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# Sequential $\boldsymbol{\varepsilon}$-Contamination with Learning*

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**Abstract**

The $\varepsilon$-contamination has been studied extensively as a convenient and operational specification of Knightian uncertainty. However, it is formulated in a static, one-shot economic environment. This paper extends this concept into a dynamic and sequential framework, allowing learning and guaranteeing time consistency of intertemporal decision. We develop the theory of the *rectangular* $\boldsymbol{\varepsilon}$-contamination, which can be represented by a sequence of $\varepsilon$'s that "contaminates" the conditional principal probability measure. We then compare this sequential (thus closed-loop) rectangular $\boldsymbol{\varepsilon}$-contamination with the initial-period one-shot (thus open-loop) $\varepsilon$-contamination, which is a straightforward extension of the static $\varepsilon$-contamination.

**JEL codes**: C61, D81, D83

**Keywords**: Knightian uncertainty, Open-loop $\varepsilon$-contamination, Rectangular $\boldsymbol{\varepsilon}$-contamination, Learning

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# 1 Introduction

The $\varepsilon$-contamination has been studied extensively as a convenient and operational specification of Knightian uncertainty. In the $\varepsilon$-contamination framework, the decision-maker is assumed to be $(1-\varepsilon) \times 100\%$-certain that she faces a particular probability measure, which may be called a "principal" probability measure, but with $\varepsilon \times 100\%$-fear she feels completely ignorant so that she may think she faces the worst case. This concept is applied to analyze the effect of Knightian uncertainty on the economic agent's behavior such as search (Nishimura and Ozaki, 2004), asset pricing (Epstein and Wang, 1994), voting (Chu and Liu, 2002) and learning (Nishimura and Ozaki, 2017, Chapter 14). Also, it has a simple and intuitive axiomatic foundation (Nishimura and Ozaki, 2006).[1] The $\varepsilon$-contamination also comes up in the statistics literature on robustness (for example, see Berger, 1985).

The $\varepsilon$-contamination described above, however, is formulated in a static, one-shot economic environment. Since many economic problems are dynamic and sequential in nature, we need a dynamic and sequential version of the $\varepsilon$-contamination. In particular, it is desirable to formulate sequential $\varepsilon$-contamination exhibiting time consistency of the intertemporal choices, since it is customarily assumed in many practical applications. It is well-known (Epstein and Schneider, 2003) that Knightian uncertainty should exhibit the *rectangularity* property in order to guarantee time consistency of intertemporal decision-making in the maxmin expected utility model under Knightian uncertainty (Gilboa and Schmeidler, 1989). Thus, we formulate "time-consistent" or *rectangular* $\varepsilon$-contamination as a dynamic extension of static, one-shot $\varepsilon$-contamination.

There is another issue in the dynamic formulation, which is *learning*. The issue can be explained by the analogy of the issue of learning under traditional framework of no Knightian uncertainty. Suppose that the decision-maker faces a particular probability measure. If she thinks she knows all parameters of the probability measure, she has no need to learn from new observation. In contrast, if she is uncertain about some of the parameters, she wants to learn about these uncertain parameters from new observation.

The situation is the same in the case of Knightian uncertainty, in which the decision-maker faces a *set* of probability measures. If she thinks she knows all parameters of the probability measures in the set, she has no need to learn from new observation. However, if she is uncertain about some of the parameters, she wants to learn about these uncertain parameters of the probability measures in the set from new observation. In this paper, we take the latter approach, and formulates the rectangular $\varepsilon$-contamination *with learning*.

[1] Nishimura and Ozaki (2006) treat $\varepsilon$ as exogenously given. See also Kopylov (2009) for another axiomatization from a different perspective in which $\varepsilon$ is a preference parameter that is endogenously derived.

We then compare the sequential *rectangular* $\varepsilon$-contamination with the initial-period one-shot $\varepsilon$-contamination in which $\varepsilon$-contamination is applied all at once in the initial period with respect to all possible evolutions of probability measures from the initial period to the last. The initial-period one-shot $\varepsilon$-contamination is a straightforward extension of the static, one-shot $\varepsilon$-contamination, though it may cause time *in*consistency.[2]

The results are remarkable. The *rectangular* $\varepsilon$-contamination can be represented by a sequence of $\varepsilon$'s that "contaminates" the conditional "principal" probability measure, which has a simple closed form that is dependent on the initial $\varepsilon$. The *rectangular* $\varepsilon$-contamination only slightly dilates the initial-period one-shot $\varepsilon$-contamination so as to make it satisfy the rectangularity. However, the both sets become exactly the same *a posteriori* when updated by generalized Bayes' rule, which is well-known updating rule for Knightian uncertainty.

The organization of the paper is as follows. In Section 2, we present the model and explain the main results in a two-period model. There we also present an example in which the initial-period one-shot $\varepsilon$-contamination is not rectangular, so that it does not guarantee time-consistency of decision-making. Section 3 extends the analyses of Section 2 to arbitrarily finitely many periods, and obtain the main results. Section 4 contains some concluding remarks.

# 2 Two-Period Models

## 2.1 Notations

This section exclusively considers the two-period model to illustrate the basic results of this paper in an intuitive way. The model will be extended to an arbitrary finite-horizon model in Section 3. The following notations draw on Chapter 14 of Nishimura and Ozaki (2017) at the outset, and then, they will be further simplified for the future use.

Let $S$ be a state space for each single period and let $\Omega := S \times S$ be the whole state space. A generic element of $\Omega$ is denoted by $(s_1, s_2)$.

The information structure, which represents the basis of the decision-maker's view of the world, is exogenously given by a filtration $\mathcal{F} := \langle \mathcal{F}_t \rangle_{t=0,1,2}$. Let $m, n \geq 2$ and let $\langle E_i \rangle_{i=1}^m$ and $\langle F_j \rangle_{j=1}^n$ be two finite partitions of $S$. Throughout this section, we fix these two partitions. We assume that $\mathcal{F}_1$ is represented by a finite partition of $\Omega$ of the form: $\langle E_i \times S \rangle_i$, and that $\mathcal{F}_2$ is represented by a finite partition of $\Omega$ of the form: $\langle E_i \times F_j \rangle_{i,j}$.

We abuse a notation by using a partition also to denote the algebra generated by that partition on $S$ and $\Omega$. By this convention, $\mathcal{F}_1$ and $\mathcal{F}_2$ are

[2] In the control theory terminology, the initial-period one-shot $\varepsilon$-contamination is an "open-loop" one, while sequential *rectangular* $\varepsilon$-contamination is a "closed-loop" one.

the algebras on $\Omega$ and it holds that $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2$, where $\mathcal{F}_0 := \{\phi, \Omega\}$. Thus, information increases as time goes by.

Consider a measurable space $(\Omega, \mathcal{F}_i)$ and let $\mathscr{M}(\Omega, \mathcal{F}_i)$ be the space of all probability charges on it ($i = 1, 2$). Given $p \in \mathscr{M}(\Omega, \mathcal{F}_2)$, we denote by $p|_1$ ($= p|\mathcal{F}_1$) its restriction on $(\Omega, \mathcal{F}_1)$. Although $p|_1$ is formally a charge on $\Omega$, it can be naturally regarded as the one on the measurable space, $(S, \langle E_i \rangle_i)$, and in that case, $p|_1(\cdot) = p(\cdot \times S)$. Thus viewed, $p|_1$ can be considered as the *first-period marginal probability charge* of $p$. We henceforth write $p|_1(E_i)$ simply as $p_i$ for all $i \leq m$.

Given $p \in \mathscr{M}(\Omega, \mathcal{F}_2)$, $i \leq m$ and $E_i$ satisfying $p(E_i \times S) > 0$, we denote by $p|_{E_i}(\cdot)$ the "posterior" probability charge on $(S, \langle F_j \rangle_j)$ conditional on the occurrence of $E_i \times S$. Here, the adjective "posterior" signifies the fact that this is a probability charge the decision-maker obtains after she made an observation, $E_i$, in the first period (and when she updates based on it). That is, $(\forall i, j)\ p|_{E_i}(F_j) := p(E_i \times F_j)/p(E_i \times S)$. With this conventional wording, we henceforth write $p|_{E_i}(F_j)$ simply as $p_{ij}$ for all $i \leq m$ and $j \leq n$. In a casual word, $p_{ij}$ is the *conditional probability charge* of $F_j$ *given* $E_i$.

Finally, given $p \in \mathscr{M}(\Omega, \mathcal{F}_2)$, we henceforth write $p(E_i \times F_j)$ simply as $p_{i,j}$ for all $i \leq m$ and $j \leq n$.

### 2.1.1 The Decomposition of A Probability Charge

So far, we have defined three real numbers: $p_i$, $p_{ij}$, and $p_{i,j}$, for each $p \in \mathscr{M}(\Omega, \mathcal{F}_2)$ and each $i$ and $j$. (Note that $p_{ij}$ and $p_{i,j}$ are totally different objects.)

By means of all these notations, an important result called the "decomposition of a probability charge in terms of its conditional and marginal" is stated as follows: Given any probability charge $p \in \mathscr{M}(\Omega, \mathcal{F}_2)$, $p$ can be written as

$$(\forall i, j) \quad p_{i,j} = p_i \cdot p_{ij} \tag{1}$$

as far as $p(E_i \times S) > 0$.

Conversely, given any list (vector) of first-period marginals of $p \in \mathscr{M}(\Omega, \mathcal{F}_2)$ (where each marginal is identified with an element of $\mathscr{M}(S, \langle E_i \rangle_i)$), $(p_i)_i := (p_1, p_2, \ldots, p_m)$, as well as any set of lists (vectors) of well-defined conditionals (where any element of the set is similarly identified with a list of elements of $\mathscr{M}(S, \langle F_j \rangle_j)$),

$$\{(p_{ij})_j\}_i := \{(p_{11}, p_{12}, \ldots, p_{1n}), (p_{21}, p_{22}, \ldots, p_{2n}), \ldots, (p_{m1}, p_{m2}, \ldots, p_{mn})\},$$

the right-hand side of Equation (1) "defines" a probability charge $p \in \mathscr{M}(\Omega, \mathcal{F}_2)$ with $i$ and $j$ varying.

This decomposition (that is, Equation (1)) will be used repeatedly in what follows.

## 2.2 Knightian Uncertainty and Rectangularity

A nonempty subset $\mathscr{P}$ of $\mathscr{M}(\Omega, \mathcal{F}_2)$ is called *Knightian uncertainty*.

Given any Knightian uncertainty, $\mathscr{P}$, its *first-period marginal Knightian uncertainty*, denoted by $\mathscr{P}|_1$, is the nonempty subset of $\mathscr{M}(S, \langle E_i \rangle_i)$ that is defined by

$$\mathscr{P}|_1 := \{\, p|_1 \,|\, p \in \mathscr{P} \,\} \ ,$$

where $p|_1$ is the first-period marginal probability charge of $p$ defined in the previous subsection and it is written in the conventional wording so that $p|_1 \in \mathscr{M}(S, \langle E_i \rangle_i)$.

Next, let $\mathscr{P}$ be Knightian uncertainty, suppose that $E \in \langle E_i \rangle_i$ was observed in the first period, and suppose that *every* probability charge in $\mathscr{P}$ is updated by Bayes' rule. As a result of this procedure, we obtain $\mathscr{P}|_E \subseteq \mathscr{M}(S, \langle F_j \rangle_j)$ which is defined by

$$\mathscr{P}|_E := \{\, p|_E(\cdot) \,|\, p \in \mathscr{P} \,\} \ ,$$

where $p|_E(\cdot)$ is the "posterior" probability charge defined in the previous subsection. Note that $\mathscr{P}|_E = \phi_{GB}(\mathscr{P}, E)$ by the notation of Nishimura and Ozaki (2017, Chapter 14), where "GB" abbreviates "generalized Bayes." The set $\mathscr{P}|_E$ may be thought of as the state of uncertainty in the second period after the observation $E$ was made in the first period.

Knightian uncertainty $\mathscr{P}$ is *rectangular* by definition if for any $p', p'' \in \mathscr{P}$, $(p'_i \cdot p''_{ij})_{i,j} \in \mathscr{P}$, where $p'$ is decomposed into $(\forall i,j)\ p'_{i,j} = p'_i \cdot p'_{ij}$, $p''$ is decomposed into $(\forall i,j)\ p''_{i,j} = p''_i \cdot p''_{ij}$ and $(p'_i \cdot p''_{ij})_{i,j}$ defines a probability charge on $\mathscr{M}(\Omega, \mathcal{F}_2)$ by Equation (1). The concept of rectangularity was introduced by Epstein and Schneider (2003).

A novelty of the rectangularity is that the next proposition holds whenever it is satisfied. To state the proposition precisely, for any real-valued $\mathcal{F}_2$-measurable function $u$ on $\Omega$, we denote by $E^p[u]$ its mathematical expectation with respect to $p \in \mathscr{M}(\Omega, \mathcal{F}_2)$.

In the next proposition, $u_i$ denotes arbitrary real-valued $\mathcal{F}_i$-measurable function on $\Omega$ $(i = 1, 2)$ and $E$ is an arbitrary element of the partition, $\langle E_i \rangle_i$. Note here that because a function $u_1$ and the second expectation in the right-hand side of (2) below are both $\mathcal{F}_1$-measurable, the first expectation in the right-hand side of (2) is well-defined with respect to the first-period marginal probability charge, $p'$.

Also, note that the expectation $E^{p''}[u_2]$ in the right-hand side of (2) is the "conditional expectation." That is, to be more precise, it is equal to $E^{p''}[u_2 \,|\langle E_i \times S \rangle_i](s_1, s_2)$ with $s_1 \in E$. The $\mathcal{F}_1$-measurability mentioned in the previous paragraph is with respect to the argument, $s_1$.[3]

[3] More precisely, it is with respect to $(s_1, s_2)$.

**Proposition 1** *Let $\mathscr{P}$ be a weak $^*$compact subset of $\mathscr{M}(\Omega, \mathcal{F}_2)$. Then, both $\mathscr{P}|_1$ and $\mathscr{P}|_E$ are also weak $^*$compact. Thus, all the minima in (2) exist. Furthermore, if $\mathscr{P}$ is rectangular, we have*

$$\min_{p \in \mathscr{P}} E^p [u_1 + u_2] = \min_{p' \in \mathscr{P}|_1} E^{p'} \left[ u_1 + \min_{p'' \in \mathscr{P}|_E} E^{p''} [u_2] \right] . \tag{2}$$

**Proof** First, because a set of probability charges with the common finite support can be identified as a subset of the finite-dimensional Euclidean space, the weak $^*$ compactness of $\mathscr{P}$ implies that so are $\mathscr{P}|_1$ and $\mathscr{P}|_E$.

Second, given any $p = (p_{i,j})_{i,j} \in \mathscr{P}$, note that (1) implies that $p$ can be written as $p = (p'_i \cdot p''_{ij})_{i,j}$, where $(p'_i)_i \in \mathscr{P}|_1$ is the list of the first-period marginals of $p$ and $(p''_{ij})_j \in \mathscr{P}|_E$ is the list of well-defined conditionals when $E = E_i$ is observed in the first period.

Third, by the law of iterated expectations, and by the remark made right before the statement of the proposition, we obtain

$$E^p [u_2] = E^{p'} \left[ E^{p''} [u_2] \right] = E^{p'} \left[ E^{p''} [u_2 \,| \langle E_i \times S \rangle_i ] \, (s_1, s_2) \right] ,$$

where the outer expectations of the middle and right terms aggregate with respect to $s_1$. Therefore, we obtain

$$E^p [u_1 + u_2] = E^{p'} \left[ u_1 + E^{p''} [u_2] \right] .$$

Fourth, we show that "$\geq$" holds in (2). The equality in the previous paragraph immediately implies that for any $p \in \mathscr{P}$,

$$E^p [u_1 + u_2] \geq \min_{p' \in \mathscr{P}|_1} E^{p'} \left[ u_1 + \min_{p'' \in \mathscr{P}|_E} E^{p''} [u_2] \right] ,$$

which proves the claim. We remark that we did not use the rectangularity of $\mathscr{P}$.

Fifth and Finally, we show "$\leq$" holds in (2). To this end, on the contrary, assume that $>$ holds there. By the compactness of the relevant sets, there exist $p'^* = (p'^*_i)_i \in \mathscr{P}|_1$ and $p''^* = (p''^*_{ij})_j \in \mathscr{P}|_E$ that attain the minima in the right-hand side of (2). *By the rectangularity of $\mathscr{P}$*, $p^* := (p^*_{i,j})_{i,j} := (p'^*_i \times p''^*_{ij})_{i,j}$ must be contained by $\mathscr{P}$. Thus,

$$\begin{aligned}
\min_{p \in \mathscr{P}} E^p [u_1 + u_2] &> \min_{p' \in \mathscr{P}|_1} E^{p'} \left[ u_1 + \min_{p'' \in \mathscr{P}|_E} E^{p''} [u_2] \right] \\
&= E^{p'^*} \left[ u_1 + E^{p''^*} [u_2] \right] \\
&= E^{p^*} [u_1 + u_2] \\
&\geq \min_{p \in \mathscr{P}} E^p [u_1 + u_2] ,
\end{aligned}$$

where we invoked the law of iterated expectations again. This is a contradiction we desire. □

## 2.3 The Initial-Period One-Shot $\varepsilon$-Contamination

This subsection defines the initial-period one-shot $\varepsilon$-contamination for the two-period model presented in the previous subsection. This is a straightforward extension of the static one-shot $\varepsilon$-contamination.

Formally, let $p^0$ be a probability charge on $(\Omega, \mathcal{F}_2)$ such that $(\forall i)\ p_i^0 > 0$, and let $\varepsilon \in (0,1)$. We may call this probability charge as a "principal" probability charge. We assume that the decision-maker's view about the world is represented by the initial-period one-shot *$\varepsilon$-contamination of* $p^0$, denoted by $\left\{p^0\right\}^\varepsilon$, which is defined by

$$\left\{p^0\right\}^\varepsilon := \left\{\ (1-\varepsilon)p^0 + \varepsilon q \,\middle|\, q \in \mathscr{M}(\Omega, \mathcal{F}_2)\ \right\}\ . \tag{3}$$

As suggested in the footnote of Introduction, this can also be called "open-loop" *$\varepsilon$-contamination*. For notational simplicity, we hereafter omit the adjective "initial-period one-shot" and simply call $\left\{p^0\right\}^\varepsilon$ as the *$\varepsilon$-contamination of* $p^0$ so long as it does not cause any confusion.

Clearly, $\left\{p^0\right\}^\varepsilon \subseteq \mathscr{M}(\Omega, \mathcal{F}_2)$. It is easy to observe that

$$\left\{p^0\right\}^\varepsilon\Big|_1 = \left\{p^0|_1\right\}^\varepsilon\ . \tag{4}$$

We change Definition (3) into a slightly more convenient equivalent form, which can be utilized to define rectangular $\varepsilon$-contamination in the next subsection.

Firstly, note that any element $p \in \mathscr{M}(\Omega, \mathcal{F}_2)$ can be alternatively expressed as an $(m \times n)$-dimensional vector, $p$, as

$$p = (p_{1,1}, p_{1,2}, \ldots, p_{1,n}; p_{2,1}, p_{2,2}, \ldots, p_{2,n}; \ldots; p_{m,1}, p_{m,2}, \ldots, p_{m,n})$$

which satisfies $p \in [0,1]^{m\times n}$ and $\sum_{i=1}^m \sum_{j=1}^n p_{i,j} = 1$. For simplicity, we write this as $p = \left(p_{i,j}\right)_{i,j}$ (while we already used this notations in some occasions). As is apparent in the above formulation, $p$ includes all possible "evolutions" of probability charges over two periods.

Secondly, by the use of this notation, Definition (3) will become

$$\left\{p^0\right\}^\varepsilon = \left\{ (1-\varepsilon)\left(p_{i,j}^0\right)_{i,j} + \varepsilon\left(q_{i,j}\right)_{i,j} \,\middle|\, \left(q_{i,j}\right)_{i,j} \in [0,1]^{m\times n} \right. \\ \left. \text{and } \textstyle\sum_{i,j} q_{i,j} = 1 \right\}. \tag{5}$$

Now, for any $q \in \mathscr{M}(\Omega, \mathcal{F}_2)$, define $(\forall i,j)\quad \delta_{i,j} := \varepsilon[-p_{i,j}^0 + q_{i,j}]$. then, it can be easily verified that the requirement that $(\forall i,j)\ q_{i,j} \in [0,1]$ and $\sum_{i,j} q_{i,j} = 1$ is equivalent to the requirement that $(\forall i,j)\ \delta_{i,j} \in [-\varepsilon p_{i,j}^0, \varepsilon(1 - p_{i,j}^0)]$ and $\sum_{i,j} \delta_{i,j} = 0$. Therefore, (5) is further rewritten as, since by (1), $p_{i,j}^0 = p_i^0 \cdot p_{ij}^0$,

$$\{p^0\}^{\varepsilon} = \left\{ \left(p_i^0 \cdot p_{ij}^0 + \delta_{i,j}\right)_{i,j} \,\middle|\, (\forall i,j)\ \delta_{i,j} \in [\underline{\delta}_{i,j}, \bar{\delta}_{i,j}] \text{ and } \textstyle\sum_{i,j} \delta_{i,j} = 0 \right\} \tag{6}$$

where

$$\underline{\delta}_{i,j} := -\varepsilon p_{i,j}^0 \quad \text{and} \quad \bar{\delta}_{i,j} := \varepsilon(1 - p_{i,j}^0)\,. \tag{7}$$

Thus, the initial-period one-shot (open-loop) $\varepsilon$-contamination is represented by (6) and (7). This suggests a neat formulation of rectangular $\varepsilon$-contamination, which will be defined formally in the next subsection.

### 2.4 The Rectangular $\boldsymbol{\varepsilon}$-Contamination

Let $p^0 \in \mathscr{M}(\Omega, \mathcal{F}_2)$ be such that $(\forall i)\ p_i^0 > 0$. Now, let $\boldsymbol{\varepsilon}$ be a lengthy real vector defined by $\boldsymbol{\varepsilon} := (\underline{\varepsilon}_i; \bar{\varepsilon}_i; \underline{\varepsilon}_{ij}; \bar{\varepsilon}_{ij})_{i,j}$, where we assume that $(\forall i,j)\ -p_i^0 \leq \underline{\varepsilon}_i \leq 0 \leq \bar{\varepsilon}_i \leq 1 - p_i^0$ and $-p_{ij}^0 \leq \underline{\varepsilon}_{ij} \leq 0 \leq \bar{\varepsilon}_{ij} \leq 1 - p_{ij}^0$.

Then, we use $\boldsymbol{\varepsilon}$ to define the *rectangular $\boldsymbol{\varepsilon}$-contamination*, $\{p^0\}^{rec\boldsymbol{\varepsilon}}$, by

$$\begin{aligned}\{p^0\}^{rec\boldsymbol{\varepsilon}} := \Big\{ \left((p_i^0 + \varepsilon_i)(p_{ij}^0 + \varepsilon_{ij})\right)_{i,j} \,\Big|\, & (\forall i)\ \varepsilon_i \in [\underline{\varepsilon}_i, \bar{\varepsilon}_i];\ \textstyle\sum_i \varepsilon_i = 0; \\ & (\forall i,j)\ \varepsilon_{ij} \in [\underline{\varepsilon}_{ij}, \bar{\varepsilon}_{ij}] \ \text{ and } \ (\forall i)\ \textstyle\sum_j \varepsilon_{ij} = 0 \Big\}\,. \end{aligned} \tag{8}$$

Note that the restrictions imposed on the range within which $\varepsilon_i$ and $\varepsilon_{ij}$ may move around are necessary for $(p_i^0 + \varepsilon_i)_i$ and $(\forall i)\ (p_{ij}^0 + \varepsilon_{ij})_j$ to be probability charges as well as for $p^0$ to be included in $\{p^0\}^{rec\boldsymbol{\varepsilon}}$. In contrast with the "open-loop" initial-period one-shot $\varepsilon$-contamination, the rectangular $\boldsymbol{\varepsilon}$-contamination allows the change in Knightian uncertainty after an observation in the first period because $\varepsilon$ may depend on $i$. Thus, it can be described as the "cloed-loop" $\boldsymbol{\varepsilon}$-contamination (compare (8) and (6)).

The rectangular $\boldsymbol{\varepsilon}$-contamination defined above is in fact rectangular as its name suggests, which is shown in the following proposition.

**Proposition 2** *The rectangular $\boldsymbol{\varepsilon}$-contamination defined by (8) is rectangular.*

**Proof** First, observe that, for any $p \in \{p^0\}^{rec\boldsymbol{\varepsilon}}$ and for any $i,j$, $p_i = p_i^0 + \varepsilon_i$ and $p_{ij} = p_{ij}^0 + \varepsilon_{ij}$ because $(\forall i)\ \sum_j \varepsilon_{ij} = 0$ by assumption.

Second, to complete the proof, let $p', p'' \in \{p^0\}^{rec\boldsymbol{\varepsilon}}$. Then, $p'$ can be written as $\Big((p_i^0 + \varepsilon_i')(p_{ij}^0 + \varepsilon_{ij}')\Big)_{i,j}$ for some $(\varepsilon_i')_i$ and $(\varepsilon_{ij}')_{ij}$ satisfying (8), and $p''$ can be so as $\Big((p_i^0 + \varepsilon_i'')(p_{ij}^0 + \varepsilon_{ij}'')\Big)_{i,j}$ for some $(\varepsilon_i'')_i$ and $(\varepsilon_{ij}'')_{ij}$ satisfying (8). Then, from the first paragraph, we conclude that

$$(p_i' \cdot p_{ij}'')_{i,j} = \left((p_i^0 + \varepsilon_i')(p_{ij}^0 + \varepsilon_{ij}'')\right)_{i,j}$$

holds. Because $(\varepsilon_i')_i$ and $(\varepsilon_{ij}'')_{ij}$ satisfy all the requirements in (8), $(p_i' \cdot p_{ij}'')_{i,j} \in \mathscr{P}$ and the proof is complete. $\square$

The main objective of this subsection is to scrutinize the relation between the (initial-period one-shot or "open-loop") $\varepsilon$-contamination introduced in the previous subsection and the ("closed-loop") rectangular $\boldsymbol{\varepsilon}$-contamination introduced right above.

To this end, we set the vector introduced at the start of this subsection, $\boldsymbol{\varepsilon} = (\underline{\varepsilon}_i; \bar{\varepsilon}_i; \underline{\varepsilon}_{ij}; \bar{\varepsilon}_{ij})_{i,j}$, as follows: $(\forall i, j)$

$$\underline{\varepsilon}_i := -\varepsilon p_i^0;\ \bar{\varepsilon}_i := \varepsilon(1-p_i^0);\ \underline{\varepsilon}_{ij} := \frac{-\varepsilon p_{ij}^0}{(1-\varepsilon)p_i^0+\varepsilon};\ \bar{\varepsilon}_{ij} := \frac{\varepsilon(1-p_{ij}^0)}{(1-\varepsilon)p_i^0+\varepsilon}\,, \quad (9)$$

where $\varepsilon$ is the one with which the "open-loop" $\varepsilon$-contamination is defined in (3), or in (6) and (7).

Note that the specification of $\boldsymbol{\varepsilon}$ by (9) satisfies all the inequalities we assumed in the first paragraph of this subsection. *We hereafter consider the recursive $\boldsymbol{\varepsilon}$-contamination characterized by this $\boldsymbol{\varepsilon}$, that is, by (9).*

**Proposition 3 (First-Period Marginals)** *The first-period marginal Knightian uncertainty of the rectangular $\boldsymbol{\varepsilon}$-contamination ((8) and (9)) and that of the $\varepsilon$-contamination ((6) and (7)) coincide. That is,*

$$\left\{p^0\right\}^{rec\varepsilon}\big|_1 = \left\{p^0\right\}^{\varepsilon}\big|_1 .$$

**Proof** First note that

$$\left\{p^0\right\}^{rec\varepsilon}\big|_1 = \left\{ \left(p_i^0 + \varepsilon_i\right)_i \,\middle|\, (\forall i)\, \varepsilon_i \in [\underline{\varepsilon}_i, \bar{\varepsilon}_i] \text{ and } \textstyle\sum_i \varepsilon_i = 0 \right\}.$$

Therefore, the result follows from (4) and (6) by noting that $(\forall i)$ $\sum_j \underline{\delta}_{i,j} = \underline{\varepsilon}_i$ and $\sum_j \bar{\delta}_{i,j} = \bar{\varepsilon}_i$. □

This proposition suggests that the "bounds" $\underline{\varepsilon}_i$ and $\bar{\varepsilon}_i$ that appear in the recursive $\boldsymbol{\varepsilon}$-contamination are "tight."

The next proposition shows that the recursive $\boldsymbol{\varepsilon}$-contamination is at least as large as the "open-loop" $\varepsilon$-contamination.

**Proposition 4** *It holds that $\left\{p^0\right\}^{\varepsilon} \subseteq \left\{p^0\right\}^{rec\varepsilon}$.*

**Proof** Let $p \in \left\{p^0\right\}^{\varepsilon}$ and write it as $p = (p_{i,j}^0 + \delta_{i,j})_{i,j}$ with some $(\delta_{i,j})_{i,j}$ that satisfies the requirements stated in (6).

By definition, it follows that $p|_1 \in \left\{p^0\right\}^{\varepsilon}\big|_1$. This, (6), and (7) mean that we can write $p|_1$ as $p|_1 = \left(p_i^0 + \delta_i'\right)_i$ with some $(\delta_i')_i$ such that $(\forall i)\ -\varepsilon p_i^0 \leq \delta_i' \leq \varepsilon(1-p_i^0)$ and $\sum_i \delta_i' = 0$. That is, let $(\forall i)$ $\delta_i' := \sum_j \delta_{i,j}$.

For each $i$, define $\varepsilon_i$ by $\varepsilon_i := \delta_i'$. Then, by definition and the previous paragraph, it holds that $\sum_i \varepsilon_i = 0$, that $\underline{\varepsilon}_i = -\varepsilon p_i^0 \leq \delta_i' = \varepsilon_i$ and that $\bar{\varepsilon}_i = \varepsilon(1-p_i^0) \geq \delta_i' = \varepsilon_i$, where $(\forall i)$ $\underline{\varepsilon}_i$ and $\bar{\varepsilon}_i$ are defined by (9). Thus, all the requirements for $\varepsilon_i$ in (8) are now met.

Next, for each $i, j$, define $\varepsilon_{ij}$ by $\varepsilon_{ij} := (\delta_{i,j} - \delta'_i p^0_{ij})/(p^0_i + \delta'_i)$. Then, $(\forall i)$ $\sum_j \varepsilon_{ij} = (\sum_j \delta_{i,j} - \delta'_i \sum_j p^0_{ij})/(p^0_i + \delta'_i) = (\delta'_i - \delta'_i)/(p^0_i + \delta'_i) = 0$, where we used the definition of $\delta'_i$ and the fact that $(\forall i)$ $p^0_{ij}$ is a (conditional) charge.

Let $\underline{\varepsilon}_{ij}$ and $\bar{\varepsilon}_{ij}$ be as defined in (9). First, we show that $(\forall i, j)$ $\varepsilon_{ij} \geq \underline{\varepsilon}_{ij}$. To this end, note that $\partial \varepsilon_{ij}/\partial \delta'_i = (-p^0_{i,j} - \delta_{i,j})/(p^0_i + \delta'_i)^2 < 0$, where the numerator must be negative because $-p^0_{i,j} - \delta_{i,j} \leq -p^0_{i,j} - (-\varepsilon p^0_{i,j}) = (\varepsilon - 1)p^0_{i,j} < 0$ since $\delta_{i,j} \geq -\varepsilon p^0_{i,j}$ and $\varepsilon < 1$. Therefore, $\varepsilon_{ij}$ attains its lower bound when $\delta'_i$ attains its upper bound. We thus obtain $(\forall i, j)$

$$
\begin{aligned}
\varepsilon_{ij} &\geq \frac{-\varepsilon p^0_{i,j} - \varepsilon(1 - p^0_i)p^0_{ij}}{p^0_i + \varepsilon(1 - p^0_i)} \\
&= \frac{-\varepsilon p^0_{i,j} - \varepsilon p^0_{ij} + \varepsilon p^0_{i,j}}{p^0_i + \varepsilon(1 - p^0_i)} \\
&= \frac{-\varepsilon p^0_{ij}}{(1-\varepsilon)p^0_i + \varepsilon} = \underline{\varepsilon}_{ij} \,.
\end{aligned}
$$

Second, we show that $(\forall i, j)$ $\varepsilon_{ij} \leq \bar{\varepsilon}_{ij}$, note that $\varepsilon_{ij} = (\delta_{i,j} - \sum_\ell \delta_{i,\ell}\, p^0_{ij})/(p^0_i + \sum_\ell \delta_{i,\ell})$, and hence that $\partial \varepsilon_{ij}/\partial \delta_{i,j} = \sum_{\ell \neq j}(p^0_{i,\ell} + \delta_{i,\ell})/(p^0_i + \sum_\ell \delta_{i,\ell})^2 > 0$.[4] Therefore, $\varepsilon_{ij}$ attains its maximum when $\delta_{i,j}$ is maximal, that is, when $\delta_{i,j} = \varepsilon(1 - p^0_{i,j})$. However, this occurs precisely only when $\delta_{i,\ell} = -\varepsilon p^0_{i,\ell}$ for $\ell \neq j$ because if otherwise, the two requirements that $\sum_{i,j} \delta_{i,j} = 0$ and that $(\forall i, j)$ $\delta_{i,j} \geq -\varepsilon p^0_{i,j}$ cannot be satisfied simultaneously. Therefore, at this time, it holds that $\sum_\ell \delta_{i,\ell} = \varepsilon(1 - p^0_i)$, and we obtain

$$
\begin{aligned}
\varepsilon_{ij} &\leq \frac{\varepsilon(1 - p^0_{i,j}) - \varepsilon(1 - p^0_i)p^0_{ij}}{p^0_i + \varepsilon(1 - p^0_i)} \\
&= \frac{\varepsilon(1 - p^0_{ij})}{(1-\varepsilon)p^0_i + \varepsilon} = \bar{\varepsilon}_{ij} \,.
\end{aligned}
$$

So far, we have found $(\varepsilon_i)_i$ and $(\varepsilon_{ij})_{i,j}$ that satisfies (8) and (9).

Finally, it suffices to verify that for these $(\varepsilon_i)_i$ and $(\varepsilon_{ij})_{i,j}$, it holds that $p^0_{i,j} + \delta_{i,j} = (p^0_i + \varepsilon_i)(p^0_{ij} + \varepsilon_{ij})$ for each $i$ and $j$. But, this is immediate from the definitions of $\varepsilon_i$ and $\varepsilon_{ij}$: $(\forall i, j)$

$$
\begin{aligned}
& (p^0_i + \varepsilon_i)(p^0_{ij} + \varepsilon_{ij}) \\
=\ & p^0_{i,j} + \varepsilon_i p^0_{ij} + (p^0_i + \varepsilon_i)\varepsilon_{ij} \\
=\ & p^0_{i,j} + \delta'_i p^0_{ij} + (p^0_i + \delta'_i)\frac{\delta_{i,j} - \delta'_i p^0_{ij}}{p^0_i + \delta'_i}
\end{aligned}
$$

[4]We may assume the strict positivity here because the numerator being zero will take place only when $p_{i,j} = 1$ for some $i$ and $j$, which implies, together with the assumption that $p \in \{p^0\}^\varepsilon$, that $p^0 = (0, \ldots, 0, 1, 0, \ldots, 0)$, meaning that $p^0$ represents no risk, which we excluded in the start of subsection 2.4.

$$
\begin{aligned}
&= p^0_{i,j} + \delta'_i p^0_{ij} + \delta_{i,j} - \delta'_i p^0_{ij} \\
&= p^0_{i,j} + \delta_{i,j}\,,
\end{aligned}
$$

which completes the proof. □

However, it does not hold that $\{p^0\}^\varepsilon \supseteq \{p^0\}^{rec\varepsilon}$ as the next example shows. Thus, this example shows that the initial-period one-shot ("open-loop") $\varepsilon$-contamination may not be rectangular so that it may not guarantee time consistency of intertemporal decisions.

**Example 2.1** Let $S := \{T, B\}$. Then, we have $\Omega = \{TT, TB, BT, BB\}$. Let $p^0$ be a probability charge on $(\Omega, 2^\Omega)$ defined by $p^0(\{TT\}) = p^0(\{TB\}) = p^0(\{BT\}) = p^0(\{BB\}) = 1/4$ and we consider $\{p^0\}^\varepsilon$ for an arbitrary $\varepsilon \in (0,1)$. We may write $p^0(\{TT\})$ as $p^0_{T,T}$, and so on.

We see that $p = \left(\frac{1}{4}+\frac{\varepsilon}{4}, \frac{1}{4}+\frac{\varepsilon}{4}, \frac{1}{4}-\frac{\varepsilon}{4}, \frac{1}{4}-\frac{\varepsilon}{4}\right) \in \{p^0\}^\varepsilon$ (let $q := \left(\frac{1}{2}, \frac{1}{2}, 0, 0\right)$), and that $p' = \left(\frac{1}{4}+\frac{\varepsilon}{4}, \frac{1}{4}-\frac{\varepsilon}{4}, \frac{1}{4}+\frac{\varepsilon}{4}, \frac{1}{4}-\frac{\varepsilon}{4}\right) \in \{p^0\}^\varepsilon$ (let $q := \left(\frac{1}{2}, 0, \frac{1}{2}, 0\right)$).

From $p$, we can compute the first-period marginal of $p$, $p|_1$, as $p|_1 = (p_T, p_B) = \left(\frac{1}{2}+\frac{\varepsilon}{2}, \frac{1}{2}-\frac{\varepsilon}{2}\right)$, and the conditionals of $p$, $p_{TT}$ and so on, as $p_{TT} = p_{TB} = p_{BT} = p_{BB} = 1/2$. Similarly, from $p'$, we can compute $p'|_1 = (p'_T, p'_B) = \left(\frac{1}{2}, \frac{1}{2}\right)$, $p'_{TT} = \frac{1}{2}+\frac{\varepsilon}{2}$, $p'_{TB} = \frac{1}{2}-\frac{\varepsilon}{2}$, $p'_{BT} = \frac{1}{2}+\frac{\varepsilon}{2}$ and $p'_{BB} = \frac{1}{2}-\frac{\varepsilon}{2}$.

These computations show that $(p_T p'_{TT}, p_T p'_{TB}, p_B p'_{BT}, p_B p'_{BB}) =$

$$
\left(\frac{1}{4}+\frac{\varepsilon}{2}+\frac{\varepsilon^2}{4}, \frac{1}{4}-\frac{\varepsilon^2}{4}, \frac{1}{4}-\frac{\varepsilon^2}{4}, \frac{1}{4}-\frac{\varepsilon}{2}+\frac{\varepsilon^2}{4}\right)
$$

would be an element of $\{p^0\}^\varepsilon$, if it were rectangular. But it is not, in fact, because $p_B p'_{BB} = \frac{1}{4}-\frac{\varepsilon}{2}+\frac{\varepsilon^2}{4} < \frac{1}{4}-\frac{\varepsilon}{4} = (1-\varepsilon)p^0_{B,B}$, which is the minimum value $p_{B,B}$ can take on as long as $p$ belongs to $\{p^0\}^\varepsilon$. □

Let $\mathscr{P} \subseteq \mathscr{M}(\Omega, \mathcal{F}_2)$ be any Knightian uncertainty. Then, consider *rectangular* Knightian uncertainty containing $\mathscr{P}$ that is *minimal*. We call it the *rectangular-hull* of $\mathscr{P}$, if any, and denote it by $\mathrm{rect}\mathscr{P}$. That is, if $\mathscr{P}'$ is rectangular and containing $\mathscr{P}$, then $\mathrm{rect}\mathscr{P} \subseteq \mathscr{P}'$.

Our next result is concerned with the rectangular-hull of the $\varepsilon$-contamination.

**Proposition 5 (Rectangular-Hull)** *For any $p^0 \in \mathscr{M}(\Omega, \mathcal{F}_2)$ and any $\varepsilon \in (0,1)$, it holds that* $\mathrm{rect}\left(\{p^0\}^\varepsilon\right) = \{p^0\}^{rec\varepsilon}$.

**Proof** Because $\{p^0\}^{rec\varepsilon}$ is rectangular (Proposition 2), it suffices to prove that any rectangular set including $\{p^0\}^\varepsilon$ contains $\{p^0\}^{rec\varepsilon}$.

Let $p^0 \in \mathscr{M}(\Omega, \mathcal{F}_2)$, let $\varepsilon \in (0,1)$, and let $i$ and $j$ be arbitrarily fixed below.

First, note that there exists $p' \in \{p^0\}^\varepsilon$ such that $p'_i = (1-\varepsilon)p^0_i + \varepsilon$, which is the maximum value the first-period marginal, $p_i$, can assume subject to $p \in \{p^0\}^\varepsilon$. To do this, we can let $(q_{i,j})_{i,j}$ be such that $\sum_\ell q_{i,\ell} = 1$.

Second, note that there exists $p'' \in \{p^0\}^\varepsilon$ such that

$$p''_{ij} = \frac{(1-\varepsilon)p^0_{i,j} + \varepsilon}{(1-\varepsilon)p^0_i + \varepsilon} \,,$$

which is the maximum value the conditional, $p_{ij}$, can assume subject to $p \in \{p^0\}^\varepsilon$. To do this, we can let $(q_{i,j})_{i,j}$ be such that $q_{i,j} = 1$.

The above two paragraphs show that $p'_i \cdot p''_{ij} = (1-\varepsilon)p^0_{i,j} + \varepsilon$ is a value $p_{i,j}$ can assume as long as $p$ is an element of any rectangular set that contains $\{p^0\}^\varepsilon$. Here, some computations exhibit

$$\begin{aligned} p'_i \cdot p''_{ij} &= (1-\varepsilon)p^0_{i,j} + \varepsilon = \left(p^0_i + \varepsilon(1-p^0_i)\right)\left(p^0_{ij} + \frac{\varepsilon(1-p^0_{ij})}{(1-\varepsilon)p^0_i + \varepsilon}\right) \\ &= (p^0_i + \bar{\varepsilon}_i)(p^0_{ij} + \bar{\varepsilon}_{ij}) \,, \end{aligned}$$

where $\bar{\varepsilon}_i$ and $\bar{\varepsilon}_{ij}$ are defined by (9).

Similarly, note that there exists $p''' \in \{p^0\}^\varepsilon$ such that $p'''_i = (1-\varepsilon)p^0_i$, which is the minimum value the first-period marginal, $p_i$, can assume subject to $p \in \{p^0\}^\varepsilon$. (Let $(q_{i,j})_{i,j}$ be such that $(\forall \ell)\, q_{i,\ell} = 0$.) Also note that there exists $p'''' \in \{p^0\}^\varepsilon$ such that

$$p''''_{ij} = \frac{(1-\varepsilon)p^0_{i,j}}{(1-\varepsilon)p^0_i + \varepsilon} \,,$$

which is the minimum value the conditional, $p_{ij}$, can assume subject to $p \in \{p^0\}^\varepsilon$. (Let $(q_{i,j})_{i,j}$ be such that $\sum_{\ell \neq j} q_{i,\ell} = 1$.)

Therefore, by a similar reasoning as above, $p'''_i \cdot p''''_{ij}$ is a value $p_{i,j}$ can assume as long as $p$ is an element of any rectangular set that contains $\{p^0\}^\varepsilon$. Here, some computations exhibit

$$\begin{aligned} p'''_i \cdot p''''_{ij} &= (1-\varepsilon)p^0_i \cdot \frac{(1-\varepsilon)p^0_{i,j}}{(1-\varepsilon)p^0_i + \varepsilon} = (1-\varepsilon)p^0_i \cdot \left(p^0_{ij} - \frac{\varepsilon p^0_{ij}}{(1-\varepsilon)p^0_i + \varepsilon}\right) \\ &= (p^0_i + \underline{\varepsilon}_i)(p^0_{ij} + \underline{\varepsilon}_{ij}) \,, \end{aligned}$$

where $\underline{\varepsilon}_i$ and $\underline{\varepsilon}_{ij}$ are defined by (9).

By the first paragraph, we know that $\mathrm{rect}\left(\{p^0\}^\varepsilon\right) \subseteq \{p^0\}^{rec\varepsilon}$. Furthermore, the arguments so far show that we can always find a probability charge in any rectangular set containing $\{p^0\}^\varepsilon$ that achieves the "upper rim" of $\{p^0\}^{rec\varepsilon}$ for arbitrary $i$ and $j$ and (possibly) another probability charge in such a set that achieves the "lower rim" of $\{p^0\}^{rec\varepsilon}$ for arbitrary $i$ and $j$. This fact proves that both sets are identical. $\square$

We close this subsection by presenting another intuitive and convenient expression of the rectangular $\boldsymbol{\varepsilon}$-contamination, which we call the $\varepsilon$-$\boldsymbol{\varepsilon}'$ contamination. That is, the rectangular $\boldsymbol{\varepsilon}$-contamination can be characterized as the $\varepsilon$-contamination of the "principal" probability charge before making observation and its $\boldsymbol{\varepsilon'}$-contamination after making observation, where $\boldsymbol{\varepsilon'}$ has a simple formula that depends on $\varepsilon$.

Let $p \in \mathscr{M}(\Omega, \mathcal{F}_2)$, let $\varepsilon \in (0,1)$, let the "bounds" $(\underline{\varepsilon}_i; \bar{\varepsilon}_i; \underline{\varepsilon}_{ij}; \bar{\varepsilon}_{ij})_{i,j}$ be related with $\varepsilon$ by (9), and let the rectangular $\boldsymbol{\varepsilon}$-contamination is defined by (8). Then, we have

**Proposition 6 ($\varepsilon$-$\boldsymbol{\varepsilon'}$ Contamination)** *It holds that*

$$\{p^0\}^{rec\varepsilon} = \left\{ \left(\left((1-\varepsilon)p_i^0 + \varepsilon q_i\right) \cdot \left((1-\varepsilon_i')p_{ij}^0 + \varepsilon_i' q_{ij}\right)\right)_{i,j} \,\middle|\, (\forall i)\, q_i \in [0,1]; \right.$$
$$\left. \textstyle\sum_i q_i = 1;\ (\forall i,j)\, q_{ij} \in [0,1] \text{ and } (\forall i)\ \sum_j q_{ij} = 1 \right\}, \quad (10)$$

*where for each $i$, $\varepsilon_i'$ is defined by*

$$\varepsilon_i' := \frac{\varepsilon}{(1-\varepsilon)p_i^0 + \varepsilon}\,.$$

*Furthermore, $(\forall i)\ \varepsilon_i' > \varepsilon$.*

**Proof** First, note that we have $(\forall i)$ $(1-\varepsilon)p_i^0 + \varepsilon q_i = p_i^0 + \varepsilon(q_i - p_i^0)$. Then, it is immediate that $\varepsilon_i := \varepsilon(q_i - p_i^0)$ satisfies all the requirements in (8) and (9) by (10).

Second, note that $(\forall i,j)$ $(1-\varepsilon_i')p_{ij}^0 + \varepsilon_i' q_{ij} = p_{ij}^0 + \varepsilon_i'(q_{ij} - p_{ij}^0)$. Then, it is immediate that $\varepsilon_{ij} := \varepsilon_i'(q_{ij} - p_{ij}^0)$ satisfies all the requirements in (8) and (9) by (10) and the definition of $\varepsilon_i'$.

Finally, the last claim follows from the fact that $(1-\varepsilon)p_i^0 + \varepsilon < 1$ for each $i$, which holds because $(\forall i)$ $p_i^0 < 1$ since we assume that $(\forall i)$ $p_i^0 > 0$ (see the start of Subsection 2.4). □

Proposition 6 (in particular, its claim that $(\forall i)\, \varepsilon_i' > \varepsilon$) suggests that ambiguity dilates upon learning, which is a remarkable property of ambiguity. In light of (2) of Proposition 1, this is almost saying that "$\mathscr{P}|_1 \subsetneq \mathscr{P}|_E$."[5]

Note that Equations (10) and (2) clearly show that time-consistency (rectangularity) and dilation of ambiguity upon learning are completely consistent.

## 2.5 Learning with Rectangular $\boldsymbol{\varepsilon}$-Contamination

This subsection studies learning procedure when the uncertainty is characterized by the *rectangular* $\boldsymbol{\varepsilon}$-contamination. The learning when the uncertainty is characterized by the "open-loop" $\varepsilon$-contamination was extensively studied by Nishimura and Ozaki (2017, Chapter 14).

[5] This is not exactly right though, because we need to assume the condition concerning "value of information." For the complete details, see Nishimura and Ozaki (2017, Chapter 14).

Now, let $p^0 \in \mathscr{M}(\Omega, \mathcal{F}_2)$, let $\varepsilon \in (0,1)$, and let $\{p^0\}^{rec\varepsilon}$ be the rectangular $\boldsymbol{\varepsilon}$-contamination defined by (8). Also, let $E_i \in \langle E_k \rangle_k$ for some $i$. Then, the definition of the second-period marginal uncertainty and Proposition 6 imply

$$\{p^0\}^{rec\varepsilon}\big|_{E_i} = \left\{ \left((1-\varepsilon_i')p^0_{ij} + \varepsilon_i' q_{ij}\right)_j \,\middle|\, (\forall j)\, q_{ij} \in [0,1] \text{ and } \textstyle\sum_j q_{ij} = 1 \right\}, \tag{11}$$

where $\varepsilon_i'$ is as define in Proposition 6.

The next proposition is one of the main results of this paper. This proposition shows that "posterior" Knightian uncertainty of the rectangular $\boldsymbol{\varepsilon}$-contamination *after observation* is the same as the "posterior" of the (initial-period one-shot) $\varepsilon$-contamination.

**Proposition 7 ("Posteriors")** *For any $i$ and for any $E_i \in \langle E_k \rangle_k$,*

$$\{p^0\}^{rec\varepsilon}\big|_{E_i} = \{p^0\}^{\varepsilon}\big|_{E_i} .$$

**Proof** Note that it holds that

$$\{p^0\}^{\varepsilon}\big|_{E_i} = \{p^0|_{E_i}(\cdot)\}^{\varepsilon_i'} = \left\{\left(p^0_{ij}\right)_j\right\}^{\varepsilon_i'}\big|_{E_i} = \{p^0\}^{rec\varepsilon}\big|_{E_i} .$$

Here, the first equality follows from Theorem 14.5.1 of Nishimura and Ozaki (2017, Chapter 14)[6]; the second equality is simple manipulation; and the third equality holds by (11) and the definition of $\varepsilon_i'$. Thus the proof is complete. $\square$

# 3 Arbitrarily-Finite-Horizon Models

The purpose of this section is to show the results obtained for the two-period models in the previous section hold true in arbitrarily-finite-horizon models, and thus they are general results.

## 3.1 (Slightly Heavy) Notations

Let $T \in \mathbb{N}\backslash\{0,1\}$ be a length of a finite horizon, and for each $t \in \{1,2,\dots,T\}$, $n_t\,(\geq 2)$ be a number of elements of each finite partition of $S$; that is, let $\langle E_{1,i_1}\rangle_{i_1=1}^{n_1}$, $\langle E_{2,i_2}\rangle_{i_2=1}^{n_2}$, $\dots$ , $\langle E_{T,i_T}\rangle_{i_T=1}^{n_T}$ be finite partitions of $S$, which are fixed throughout the rest of the paper. We identify each of $(S, \langle E_{1,i_1}\rangle_{i_1=1}^{n_1})$, $(S^2, \langle E_{1,i_1} \times E_{2,i_2}\rangle_{i_1=1}^{n_1}{}_{i_2=1}^{n_2})$, $\dots$ , $(S^T, \langle E_{1,i_1} \times E_{2,i_2} \times \cdots \times E_{T,i_T}\rangle_{i_1=1}^{n_1}{}_{i_2=1}^{n_2} \cdots {}_{i_T=1}^{n_T})$ with each of the measurable spaces, $(\Omega, \mathcal{F}_1)$, $(\Omega, \mathcal{F}_2)$, $\dots$ , $(\Omega, \mathcal{F}_T)$, exactly as we did in the previous section. (Note that $\Omega := S^T$.)

Let $\mathscr{M}(\Omega, \mathcal{F}_T)$ be the space of all probability charges on $(\Omega, \mathcal{F}_T)$. Given $p \in \mathscr{M}(\Omega, \mathcal{F}_T)$, we write the *joint probability charge*, $p(E_{1,i_1} \times E_{2,i_2} \times \cdots \times$

[6] Note that $\varepsilon'$ there is equal to $\varepsilon_i'$ here.

$E_{T,i_T})$, simply as $p_{i_1,i_2,\ldots,i_T}$, where for each $t \le T$, $E_{t,i_t} \in \langle E_{t,i_t}\rangle_{i_t=1}^{n_t}$. For each $p \in \mathscr{M}(\Omega, \mathcal{F}_T)$, the *first-t-period marginal* of $p$ is denoted and defined by[7]

$$(\forall t)(\forall i_1, i_2, \ldots, i_t)\;\; p_{i_1\, i_2\, \ldots\, i_t} := p(E_{1,i_1} \times E_{2,i_2} \times \cdots \times E_{t,i_t} \times S \times \cdots \times S)\,.$$

Obviously, the first-$T$-period marginal is identical to the joint probability charge.

Next, the *one-period-ahead conditional* of $p$ *given* $E_{1,i_1} \times E_{2,i_2} \times \cdots \times E_{t,i_t}$ is denoted and defined by

$$(\forall t \le T-1)(\forall i_1, i_2, \ldots, i_t)\;\; p^+_{i_{t+1}|i_1\, i_2\, \ldots\, i_t} := \frac{p_{i_1\, i_2\, \ldots\, i_t\, i_{t+1}}}{p_{i_1\, i_2\, \ldots\, i_t}}\,,$$

where the probability charges in the numerator and denominator are the first-some-appropriate-period marginals defined above.[8]

For any $p \in \mathscr{M}(\Omega, \mathcal{F}_T)$, we denote its first-$t$-period marginal *charge* in $\mathscr{M}(\Omega, \mathcal{F}_t)$ (*not* a single number) by $p_t$,[9] as well as its one-period-ahead conditional *charge* in $\mathscr{M}(S, \langle E_{t+1,i_{t+1}}\rangle_{i_{t+1}})$ (*not* a single number) given $E_{1,i_1} \times E_{2,i_2} \times \cdots \times E_{t,i_t}$ by $p^+(\cdot | E_{1,i_1} \times E_{2,i_2} \times \cdots \times E_{t,i_t})$.

### 3.1.1 The Decomposition of A Probability Charge

For each $p \in \mathscr{M}(\Omega, \mathcal{F}_T)$, its decomposition into its marginal and (one-period-ahead) conditional is now represented as follows:

$$(\forall t \le T-1) \quad p_{i_1\, i_2\, \ldots\, i_t\, i_{t+1}} = p_{i_1\, i_2\, \ldots\, i_t} \cdot p^+_{i_{t+1}|i_1\, i_2\, \ldots\, i_t}\,, \tag{12}$$

where the left-hand side in (12) is $p$'s fisrt-$(t+1)$-period marginal, while its right-hand side is the product of $p$'s fisrt-$t$-period marginal and its one-period-ahead marginal given the first $t$ observations.

These decompositions (that is, Equations (12)) will be used repeatedly in what follows.

## 3.2 Knightian Uncertainty and Rectangularity

A nonempty subset $\mathscr{P}$ of $\mathscr{M}(\Omega, \mathcal{F}_T)$ is called *Knightian uncertainty*.

Given any Knightian uncertainty, $\mathscr{P}$, and any $t \le T-1$, its *first-t-period marginal Knightian uncertainty*, denoted by $\mathscr{P}_t$, is the nonempty subset of $\mathscr{M}(\Omega, \mathcal{F}_t)$ that is defined by

$$\mathscr{P}_t := \{\, p_t \,|\, p \in \mathscr{P}\,\}\,,$$

[7]When $T = 2$, $p_{i_1} = p|_1(E_{1,i_1})$, where the right-hand side was introduced in the previous section.

[8]When $T = 2$, $p^+$ is well-defined only when $t = 1$ and $p^+_{i_2|i_1} = p_{i_1 i_2}$, where the right-hand side was introduced in the previous section.

[9]Note that it is only when $t = 1$ that $p|_t = p_t$ holds, where the former appeared in the previous section.

where $p_t$ is the first-$t$-period marginal probability charge defined in the previous subsection.[10]

Again, let $\mathscr{P}$ be Knightian uncertainty, $t \leq T-1$ and suppose that $E_1 \times \cdots \times E_t \in \langle E_{1,i_1} \times \cdots \times E_{t,i_t} \rangle_{i_1,\dots,i_t}$ has been observed in the first $t$ periods, and that *every* probability charge in $\mathscr{P}_{t+1}$ is updated by Bayes' rule. As a result of this procedure, we obtain the *one-period-ahead conditional* Knightian uncertanty, denoted $\mathscr{P}^+|_{E_1 \times \cdots \times E_t}$, which is a subset of $\mathscr{M}(S, \langle E_{t+1,i_{t+1}} \rangle_{i_{t+1}})$. That is,

$$\mathscr{P}^+|_{E_1 \times \cdots \times E_t} := \left\{ p^+(\cdot | E_1 \times \cdots \times E_t) \,|\, p \in \mathscr{P} \right\} ,$$

where $p^+$ is the one-period-ahead probability charge defined in the previous subsection.

Knightian uncertainty, $\mathscr{P}$, is *rectangular* by definition if for any $p', p'' \in \mathscr{P}$, it holds that

$$(\forall t \leq T-1) \qquad \left( p'_{i_1\, i_2 \dots i_t} \cdot p''^{+}_{i_{t+1} | i_1\, i_2 \dots i_t} \right)_{i_1, i_2, \dots, i_t, i_{t+1}} \in \mathscr{P}_{t+1} \, .$$

Note that if Knighian uncertainty is a singleton (that is, if it is a risk), then it is clearly rectangular in view of (12).

In order to state an important result, let $t \leq T$ and denote by $u_t$ an arbitrary real-valued function on $\Omega$ that is $\mathcal{F}_t$-measurable. As we did in the previous section, we denote by $E^p[u]$ the mathematical expectation of such a function with respect to a probability charge $p \in \mathscr{M}(\Omega, \mathcal{F}_T)$.

The next proposition states that rectangular Knightian uncertainty implies that *the iterated or sequential maxmin preference, which is a dynamic extension of the atemporal preference à la Gilboa and Schmeidler (1989) is identified with the one-shot "open-loop" maxmin preference*.

**Proposition 8 (Epstein-Schneider, 2003)** *Let $\mathscr{P}$ be rectangular Knightian uncertainty that is weak $^*$compact. Then,*

$$\begin{aligned}
& \min_{p \in \mathscr{P}} E^p \left[ \textstyle\sum_{i=1}^T u_i \right] \\
= \; & \min_{p' \in \mathscr{P}_1} E^{p'} \left[ u_1 + \min_{p'' \in \mathscr{P}^+|_{E_1}} E^{p''} \left[ \textstyle\sum_{i=2}^T u_i \right] \right] \\
= \; & \cdots\cdots\cdots \\
= \; & \min_{p' \in \mathscr{P}_1} E^{p'} \left[ u_1 + \min_{p'' \in \mathscr{P}^+|_{E_1}} E^{p''} \left[ u_2 + \min_{p^{(3)} \in \mathscr{P}^+|_{E_1 \times E_2}} E^{p^{(3)}} \left[ u_3 + \cdots \right.\right.\right. \\
& \left.\left.\left. \min_{p^{(T)} \in \mathscr{P}^+|_{E_1 \times \cdots \times E_{T-1}}} E^{p^{(T)}} [u_T] \cdots \right]\right]\right] ,
\end{aligned}$$

*where $p^{(t)}$ is an abbreviation of $p^{''\cdots'}$ (t primes), which is a generic probability charge which is relevant there.*

[10] Note that it is only when $t=1$ that $\mathscr{P}|_t = \mathscr{P}_t$ holds. See the previous footnote.

**Proof** We only prove the equation:

$$\begin{aligned}
&\min_{p\in\mathscr{P}} E^p\left[\textstyle\sum_{i=1}^T u_i\right]\\
= &\min_{p'\in\mathscr{P}_1} E^{p'}\Big[u_1+\min_{p''\in\mathscr{P}^+|_{E_1}} E^{p''}\Big[u_2+\min_{p^{(3)}\in\mathscr{P}^+|_{E_1\times E_2}} E^{p^{(3)}}\Big[u_3+\cdots\\
&\min_{p^{(T)}\in\mathscr{P}^+|_{E_1\times\cdots\times E_{T-1}}} E^{p^{(T)}}[u_T]\cdots\Big]\Big]\Big].
\end{aligned}$$

The other ones can be proved very similarly. Also, we only prove that "$\leq$" holds because the other direction of the inequality can be proved almost the same way as Proposition 1 without invoking the rectangularity.

To this end, assume that $>$ holds there on the contrary. By the compactness of the relevant Knightian Uncertainty, which is guaranteed by the assumed weak $^*$compactness of $\mathscr{P}$, there exists a sequence of probability charges such that $p'^* \in \mathscr{P}_1, p''^* \in \mathscr{P}^+|_{E_1}, \ldots, p^{(T)*} \in \mathscr{P}^+|_{E_1\times\cdots\times E_{T-1}}$, each of which attains the corresponding minimum.

Here, note that by the decomposition of a risk, for any sequence of observations, $E_1, \ldots, E_T$, it holds that

$$\begin{aligned}
p^*_{i_1,i_2,\ldots,i_T} = p^*_{i_1 i_2\ldots i_T} &:= p^{(T-1)*}_{i_1 i_2\ldots i_{T-1}}\cdot p^{(T)*}_{i_T|i_1 i_2\ldots i_{T-1}}\\
&= p^{(T-2)*}_{i_1 i_2\ldots i_{T-2}}\cdot p^{(T-1)*}_{i_{T-1}|i_1 i_2\ldots i_{T-2}}\cdot p^{(T)*}_{i_T|i_1 i_2\ldots i_{T-1}}\\
&= \ldots\ \ldots\ldots\ \ldots\\
&= p'^*_{i_1}\cdot p''^*_{i_2|i_1}\cdots p^{(T)*}_{i_T|i_1 i_2\ldots i_{T-1}}
\end{aligned}$$

(see the Equation (12)).

By exactly the same logic as Proposition 1, we reached the contradiction because $\left(p^*_{i_1,i_2,\ldots,i_T}\right)_{i_1,i_2,\ldots,i_T} \in \mathscr{P}$ *by the rectangularity.* $\square$

### 3.3 The Rectangular $\boldsymbol{\varepsilon}$-Contamination

Let $p^0 \in \mathscr{M}(\Omega, \mathcal{F}_T)$ and let $\varepsilon \in (0,1)$. Then, the *(initial-period one-shot) $\varepsilon$-contamination* of $p^0$ is denoted and defined by

$$\left\{p^0\right\}^{\varepsilon} := \left\{ \, (1-\varepsilon)p^0 + \varepsilon q \,\middle|\, q \in \mathscr{M}(\Omega, \mathcal{F}_T) \, \right\} . \tag{13}$$

Use the same $p^0$ and $\varepsilon$ to define $\underline{\varepsilon}_{i_1} := -\varepsilon p^0_{i_1}$, $\bar{\varepsilon}_{i_1} := \varepsilon(1-p^0_{i_1})$, and $(\forall t \in \{2, \ldots, T\})$

$$\underline{\varepsilon}_{i_1 \ldots i_t} := \frac{-\varepsilon p^{0+}_{i_t|i_1\ldots i_{t-1}}}{(1-\varepsilon)p^0_{i_1\ldots i_{t-1}} + \varepsilon} \quad \text{and} \quad \bar{\varepsilon}_{i_1\ldots i_t} := \frac{\varepsilon(1 - p^{0+}_{i_t|i_1\ldots i_{t-1}})}{(1-\varepsilon)p^0_{i_1\ldots i_{t-1}} + \varepsilon}, \tag{14}$$

where $p^{0+}$ is the one-period-ahead conditional of $p^0$ defined in Section 3.1. Then, the *rectangular $\boldsymbol{\varepsilon}$-contamination* of $p^0$ is defined by[11]

$$\begin{aligned}\left\{p^0\right\}^{rec\boldsymbol{\varepsilon}} := \Bigg\{ \Big( (p^0_{i_1} + \varepsilon_{i_1})(p^{0+}_{i_2|i_1} + \varepsilon_{i_1 i_2}) \cdots (p^{0+}_{i_T|i_1\ldots i_{T-1}} + \varepsilon_{i_1\ldots i_T}) \Big)_{i_1,\ldots,i_T} \,\Bigg| \\ (\forall i_1)\ \varepsilon_{i_1} \in [\underline{\varepsilon}_{i_1}, \bar{\varepsilon}_{i_1}];\ \textstyle\sum_{i_1} \varepsilon_{i_1} = 0;\ \ldots\ ; \\ (\forall i_1, \ldots, i_T)\ \varepsilon_{i_1\ldots i_T} \in [\underline{\varepsilon}_{i_1\ldots i_T}, \bar{\varepsilon}_{i_1\ldots i_T}] \\ \text{and} \ \ (\forall i_1, \ldots, i_{T-1}) \textstyle\sum_{i_T} \varepsilon_{i_1\ldots i_{T-1} i_T} = 0 \Bigg\} . \end{aligned} \tag{15}$$

**Proposition 9** *The rectangular $\boldsymbol{\varepsilon}$-contamination is rectangular.*

**Proof** The proof can be conducted very closely following the proof for the case where $T = 2$ (Proposition 2). Therefore, it is omitted. □

**Proposition 10** *It holds that $\left\{p^0\right\}^{\varepsilon} \subseteq \left\{p^0\right\}^{rec\boldsymbol{\varepsilon}}$.*

**Proof** When $T = 2$, the claim holds true (Proposition 4).

Now, let $\left\{p^0\right\}^{\varepsilon}$ be the initial-period one-shot $\varepsilon$-contamination with $T = 3$ and use the same $p^0$ and $\varepsilon$ to define $p_2 := \Big( (p^0_{i_1} + \varepsilon_{i_1})(p^{0+}_{i_2|i_1} + \varepsilon_{i_1 i_2}) \Big)_{i_1 i_2} \in \mathscr{M}(S^2, \langle E_{1,i_1} \times E_{2,i_2} \rangle_{i_1,i_2})$. By the way of the construction of $\left\{p^0\right\}^{rec\boldsymbol{\varepsilon}}$ with $T = 3$ from $p_2$, the claim for $T = 3$ can be proved by very closely following the proof for the case where $T = 2$ by letting $\varepsilon_{i_1 i_2 i_3} := (\delta_{i_1,i_2,i_3} - \delta_{i_1,i_2} p^{0+}_{i_3|i_1 i_2})/(p^0_{i_1 i_2} + \delta_{i_1,i_2})$, where $\delta_{i_1,i_2} := \sum_{i_3} \delta_{i_1,i_2,i_3}$ and $p^0_{i_1 i_2} := \sum_{i_3} p^0_{i_1 i_2 i_3}$ for some $\delta_{i_1,i_2,i_3}$. Thus, we omit the details of the proof.

For $T \geq 4$, repeat the procedure briefly described in the previous paragraph. □

[11] We use a boldface letter to express the epsilon because it is a vector, not a single number.

**Proposition 11** *It holds that*

$$\{p^0\}^{rec\varepsilon} = \Big\{ \Big( \big((1-\varepsilon^1)p^0_{i_1} + \varepsilon^1 q_{i_1}\big) \big((1-\varepsilon^2_{i_1})p^{0+}_{i_2|i_1} + \varepsilon^2_{i_1} q_{i_1 i_2}\big) \cdots \\ \big((1-\varepsilon^T_{i_1\ldots i_{T-1}})p^{0+}_{i_T|i_1\ldots i_{T-1}} + \varepsilon^T_{i_1\ldots i_{T-1}} q_{i_1 i_2\ldots i_T}\big)\Big)_{i_1,i_2,\ldots,i_T} \Big| \\ (\forall i_1)\ q_{i_1} \in [0,1];\ \textstyle\sum_{i_1} q_{i_1} = 1; \ldots;\ (\forall i_1,\ldots,i_T)\ q_{i_1\ldots i_T} \in [0,1] \\ \text{and}\ \ (\forall i_1,\ldots,i_{T-1})\ \textstyle\sum_{i_T} q_{i_1\ldots i_{T-1} i_T} = 1 \Big\}, \quad (16)$$

*where $\varepsilon^1 := \varepsilon$ that is the one defining the initial-period one-shot $\varepsilon$-contamination, and for each $t \in \{2,\ldots,T\}$ and each $(i_1,\ldots,i_{t-1})$, $\varepsilon^t_{i_1\ldots i_{t-1}}$ is defined by*

$$\varepsilon^t_{i_1\ldots i_{t-1}} := \frac{\varepsilon}{(1-\varepsilon)p^0_{i_1\ldots i_{t-1}} + \varepsilon}.$$

*Furthermore, $(\forall i_1,\ldots,i_{T-1})$ $\varepsilon^1 < \varepsilon^2_{i_1} < \cdots < \varepsilon^T_{i_1\ldots i_{T-1}}$, unless $E_t = S$.*

**Proof** The first half of the claim can be proved by very closely following the proof for the case when $T = 2$, and hence, the details of the proof is omitted.

To show that it holds that, for any sequence of observations, $\varepsilon^1 < \varepsilon^2_{i_1} < \cdots < \varepsilon^T_{i_1\ldots i_{T-1}}$, simply note that $p^0_{i_1\ldots i_{t-1}} > p^0_{i_1\ldots i_{t-1} i_t}$ unless $E_t = S$ by the definition of marginals. □

### 3.4 Learning with Rectangular $\varepsilon$-Contamination

The one-period-ahead conditional Knightian uncertainty of any rectangular $\varepsilon$-contamination has a very convenient form.

To precisely observe this, let $p^0 \in \mathscr{M}(\Omega, \mathcal{F}_T)$ and let $\varepsilon \in (0,1)$ as usual. First, note that for any $t \le T$, Proposition 11 and the definition of the marginal imply that

$$\{p^0\}^{rec\varepsilon}_t = \Big( \big((1-\varepsilon^1)p^0_{i_1} + \varepsilon^1 q_{i_1}\big) \big((1-\varepsilon^2_{i_1})p^{0+}_{i_2|i_1} + \varepsilon^2_{i_1} q_{i_1 i_2}\big) \cdots \\ \big((1-\varepsilon^t_{i_1\ldots i_{t-1}})p^{0+}_{i_t|i_1\ldots i_{t-1}} + \varepsilon^t_{i_1\ldots i_{t-1}} q_{i_1 i_2\ldots i_t}\big)\Big)_{i_1,i_2,\ldots,i_t},$$

where $q's$ satisfies the conditions imposed in (16) and $\varepsilon^t_{i_1\ldots i_{t-1}}$ is defined in the previous subsection.

Then, the definition of the one-period-ahead conditional in turn implies that for any $t \le T$ and any $E_1 \times \cdots \times E_{t-1}$,

$$\left(\{p^0\}^{rec\varepsilon}\right)^+ |_{E_1\times\cdots\times E_{t-1}} = \Big\{ (1-\varepsilon^t_{i_1\ldots i_{t-1}})p^{0+}(\cdot|E_1\times\cdots\times E_{t-1}) + \varepsilon^t_{i_1\ldots i_{t-1}} q \ \Big| \\ q \in \mathscr{M}(S, \langle E_{t,i_t}\rangle_{i_t}) \Big\}. \quad (17)$$

Here, note that the resemblance between (11) and (17), which strongly suggests that the next proposition holds true.

**Proposition 12** *For any $t \leq T$ and any $E_1 \times \cdots \times E_{t-1}$, it holds that*

$$\left(\{p^0\}^{rec\varepsilon}\right)^+ \big|_{E_1 \times \cdots \times E_{t-1}} = \left(\{p^0\}^{\varepsilon}\right)^+ \big|_{E_1 \times \cdots \times E_{t-1}} .$$

**Proof** The way of the construction of the one-period-ahead conditional allows us to mimic the proof for the case where $T = 2$ (Proposition 7), and hence we omit the proof. $\square$

In order to state our final result on the learning behavior with the rectangular $\boldsymbol{\varepsilon}$-contamination, we largely simplify the dynamic structure underlying our model.[12]

Let $(\Omega, \mathcal{F}_T) := (S^T, \otimes_{t=1}^T \langle E_i \rangle_{i=1}^n)$, where $\otimes_{t=1}^T \langle E_i \rangle_{i=1}^n$ is the $T$-time self-direct-product of the identical finite partition of $S$, $\langle E_i \rangle_{i=1}^n$. Also, assume that $p^0 := p^{00} \otimes p^{00} \otimes \cdots \otimes p^{00}$, which is the $T$-time self-direct-product of some $p^{00} \in \mathscr{M}(S, \langle E_i \rangle_{i=1}^n)$.

Then, we can prove the following proposition.

**Proposition 13** *For any $t \leq T$ and any $E_{i_1} \times \ldots \times E_{i_{t-1}} \times E_{i_t}$ such that $E_{i_t} \neq S$, it holds that*

$$\left(\{p^0\}^{rec\varepsilon}\right)^+ \big|_{E_{i_1} \times \cdots \times E_{i_{t-1}}} \subsetneq \left(\{p^0\}^{rec\varepsilon}\right)^+ \big|_{E_{i_1} \times \cdots \times E_{i_{t-1}} \times E_{i_t}} .$$

**Proof** In view of the Equation (17) and the current underlying stochastic structure, the left-hand side of the inclusion in the proposition turns out to be

$$\left\{ (1 - \varepsilon^t_{i_1 \ldots i_{t-1}}) p^{00} + \varepsilon^t_{i_1 \ldots i_{t-1}} q \;\middle|\; q \in \mathscr{M}(S, \langle E_i \rangle_i) \right\},$$

while its right-hand side turns out to be

$$\left\{ (1 - \varepsilon^{t+1}_{i_1 \ldots i_t}) p^{00} + \varepsilon^{t+1}_{i_1 \ldots i_t} q \;\middle|\; q \in \mathscr{M}(S, \langle E_i \rangle_i) \right\}.$$

Then, the result follows because $\varepsilon^t_{i_1 \ldots i_{t-1}} < \varepsilon^{t+1}_{i_1 \ldots i_t}$ by Proposition 11. $\square$

The last proposition exhibits that an active learning by observing the occurrence of an event always dilates the degree of uncertainty whenever Knightian uncertainty is specified by the rectangular $\boldsymbol{\varepsilon}$-contamination and underlying stochastic structure is as in the proposition.[13]

As we already claimed with respect to two-period models, we must emphasize again that Proposition 8 (in particular, its last equation), the first equation of subsection 3.4 as well as Proposition 13 show that time-consistent (rectangularity) and dilation of ambiguity upon learning are completely consistent.

[12] We can dispense with this restriction by paying a cost that the conclusion of the next proposition holds only when the complicated conditions are all met.

[13] Shishkin ® Ortoleva (2019) try to measure the degree of the dilation of ambiguity upon learning in a more general framework.

## 4 Concluding Remarks

The analyses conducted in this paper strongly suggests a promising direction of a new research. An arbitrarily-finite-horizon model may be extended to the infinite-horizon model, where the rectangularity generates the "recursive" $\varepsilon$-contamination, with or without learning.

Then, the dynamic recursive preference represented by the Koopmans-type equation for the maxmin behavior may allow the application of dynamic programming techniques and such a model could be a benchmark for more applied research where the pessimism characterized in the form of the $\varepsilon$-contamination may play a central role.

A perfect-knowledge assumption with respect to the range of parameter values, that is, the rectangular (recursive) $\varepsilon$-contamination *without* learning may facilitate the analyses largely by its stationary structure of some kind. By using such a model, we may provide an operational model that could have a potential power to explain seemingly complicated economic phenomena.

## References


[1] Berger, J. O. (1985): *Statistical Decision Theory and Bayesian Analysis* (Second Edition), Springer-Verlag.

[2] Chu, C. Y. C. and W.-F. Liu (2002): "Knightian uncertainty and the voting paradox," mimeo., University of Washington.

[3] Epstein, L. G. and M. Schneider (2003): "Recursive multiple-priors," *Journal of Economic Theory* 113, 1-31.

[4] Epstein, L. and T. Wang (1994): "Intertemporal asset pricing and Knightian uncertainty". *Econometrica*, 62, 283-382, 1994.

[5] Gilboa, I., and D. Schmeidler (1989): "Maxmin expected utility with non-unique prior," *Journal of Mathematical Economics* 18, 141-153.

[6] Kopylov, I. (2009): "Choice deferral and ambiguity aversion", *Theoretical Economics*, 4, 199-225.

[7] Nishimura, K. G., and H. Ozaki (2006): "An axiomatic approach to $\varepsilon$-contamination," *Economic Theory* 27, 333-340.

[8] Nishimura, K. G., and H. Ozaki (2017): *Economics of Pessimism and Optimism: Theory of Knightian Uncertaonty and Its Applications*, Springer.

[9] Shishkin, D. ® Ortoleva, P. (2019): "Ambiguous information and dilation: an experiment," mimeo. http://pietroortoleva.com/