, in order to allow flexible interactions among populations and also to be complementary to the result in [8], we focus on the problems with small T. Requiring small T is a reasonable trade-off for quadratic interactions by considering the fact that, even for a deterministic LQ-problem, the relevant Riccati equation may diverge within a finite time. After the analysis for small T, we provide another solution which allows general T but requires the couplings between FSDE and BSDE are small enough.

Theorem 3.4. *Under Assumption* **(MFG-a)**, *there exists some positive constant c depending only on (L,λ) such that, for any $T\le c$, the system of FBSDEs* (3.19) *(and hence the matching problem* (3.2)*) is solvable for any $\xi^1,\xi^2\in\mathbb{L}^2(\Omega,\mathcal{F}_0,\mathbb{P};\mathbb{R}^d)$.*

Proof. For any $n\in\mathbb{N}$ and $\mu\in\mathcal{P}_2(\mathbb{R}^d)$, let us define $\phi_n\circ\mu$ as a push-forward of μ by the map $\mathbb{R}^d\ni x\mapsto\dfrac{nx}{\max(M_2(\mu),n)}$. In other words, for any random variable X with $\mathcal{L}(X)=\mu$, the law of $\dfrac{nX}{\max(M_2(\mu),n)}$ is given by $\phi_n\circ\mu$. Obviously, $M_2(\phi_n\circ\mu)\le n$ and the map $\mu\mapsto\phi_n\circ\mu$ is continuous with respect to W_2-distance. Using this map, we introduce a sequence of approximated functions

$$\begin{aligned}&(b^n_{i,0},b^n_{i,2},\sigma^n_{i,0})(t,\mu,\nu):=(b_{i,0},b_{i,2},\sigma_{i,0})(t,\phi_n\circ\mu,\phi_n\circ\nu)\ ,\\&f^n_i(t,x,\mu,\nu,\alpha):=f_i(t,x,\phi_n\circ\mu,\phi_n\circ\nu,\alpha),\quad g^n_i(x,\mu,\nu):=g_i(x,\phi_n\circ\mu,\phi_n\circ\nu),\end{aligned}$$

and accordingly define

$$\begin{aligned}&b^n_i(t,x,\mu,\nu,\alpha):=b^n_{i,0}(t,\mu,\nu)+b_{i,1}(t,\mu,\nu)x+b^n_{i,2}(t,\mu,\nu)\alpha\ ,\\&\sigma^n_i(t,x,\mu,\nu):=\sigma^n_{i,0}(t,\mu,\nu)+\sigma_{i,1}(t,\mu,\nu)x\ ,\\&H^n_i(t,x,\mu,\nu,y,z,\alpha):=\langle b^n_i(t,x,\mu,\nu,\alpha),y\rangle+\mathrm{tr}[\sigma^n_i(t,x,\mu,\nu)^\top z]+f^n_i(t,x,\mu,\nu,\alpha)\ ,\end{aligned}$$

for any $(t,x,\mu,\nu,y,z,\alpha)\in[0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)^2\times\mathbb{R}^d\times\mathbb{R}^{d\times d}\times A_i$. Since $\partial_\alpha H^n_i(t,x,\mu,\nu,y,z,\alpha)=b_{i,2}(t,\phi_n\circ\mu,\phi_n\circ\nu)^\top y+\partial_\alpha f_i(t,x,\phi_n\circ\mu,\phi_n\circ\nu,\alpha)$, the minimizer given as a solution to the variational inequality (3.5) satisfies

$$\hat\alpha^n_i(t,x,\mu,\nu,y)=\hat\alpha_i(t,x,\phi_n\circ\mu,\phi_n\circ\nu,y)\ ,\tag{3.23}$$

where $\hat\alpha_i$ is the minimizer of the original Hamiltonian H_i. The regularization for $b_{i,2}$ is done solely to obtain the simple expression (3.23) for the minimizer.

The new coefficient functions $(b^n_i,\sigma^n_i,f^n_i,g^n_i)$ clearly satisfy **(MFG-a,b)** for each n. Thus Theorem 3.3 guarantees that there exists a solution to the following system of FBSDEs of MKV-type with $i,j\in\{1,2\}$, $j\neq i$:

$$\begin{aligned}dX^{i,n}_t&=b^n_i(t,X^{i,n}_t,\mathcal{L}(X^{i,n}_t),\mathcal{L}(X^{j,n}_t),\hat\alpha^n_i(t,X^{i,n}_t,\mathcal{L}(X^{i,n}_t),\mathcal{L}(X^{j,n}_t),Y^{i,n}_t))dt\\&\quad+\sigma^n_i(t,X^{i,n}_t,\mathcal{L}(X^{i,n}_t),\mathcal{L}(X^{j,n}_t))dW^i_t\ ,\\dY^{i,n}_t&=-\partial_x H^n_i(t,X^{i,n}_t,\mathcal{L}(X^{i,n}_t),\mathcal{L}(X^{j,n}_t),Y^{i,n}_t,Z^{i,n}_t,\hat\alpha^n_i(t,X^{i,n}_t,\mathcal{L}(X^{i,n}_t),\mathcal{L}(X^{j,n}_t),Y^{i,n}_t))dt+Z^{i,n}_t dW^i_t\ ,\end{aligned}\tag{3.24}$$

with $X^{i,n}_0=\xi^i$ and $Y^{i,n}_T=\partial_x g^n_i(X^{i,n}_T,\mathcal{L}(X^{i,n}_T),\mathcal{L}(X^{j,n}_T))$. Treating $(\mathcal{L}(X^{i,n}_t))_{1\le i\le 2}$ as inputs, we can see that there exist some constants $C=C(L,\lambda)$ and $C'=C'(L,\lambda,K)$ such that

$$|Y^{i,n}_t|\le C\Big(|X^{i,n}_t|+\sup_{s\in[t,T]}\sum_{j=1}^2 M_2(\mathcal{L}(X^{j,n}_s))\Big)+C'.$$

from the growth estimate in Lemma 3.2. It then follows from Lemma 3.1 that

$$|\hat\alpha^n_i(t,X^{i,n}_t,\mathcal{L}(X^{i,n}_t),\mathcal{L}(X^{j,n}_t),Y^{i,n}_t)|\le C\Big(|X^{i,n}_t|+\sup_{s\in[t,T]}\sum_{j=1}^2 M_2(\mathcal{L}(X^{j,n}_s))\Big)+C',$$

uniformly in n. Then, for any $t\in[0,T]$, it is easy to check that

$$\begin{aligned}\mathbb{E}\big[|X_t^{i,n}|^2\big] &\le C\mathbb{E}\Big[|\xi^i|^2+\int_0^t\big[|b_i(s,X_s^{i,n},\mathcal{L}(X_s^{i,n}),\mathcal{L}(X_s^{j,n}),\hat{\alpha}_i^n(t))|^2+|\sigma_i(t,X_t^{i,n},\mathcal{L}(X_t^{i,n}),\mathcal{L}(X_t^{j,n}))|^2dt\Big]\\ &\le C'+C\Big(||\xi^i||_2^2+T\sup_{s\in[0,T]}\sum_{j=1}^2 M_2(\mathcal{L}(X_s^{j,n}))^2\Big)+C\int_0^t\sum_{j=1}^2\mathbb{E}\big[|X_s^{j,n}|^2\big]ds\ ,\end{aligned}\tag{3.25}$$

with $C=C(L,\lambda)$. Applying Gronwall's inequality to the summation over $1\le i\le 2$, we get

$$\sup_{t\in[0,T]}\sum_{i=1}^2\mathbb{E}\big[|X_t^{i,n}|^2\big]\le C'+C\Big(||\boldsymbol{\xi}||_2^2+T\sup_{t\in[0,T]}\sum_{i=1}^2 M_2(\mathcal{L}(X_t^{i,n}))^2\Big)\ ,$$

with $\boldsymbol{\xi}:=(\xi^1,\xi^2)$. Therefore, there exists a constant c depending only on (L,λ) such that, for any $T\le c$,

$$\sup_{t\in[0,T]}\sum_{i=1}^2\mathbb{E}\big[|X_t^{i,n}|^2\big]\le C'\big(1+||\boldsymbol{\xi}||_2^2\big)\tag{3.26}$$

uniformly in n.

Let us assume $T\le c$ in the remainder. From (3.26), we can show straightforwardly that $\mathbb{E}\big[|X_t^{i,n}-X_s^{i,n}|^2\big]\le C|t-s|$ and $\mathbb{E}\big[\sup_{t\in[0,T]}|X_t^{i,n}|^4|\mathcal{F}_0\big]^{\frac12}\le C\big(1+|\xi^i|^2\big)$ hold uniformly in n. Just as in (3.22), we have for any $a\ge 1$,

$$\sup_{n\ge1}\mathbb{E}\Big[\sup_{t\in[0,T]}|X_t^{i,n}|^2\mathbf{1}_{\{\sup_{t\in[0,T]}|X_t^{i,n}|\ge a\}}\Big]\le C\Big(a^{-1}+\mathbb{E}\big[|\xi^i|^2\mathbf{1}_{\{|\xi^i|\ge\sqrt{a}\}}\big]\Big)^{\frac12}.$$

Hence, combined with the equicontinuity, we conclude that $(\mathcal{L}(X_t^{1,n})_{t\in[0,T]},\mathcal{L}(X_t^{2,n})_{t\in[0,T]})_{n\ge1}$ is a relatively compact subset of $\mathcal{C}([0,T];\mathcal{P}_2(\mathbb{R}^d))^2$. Therefore, there exists some $(\boldsymbol{\mu}^1,\boldsymbol{\mu}^2)\in\mathcal{C}([0,T];\mathcal{P}_2(\mathbb{R}^d))^2$ such that, upon extracting some subsequence (still denoted by n),

$$\lim_{n\to\infty}\sup_{t\in[0,T]}W_2(\mathcal{L}(X_t^{i,n}),\mu_t^i)=0,\ i\in\{1,2\}\ .$$

Let us define $(X_t^i,Y_t^i,Z_t^i)_{t\in[0,T],1\le i\le2}$ as the solutions to the FBSDEs (3.7) with those $(\boldsymbol{\mu}^1,\boldsymbol{\mu}^2)$ as inputs. The convergence $\mathbb{E}\big[\sup_{t\in[0,T]}|X_t^{i,n}-X_t^i|^2\big]\to0$ as $n\to\infty$ can be shown by the stability result in Lemma 3.2. Note that, thanks to the boundedness of (3.26), there exists $n_0\in\mathbb{N}$ such that we can replace all the approximated coefficients $(b_i^n,\sigma_i^n,f_i^n,g_i^n,\hat{\alpha}_i^n)$ by the original ones $(b_i,\sigma_i,f_i,g_i,\hat{\alpha}_i)$ for any $n\ge n_0$. The convergence $\mathbb{E}\big[\sup_{t\in[0,T]}|X_t^{i,n}-X_t^i|^2\big]\to0$ then follows easily by the dominated convergence theorem. By the inequality $\sup_{t\in[0,T]}W_2(\mathcal{L}(X_t^i),\mu_t^i)\le\sup_{t\in[0,T]}\big(W_2(\mathcal{L}(X_t^i),\mathcal{L}(X_t^{i,n}))+W_2(\mathcal{L}(X_t^{i,n}),\mu_t^i)\big)$, the above convergence implies $\boldsymbol{\mu}^i=\mathcal{L}(X_t^i)_{t\in[0,T]}$ $1\le i\le2$. Therefore, $(X_t^i,Y_t^i,Z_t^i)_{t\in[0,T],1\le i\le2}$ is actually a wanted solution to the system of FBSDEs (3.19). □

Another simple method to allow the quadratic cost functions is making the couplings between FSDE and BSDE small enough.

Theorem 3.5. *Under Assumption* **(MFG-a)** *and a given T, the system of FBSDEs* (3.19) *(and hence the matching problem* (3.2)*) is solvable for any $\xi^1,\xi^2\in\mathbb{L}^2(\Omega,\mathcal{F}_0,\mathbb{P};\mathbb{R}^d)$ if $\lambda^{-1}||b_{i,2}||_\infty$, $1\le i\le2$ are small enough.*

Proof. By the growth estimate for $\hat{\alpha}_i$ in (3.6), it is straightforward to check that the term involving

$\sup_{s\in[0,T]} M_2(\mathcal{L}(X_s^{j,n}))^2$ in (3.25) is proportional to $\lambda^{-1}||b_{i,2}||_\infty$. Thus, by making $\lambda^{-1}||b_{i,2}||_\infty$ small enough for a given T, we obtain the same estimate (3.26). The remaining procedures for the proof are exactly the same as in Theorem 3.4. $\square$

Remark 3.3. *As one can see, there is no difficulty to generalize all the analyses in Section 3 for any finite number of populations $1 \leq i \leq m$. It results in a search for a fixed point in the map $\mathcal{C}([0,T];\mathcal{P}_2(\mathbb{R}^d))^m \ni (\boldsymbol{\mu}^i)_{i=1}^m \mapsto (\mathcal{L}(X_t^i)_{t\in[0,T]})_{i=1}^m \in \mathcal{C}([0,T];\mathcal{P}_2(\mathbb{R}^d))^m$, which can be done in the same way.*

4 Games among Cooperative Populations

In this section, we try to establish the existence of a Nash equilibrium between two competing populations within each of which the agents share the same cost functions as well as the coefficient functions of the state dynamics. The difference from the situation studied in Section 3 is that the agents within each population now cooperate by using the common feedback strategy, say, under the command of a central planner. This results in a control problem of McKean-Vlasov type in the large population limit. See Chapter 6 in [10] to understand the details how the large population limit of cooperative agents induces a control problem of MKV type. The current problem has been discussed in Section 3 in Bensoussan et.al.[3] in the name *Nash Mean Field Type Control Problem*, where the necessary conditions of the optimality are provided in the form of a master equation. In this section, we adopt the probabilistic approach developed in Carmona & Delarue (2015) [9] for single population, and then provide several sets of sufficient conditions for the existence of an equilibrium.

4.1 Definition of Nash Mean Field Type Control Problem

Let us first formulate the problem to be studied in this section.
(i) Fix any two deterministic flows of probability measures $(\boldsymbol{\mu}^i = (\mu_t^i)_{t\in[0,T]})_{i\in\{1,2\}}$ given on $\mathbb{R}^d$.
(ii) Solve the two optimal control problems of McKean-Vlasov type

$$\inf_{\boldsymbol{\alpha}^1\in\mathbb{A}_1} J_1^{\boldsymbol{\mu}^2}(\boldsymbol{\alpha}^1), \quad \inf_{\boldsymbol{\alpha}^2\in\mathbb{A}_2} J_2^{\boldsymbol{\mu}^1}(\boldsymbol{\alpha}^2) \tag{4.1}$$

over some admissible strategies $\mathbb{A}_i$ ($i \in \{1,2\}$), where

$$J_1^{\boldsymbol{\mu}^2}(\boldsymbol{\alpha}^1) := \mathbb{E}\Big[\int_0^T f_1(t, X_t^1, \mathcal{L}(X_t^1), \mu_t^2, \alpha_t^1)dt + g_1(X_T^1, \mathcal{L}(X_T^1), \mu_T^2)\Big] ,$$
$$J_2^{\boldsymbol{\mu}^1}(\boldsymbol{\alpha}^2) := \mathbb{E}\Big[\int_0^T f_2(t, X_t^2, \mathcal{L}(X_t^2), \mu_t^1, \alpha_t^2)dt + g_2(X_T^2, \mathcal{L}(X_T^2), \mu_T^1)\Big] ,$$

subject to the d-dimensional diffusion dynamics of McKean-Vlasov type:

$$dX_t^1 = b_1(t, X_t^1, \mathcal{L}(X_t^1), \mu_t^2, \alpha_t^1)dt + \sigma_1(t, X_t^1, \mathcal{L}(X_t^1), \mu_t^2)dW_t^1 ,$$
$$dX_t^2 = b_2(t, X_t^2, \mathcal{L}(X_t^2), \mu_t^1, \alpha_t^2)dt + \sigma_2(t, X_t^2, \mathcal{L}(X_t^2), \mu_t^1)dW_t^2 ,$$

for $t \in [0,T]$ with $\big(X_0^i = \xi^i \in \mathbb{L}^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d)\big)_{1\leq i\leq 2}$. For each population $i \in \{1,2\}$, we suppose, as before, that $\mathbb{A}_i$ is the set of A_i-valued $\mathbb{F}^i$-progressively measurable processes $\boldsymbol{\alpha}^i$ satisfying $\mathbb{E}\int_0^T |\alpha_t^i|^2 dt < \infty$ and $A_i \subset \mathbb{R}^k$ is closed and convex.

(iii) Find a pair of probability flows $(\boldsymbol{\mu}^1, \boldsymbol{\mu}^2)$ as a solution to the matching problem:

$$\forall t \in [0,T], \quad \mu_t^1 = \mathcal{L}(\hat{X}_t^{1,\boldsymbol{\mu}^2}), \quad \mu_t^2 = \mathcal{L}(\hat{X}_t^{2,\boldsymbol{\mu}^1}) \ , \tag{4.2}$$

where $(\hat{X}^{i,\boldsymbol{\mu}^j})_{i\in\{1,2\},j\neq i}$ are the solutions to the optimal control problems in (ii).

4.2 Optimization for given flows of probability measures

In this subsection, we consider the step (ii) in the above formulation. Before giving the set of main assumptions, let us mention the notion of differentiability for functions defined on the space of probability measures. We adopt the notion of L-differentiability used in [9], which was first introduced by Lions in his lecture at the *College de France* (see the lecture notes summarized in [6]), where the differentiation is based on the *lifting* of functions $\mathcal{P}_2(\mathbb{R}^d) \ni \mu \to u(\mu)$ to functions $\widetilde{u}$ defined on a Hilbert space $\mathbb{L}^2(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^d)$ by $\widetilde{u}(X) := u(\mathcal{L}(X))$ with $X \in \mathbb{L}^2(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^d)$ over some probability space $(\Omega,\mathcal{F},\mathbb{P})$ with Ω being a Polish space and $\mathbb{P}$ an atomless probability measure.

Definition 4.1. *(Definition 5.22 in [10]) A function u on $\mathcal{P}_2(\mathbb{R}^d)$ is said to be L-differentiable at $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ if there exists a random variable X_0 with law μ_0 such that the lifted function $\widetilde{u}$ is Frechet differentiable at X_0.*

By Proposition 5.24 [10], if u is L-differentiable at μ_0 in the sense of Definition 4.1, then $\widetilde{u}$ is differentiable at any X_0' with $\mathcal{L}(X_0') = \mu_0$ and the law of the pair $(X_0', D\widetilde{u}(X_0'))$ is independent of the choice of the random variable X_0'. Thus the L-derivative may be denoted by $\partial_\mu u(\mu_0)(\cdot) : \mathbb{R}^d \ni x \mapsto \partial_\mu u(\mu_0)(x) \in \mathbb{R}^d$, which is uniquely defined μ_0-almost everywhere on $\mathbb{R}^d$. It satisfies, according to the definition, that:

$$u(\mu) = u(\mu_0) + \mathbb{E}\big[\langle X - X_0, \partial_\mu u(\mathcal{L}(X_0))(X_0)\rangle\big] + o(||X - X_0||_2) \ ,$$

whenever the random variables X and X_0 have the distributions $\mathcal{L}(X) = \mu,\ \mathcal{L}(X_0) = \mu_0$. For example, if the function u is of the form $u(\mu) := \int_{\mathbb{R}^d} h(x)\mu(dx)$ for some function $h : \mathbb{R}^d \to \mathbb{R}$, we have $\widetilde{u}(X) = \mathbb{E}[h(X)]$ by a random varibale X with $\mathcal{L}(X) = \mu$. If the function h is differentiable, the definition of L-derivative implies that $\partial_\mu u(\mu)(\cdot) = \partial_x h(\cdot)$. For details of L-derivatives, their regularity properties and examples, see Section 6 in [6] and Chapter 5 in [10]. We now give the main assumptions in this section:[3]

Assumption 4.1. (MFTC-a) *$L, K \geq 0$ and $\lambda > 0$ are some constants. For $1 \leq i \leq 2$, the measurable functions $b_i : [0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)^2\times A_i \to \mathbb{R}^d$, $\sigma_i : [0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)^2 \to \mathbb{R}^{d\times d}$, $f_i : [0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)^2\times A_i \to \mathbb{R}$, and $g_i : \mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)^2 \to \mathbb{R}$ satisfy the following conditions:*
(A1) *The functions b_i and σ_i are affine in $(x,\alpha,\bar{\mu})$ in the sense that, for any $(t,x,\mu,\nu,\alpha) \in [0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)^2\times A_i$,*

$$\begin{aligned}
b_i(t,x,\mu,\nu,\alpha) &:= b_{i,0}(t,\nu) + b_{i,1}(t,\nu)x + \bar{b}_{i,1}(t,\nu)\bar{\mu} + b_{i,2}(t,\nu)\alpha \ , \\
\sigma_i(t,x,\mu,\nu) &:= \sigma_{i,0}(t,\nu) + \sigma_{i,1}(t,\nu)x + \bar{\sigma}_{i,1}(t,\nu)\bar{\mu} \ ,
\end{aligned}$$

where $\bar{\mu} := \int_{\mathbb{R}^d} x\mu(dx)$, and $b_{i,0}, b_{i,1}, \bar{b}_{i,1}, b_{i,2}, \sigma_{i,0}, \sigma_{i,1}$ and $\bar{\sigma}_{i,1}$ defined on $[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$ are $\mathbb{R}^d$, $\mathbb{R}^{d\times d}$, $\mathbb{R}^{d\times d}$, $\mathbb{R}^{d\times k}$, $\mathbb{R}^{d\times d}$, $\mathbb{R}^{d\times d\times d}$ and $\mathbb{R}^{d\times d\times d}$-valued measurable functions, respectively.
(A2) *For any $t \in [0,T]$, the functions $\mathcal{P}_2(\mathbb{R}^d) \ni \nu \mapsto (b_{i,0}, b_{i,1}, \bar{b}_{i,1}, b_{i,2}, \sigma_{i,0}, \sigma_{i,1}, \bar{\sigma}_{i,1})(t,\nu)$ are continuous in W_2-distance. Moreover for any $(t,\nu) \in [0,T]\times\mathcal{P}_2(\mathbb{R}^d)$,*

$$\begin{aligned}
&|b_{i,0}(t,\nu)|, |\sigma_{i,0}(t,\nu)| \leq K + LM_2(\nu) \ , \\
&|b_{i,1}(t,\nu)|, |\bar{b}_{i,1}(t,\nu)|, |b_{i,2}(t,\nu)|, |\sigma_{i,1}(t,\nu)|, |\bar{\sigma}_{i,1}(t,\nu)| \leq L \ .
\end{aligned}$$

[3] We slightly abuse the notation to lighten the expression.

(A3) *For any* $t \in [0,T], x, x' \in \mathbb{R}^d, \mu, \mu', \nu \in \mathcal{P}_2(\mathbb{R}^d)$ *and* $\alpha, \alpha' \in A_i$*, the functions* f_i *and* g_i *satisfy the quadratic growth conditions*

$$|f_i(t,x,\mu,\nu,\alpha)| \le K + L\big(|x|^2 + |\alpha|^2 + M_2(\mu)^2 + M_2(\nu)^2\big) \ ,$$
$$|g_i(x,\mu,\nu)| \le K + L\big(|x|^2 + M_2(\mu)^2 + M_2(\nu)^2\big) \ ,$$

and the local Lipschitz continuity

$$\begin{aligned}&|f_i(t,x',\mu',\nu,\alpha') - f_i(t,x,\mu,\nu,\alpha)| + |g_i(x',\mu',\nu) - g_i(x,\mu,\nu)| \\ &\le \Big(K + L\big[|(x',\alpha')| + |(x,\alpha)| + M_2(\mu') + M_2(\mu) + M_2(\nu)\big]\Big)\big[|(x',\alpha') - (x,\alpha)| + W_2(\mu',\mu)\big] \ .\end{aligned}$$

(A4) *The functions* f_i *and* g_i *are once continuously differentiable in* (x,α) *and* x *respectively, and their derivatives are L-Lipschitz continuous with respect to* (x,α,μ) *and* (x,μ) *i.e.*

$$\begin{aligned}&|\partial_{(x,\alpha)} f_i(t,x',\mu',\nu,\alpha') - \partial_{(x,\alpha)} f_i(t,x,\mu,\nu,\alpha)| + |\partial_x g_i(x',\mu',\nu) - \partial_x g_i(x,\mu,\nu)| \\ &\le L\big(|x'-x| + |\alpha'-\alpha| + W_2(\mu',\mu)\big) \ ,\end{aligned}$$

for any $t \in [0,T], x, x' \in \mathbb{R}^d, \mu, \mu', \nu \in \mathcal{P}_2(\mathbb{R}^d), \alpha, \alpha' \in A_i$*. The derivatives also satisfy the growth condition*

$$|\partial_{(x,\alpha)} f_i(t,x,\mu,\nu,\alpha)| + |\partial_x g_i(x,\mu,\nu)| \le K + L\big(|x| + |\alpha| + M_2(\mu) + M_2(\nu)\big) \ .$$

Moreover, the derivatives $\partial_{(x,\alpha)} f_i$ *and* $\partial_x g_i$ *are continuous also in* ν *with respect to the* W_2*-distance.*
(A5) *The functions* f_i *and* g_i *are L-differentiable with respect to the first measure argument* μ *and they satisfy that, for any* $t \in [0,T], x, x' \in \mathbb{R}^d, \mu, \mu', \nu \in \mathcal{P}_2(\mathbb{R}^d), \alpha, \alpha' \in A_i$ *and any random variables* X, X' *with* $\mathcal{L}(X) = \mu$*,* $\mathcal{L}(X') = \mu'$*, L-Lipschitz continuity in* $\mathbb{L}^2$ *i.e.,*

$$\begin{aligned}&||\partial_\mu f_i(t,x',\mu',\nu,\alpha')(X') - \partial_\mu f_i(t,x,\mu,\nu,\alpha)(X)||_2 + ||\partial_\mu g_i(x',\mu',\nu)(X') - \partial_\mu g_i(x,\mu,\nu)(X)||_2 \\ &\le L\big(|x'-x| + |\alpha'-\alpha| + ||X'-X||_2\big) \ ,\end{aligned}$$

as well as the following growth condition:

$$||\partial_\mu f_i(t,x,\mu,\nu,\alpha)(X)||_2 + ||\partial_\mu g_i(x,\mu,\nu)(X)||_2 \le K + L\big(|x| + |\alpha| + M_2(\mu) + M_2(\nu)\big) \ .$$

Moreover, the maps $\mathcal{P}_2(\mathbb{R}^d) \ni \nu \mapsto \partial_\mu f_i(t,x,\mu,\nu,\alpha)(v)$ *and* $\mathcal{P}_2(\mathbb{R}^d) \ni \nu \mapsto \partial_\mu g_i(x,\mu,\nu)(v)$ *are continuous with respect to* W_2*-distance for any* $(t,x,\mu,\alpha) \in [0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A_i$*, and* μ*-a.e.* $v \in \mathbb{R}^d$*.*
(A6) *For any* $t \in [0,T], x, x' \in \mathbb{R}^d, \mu, \mu', \nu \in \mathcal{P}_2(\mathbb{R}^d), \alpha, \alpha' \in A_i$*, and any random variables* X, X' *with* $\mathcal{L}(X) = \mu, \mathcal{L}(X') = \mu'$*, the functions* f_i *and* g_i *satisfy the convexity relations:*

$$\begin{aligned}&f_i(t,x',\mu',\nu,\alpha') - f_i(t,x,\mu,\nu,\alpha) - \langle (x'-x,\alpha'-\alpha), \partial_{(x,\alpha)} f_i(t,x,\mu,\nu,\alpha)\rangle \\ &\quad -\mathbb{E}\big[\langle X'-X, \partial_\mu f_i(t,x,\mu,\nu,\alpha)(X)\rangle\big] \ge \lambda|\alpha'-\alpha|^2 \ , \\ &g_i(x',\mu',\nu) - g_i(x,\mu,\nu) - \langle x'-x, \partial_x g_i(x,\mu,\nu)\rangle - \mathbb{E}\big[\langle X'-X, \partial_\mu g_i(x,\mu,\nu)(X)\rangle\big] \ge 0 \ .\end{aligned}$$

Remark 4.1. *By Lemma 3.3 in [9], the Lipschitz continuity in* **(A5)** *above implies that we can modify* $\partial_\mu f_i(t,x,\mu,\nu,\alpha)(\cdot)$ *and* $\partial_\mu g_i(x,\mu,\nu)(\cdot)$ *on a* μ*-negligible set in such a way that,* $\forall v, v' \in \mathbb{R}^d$

$$|\partial_\mu f_i(t,x,\mu,\nu,\alpha)(v') - \partial_\mu f_i(t,x,\mu,\nu,\alpha)(v)| \le L|v'-v| \ ,$$
$$|\partial_\mu g_i(x,\mu,\nu)(v') - \partial_\mu g_i(x,\mu,\nu)(v)| \le L|v'-v| \ ,$$

for any $(t,x,\mu,\nu,\alpha)\in[0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)^2\times A_i$. In the remainder of the work, we always use these Lipschitz continuous versions.

As before, we first consider the optimal control problem (4.1) for given deterministic flows of probability measures. The Hamiltonian for each population $H_i:[0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)^2\times\mathbb{R}^d\times\mathbb{R}^{d\times d}\times A_i\ni(t,x,\mu,\nu,y,z,\alpha)\mapsto H_i(t,x,\mu,\nu,y,z,\alpha)\in\mathbb{R}$ and its minimizer $\hat{\alpha}_i:[0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)^2\times\mathbb{R}^d\ni(t,x,\mu,\nu,y)\mapsto\hat{\alpha}_i(t,x,\mu,\nu,y)\in A_i$ are defined in the same way as (3.3) and (3.4) with the coefficients replaced by those given in the current section.

Lemma 4.1. *Under Assumption* **(MFTC-a)***, for all $(t,x,\mu,\nu,y)\in[0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)^2\times\mathbb{R}^d$, there exists a unique minimizer $\hat{\alpha}_i(t,x,\mu,\nu,y)$ of $H_i^{(r)}$, where the map $[0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)^2\times\mathbb{R}^d\ni(t,x,\mu,\nu,y)\mapsto\hat{\alpha}_i(t,x,\mu,\nu,y)\in A_i$ is measurable. There exist constants C depending only on (L,λ) and C' depending additionally on K such that, for any $t\in[0,T],x,x',y,y'\in\mathbb{R}^d,\mu,\nu\in\mathcal{P}_2(\mathbb{R}^d)$,*

$$
\begin{aligned}
&|\hat{\alpha}_i(t,x,\mu,\nu,y)|\le C'+C\big(|x|+|y|+M_2(\mu)+M_2(\nu)\big)\\
&|\hat{\alpha}_i(t,x,\mu,\nu,y)-\hat{\alpha}_i(t,x',\mu,\nu,y')|\le C\big(|x-x'|+|y-y'|\big)\ .
\end{aligned}
$$

Moreover, for any $(t,x,y)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^d$, the map $\mathcal{P}_2(\mathbb{R}^d)^2\ni(\mu,\nu)\mapsto\hat{\alpha}_i(t,x,\mu,\nu,y)$ is continuous with respect to W_2-distance:

$$
\begin{aligned}
&|\hat{\alpha}_i(t,x,\mu,\nu,y)-\hat{\alpha}_i(t,x,\mu',\nu',y)|\\
&\le(2\lambda)^{-1}\Big(LW_2(\mu,\mu')+|b_{i,2}(t,\nu)-b_{i,2}(t,\nu')||y|+|\partial_\alpha f_i(t,x,\mu',\nu,\hat{\alpha}_i)-\partial_\alpha f_i(t,x,\mu',\nu',\hat{\alpha}_i)|\Big)
\end{aligned}
$$

where $\hat{\alpha}_i:=\hat{\alpha}_i(t,x,\mu,\nu,y)$.

Proof. It can be shown exactly in the same way as Lemma 3.1. □

The control problem (4.1) for each population $1\le i\le 2$ with a given flow of probability measure $\boldsymbol{\mu}^j\in\mathcal{C}([0,T];\mathcal{P}_2(\mathbb{R}^d)), j\neq i$ is actually the special case studied in [9] and Section 6.4 in [10]. In fact, we have removed the control $\boldsymbol{\alpha}_i$ dependency from the diffusion coefficient σ_i.[4] The relevant adjoint equations for the optimal control problem of MKV-type (4.1) are given by with $i,j\in\{1,2\},j\neq i$:

$$
\begin{aligned}
dX_t^i&=b_i(t,X_t^i,\mathcal{L}(X_t^i),\mu_t^j,\hat{\alpha}_i(t,X_t^i,\mathcal{L}(X_t^i),\mu_t^j,Y_t^i))dt+\sigma_i(t,X_t^i,\mathcal{L}(X_t^i),\mu_t^j)dW_t^i\ ,\\
dY_t^i&=-\partial_x H_i(t,X_t^i,\mathcal{L}(X_t^i),\mu_t^j,Y_t^i,Z_t^i,\hat{\alpha}_i(t,X_t^i,\mathcal{L}(X_t^i),\mu_t^j,Y_t^i))dt\\
&\quad-\widetilde{\mathbb{E}}\big[\partial_\mu H_i(t,\widetilde{X}_t^i,\mathcal{L}(X_t^i),\mu_t^j,\widetilde{Y}_t^i,\widetilde{Z}_t^i,\hat{\alpha}_i(t,\widetilde{X}_t^i,\mathcal{L}(X_t^i),\mu_t^j,\widetilde{Y}_t^i))(X_t^i)\big]dt+Z_t^i dW_t^i\ ,
\end{aligned}
\tag{4.3}
$$

with $X_0^i=\xi^i\in\mathbb{L}^2(\Omega,\mathcal{F}_0,\mathbb{P};\mathbb{R}^d)$ and $Y_T^i=\partial_x g_i(X_T^i,\mathcal{L}(X_T^i),\mu_T^j)+\widetilde{\mathbb{E}}\big[\partial_\mu g_i(\widetilde{X}_T^i,\mathcal{L}(X_T^i),\mu_T^j)(X_T^i)\big]$. Here, $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$ denotes a copy of $(\Omega,\mathcal{F},\mathbb{P})$ and every random variable with tilde, such as $\widetilde{X}$, denotes a clone of X on $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$. The expectation under $\widetilde{\mathbb{P}}$ is denoted by $\widetilde{\mathbb{E}}$. More explicitly, one can write (4.3) as

$$
\begin{aligned}
dX_t^i&=\big(b_{i,0}(t,\mu_t^j)+b_{i,1}(t,\mu_t^j)X_t^i+\bar{b}_{i,1}(t,\mu_t^j)\mathbb{E}[X_t^i]+b_{i,2}(t,\mu_t^j)\hat{\alpha}_i(t,X_t^i,\mathcal{L}(X_t^i),\mu_t^j,Y_t^i)\big)dt\\
&\quad+\big(\sigma_{i,0}(t,\mu_t^j)+\sigma_{i,1}(t,\mu_t^j)X_t^i+\bar{\sigma}_{i,1}(t,\mu_t^j)\mathbb{E}[X_t^i]\big)dW_t^i\ ,\\
dY_t^i&=-\big(b_{i,1}(t,\mu_t^j)^\top Y_t^i+\sigma_{i,1}(t,\mu_t^j)^\top Z_t^i+\partial_x f_i(t,X_t^i,\mathcal{L}(X_t^i),\mu_t^j,\hat{\alpha}_i(t,X_t^i,\mathcal{L}(X_t^i),\mu_t^j,Y_t^i))\big)dt\\
&\quad-\big(\bar{b}_{i,1}(t,\mu_t^j)^\top\mathbb{E}[Y_t^i]+\bar{\sigma}_{i,1}(t,\mu_t^j)^\top\mathbb{E}[Z_t^i]+\widetilde{\mathbb{E}}\big[\partial_\mu f_i(t,\widetilde{X}_t^i,\mathcal{L}(X_t^i),\mu_t^j,\hat{\alpha}_i(t,\widetilde{X}_t^i,\mathcal{L}(X_t^i),\mu_t^j,\widetilde{Y}_t^i))(X_t^i)\big]dt\\
&\quad+Z_t^i dW_t^i\ ,
\end{aligned}
$$

[4] It then makes possible to derive the stability relation irrespective of the size of Lipschitz constant for Z.

which is a $C(L,\lambda)$-Lipschitz FBSDE of McKean-Vlasov type. Note that due to Lemma 4.1, $\hat{\alpha}_i$ is Lipschitz continuous not only in (X^i, Y^i) but also in $\mathcal{L}(X^i)$. For each $i \in \{1,2\}$, it is important to notice that the Lipschitz constant is independent of the given flow $\boldsymbol{\mu}^j, j \neq i$. Since Assumption **(MFTC-a)** satisfies every solvability condition used in [9], we have the following results:[5]

Theorem 4.1. *Under Assumption* **(MFTC-a)**, *the adjoint FBSDE* (4.3) *of each* $i \in \{1,2\}$ *has a unique solution* $(\hat{X}^i_t, \hat{Y}^i_t, \hat{Z}^i_t)_{t\in[0,T]} \in \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{H}^2$ *for any flow* $\boldsymbol{\mu}^j \in \mathcal{C}([0,T];\mathcal{P}_2(\mathbb{R}^d))$ *and any initial condition* $\xi^i \in \mathbb{L}^2(\Omega,\mathcal{F}_0,\mathbb{P};\mathbb{R}^d)$. *If we set* $\hat{\boldsymbol{\alpha}}_i = \big(\hat{\alpha}^i_t = \hat{\alpha}_i(t,\hat{X}^i_t,\mathcal{L}(X^i_t),\mu^j_t,\hat{Y}^i_t)\big)_{t\in[0,T]}$, *then it gives the optimal control. In particular, the inequality* $J_i^{\boldsymbol{\mu}^j}(\hat{\boldsymbol{\alpha}}^i) + \lambda\mathbb{E}\int_0^T|\beta^i_t - \hat{\alpha}^i_t|^2dt \le J_i^{\boldsymbol{\mu}^j}(\boldsymbol{\beta}^i)$ *holds for any* $\boldsymbol{\beta}^i \in \mathbb{A}_i$.

Proof. This is the direct result of Theorem 4.7 (sufficiency) and Theorem 5.1 (unique solvability) by Carmona & Delarue (2015)[9], where the sufficiency is proved in a parallel way to Theorem 3.1, and the unique solvability is based on *the continuation method* developed by Peng & Wu (1999) [30]. □

Lemma 4.2. *Under the same conditions used in Theorem 4.1, for any* $t \in [0,T]$ *and any* $\xi^i \in \mathbb{L}^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^d)$, *there exists a unique solution, denoted by* $(X^{i,t,\xi^i}_s, Y^{i,t,\xi^i}_s, Z^{i,t,\xi^i}_s)_{t\le s\le T}$, *of* (4.3) *on* $[t,T]$ *with* $X^{i,t,\xi^i}_t = \xi^i$ *as initial condition. Moreover, for any* $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, *there exists a measurable mapping* $u_i^{\boldsymbol{\mu}^j}(t,\cdot,\mu): \mathbb{R}^d \ni x \mapsto u_i^{\boldsymbol{\mu}^j}(t,x,\mu)$ *with* $Y^{i,t,\xi^i}_t = u_i^{\boldsymbol{\mu}^j}(t,\xi^i,\mathcal{L}(\xi^i))$ $\mathbb{P}$*-a.s. such that, for any* $\xi^i, \xi'^{,i} \in \mathbb{L}^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^d)$,

$$\mathbb{E}\big[|u_i^{\boldsymbol{\mu}^j}(t,\xi^i,\mathcal{L}(\xi^i)) - u_i^{\boldsymbol{\mu}^j}(t,\xi'^{,i},\mathcal{L}(\xi'^{,i}))|^2\big]^{\frac{1}{2}} \le C\mathbb{E}\big[|\xi^i - \xi'^{,i}|^2\big]^{\frac{1}{2}}\ , \tag{4.4}$$

with some constant C *depending only on* L *and* λ.

Proof. This is a direct result of Lemma 5.6 in [9]. The Lipschitz constant can be read from the stability estimate used in the continuation method (Lemma 5.5 in [9]), which is dependent only on the Lipschitz constant of the FBSDE. □

Remark 4.2. . *Note that, due to the uniqueness of the solution, we have for any* $t \in [0,T]$, $\hat{Y}^i_t = Y^{i,0,\xi^i}_t = Y^{i,t,\hat{X}^i_t}_t = u_i^{\boldsymbol{\mu}^j}(t,\hat{X}^i_t,\mathcal{L}(\hat{X}^i_t))$ $\mathbb{P}$*-a.s. Moreover, once again by Lemma 3.3 in [9], for any* $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, *there exists a version* $\mathbb{R}^d \ni x \mapsto u_i^{\boldsymbol{\mu}^j}(t,x,\mu)$ *in* $\mathbb{L}^2(\mathbb{R}^d,\mu)$ *that is Lipschitz continuous with the same Lipschitz constant* C *used in* (4.4) *i.e.,* $|u_i^{\boldsymbol{\mu}^j}(t,x,\mu) - u_i^{\boldsymbol{\mu}^j}(t,x',\mu)| \le C|x-x'|$ *for any* $x,x' \in \mathbb{R}^d$. *In the remainder, we always use this Lipschitz version and often adopt a simpler notation* u_i *without the superscript* $\boldsymbol{\mu}^j$.

Making use of the Lipschitz continuity in Lemma 4.2, we can derive the stability relation.

Lemma 4.3. *Suppose that the two set of functions* (b_i,σ_i,f_i,g_i) *and* $(b'_i,\sigma'_i,f'_i,g'_i)$ *satisfy Assumption* **(MFTC-a)**. *For given inputs* $\xi^i, \xi'^{,i} \in \mathbb{L}^2(\Omega,\mathcal{F}_0,\mathbb{P};\mathbb{R}^d)$ *and* $\boldsymbol{\mu}^j, \boldsymbol{\mu}'^{,j} \in \mathcal{C}([0,T];\mathcal{P}_2(\mathbb{R}^d))$, *let us denote the corresponding solution to* (4.3) *by* $(X^i_t,Y^i_t,Z^i_t)_{t\in[0,T]}$ *and* $(X'^{,i}_t,Y'^{,i}_t,Z'^{,i}_t)_{t\in[0,T]}$, *respec-*

[5] In [9], $A = \mathbb{R}^k$ is assumed. However, there is no difficulty for extending a general closed and convex subset $A \subset \mathbb{R}^k$, which is actually the case studied in Chapter 6 in [10].

tively. Then, there exists a constant C depending only on (L,λ) such that

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T]}|X_t^i-X_t^{\prime,i}|^2+\sup_{t\in[0,T]}|Y_t^i-Y_t^{\prime,i}|^2+\int_0^T|Z_t^i-Z_t^{\prime,i}|^2dt\Big]\\
&\le C\mathbb{E}\Big\{|\xi^i-\xi^{\prime,i}|^2+|\partial_x g_i(X_T^i,\mathcal{L}(X_T^i),\mu_T^j)-\partial_x g_i'(X_T^i,\mathcal{L}(X_T^i),\mu_T^{\prime,j})|^2\\
&\qquad\qquad+\widetilde{\mathbb{E}}\big[|\partial_\mu g_i(\widetilde{X}_T^i,\mathcal{L}(X_T^i),\mu_T^j)(X_T^i)-\partial_\mu g_i'(\widetilde{X}_T^i,\mathcal{L}(X_T^i),\mu_T^{j,\prime})(X_T^i)|^2\big]\\
&+\int_0^T|b_i(t,X_t^i,\mathcal{L}(X_t^i),\mu_t^j,\hat{\alpha}_i(t,X_t^i,\mathcal{L}(X_t^i),\mu_t^j,Y_t^i))-b_i'(t,X_t^i,\mathcal{L}(X_t^i),\mu_t^{\prime,j},\hat{\alpha}_i'(t,X_t^i,\mathcal{L}(X_t^i),\mu_t^{\prime,j},Y_t^i))|^2dt\\
&+\int_0^T|\sigma_i(t,X_t^i,\mathcal{L}(X_t^i),\mu_t^j)-\sigma_i'(t,X_t^i,\mathcal{L}(X_t^i),\mu_t^{\prime,j})|^2dt\\
&+\int_0^T\Big(|\partial_x H_i(t,X_t^i,\mathcal{L}(X_t^i),\mu_t^j,Y_t^i,Z_t^i,\hat{\alpha}_i(t,X_t^i,\mathcal{L}(X_t^i),\mu_t^j,Y_t^i))\\
&\qquad\qquad-\partial_x H_i'(t,X_t^i,\mathcal{L}(X_t^i),\mu_t^{\prime,j},Y_t^i,Z_t^i,\hat{\alpha}_i'(t,X_t^i,\mathcal{L}(X_t^i),\mu_t^{\prime,j},Y_t^i))|^2\Big)dt\\
&+\int_0^T\widetilde{\mathbb{E}}\Big[|\partial_\mu H_i(t,\widetilde{X}_t^i,\mathcal{L}(X_t^i),\mu_t^j,\widetilde{Y}_t^i,\widetilde{Z}_t^i,\hat{\alpha}_i(t,\widetilde{X}_t^i,\mathcal{L}(X_t^i),\mu_t^j,\widetilde{Y}_t^i))(X_t^i)\\
&\qquad\qquad-\partial_\mu H_i'(t,\widetilde{X}_t^i,\mathcal{L}(X_t^i),\mu_t^{\prime,j},\widetilde{Y}_t^i,\widetilde{Z}_t^i,\hat{\alpha}_i'(t,\widetilde{X}_t^i,\mathcal{L}(X_t^i),\mu_t^{\prime,j},\widetilde{Y}_t^i))(X_t^i)|^2\Big]dt\Big\}\ , \qquad (4.5)
\end{aligned}
$$

where the functions $H_i,H_i':[0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)^2\times\mathbb{R}^d\times\mathbb{R}^{d\times d}\times A_i\to\mathbb{R}$ are the Hamiltonians associated with the coefficients (b_i,σ_i,f_i) and (b_i',σ_i',f_i'), respectively, and $\hat{\alpha}_i,\hat{\alpha}_i'$ are their minimizers.

In particular, there is another constant C' depending additionally on K such that

$$
\mathbb{E}\Big[\sup_{t\in[0,T]}|X_t^i|^2+\sup_{t\in[0,T]}|Y_t^i|^2+\int_0^T|Z_t^i|^2dt\Big]\le C\Big(||\xi^i||_2^2+\sup_{t\in[0,T]}M_2(\mu_t^j)^2\Big)+C', \qquad (4.6)
$$

and, for any $t\in[0,T]$,

$$
|Y_t^i|\le C\Big(||\xi^i||_2+|X_t^i|+\sup_{s\in[0,T]}M_2(\mu_s^j)\Big)+C',\ \ \mathbb{P}\text{-a.s.} \qquad (4.7)
$$

Proof. It can be proved in the same way as Lemma 3.2. For small $T\le c$, where c is dependent only on (L,λ), using the inequality $W_2(\mathcal{L}(X),\mathcal{L}(Y))^2\le\mathbb{E}|X-Y|^2$, one can show the stability relation (4.5) exactly in the same way as in the standard Lipschitz FBSDE of non-MKV type. For general T, we can connect the short-term estimate by the same technique adopted in the proof of Lemma 3.2. Here, we make use of the Lipschitz continuity in Lemma 4.2.

As for the growth conditions, we get, by the same arguments used to derive (3.18),

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{t\in[0,T]}|X_t^i|^2+\sup_{t\in[0,T]}|Y_t^i|^2+\int_0^T|Z_t^i|^2dt\Big]\\
&\le C\Big(||\xi^i||_2^2+|\partial_x g_i(0,\delta_0,\mu_T^j)|^2+|\partial_\mu g_i(0,\delta_0,\mu_T^j)(0)|^2\\
&+\int_0^T\big(|b_i(t,0,\delta_0,\mu_t^j,\hat{\alpha}_i(t,0,\delta_0,\mu_t^j,0))|^2+|\sigma_i(t,0,\delta_0,\mu_t^j)|^2\big)dt\\
&+\int_0^T\big(|\partial_x f_i(t,0,\delta_0,\mu_t^j,\hat{\alpha}_i(t,0,\delta_0,\mu_t^j,0))|^2+|\partial_\mu f_i(t,0,\delta_0,\mu_t^j,\hat{\alpha}_i(t,0,\delta_0,\mu_t^j,0))(0)|^2\big)dt\Big), \qquad (4.8)
\end{aligned}
$$

where δ_0 denotes the distribution with Dirac mass at the origin. (4.6) now easily follows. Finally, since $Y_t^i = Y_t^{i,t,X_t^i}$, we have $||u_i(t, X_t^i, \mathcal{L}(X_t^i))||_2 \le C\Big(||\xi^i||_2 + \sup_{t\in[0,T]} M_2(\mu_t^j)\Big) + C'$ from (4.6). By the Lipschitz continuity in Remark 4.2 and the estimate in (4.6), we get

$$|u_i(t,0,\mathcal{L}(X_t^i))| \le ||u_i(t, X_t^i, \mathcal{L}(X_t^i))||_2 + C||X_t^i||_2 \le C\Big(||\xi^i||_2 + \sup_{t\in[0,T]} M_2(\mu_t^j)\Big) + C' \ .$$

Using the Lipschitz continuity in Remark 4.2 once again, we get the desired estimate (4.7). □

4.3 Nash MFTC equilibrium under additional boundedness

In preceding subsections, we have seen that, for given flows of probability measures $\boldsymbol{\mu}^1, \boldsymbol{\mu}^2 \in \mathcal{C}([0,T];\mathcal{P}_2(\mathbb{R}^d))$, the solution to each optimal control problem of (4.1) is characterized by the uniquely solvable FBSDE (4.3). It follows that finding a solution to a matching problem (4.2) is equivalent to find a solution to the coupled systems of FBSDEs of MKV-type: for $i, j \in \{1,2\}, j \neq i$,

$$\begin{aligned} dX_t^i &= b_i(t, X_t^i, \mathcal{L}(X_t^i), \mathcal{L}(X_t^j), \hat{\alpha}_i(t, X_t^i, \mathcal{L}(X_t^i), \mathcal{L}(X_t^j), Y_t^i))dt + \sigma_i(t, X_t^i, \mathcal{L}(X_t^i), \mathcal{L}(X_t^j))dW_t^i, \\ dY_t^i &= -\partial_x H_i(t, X_t^i, \mathcal{L}(X_t^i), \mathcal{L}(X_t^j), Y_t^i, Z_t^i, \hat{\alpha}_i(t, X_t^i, \mathcal{L}(X_t^i), \mathcal{L}(X_t^j), Y_t^i))dt \\ &\quad -\widetilde{\mathbb{E}}\big[\partial_\mu H_i(t, \widetilde{X}_t^i, \mathcal{L}(X_t^i), \mathcal{L}(X_t^j), \widetilde{Y}_t^i, \widetilde{Z}_t^i, \hat{\alpha}_i(t, \widetilde{X}_t^i, \mathcal{L}(X_t^i), \mathcal{L}(X_t^j), \widetilde{Y}_t^i))(X_t^i)\big]dt + Z_t^i dW_t^i, \end{aligned} \tag{4.9}$$

with $X_0^i = \xi^i \in \mathbb{L}^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d)$ and $Y_T^i = \partial_x g_i(X_T^i, \mathcal{L}(X_T^i), \mathcal{L}(X_T^j)) + \widetilde{\mathbb{E}}\big[\partial_\mu g_i(\widetilde{X}_T^i, \mathcal{L}(X_T^i), \mathcal{L}(X_T^j))(X_T^i)\big]$.

In this subsection, we prove the existence of a solution to the system of FBSDEs (4.9) under the additional assumption.

Assumption 4.2. (MFTC-b) *For each $1 \le i \le 2$, there exists some constant Λ and some point $0_{A_i} \in A_i$ such that, for any $t \in [0,T]$ and any $\nu \in \mathcal{P}_2(\mathbb{R}^d)$,*

$$\begin{aligned} &|b_{i,0}(t,\nu)|, |\sigma_{i,0}(t,\nu)| \le \Lambda, \\ &|\partial_{(x,\alpha)} f_i(t, 0, \delta_0, \nu, 0_{A_i})|, |\partial_x g_i(0, \delta_0, \nu)| \le \Lambda, \\ &|\partial_\mu f_i(t, 0, \delta_0, \nu, 0_{A_i})(0)|, |\partial_\mu g_i(0, \delta_0, \nu)(0)| \le \Lambda \ . \end{aligned}$$

Here is the main result of this subsection.

Theorem 4.2. *Under Assumptions* **(MFTC-a,b)***, the system of FBSDEs* (4.9) *(and hence the matching problem* (4.2)*) is solvable for any $\xi^1, \xi^2 \in \mathbb{L}^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d)$.*

Proof. We let, with $1 \le i \le 2$, $(X_t^{i,\boldsymbol{\mu}^j}, Y_t^{i,\boldsymbol{\mu}^j}, Z_t^{i,\boldsymbol{\mu}^j})_{t\in[0,T]}$ denote the solution to the FBSDE (4.3) for a given flow $\boldsymbol{\mu}^j \in \mathcal{C}([0,T];\mathcal{P}_2(\mathbb{R}^d))$ $j \neq i$ and the initial condition $X_0^{i,\boldsymbol{\mu}^j} = \xi^i$. By Theorem 4.1, we can define a map:

$$\Phi : \mathcal{C}([0,T];\mathcal{P}_2(\mathbb{R}^d))^2 \ni (\boldsymbol{\mu}^1, \boldsymbol{\mu}^2) \mapsto \big(\mathcal{L}(X_t^{1,\boldsymbol{\mu}^2})_{t\in[0,T]}, \mathcal{L}(X_t^{2,\boldsymbol{\mu}^2})_{t\in[0,T]}\big) \in \mathcal{C}([0,T];\mathcal{P}_2(\mathbb{R}^d))^2 \ .$$

It is easy to see that the solvability of the system of FBSDEs with McKean-Vlasov type (4.9) is equivalent to the existence of a fixed point of the map Φ. As in Theorem 3.3, we equip the linear space $\mathcal{C}([0,T];\mathcal{M}_f^1(\mathbb{R}^d))^2$ with the supremum of the Kantorovich-Rubinstein norm (3.20) so that we can apply Schauder FPT (Theorem 3.2).

We start from studying a priori estimates. By the estimate in (3.6), we get

$$|\hat{\alpha}_i(t,x,\mu,\nu,y)| \le \lambda^{-1}\big(|b_{i,2}(t,\nu)||y| + |\partial_\alpha f_i(t,x,\mu,\nu,0_{A_i})|\big) + |0_{A_i}|,$$

and hence $|\hat{\alpha}_i(t,0,\delta_0,\mu_t^j,0)| \le \lambda^{-1}\Lambda + |0_{A_i}| \le C(\lambda,\Lambda)$ uniformly in $\boldsymbol{\mu}^j$. The estimate (4.8) then implies that $\mathbb{E}\big[\sup_{t\in[0,T]} |Y_t^{i,\boldsymbol{\mu}^j}|^2\big] \le C\big(1 + ||\xi^i||_2^2\big)$ with C independent of $\boldsymbol{\mu}^j$. From the last part of the proof for Lemma 4.3, we get, for any $t \in [0,T]$,

$$|Y_t^{i,\boldsymbol{\mu}^j}| \le C\big(1 + ||\xi^i||_2 + |X_t^{i,\boldsymbol{\mu}^j}|\big), \quad \mathbb{P}\text{-a.s.}$$

and hence $|\hat{\alpha}_i(t, X_t^{i,\boldsymbol{\mu}^j}, \mathcal{L}(X_t^{i,\boldsymbol{\mu}^j}), \mu_t^j, Y_t^{i,\boldsymbol{\mu}^j})| \le C\big(1 + ||\xi^i||_2 + |X_t^{i,\boldsymbol{\mu}^j}| + M_2(\mathcal{L}(X_t^{i,\boldsymbol{\mu}^j}))\big)$ uniformly in $\boldsymbol{\mu}^j$. Thus it is straightforward to see that there exists some constant C independent of $\boldsymbol{\mu}^j$ such that $\mathbb{E}\big[\sup_{t\in[0,T]} |X_t^{i,\boldsymbol{\mu}^j}|^2\big] \le C$ and

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |X_t^{i,\boldsymbol{\mu}^j}|^4|\mathcal{F}_0\Big]^{\frac{1}{2}} \le C\big(1 + |\xi^i|^2\big), \quad W_2(\mathcal{L}(X_t^{i,\boldsymbol{\mu}^j}), \mathcal{L}(X_s^{i,\boldsymbol{\mu}^j})) \le C|t-s|^{\frac{1}{2}}, \ \forall t, s, \in [0,T].$$

Therefore, just repeating the arguments used in the proof for Theorem 3.3, we can show that Φ is a self-map on a closed and convex subset $\mathcal{E}$ of $\mathcal{C}([0,T]; \mathcal{M}_f^1(\mathbb{R}^d))^2$,

$$\begin{aligned}\mathcal{E} := \Big\{&(\boldsymbol{\mu}^1, \boldsymbol{\mu}^2) \in \mathcal{C}([0,T]; \mathcal{P}_2(\mathbb{R}^d))^2; \\ &\forall a \ge 1, 1 \le i \le 2, \quad \sup_{t\in[0,T]} \int_{|x|\ge a} |x|^2 \mu_t^i(dx) \le C\Big(a^{-1} + \mathbb{E}\big[|\xi^i|^2 \mathbf{1}_{\{|\xi^i| \ge \sqrt{a}\}}\Big)^{\frac{1}{2}}\Big\}, \end{aligned} \tag{4.10}$$

with some constant C and that $\Phi(\mathcal{E})$ is a relatively compact subset of $\mathcal{C}([0,T]; \mathcal{P}_2(\mathbb{R}^d))^2$. The continuity of the map Φ can be shown by Lemma 4.3 just as in Theorem 3.3. Schauder FPT now guarantees the existence of a fixed point for map Φ, which then establishes the existence of solution to the system of FBSDEs (4.9). □

4.4 Nash MFTC equilibrium for small T or small coupling

Here is the main result of this section.

Theorem 4.3. *Under Assumption* **(MFTC-a)***, there exists some positive constant c depending only on (L, λ) such that, for any $T \le c$, the system of FBSDEs* (4.9) *(and hence the matching problem* (4.2)*) is solvable for any $\xi^1, \xi^2 \in \mathbb{L}^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d)$.*

Proof. As in the proof for Theorem 3.4, we use the push-forward $\phi_n \circ \mu$ of the measure $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ defined by the map $\mathbb{R}^d \ni x \mapsto \dfrac{nx}{\max(M_2(\mu), n)}$. For eacn $n \in \mathbb{N}$, we introduce the approximated coefficient functions by

$$\begin{aligned}&(b_{i,0}^n, b_{i,2}^n, \sigma_{i,0}^n)(t,\nu) := (b_{i,0}, b_{i,2}, \sigma_{i,0})(t, \phi_n \circ \nu), \\ &f_i^n(t,x,\mu,\nu,\alpha) := f_i(t,x,\mu,\phi_n \circ \nu, \alpha), \quad g_i^n(x,\mu,\nu) := g_i(x,\mu,\phi_n\circ\nu) \ ,\end{aligned}$$

and accordingly define

$$\begin{aligned}&b_i^n(t,x,\mu,\nu,\alpha) := b_{i,0}^n(t,\nu) + b_{i,1}(t,\nu)x + \bar{b}_{i,1}(t,\nu)\bar{\mu} + b_{i,2}^n(t,\nu)\alpha \ , \\ &\sigma_i^n(t,x,\mu,\nu) := \sigma_{i,0}^n(t,\nu) + \sigma_{i,1}(t,\nu)x + \bar{\sigma}_{i,1}(t,\nu)\bar{\mu} \ , \\ &H_i^n(t,x,\mu,\nu,y,z,\alpha) := \langle b_i^n(t,x,\mu,\nu,\alpha), y\rangle + \mathrm{tr}[\sigma_i^n(t,x,\mu,\nu)^\top z] + f_i^n(t,x,\mu,\nu,\alpha) \ .\end{aligned}$$

It is obvious to see that the approximated coefficients $(b_i^n, \sigma_i^n, f_i^n, g_i^n)$ satisfy every condition in Assumptions **(MFTC-a,b)**. Moreover, the minimizer $\hat{\alpha}_i^n$ of H_i^n is given by

$$\hat{\alpha}_i^n(t,x,\mu,\nu,y) = \hat{\alpha}_i(t,x,\mu,\phi_n \circ \nu, y) \ ,$$

where $\hat{\alpha}_i$ is the minimizer of the original Hamiltonian. The regularization for $b_{i,2}$ is done solely to obtain the simple expression for $\hat{\alpha}_i^n$ as above. By Theorem 4.2, for eacn $n \in \mathbb{N}$, there exists a solution $(X_t^{i,n}, Y_t^{i,n}, Z_t^{i,n})_{t\in[0,T]}, 1 \leq i \leq 2$ to the system of FBSDEs (4.9) with the approximated coefficient functions $(b_i^n, \sigma_i^n, f_i^n, g_i^n)_{1\leq i\leq 2}$. By the estimate (4.7), there exist constants C depending only on (L, λ) and C' depending additionally on K such that, for any $t \in [0,T]$, $|Y_t^{i,n}| \leq C\Big(||\xi^i||_2 + |X_t^{i,n}| + \sup_{s\in[0,T]} M_2(\mathcal{L}(X_s^{j,n}))\Big) + C'$, $\mathbb{P}$-a.s. uniformly in n. Lemma 4.1 then implies that $\hat{\alpha}_i^n(t) := \hat{\alpha}_i^n(t, X_t^{i,n}, \mathcal{L}(X_t^{i,n}), \mathcal{L}(X_t^{j,n}), Y_t^{i,n})$ satisfies $|\hat{\alpha}_i^n(t)| \leq C\Big(||\xi^i||_2 + |X_t^{i,n}| + M_2(\mathcal{L}(X_t^{i,n})) + \sup_{s\in[0,T]} M_2(\mathcal{L}(X_s^{j,n}))\Big) + C'$. Thus, for any $t \in [0,T]$,

$$\begin{aligned}
\mathbb{E}\big[|X_t^{i,n}|^2\big] &\leq C\mathbb{E}\Big[|\xi^i|^2 + \int_0^t \big[|b_i^n(s, X_s^{i,n}, \mathcal{L}(X_s^{i,n}), \mathcal{L}(X_s^{j,n}), \hat{\alpha}_i^n(s))|^2 \\
&\qquad\qquad + |\sigma_i^n(s, X_s^{i,n}, \mathcal{L}(X_s^{i,n}), \mathcal{L}(X_s^{j,n}))|^2\big] ds\Big] \\
&\leq C\Big(||\xi^i||_2^2 + T \sup_{s\in[0,T]} M_2(\mathcal{L}(X_s^{j,n}))^2 + \int_0^t \sum_{j=1}^2 \mathbb{E}[|X_s^{j,n}|^2] ds\Big) + C'. \qquad (4.11)
\end{aligned}$$

Hence Gronwall's inequality gives $\sum_{i=1}^2 \sup_{t\in[0,T]} \mathbb{E}\big[|X_t^{i,n}|^2\big] \leq C' + CT \sum_{i=1}^2 \sup_{t\in[0,T]} M_2(\mathcal{L}(X_t^{i,n}))^2$, where C' now depends also on $||\boldsymbol{\xi}||_2$. Therefore there exists some constant c depending only on (L, λ) such that, for any $T \leq c$,

$$\sum_{i=1}^2 \sup_{t\in[0,T]} \mathbb{E}\big[|X_t^{i,n}|^2\big] \leq C(L, \lambda, K, ||\boldsymbol{\xi}||_2) \qquad (4.12)$$

uniformly in n. For such $T \leq c$, using the estimate (4.12), we get by the standard technique that

$$\begin{aligned}
&W_2(\mathcal{L}(X_t^{i,n}), \mathcal{L}(X_s^{i,n})) \leq C(L, \lambda, K, ||\boldsymbol{\xi}||_2)|t-s|^{\frac{1}{2}} , \\
&\mathbb{E}\Big[\sup_{t\in[0,T]} |X_t^{i,n}|^4 |\mathcal{F}_0\Big]^{\frac{1}{2}} \leq C(L, \lambda, K, ||\boldsymbol{\xi}||_2)\big(1 + |\xi^i|^2\big) ,
\end{aligned}$$

uniformly in n. We thus see that $(\mathcal{L}(X_t^{1,n})_{t\in[0,T]}, \mathcal{L}(X_t^{2,n})_{t\in[0,T]})_{n\geq 1}$ is a relatively compact subset of $\mathcal{C}([0,T]; \mathcal{P}_2(\mathbb{R}^d))^2$. Upon extracting some subsequence, there exist $\boldsymbol{\mu}^1, \boldsymbol{\mu}^2 \in \mathcal{C}([0,T]; \mathcal{P}_2(\mathbb{R}^d))$ such that $\lim_{n\to\infty} \sup_{t\in[0,T]} W_2(\mathcal{L}(X_t^{i,n}), \mu_t^i) = 0, 1 \leq i \leq 2$. By letting $(X_t^i, Y_t^i, Z_t^i)_{t\in[0,T], 1\leq i\leq 2}$ denote the solution to the FBSDE (4.3) with the flows $(\boldsymbol{\mu}^1, \boldsymbol{\mu}^2)$, we can prove that $(X_t^i, Y_t^i, Z_t^i)_{t\in[0,T], 1\leq i\leq 2}$ is actually a solution to (4.9) by the stability estimate in Lemma 4.3 and the same arguments used in the proof for Theorem 3.4. □

As in Section 3, it is possible to guarantee the existence of an equilibrium for a given T with quadratic cost functions by making the coupling between FSDE and BSDE small enough.

Theorem 4.4. *Under Assumption* **(MFTC-a)** *and a given T, the system of FBSDEs* (4.9) *(and hence he matching problem* (4.2)*) is solvable for any $\xi^1, \xi^2 \in \mathbb{L}^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d)$ if $\lambda^{-1}||b_{i,2}||_\infty$, $1 \leq i \leq 2$ are small enough.*

Proof. As in the proof of Theorem 3.5, the term involving $\sup_{t\in[0,T]} M_2(\mathcal{L}(X_s^{j,n}))^2$ in (4.11) is proportional to $\lambda^{-1}||b_{i,2}||_\infty$. Hence, if we make this factor small enough, we obtains the estimate (4.12) for a given T. The remaining arguments are the same as in the proof for Theorem 4.3. □

Remark 4.3. *There is no difficulty to generalize all the analyses in Section 4 for handling any finite number of populations* $1 \leq i \leq m$.

5 Games among Cooperative and non-Cooperative Populations

As a natural extension of Sections 3 and 4, we now study a Nash equilibrium with two populations, where the agents in the first population (P-1) cooperate by adopting the same feedback strategy while each agent in the second population (P-2) competes with every other agent. As before, we assume that the agents in each population share the same cost functions as well as the coefficient functions of their state dynamics. Let us call the large population limit of this problem as *Nash MFTC-MFG Problem*.

One of the motives to study this problem is to treat a situation, for example, where a large number of oil producers are competing to maximize their profits while a part of them are members of a certain association, such as OPEC, cooperating within the group to maintain a favorable level of oil price. Since the analysis can be generalized to any finite number of populations, it may have a wide scope of application.

5.1 Definition of Nash MFTC-MFG problem

We formulate the problem in the following way.
(i) Fix any two deterministic flows of probability measures $(\boldsymbol{\mu}^i = (\mu_t^i)_{t\in[0,T]})_{i\in\{1,2\}}$ given on $\mathbb{R}^d$.
(ii) Solve the two optimal control problems

$$\inf_{\boldsymbol{\alpha}^1\in\mathbb{A}_1} J_1^{\boldsymbol{\mu}^2}(\boldsymbol{\alpha}^1), \quad \inf_{\boldsymbol{\alpha}^2\in\mathbb{A}_2} J_2^{\boldsymbol{\mu}^2,\boldsymbol{\mu}^1}(\boldsymbol{\alpha}_2) \tag{5.1}$$

over some admissible strategies $\mathbb{A}_i$ $(i \in \{1,2\})$, where

$$J_1^{\boldsymbol{\mu}^2}(\boldsymbol{\alpha}^1) := \mathbb{E}\Big[\int_0^T f_1(t, X_t^1, \mathcal{L}(X_t^1), \mu_t^2, \alpha_t^1)dt + g_1(X_T^1, \mathcal{L}(X_T^1), \mu_T^2)\Big] ,$$
$$J_2^{\boldsymbol{\mu}^2,\boldsymbol{\mu}^1}(\boldsymbol{\alpha}^2) := \mathbb{E}\Big[\int_0^T f_2(t, X_t^2, \mu_t^2, \mu_t^1, \alpha_t^2)dt + g_2(X_T^2, \mu_T^2, \mu_T^1)\Big] ,$$

subject to the dynamic constraints

$$dX_t^1 = b_1(t, X_t^1, \mathcal{L}(X_t^1), \mu_t^2, \alpha_t^1)dt + \sigma_1(t, X_t^1, \mathcal{L}(X_t^1), \mu_t^2)dW_t^1 ,$$
$$dX_t^2 = b_2(t, X_t^2, \mu_t^2, \mu_t^1, \alpha_t^2)dt + \sigma_2(t, X_t^2, \mu_t^2, \mu_t^1)dW_t^2 ,$$

for $t \in [0,T]$ with $\big(X_0^i = \xi^i \in \mathbb{L}^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d)\big)_{1\leq i\leq 2}$. Notice that the first control problem is of McKean-Vlasov type which represents the large population limit of cooperative agents. For each population $i \in \{1,2\}$, we suppose that $\mathbb{A}_i$ is the set of A_i-valued $\mathbb{F}^i$-progressively measurable processes $\boldsymbol{\alpha}^i$ satisfying $\mathbb{E}\int_0^T |\alpha_t^i|^2 dt < \infty$ and $A_i \subset \mathbb{R}^k$ is closed and convex, as before.
(iii) Find a pair of probability flows $(\boldsymbol{\mu}^1, \boldsymbol{\mu}^2)$ as a solution to the matching problem:

$$\forall t \in [0,T], \quad \mu_t^1 = \mathcal{L}(\hat{X}_t^{1,\boldsymbol{\mu}^2}), \quad \mu_t^2 = \mathcal{L}(\hat{X}_t^{2,\boldsymbol{\mu}^2,\boldsymbol{\mu}^1}) , \tag{5.2}$$

where $(\hat{X}^{1,\boldsymbol{\mu}^2})$ and $(\hat{X}^{2,\boldsymbol{\mu}^2,\boldsymbol{\mu}^1})$ are the solutions to the optimal control problems in (ii).

Throughout Section 5, the major assumptions for the coefficients $(b_1, \sigma_1, f_1, g_1)$ of the first population (P-1) are given by **(MFTC-a)**, and those for the coefficients $(b_2, \sigma_2, f_2, g_2)$ of the second population (P-2) are given by **(MFG-a)**. We have already learned from Theorems 3.1 and 4.1 that the solution to each of the optimal control problems in (5.1) for given deterministic flows $\boldsymbol{\mu}^1, \boldsymbol{\mu}^2 \in \mathcal{C}([0,T];\mathcal{P}_2(\mathbb{R}^d))$ is characterized by the uniquely solvable FBSDEs,

$$\begin{aligned} dX_t^1 &= b_1(t, X_t^1, \mathcal{L}(X_t^1), \mu_t^2, \hat\alpha_1(t, X_t^1, \mathcal{L}(X_t^1), \mu_t^2, Y_t^1))dt + \sigma_1(t, X_t^1, \mathcal{L}(X_t^1), \mu_t^2)dW_t^1 \ , \\ dY_t^1 &= -\partial_x H_1(t, X_t^1, \mathcal{L}(X_t^1), \mu_t^2, Y_t^1, Z_t^1, \hat\alpha_1(t, X_t^1, \mathcal{L}(X_t^1), \mu_t^2, Y_t^1))dt \\ &\quad -\widetilde{\mathbb{E}}\big[\partial_\mu H_1(t, \widetilde{X}_t^1, \mathcal{L}(X_t^1), \mu_t^2, \widetilde{Y}_t^1, \widetilde{Z}_t^1, \hat\alpha_1(t, \widetilde{X}_t^1, \mathcal{L}(X_t^1), \mu_t^2, \widetilde{Y}_t^1))(X_t^1)\big]dt + Z_t^1 dW_t^1 \ , \end{aligned} \tag{5.3}$$

with $X_0^1 = \xi^1$ and $Y_T^1 = \partial_x g_1(t, X_T^1, \mathcal{L}(X_T^1), \mu_T^2) + \widetilde{\mathbb{E}}\big[\partial_\mu g_1(\widetilde{X}_T^1, \mathcal{L}(X_T^1), \mu_T^2)(X_T^1)\big]$, and

$$\begin{aligned} dX_t^2 &= b_2(t, X_t^2, \mu_t^2, \mu_t^1, \hat\alpha_2(t, X_t^2, \mu_t^2, \mu_t^1, Y_t^2))dt + \sigma_2(t, X_t^2, \mu_t^2, \mu_t^1)dW_t^2 \ , \\ dY_t^2 &= -\partial_x H_2(t, X_t^2, \mu_t^2, \mu_t^1, Y_t^2, Z_t^2, \hat\alpha_2(t, X_t^2, \mu_t^2, \mu_t^1, Y_t^2))dt + Z_t^2 dW_t^2 \ , \end{aligned} \tag{5.4}$$

with $X_0^2 = \xi^2$ and $Y_T^2 = \partial_x g_2(X_T^2, \mu_T^2, \mu_T^1)$, respectively. Here, the Hamiltonian H_i and its minimizer $\hat\alpha_i$ are defined as before using the corresponding coefficients (b_i, σ_i, f_i).

5.2 MFTC-MFG equilibrium under additional boundedness

In order to establish the existence of an equilibrium (5.2), we have to show the existence of a solution to the following coupled system of FBSDEs:

$$\begin{aligned} dX_t^1 &= b_1(t, X_t^1, \mathcal{L}(X_t^1), \mathcal{L}(X_t^2), \hat\alpha_1(t, X_t^1, \mathcal{L}(X_t^1), \mathcal{L}(X_t^2), Y_t^1))dt + \sigma_1(t, X_t^1, \mathcal{L}(X_t^1), \mathcal{L}(X_t^2))dW_t^1 \ , \\ dY_t^1 &= -\partial_x H_1(t, X_t^1, \mathcal{L}(X_t^1), \mathcal{L}(X_t^2), Y_t^1, Z_t^1, \hat\alpha_1(t, X_t^1, \mathcal{L}(X_t^1), \mathcal{L}(X_t^2), Y_t^1))dt \\ &\quad -\widetilde{\mathbb{E}}\big[\partial_\mu H_1(t, \widetilde{X}_t^1, \mathcal{L}(X_t^1), \mathcal{L}(X_t^2), \widetilde{Y}_t^1, \widetilde{Z}_t^1, \hat\alpha_1(t, \widetilde{X}_t^1, \mathcal{L}(X_t^1), \mathcal{L}(X_t^2), \widetilde{Y}_t^1))(X_t^1)\big]dt + Z_t^1 dW_t^1 \ , \end{aligned}$$

with $X_0^1 = \xi^1$ and $Y_T^1 = \partial_x g_1(t, X_T^1, \mathcal{L}(X_T^1), \mathcal{L}(X_T^2)) + \widetilde{\mathbb{E}}\big[\partial_\mu g_1(\widetilde{X}_T^1, \mathcal{L}(X_T^1), \mathcal{L}(X_T^2))(X_T^1)\big]$, and

$$\begin{aligned} dX_t^2 &= b_2(t, X_t^2, \mathcal{L}(X_t^2), \mathcal{L}(X_t^1), \hat\alpha_2(t, X_t^2, \mathcal{L}(X_t^2), \mathcal{L}(X_t^1), Y_t^2))dt + \sigma_2(t, X_t^2, \mathcal{L}(X_t^2), \mathcal{L}(X_t^1))dW_t^2 \ , \\ dY_t^2 &= -\partial_x H_2(t, X_t^2, \mathcal{L}(X_t^2), \mathcal{L}(X_t^1), Y_t^2, Z_t^2, \hat\alpha_2(t, X_t^2, \mathcal{L}(X_t^2), \mathcal{L}(X_t^1), Y_t^2))dt + Z_t^2 dW_t^2 \ , \end{aligned} \tag{5.5}$$

with $X_0^2 = \xi^2$ and $Y_T^2 = \partial_x g_2(X_T^2, \mathcal{L}(X_T^2), \mathcal{L}(X_T^1))$.

In this section, our goal is to prove the following result.

Theorem 5.1. *Under Assumptions* **(MFTC-a,b)** *for the coefficients* $(b_1, \sigma_1, f_1, g_1)$ *and Assumptions* ***(MFG-a,b)*** *for the coefficients* $(b_2, \sigma_2, f_2, g_2)$*, the system of FBSDEs* (5.5) *(and hence the matching problem* (5.2)*) is solvable for any* $\xi^1, \xi^2 \in \mathbb{L}^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d)$.

Proof. We let $(X_t^{1,\boldsymbol{\mu}^2}, Y_t^{1,\boldsymbol{\mu}^2}, Z_t^{1,\boldsymbol{\mu}^2})_{t\in[0,T]}$ and $(X_t^{2,\boldsymbol{\mu}^2,\boldsymbol{\mu}^1}, Y_t^{2,\boldsymbol{\mu}^2,\boldsymbol{\mu}^1}, Z_t^{2,\boldsymbol{\mu}^2,\boldsymbol{\mu}^1})_{t\in[0,T]}$ denote the solutions to the FBSDE (5.3) and (5.4) respectively for given flows of probability measures $(\boldsymbol{\mu}^1, \boldsymbol{\mu}^2)$. By defining the map Φ as

$$\Phi : \mathcal{C}([0,T];\mathcal{P}_2(\mathbb{R}^d))^2 \ni (\boldsymbol{\mu}^1, \boldsymbol{\mu}^2) \mapsto \big(\mathcal{L}(X_t^{1,\boldsymbol{\mu}^2})_{t\in[0,T]}, \mathcal{L}(X_t^{2,\boldsymbol{\mu}^2,\boldsymbol{\mu}^1})_{t\in[0,T]}\big) \in \mathcal{C}([0,T];\mathcal{P}_2(\mathbb{R}^d))^2 \ , \tag{5.6}$$

the claim is proved once we find a fixed point of the map Φ.

It is the direct result of Theorem 4.2 for (P-1) and Theorem 3.3 for (P-2) that there exists a constant C independent of $\boldsymbol{\mu}^1$ and $\boldsymbol{\mu}^2$ such that, for any $t, s \in [0,T]$,

$$\begin{aligned} &\mathbb{E}\big[\sup_{t\in[0,T]} |X_t^{1,\boldsymbol{\mu}^2}|^4 | \mathcal{F}_0\big]^{\frac12} \le C\big(1 + |\xi^1|^2\big), \quad \mathbb{E}\big[\sup_{t\in[0,T]} |X_t^{2,\boldsymbol{\mu}^2,\boldsymbol{\mu}^1}|^4 | \mathcal{F}_0\big]^{\frac12} \le C\big(1 + |\xi^2|^2\big), \\ &W_2(\mathcal{L}(X_t^{1,\boldsymbol{\mu}^2}), \mathcal{L}(X_s^{1,\boldsymbol{\mu}^2})) \le C|t-s|^{\frac12} \ , \quad W_2(\mathcal{L}(X_t^{2,\boldsymbol{\mu}^2,\boldsymbol{\mu}^1}), \mathcal{L}(X_s^{2,\boldsymbol{\mu}^2,\boldsymbol{\mu}^1})) \le C|t-s|^{\frac12}. \end{aligned} \tag{5.7}$$

Thus we can show that, for the same form of closed and convex subset $\mathcal{E}$ of $C([0,T];\mathcal{M}_f^1(\mathbb{R}^d))^2$ in (4.10) with sufficiently large C, that Φ maps $\mathcal{E}$ into itself and also that $\Phi(\mathcal{E})$ is a relatively compact subset of $\mathcal{C}([0,T];\mathcal{P}_2(\mathbb{R}^d))^2$. The continuity of the map Φ can be shown by Lemmas 4.3 and 3.2 just as in Theorems 4.2 and 3.3. By Schauder FPT, the claim is proved. □

5.3 MFTC-MFG equilibrium for small T or small coupling

We now give the main result of Section 5.

Theorem 5.2. *Under Assumption* **(MFTC-a)** *for the coefficients* (b_1,σ_1,f_1,g_1) *and Assumption* **(MFG-a)** *for the coefficients* (b_2,σ_2,f_2,g_2)*, there exists some positive constant c depending only on (L,λ) such that, for any $T\le c$, the system of FBSDEs* (5.5) *(and hence matching problem* (5.2)*) is solvable for any $\xi^1,\xi^2\in\mathbb{L}^2(\Omega,\mathcal{F}_0,\mathbb{P};\mathbb{R}^d)$.*

Proof. Let us introduce the approximated functions $(b_1^n,\sigma_1^n,f_1^n,g_1^n)_{n\ge1}$ as in Theorem 4.3 and also $(b_2^n,\sigma_2^n,f_2^n,g_2^n)_{n\ge1}$ as in Theorem 3.4, which satisfy Assumptions **(MFTC-a,b)** and Assumptions **(MFG-a,b)** for each n, respectively. Theorem 5.1 then guarantees that there exists a solution to the system of FBSDEs (5.5) with the approximated functions $(b_i^n,\sigma_i^n,f_i^n,g_i^n)_{1\le i\le 2}$ for each n. We let $(X_t^{i,n},Y_t^{i,n},Z_t^{i,n})_{t\in[0,T]}, 1\le i\le 2$ denote the corresponding solution.

Since inequalities (4.11) and (3.25) still hold, we can show that there exist constants C depending only on (L,λ) and C' depending additionally on K such that,

$$\mathbb{E}\big[|X_t^{1,n}|^2\big]\le C\Big(||\xi^1||_2^2+T\sup_{s\in[0,T]}M_2(\mathcal{L}(X_s^{2,n}))^2+\int_0^t\mathbb{E}\sum_{i=1}^2\big[|X_s^{i,n}|^2\big]ds\Big)+C'\ ,$$
$$\mathbb{E}\big[|X_t^{2,n}|^2\big]\le C\Big(||\xi^2||_2^2+T\sup_{s\in[0,T]}\sum_{i=1}^2M_2(\mathcal{L}(X_s^{i,n}))^2+\int_0^t\sum_{i=1}^2\mathbb{E}\big[|X_s^{i,n}|^2\big]ds\Big)+C'\ .$$

Hence we get, by Gronwall's inequality, that

$$\sup_{t\in[0,T]}\sum_{i=1}^2\mathbb{E}\big[|X_t^{i,n}|^2\big]\le C\Big(||\boldsymbol{\xi}||_2^2+T\sup_{s\in[0,T]}\sum_{i=1}^2M_2(\mathcal{L}(X_s^{i,n}))^2\Big)+C'.$$

Therefore there exists a positive constant c depending only on (L,λ) such that, for any $T\le c$,

$$\sup_{t\in[0,T]}\sum_{i=1}^2\mathbb{E}\big[|X_t^{i,n}|^2\big]\le C'\big(1+||\boldsymbol{\xi}||_2^2\big)\ .$$

Using the linear growth property of $\hat{\alpha}_i^n$ in $|X_t^{i,n}|$, we can show that $(\mathcal{L}(X_t^{1,n})_{t\in[0,T]},\mathcal{L}(X_t^{2,n})_{t\in[0,T]})_{n\ge1}$ is a relatively compact subset of $\mathcal{C}([0,T];\mathcal{P}_2(\mathbb{R}^d))^2$. The remaining arguments proceed in exactly the same way as in the proofs for Theorems 4.3 and 3.4. □

Theorem 5.3. *Under Assumption* **(MFTC-a)** *for the coefficients* (b_1,σ_1,f_1,g_1) *and Assumption* **(MFG-a)** *for the coefficients* (b_2,σ_2,f_2,g_2) *and a given T, the system of FBSDEs* (5.5) *(and hence matching problem* (5.2)*) is solvable for any $\xi^1,\xi^2\in\mathbb{L}^2(\Omega,\mathcal{F}_0,\mathbb{P};\mathbb{R}^d)$ if $\lambda^{-1}||b_{i,2}||_\infty$, $1\le i\le 2$ are small enough.*

Proof. The claim can be proved in a completely parallel way to Theorems 3.5 and 4.4. □

Remark 5.1. *As in Sections 3 and 4, the analysis can be easily extended for the situation with any finite number of cooperative and non-cooperative populations.*

6 Approximate Equilibrium for MFG with Finite Agents

In the remaining sections, we investigate quantitative relationships between the solutions to the mean field games obtained in the previous three sections and those to their associated games with finite number of agents. We make use of the techniques developed in [34, 4, 6, 8, 9] and in particular Chapter 6 in [11] with appropriate generalizations to fit our situation. First, in this section, we shall study the finite agent problem associated with the multi-population mean field game solved in Section 3. Throughout the section, we assume that the conditions used either in Theorem 3.4 or Theorem 3.5 are satisfied. We let $(\boldsymbol{\mu}^1, \boldsymbol{\mu}^2) \in \mathcal{C}([0,T]; \mathcal{P}_2(\mathbb{R}^d))^2$ denote a solution to the matching problem (3.2).

6.1 Convergence of approximate optimal controls

For each population $1 \leq i \leq 2$, we suppose that there are N_i agents who are labeled by p. Let us first introduce N_i independent and identically distributed (i.i.d.) copies of the state process in the mean field setup:

$$d\underline{X}_t^{i,p} = b_i(t, \underline{X}_t^{i,p}, \mu_t^i, \mu_t^j, \underline{\hat{\alpha}}_t^{i,p})dt + \sigma_i(t, \underline{X}_t^{i,p}, \mu_t^i, \mu_t^j)dW_t^{i,p} \ , \tag{6.1}$$

for $1 \leq p \leq N_i$, $j \neq i$, $t \in [0,T]$ with $\underline{X}_0^{i,p} = \xi^{i,p}$, and

$$\underline{\hat{\alpha}}_t^{i,p} := \hat{\alpha}_i(t, \underline{X}_t^i, \mu_t^i, \mu_t^j, u_i(t, \underline{X}_t^i))$$

for any $t \in [0,T]$. Here, $(\xi^{i,p})_{1 \leq p \leq N_i}$ is the set of i.i.d random variables with $\mathcal{L}(\xi^{i,p}) = \mu_0^i$, and $(\boldsymbol{W}^{i,p} = (W_t^{i,p})_{t \in [0,T]})_{1 \leq p \leq N_i}$ are independent standard Brownian motions, which are also independent from $(\xi^{i,p})_{1 \leq p \leq N_i}$. Moreover, they are assumed to be independent from those in the other population. In other words, all of the set $(\xi^{i,p}, \boldsymbol{W}^{i,p})_{1 \leq p \leq N_i, 1 \leq i \leq 2}$ are assumed to be independent. u_i is the decoupling field given in Theorem 3.1 associated with the equilibrium flows of probability measures $(\boldsymbol{\mu}^i, \boldsymbol{\mu}^j)$. $\hat{\alpha}_i$ is the minimizer of the Hamiltonian for the population i defined in (3.4). By construction, $(\underline{\boldsymbol{X}}^{i,p})_{1 \leq p \leq N_i}$ are i.i.d. processes satisfying $\mathcal{L}(X_t^{i,p}) = \mu_t^i$, $\forall t \in [0,T]$. We denote the empirical distribution for $(\underline{\boldsymbol{X}}^{i,p})_{1 \leq p \leq N_i}$ by

$$\underline{\mu}_t^i := \frac{1}{N_i} \sum_{p=1}^{N_i} \delta_{\underline{X}_t^{i,p}} \ .$$

In the remainder, the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is enlarged accordingly to support $(\xi^{i,p}, \boldsymbol{W}^{i,p})_{1 \leq p \leq N_i, 1 \leq i \leq 2}$ and the filtration $\mathbb{F}$ is assumed to be generated by $(\xi^{i,p}, \boldsymbol{W}^{i,p})_{1 \leq p \leq N_i, 1 \leq i \leq 2}$ with complete and right-continuous augmentation.

Lemma 6.1. *Suppose that the conditions either for Theorem 3.4 or Theorem 3.5 are satisfied. Then, for each population $1 \leq i \leq 2$, there exists some sequence $(\epsilon_{N_i})_{N_i \geq 1}$ that tends to 0 as N_i tends to ∞ and some constant C such that*

$$\sup_{t \in [0,T]} \mathbb{E}\Big[W_2(\underline{\mu}_t^i, \mu_t^i)^2\Big] \leq C\epsilon_{N_i}^2 \ .$$

Furthermore, when $\mu_0^i \in \mathcal{P}_r(\mathbb{R}^d)$ with $r > 4$, we have an explicit estimate

$$\epsilon_{N_i}^2 = N_i^{-2/\max(d,4)}(1 + \ln(N_i)\mathbf{1}_{d=4}) \ .$$

Proof. When $\mu_0^i \in \mathcal{P}_r(\mathbb{R}^d)$, $\forall r \geq 2$, it is standard to check

$$\sup_{t\in[0,T]} M_r(\mu_r^i)^r \leq \mathbb{E}\Big[\sup_{t\in[0,T]} |\underline{X}_t^{i,p}|^r\Big] \leq C\Big(1 + M_r(\mu_0^i)^r\Big)$$

with some C independent of N_i thanks to the linear growth of the coefficients in (6.1). Then, the last claim is the direct result of Theorem 5.8 and Remark 5.9 in [10].

As for the first claim, (5.19) in [10] implies

$$\lim_{N_i\to\infty} \mathbb{E}\big[W_2(\underline{\mu}_t^i, \mu_t^i)^2\big] = 0 \tag{6.2}$$

for each t. In order to prove the uniform convergence in t [6], it suffices to show that there exists a compact set $\mathcal{K} \subset \mathcal{C}([0,T];\mathbb{R})$ such that

$$\Big(\mathbb{E}[W_2(\underline{\mu}_t^i, \mu_t^i)^2]_{t\in[0,T]}\Big)_{N_i\geq 1} \subset \mathcal{K} \ .$$

In fact, if this is the case, every subsequence has a uniformly convergent subsequence, all of which converge to 0 due to the pointwise convergence in (6.2). Hence, the whole sequence must uniformly converges to 0. The boundedness can be checked by

$$\sup_{t\in[0,T]} \mathbb{E}\big[W_2(\underline{\mu}_t^i, \mu_t^i)^2\big] \leq 2 \sup_{t\in[0,T]} \Big(\frac{1}{N_i}\sum_{p=1}^{N_i} \mathbb{E}[|\underline{X}_t^{i,p}|^2] + M_2(\mu_t^i)^2\Big) \leq 4 \sup_{t\in[0,T]} M_2(\mu_t^i)^2 \leq C \ .$$

Moreover, for any $0 \leq t, s \leq T$,

$$\begin{aligned}
&\Big|\mathbb{E}\big[W_2(\underline{\mu}_t^i, \mu_t^i)^2\big] - \mathbb{E}\big[W_2(\underline{\mu}_s^i, \mu_s^i)^2\big]\Big| \leq C\mathbb{E}\big[(W_2(\underline{\mu}_t^i, \mu_t^i) - W_2(\underline{\mu}_s^i, \mu_s^i))^2\big]^{\frac{1}{2}} \\
&\leq C\Big(\mathbb{E}[W_2(\underline{\mu}_t^i, \underline{\mu}_s^i)^2] + W_2(\mu_t^i, \mu_s^i)^2\Big)^{\frac{1}{2}} \leq C\mathbb{E}\big[|\underline{X}_t^{i,p} - \underline{X}_s^{i,p}|^2\big]^{\frac{1}{2}} \leq C|t-s|^{\frac{1}{2}} \ ,
\end{aligned}$$

which implies the equicontinuity. Arzela-Ascoli theorem guarantees the desired compactness. □

Assumption 6.1. (MFG-FA) *On top of Assumption* **(MFG-a)***, either T or $(\lambda^{-1}||b_{i,2}||_\infty)_{1\leq i\leq 2}$ is small enough to satisfy the conditions for Theorem 3.4 or Theorem 3.5. Moreover, for $1 \leq i \leq 2$,*
(A1) *There exists some constant K such that*

$$|(b_{i,0}, \sigma_{i,0})(t, \mu', \nu') - (b_{i,0}, \sigma_{i,0})(t, \mu, \nu)| \leq K\big(W_2(\mu', \mu) + W_2(\nu', \nu)\big)$$

for any $t \in [0,T], \mu', \mu, \nu', \nu \in \mathcal{P}_2(\mathbb{R}^d)$, and $b_{i,1}, b_{i,2}$ as well as $\sigma_{i,1}$ are independent of the measure arguments.
(A2) *f_i and g_i are local Lipschitz continuous with respect to the measure arguments i.e., there exists some constant K for any $t \in [0,T], x \in \mathbb{R}^d, \mu', \mu, \nu', \nu \in \mathcal{P}_2(\mathbb{R}^d)$ and $\alpha \in A_i$, such that,* [7]

$$\begin{aligned}
&|(f_i, g_i)(t, x, \mu', \nu', \alpha) - (f_i, g_i)(t, x, \mu, \nu, \alpha)| \\
&\leq K\Big(1 + |x| + M_2(\mu') + M_2(\mu) + M_2(\nu') + M_2(\nu) + |\alpha|\Big)\big(W_2(\mu', \mu) + W_2(\nu', \nu)\big) \ .
\end{aligned}$$

[6] See the arguments leading to the estimate (2.15) in the proof of Theorem 2.12 in [11].

[7] Although there is no (t, α) dependence in g_i, we slightly abuse the notation to save the space. Note that the local Lipschitz property for the arguments (x, α) follows from **(MFG-a)**.

For $1 \le i, j \le 2$, $j \ne i$, $1 \le p \le N_i$, let us consider the following state dynamics.

$$dX_t^{i,p} = b_i(t, X_t^{i,p}, \overline{\mu}_t^i, \overline{\mu}_t^j, \hat{\alpha}_t^{i,p})dt + \sigma_i(t, X_t^{i,p}, \overline{\mu}_t^i, \overline{\mu}_t^j)dW_t^{i,p} \ , \tag{6.3}$$

for $t \in [0, T]$ with $X_0^{i,p} = \xi^{i,p}$. Here, $\overline{\mu}_t^i := \frac{1}{N_i}\sum_{p=1}^{N_i} \delta_{X_t^{i,p}}$ is the empirical distribution and

$$\hat{\alpha}_t^{i,p} := \hat{\alpha}_i(t, X_t^{i,p}, \mu_t^i, \mu_t^j, u_i(t, X_t^{i,p})) \ .$$

Since we have $W_2(\overline{\mu}_t^i, \overline{\mu}_t'^{,i})^2 \le \frac{1}{N_i}\sum_{p=1}^{N_i}|X_t^{i,p} - X_t'^{,i,p}|^2$, Assumption **(MFG-FA) (A1)**, the Lipschitz continuity of $\hat{\alpha}_i$ and that of the decoupling field u_i make (6.3) an (N_1+N_2)-dimensional standard Lipschitz SDE. $(\boldsymbol{X}^{i,p})_{1\le i\le 2, 1\le p\le N_i}$ correspond to the state processes of the agents who adopt the feedback control function $[0,T]\times\mathbb{R}^d \ni (t,x) \mapsto \hat{\alpha}_i(t, x, \mu_t^i, \mu_t^j, u_i(t,x))$. The cost functional for the agent p in the ith population is given by

$$J_i^{N_i, N_j, p}(\hat{\boldsymbol{\alpha}}^{i,(N_i)}, \hat{\boldsymbol{\alpha}}^{j,(N_j)}) := \mathbb{E}\Big[\int_0^T f_i(t, X_t^{i,p}, \overline{\mu}_t^i, \overline{\mu}_t^j, \hat{\alpha}_t^{i,p})dt + g_i(X_T^{i,p}, \overline{\mu}_T^i, \overline{\mu}_T^j)\Big] \ .$$

On the other hand, the optimal cost functional for the mean field game in Section 3 is

$$J_i^p := \mathbb{E}\Big[\int_0^T f_i(t, \underline{X}_t^{i,p}, \mu_t^i, \mu_t^j, \underline{\hat{\alpha}}_t^{i,p})dt + g_i(\underline{X}_T^{i,p}, \mu_T^i, \mu_T^j)\Big] \ .$$

Lemma 6.2. *Under Assumption* **(MFG-FA)***, for all* $1 \le i \le 2$, $N_i \in \mathbb{N}$, $1 \le p \le N_i$*, there exists some constant* C *independent of* $(N_i)_{i=1}^2$ *such that,*

$$|J_i^{N_i, N_j, p}(\hat{\boldsymbol{\alpha}}^{i,(N_i)}, \hat{\boldsymbol{\alpha}}^{j,(N_j)}) - J_i^p| \le C\sum_{j=1}^2 \epsilon_{N_j} \ ,$$

where ϵ_{N_j} *is the one given in Lemma 6.1.*

Proof. It suffices to check the case with $(p=1)$. By Lipschitz continuity and the triangle inequality, we have $\mathbb{E}\big[\sup_{s\in[0,t]}|X_s^{i,1} - \underline{X}_s^{i,1}|^2\big] \le C\int_0^t \mathbb{E}\Big[|X_s^{i,1} - \underline{X}_s^{i,1}|^2 + \sum_{i=1}^2 W_2(\overline{\mu}_t^i, \underline{\mu}_t^i)^2 + \sum_{i=1}^2 W_2(\underline{\mu}_t^i, \mu_t^i)^2\Big]ds$. Applying Gronwall's inequality after summing over $1 \le i \le 2$, we get

$$\sum_{i=1}^2 \mathbb{E}\big[\sup_{t\in[0,T]}|X_t^{i,1} - \underline{X}_t^{i,1}|^2\big] \le C\sum_{i=1}^2 \sup_{t\in[0,T]}\mathbb{E}\big[W_2(\underline{\mu}_t^i, \mu_t^i)^2\big] \le C\sum_{i=1}^2 \epsilon_{N_i}^2 \ .$$

Hence the triangle inequality implies that $\sup_{t\in[0,T]}\mathbb{E}\big[W_2(\overline{\mu}_t^i, \mu_t^i)^2\big] \le C\sum_{j=1}^2 \epsilon_{N_j}^2$. We also see $\mathbb{E}\big[\sup_{t\in[0,T]}|X_t^{i,1}|^2\big] \le C\Big(1 + \mathbb{E}\big[\sup_{t\in[0,T]}|\underline{X}_t^{i,1}|^2\big]\Big) \le C$. Using local Lipschitz continuity, it is straightforward to conclude

$$|J_i^{N_i, N_j, 1}(\hat{\boldsymbol{\alpha}}^{i,(N_i)}, \hat{\boldsymbol{\alpha}}^{j,(N_j)}) - J_i^1| \le C\Big(1 + \sup_{t\in[0,T]}\mathbb{E}\Big[|X_t^{i,1}|^2 + |\underline{X}_t^{i,1}|^2 + \sum_{j=1}^2 M_2(\overline{\mu}_t^j)^2\Big]^{\frac{1}{2}}\Big)$$
$$\times \sup_{t\in[0,T]}\mathbb{E}\Big[|X_t^{i,1} - \underline{X}_t^{i,1}|^2 + \sum_{j=1}^2 W_2(\overline{\mu}_t^j, \mu_t^j)^2\Big]^{\frac{1}{2}} \le C\sum_{j=1}^2 \epsilon_{N_j} \ .$$

□

Remark 6.1. *From the above analysis, we see that* **(MFG-FA) (A2)** *is unnecessary if we only* *need the convergence* $J_i^{N_i,N_j,p}(\hat{\boldsymbol{\alpha}}^{i,(N_i)},\hat{\boldsymbol{\alpha}}^{j,(N_j)})\to J_i^p$ *when* $N_1,N_2\to\infty$*.* **(A2)** *is just used to derive the explicit order of convergence in terms of* $(\epsilon_{N_i})_{i=1}^2$*.*

6.2 Approximate Nash Equilibrium

In order to investigate an approximate Nash equilibrium, we suppose that one agent deviates from the feedback control function $[0,T]\times\mathbb{R}^d\ni(t,x)\mapsto\hat{\alpha}_i(t,x,\mu_t^i,\mu_t^j,u_i(t,x))$. By symmetry, we may assume that this is the first agent in the ith population. The state dynamics of the agents is now given by, $1\le i,j\le 2, j\ne i$, $1\le p\le N_i$,

$$dU_t^{i,p}=b_i(t,U_t^{i,p},\overline{\nu}_t^i,\overline{\nu}_t^j,\beta_t^{i,p})dt+\sigma_i(t,U_t^{i,p},\overline{\nu}_t^i,\overline{\nu}_t^j)dW_t^{i,p}\ ,$$

for $t\in[0,T]$ with $U_0^{i,p}=\xi^{i,p}$. $\boldsymbol{\beta}^{i,1}\in\mathbb{H}^2$ is any A_i-valued $\mathbb{F}$-progressively measurable process, and

$$\begin{aligned}\beta_t^{i,p}&:=\hat{\alpha}_i(t,U_t^{i,p},\mu_t^i,\mu_t^j,u_i(t,U_t^{i,p})),\ 2\le p\le N_i\ ,\\ \beta_t^{j,q}&:=\hat{\alpha}_j(t,U_t^{j,q},\mu_t^j,\mu_t^i,u_j(t,U_t^{j,q})),\ 1\le q\le N_j\ .\end{aligned}\tag{6.4}$$

$\overline{\nu}_t^i:=\frac{1}{N_i}\sum_{p=1}^{N_i}\delta_{U_t^{i,p}}$, $1\le i\le 2$ is the empirical distribution. This is an (N_1+N_2)-dimensional Lipschitz SDE and hence well-defined. The cost functional associated with the deviating agent is given by

$$J_i^{N_i,N_j,1}(\boldsymbol{\beta}^{i,1},\hat{\boldsymbol{\alpha}}^{i,(N_i)^{-1}},\hat{\boldsymbol{\alpha}}^{j,(N_j)}):=\mathbb{E}\Big[\int_0^T f_i(t,U_t^{i,1},\overline{\nu}_t^i,\overline{\nu}_t^j,\beta_t^{i,1})dt+g_i(U_T^{i,1},\overline{\nu}_T^i,\overline{\nu}_T^j)\Big]\ .$$

Remark 6.2. *As we can see from the above definition of control strategies, we shall focus on the approximate Nash equilibrium in the sense of the closed loop framework. The analysis for the open loop framework can be done in almost the same (actually slightly simpler) manner, which just requires to replace the feedback forms in* (6.4) *by*

$$\beta_t^{i,p}=\underline{\hat{\alpha}}_t^{i,p},\quad 2\le p\le N_i,\quad \beta_t^{j,q}=\underline{\hat{\alpha}}_t^{j,q},\quad 1\le q\le N_j\ .$$

Here is the main result of this section.

Theorem 6.1. *Under Assumption* **(MFG-FA)** *with sufficiently large* $(N_i)_{i=1}^2$*, the feedback control functions* $\big([0,T]\times\mathbb{R}^d\ni(t,x)\mapsto\hat{\alpha}_i(t,x,\mu_t^i,\mu_t^j,u_i(t,x))\big)_{1\le i,j\le 2,j\ne i}$ *form an* $(\sum_{j=1}^2\varepsilon_{N_j})$*-approximate Nash equilibrium i.e., there exists some constant* C *independent of* $(N_i)_{i=1}^2$ *such that*

$$J_i^{N_i,N_j,1}(\boldsymbol{\beta}^{i,1},\hat{\boldsymbol{\alpha}}^{i,(N_i)^{-1}},\hat{\boldsymbol{\alpha}}^{j,(N_j)})\ge J_i^{N_i,N_j,1}(\hat{\boldsymbol{\alpha}}^{i,(N_i)},\hat{\boldsymbol{\alpha}}^{j,(N_j)})-C\sum_{j=1}^2\varepsilon_{N_j}$$

for any A_i*-valued* $\mathbb{F}$*-progressively measurable processe* $\boldsymbol{\beta}^{i,1}\in\mathbb{H}^2$*, where* $\big(\varepsilon_{N_j}:=\max(\epsilon_{N_j},N_j^{-\frac{1}{2}})\big)_{1\le j\le 2}$*.*

Proof. **(first step)** Let us introduce another dynamics

$$d\overline{U}_t^{i,1}=b_i(t,\overline{U}_t^{i,1},\mu_t^i,\mu_t^j,\beta_t^{i,1})dt+\sigma_i(t,\overline{U}_t^{i,1},\mu_t^i,\mu_t^j)dW_t^{i,1}$$

for $t\in[0,T]$ with $U_0^{i,1}=\xi^{i,1}$. It is immediate to see that the following estimate holds:

$$\mathbb{E}\big[\sup_{t\in[0,T]}|\overline{U}_t^{i,1}|^2\big]\le C\big(1+||\beta^{i,1}||_{\mathbb{H}^2}^2\big).\tag{6.5}$$

The associated cost functional

$$\overline{J}_i^1(\boldsymbol{\beta}^{i,1}) := \mathbb{E}\Big[\int_0^T f_i(t, \overline{U}_t^{i,1}, \mu_t^i, \mu_t^j, \beta_t^{i,1})dt + g_i(\overline{U}_T^{i,1}, \mu_T^i, \mu_T^j)\Big]$$

satisfies, by Theorem 3.1, an inequality

$$\overline{J}_i^1(\boldsymbol{\beta}^{i,1}) \geq J_i^1 + \lambda \mathbb{E}\int_0^T |\beta_t^{i,1} - \underline{\hat{\alpha}}_t^{i,1}|^2 dt \ . \tag{6.6}$$

(second step) By the linear growth property of the coefficients, we get

$$\begin{aligned}
\mathbb{E}\Big[|U_t^{i,p}|^2\Big] &\leq C\Big(1 + \int_0^t \mathbb{E}\Big[|U_s^{i,p}|^2 + \frac{1}{N_i}\sum_{p=1}^{N_i}|U_s^{i,p}|^2 + \frac{1}{N_j}\sum_{q=1}^{N_j}|U_s^{j,q}|^2 + \mathbf{1}_{\{p=1\}}|\beta_s^{i,1}|^2\Big]ds\Big) \ , \\
\mathbb{E}\Big[|U_t^{j,q}|^2\Big] &\leq C\Big(1 + \int_0^t \mathbb{E}\Big[|U_s^{j,q}|^2 + \frac{1}{N_i}\sum_{p=1}^{N_i}|U_s^{i,p}|^2 + \frac{1}{N_j}\sum_{q=1}^{N_j}|U_s^{j,q}|^2\Big]ds\Big) \ ,
\end{aligned}$$

Taking the average and the applying Gronwall's inequality, we get

$$\sup_{t\in[0,T]}\Big(\frac{1}{N_i}\sum_{p=1}^{N_i}\mathbb{E}[|U_t^{i,p}|^2] + \frac{1}{N_j}\sum_{q=1}^{N_j}\mathbb{E}[|U_t^{j,q}|^2]\Big) \leq C\Big(1 + \frac{1}{N_i}||\beta^{i,1}||_{\mathbb{H}^2}^2\Big). \tag{6.7}$$

Using the above estimate and Burkholder-Davis-Gundy (BDG) inequality, we get

$$\begin{aligned}
\mathbb{E}\Big[\sup_{t\in[0,T]}|U_t^{i,p}|^2\Big] &\leq C\Big(1 + (\frac{1}{N_i} + \mathbf{1}_{\{p=1\}})||\beta^{i,1}||_{\mathbb{H}^2}^2\Big) \ , \\
\mathbb{E}\Big[\sup_{t\in[0,T]}|U_t^{j,q}|^2\Big] &\leq C\Big(1 + \frac{1}{N_i}||\beta^{i,1}||_{\mathbb{H}^2}^2\Big) \ .
\end{aligned} \tag{6.8}$$

By similar calculation, we see

$$\mathbb{E}\big[|U_t^{i,p} - \underline{X}_t^{i,p}|^2\big] \leq C\mathbb{E}\int_0^t \mathbb{E}\Big[|U_s^{i,p} - \underline{X}_s^{i,p}|^2 + \sum_{j=1}^2 W_2(\overline{\nu}_s^j, \mu_s^j)^2 + \mathbf{1}_{\{p=1\}}|\beta_s^{i,1} - \underline{\hat{\alpha}}_s^{i,1}|^2\Big]ds \ .$$

Combining the same estimate for the jth population, we get from Gronwall's inequality that

$$\begin{aligned}
&\sup_{t\in[0,T]}\Big(\frac{1}{N_i}\sum_{p=1}^{N_i}\mathbb{E}\big[|U_t^{i,p} - \underline{X}_t^{i,p}|^2\big] + \frac{1}{N_j}\sum_{p=1}^{N_j}\mathbb{E}\big[|U_t^{j,q} - \underline{X}_t^{j,q}|^2\big]\Big) \\
&\leq C\Big(\sum_{i=1}^2 \sup_{t\in[0,T]}\mathbb{E}[W_2(\underline{\mu}_t^i, \mu_t^i)^2] + \frac{1}{N_i}\mathbb{E}\int_0^T |\beta_t^{i,1} - \underline{\hat{\alpha}}_t^{i,1}|^2 dt\Big) \\
&\leq C\Big((1 + ||\beta^{i,1}||_{\mathbb{H}^2}^2)\varepsilon_{N_i}^2 + \varepsilon_{N_j}^2\Big) \ .
\end{aligned}$$

Here, we have used $||\underline{\hat{\alpha}}^{i,1}||^2_{\mathbb{H}^2} \le C$ and the result of Lemma 6.1. By the triangle inequality,

$$\begin{aligned}\sup_{t\in[0,T]}\sum_{i=1}^2 \mathbb{E}\big[W_2(\overline{\nu}^i_t,\mu^i_t)^2\big] &\le 2\sup_{t\in[0,T]}\sum_{i=1}^2\Big(\mathbb{E}\big[W_2(\overline{\nu}^i_t,\underline{\mu}^i_t)^2\big]+\mathbb{E}\big[W_2(\underline{\mu}^i_t,\mu^i_t)^2\big]\Big)\\ &\le C\Big((1+||\beta^{i,1}||^2_{\mathbb{H}^2})\varepsilon^2_{N_i}+\varepsilon^2_{N_j}\Big) \end{aligned}\tag{6.9}$$

holds. Similarly, we have

$$\mathbb{E}\big[\sup_{s\in[0,t]}|U^{i,1}_s-\overline{U}^{i,1}_s|^2\big]\le C\mathbb{E}\int_0^t\Big[|U^{i,1}_s-\overline{U}^{i,1}_s|^2+W_2(\overline{\nu}^i_s,\mu^i_s)^2+W_2(\overline{\nu}^j_s,\mu^j_s)^2\Big]ds$$

and hence from (6.9) we get

$$\mathbb{E}\big[\sup_{t\in[0,T]}|U^{i,1}_t-\overline{U}^{i,1}_t|^2\big]^{\frac{1}{2}}\le C\Big((1+||\beta^{i,1}||_{\mathbb{H}^2})\varepsilon_{N_i}+\varepsilon_{N_j}\Big)\ . \tag{6.10}$$

(third step) Finally, we get from the local Lipschitz continuity of the cost functions,

$$\begin{aligned}&|J_i^{N_i,N_j,1}(\boldsymbol{\beta}^{i,1},\hat{\boldsymbol{\alpha}}^{i,(N_i)^{-1}},\hat{\boldsymbol{\alpha}}^{j,(N_j)})-\overline{J}^1_i(\boldsymbol{\beta}^{i,1})|\\ &=\Big|\mathbb{E}\Big[\int_0^T\big(f_i(t,U^{i,1}_t,\overline{\nu}^i_t,\overline{\nu}^j_t,\beta^{i,1}_t)-f_i(t,\overline{U}^{i,1}_t,\mu^i_t,\mu^j_t,\beta^{i,1}_t)\big)dt+g_i(U^{i,1}_T,\overline{\nu}^i_T,\overline{\nu}^j_T)-g_i(\overline{U}^{i,1}_T,\mu^i_T,\mu^j_T)\Big]\Big|\\ &\le C\Big(1+\sup_{t\in[0,T]}\mathbb{E}\Big[|U^{i,1}_t|^2+|\overline{U}^{i,1}_t|^2+M_2(\overline{\nu}^i_t)^2+M_2(\overline{\nu}^j_t)^2\Big]^{\frac{1}{2}}+||\beta^{i,1}||_{\mathbb{H}^2}\Big)\\ &\qquad\times\Big(\sup_{t\in[0,T]}\mathbb{E}\big[|U^{i,1}_t-\overline{U}^{i,1}_t|^2\big]^{\frac{1}{2}}+\sum_{i=1}^2\sup_{t\in[0,T]}\mathbb{E}\big[W_2(\overline{\nu}^i_t,\mu^i_t)^2\big]^{\frac{1}{2}}\Big)\ .\end{aligned}$$

From (6.5), (6.7), (6.8), (6.9) and (6.10), we get

$$\begin{aligned}&|J_i^{N_i,N_j,1}(\boldsymbol{\beta}^{i,1},\hat{\boldsymbol{\alpha}}^{i,(N_i)^{-1}},\hat{\boldsymbol{\alpha}}^{j,(N_j)})-\overline{J}^1_i(\boldsymbol{\beta}^{i,1})|\\ &\qquad\le C\big(1+||\beta^{i,1}||_{\mathbb{H}^2}\big)\Big((1+||\beta^{i,1}||_{\mathbb{H}^2})\varepsilon_{N_i}+\varepsilon_{N_j}\Big)\ .\end{aligned}$$

By the estimate in Lemma 6.2, (6.6) and the fact that $||\underline{\hat{\alpha}}^{i,1}||^2_{\mathbb{H}} \le C$, we see

$$\begin{aligned}&J_i^{N_i,N_j,1}(\boldsymbol{\beta}^{i,1},\hat{\boldsymbol{\alpha}}^{i,(N_i)^{-1}},\hat{\boldsymbol{\alpha}}^{j,(N_j)})-J_i^{N_i,N_j,1}(\hat{\boldsymbol{\alpha}}^{i,(N_i)},\hat{\boldsymbol{\alpha}}^{j,(N_j)})\\ &\quad\ge\lambda||\beta^{i,1}-\underline{\hat{\alpha}}^{i,1}||^2_{\mathbb{H}^2}-C\big(1+||\beta^{i,1}||_{\mathbb{H}^2}\big)\Big((1+||\beta^{i,1}||_{\mathbb{H}^2})\varepsilon_{N_i}+\varepsilon_{N_j}\Big)\\ &\quad\ge(\lambda-C\sum_{j=1}^2\varepsilon_{N_j})||\beta^{i,1}-\underline{\hat{\alpha}}^{i,1}||^2_{\mathbb{H}^2}-C\sum_{j=1}^2\varepsilon_{N_j}\ .\end{aligned}$$

For large N_1 and N_2 with $C\sum_{j=1}^2\varepsilon_{N_j}\le\lambda$, we get the desired result. □

Remark 6.3. *The above analysis can be generalized straightforwardly to the setup with any finite number of populations,* $1\le i\le m$.

7 Approximate Equilibrium for MFTC with Finite Agents

In this section, we shall show how the solution to the Nash MFTC problem studied in Section 4 can provide an approximate Nash equilibrium among the two competing populations of finite agents who are cooperative within each population. In the last section dealing with the non-cooperative agents, the effect to the interactions from the agent deviating from the optimal strategy was shown to vanish in the large population limit. This does not happen in the current case, because all the agents in one population adopt the common strategy different from the optimal one. We shall see that this feature requires us more stringent assumptions to obtain an approximate Nash equilibrium.

Throughout the section, we assume that the conditions used either in Theorem 4.3 or Theorem 4.4 are satisfied. We let $(\boldsymbol{\mu}^1, \boldsymbol{\mu}^2) \in \mathcal{C}([0,T]; \mathcal{P}_2(\mathbb{R}^d))^2$ denote a solution to the matching problem (4.2). Moreover, unless otherwise stated, we use the same notation in the last section.

7.1 Convergence of approximate optimal controls

For each population $1 \le i \le 2$, we first consider (N_i) i.i.d. copies of the sate process in the mean field setup

$$d\underline{X}_t^{i,p} = b_i(t, \underline{X}_t^{i,p}, \mu_t^i, \mu_t^j, \underline{\hat{\alpha}}_t^{i,p})dt + \sigma_i(t, \underline{X}_t^{i,p}, \mu_t^i, \mu_t^j)dW_t^{i,p},$$

for $1 \le p \le N_i$, $j \neq i$, $t \in [0,T]$ with $\underline{X}_0^{i,p} = \xi^{i,p}$, and

$$\underline{\hat{\alpha}}_t^{i,p} := \hat{\alpha}_i(t, \underline{X}_t^{i,p}, \mu_t^i, \mu_t^j, u_i(t, \underline{X}_t^{i,p}, \mu_t^i)) \ ,$$

where u_i is the function defined in Lemma 4.2 associated with the equilibrium flows of probability measures $(\boldsymbol{\mu}^i, \boldsymbol{\mu}^j)$. As in the last section, $(\xi^{i,p}, \boldsymbol{W}^{i,p})_{1 \le p \le N_i, 1 \le i \le 2}$ are assumed to be independent with $\mathcal{L}(\xi^{i,p}) = \mu_0^i$. By construction, $(\underline{\boldsymbol{X}}^{i,p})_{1 \le p \le N_i}$ are i.i.d. processes satisfying $\mathcal{L}(\underline{X}_t^{i,p}) = \mu_t^i$, $\forall t \in [0,T]$. $\underline{\mu}_t^i$ denotes the empirical distribution of $(\underline{X}_t^{i,p})_{1 \le p \le N_i}$.

Lemma 7.1. *Suppose that the conditions either for Theorem 4.3 or Theorem 4.4 are satisfied. Then, for each population $1 \le i \le 2$, there exists some sequence $(\epsilon_{N_i})_{N_i \ge 1}$ that tends to 0 as N_i tends to ∞ and some constant C such that*

$$\sup_{t \in [0,T]} \mathbb{E}\Big[W_2(\underline{\mu}_t^i, \mu_t^i)^2\Big] \le C\epsilon_{N_i}^2 \ .$$

Furthermore, when $\mu_0^i \in \mathcal{P}_r(\mathbb{R}^d)$ with $r > 4$, we have an explicit estimate

$$\epsilon_{N_i}^2 = N_i^{-2/\max(d,4)}(1 + \ln(N_i)\mathbf{1}_{d=4}) \ .$$

Proof. It can be proved in the same way as Lemma 6.1. □

Let us introduce the following assumptions.

Assumption 7.1. (MFTC-FA-a) *On top of Assumption* **(MFTC-a)**, *either T or $(\lambda^{-1}||b_{i,2}||_\infty)_{1 \le i \le 2}$ is small enough to satisfy the conditions for Theorem 4.3 or Theorem 4.4. Moreover, for $1 \le i \le 2$,*
(A1) *There exists some constant K such that*

$$|(b_{i,0}, \sigma_{i,0})(t, \nu') - (b_{i,0}, \sigma_{i,0})(t, \nu)| \le KW_2(\nu', \nu)$$

for any $t \in [0,T]$, $\nu', \nu \in \mathcal{P}_2(\mathbb{R}^d)$, and $b_{i,1}, \bar{b}_{i,1}, b_{i,2}, \sigma_{i,1}$ as well as $\bar{\sigma}_{i,1}$ are independent of the measure argument.

(A2) *f_i and g_i are local Lipschitz continuous with respect to the second measure argument i.e., there exists some constant K for any $t \in [0,T], x \in \mathbb{R}^d, \mu, \nu', \nu \in \mathcal{P}_2(\mathbb{R}^d)$ and $\alpha \in A_i$, such that*

$$\begin{aligned}&|(f_i, g_i)(t, x, \mu, \nu', \alpha) - (f_i, g_i)(t, x, \mu, \nu, \alpha)| \\ &\quad \le K\Big(1 + |x| + M_2(\mu) + M_2(\nu') + M_2(\nu) + |\alpha|\Big) W_2(\nu', \nu) \ .\end{aligned}$$

Assumption 7.2. (MFTC-FA-b) *On top of Assumption* **(MFTC-FA-a)***, for $1 \le i \le 2$, $b_{i,0}$ and $\sigma_{i,0}$ are independent of the measure argument.*

Remark 7.1. *Assumption* **(MFTC-FA-b)** *will be used in the last part where we prove the property of the approximate Nash equilibrium. Under this stringent assumption, the mutual interactions among the agents belonging to the different populations are induced only through the cost functions and can appear only in their control strategies.*

As in the last section, we introduce for $1 \le i, j \le 2$, $j \ne i$, $1 \le p \le N_i$ the state dynamics

$$dX_t^{i,p} = b_i(t, X_t^{i,p}, \overline{\mu}_t^i, \overline{\mu}_t^j, \hat{\alpha}_t^{i,p})dt + \sigma_i(t, X_t^{i,p}, \overline{\mu}_t^i, \overline{\mu}_t^j)dW_t^{i,p} \ ,$$

for $t \in [0,T]$ with $X_0^{i,p} = \xi^{i,p}$. Here, $\overline{\mu}_t^i := \frac{1}{N_i}\sum_{p=1}^{N_i} \delta_{X_t^{i,p}}$ and

$$\hat{\alpha}_t^{i,p} := \hat{\alpha}_i(t, X_t^{i,p}, \mu_t^i, \mu_t^j, u_i(t, X_t^{i,p}, \mu_t^i)) \ .$$

Under Assumption **(MFTC-FA-a)**, it is an $(N_1 + N_2)$-dimensional Lipschitz SDE and hence is well-defined. This corresponds to the situation where all the agents in each population adopt the common feedback control given by the solution to the problem in Section 4. Let us write the cost functional for the agent in the ith population as

$$J_i^{N_i,N_j}(\hat{\boldsymbol{\alpha}}^{i,(N_i)}, \hat{\boldsymbol{\alpha}}^{j,(N_j)}) := \mathbb{E}\Big[\int_0^T f_i(t, X_t^{i,p}, \overline{\mu}_t^i, \overline{\mu}_t^j, \hat{\alpha}_t^{i,p})dt + g_i(X_T^{i,p}, \overline{\mu}_T^i, \overline{\mu}_T^j)\Big] \ .$$

The corresponding cost functional in the mean field problem is given by

$$J_i := \mathbb{E}\Big[\int_0^T f_i(t, \underline{X}_t^{i,p}, \mu_t^i, \mu_t^j, \underline{\hat{\alpha}}_t^{i,p})dt + g_i(\underline{X}_T^{i,p}, \mu_T^i, \mu_T^j)\Big] \ .$$

where $1 \le p \le N_i$ is arbitrary in both cases. With the word cooperative, we mean that the agents use the common feedback control function. Hence, even when we consider general strategy later, all the cost functionals among the agents within each population are the same.

Lemma 7.2. *Under Assumption* **(MFTC-FA-a)***, for all $1 \le i \le 2$, $N_i \in \mathbb{N}$, there exists some constant C independent of $(N_i)_{i=1}^2$ such that,*

$$|J_i^{N_i,N_j}(\hat{\boldsymbol{\alpha}}^{i,(N_i)}, \hat{\boldsymbol{\alpha}}^{j,(N_j)}) - J_i| \le C\sum_{j=1}^2 \epsilon_{N_j} \ ,$$

where ϵ_{N_j} is the one given in Lemma 7.1.

Proof. It is straightforward to get the estimate

$$\sum_{i=1}^2 \mathbb{E}\big[\sup_{t\in[0,T]} |X_t^{i,p} - \underline{X}_t^{i,p}|^2\big] \le C\sum_{i=1}^2 \sup_{t\in[0,T]} \mathbb{E}\big[W_2(\overline{\mu}_t^i, \mu_t^i)^2\big] \le C\sum_{i=1}^2 \epsilon_{N_i}^2 \ ,$$

and $\mathbb{E}\big[\sup_{t\in[0,T]}|X_t^{i,p}|^2\big]+\mathbb{E}\big[\sup_{t\in[0,T]}|\underline{X}_t^{i,p}|^2\big]\leq C$. Using the local Lipschitz continuity for f_i and g_i, we can prove the convergence of the cost functional exactly in the same way as in Lemma 6.2. □

7.2 Approximate Nash Equilibrium

We now consider the general state dynamics for the agents $1\leq i,j\leq 2$, $j\neq i$, $1\leq p\leq N_i$,

$$dU_t^{i,p}=b_i(t,U_t^{i,p},\overline{\nu}_t^i,\overline{\nu}_t^j,\beta_t^{i,p})dt+\sigma_i(t,U_t^{i,p},\overline{\nu}_t^i,\overline{\nu}_t^j)dW_t^{i,p}$$

with $t\in[0,T]$, $U_0^{i,p}=\xi^{i,p}$ and $\boldsymbol{\beta}^{i,p}\in\mathbb{H}^2$ is an A_i-valued $\mathbb{F}$-progressively measurable process. Since we suppose that the agents are cooperative within each population, we force the set of strategies $(\boldsymbol{\beta}^{i,p})_{1\leq p\leq N_i}$ to satisfy the condition so that $(\xi^{i,p},\boldsymbol{\beta}^{i,p},\boldsymbol{W}^{i,p})_{1\leq p\leq N_i}$ is exchangeable i.e. the distribution is invariant under the permutation with p. As before $\overline{\nu}_t^i:=\frac{1}{N_i}\sum_{p=1}^{N_i}\delta_{U_t^{i,p}}$ denotes the empirical distribution. The associated cost functional for the ith population is now given by, with any $1\leq p\leq N_i$,

$$J_i^{N_i,N_j}(\boldsymbol{\beta}^{i,(N_i)},\boldsymbol{\beta}^{j,(N_j)}):=\mathbb{E}\Big[\int_0^T f_i(t,U_t^{i,p},\overline{\nu}_t^i,\overline{\nu}_t^j,\beta_t^{i,p})dt+g_i(U_T^{i,p},\overline{\nu}_T^i,\overline{\nu}_T^j)\Big]\ .$$

Let us also introduce $(\underline{Y}_t^{i,p},\underline{Z}_t^{i,p})_{t\in[0,T]}$ the solution to (4.9) associated with the forward component $(\underline{X}_t^{i,p})_{t\in[0,T]}$:

$$\begin{aligned}d\underline{Y}_t^{i,p}\;&=\;-\partial_x H_i(t,\underline{X}_t^{i,p},\mu_t^i,\mu_t^j,\underline{Y}_t^{i,p},\underline{Z}_t^{i,p},\hat{\alpha}_i(t,\underline{X}_t^{i,p},\mu_t^i,\mu_t^j,\underline{Y}_t^{i,p}))dt\\&\quad-\widetilde{\mathbb{E}}\big[\partial_\mu H_i(t,\widetilde{\underline{X}}_t^{i,p},\mu_t^i,\mu_t^j,\widetilde{\underline{Y}}_t^{i,p},\widetilde{\underline{Z}}_t^{i,p},\hat{\alpha}_i(t,\widetilde{\underline{X}}_t^{i,p},\mu_t^i,\mu_t^j,\widetilde{\underline{Y}}_t^{i,p}))(\underline{X}_t^{i,p})\big]dt+\underline{Z}_t^{i,p}dW_t^{i,p},\end{aligned}\quad(7.1)$$

with $\underline{Y}_T^{i,p}=\partial_x g_i(\underline{X}_T^{i,p},\mu_T^i,\mu_T^j)+\widetilde{\mathbb{E}}\big[\partial_\mu g_i(\widetilde{\underline{X}}_T^{i,p},\mu_T^i,\mu_T^j)(\underline{X}_T^{i,p})\big]$. Note that, $\underline{Y}_t^{i,p}=u_i(t,\underline{X}_t^{i,p},\mu_t^i)$ a.s. for any $t\in[0,T]$.

Proposition 7.1. *Under Assumption* **(MFTC-FA-a)**, *for $1\leq i\leq 2$, $N_i\in\mathbb{N}$, there exists some constant C independent of $(N_i)_{i=1}^2$ such that*

$$\begin{aligned}J_i^{N_i,N_j}(\boldsymbol{\beta}^{i,(N_i)},\boldsymbol{\beta}^{j,(N_j)})-J_i&\geq\lambda\mathbb{E}\int_0^T|\beta_t^{i,p}-\underline{\hat{\alpha}}_t^{i,p}|^2dt\\&-C\Big(1+\sup_{t\in[0,T]}\mathbb{E}\big[|U_t^{i,p}-\underline{X}_t^{i,p}|^2\big]^{\frac{1}{2}}+\mathbb{E}\Big[\int_0^T|\beta_t^{i,p}-\underline{\hat{\alpha}}_t^{i,p}|^2dt\Big]^{\frac{1}{2}}\Big)\varepsilon_{N_i}\\&+\mathbb{E}\Big[\int_0^T\big(H_i(t,U_t^{i,p},\overline{\nu}_t^i,\overline{\nu}_t^j,\underline{Y}_t^{i,p},\underline{Z}_t^{i,p},\beta_t^{i,p})-H_i(t,U_t^{i,p},\overline{\nu}_t^i,\mu_t^j,\underline{Y}_t^{i,p},\underline{Z}_t^{i,p},\beta_t^{i,p})\big)dt\\&\qquad+g_i(U_T^{i,p},\overline{\nu}_T^i,\overline{\nu}_T^j)-g_i(U_T^{i,p},\overline{\nu}_T^i,\mu_T^j)\Big]\ ,\end{aligned}$$

where $\varepsilon_{N_i}:=\max(N_i^{-\frac{1}{2}},\epsilon_{N_i})$, and $1\leq p\leq N_i$ is arbitrary.

Proof. We can show the claim by following the same arguments used in the proof for Theorem 6.16 [11]. Since it is rather technical and lengthy, we give the details in Appendix A. □

For investigating the approximate Nash equilibrium property, we now suppose that the agents in the ith population use general strategy $(\boldsymbol{\beta}^{i,p})_{1\leq p\leq N_i}$ under the restriction that $(\xi^{i,p},\boldsymbol{\beta}^{i,p},\boldsymbol{W}^{i,p})_{1\leq p\leq N_i}$ is exchangeable, and that the agents in the jth population $1\leq q\leq N_j$ adopt the strategy

$$\beta_t^{j,q}:=\hat{\alpha}_j(t,U_t^{j,q},\mu_t^j,\mu_t^i,u_j(t,U_t^{j,q},\mu_t^j))$$

for any $t \in [0,T]$. The cost functional for the ith population is now given by

$$J_i^{N_i,N_j}(\boldsymbol{\beta}^{i,(N_i)}, \hat{\boldsymbol{\alpha}}^{j,(N_j)}) = \mathbb{E}\Big[\int_0^T f_i(t, U_t^{i,p}, \overline{\nu}_t^i, \overline{\nu}_t^j, \beta_t^{i,p})dt + g_i(U_T^{i,p}, \overline{\nu}_T^i, \overline{\nu}_T^j)\Big]$$

with the above specified control strategies. We now proceed as in the proof for Theorem 6.1. The crucial problem is the term $\mathbb{E}\big[W_2(\overline{\nu}_t^j, \mu_t^j)^2\big]^{\frac{1}{2}}$ arising from the last line in the estimate of Proposition 7.1. Although this term is suppressed in the non-cooperative game as in (6.9), it does not happen in the current situation. The deviation from the strategy $\hat{\alpha}_i$ for the agents in the ith population produces the term $||\beta^{i,1} - \underline{\hat{\alpha}}||_{\mathbb{H}^2}$ with no suppression of ε_{N_i} in the estimate for $\mathbb{E}\big[W_2(\overline{\nu}_t^j, \mu_t^j)^2\big]^{\frac{1}{2}}$. This is why we need Assumption **(MFTC-FA-b)**.

Theorem 7.1. *Under Assumption* **(MFTC-FA-b)** *with sufficiently large* $(N_i)_{i=1}^2$*, the feedback control functions* $\big([0,T] \times \mathbb{R}^d \ni (t,x) \mapsto \hat{\alpha}_i(t,x,\mu_t^i,\mu_t^j,u_i(t,x,\mu_t^i))\big)_{1 \le i,j \le 2, j \ne i}$ *form an* $(\sum_{j=1}^2 \varepsilon_{N_j})$*-approximate Nash equilibrium i.e., there exists some constant* C *independent of* $(N_i)_{i=1}^2$ *such that*

$$J_i^{N_i,N_j}(\boldsymbol{\beta}^{i,(N_i)}, \hat{\boldsymbol{\alpha}}^{j,(N_j)}) \ge J_i^{N_i,N_j}(\hat{\boldsymbol{\alpha}}^{i,(N_i)}, \hat{\boldsymbol{\alpha}}^{j,(N_j)}) - C\sum_{j=1}^2 \varepsilon_{N_j}$$

for any A_i*-valued* $\mathbb{F}$*-progressively measurable processes* $(\boldsymbol{\beta}^{i,p} \in \mathbb{H}^2)_{1 \le p \le N_i}$ *so that* $(\xi^{i,p}, \boldsymbol{\beta}^{i,p}, \boldsymbol{W}^{i,p})_{1 \le p \le N_i}$ *is exchangeable. Here,* $\big(\varepsilon_{N_j} := \max(N_j^{-\frac{1}{2}}, \epsilon_{N_j})\big)_{1 \le j \le 2}$.

Proof. Under **(MFTC-FA-b)**, we can write the coefficients for both populations $1 \le j \le 2$ as,

$$\begin{aligned} b_j(t,x,\mu,\alpha) &= b_{j,0}(t) + b_{j,1}(t)x + \bar{b}_{j,1}(t)\bar{\mu} + b_{j,2}(t)\alpha \\ \sigma_j(t,x,\mu) &= \sigma_{j,0}(t) + \sigma_{j,1}(t)x + \bar{\sigma}_{j,1}(t)\bar{\mu}\ . \end{aligned} \tag{7.2}$$

For the ith population, we get

$$\mathbb{E}\big[\sup_{s \in [0,t]} |U_s^{i,p}|^2\big] \le C\mathbb{E}\Big[|\xi^{i,p}|^2 + \int_0^t \big(1 + |U_s^{i,p}|^2 + M_2(\overline{\nu}_s^i)^2 + |\beta_s^{i,p}|^2\big)ds\Big]\ ,$$

which yields $\mathbb{E}\big[\sup_{t \in [0,T]} |U_t^{i,p}|^2\big] \le C\big(1 + ||\beta^{i,p}||_{\mathbb{H}^2}^2\big)$. We also get $\mathbb{E}\big[\sup_{t \in [0,T]} |U_t^{j,q}|^2\big] \le C$ for the jth population. Similarly, we see for the ith population,

$$\mathbb{E}\big[\sup_{s \in [0,t]} |U_s^{i,p} - \underline{X}_s^{i,p}|^2\big] \le C\mathbb{E}\int_0^t \Big[|U_s^{i,p} - \underline{X}_s^{i,p}|^2 + W_2(\overline{\nu}_s^i, \underline{\mu}_s^i)^2 + W_2(\underline{\mu}_s^i, \mu_s^i)^2 + |\beta_s^{i,p} - \underline{\hat{\alpha}}_s^{i,p}|^2\Big]ds,$$

and then, by Gronwall's inequality, we get

$$\begin{aligned} \mathbb{E}\big[\sup_{t \in [0,T]} |U_t^{i,p} - \underline{X}_t^{i,p}|^2\big] &\le C\Big(\sup_{t \in [0,T]} \mathbb{E}\big[W_2(\underline{\mu}_t^i, \mu_t^i)^2\big] + ||\beta^{i,p} - \underline{\hat{\alpha}}^{i,p}||_{\mathbb{H}^2}^2\Big) \\ &\le C\big(\varepsilon_{N_i}^2 + ||\beta^{i,p} - \underline{\hat{\alpha}}^{i,p}||_{\mathbb{H}^2}^2\big)\ . \end{aligned}$$

Similarly, we have

$$\mathbb{E}\big[\sup_{t \in [0,T]} |U_t^{j,q} - \underline{X}_t^{j,q}|^2\big] \le C\Big(\sup_{t \in [0,T]} \mathbb{E}\big[W_2(\underline{\mu}_t^j, \mu_t^j)^2\big]\Big) \le C\varepsilon_{N_j}^2\ .$$

In particular, by the triangle inequality, $\sup_{t \in [0,T]} \mathbb{E}\big[W_2(\overline{\nu}_t^j, \mu_t^j)^2\big] \le C\varepsilon_{N_j}^2$ holds.

From (7.2), it is easy to see

$$\begin{aligned}
&|H_i(t, U_t^{i,p}, \overline{\nu}_t^i, \overline{\nu}_t^j, \underline{Y}_t^{i,p}, \underline{Z}_t^{i,p}, \beta_t^{i,p}) - H_i(t, U_t^{i,p}, \overline{\nu}_t^i, \mu_t^j, \underline{Y}_t^{i,p}, \underline{Z}_t^{i,p}, \beta_t^{i,p})| \\
&= |f_i(t, U_t^{i,p}, \overline{\nu}_t^i, \overline{\nu}_t^j, \beta_t^{i,p}) - f_i(t, U_t^{i,p}, \overline{\nu}_t^i, \mu_t^j, \beta_t^{i,p})| \\
&\leq C(1 + |U_t^{i,p}| + |M_2(\overline{\nu}_t^i)| + |M_2(\overline{\nu}_t^j)| + |\beta_t^{i,p}|) W_2(\overline{\nu}_t^j, \mu_t^j)
\end{aligned} \tag{7.3}$$

Using similar estimate for g_i and exchangeability, we get from Proposition 7.1,

$$\begin{aligned}
&J_i^{N_i,N_j}(\boldsymbol{\beta}^{i,(N_i)}, \hat{\boldsymbol{\alpha}}^{j,(N_j)}) - J_i \\
&\geq \lambda \mathbb{E} \int_0^T |\beta_t^{i,p} - \underline{\hat{\alpha}}_t^{i,p}|^2 dt - C\big(1 + ||\beta^{i,p} - \underline{\hat{\alpha}}^{i,p}||_{\mathbb{H}^2}\big)\varepsilon_{N_i} \\
&\quad -C\Big(1 + \sup_{t\in[0,T]} \mathbb{E}\big[|U_t^{i,p}|^2 + |U_t^{j,q}|^2\big]^{\frac{1}{2}} + ||\beta^{i,p}||_{\mathbb{H}^2}\Big) \sup_{t\in[0,T]} \mathbb{E}\big[W_2(\overline{\nu}_t^j, \mu_t^j)^2\big]^{\frac{1}{2}} \\
&\geq \lambda \mathbb{E} \int_0^T |\beta_t^{i,p} - \underline{\hat{\alpha}}_t^{i,p}|^2 dt - C\big(1 + ||\beta^{i,p} - \underline{\hat{\alpha}}^{i,p}||_{\mathbb{H}^2}\big)\varepsilon_{N_i} - C\Big(1 + ||\beta^{i,p}||_{\mathbb{H}^2}\Big)\varepsilon_{N_j} \ .
\end{aligned}$$

Since $||\underline{\hat{\alpha}}^{i,p}||_{\mathbb{H}^2} \leq C$, and using the fact that

$$||\beta^{i,p} - \underline{\hat{\alpha}}^{i,p}||_{\mathbb{H}^2}\varepsilon_{N_i} \leq C\big(||\beta^{i,p} - \underline{\hat{\alpha}}^{i,p}||_{\mathbb{H}^2}^2 \varepsilon_{N_i} + \varepsilon_{N_i}\big)$$

we get

$$J_i^{N_i,N_j}(\boldsymbol{\beta}^{i,(N_i)}, \hat{\boldsymbol{\alpha}}^{j,(N_j)}) - J_i \geq \big(\lambda - C\sum_{j=1}^{2} \varepsilon_{N_j}\big)||\beta^{i,p} - \underline{\hat{\alpha}}^{i,p}||_{\mathbb{H}^2}^2 - C\sum_{j=1}^{2} \varepsilon_{N_j} \ .$$

We now get the desired estimate from Lemma 7.2 for sufficiently large N_1 and N_2. □

Remark 7.2. *We have investigated the approximate Nash equilibrium in the closed loop framework. The analysis for the open loop framework can be done in a quite similar manner as explained in Remark 6.2. Generalization to an arbitrary number of populations $1 \leq i \leq m$ can be done straightforwardly.*

8 Approximate Equilibrium for MFTC-MFG with Finite Agents

In this section, we shall see how the solution to the Nash MFTC-MFG problem studied in Section 5 can provide an approximate Nash equilibrium among the two competing populations of finite agents, where the agents in the first population are cooperative but those in the second population are not. As we have seen in the last section, the effect of deviation from the optimal strategy in the first population will not be suppressed by ε_{N_1}. In order to obtain an approximate Nash equilibrium, this feature implies that we have to cut the direct interaction with the first population in the state dynamics of the second one. On the other hand, the agents in the second population are non-cooperative. Hence, the effect of the deviation of the single agent in the second population will be suppressed by ε_{N_2}. This suggests that we may include the direct interaction with the second population in the state dynamics of the first one.

Throughout the section, we assume that the conditions used either in Theorem 5.2 or Theorem 5.3 are satisfied. We let $(\boldsymbol{\mu}^1, \boldsymbol{\mu}^2) \in \mathcal{C}([0,T]; \mathcal{P}_2(\mathbb{R}^d))^2$ denote a solution to the matching problem (5.2). Moreover, unless otherwise stated, we use the same notation in the last two sections.

8.1 Convergence of approximate optimal controls

For each population $1 \le i \le 2$, we give i.i.d. copies of the state process in the mean field setup:

$$d\underline{X}_t^{i,p} = b_i(t, \underline{X}_t^{i,p}, \mu_t^i, \mu_t^j, \underline{\hat{\alpha}}_t^{i,p})dt + \sigma_i(t, \underline{X}_t^{i,p}, \mu_t^i, \mu_t^j)dW_t^{i,p} \ ,$$

for $1 \le p \le N_i$, $j \ne i$, $t \in [0, T]$ with $X_0^{i,p} = \xi^{i,p}$, and

$$\begin{aligned}
\underline{\hat{\alpha}}^{1,p} &:= \hat{\alpha}_1(t, \underline{X}_t^{1,p}, \mu_t^1, \mu_t^2, u_1(t, \underline{X}_t^{1,p}, \mu_t^1)) \ , \\
\underline{\hat{\alpha}}^{2,p} &:= \hat{\alpha}_2(t, \underline{X}_t^{2,p}, \mu_t^2, \mu_t^1, u_2(t, \underline{X}_t^{2,p})) \ ,
\end{aligned}$$

where u_1 is the master field defined in Lemma 4.2 applied to (5.3) and u_2 the decoupling field defined in Theorem 3.1 applied to (5.4) where the equilibrium flow of probability measures $(\boldsymbol{\mu}^1, \boldsymbol{\mu}^2)$ are used as inputs in both cases. As before, $(\xi^{i,p}, \boldsymbol{W}^{i,p})_{1 \le p \le N_i, 1 \le i \le 2}$ are assumed to be independent with $\mathcal{L}(\xi^{i,p}) = \mu_0^i$. By construction, $(\underline{\boldsymbol{X}}^{i,p})_{1 \le p \le N_i}$ are i.i.d. processes satisfying $\mathcal{L}(\underline{X}_t^{i,p}) = \mu_t^i$, $\forall t \in [0, T]$. $\underline{\mu}_t^i := \frac{1}{N_i} \sum_{p=1}^{N_i} \delta \underline{X}_t^{i,p}$ denotes the empirical distribution of $(\underline{X}_t^{i,p})_{1 \le p \le N_i}$.

Lemma 8.1. *Suppose that the conditions either for Theorem 5.2 or Theorem 5.3 are satisfied. Then, for each population $1 \le i \le 2$, there exists some sequence $(\epsilon_{N_i})_{N_i \ge 1}$ that tends to 0 as N_i tends to ∞ and some constant C such that*

$$\sup_{t \in [0,T]} \mathbb{E}\Big[W_2(\underline{\mu}_t^i, \mu_t^i)^2\Big] \le C\epsilon_{N_i}^2 \ .$$

Furthermore, when $\mu_0^i \in \mathcal{P}_r(\mathbb{R}^d)$ with $r > 4$, we have an explicit estimate

$$\epsilon_{N_i}^2 = N_i^{-2/\max(d,4)}(1 + \ln(N_i)\mathbf{1}_{d=4}) \ .$$

Proof. It is the direct result of Lemma 6.1 and Lemma 7.1. □

We introduce the following assumptions.

Assumption 8.1. (MFTC-MFG-FA-a) *On top of Assumption* **(MFTC-a)** *for the coefficients $(b_1, \sigma_1, f_1, g_1)$ and Assumption* **(MFG-a)** *for the coefficients $(b_2, \sigma_2, f_2, g_2)$, either T or $(\lambda^{-1}||b_{i,2}||_\infty)_{1 \le i \le 2}$ is small enough to satisfy the conditions for Theorem 5.2 or Theorem 5.3. Moreover, the coefficients $(b_1, \sigma_1, f_1, g_1)$ satisfy* **(MFTC-FA-a) (A1-A2)***, and $(b_2, \sigma_2, f_2, g_2)$ satisfy* **(MFG-FA) (A1-A2)**.

Assumption 8.2. (MFTC-MFG-FA-b) *On top of Assumption* **(MFTC-MFG-FA-a)***, the coefficients $b_{2,0}$ and $\sigma_{2,0}$ are independent of the second measure argument, i.e. $(b_{2,0}, \sigma_{2,0})(t, \mu, \nu) = (b_{2,0}, \sigma_{2,0})(t, \mu)$.*

For $1 \le i, j \le 2$, $j \ne i$, $1 \le p \le N_i$, we introduce the state processes

$$dX_t^{i,p} = b_i(t, X_t^{i,p}, \overline{\mu}_t^i, \overline{\mu}_t^j, \hat{\alpha}_t^{i,p})dt + \sigma_i(t, X_t^{i,p}, \overline{\mu}_t^i, \overline{\mu}_t^j)dW_t^{i,p} \ ,$$

for $t \in [0, T]$ with $X_0^{i,p} = \xi^{i,p}$. Here, $\overline{\mu}_t^i := \frac{1}{N_i} \sum_{p=1}^{N_i} \delta_{X_t^{i,p}}$ denotes the empirical distribution and

$$\begin{aligned}
\hat{\alpha}_t^{1,p} &:= \hat{\alpha}_1(t, X_t^{1,p}, \mu_t^1, \mu_t^2, u_1(t, X_t^{1,p}, \mu_t^1)) \ , \\
\hat{\alpha}_t^{2,p} &:= \hat{\alpha}_2(t, X_t^{2,p}, \mu_t^2, \mu_t^1, u_2(t, X_t^{2,p})) \ .
\end{aligned}$$

Under Assumption **(MFTC-MFG-FA-a)**, it is an $(N_1 + N_2)$-dimensional Lipschitz SDE and hence is well-defined. The corresponding cost functional for any agent p in the first population is given by

$$J_1^{N_1,N_2}(\hat{\boldsymbol{\alpha}}^{1,(N_1)},\hat{\boldsymbol{\alpha}}^{2,(N_2)}) := \mathbb{E}\Big[\int_0^T f_1(t,X_t^{1,p},\overline{\mu}_t^1,\overline{\mu}_t^2,\hat{\alpha}_t^{1,p})dt + g_1(X_T^{1,p},\overline{\mu}_T^1,\overline{\mu}_T^2)\Big] ,$$

and the for the agent q in the second population,

$$J_2^{N_2,N_1,q}(\hat{\boldsymbol{\alpha}}^{2,(N_2)},\hat{\boldsymbol{\alpha}}^{1,(N_1)}) := \mathbb{E}\Big[\int_0^T f_2(t,X_t^{2,q},\overline{\mu}_t^2,\overline{\mu}_t^1,\hat{\alpha}_t^{2,q})dt + g_2(X_T^{2,q},\overline{\mu}_T^2,\overline{\mu}_T^1)\Big] .$$

We also introduce the optimal cost functionals in the mean field setup:

$$J_1 := \mathbb{E}\Big[\int_0^T f_1(t,\underline{X}_t^{1,p},\mu_t^1,\mu_t^2,\underline{\hat{\alpha}}_t^{1,p})dt + g_1(\underline{X}_T^{1,p},\mu_T^1,\mu_T^2)\Big] ,$$

$$J_2^q := \mathbb{E}\Big[\int_0^T f_2(t,\underline{X}_t^{2,q},\mu_t^2,\mu_t^1,\underline{\hat{\alpha}}_t^{2,q})dt + g_2(\underline{X}_T^{2,q},\mu_T^2,\mu_T^1)\Big] .$$

Remark 8.1. *Under the given control strategy, the value of J_2^q is independent of q. However, since each agent in the second population can choose his/her own strategy in general, we need to specify the agent when we discuss the approximate Nash equilibrium later. This is why we keep the index q in the cost functional.*

Lemma 8.2. *Under Assumption* **(MFTC-MFG-FA-a)***, for any $N_1,N_2\in\mathbb{N}$ and $1\le q\le N_2$, there exists some constant C independent of $(N_i)_{i=1}^2$ such that,*

$$|J_1^{N_1,N_2}(\hat{\boldsymbol{\alpha}}^{1,(N_1)},\hat{\boldsymbol{\alpha}}^{2,(N_2)}) - J_1| \le C\sum_{j=1}^2 \epsilon_{N_j},\quad |J_2^{N_2,N_1,q}(\hat{\boldsymbol{\alpha}}^{2,(N_2)},\hat{\boldsymbol{\alpha}}^{1,(N_1)}) - J_2^q| \le C\sum_{j=1}^2 \epsilon_{N_j}$$

where ϵ_{N_j} is the one given in Lemma 8.1.

Proof. As in the last two sections, we can show, by the same arguments,

$$\sum_{i=1}^2 \mathbb{E}\big[\sup_{t\in[0,T]}|X_t^{i,p}-\underline{X}_t^{i,p}|^2\big] \le C\sum_{i=1}^2 \sup_{t\in[0,T]}\mathbb{E}\big[W_2(\overline{\mu}_t^i,\mu_t^i)^2\big] \le C\sum_{i=1}^2 \epsilon_{N_i}^2 ,$$

and $\mathbb{E}\big[\sup_{t\in[0,T]}|X_t^{i,p}|^2\big]+\mathbb{E}\big[\sup_{t\in[0,T]}|\underline{X}_t^{i,p}|^2\big]\le C$. Thus the convergence of the cost functionals is the direct result of Lemma 6.2 and Lemma 7.2. □

8.2 Approximate Nash Equilibrium

We now consider the general state dynamics for the agents $1\le i,j\le 2$, $j\neq i$, $1\le p\le N_i$,

$$dU_t^{i,p} = b_i(t,U_t^{i,p},\overline{\nu}_t^i,\overline{\nu}_t^j,\beta_t^{i,p})dt + \sigma_i(t,U_t^{i,p},\overline{\nu}_t^i,\overline{\nu}_t^j)dW_t^{i,p}$$

with $t\in[0,T]$, $U_0^{i,p}=\xi^{i,p}$ and $\boldsymbol{\beta}^{i,p}\in\mathbb{H}^2$ is an A_i-valued $\mathbb{F}$-progressively measurable process. For the first population, we impose the condition so that $(\xi^{1,p},\boldsymbol{\beta}^{1,p},\boldsymbol{W}^{1,p})_{1\le p\le N_1}$ is exchangeable. As usual, $\overline{\nu}_t^i := \frac{1}{N_i}\sum_{p=1}^{N_i}\delta_{U_t^{i,p}}$ denotes the empirical distribution. We shall investigate the following two situations:

(setup-1): The agents in the first population adopt the general exchangeable strategy $(\boldsymbol{\beta}^{1,p})_{1\le p\le N_1}$

and the agents in the second population adopt, for any $t \in [0,T]$,

$$\beta_t^{2,q} := \hat{\alpha}_2(t, U_t^{2,q}, \mu_t^2, \mu_t^1, u_2(t, U_t^{2,q})) \ , 1 \le q \le N_2.$$

(setup-2): The first agent in the second population adopts the general strategy $\boldsymbol{\beta}^{2,1}$, and the remaining agents in the second population as well as the agents in the first population adopt, for any $t \in [0,T]$,

$$\begin{aligned} \beta_t^{1,p} &:= \hat{\alpha}_1(t, U_t^{1,p}, \mu_t^1, \mu_t^2, u_1(t, U_t^{1,p}, \mu_t^1)) \ , 1 \le p \le N_1, \\ \beta_t^{2,q} &:= \hat{\alpha}_2(t, U_t^{2,q}, \mu_t^2, \mu_t^1, u_2(t, U_t^{2,q})) \ , 2 \le q \le N_2 \ . \end{aligned}$$

The cost functional for any agent in the first population in **(setup-1)** is given by

$$J_1^{N_1,N_2}(\boldsymbol{\beta}^{1,(N_1)}, \hat{\boldsymbol{\alpha}}^{2,(N_2)}) := \mathbb{E}\Big[\int_0^T f_1(t, U_t^{1,p}, \overline{\nu}_t^1, \overline{\nu}_t^2, \beta_t^{1,p})dt + g_1(U_T^{1,p}, \overline{\nu}_T^1, \overline{\nu}_T^2)\Big]$$

with an arbitrary $1 \le p \le N_1$. On the other hand, the cost functional for the first agent in the second population in **(setup-2)** is given by

$$J_2^{N_2,N_1,1}(\boldsymbol{\beta}^{2,1}, \hat{\boldsymbol{\alpha}}^{2,(N_2)^{-1}}, \hat{\boldsymbol{\alpha}}^{1,(N_1)}) := \mathbb{E}\Big[\int_0^T f_2(t, U_t^{2,1}, \overline{\nu}_t^2, \overline{\nu}_t^1, \beta_t^{2,1})dt + g_2(U_t^{2,1}, \overline{\nu}_T^2, \overline{\nu}_T^1)\Big] \ .$$

The main result of this section is as follows.

Theorem 8.1. *Under Assumption* **(MFTC-MFG-FA-b)** *with sufficiently large $(N_i)_{i=1}^2$, the feedback control functions $\big([0,T]\times\mathbb{R}^d \ni (t,x) \mapsto \hat{\alpha}_1(t,x,\mu_t^1,\mu_t^2,u_1(t,x,\mu_t^1))\big)$ for the first population and $\big([0,T]\times\mathbb{R}^d \ni (t,x) \mapsto \hat{\alpha}_2(t,x,\mu_t^2,\mu_t^1,u_2(t,x))\big)$ for the second one form an $(\sum_{j=1}^2 \varepsilon_{N_j})$-approximate Nash equilibrium i.e., there exists some constant C independent of $(N_i)_{i=1}^2$ such that*

$$J_1^{N_1,N_2}(\boldsymbol{\beta}^{1,(N_1)}, \hat{\boldsymbol{\alpha}}^{2,(N_2)}) \ge J_1^{N_1,N_2}(\hat{\boldsymbol{\alpha}}^{1,(N_1)}, \hat{\boldsymbol{\alpha}}^{2,(N_2)}) - C\sum_{j=1}^2 \varepsilon_{N_j}$$

under **(setup-1)***, and also that*

$$J_2^{N_2,N_1,1}(\boldsymbol{\beta}^{2,1}, \hat{\boldsymbol{\alpha}}^{2,(N_2)^{-1}}, \hat{\boldsymbol{\alpha}}^{1,(N_1)}) \ge J_2^{N_2,N_1,1}(\hat{\boldsymbol{\alpha}}^{2,(N_2)}, \hat{\boldsymbol{\alpha}}^{1,(N_1)}) - C\sum_{j=1}^2 \varepsilon_{N_j}$$

*under **(setup-2)**. In both cases, $\big(\varepsilon_{N_j} := \max(N_j^{-\frac{1}{2}}, \epsilon_{N_j})\big)_{1\le j\le 2}$.*

Proof. **(first step)**: Let us first prove the claim under **(setup-1)**. In contrast to the assumptions used in Theorem 7.1, the agents in the first population now have direct interactions with those in the second population in their state processes. Applying the result of Proposition 7.1 to the first population, we get

$$
\begin{aligned}
J_1^{N_1,N_2}(\boldsymbol{\beta}^{1,(N_1)},\hat{\boldsymbol{\alpha}}^{2,(N_2)})-J_1 &\geq \lambda\mathbb{E}\int_0^T|\beta_t^{1,p}-\underline{\hat{\alpha}}_t^{1,p}|^2dt\\
&-C\Big(1+\sup_{t\in[0,T]}\mathbb{E}\big[|U_t^{1,p}-\underline{X}_t^{1,p}|^2\big]^{\frac{1}{2}}+\mathbb{E}\big[\int_0^T|\beta_t^{1,p}-\underline{\hat{\alpha}}_t^{1,p}|^2dt\big]^{\frac{1}{2}}\Big)\varepsilon_{N_1}\\
&+\mathbb{E}\Big[\int_0^T\big(H_1(t,U_t^{1,p},\overline{\nu}_t^1,\overline{\nu}_t^2,\underline{Y}_t^{1,p},\underline{Z}_t^{1,p},\beta_t^{1,p})-H_1(t,U_t^{1,p},\overline{\nu}_t^1,\mu_t^2,\underline{Y}_t^{1,p},\underline{Z}_t^{1,p},\beta_t^{1,p})\big)dt\\
&\qquad+g_1(U_T^{1,p},\overline{\nu}_T^1,\overline{\nu}_T^2)-g_1(U_T^{1,p},\overline{\nu}_T^1,\mu_T^2)\Big]\ ,
\end{aligned}
$$

where $(\underline{Y}^{1,p},\underline{Z}^{1,p})$ is defined in the same way as (7.1). Since (b_2,σ_2) are independent from the second measure argument, it is straightforward to confirm that $\mathbb{E}\big[\sup_{t\in[0,T]}|U_t^{2,q}|^2\big]\leq C$ for any q. Then we get $\mathbb{E}\big[\sup_{t\in[0,T]}|U_t^{1,p}|^2\big]\leq C\big(1+||\beta^{1,p}||^2_{\mathbb{H}^2}\big)$ for any p. Moreover, it is immediate to obtain, for any q,

$$
\mathbb{E}\big[\sup_{t\in[0,T]}|U_t^{2,q}-\underline{X}_t^{2,q}|^2\big]\leq C\sup_{t\in[0,T]}\mathbb{E}\big[W_2(\underline{\mu}_t^2,\mu_t^2)^2\big]\leq C\varepsilon_{N_2}^2\ .
$$

In particular, this also implies $\sup_{t\in[0,T]}\mathbb{E}\big[W_2(\overline{\nu}_t^2,\mu_t^2)^2\big]\leq C\varepsilon_{N_2}^2$. Since, for any p,

$$
\mathbb{E}\big[\sup_{s\in[0,t]}|U_s^{1,p}-\underline{X}_s^{1,p}|^2\big]\leq C\mathbb{E}\int_0^t\Big[|U_s^{1,p}-\underline{X}_s^{1,p}|^2+\sum_{j=1}^2W_2(\overline{\nu}_s^j,\mu_s^j)^2+|\beta_s^{1,p}-\underline{\hat{\alpha}}_s^{1,p}|^2\Big]ds
$$

we get, from the last estimate and the triangle inequality,

$$
\mathbb{E}\big[\sup_{t\in[0,T]}|U_t^{1,p}-\underline{X}_t^{1,p}|^2\big]\leq C\Big(\sum_{j=1}^2\varepsilon_{N_j}^2+||\beta^{1,p}-\underline{\hat{\alpha}}^{1,p}||^2_{\mathbb{H}^2}\Big)\ .
$$

By the standard calculation, we see

$$
\begin{aligned}
&\Big|\mathbb{E}\Big[\int_0^T\big(H_1(t,U_t^{1,p},\overline{\nu}_t^1,\overline{\nu}_t^2,\underline{Y}_t^{1,p},\underline{Z}_t^{1,p},\beta_t^{1,p})-H_1(t,U_t^{1,p},\overline{\nu}_t^1,\mu_t^2,\underline{Y}_t^{1,p},\underline{Z}_t^{1,p},\beta_t^{1,p})\big)dt\\
&\qquad+g_1(U_T^{1,p},\overline{\nu}_T^1,\overline{\nu}_T^2)-g_1(U_T^{1,p},\overline{\nu}_T^1,\mu_T^2)\Big]\Big|\\
&\leq C\Big(1+\sup_{t\in[0,T]}\mathbb{E}\Big[\sum_{i=1}^2|U_t^{i,p}|^2+|\underline{Y}_t^{1,p}|^2\Big]^{\frac{1}{2}}+\mathbb{E}\Big[\int_0^T\big[|\underline{Z}_t^{1,p}|^2+|\beta_t^{1,p}|^2\big]dt\Big]^{\frac{1}{2}}\Big)\sup_{t\in[0,T]}\mathbb{E}\big[W_2(\overline{\nu}_t^2,\mu_t^2)^2\big]^{\frac{1}{2}}\\
&\leq C\big(1+||\beta^{1,p}||_{\mathbb{H}^2}\big)\varepsilon_{N_2}\ .
\end{aligned}
$$

Since $||\underline{\hat{\alpha}}^{1,p}||_{\mathbb{H}^2}\leq C$, we get

$$
\begin{aligned}
&J_1^{N_1,N_2}(\boldsymbol{\beta}^{1,(N_1)},\hat{\boldsymbol{\alpha}}^{2,(N_2)})-J_1\\
&\quad\geq\lambda||\beta^{1,p}-\underline{\hat{\alpha}}^{1,p}||^2_{\mathbb{H}^2}-C\Big(1+||\beta_t^{1,p}-\underline{\hat{\alpha}}^{1,p}||_{\mathbb{H}^2}\Big)\sum_{j=1}^2\varepsilon_{N_j}\\
&\quad\geq\Big(\lambda-C\sum_{j=1}^2\varepsilon_{N_j}\Big)||\beta^{1,p}-\underline{\hat{\alpha}}^{1,p}||^2_{\mathbb{H}^2}-C\sum_{j=1}^2\varepsilon_{N_j}\ .
\end{aligned}
$$

Now Lemma 8.2 gives the desired estimate.

(second step): Let us now prove the claim under **(setup-2)**. By putting $i = 2$ and $j = 1$, all of the arguments in the proof for Theorem 6.1 work as they are. In fact, due to the independence of (b_2, σ_2) from the second measure argument, some of the estimates become slightly simpler. In particular, (6.8) and (6.9) hold with $(i = 2, j = 1)$. The estimate (6.10) is now given by

$$\mathbb{E}\big[\sup_{t\in[0,T]} |U_T^{2,1} - \overline{U}_t^{2,1}|^2\big]^{\frac{1}{2}} \le C(1 + ||\beta^{2,1}||_{\mathbb{H}^2})\varepsilon_{N_2}$$

without the term ε_{N_1}. The estimate for $|J_2^{N_2,N_1,1}(\boldsymbol{\beta}^{2,1}, \hat{\boldsymbol{\alpha}}^{2,(N_2)^{-1}}, \hat{\boldsymbol{\alpha}}^{1,(N_1)}) - \overline{J}_2^1(\boldsymbol{\beta}^{2,1})|$ is given by exactly the same formula as in **(third step)** of the proof for Theorem 6.1 with $(i = 2, j = 1)$. Now, combining the result in Lemma 8.2, we get the desired estimate. □

Remark 8.2. *We have investigated the approximate Nash equilibrium in the closed loop framework. The analysis for the open loop framework can be done in a quite similar manner as explained in Remark 6.2. Generalization to an arbitrary number of populations $1 \le i \le m$ can be done straightforwardly, but the direct interactions in the state processes must be carefully arranged. The empirical distribution of the states of the agents who are in a cooperative population must not appear in the coefficients of the state process of the agents in any other populations. The empirical distribution can appear only in the control strategies indirectly induced by the interactions in the cost functions. On the other hand, the distribution of the states of the agents who are in a non-cooperative population can directly appear in the coefficients of the state processes of the agents in any populations.*

9 Conclusion and Discussion

In this work, we have systematically investigated mean field games and mean field type control problems with multiple populations for three different situations: (i) every agent is non-cooperative, (ii) the agents within each population are cooperative, and (iii) the agents in some populations are cooperative but not in the other populations. The relevant adjoint equations were shown to be given in terms of a coupled system of forward-backward stochastic differential equations of McKean-Vlasov type. In each case, we have provided several sets of sufficient conditions for the existence of an equilibrium, in particular the one which allows the cost functions of quadratic growth both in the state variable as well as its distribution so that it is applicable to some of the popular setups of linear quadratic problems. In the second half of the paper, under additional assumptions, we have proved that each solution to the mean field problems solved in the first half of the paper actually provides an approximate Nash equilibrium for the corresponding game with a large but finite number of agents.

As future works, we may study similar problems by adopting HJB type approach using so-called quadratic growth BSDEs as in [10], where the backward component directly represents the value function. Although we need the boundedness of the coefficients and the non-degeneracy for the diffusion function, the resultant boundedness of the solution to the BSDEs will make the analysis simpler. When each agent is subject to independent random Poisson measure, we may use the recent developments of the quadratic growth BSDEs with jumps such as in [29, 23, 18]. Finally, developing an efficient numerical method for mean field games and mean field type control problems remains as a very important issue. For a general problem, due to its infinite dimensionality, machine learning techniques (such as in [16]) are promising candidates. If the problem can be approximated by a linear quadratic setup, its solution may help to accelerate the speed of convergence for the learning process in the spirit of the work [19].

A Proof for Proposition 7.1

In the following, we use the Landau notation $\mathcal{O}(\cdot)$ in the sense that $|\mathcal{O}(x)| \le C|x|$ with some constant C independent of the population sizes (N_1, N_2). Let us define

$$
\begin{aligned}
T_1^{i,p} &:= \mathbb{E}\big[\langle U_T^{i,p} - \underline{X}_T^{i,p}, \underline{Y}_T^{i,p}\rangle\big] + \mathbb{E}\Big[\int_0^T \big[f_i(t, U_t^{i,p}, \overline{\nu}_t^i, \overline{\nu}_t^j, \beta_t^{i,p}) - f_i(t, \underline{X}_t^{i,p}, \mu_t^i, \mu_t^j, \underline{\hat{\alpha}}_t^{i,p})\big]dt\Big] ,\\
T_{2,1}^{i,p} &:= \mathbb{E}\big[g_i(U_T^{i,p}, \overline{\nu}_T^i, \overline{\nu}_T^j) - g_i(\underline{X}_T^{i,p}, \mu_T^i, \mu_T^j)\big] ,\\
T_{2,2}^{i,p} &:= \mathbb{E}\big[\langle U_T^{i,p} - \underline{X}_T^{i,p}, \partial_x g_i(\underline{X}_T^{i,p}, \mu_T^i, \mu_T^j)\rangle\big] ,\\
T_{2,3}^{i,p} &:= \mathbb{E}\widetilde{\mathbb{E}}\big[\langle \widetilde{U}_T^{i,p} - \widetilde{\underline{X}}_T^{i,p}, \partial_\mu g_i(\underline{X}_T^{i,p}, \mu_T^i, \mu_T^j)(\widetilde{\underline{X}}_T^{i,p})\big] ,
\end{aligned}
$$

which satisfy $J_i^{N_i,N_j}(\boldsymbol{\beta}^{i,(N_i)}, \boldsymbol{\beta}^{j,(N_j)}) - J_i = T_1^{i,p} + T_2^{i,p}$ with $T_2^{i,p} := T_{2,1}^{i,p} - T_{2,2}^{i,p} - T_{2,3}^{i,p}$.

A.1 Estimate for $T_2^{i,p}$

Consider the difference

$$
\begin{aligned}
&\Big|\mathbb{E}\widetilde{\mathbb{E}}\big[\langle \widetilde{U}_T^{i,p} - \widetilde{\underline{X}}_T^{i,p}, \partial_\mu g_i(\underline{X}_T^{i,p}, \mu_T^i, \mu_T^j)(\widetilde{\underline{X}}_T^{i,p})\big] - \frac{1}{N_i}\sum_{q=1}^{N_i} \widetilde{\mathbb{E}}\big[\langle \widetilde{U}_T^{i,p} - \widetilde{\underline{X}}_T^{i,p}, \partial_\mu g_i(\widetilde{\underline{X}}_T^{i,q}, \mu_T^i, \mu_T^j)(\widetilde{\underline{X}}_T^{i,p})\big]\Big|\\
&\le \mathbb{E}\big[|U_T^{i,p} - \underline{X}_T^{i,p}|^2\big]^{\frac12} \widetilde{\mathbb{E}}\Big[\Big|\mathbb{E}\big[\partial_\mu g_i(\underline{X}_T^{i,p}, \mu_T^i, \mu_T^j)(\widetilde{\underline{X}}_T^{i,p})\big] - \frac{1}{N_i}\sum_{q=1}^{N_i} \partial_\mu g_i(\widetilde{\underline{X}}_T^{i,q}, \mu_T^i, \mu_T^j)(\widetilde{\underline{X}}_T^{i,p})\Big|^2\Big]^{\frac12}\\
&\le C\mathbb{E}\big[|U_T^{i,p} - \underline{X}_T^{i,p}|^2\big]^{\frac12} N_i^{-\frac12}.
\end{aligned}
\tag{A.1}
$$

The last estimate is from the law of large numbers with the finite second moment of $\partial_\mu g_i$-term and the independence of $(\widetilde{\underline{X}}_T^{i,q})_{1\le q\le N_i}$. Taking the average in p, we get,

$$
\begin{aligned}
\frac{1}{N_i}\sum_{p=1}^{N_i} T_{2,3}^{i,p} &= \mathbb{E}\big[|U_T^{i,1} - \underline{X}_T^{i,1}|^2\big]^{\frac12}\mathcal{O}(N_i^{-\frac12}) + \frac{1}{N_i^2}\sum_{p,q=1}^{N_i} \mathbb{E}\big[\langle U_T^{i,p} - \underline{X}_T^{i,p}, \partial_\mu g_i(\underline{X}_T^{i,q}, \mu_T^i, \mu_T^j)(\underline{X}_T^{i,p})\big]\\
&= \mathbb{E}\big[|U_T^{i,1} - \underline{X}_T^{i,1}|^2\big]^{\frac12}\mathcal{O}(N_i^{-\frac12}) + \frac{1}{N_i}\sum_{p=1}^{N_i} \mathbb{E}\widetilde{\mathbb{E}}\big[\langle U_T^{i,\theta} - \underline{X}_T^{i,\theta}, \partial_\mu g_i(\underline{X}_T^{i,p}, \mu_T^i, \mu_T^j)(\underline{X}_T^{i,\theta})\big] ,
\end{aligned}
$$

where θ is a random variable on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ with uniform distribution on the set $\{1, \cdots, N_i\}$. Using Lemma 7.1 and the Lipschitz property of $\partial_\mu g_i$ with respect to the first measure argument, we get

$$
\frac{1}{N_i}\sum_{p=1}^{N_i} T_{2,3}^{i,p} = \mathbb{E}\big[|U_T^{i,1} - \underline{X}_T^{i,1}|^2\big]^{\frac12}\mathcal{O}(\varepsilon_{N_i}) + \frac{1}{N_i}\sum_{p=1}^{N_i} \mathbb{E}\widetilde{\mathbb{E}}\big[\langle U_T^{i,\theta} - \underline{X}_T^{i,\theta}, \partial_\mu g_i(\underline{X}_T^{i,p}, \underline{\mu}_T^i, \mu_T^j)(\underline{X}_T^{i,\theta})\big] . \tag{A.2}
$$

Using Lemma 7.1 also for $T_{2,2}^{i,p}, T_{2,3}^{i,p}$, we obtain

$$
\begin{aligned}
\frac{1}{N_i}\sum_{p=1}^{N_i} T_2^{i,p} &= \frac{1}{N_i}\sum_{p=1}^{N_i}\Big\{\mathbb{E}\big[g_i(U_T^{i,p}, \overline{\nu}_T^i, \mu_T^j) - g_i(\underline{X}_T^{i,p}, \underline{\mu}_T^i, \mu_T^j)\big]\\
&\quad - \mathbb{E}\big[\langle U_T^{i,p} - \underline{X}_T^{i,p}, \partial_x g_i(\underline{X}_T^{i,p}, \underline{\mu}_T^i, \mu_T^j)\rangle\big] - \mathbb{E}\widetilde{\mathbb{E}}\big[\langle U_T^{i,\theta} - \underline{X}_T^{i,\theta}, \partial_\mu g_i(\underline{X}_T^{i,p}, \underline{\mu}_T^i, \mu_T^j)(\underline{X}_T^{i,\theta})\rangle\big]\Big\}\\
&\quad + \Big(1 + \mathbb{E}\big[|U_T^{i,1} - \underline{X}_T^{i,1}|^2\big]^{\frac12}\Big)\mathcal{O}(\varepsilon_{N_i}) + \frac{1}{N_i}\sum_{p=1}^{N_i} \mathbb{E}\big[g_i(U_T^{i,p}, \overline{\nu}_T^i, \overline{\nu}_T^j) - g_i(U_T^{i,p}, \overline{\nu}_T^i, \mu_T^j)\big] .
\end{aligned}
$$

Using the fact that the conditional law of $U_T^{i,\theta}$ (respectively $\underline{X}_T^{i,\theta}$) under $\widetilde{\mathbb{P}}$ is given by the empirical distribution $\overline{\nu}_T^i$ (respectively $\underline{\mu}_T^i$), the convexity in **(MFTC-a) (A6)** implies

$$\frac{1}{N_i}\sum_{p=1}^{N_i} T_2^{i,p} \quad \geq \quad \Big(1+\mathbb{E}\big[|U_T^{i,1}-\underline{X}_T^{i,1}|^2\big]^{\frac{1}{2}}\Big)\mathcal{O}(\varepsilon_{N_i}) + \frac{1}{N_i}\sum_{p=1}^{N_i}\mathbb{E}\big[g_i(U_T^{i,p},\overline{\nu}_T^i,\overline{\nu}_T^j) - g_i(U_T^{i,p},\overline{\nu}_T^i,\mu_T^j)\big] \ .$$

A.2 Estimate for $T_1^{i,p}$

Using Ito formula to evaluate $\mathbb{E}\big[\langle U_T^{i,p}-\underline{X}_T^{i,p},\underline{Y}_T^{i,p}\rangle\big]$, we can rewrite $T_1^{i,p}$ as

$$\begin{aligned} T_1^{i,p} \quad &= \quad \mathbb{E}\int_0^T \Big\{H_i(t,U_t^{i,p},\overline{\nu}_t^i,\overline{\nu}_t^j,\underline{Y}_t^{i,p},\underline{Z}_t^{i,p},\beta_t^{i,p}) - H_i(t,\underline{X}_t^{i,p},\mu_t^i,\mu_t^j,\underline{Y}_t^{i,p},\underline{Z}_t^{i,p},\underline{\hat{\alpha}}_t^{i,p}) \\ &\qquad -\langle U_t^{i,p}-\underline{X}_t^{i,p},\partial_x H_i(t,\underline{X}_t^{i,p},\mu_t^i,\mu_t^j,\underline{Y}_t^{i,p},\underline{Z}_t^{i,p},\underline{\hat{\alpha}}_t^{i,p})\rangle \\ &\qquad -\widetilde{\mathbb{E}}\big[\langle \widetilde{U}_t^{i,p}-\widetilde{\underline{X}}_t^{i,p},\partial_\mu H_i(t,\underline{X}_t^{i,p},\mu_t^i,\mu_t^j,\underline{Y}_t^{i,p},\underline{Z}_t^{i,p},\underline{\hat{\alpha}}_t^{i,p})(\widetilde{\underline{X}}_t^{i,p})\rangle\big]\Big\}dt \\ &=: \quad T_{1,1}^{i,p} - T_{1,2}^{i,p} - T_{1,3}^{i,p} \ . \end{aligned}$$

Using local Lipschitz continuity of H_i with respect to the first measure argument, the estimate

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|\underline{X}_t^{i,p}|^2 + \sup_{t\in[0,T]}|\underline{Y}_t^{i,p}|^2 + \int_0^T(|\underline{Z}_t^{i,p}|^2 + |\underline{\hat{\alpha}}_t^{i,p}|^2)dt\Big] \leq C \ ,$$

and the result in Lemma 7.1, we get

$$T_{1,1}^{i,p} = \mathbb{E}\int_0^T \Big\{H_i(t,U_t^{i,p},\overline{\nu}_t^i,\overline{\nu}_t^j,\underline{Y}_t^{i,p},\underline{Z}_t^{i,p},\beta_t^{i,p}) - H_i(t,\underline{X}_t^{i,p},\underline{\mu}_t^i,\mu_t^j,\underline{Y}_t^{i,p},\underline{Z}_t^{i,p},\underline{\hat{\alpha}}_t^{i,p})\Big\}dt + \mathcal{O}(\varepsilon_{N_i}) \ .$$

We also get by similar calculation that

$$\begin{aligned} T_{1,2}^{i,p} \quad &= \quad \mathbb{E}\int_0^T \big[\langle U_t^{i,p}-\underline{X}_t^{i,p},\partial_x H_i(t,\underline{X}_t^{i,p},\underline{\mu}_t^i,\mu_t^j,\underline{Y}_t^{i,p},\underline{Z}_t^{i,p},\underline{\hat{\alpha}}_t^{i,p})\rangle\big]dt \\ &\qquad + \sup_{t\in[0,T]}\mathbb{E}\big[|U_t^{i,p}-\underline{X}_t^{i,p}|^2\big]^{\frac{1}{2}}\mathcal{O}(\varepsilon_{N_i}) \ . \end{aligned}$$

By the same arguments used in $(A.1)$, we get

$$\begin{aligned} T_{1,3}^{i,p} \quad &= \quad \frac{1}{N_i}\sum_{q=1}^{N_i}\int_0^T \mathbb{E}\big[\langle U_t^{i,p}-\underline{X}_t^{i,p},\partial_\mu H_i(t,\underline{X}_t^{i,q},\mu_t^i,\mu_t^j,\underline{Y}_t^{i,q},\underline{Z}_t^{i,q},\underline{\hat{\alpha}}_t^{i,q})(\underline{X}_t^{i,p})\rangle\big]dt \\ &\qquad + \sup_{t\in[0,T]}\mathbb{E}\big[|U_t^{i,1}-\underline{X}_t^{i,1}|^2\big]^{\frac{1}{2}}\mathcal{O}(N_i^{-\frac{1}{2}}) \ , \end{aligned}$$

and then same analysis used for $(A.2)$ gives

$$\begin{aligned} \frac{1}{N_i}\sum_{p=1}^{N_i} T_{1,3}^{i,p} \quad &= \quad \frac{1}{N_i}\sum_{p=1}^{N_i}\int_0^T \mathbb{E}\widetilde{\mathbb{E}}\big[\langle U_t^{i,\theta}-\underline{X}_t^{i,\theta},\partial_\mu H_i(t,\underline{X}_t^{i,p},\underline{\mu}_t^i,\mu_t^j,\underline{Y}_t^{i,p},\underline{Z}_t^{i,p},\underline{\hat{\alpha}}_t^{i,p})(\underline{X}_t^{i,\theta})\rangle\big]dt \\ &\qquad + \sup_{t\in[0,T]}\mathbb{E}\big[|U_t^{i,1}-\underline{X}_t^{i,1}|^2\big]^{\frac{1}{2}}\mathcal{O}(\varepsilon_{N_i}) \ . \end{aligned}$$

Finally, we can see that

$$\Big|\mathbb{E}\int_0^T \big[\langle \beta_t^{i,p}-\underline{\hat{\alpha}}_t^{i,p}, \partial_\alpha H_i(t,\underline{X}_t^{i,p},\underline{\mu}_t^i,\mu_t^j,\underline{Y}_t^{i,p},\underline{Z}_t^{i,p},\underline{\hat{\alpha}}_t^{i,p}) - \partial_\alpha H_i(t,\underline{X}_t^{i,p},\mu_t^i,\mu_t^j,\underline{Y}_t^{i,p},\underline{Z}_t^{i,p},\underline{\hat{\alpha}}_t^{i,p})\rangle\big]dt\Big|$$
$$\le C\mathbb{E}\Big[\int_0^T |\beta_t^{i,p}-\underline{\hat{\alpha}}_t^{i,p}|^2 dt\Big]^{\frac{1}{2}}\varepsilon_{N_i}\ .$$

Using the optimality condition, exchangeability, and the results obtained above, we get

$$\frac{1}{N_i}\sum_{p=1}^{N_i} T_1^{i,p} \ge \frac{1}{N_i}\sum_{p=1}^{N_i}(T_{1,1}^{i,p}-T_{1,2}^{i,p}-T_{1,3}^{i,p}) - \frac{1}{N_i}\sum_{p=1}^{N_i}\mathbb{E}\int_0^T \big[\langle \beta_t^{i,p}-\underline{\hat{\alpha}}_t^{i,p}, \partial_\alpha H_i(t,\underline{X}_t^{i,p},\mu_t^i,\mu_t^j,\underline{Y}_t^{i,p},\underline{Z}_t^{i,p},\underline{\hat{\alpha}}_t^{i,p})\rangle\big]dt$$
$$= \frac{1}{N_i}\sum_{p=1}^{N_i}\mathbb{E}\int_0^T\Big\{H_i(t,U_t^{i,p},\overline{\nu}_t^i,\mu_t^j,\underline{Y}_t^{i,p},\underline{Z}_t^{i,p},\beta_t^{i,p}) - H_i(t,\underline{X}_t^{i,p},\underline{\mu}_t^i,\mu_t^j,\underline{Y}_t^{i,p},\underline{Z}_t^{i,p},\underline{\hat{\alpha}}_t^{i,p})$$
$$-\langle U_t^{i,p}-\underline{X}_t^{i,p}, \partial_x H_i(t,\underline{X}_t^{i,p},\underline{\mu}_t^i,\mu_t^j,\underline{Y}_t^{i,p},\underline{Z}_t^{i,p},\underline{\hat{\alpha}}_t^{i,p})\rangle - \widetilde{\mathbb{E}}\big[\langle U_t^{i,\theta}-\underline{X}_t^{i,\theta}, \partial_\mu H_i(t,\underline{X}_t^{i,p},\underline{\mu}_t^i,\mu_t^j,\underline{Y}_t^{i,p},\underline{Z}_t^{i,p},\underline{\hat{\alpha}}_t^{i,p})(\underline{X}_t^{i,\theta})\rangle\big]$$
$$-\langle \beta_t^{i,p}-\underline{\hat{\alpha}}_t^{i,p}, \partial_\alpha H_i(t,\underline{X}_t^{i,p},\underline{\mu}_t^i,\mu_t^j,\underline{Y}_t^{i,p},\underline{Z}_t^{i,p},\underline{\hat{\alpha}}_t^{i,p})\rangle\Big\}dt$$
$$+\Big(1+\sup_{t\in[0,T]}\mathbb{E}\big[|U_t^{i,1}-\underline{X}_t^{i,1}|^2\big]^{\frac{1}{2}} + \mathbb{E}\Big[\int_0^T|\beta_t^{i,1}-\underline{\hat{\alpha}}_t^{i,1}|^2dt\Big]^{\frac{1}{2}}\Big)\mathcal{O}(\varepsilon_{N_i})$$
$$+\mathbb{E}\int_0^T\big[H_i(t,U_t^{i,1},\overline{\nu}_t^i,\overline{\nu}_t^j,\underline{Y}_t^{i,1},\underline{Z}_t^{i,1},\beta_t^{i,1}) - H_i(t,U_t^{i,1},\overline{\nu}_t^i,\mu_t^j,\underline{Y}_t^{i,1},\underline{Z}_t^{i,1},\beta_t^{i,1})\big]dt$$
$$\ge \lambda\mathbb{E}\int_0^T|\beta_t^{i,1}-\underline{\hat{\alpha}}_t^{i,1}|^2dt$$
$$+\Big(1+\sup_{t\in[0,T]}\mathbb{E}\big[|U_t^{i,1}-\underline{X}_t^{i,1}|^2\big]^{\frac{1}{2}} + \mathbb{E}\Big[\int_0^T|\beta_t^{i,1}-\underline{\hat{\alpha}}_t^{i,1}|^2dt\Big]^{\frac{1}{2}}\Big)\mathcal{O}(\varepsilon_{N_i})$$
$$+\mathbb{E}\int_0^T\big[H_i(t,U_t^{i,1},\overline{\nu}_t^i,\overline{\nu}_t^j,\underline{Y}_t^{i,1},\underline{Z}_t^{i,1},\beta_t^{i,1}) - H_i(t,U_t^{i,1},\overline{\nu}_t^i,\mu_t^j,\underline{Y}_t^{i,1},\underline{Z}_t^{i,1},\beta_t^{i,1})\big]dt\ .$$

A.3 Final Step

By exchangeability, we have

$$J_i^{N_i,N_j}(\boldsymbol{\beta}^{i,(N_i)},\boldsymbol{\beta}^{j,(N_j)}) - J_i = T_1^{i,p}+T_2^{i,p} = \frac{1}{N_i}\sum_{p=1}^{N_i}(T_1^{i,p}+T_2^{i,p})$$
$$\ge \lambda\mathbb{E}\int_0^T|\beta_t^{i,1}-\underline{\hat{\alpha}}_t^{i,1}|^2dt + \Big(1+\sup_{t\in[0,T]}\mathbb{E}\big[|U_t^{i,1}-\underline{X}_t^{i,1}|^2\big]^{\frac{1}{2}} + \mathbb{E}\Big[\int_0^T|\beta_t^{i,1}-\underline{\hat{\alpha}}_t^{i,1}|^2dt\Big]^{\frac{1}{2}}\Big)\mathcal{O}(\varepsilon_{N_i})$$
$$+\mathbb{E}\int_0^T\big[H_i(t,U_t^{i,1},\overline{\nu}_t^i,\overline{\nu}_t^j,\underline{Y}_t^{i,1},\underline{Z}_t^{i,1},\beta_t^{i,1}) - H_i(t,U_t^{i,1},\overline{\nu}_t^i,\mu_t^j,\underline{Y}_t^{i,1},\underline{Z}_t^{i,1},\beta_t^{i,1})\big]dt$$
$$+\mathbb{E}\big[g_i(U_T^{i,1},\overline{\nu}_T^i,\overline{\nu}_T^j) - g_i(U_T^{i,1},\overline{\nu}_T^i,\mu_T^j)\big]\ .$$

This gives the desired result.

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