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Abstract

This paper presents a new asset pricing model incorporating fundamental uncertainties by choice of a probability measure. This approach is novel in that we incorporate uncertainties on Brownian motions describing risks into the existing asset pricing model. Particularly, we show extensions of interest rate models to the ones with uncertainties on the Brownian motions, which make the yield curve reflect not only economic factors but also views of the market participants on the Brownian motions. Such yield curve models are especially important in yield curve trading of hedge funds as well as monetary policy making of central banks under low interest rate environments observed after the global financial crisis, in which yield curves are less affected by economic factors since they are controlled by the central banks, but are driven mainly by sentiments of market participants.

Firstly, to model aggressive (positive)/conservative (cautious) attitudes towards such fundamental uncertainties, we consider a sup-inf/inf-sup problem on the utility of a representative agent with respect to uncertainties over Brownian motions, i.e. fundamental market risks, by choice of a probability measure. Secondly, we show that the problem is solved via a backward-stochastic differential equations (BSDEs) approach. Then, under a probability measure determined by solving the sup-inf/inf-sup problem, we propose interest rate models with those uncertainties and explicitly obtain their term structures of interest rates. Particularly, we present two approaches to solving the relevant coupled forward-backward stochastic differential equations (FBSDEs) to obtain expressions of the equilibrium interest rate and the term structure of interest rates. In detail, the first approach is by comparison theorems, and the second approach is to predetermine the signs of the volatilities of the BSDE in the coupled system and confirm them by explicitly solving the separated BSDE. Finally, we present concrete examples with numerical experiments.

1 Introduction

This paper presents a new asset pricing model incorporating fundamental uncertainties by choice of a probability measure. This approach is novel in that we incorporate uncertainties

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on Brownian motions describing risks into the existing asset pricing model. Particularly, we show extensions of interest rate models to the ones with uncertainties on the Brownian motions, which make the yield curve reflect not only economic factors but also views on the Brownian motions of the market participants. Such yield curve models are especially important in yield curve trading of hedge funds as well as monetary policy making of central banks under low interest rate environments observed after the global financial crisis, in which yield curves are less affected by economic factors since they are controlled by the central banks, but are driven mainly by sentiments of market participants. Specifically, we model the market participants' aggressive (positive)/conservative (cautious) attitudes towards uncertainties by a sup-inf/inf-sup problem on an agent's utility by choice of a probability measure. The problem reduces to solving a forward-backward stochastic differential equations. In the theory of asset pricing with ambiguity, such as the methods of robust control in Hansen and Sargent [9] and the pricing with ambiguity in Chen and Epstein [5], only conservative attitudes of the agent towards risks has been considered. However, in practice, aggressive attitudes of the market about some risks are also reflected in asset prices. Moreover, the term structure of interest rates in the recent global low interest environments are driven by those optimistic and pessimistic sentiment of the market (e.g. see Nishimura-Sato-Takahashi [20]). As far as we know, there has not been an asset pricing model considering both conservative and aggressive sides. Our approach is also new in taking the both sides into account by means of the change in probability measures.

"Market sentiment", which is modeled as attitudes towards fundamental uncertainties, that is, uncertainties about fundamental market risks represented by Brownian motions in this paper, is frequently considered as an important factor for a determinant of asset market prices. After the global financial crisis in 2007-2009, the financial markets have been more driven by market sentiment rather than economic events. Particularly, in the global recession after the crisis, central banks, such as Federal Reserve, European Central Bank, and Bank of Japan, conducted monetary easing to lower interest rates (see Joyce et al. [11], Szczerbowicz [25], and Ueda [26] for instance). Bank of Japan started yield curve control as a monetary easing policy in 2016, which aims to control long-end interest rates, which had been untouched by the central bank and determined solely by supply and demand in the bond market (e.g. Bank of Japan [1]). The quantitative and qualitative monetary easing with yield curve control by Bank of Japan gathered attention from other central banks as a new method of monetary easing in low interest rate environments. In the low interest rate environments in which the short-end of the yield curve is flat, the long-end of the yield curve is driven by market sentiment and hedge funds actively trade betting on movements of the long-end of the yield curve (e.g. McGeever [15]). Thus, it is important for central banks as well as hedge funds to use interest rate models incorporating such sentiment, which is not explicitly utilized in traditional term structure models of interest rates. This paper develops a foundation and provides concrete examples of term structure models of interests with market sentiment, i.e. conservative or aggressive attitudes towards fundamental uncertainties (uncertainties over fundamental market risks).

Particularly, an idea behind the models is as follows: In an economic model where risks are expressed by Brownian motions (a multi-dimensional Brownian motion), there may exist a fundamental uncertainty, which is an uncertainty about a risk of each Brownian motion and is represented by a stochastic process λ_j for the uncertainty corresponding to the j-th risk (Brownian motion B_j). When there is a fundamental uncertainty about the j-th risk, we only know the true j-th risk is one of $\{B_j^\lambda; \lambda_j \in \Lambda_j\}$ with $B_{j,t}^\lambda := B_{j,t} - \int_0^t \lambda_{j,s} ds$, $0 \le t < \infty$ given some set Λ_j, and we cannot tell which is the true one. In contrast, if $\lambda_j \equiv 0$ (i.e. $B_j^\lambda = B_j$), there is no fundamental uncertainty about the j-th risk. Here, B_j^λ is a Brownian motion under a new probability measure induced by $(\lambda_j)_j$ through *a change of a probability measure.* (See the details for the following sections.) Further, in order to model aggressive/conservative attitudes towards fundamental uncertainties, we consider a sup-inf/inf-sup problem on a representative agent's utility, who optimally choses its probability measure through minimizing the utility with respect to the fundamental uncertainty (λ_1) for the first Brownian motion (B_1) while maximizing it with respect to the fundamental uncertainty (λ_2) for the second Brownian motion (B_2). Here, we assume that the agent has a conservative (cautious) attitude towards taking a risk of one Brownian motion say B_1 and an aggressive (positive) attitude towards a risk of a different Brownian motion B_2, while there are no fundamental uncertainties about the risks associated with the other Brownian motions. Hereafter, we call these two independent Brownian motions market risks, and also refer to the fundamental uncertainties λ_j, $j = 1, 2$, as uncertainties over the market risks.

For example, B_1 and B_2 can be taken as Brownian motions associated with foreign and domestic news, respectively. In such a case, the market, which is deemed to be the agent, is cautious about taking risks related to the foreign news, while it has strong views on risks associated with the domestic news, therefore is willing to take those risks aggressively. Nishimura-Sato-Takahashi [20] estimates a term structure model with explicit sentiment factors in a period including the global financial crisis, in which market confidence was substantially eroded. In particular, it makes use of a large text data of market news to obtain observations associated with sentiment factors. In the work, the market is cautious about foreign risks, such as news on fiscal conditions of foreign countries and foreign exchange rates, while it is aggressive about domestic risks, such as domestic business conditions. In another instance, B_1 and B_2 can be taken as Brownian motions related to foreign exchange and stock news, respectively. This implies that the market is bearish about the views on foreign exchange risks, however is bullish about taking equity risks.

Specifically, we take a BSDE approach to solve a sup-inf/inf-sup problem arising from a model that incorporates conservative and aggressive attitudes towards risks. Then, we consider asset pricing under the probability measure of the agent, which is described as a result of the sup-inf/inf-sup problem, and show that the interest rate is obtained by solving a system of FBSDEs. To the best of our knowledge, this is the first attempt to rigorously develop interest rate models with fundamental uncertainties based on a BSDE approach. Also, we provide cases in which the system of FSBDEs reduces to a combination of a BSDE and forward SDEs.

Furthermore, we present an example of the interest rate model with fundamental uncertainties, which is deterministic without these uncertainties. In this case, the uncertainties are the only sources of the randomness in the short rate and the yield curve movements. Such a model is particularly important in low interest rate environments, in which the yield curve moves mainly by uncertainties over market risks. For instance, Bank of Japan introduced yield curve control in their monetary policy in 2016 for further monetary easing in a low interest rate environment (e.g. Bank of Japan [1]). Also, in low interest rate environments, hedge funds actively trade on yield curves (e.g. McGeever[15]). This model helps central banks effectively control yield curves and hedge funds trade on the movements of the curves.

While the theories of robust control (e.g. Hansen and Sargent [9]) and ambiguity in Chen and Epstein [5] consider the conservative side in their utility maximization problems, our study takes both aggressive (positive) and conservative (cautious) attitudes of the representative agent into pricing and derive expressions of an equilibrium interest rate and its yield curve under uncertainties over different market risks. Moreover, we present conditions with which the system of FBSDEs reduces to a combination of a BSDE and forward SDEs, and solve the system to obtain explicit expressions of the equilibrium interest rate and market price of risks for concrete utilities. Furthermore, we consider cases in which fundamental uncertainties increase as the market risks grow, which is expressed by the ranges of the representative agent's uncertainties about Brownian motions varying in accordance with the state-variable processes describing the risks.

For other financial applications of optimal control, Gao et al.[7] consider a problem of hedging risk exposure to imperfectly liquid stock by investing in put options. Chronopoulos et al.[2] propose an analytical real options framework that incorporates major components relevant to cybersecurity practice. Gao et al.[6] investigate an uncertain stock model by uncertain differential equations involved by a Liu process. Wu and Chung [27] propose a new approach for options trading based on Kelly criterion. Calafiore [3] deals with optimal allocation of a portfolio by a data-driven approach computing the portfolio composition directly from historical data by a min-max based portfolio selection rule. Mukuddem-Petersen and Petersen [17] consider optimal risk management of banks in a stochastic dynamic setting. Yiu et al. [28] investigate an optimal portfolio selection problem with regime switching and value-at-risk constraint. Zhang et al. [30] study American option pricing by applying augmented Lagrangian method to the corresponding variational inequality problem. Saito and Takahashi [22] examine derivatives pricing under market impact through a stochastic control problem solved by an asymptotic method. Saito and Takahashi [23] consider a linear quadratic stochastic differential game to investigate trading behaviors of three types of players in high frequency stock markets.

This paper is organized as follows. After Section 2 introduces a model with fundamental uncertainties, which is a sup-inf/inf-sup problem on a stochastic differential utility. Section 3 proves existence and uniqueness of a solution of BSDEs associated with the sup-inf/inf-sup problem. Section 4 shows that the sup-inf/inf-sup problem reduces to solving of the associated BSDE with stochastic Lipschitz coefficients. Section 5 presents expressions of an equilibrium interest rate and yield curve. Section 6 provides a case with numerical experiments, in which explicit expressions of the equilibrium interest rate are obtained by comparison theorems. Section 7 shows that coupled FBSDEs under the agent's probability measure are separated into forward SDEs and a BSDE under certain conditions, and the expressions of the equilibrium interest rate and the term structure of interest rates are obtained for three types of stochastic differential utilities. Finally, Section 8 concludes. Appendix shows the proofs and derivations omitted in the main text.

2 Sup-inf/inf-sup problem with respect to fundamental uncertainties

This section introduces a sup-inf problem arising from a model with fundamental uncertainties, which incorporates conservative and aggressive attitudes towards risks.

First, we suppose that a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, P)$ and a d-dimensional Brownian motion $B = (B_1, \ldots, B_d)$ $(d \geq 2)$ are given, where $\{\mathcal{F}_t\}_{0\leq t\leq T}$ is the augmentation of the natural filtration generated by B, and we call P the physical measure, hereafter. Next, for a $\mathcal{R}^2$-valued $\{\mathcal{F}_t\}$-progressively measurable processes $\lambda = (\lambda_1, \lambda_2)$, where $\mathcal{Z}_t(\lambda)$ defined by

$$\mathcal{Z}_t(\lambda) := \exp\left\{\sum_{j=1}^{2}\int_0^t \lambda_{j,s}dB_{j,s} - \sum_{j=1}^{2}\frac{1}{2}\int_0^t \lambda_{j,s}^2 ds\right\} \tag{1}$$

is a martingale, we define a probability measure P^{λ_1,λ_2} by

$$P^{\lambda_1,\lambda_2}(A) := E[\mathcal{Z}_T(\lambda)1_A];\ A \in \mathcal{F}_T, \tag{2}$$

where, λ_1 and λ_2 stand for uncertainties over the risks associated with Brownian motions B_1 and B_2, respectively. We also define a set Λ as

$$\Lambda = \{(\lambda_1, \lambda_2); \mathcal{Z}(\lambda) \text{ is a martingale and } |\lambda_{j,t}| \leq |\bar{\lambda}_j(t, X_t)|,\ 0 \leq t \leq T,\ j = 1, 2\}, \tag{3}$$

where $\bar{\lambda}_j : [0, T] \times \mathcal{R}^l \to \mathcal{R}$ $(l \geq d)$, $j = 1, 2$, are measurable functions and $\mathcal{R}^l$-valued stochastic process X is a state variable satisfying a stochastic differential equation (SDE): with $\mu_x, \sigma_{x,j} : \mathcal{R}^l \to \mathcal{R}^l$, $j = 1, 2, \ldots, d$,

$$dX_t = \mu_x(X_t)dt + \sum_{j=1}^{d}\sigma_{x,j}(X_t)dB_{j,t}. \tag{4}$$

Hereafter, we assume that $\bar{\lambda}_j$ $(j = 1, 2)$ and X are exogenously given and that SDE (4) has a unique strong solution.

Then, a representative agent who has a conservative (aggressive) view on Brownian motion $B_1(B_2)$ supposes the worst (best) case. Thus, the agent implements optimization with respect to λ_j $(j = 1, 2)$, that is, minimize (maximize) its utility with respect to $\lambda_1(\lambda_2)$. In contrast, the agent has no uncertainties over risks represented by Brownian motions B_j, $j = 3, \cdots, d$, so that we have $\lambda_j \equiv 0$. Then, $B_{1,t}^{\lambda_1,\lambda_2} = B_{1,t} - \int_0^t \lambda_{1,s}ds$, $B_{2,t}^{\lambda_1,\lambda_2} = B_{2,t} - \int_0^t \lambda_{2,s}ds$ and $B_j^{\lambda_1,\lambda_2} = B_j$ for $j = 3, \cdots, d$ are Brownian motions under the probability measure P^{λ_1,λ_2} generated by a martingale $\mathcal{Z}(\lambda)$ with $\lambda = (\lambda_1, \lambda_2, 0, \cdots, 0)$.

More concretely, given a consumption process c, an agent with a standard utility solves the following problem:

$$\sup_{\lambda_2\in\Lambda_2}\inf_{\lambda_1\in\Lambda_1(\lambda_2)} E^{P^{\lambda_1,\lambda_2}}\left[\int_0^T e^{-\beta t}u(c_t)dt\right],\ (\beta > 0), \tag{5}$$

or

$$\inf_{\lambda_1\in\Lambda_1} \sup_{\lambda_2\in\Lambda_2(\lambda_1)} E^{P^{\lambda_1,\lambda_2}}\left[\int_0^T e^{-\beta t}u(c_t)dt\right],\ (\beta>0), \tag{6}$$

where conservative and aggressive attitudes are expressed by $\inf_{\lambda_1}$ and $\sup_{\lambda_2}$, respectively. Here, for $(j,k=1,2,\ k\neq j)$ we define Λ_j and $\Lambda_j(\lambda_k)$ respectively as

$$\Lambda_j=\{\lambda_j;|\lambda_{j,t}|\leq|\bar{\lambda}_{j,t}(X_t)|,\ 0\leq t\leq T\},\ j=1,2, \tag{7}$$

and

$$\Lambda_j(\lambda_k)=\{\lambda_j;|\lambda_{j,t}|\leq|\bar{\lambda}_{j,t}(X_t)|,\ 0\leq t\leq T,\text{ and }\mathcal{Z}((\lambda_1,\lambda_2))\text{ is a martingale for given }\lambda_k\in\Lambda_k\}$$
$$\text{for }(j,k)=(1,2),(2,1). \tag{8}$$

More generally, let us define a representative agent's stochastic differential utility (SDU, continuous-time version of recursive utility. For an introduction of SDU, see Section 1.3 in Ma and Yong [14] for instance) Y^{λ_1,λ_2} as follows: with an aggregator $g:[0,T]\times\mathcal{C}([0,T]\to\mathcal{R}^d)\times\mathcal{R}^l\times\mathcal{R}\to\mathcal{R}$,

$$Y_t^{\lambda_1,\lambda_2}=E^{P^{\lambda_1,\lambda_2}}\left[\xi+\int_t^T g(s,B,X_s,Y_s^{\lambda_1,\lambda_2})ds\middle|\mathcal{F}_t\right], \tag{9}$$

where ξ is a bounded $\mathcal{F}_T$-measurable random variable.

Next, let us set $J(\lambda_1,\lambda_2)$ as

$$J(\lambda_1,\lambda_2)=Y_0^{\lambda_1,\lambda_2},\ (\lambda_1,\lambda_2)\in\Lambda. \tag{10}$$

Then, we consider the following sup-inf and inf-sup problems:

- (sup-inf problem)

$$\begin{aligned}&\sup_{\lambda_2\in\Lambda_2}\inf_{\lambda_1\in\Lambda_1(\lambda_2)} J(\lambda_1,\lambda_2)\\ &=\sup_{\lambda_2\in\Lambda_2}\inf_{\lambda_1\in\Lambda_1(\lambda_2)} E^{P^{\lambda_1,\lambda_2}}\left[\xi+\int_0^T g(s,B,X_s,Y_s^{\lambda_1,\lambda_2})ds\right],\end{aligned} \tag{11}$$

- (inf-sup problem)

$$\begin{aligned}&\inf_{\lambda_1\in\Lambda_1}\sup_{\lambda_2\in\Lambda_2(\lambda_1)} J(\lambda_1,\lambda_2)\\ &=\inf_{\lambda_1\in\Lambda_1}\sup_{\lambda_2\in\Lambda_2(\lambda_1)} E^{P^{\lambda_1,\lambda_2}}\left[\xi+\int_0^T g(s,B,X_s,Y_s^{\lambda_1,\lambda_2})ds\right],\end{aligned} \tag{12}$$

where the representative agent is conservative about the uncertainty about B_1 and minimizes the stochastic differential utility $J(\lambda_1,\lambda_2)$ with respect to λ_1, and is aggressive about the uncertainty about B_2 and maximizes $J(\lambda_1,\lambda_2)$ with respect to λ_2.

3 BSDEs for the model with fundamental uncertainties

In this section, we discuss backward stochastic differential equations (BSDEs) associated with a model with fundamental uncertainties introduced in the previous section. Particularly, we provide existence and uniqueness of solutions of those BSDEs under certain conditions in Propositions 1 and 2.

First, since $\mathcal{Z}^{\lambda_1,\lambda_2}$, $(\lambda_1,\lambda_2)\in\Lambda$ is a P-martingale, applying Girsanov's theorem, we can define a d-dimensional Brownian motion under P^{λ_1,λ_2}, $B^{\lambda_1,\lambda_2}=(B_1^{\lambda_1,\lambda_2},\dots,B_d^{\lambda_1,\lambda_2})$, by

$$\begin{aligned} B_{1,t}^{\lambda_1,\lambda_2} &= B_{1,t}-\int_0^t \lambda_{1,s}ds,\\ B_{2,t}^{\lambda_1,\lambda_2} &= B_{2,t}-\int_0^t \lambda_{2,s}ds,\\ B_{j,t}^{\lambda_1,\lambda_2} &= B_{j,t}\ (3\le j\le d). \end{aligned} \tag{13}$$

Then, Y^{λ_1,λ_2} in (9) is characterized as a unique solution of the following BSDE:

$$\begin{aligned} dY_t^{\lambda_1,\lambda_2} &= -g(t,B,X_t,Y_t^{\lambda_1,\lambda_2})dt+\sum_{j=1}^d Z_{j,t}^{\lambda_1,\lambda_2}dB_{j,t}^{\lambda_1,\lambda_2}\\ &= -\left(g(t,B,X_t,Y_t^{\lambda_1,\lambda_2})+\lambda_{1,t}Z_{1,t}^{\lambda_1,\lambda_2}+\lambda_{2,t}Z_{2,t}^{\lambda_1,\lambda_2}\right)dt+\sum_{j=1}^d Z_{j,t}^{\lambda_1,\lambda_2}dB_{j,t},\ Y_T^{\lambda_1,\lambda_2}=\xi. \end{aligned} \tag{14}$$

Namely, $Y_t^{\lambda_1,\lambda_2}$ is expressed as

$$\begin{aligned} Y_t^{\lambda_1,\lambda_2} &= \xi+\int_t^T g(s,B,X_s,Y_s^{\lambda_1,\lambda_2})ds-\sum_{j=1}^d\int_t^T Z_{j,s}^{\lambda_1,\lambda_2}dB_{j,s}^{\lambda_1,\lambda_2}\\ &= \int_t^T\left(g(s,B,X_s,Y_s^{\lambda_1,\lambda_2})+\lambda_{1,s}Z_{1,s}^{\lambda_1,\lambda_2}+\lambda_{2,s}Z_{2,s}^{\lambda_1,\lambda_2}\right)ds-\sum_{j=1}^d\int_t^T Z_{j,s}^{\lambda_1,\lambda_2}dB_{j,s}. \end{aligned} \tag{15}$$

In particular, we note that by taking a conditional expectation under P^{λ_1,λ_2} in both sides of the first equality in (15), we obtain (9) if the Itô integral is a martingale.

The next proposition states that BSDE (14) ((15)) with stochastic Lipschitz coefficients λ_1 and λ_2 has a unique solution under certain conditions.

Proposition 1. *Suppose that SDE (4) has a unique strong solution and* $g:[0,T]\times\mathcal{C}([0,T]\to\mathcal{R}^d)\times\mathcal{R}^l\times\mathcal{R}\to\mathcal{R}$ *satisfies the following conditions: (i)* $g(t,\omega,x,0)$ *is bounded. (ii) There exists a constant* $L>0$ *such that*

$$\begin{aligned} &|g(t,\omega,x,y)-g(t,\omega,x,y')|\le L|y-y'|,\\ &\forall y,y'\in\mathcal{R},x\in\mathcal{R}^l,\omega\in\mathcal{C}([0,T]\to\mathcal{R}^d),t\in[0,T]. \end{aligned} \tag{16}$$

Suppose also that an exponential local martingale with progressively measurable processes λ_j $(j = 1, 2)$,

$$\exp\left(-\frac{1}{2}\sum_{j=1}^{2}\int_0^T \lambda_{j,s}^2 ds + \sum_{j=1}^{2}\int_0^T \lambda_{j,s} dB_{j,s}\right) \tag{17}$$

is a martingale and

$$E\left[\sup_{0\le s\le T}|\lambda_s|^4\right] < \infty. \tag{18}$$

Then, BSDE

$$\begin{aligned} dY_t^{\lambda_1,\lambda_2} &= -\left(g(t, B, X_t, Y_t^{\lambda_1,\lambda_2}) + \lambda_{1,t}Z_{1,t}^{\lambda_1,\lambda_2} + \lambda_{2,t}Z_{2,t}^{\lambda_1,\lambda_2}\right)dt + \sum_{j=1}^{d} Z_{j,t}^{\lambda_1,\lambda_2} dB_{j,t}, \\ Y_T^{\lambda_1,\lambda_2} &= \xi, \end{aligned} \tag{19}$$

with a bounded $\mathcal{F}_T$-measurable random variable ξ has a unique solution $(Y^{\lambda_1,\lambda_2}, Z^{\lambda_1,\lambda_2})$ such that

$$E\left[\int_0^T |Z_s^{\lambda_1,\lambda_2}|^2 ds\right] < \infty \tag{20}$$

and Y^{λ_1,λ_2} is uniformly bounded with respect to $(t, \omega) \in [0, T] \times \mathcal{C}([0, T] \to \mathcal{R}^d)$.

Proof. See Appendix A.

Also, in the next proposition, we prove existence and uniqueness of a solution for a BSDE that contains $|Z_1^{\lambda_1,\lambda_2}|$ and $|Z_2^{\lambda_1,\lambda_2}|$ in the driver. The BSDE appears in solving *sup-inf problem* (11) and *inf-sup problem* (12) in Section 4, where λ_1 and λ_2 in (14) ((15)) are given by $-|\bar{\lambda}_1(X)|sgn(Z_1^{\lambda_1,\lambda_2})$ and $|\bar{\lambda}_2(X)|sgn(Z_2^{\lambda_1,\lambda_2})$, respectively,

Proposition 2. *Suppose that SDE (4) has a unique strong solution and $g : [0, T]\times\mathcal{C}([0, T] \to \mathcal{R}^d) \times \mathcal{R}^l \times \mathcal{R} \to \mathcal{R}$ satisfies the following conditions: (i) $g(t, \omega, x, 0)$ is bounded. (ii) There exists a constant $L > 0$ such that*

$$|g(t, \omega, x, y) - g(t, \omega, x, y')| \le L|y - y'|,$$
$$\forall y, y' \in \mathcal{R}, x \in \mathcal{R}^l, \omega \in \mathcal{C}([0, T] \to \mathcal{R}^d), t \in [0, T].$$

If a weak version of Novikov's condition (e.g. Corollary 3.5.14 in Karatzas and Shreve [12]); there exists a partition of $[0, T]$, $0 = t_0 < t_1 < \cdots < t_N = T$, such that

$$E\left[\exp\left(\sum_{j=1}^{2}\frac{1}{2}\int_{t_{n-1}}^{t_n} \bar{\lambda}_{j,s}(X_s)^2 ds\right)\right] < \infty, \text{ for all } 1 \le n \le N. \tag{21}$$

is satisfied for $\bar{\lambda}_1$ and $\bar{\lambda}_2$ and

$$E\left[\sup_{0\le s\le T}|\bar{\lambda}_s|^4\right] < \infty, \tag{22}$$

BSDE

$$dY_t^{\lambda_1^*,\lambda_2^*} = -\left(g(t,B,X_t,Y_t^{\lambda_1^*,\lambda_2^*}) - |\bar{\lambda}_{1,t}(X_t)||Z_{1,t}^{\lambda_1^*,\lambda_2^*}| + |\bar{\lambda}_{2,t}(X_t)||Z_{2,t}^{\lambda_1^*,\lambda_2^*}|)\right)dt + \sum_{j=1}^{d} Z_{j,t}^{\lambda_1^*,\lambda_2^*} dB_{j,t},$$
$$Y_T^{\lambda_1^*,\lambda_2^*} = \xi, \tag{23}$$

with a bounded $\mathcal{F}_T$-measurable random variable ξ has a unique solution $(Y^{\lambda_1^*,\lambda_2^*}, Z^{\lambda_1^*,\lambda_2^*})$ such that

$$E\left[\int_0^T |Z_s^{\lambda_1^*,\lambda_2^*}|^2 ds\right] < \infty \tag{24}$$

and $Y^{\lambda_1^*,\lambda_2^*}$ is uniformly bounded with respect to $(t,\omega) \in [0,T] \times \mathcal{C}([0,T] \to \mathcal{R}^d)$.

Proof. See Appendix B.

3.1 Examples of the state-variable process

In the following, we present two examples of a state-variable process in which the assumptions (17) & (18) in Proposition 1 and (21) & (22) in Proposition 2 are satisfied.

Example 1. (Ornstein-Uhlenbeck process)

Suppose that $\bar{\lambda}_j(t,X_t) = \tilde{\lambda}_{j,t}X_t,\ j = 1,2$ for some $\mathcal{R}^{1\times l}$-valued bounded deterministic functions $\tilde{\lambda}_{j,t} = (\tilde{\lambda}_{j,t}^{(1)} \dots \tilde{\lambda}_{j,t}^{(l)})$, where $\tilde{\lambda}_{j,t}^{(1)}, \dots, \tilde{\lambda}_{j,t}^{(l)} > 0$, and X_t is a $\mathcal{R}^l$-valued Ornstein-Uhlenbeck process following a SDE:

$$dX_t = (K_{1,t}X_t + K_{2,t})dt + \Sigma_{x,t}dB_t, \tag{25}$$

where $K_{1,t}, K_{2,t}, \Sigma_{x,t}$ are bounded deterministic functions on $[0,T]$ taking their values on $\mathcal{R}^{l\times l}, \mathcal{R}^{l\times 1}, \mathcal{R}^{l\times d}$, respectively. Then, the moment conditions (18) and (22) are clearly satisfied. Moreover, an exponential local martingale (17) is a martingale, which is proved as follows.

First, X_t is expressed as (e.g. Eq. 5.6.6 in p.354 in Karatzas and Shreve [12])

$$X_t = \Phi_t\left[X_0 + \int_0^t \Phi^{-1}(s)K_{2,s}ds + \int_0^t \Phi^{-1}(s)\Sigma_x(s)dB_s\right], \tag{26}$$

where $\Phi(t)$ is the fundamental solution of the $l \times l$ matrix ODE:

$$\dot{\Phi}(t) = K_{1,t}\Phi(t),\ \Phi(0) = I. \tag{27}$$

Let $H_t(\omega) = (\lambda_{1,t}(\omega)\ \lambda_{2,t}(\omega))$. Note that

$$\begin{aligned}
&|H_t(\omega)| \\
&\leq |\lambda_{1,t}(\omega)| + |\lambda_{2,t}(\omega)| \\
&\leq |\tilde{\lambda}_{1,t}||X_t| + |\tilde{\lambda}_{2,t}||X_t| \\
&\leq \left(|\tilde{\lambda}_{1,t}| + |\tilde{\lambda}_{2,t}|\right) \sup_{0\leq s\leq t} |X_s|.
\end{aligned} \tag{28}$$

Since

$$\begin{aligned}\sup_{0\le s\le t}|X_s| &\le \sup_{0\le s\le t}\left(|\Phi_s X_0| + \left|\Phi_s\int_0^s \Phi^{-1}(v)K_{2,v}dv\right|\right) + \sup_{0\le s\le t}\left|\Phi_s\int_0^s \Phi^{-1}(v)\sigma_x(v)dB_v\right| \\ &\le \sup_{0\le s\le t}\left(|\Phi_s X_0| + \left|\Phi_s\int_0^s \Phi^{-1}(v)K_{2,v}dv\right|\right) \\ &\quad + \sup_{0\le s\le t}\sum_{1\le i,j\le n}|\Phi_{s,i,j}|\sup_{0\le s\le t}\left|\int_0^s \Phi^{-1}(v)\sigma_x(v)dB_v\right|, \qquad (29)\end{aligned}$$

setting

$$k = \left(|\tilde{\lambda}_{1,t}| + |\tilde{\lambda}_{2,t}|\right)\max\left(\sup_{0\le s\le t}\left(|\Phi_s X_0| + \left|\Phi_s\int_0^s \Phi^{-1}(v)K_{2,v}dv\right|\right), \sup_{0\le s\le t}\sum_{1\le i,j\le n}|\Phi_{s,i,j}|\right), \qquad (30)$$

we have

$$|H_t(\omega)| \le k\left(1 + \sup_{0\le s\le t}\left|\int_0^s \Phi^{-1}(v)\sigma_x(v)dB_v\right|\right). \qquad (31)$$

Since $\Phi^{-1}(t)\sigma_x(t)$ is a bounded deterministic function on $[0,T]$, by Lemma 15.5.7 in Cohen and Elliott [4], for each $0 \le s \le T$, there exists a positive constant $a(s)$ such that

$$E\left[\exp\left(a(s)\left|\int_0^s \Phi^{-1}(v)\sigma_x(v)dB_v\right|^2\right)\right] < \infty. \qquad (32)$$

Together with the fact that $\{\int_0^s \Phi^{-1}(v)\sigma_x(v)dB_v\}_{0\le s\le T}$ is a martingale, by Example 15.5.6 in Cohen and Elliott [4], the exponential local martingale (17) is a martingale. We also remark that a weak version of Novikov's condition (21) is satisfied. (See Examples 15.5.6 and 15.5.3 in Cohen and Elliott [4].) □

Example 2. (Square-root process)

Suppose that $|\lambda_j| \le \bar{\lambda}_j(t, X_t) = \tilde{\lambda}_{j,t}\sqrt{X_{j,t}},\ j = 1,2$ for some bounded deterministic functions $\tilde{\lambda}_{j,t} > 0$, and X_t is a $\mathcal{R}^2$-valued square root process following a SDE:

$$dX_{j,t} = (a_{j,t} - b_{j,t}X_{j,t})dt + \sigma_{x,j,t}\sqrt{X_{j,t}}dB_{j,t},\ X_{j,0} = x_j > 0,\ j = 1,2, \qquad (33)$$

where $a_{j,t}, b_{j,t}, \sigma_{x,j,t} : [0,T] \to \mathcal{R}$ are bounded functions with $a_{j,t}, b_{j,t} > 0,\ \sigma_{x,j,t} > c_0,\ 0 \le t \le T,\ j = 1,2$ for some $c_0 > 0$, and B_1, B_2 are independent. Then, a weak version of Novikov's condition (21) is satisfied and exponential local martingale (17) is a martingale, which follows from Theorem 3.2 in Shirakawa [24]. Moreover, the moment conditions (18) and (22) are satisfied by Eq. 5.3.17 in Problem 5.3.15 in Karatzas and Shreve [12]. Notice that existence and uniqueness of a strong solution of SDE (33) follow from Theorems 4.1.1, 4.2.3, and 4.2.4 in Ikeda and Watanabe [10] and Proposition 5.2.13 in Karatzas and Shreve [12].

4 Solution of sup-inf/inf-sup problem

We recall that λ_1 and λ_2 represent deviation of the representative agent's probability measure P^{λ_1,λ_2} from the physical measure P. Specifically, the relation between Brownian motions B_1 and B_2 under the physical measure P and Brownian motions $B_1^{\lambda_1,\lambda_2}$ and $B_2^{\lambda_1,\lambda_2}$ under the probability measure P^{λ_1,λ_2} is

$$\begin{aligned} dB_{1,t} &= dB_{1,t}^{\lambda_1,\lambda_2} + \lambda_{1,t}dt, \\ dB_{2,t} &= dB_{2,t}^{\lambda_1,\lambda_2} + \lambda_{2,t}dt, \end{aligned} \tag{34}$$

as described in (13).

Taking the conditional expectations under P^{λ_1,λ_2} with respect to $\mathcal{F}_t$ in both sides of (34), we obtain

$$\begin{aligned} E^{P^{\lambda_1,\lambda_2}}[dB_{1,t}|\mathcal{F}_t] &= \lambda_{1,t}dt, \\ E^{P^{\lambda_1,\lambda_2}}[dB_{2,t}|\mathcal{F}_t] &= \lambda_{2,t}dt, \end{aligned} \tag{35}$$

which implies that under the probability measure P^{λ_1,λ_2}, $dB_{1,t}$ and $dB_{2,t}$ are expected as $\lambda_{1,t}dt$ and $\lambda_{2,t}dt$, respectively.

Thus, the sup-inf (inf-sup) problem in (11)((12)) is considered to be an optimization to determine the views on Brownian motions B_1 and B_2 so that the SDU is minimized with respect to λ_1 for given λ_2, and maximized with respect to λ_2 for given λ_1. In other words, the representative agent is most conservative through the view λ_1 on B_1 for given λ_2, and at the same time, most aggressive through the view λ_2 on B_2 for given λ_1.

For example, X_1 and X_2 can be taken as foreign and domestic market news, respectively. In this case, the market, which is considered to be the representative agent, is cautious about risks related to the foreign news and becomes conservative about taking those risks, while the market has a strong view on risks related to the domestic news and is willing to take those risks aggressively. Similarly, X_1 and X_2 can be taken as foreign exchange and stock news, respectively, which implies that the market is cautious about the foreign exchange risks, however is bullish on the stock news. In either case, the representative agent becomes most conservative through the view λ_1 on Brownian motion B_1 driving X_1, but most aggressive through the view λ_2 on Brownian motion B_2 driving X_2.

This section shows that under certain conditions, sup-inf problem (11) is equivalent to inf-sup problem (12), and these problems are solved by finding a solution of BSDE (23), which is summarized in the following theorem.

Theorem 1. *Let*

$$\lambda_{j,t}^* = (-1)^j |\bar{\lambda}_{j,t}(X_t)| sgn(Z_{j,t}^{\lambda_1^*,\lambda_2^*}), \ (j = 1, 2), \tag{36}$$

and $Y_t^{\lambda_1^*,\lambda_2^*}$ *be a unique solution of BSDE (23). Suppose that a weak version of Novikov's condition (21) is satisfied for* $\bar{\lambda}_j(X), \ j = 1, 2,$ *and*

$$E\left[\sup_{0\le s\le T} |\bar{\lambda}_s|^4\right] < \infty. \tag{37}$$

Then, $(\lambda_1^*, \lambda_2^*) \in \Lambda_1 \times \Lambda_2$ *attains the sup-inf in the problem (11), as well as the inf-sup in the problem (12).*

Remark 1. *In Examples 1 & 2 and Theorem 1, a weak version of Novikov's condition (21) is satisfied for* $\bar{\lambda}_j(X)$*,* $j = 1, 2$*. This condition guarantees that for all* $\lambda = (\lambda_1, \lambda_2)$ *with* $|\lambda_{j,t}| \leq |\bar{\lambda}_j(t, X_t)|$*,* $0 \leq t \leq T$*,* $j = 1, 2$*,* $\{\mathcal{Z}_t(\lambda)\}_{0 \leq t \leq T}$ *is a martingale. Thus, by (7) and (8),* $\Lambda_1(\lambda_2) = \Lambda_1$*,* $\Lambda_2(\lambda_1) = \Lambda_2$*. Also, for* Λ *defined by (3), we have* $\Lambda = \Lambda_1 \times \Lambda_2$*.*

Moreover, by Girsanov's theorem, P^{λ_1, λ_2} *in (2) is well-defined as a probability measure and* B^{λ_1, λ_2} *in (13) is a d-dimensional Brownian motion under* P^{λ_1, λ_2} *for all* $\lambda \in \Lambda = \Lambda_1 \times \Lambda_2$*.*

We also note that, in Example 1, the control sets Λ_1 *and* Λ_2 *are driven by the common stochastic process* X*, while in Example 2,* Λ_1 *and* Λ_2 *depend on stochastic processes* X_1 *and* X_2*, which are independent, respectively.*

Remark 2. *Suppose that there are two agents C and D with a common utility function. The agent C is conservative on* B_1 *with no fundamental uncertainty on* B_2*, i.e.* $\bar{\lambda}_2 \equiv 0$*, while the agent D is aggressive on* B_2 *with no fundamental uncertainty on* B_1*, i.e.* $\bar{\lambda}_1 \equiv 0$*. Then, Theorem 1 can be considered to guarantee a Nash equilibrium for a two-person game, in which each agent reflects his/her own view on the risk associated with a Brownian motion* B_1 *or* B_2 *through optimally choosing the process* λ_1 *or* λ_2*:*

$$\inf_{\lambda_1 \in \Lambda_1} \sup_{\lambda_2 \in \Lambda_2(\lambda_1)} J(\lambda_1, \lambda_2) = \sup_{\lambda_2 \in \Lambda_2(\lambda_1)} \inf_{\lambda_1 \in \Lambda_1} J(\lambda_1, \lambda_2).$$

Proof.

Since $\Lambda_1(\lambda_2) = \Lambda_1$ and $\Lambda_2(\lambda_1) = \Lambda_2$ by a weak version of Novikov's condition (21) as in Remark 1, in the following, we show that

$$Y_0^{\lambda_1^*, \lambda_2^*} = \sup_{\lambda_2 \in \Lambda_2} \inf_{\lambda_1 \in \Lambda_1} Y_0^{\lambda_1, \lambda_2} = \inf_{\lambda_1 \in \Lambda_1} \sup_{\lambda_2 \in \Lambda_2} Y_0^{\lambda_1, \lambda_2}. \tag{38}$$

To prove this, it suffices to show that $(\lambda_1^*, \lambda_2^*) \in \Lambda_1 \times \Lambda_2$ is a saddle point of $J(\lambda_1, \lambda_2)$, meaning that

$$J(\lambda_1^*, \lambda_2) \leq J(\lambda_1^*, \lambda_2^*) \leq J(\lambda_1, \lambda_2^*), \forall \lambda_1 \in \Lambda_1, \lambda_2 \in \Lambda_2. \tag{39}$$

When $(\lambda_1^*, \lambda_2^*)$ is a saddle point, we have

$$\inf_{\lambda_1 \in \Lambda_1} \sup_{\lambda_2 \in \Lambda_2} J(\lambda_1, \lambda_2) \leq \sup_{\lambda_2 \in \Lambda_2} J(\lambda_1^*, \lambda_2) = J(\lambda_1^*, \lambda_2^*) = \inf_{\lambda_1 \in \Lambda_1} J(\lambda_1, \lambda_2^*) \leq \sup_{\lambda_2 \in \Lambda_2} \inf_{\lambda_1 \in \Lambda_1} J(\lambda_1, \lambda_2), \tag{40}$$

which is combined with the opposite relation

$$\inf_{\lambda_1 \in \Lambda_1} \sup_{\lambda_2 \in \Lambda_2} J(\lambda_1, \lambda_2) \geq \sup_{\lambda_2 \in \Lambda_2} \inf_{\lambda_1 \in \Lambda_1} J(\lambda_1, \lambda_2), \tag{41}$$

(40) holds as an equality.

(41) is proved as follows.

For all $(\lambda_1, \lambda_2) \in \Lambda = \Lambda_1 \times \Lambda_2$,

$$\sup_{\lambda_2 \in \Lambda_2} J(\lambda_1, \lambda_2) \geq J(\lambda_1, \lambda_2) \geq \inf_{\lambda_1 \in \Lambda_1} J(\lambda_1, \lambda_2). \tag{42}$$

For any $\epsilon > 0$, there exists $\hat{\lambda}_1 \in \Lambda_1$ such that

$$\inf_{\lambda_1 \in \Lambda_1} \sup_{\lambda_2 \in \Lambda_2} J(\lambda_1, \lambda_2) + \epsilon \geq \sup_{\lambda_2 \in \Lambda_2} J(\hat{\lambda}, \lambda_2). \tag{43}$$

Then, by (42), we have

$$\inf_{\lambda_1 \in \Lambda_1} \sup_{\lambda_2 \in \Lambda_2} J(\lambda_1, \lambda_2) + \epsilon \geq \inf_{\lambda_1 \in \Lambda_1} J(\lambda_1, \lambda_2), \ \forall \lambda_2 \in \Lambda_2, \tag{44}$$

and thus

$$\inf_{\lambda_1 \in \Lambda_1} \sup_{\lambda_2 \in \Lambda_2} J(\lambda_1, \lambda_2) + \epsilon \geq \sup_{\lambda_2 \in \Lambda_2} \inf_{\lambda_1 \in \Lambda_1} J(\lambda_1, \lambda_2). \tag{45}$$

Since (45) holds for all $\epsilon > 0$, we obtain (41).

Next, we show the second inequality in (39),

$$J(\lambda_1^*, \lambda_2^*) - J(\lambda_1, \lambda_2^*) = Y_0^{\lambda_1^*, \lambda_2^*} - Y_0^{\lambda_1, \lambda_2^*} \leq 0. \tag{46}$$

Note that BSDE (23) is rewritten as

$$\begin{aligned} dY_t^{\lambda_1^*, \lambda_2^*} &= -\left(g(t, B, X_t, Y_t^{\lambda_1^*, \lambda_2^*}) + \lambda_{1,t}^* Z_{1,t}^{\lambda_1^*, \lambda_2^*} + \lambda_{2,t}^* Z_{2,t}^{\lambda_1^*, \lambda_2^*} \right) dt + \sum_{j=1}^d Z_{j,t}^{\lambda_1^*, \lambda_2^*} dB_{j,t}, \\ Y_T^{\lambda_1^*, \lambda_2^*} &= \xi. \end{aligned} \tag{47}$$

Here, uniqueness and existence of a solution of BSDE (47) are guaranteed by Proposition 2.

Also note that $Y_t^{\lambda_1, \lambda_2^*}$ is a unique solution of a BSDE

$$\begin{aligned} dY_t^{\lambda_1, \lambda_2^*} &= -\left(g(t, B, X_t, Y_t^{\lambda_1, \lambda_2^*}) + \lambda_{1,t} Z_{1,t}^{\lambda_1, \lambda_2^*} + \lambda_{2,t}^* Z_{2,t}^{\lambda_1, \lambda_2^*} \right) dt + \sum_{j=1}^d Z_{j,t}^{\lambda_1, \lambda_2^*} dB_{j,t}, \\ Y_T^{\lambda_1, \lambda_2^*} &= \xi, \end{aligned} \tag{48}$$

in which existence and uniqueness of a solution are guaranteed by Proposition 1 as in the following discussion. For any $(\lambda_1, \lambda_2^*) \in \Lambda$, $\mathcal{Z}^{\lambda_1, \lambda_2^*}$ is a P-martingale since a weak version of Novikov's condition holds for (λ_1, λ_2^*), which is due to a weak Novikov's condition (21) on $(\bar{\lambda}_1(X), \bar{\lambda}_2(X))$ and the fact that $|\lambda_{1,t}| \leq |\bar{\lambda}_1(X_t)|$ and $|\lambda_{2,t}^*| \leq |\bar{\lambda}_2(X_t)|$ for $0 \leq t \leq T$. Similarly, the condition (18) in Proposition 1 holds for any $(\lambda_1, \lambda_2^*) \in \Lambda$ due to the assumption (37) on the moment of $(\bar{\lambda}_1(X), \bar{\lambda}_2(X))$.

By (47) and (48), we have

$$\begin{aligned} &d(Y_t^{\lambda_1^*, \lambda_2^*} - Y_t^{\lambda_1, \lambda_2^*}) \\ &= -b_t (Y_t^{\lambda_1^*, \lambda_2^*} - Y_t^{\lambda_1, \lambda_2^*}) dt - (\lambda_{1,t}^* - \lambda_{1,t}) Z_1^{\lambda_1^*, \lambda_2^*} dt \\ &\quad + (Z_{1,t}^{\lambda_1^*, \lambda_2^*} - Z_{1,t}^{\lambda_1, \lambda_2^*})(dB_{1,t} - \lambda_{1,t} dt) + (Z_{1,t}^{\lambda_1^*, \lambda_2^*} - Z_{1,t}^{\lambda_1, \lambda_2^*})(dB_{2,t} - \lambda_{2,t}^* dt) \\ &\quad + \sum_{j=3}^d (Z_{j,t}^{\lambda_1^*, \lambda_2^*} - Z_{j,t}^{\lambda_1, \lambda_2^*}) dB_{j,t} \\ &= -b_t (Y_t^{\lambda_1^*, \lambda_2^*} - Y_t^{\lambda_1, \lambda_2^*}) dt - (\lambda_{1,t}^* - \lambda_{1,t}) Z_{1,t}^{\lambda_1^*, \lambda_2^*} dt + \sum_{j=1}^d (Z_{j,t}^{\lambda_1^*, \lambda_2^*} - Z_{j,t}^{\lambda_1, \lambda_2^*}) dB_{j,t}^{\lambda_1, \lambda_2^*}, \end{aligned} \tag{49}$$

where

$$b_t = -\frac{g(t, B, X_t, Y_t^{\lambda_1^*, \lambda_2^*}) - g(t, B, X_t, Y_t^{\lambda_1, \lambda_2^*})}{Y_t^{\lambda_1^*, \lambda_2^*} - Y_t^{\lambda_1, \lambda_2^*}} 1_{\{Y_t^{\lambda_1^*, \lambda_2^*} - Y_t^{\lambda_1, \lambda_2^*} \neq 0\}}, \tag{50}$$

$B^{\lambda_1, \lambda_2^*} = (B_1^{\lambda_1, \lambda_2^*}, \ldots, B_d^{\lambda_1, \lambda_2^*})$ is a d-dimensional Brownian motion under $P^{\lambda_1, \lambda_2^*}$ defined as in (13).

Set $\bar{Y}_t = e^{\int_0^t b_u du}(Y_t^{\lambda_1^*, \lambda_2^*} - Y_t^{\lambda_1, \lambda_2^*})$, $\bar{Z}_{j,t} = e^{\int_0^t b_u du}(Z_{j,t}^{\lambda_1^*, \lambda_2^*} - Z_{j,t}^{\lambda_1, \lambda_2^*})$, $j = 1, \ldots, d$.

Then, we have

$$d\bar{Y}_t = -(\lambda_{1,t}^* - \lambda_{1,t}) Z_{1,t}^{\lambda_1^*, \lambda_2^*} e^{\int_0^t b_u du} dt + \sum_{j=1}^d \bar{Z}_{j,t} dB_{j,t}^{\lambda_1, \lambda_2^*}, \tag{51}$$

and thus

$$\bar{Y}_0 = \int_0^T (\lambda_{1,s}^* - \lambda_{1,s}) Z_1^{\lambda_1^*, \lambda_2^*} e^{\int_0^s b_u du} ds - \sum_{j=1}^d \int_0^T \bar{Z}_{j,s} dB_{j,s}^{\lambda_1, \lambda_2^*}. \tag{52}$$

Next, we note that

$$\left\{ \sum_{j=1}^d \int_0^t \bar{Z}_{j,s} dB_{j,s}^{\lambda_1, \lambda_2^*} \right\}_{0 \le t \le T} \tag{53}$$

is a $P^{\lambda_1, \lambda_2^*}$-martingale by the following discussion.

Firstly, by (51)

$$\sum_{j=1}^d \int_0^t \bar{Z}_{j,s} dB_{j,s}^{\lambda_1, \lambda_2^*} = \bar{Y}_t - \bar{Y}_0 + \int_0^t (\lambda_{1,s}^* - \lambda_{1,s}) Z_{1,s}^{\lambda_1^*, \lambda_2^*} e^{\int_0^s b_u du} ds. \tag{54}$$

We first take the increasing sequence of stopping times $\{\tau_n\}_{n \in \mathbf{N}}$ such that $\tau_n = T$ for sufficiently large n, $P^{\lambda_1, \lambda_2^*}$-a.s. and local martingale (53) stopped by τ_n is a martingale for all $n \in \mathbf{N}$. More concretely, we set

$$\tau_n = \inf \left\{ 0 \le t \le T; \sum_{j=1}^d \int_0^t \bar{Z}_{j,s}^2 ds \ge n \right\}, \ n \in \mathbf{N}. \tag{55}$$

Here, if $\{0 \le t \le T; \sum_{j=1}^d \int_0^t \bar{Z}_{j,s}(\omega)^2 ds \ge n\} = \emptyset$, we set $\tau_n(\omega) = T$.

Since

$$E^{\lambda_1, \lambda_2^*} \left[\sum_{j=1}^d \int_0^t \bar{Z}_{j,s\wedge\tau_n}^2 ds \right] = E^{\lambda_1, \lambda_2^*} \left[\sum_{j=1}^d \int_0^{t\wedge\tau_n} \bar{Z}_{j,s}^2 ds \right] \le n \tag{56}$$

and

$$\sum_{j=1}^d \int_0^t \bar{Z}_{j,s\wedge\tau_n} dB_{j,s}^{\lambda_1, \lambda_2^*} = \sum_{j=1}^d \int_0^{t\wedge\tau_n} \bar{Z}_{j,s} dB_{j,s}^{\lambda_1, \lambda_2^*}, \tag{57}$$

$$\left\{\sum_{j=1}^{d}\int_0^{t\wedge\tau_n}\bar{Z}_{j,s}dB_{j,s}^{\lambda_1,\lambda_2^*}\right\}_{0\le t\le T} \tag{58}$$

is a $P^{\lambda_1,\lambda_2^*}$-martingale.

Then, we observe that for all $0\le t_1\le t_2\le T$,

$$\begin{aligned}\lim_{n\to\infty}E^{\lambda_1,\lambda_2^*}\left[\sum_{j=1}^{d}\int_0^{t_2\wedge\tau_n}\bar{Z}_{j,s}dB_{j,s}^{\lambda_1,\lambda_2^*}\middle|\mathcal{F}_{t_1}\right] &= \lim_{n\to\infty}\sum_{j=1}^{d}\int_0^{t_1\wedge\tau_n}\bar{Z}_{j,s}dB_{j,s}^{\lambda_1,\lambda_2^*}\\ &= \sum_{j=1}^{d}\int_0^{t_1}\bar{Z}_{j,s}dB_{j,s}^{\lambda_1,\lambda_2^*},\end{aligned} \tag{59}$$

and

$$\begin{aligned}&\lim_{n\to\infty}E^{\lambda_1,\lambda_2^*}\left[\sum_{j=1}^{d}\int_0^{t_2\wedge\tau_n}\bar{Z}_{j,s}dB_{j,s}^{\lambda_1,\lambda_2^*}\middle|\mathcal{F}_{t_1}\right]\\ &=\lim_{n\to\infty}E^{\lambda_1,\lambda_2^*}\left[\bar{Y}_{t_2\wedge\tau_n}-\bar{Y}_0+\int_0^{t_2\wedge\tau_n}(\lambda_{1,s}^*-\lambda_{1,s})Z_{1,s}^{\lambda_1^*,\lambda_2^*}e^{\int_0^s b_u du}ds\middle|\mathcal{F}_{t_1}\right]\\ &=E^{\lambda_1,\lambda_2^*}\left[\bar{Y}_{t_2}-\bar{Y}_0+\int_0^{t_2}(\lambda_{1,s}^*-\lambda_{1,s})Z_{1,s}^{\lambda_1^*,\lambda_2^*}e^{\int_0^s b_u du}ds\middle|\mathcal{F}_{t_1}\right]\\ &=E^{\lambda_1,\lambda_2^*}\left[\sum_{j=1}^{d}\int_0^{t_2}\bar{Z}_{j,s}dB_{j,s}^{\lambda_1,\lambda_2^*}\middle|\mathcal{F}_{t_1}\right],\end{aligned} \tag{60}$$

where $E^{\lambda_1,\lambda_2^*}$ denotes the expectation under $P^{\lambda_1,\lambda_2^*}$. In the second equality in (60), we used the dominated convergence theorem and the monotone convergence theorem, since $\bar{Y}$ is uniformly bounded and $\int_0^t(\lambda_{1,s}^*-\lambda_{1,s})Z_{1,s}^{\lambda_1^*,\lambda_2^*}e^{\int_0^s b_u du}ds$ is a negative decreasing process, which is due to the following inequality

$$\lambda_{1,s}^*Z_{1,s}^{\lambda_1^*,\lambda_2^*}=-|\bar{\lambda}_{1,s}||Z_{1,s}^{\lambda_1^*,\lambda_2^*}|\le-|\lambda_{1,s}||Z_{1,s}^{\lambda_1^*,\lambda_2^*}|=-|\lambda_{1,s}Z_{1,s}^{\lambda_1^*,\lambda_2^*}|\le\lambda_1Z_{1,s}^{\lambda_1^*,\lambda_2^*}. \tag{61}$$

Thus, we have

$$E^{\lambda_1,\lambda_2^*}\left[\sum_{j=1}^{d}\int_0^{t_2}\bar{Z}_{j,s}dB_{j,s}^{\lambda_1,\lambda_2^*}\middle|\mathcal{F}_{t_1}\right]=\sum_{j=1}^{d}\int_0^{t_1}\bar{Z}_{j,s}dB_{j,s}^{\lambda_1,\lambda_2^*} \tag{62}$$

Taking the expectation with respect to $P^{\lambda_1,\lambda_2^*}$ in both sides in (52), we have

$$\bar{Y}_0=E^{\lambda_1,\lambda_2^*}\left[\int_0^T(\lambda_{1,s}^*-\lambda_{1,s})Z_{1,s}^{\lambda_1^*,\lambda_2^*}e^{\int_0^s b_u du}ds\right]. \tag{63}$$

Thus, $\bar{Y}_0\le 0$ and

$$Y_0^{\lambda_1^*,\lambda_2^*}\le Y_0^{\lambda_1,\lambda_2^*}. \tag{64}$$

The first inequality in (39) also follows in the same manner.

Therefore, $(\lambda_1^*,\lambda_2^*)$ is a saddle point of $J(\lambda_1,\lambda_2)$. □

5 Asset pricing under fundamental uncertainties

In this section, as an application of sup-inf/inf-sup problem (11)(or (12)) in Section 1, we briefly discuss asset pricing under the probability measure of the representative agent, which is given by the optimal solution of the sup-inf/inf-sup problem.

Let $\mathcal{R}^d$-valued stochastic process X be a state-variable process satisfying

$$dX_t = \mu_x(X_t)dt + \sum_{j=1}^{d} \sigma_{x,j}(X_t)dB_{j,t}, \tag{65}$$

and e be an endowment process of the representative agent, which is a $\mathcal{R}$-valued stochastic process satisfying a SDE

$$de_t = \mu_e(X_t)e_t dt + e_t \sum_{j=1}^{d} \sigma_{e,j}(X_t)dB_{j,t}, \tag{66}$$

where $\mu_x, \sigma_{x,j}, \sigma_{e,j} : \mathcal{R}^d \to \mathcal{R}^d$, $\mu_e : \mathcal{R}^d \to \mathcal{R}$, $j = 1, 2, \dots, d$.

First of all, in equilibrium for an exchange economy with a single (representative) agent, $c = e$, where c is a consumption process of the agent. In our model with fundamental uncertainties, given the representative agent's aggregator (f) and $c = e$ in equilibrium, we solve the corresponding sup-inf/inf-sup problem (11)(or (12)) replacing g in (9) with f, where $f : [0,T] \times \mathcal{R}_+ \times \mathcal{R}^d \times \mathcal{R} \to \mathcal{R}$ satisfies the following conditions: (i) $f(t,e,x,0)$ is bounded, (ii) There exists a constant $L > 0$ such that

$$\begin{aligned} &|f(t,e,x,y) - f(t,e,x,y')| \le L|y - y'|, \\ &\forall y, y' \in \mathcal{R},\ e \in \mathcal{R}_+,\ x \in \mathcal{R}^d,\ t \in [0,T], \end{aligned} \tag{67}$$

and (iii) f is continuously differentiable with respect to e and y.

Then, by Theorem 1, the optimal solution of the sup-inf/inf-sup problem $(\lambda_1^*, \lambda_2^*)$ is expressed as

$$\begin{aligned} \lambda_1^* &= -|\bar{\lambda}_1(X_t)| sgn(Z_{1,t}^{\lambda_1^*,\lambda_2^*}), \\ \lambda_2^* &= |\bar{\lambda}_2(X_t)| sgn(Z_{2,t}^{\lambda_1^*,\lambda_2^*}), \end{aligned} \tag{68}$$

where $Z_t^{\lambda_1^*,\lambda_2^*}$ is a part of a unique solution $(Y^{\lambda_1^*,\lambda_2^*}, Z^{\lambda_1^*,\lambda_2^*})$ of a BSDE:

$$\begin{aligned} dY_t^{\lambda_1^*,\lambda_2^*} &= -\left(f(t,e_t,X_t,Y_t^{\lambda_1^*,\lambda_2^*}) + \lambda_{1,s}^* Z_{1,t}^{\lambda_1^*,\lambda_2^*} + \lambda_{2,s}^* Z_{2,t}^{\lambda_1^*,\lambda_2^*} \right) dt + \sum_{j=1}^{d} Z_{j,t}^{\lambda_1^*,\lambda_2^*} dB_{j,t}, \\ &= -\left(f(t,e_t,X_t,Y_t^{\lambda_1^*,\lambda_2^*}) - |\bar{\lambda}_{1,t}(X_t)||Z_{1,t}^{\lambda_1^*,\lambda_2^*}| + |\bar{\lambda}_{2,t}(X_t)||Z_{2,t}^{\lambda_1^*,\lambda_2^*}|) \right) dt + \sum_{j=1}^{d} Z_{j,t}^{\lambda_1^*,\lambda_2^*} dB_{j,t}, \\ Y_T^{\lambda_1^*,\lambda_2^*} &= \xi. \end{aligned} \tag{69}$$

As a result, we have

$$Y_t^{\lambda_1^*,\lambda_2^*} = E^{P^{\lambda_1^*,\lambda_2^*}} \left[\xi + \int_t^T f(s,e_s,X_s,Y_s^{\lambda_1^*,\lambda_2^*}) ds \middle| \mathcal{F}_t \right], \tag{70}$$

where for all $A \in \mathcal{F}_T$,

$$P^{\lambda_1^*,\lambda_2^*}(A) := E\left[\mathcal{Z}_T^{\lambda_1^*,\lambda_2^*} 1_A\right],$$

$$\mathcal{Z}_t^{\lambda_1^*,\lambda_2^*} = \exp\left(\sum_{j=1}^2 \int_0^t \lambda_{j,s}^* dB_{j,s} - \sum_{j=1}^2 \frac{1}{2}\int_0^t \lambda_{j,s}^{*2} ds\right),\ 0 \le t \le T, \tag{71}$$

since

$$dY_t^{\lambda_1^*,\lambda_2^*} = -f(t, e_t, X_t, Y_t^{\lambda_1^*,\lambda_2^*})dt + \sum_{j=1}^d Z_{j,t}^{\lambda_1^*,\lambda_2^*} dB_{j,t}^{\lambda_1^*,\lambda_2^*},\ Y_T^{\lambda_1^*,\lambda_2^*} = \xi. \tag{72}$$

Let π be a state-price density process under $P^{\lambda_1^*,\lambda_2^*}$. In equilibrium, it holds that

$$\frac{d\pi_t}{\pi_t} = -r_t dt + \sigma_t^\pi \cdot dB_t^{\lambda_1^*,\lambda_2^*},\ \pi_0 = 1, \tag{73}$$

and

$$\pi_t = \exp\left(\int_0^t f_y(s, e_s, X_s, Y_s^{\lambda_1^*,\lambda_2^*})ds\right) f_e(t, e_t, X_t, Y_t^{\lambda_1^*,\lambda_2^*})/f_e(0, e_0, X_0, Y_t^{\lambda_1^*,\lambda_2^*}), \tag{74}$$

where r is a risk-free interest rate and $-\sigma^\pi$ is a market price of risk in equilibrium. (See Eq. (3.5) and (3.6) in Nakamura et al. [19], for instance.) Here, a subscript of f describes taking a partial derivative of f with respect to the variable, i.e. f_y and f_e are partial derivatives of f with respect to y and e, respectively with an assumption of $f_e > 0$. Hence, the equilibrium interest rate r and the market price of risk are obtained by applying Ito's formula to (309).

Next, let D be a cumulative dividend process which is RCLL (right-continuous with left limits) and a $\mathcal{R}$-valued $\{\mathcal{F}_t\}$-progressively measurable process.

Then, a risky asset (claim for dividend streams) price S_t is described under the measure $P^{\lambda_1^*,\lambda_2^*}$ as

$$S_t = E^{P^{\lambda_1^*,\lambda_2^*}}\left[\int_t^T \frac{\pi_s}{\pi_t} dD_s \middle| \mathcal{F}_t\right]. \tag{75}$$

In particular, the zero-coupon bond price $P(t,T)$ is expressed as

$$\begin{aligned} P(t,T) &= E^{P^{\lambda_1^*,\lambda_2^*}}\left[\frac{\pi_T}{\pi_t}\middle|\mathcal{F}_t\right] \\ &= E^P\left[\frac{\pi_T \mathcal{Z}_T^{\lambda_1^*,\lambda_2^*}}{\pi_t \mathcal{Z}_t^{\lambda_1^*,\lambda_2^*}}\middle|\mathcal{F}_t\right]. \end{aligned} \tag{76}$$

We also note that $P(t,T)$ in (313) is rewritten as

$$P(t,T) = E^{Q^{\lambda^*}}\left[\exp\left(-\int_t^T r_s ds\right)\middle|\mathcal{F}_t\right], \tag{77}$$

where Q^{λ^*} is a risk-neutral measure with respect to $P^{\lambda_1^*,\lambda_2^*}$:

$$Q^{\lambda^*}(A) = E^{\lambda_1^*,\lambda_2^*}[\mathcal{Z}_T^{Q^{\lambda^*}} 1_A];\ A \in \mathcal{F}_T,$$

$$\mathcal{Z}_T^{Q^{\lambda^*}} = \exp\left(-\frac{1}{2}\int_0^T |\sigma_s^\pi|^2 ds + \int_0^T \sigma_s^\pi \cdot dB_s^{\lambda_1^*,\lambda_2^*}\right). \tag{78}$$

6 Deterministic endowment volatility (Method by comparison theorems)

In this section, we deal with the case in which the signs of $Z_1^{\lambda_1^*,\lambda_2^*}$ and $Z_2^{\lambda_1^*,\lambda_2^*}$ in the system of forward SDEs (65), (66) and BSDE (69) are determined by comparison theorems. According to the uniquely determined signs of $Z_1^{\lambda_1^*,\lambda_2^*}$ and $Z_2^{\lambda_1^*,\lambda_2^*}$ in λ_1^* and λ_2^*, we can obtain an explicit expression of the short rate r by (308) and (309). As an example, we calculate the short rate r in the case of the stochastic differential/standard power utility with a log-normal endowment process and square-root state-variable processes in Section 6.1.

We consider the following SDEs for the state-variable process X and the endowment process e, as a specific case of (65) and (66). Let the dimension of Brownian motion $d = 3$ and the state-variable process $X = (X_1, X_2, X_3)$. We assume that under the physical measure P, X_1, X_2, X_3 and the endowment process e satisfy SDEs

$$\begin{cases} dX_{1,t} = \mu_1(X_{1,t})dt + \sigma_1(X_{1,t})dB_{1,t}, \ X_{1,0} = x_1, \\ dX_{2,t} = \mu_2(X_{2,t})dt + \sigma_2(X_{2,t})dB_{2,t}, \ X_{2,0} = x_2, \\ dX_{3,t} = \mu_3(X_{3,t})dt + \sigma_3(X_{3,t})dB_{3,t}, \ X_{3,0} = x_3, \\ de_t = \mu_e(X_{1,t}, X_{2,t}, X_{3,t})e_t dt + \sigma_{e,1}e_t dB_{1,t} + \sigma_{e,2}e_t dB_{2,t} + \sigma_{e,3}(X_{3,t})e_t dB_{3,t}, \ e_0 = e, \end{cases} \tag{79}$$

where $\mu_1, \mu_2, \sigma_1, \sigma_2 : \mathcal{R} \to \mathcal{R}$, $\mu_e : \mathcal{R}^3 \to \mathcal{R}$, $\mu_e(x_1, x_2, x_3)$ is decreasing with respect to x_1 and increasing with respect to x_2, $e > 0$, $\sigma_{e,1}, \sigma_{e,2} \in \mathcal{R}$, and $\sigma_{e,3} : \mathcal{R} \to \mathcal{R}$. Moreover, we assume that $\int_0^T |\mu_e(X_{1,t}, X_{2,t}, X_{3,t})|dt < \infty$. Let $f : [0,T] \times \mathcal{R}_+ \times \mathcal{R}^d \times \mathcal{R} \to \mathcal{R}$ be an aggregator satisfying conditions (i)-(iii), and ξ is a bounded $\mathcal{F}_T$-measurable random variable.

The next proposition shows that $sgn(Z_1^{\lambda_1^*,\lambda_2^*})$ and $sgn(Z_2^{\lambda_1^*,\lambda_2^*})$ in the expressions of λ_1^* and λ_2^* in (68) are uniquely determined under certain conditions.

Proposition 3. *Let X_1, X_2, X_3 and e satisfy SDEs (79). Let $v : [0,\infty) \times \mathcal{R}^3 \times \mathcal{R}^+ \to \mathcal{R}$ be a value function defined by*

$$v(t, x_1, x_2, x_3, e) = Y_t^{t,x_1,x_2,x_3,e}, \tag{80}$$

where $(Y_s^{t,x_1,x_2,x_3,e}, Z_s^{t,x_1,x_2,x_3,e})$ is a solution of a BSDE

$$\begin{aligned} dY_s = &-(f(s, e_s^{t,e}, X_{1,s}^{t,x_1}, X_{2,s}^{t,x_2}, X_{3,s}^{t,x_3}, Y_s) - |\bar{\lambda}_{1,s}(X_{1,s}^{t,x_1})||Z_{1,s}| + |\bar{\lambda}_{2,s}(X_{2,s}^{t,x_2})||Z_{2,s}|)ds \\ &+ Z_{1,s}dB_{1,s} + Z_{2,s}dB_{2,s} + Z_{3,s}dB_{3,s}, \ Y_T = \xi, \ t \le s \le T. \end{aligned} \tag{81}$$

We assume that $f_e > 0$, f is decreasing with respect to x_1 and increasing with respect to x_2, and $|\bar{\lambda}_{1,s}(x_1)|$ and $|\bar{\lambda}_{2,s}(x_2)|$ are increasing with regard to x_1, x_2, respectively. Also, we assume that $v(t, x_1, x_2, x_3, e)$ is continuously differentiable with respect to t and twice continuously differentiable with respect to x_1, x_2, x_3 and e. Moreover, we suppose $\sigma_{e,1} > 0$, $\sigma_{e,2} > 0$, $\sigma_1(X_{1,t}) \le 0$, $\sigma_2(X_{2,t}) \ge 0$.

Then, we have

$$sgn(Z_{1,t}^{\lambda_1^*,\lambda_2^*}) = +1, \ sgn(Z_{2,t}^{\lambda_1^*,\lambda_2^*}) = +1. \tag{82}$$

Proof. By a comparison theorem on SDEs (see Proposition 5.2.18 in Karatzas and Shreve [12], for example), when x_1 increases, $X_{1,s}^{t,x_1}$ increases. Since μ_e is decreasing with respect to the first variable, $e_s^{t,e}$ decreases (see Remark 3). As a result, by a slight modification of the proof of Theorem 1, a comparison theorem on a stochastic Lipschitz BSDE holds and it follows that $Y_t^{t,x_1,x_2,x_3,e}$ decreases. In detail, when x_1 increases, $e_s^{t,e}$ decreases, and consequently, the driver $f(s, e_s^{t,e}, X_{1,s}^{t,x_1}, X_{2,s}^{t,x_2}, X_{3,s}^{t,x_3}, y) - |\bar{\lambda}_{1,s}(X_{1,s}^{t,x_1})||z_1| + |\bar{\lambda}_{2,s}(X_{2,s}^{t,x_2})||z_2|$ in (81) decreases since $f_e > 0$ and $-|\bar{\lambda}_{1,s}(X_{1,s}^{t,x_1})|$ also decreases.

Similarly, when x_2 increases, both $X_{2,s}^{t,x_2}$ and $e_s^{t,e}$ increase, and then $Y_t^{t,x_1,x_2,x_3,e}$ increases. When e increases, $e_s^{t,e}$ increases, and then $Y_t^{t,x_1,x_2,x_3,e}$ increases.

Since $v(t, x_1, x_2, x_3, e)$ is continuously differentiable with respect to t and twice continuously differentiable with respect to x_1, x_2, x_3 and e, we have

$$\begin{aligned} &\partial_{x_1} v(t, x_1, x_2, x_3, e) \le 0, \\ &\partial_{x_2} v(t, x_1, x_2, x_3, e) \ge 0, \\ &\partial_e v(t, x_1, x_2, x_3, e) \ge 0. \end{aligned} \tag{83}$$

Also, by applying Ito's formula to $v(t, X_{1,t}, X_{2,t}, X_{3,t}, e_t)$ and comparing the result with (81), we have

$$\begin{pmatrix} Z_{1,t}^{\lambda_1^*,\lambda_2^*} \\ Z_{2,t}^{\lambda_1^*,\lambda_2^*} \\ Z_{3,t}^{\lambda_1^*,\lambda_2^*} \end{pmatrix} = \begin{pmatrix} \sigma_1(X_{1,t})\partial_{x_1} v(t, X_{1,t}, X_{2,t}, X_{3,t}, e_t) + \sigma_{e,1} e_t \partial_e v(t, X_{1,t}, X_{2,t}, X_{3,t}, e_t) \\ \sigma_2(X_{2,t})\partial_{x_2} v(t, X_{1,t}, X_{2,t}, X_{3,t}, e_t) + \sigma_{e,2} e_t \partial_e v(t, X_{1,t}, X_{2,t}, X_{3,t}, e_t) \\ \sigma_3(X_{3,t})\partial_{x_3} v(t, X_{1,t}, X_{2,t}, X_{3,t}, e_t) + \sigma_{e,3}(X_{3,t}) e_t \partial_e v(t, X_{1,t}, X_{2,t}, X_{3,t}, e_t) \end{pmatrix}. \tag{84}$$

By (83),

$$\begin{aligned} Z_{1,t}^{\lambda_1^*,\lambda_2^*} &= \sigma_1(X_{1,t})\partial_{x_1} v(t, X_{1,t}, X_{2,t}, X_{3,t}, e_t) + \sigma_{e,1} e_t \partial_e v(t, X_{1,t}, X_{2,t}, X_{3,t}, e_t) \ge 0, \\ Z_{2,t}^{\lambda_1^*,\lambda_2^*} &= \sigma_2(X_{2,t})\partial_{x_2} v(t, X_{1,t}, X_{2,t}, X_{3,t}, e_t) + \sigma_{e,2} e_t \partial_e v(t, X_{1,t}, X_{2,t}, X_{3,t}, e_t) \ge 0, \end{aligned} \tag{85}$$

and

$$sgn(Z_{1,t}^{\lambda_1^*,\lambda_2^*}) = +1, \;\; sgn(Z_{2,t}^{\lambda_1^*,\lambda_2^*}) = +1. \tag{86}$$

□

Remark 3. *When x_1 increases, e_t decreases by the following reasons. Let $x_1 < \tilde{x}_1$, and $X_{1,t}^{x_1}$ and $X_{1,t}^{\tilde{x}_1}$ be $X_{1,t}$ with initial values x_1 and $\tilde{x}_1$, respectively. Also, let $\tilde{e}$ and e be unique strong solutions of SDEs*

$$d\tilde{e}_t = \mu_e(X_{1,t}^{\tilde{x}_1}, X_{2,t}, X_{3,t})\tilde{e}_t dt + \sigma_{e,1,t}\tilde{e}_t dB_{1,t} + \sigma_{e,2,t}\tilde{e}_t dB_{2,t} + \sigma_{e,3}(X_{3,t})\tilde{e}_t dB_{3,t}, \tag{87}$$

$$de_t = \mu_e(X_{1,t}^{x_1}, X_{2,t}, X_{3,t}) e_t dt + \sigma_{e,1,t} e_t dB_{1,t} + \sigma_{e,2,t} e_t dB_{2,t} + \sigma_{e,3}(X_{3,t}) e_t dB_{3,t}, \tag{88}$$

with $\tilde{e}_0 = e_0 > 0$.

Then, setting $\Delta e_t = \tilde{e}_t - e_t$, we have

$$d(\Delta e_t) = \left(\mu_e(X_{1,t}^{\tilde{x}_1}, X_{2,t}, X_{3,t})\Delta e_t + \{\mu_e(X_{1,t}^{\tilde{x}_1}, X_{2,t}, X_{3,t}) - \mu_e(X_{1,t}^{x_1}, X_{2,t}, X_{3,t})\}e_t\right) dt \\ +\sigma_{e,1}\Delta e_t dB_{1,t} + \sigma_{e,2}\Delta e_t dB_{2,t} + \sigma_{e,3}(X_{3,t})\Delta e_t dB_{3,t}. \quad (89)$$

Let Γ_t be a unique strong solution of a SDE

$$d\Gamma_t = \Gamma_t(-\sigma_e \cdot dB_t - \mu_e(X_{1,t}^{\tilde{x}_1}, X_{2,t}, X_{3,t})dt + |\sigma_e|^2 dt). \quad (90)$$

Then, by Ito's formula, we have

$$d(\Gamma_t \Delta e_t) = (\mu_e(X_{1,t}^{\tilde{x}_1}, X_{2,t}, X_{3,t}) - \mu_e(X_{1,t}^{x_1}, X_{2,t}, X_{3,t}))e_t\Gamma_t dt. \quad (91)$$

Therefore, since $\Gamma_t, e_t > 0,\ 0 \leq \forall t \leq T$ and μ_e is decreasing with respect to the first variable,

$$\Delta e_t = \frac{\Gamma_0}{\Gamma_t}\Delta e_0 + \frac{1}{\Gamma_t}\int_0^t \{\mu_e(X_{1,s}^{\tilde{x}_1}, X_{2,s}, X_{3,s}) - \mu_e(X_{1,s}^{x_1}, X_{2,s}, X_{3,s})\}e_s\Gamma_s ds \leq 0, \quad (92)$$

which indicates that when x_1 increases, e decreases.

6.1 Equilibrium interest rate without fundamental uncertainties

In this subsection, we present expressions of equilibrium interest rate without fundamental uncertainties in the cases of a SDU with four different patterns of parameters with the log-normal endowment process, which correspond to stochastic differential/standard power/log-utility.

Particularly, we show that in the cases without fundamental uncertainties, the short interest rate r calculated by (308) and (309) becomes a deterministic process. We also confirm that under the model with fundamental uncertainties, a stochastic term is added to the interest rate expression in the case of the standard power utility. In other words, the fundamental uncertainties play a role in stochastic movements in the model.

Let $\beta > 0$, $\rho < 1$ and $\alpha < 1$. We consider the following SDU V that satisfies the BSDE with an aggregator $f : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$:

$$dV_t = -f(c_t, V_t)dt + \sigma_{V,t}dB_t,\ V_T = \xi, \quad (93)$$

where

$$f(c_t, V_t) = \begin{cases} \beta\frac{c_t^\rho}{\rho}(1+\alpha V_t)^{(\alpha-\rho)/\alpha} - \frac{\beta}{\rho}(1+\alpha V_t) \\ \text{(stochastic differential power utility}, \rho, \alpha \neq 0,\ \rho \neq \alpha) \\ \frac{\beta}{\rho}(c_t^\rho - 1) - \beta V_t \\ \text{(standard power utility}, \rho = \alpha \neq 0) \\ \beta(1+\alpha V_t)\left[\log c_t - \frac{\log(1+\alpha V_t)}{\alpha}\right] \\ \text{(stochastic differential log-utility}, \rho = 0, \alpha \neq 0) \\ \beta\left[\log c_t - V_t\right] \\ \text{(standard log-utility}, \rho = 0, \alpha = 0), \end{cases} \quad (94)$$

$$
\xi = \begin{cases} \frac{c_T^\alpha - 1}{\alpha} \\ \text{(stochastic differential power utility, standard power utility,} \\ \text{ stochastic differential log-utility)} \\ \log c_T \\ \text{(standard log-utility),} \end{cases} \tag{95}
$$

where B is a d-dimensional Brownian motion ($d \geq 1$) under the physical measure P, and c and V are $\mathcal{R}$-valued $\{\mathcal{F}_t\}$-progressively measurable processes. σ_V is a $\mathcal{R}^d$-valued $\{\mathcal{F}_t\}$-progressively measurable process. For the four types of stochastic differential utilities, see Appendix C. Also, for the equilibrium short rate and the term structure of interest rates without fundamental uncertainties under the stochastic differential log-utility, see Nakamura et al. [18].

Let us suppose that the endowment process e is given as

$$
\frac{de_t}{e_t} = \mu_e dt + \sigma_e \cdot dB_t \tag{96}
$$

with a constant $\mu_e \in \mathcal{R}$ and a constant vector $\sigma_e \in \mathcal{R}^d$.

Since in equilibrium, $c_t = e_t$ for all t, BSDE (93) becomes

$$
dV_t = -f(e_t, V_t)dt + \sigma_{V,t} dB_t, \ V_T = \xi. \tag{97}
$$

Here, we note that the Brownian motion B in (97) corresponds to $B^{\lambda_1^*, \lambda_2^*}$ in (69) where there are no with fundamental uncertainties.

$$
\bar{\lambda}_1(x) = \bar{\lambda}_2(x) = 0. \tag{98}
$$

Thus,

$$
\lambda_{1,t}^* = \lambda_{2,t}^* = 0, \ 0 \leq t \leq T, \tag{99}
$$

and

$$
\frac{de_t}{e_t} = \mu_e dt + \sigma_e \cdot dB_t = \mu_e dt + \sigma_e \cdot dB_t^{\lambda_1^*, \lambda_2^*}. \tag{100}
$$

By applying Ito's formula to (309), in which $Y^{\lambda_1^*, \lambda_2^*}$ is replaced with V, and comparing the drift term with (308), we obtain the equilibrium interest rate r as

$$
r_t = \beta + (1-\rho)\mu_e - \frac{|\sigma_e|^2}{2}(1-\alpha)(2-\rho) \tag{101}
$$

and the market price of risk $-\sigma^\pi$ as $(1-\alpha)\sigma_e$ in the stochastic differential power utility case, that is, when $\alpha, \rho \neq 0$ (see Appendix C for details).

Similarly, we obtain the following expressions for the remaining three cases.

- For the standard power utility case ($\rho = \alpha \neq 0$),

$$
r_t = \beta + (1-\rho)\mu_e - \frac{|\sigma_e|^2}{2}(1-\rho)(2-\rho). \tag{102}
$$

- For the stochastic differential log-utility case ($\rho = 0,\ \alpha \neq 0$),

$$r_t = \beta + \mu_e - |\sigma_e|^2(1-\alpha). \tag{103}$$

- For the standard log-utility case ($\rho = \alpha = 0$),

$$r_t = \beta + \mu_e - |\sigma_e|^2. \tag{104}$$

As we observe in (101)-(104), in the cases of the log-normal endowment process, when there is no effect of fundamental uncertainties, the short rate r is a deterministic process, in particular, with the constant coefficients for e in (96), r is a constant.

However, as we will observe in the next section, r is stochastic process even for standard utilities when there are fundamental uncertainties, since random variables λ_1^*, λ_2^* appear in the drift term of the endowment process under $P^{\lambda_1^*,\lambda_2^*}$:

$$\frac{de_t}{e_t} = (\mu_e + \sigma_{e,1}\lambda_1^* + \sigma_{e,2}\lambda_2^*)dt + \sigma_e \cdot dB_t^{\lambda_1^*,\lambda_2^*}. \tag{105}$$

Particularly, in the case of standard power utility, if we incorporate the effect of fundamental uncertainties, by applying Ito's formula to (309) as we did in (101)-(104), we obtain the term $(1-\rho)(\sigma_{e,1}\tilde{\lambda}_1\sqrt{X_{2,t}} + \sigma_{e,2}\tilde{\lambda}_2\sqrt{X_{2,t}})$ in r in (121) which contributes as a stochastic term in r. Thus, in the case of the log-normal endowment process, the fundamental uncertainties gives the sole stochastic effect on the short interest rate.

6.2 Equilibrium interest rate with fundamental uncertainties

As an application of Proposition 3, we calculate the expression of the short rate r by (308) and (309) when f is an aggregator for the standard power utility. Let $\rho < 1$ with $\rho \neq 0$, and $\beta > 0$. We consider the case of a standard power utility

$$f(e,y) = \frac{\beta}{\rho}(e^\rho - 1) - \beta y, \tag{106}$$

with an endowment process e satisfying

$$\frac{de_t}{e_t} = \mu_e dt + \sigma_{e,1}dB_{1,t} + \sigma_{e,2}dB_{2,t},\ e_0 > 0, \tag{107}$$

where $\sigma_{e,1} > 0, \sigma_{e,2} > 0$, and state-variable processes X_1 and X_2 satisfying

$$\begin{cases} dX_{1,t} = (\mu_{1,1}X_{1,t} + \mu_{1,0})dt + \sigma_1\sqrt{X_{1,t}}dB_{1,t},\ X_{1,0} = x_1 > 0, \\ dX_{2,t} = (\mu_{2,1}X_{2,t} + \mu_{2,0})dt + \sigma_2\sqrt{X_{2,t}}dB_{2,t},\ X_{2,0} = x_2 > 0, \end{cases} \tag{108}$$

where $\mu_{1,0}, \mu_{2,0} > 0$, $\mu_{1,1}, \mu_{2,1} < 0$, $\sigma_1 < 0\sigma_2 > 0$, .

We also assume $\xi = \frac{e_T^\rho - 1}{\rho}$ for the terminal condition, and

$$\bar{\lambda}_1(X_1) = \tilde{\lambda}_1\sqrt{X_{1,t}},\ \bar{\lambda}_2(X_2) = \tilde{\lambda}_2\sqrt{X_{2,t}}, \tag{109}$$

where $\tilde{\lambda}_1 < 0, \tilde{\lambda}_2 > 0$.

Then, we obtain an explicit expression of the short rate r in (308) as follows.

Let

$$\sigma_e = \begin{pmatrix} \sigma_{e,1} \\ \sigma_{e,2} \end{pmatrix},$$
$$\sigma_x(x) = \begin{pmatrix} \sigma_1\sqrt{x_1} & 0 \\ 0 & \sigma_2\sqrt{x_2} \end{pmatrix}, \tag{110}$$

$$\mu_e^*(x) = \mu_e + \sigma_{e,1}\tilde{\lambda}_1\sqrt{x_1} + \sigma_{e,2}\tilde{\lambda}_2\sqrt{x_2},$$
$$\mu_x^*(x) = \begin{pmatrix} (\mu_{1,1} + \sigma_1\tilde{\lambda}_1)x_1 + \mu_{1,0} \\ (\mu_{2,1} + \sigma_2\tilde{\lambda}_2)x_2 + \mu_{2,0} \end{pmatrix}. \tag{111}$$

Firstly, we note that $f_e > 0$ and $|\bar{\lambda}_j(x_j)|$ is increasing with respect to x_j. With some necessary modifications on the endowment process e and the state-variable process X in (107) and (108) or the aggregator f in (106) and the terminal condition ξ as in Remark 4, by applying Proposition 3, we obtain $sgn(Z_1^{\lambda_1^*,\lambda_2^*}) = +1, sgn(Z_2^{\lambda_1^*,\lambda_2^*}) = +1$.

Remark 4. *In the above example, without any modifications, the boundedness on $f(e,0)$ and ξ, and the continuous differentiability of the value function v, which are assumptions in Proposition 3, are not necessarily satisfied. One possible adjustment is that we consider bounded modifications of e and X in SDEs (107) and (108), in particular so that X does not take values in a neighborhood of 0. Then, the boundedness on $f(e,0)$ and ξ, as well as a uniform Lipschitz condition on the driver of BSDE (81), follows, and by Lemma 5.2.3 in Zhang [29], the continuous differentiability of v is obtained. Another approach is that we consider bounded modifications of $f(e,0)$ in (106) and ξ as functionals of e, and assume existence of a classical solution of PDE (A.14) without the jump component in Theorem A.9.22 in Cohen and Elliott [4], which also yields the continuous differentiability of v.*

Then, by (68), we have

$$\lambda_{1,t}^* = \tilde{\lambda}_1\sqrt{X_{1,t}},\ \lambda_{2,t}^* = \tilde{\lambda}_2\sqrt{X_{2,t}}, \tag{112}$$

and

$$dB_{1,t}^{\lambda_1^*,\lambda_2^*} = dB_{1,t} - \tilde{\lambda}_1\sqrt{X_{1,t}}dt,$$
$$dB_{2,t}^{\lambda_1^*,\lambda_2^*} = dB_{2,t} - \tilde{\lambda}_2\sqrt{X_{2,t}}dt. \tag{113}$$

Then, (107) and (108) are rewritten as

$$\frac{de_t}{e_t} = (\mu_{e,t} + \sigma_{e,1}\tilde{\lambda}_1\sqrt{X_{1,t}} + \sigma_{e,2}\tilde{\lambda}_2\sqrt{X_{2,t}})dt + \sigma_{e,1}dB_{1,t}^{\lambda_1^*,\lambda_2^*} + \sigma_{e,2}dB_{2,t}^{\lambda_1^*,\lambda_2^*},\ e_0 > 0, \tag{114}$$

$$\begin{cases} dX_{1,t} = ((\mu_{1,1} + \sigma_1\tilde{\lambda}_1)X_{1,t} + \mu_{1,0})dt + \sigma_1\sqrt{X_{1,t}}dB_{1,t}^{\lambda_1^*,\lambda_2^*},\ X_{1,0} = x_1 > 0, \\ dX_{2,t} = ((\mu_{2,1} + \sigma_2\tilde{\lambda}_2)X_{2,t} + \mu_{2,0})dt + \sigma_2\sqrt{X_{2,t}}dB_{2,t}^{\lambda_1^*,\lambda_2^*},\ X_{2,0} = x_2 > 0, \end{cases} \tag{115}$$

with $\mu_{1,0}, \mu_{2,0} > 0$, $\mu_{1,1} + \sigma_1\tilde{\lambda}_1 < 0$ and $\mu_{2,1} + \sigma_2\tilde{\lambda}_2 < 0$.

Also, $Y^{\lambda_1^*,\lambda_2^*}$ in (72) is written as

$$dY_t^{\lambda_1^*,\lambda_2^*} = -\left[\frac{\beta}{\rho}(e_t^\rho - 1) - \beta Y_t^{\lambda_1^*,\lambda_2^*}\right] dt + \sigma_{V,t} \cdot dB_t^{\lambda_1^*,\lambda_2^*},$$
$$Y_T^{\lambda_1^*,\lambda_2^*} = \frac{e_T^\rho - 1}{\rho}, \tag{116}$$

where

$$\sigma_{V,t} = \begin{pmatrix} Z_{1,t}^{\lambda_1^*,\lambda_2^*} \\ Z_{2,t}^{\lambda_1^*,\lambda_2^*} \end{pmatrix}, \ B_t^{\lambda_1^*,\lambda_2^*} = \begin{pmatrix} B_{1,t}^{\lambda_1^*,\lambda_2^*} \\ B_{2,t}^{\lambda_1^*,\lambda_2^*} \end{pmatrix}. \tag{117}$$

Next, applying Ito's formula to (309), the interest rate r in (308) is given by

$$r_t = -f_y(e_t, Y_t^{\lambda_1^*,\lambda_2^*}) - \frac{\mathcal{L}f_e(e_t, Y_t^{\lambda_1^*,\lambda_2^*})}{f_e(e_t, Y_t^{\lambda_1^*,\lambda_2^*})}. \tag{118}$$

where $\mathcal{L}f_e$ denotes the drift part of f_e, that is

$$\frac{\mathcal{L}f_e}{f_e} = \frac{f_{ee}e\mu_e^* - f_{ey}f + f_{eey}e\sigma_e\sigma_V + \frac{1}{2}f_{eee}e^2|\sigma_e|^2 + \frac{1}{2}f_{eyy}|\sigma_V|^2}{f_e}. \tag{119}$$

Thus, we obtain

$$r = \beta + (1-\rho)\mu_e^*(X_t) - \frac{1}{2}(1-\rho)(2-\rho)|\sigma_e|^2. \tag{120}$$

In particular, the short rate r in (308) is expressed as with $\rho < 1$, $\tilde{\lambda}_1 < 0$, $\tilde{\lambda}_2 > 0$, $\sigma_{e,1} > 0$ and $\sigma_{e,2} > 0$,

$$r_t = \beta + (1-\rho)\mu_e - \frac{1}{2}(1-\rho)(2-\rho)|\sigma_e|^2$$
$$+(1-\rho)(\sigma_{e,1}\tilde{\lambda}_1\sqrt{X_{1,t}} + \sigma_{e,2}\tilde{\lambda}_2\sqrt{X_{2,t}}). \tag{121}$$

By (313), the zero-coupon bond price $P(t,T)$ for maturity T at time t is given by

$$\begin{aligned} P(t,T) &= E^{P^{\lambda_1^*,\lambda_2^*}}\left[\frac{\pi_T}{\pi_t}\middle|\mathcal{F}_t\right] \\ &= E^{P^{\lambda_1^*,\lambda_2^*}}\left[\exp\left(-\int_t^T r_s ds\right)\exp\left(-\frac{1}{2}\int_t^T |\sigma_s^\pi|^2 ds + \int_t^T \sigma_s^\pi \cdot dB_s^{\lambda_1^*,\lambda_2^*}\right)\middle|\mathcal{F}_t\right] \\ &= \exp\left(-\left\{\beta + (1-\rho)\left(\mu_e - \frac{(2-\rho)}{2}|\sigma_e|^2\right)\right\}(T-t)\right) \\ &\quad \times E^{P^{\lambda_1^*,\lambda_2^*}}\left[\exp\left(-\int_t^T (1-\rho)\left(\sigma_{e,1}\tilde{\lambda}_1\sqrt{X_{1,s}} + \sigma_{e,2}\tilde{\lambda}_2\sqrt{X_{2,s}}\right)ds\right)\right. \\ &\quad\quad \left.\times \exp\left(-\frac{1}{2}\int_t^T (1-\rho)^2|\sigma_e|^2 ds - \int_t^T (1-\rho)\sigma_e \cdot dB_s^{\lambda_1^*,\lambda_2^*}\right)\middle|\mathcal{F}_t\right]. \end{aligned} \tag{122}$$

Let $Y(t, t+\tau)$ be a continuously compounded zero yield defined as

$$Y(t, t+\tau) = -\frac{1}{\tau} \log P(t, t+\tau), \tag{123}$$

where $0 < \tau \leq T - t$.

Firstly, as we observe in (121), since $(1-\rho) > 0$ and $\sigma_{e,1}\tilde{\lambda}_1 < 0$, if X_1 increases, the short interest rate r decreases. This implies that under the risk-averse representative agent, if the factor the agent is conservative about increases, then the interest rate falls. Similarly, since $\sigma_{e,2}\tilde{\lambda}_2 > 0$, if X_2 increases, then the short rate r increases, which indicates that if the factor the agent is aggressive about increases, then the interest rate rises.

Moreover, as in (122) and (123), since the continuously compounded zero yield $Y(t, t+\tau)$ depends on the movement of the short rate r in the future, which is also driven by the future movement of X, the shape of the yield curve is determined by X. In detail, if there is an exogenous shock to the factor X_1, which changes the parameters of the SDE driving X_1 in (108), and X_1 takes higher values as time passes, then the short interest rate r in the future decreases, hence, the long-end yield curve goes down, which means that there is a flattening effect to the term structure of interest rates. Conversely, if X_2 takes higher values toward the future, then the short interest rate r in the future increases, and the long-end yield curve goes higher, which means that there is a steepening effect to the term structure of interest rates. We will confirm these observations in numerical examples in the next section by shifting the mean-reversion levels and initial values of X_1 and X_2.

Remark 5. *We remark that BSDE (116) is solved as follows. Let us suppose a functional form of $Y_t^{\lambda_1^*, \lambda_2^*}$ as*

$$Y_t^{\lambda_1^*, \lambda_2^*} = \frac{A(X_t, t)e_t^{\rho} - 1}{\rho}, \tag{124}$$

where $A : \mathcal{R}^2 \times [0, T] \to \mathcal{R}$ is twice and once continuously differentiable with regard to x and t, respectively.

Then, applying Ito's formula to (124) and comparing the diffusion and the drift term with (116), we have

$$\sigma_{V,t} = A(X_t, t)e_t^{\rho} \left[\frac{1}{\rho} \frac{\partial_x A(X_t, t)}{A(X_t, t)} \sigma_x(X_t) + \sigma_e \right], \tag{125}$$

and

$$\begin{aligned}
&\beta + \partial_t A(x,t) - A(x,t) \left[\beta + \frac{\rho(1-\rho)}{2} |\sigma_e|^2 - \rho \mu_e^*(x) \right] \\
&+ \sum_{i=1}^{2} \partial_{x_i} A(x,t) \left[\mu_{x_i}^*(x) + \rho \sum_{k=1}^{2} \sigma_e^k(x) \sigma_{x_i}^k(x) \right] + \frac{1}{2} \sum_{i,j=1}^{2} \partial_{x_i x_j}^2 A(x,t) \sum_{k=1}^{2} \sigma_{x_i}^k(x) \sigma_{x_j}^k(x) = 0, \\
&A_T = 1,
\end{aligned} \tag{126}$$

with $\partial_x A(x,t) \in \mathcal{R}^2$. With some necessary modifications in Remark 6, the linear parabolic PDE (126) has a unique solution with the following stochastic expression by Feynman-Kac *formula*

$$A(x,t) = \mathbf{E}_t^A \left[e^{-\int_t^T r_s^A ds} + \beta \int_t^T e^{-\int_t^u r_s^A ds} du \right] > 0 \ (\beta > 0), \tag{127}$$

where

$$r_t^A = r^A(x_t) = \beta + \frac{\rho(1-\rho)}{2} |\sigma_e|^2 - \rho \mu_e^*(x_t). \tag{128}$$

Here, the conditional expectation $\mathbf{E}_t^A[\cdot]$ is taken under a probability measure P^A with the following SDE for x:

$$dx_t = \mu^A(x_t)dt + \sigma_x(x_t)dB_t^A, \tag{129}$$

where B^A is a d-dimensional Brownian motion under P^A and

$$\mu^A(x_t) = \left[\mu_{x_i}^*(x_t) + \rho \sum_{k=1}^{2} \sigma_e^k \sigma_{x_i}^k(x_t) \right]_{i=1,2}. \tag{130}$$

Remark 6. *The unique existence of a classical solution of PDE (126) is guaranteed by Theorem 2.9.10 in Krylov [13] if $\mu_x^*(x)$ and $\sigma_x(x)$ are smoothly modified so that those first and second derivatives satisfy a polynomial growth condition as $|x|$ tends to $\pm\infty$ and are bounded around $x = 0$ and $r^A(x) \geq 0$.*

7 Numerical examples

This section presents numerical examples of the term structure of interest rates with fundamental uncertainties. Particularly, we provide the case of the log-normal endowment process with the standard power utility case in Section 6.1. We show changes in the yield curve when parameters of the uncertainties related factors shift, which gives implications on yield curve trading by hedge funds and yield curve controls for monetary policies by central banks.

In the following numerical examples, we set the base case parameters as $\beta = 0.01$, $\rho = 0.5$, $\mu_e = 0.01$, $\sigma_{e,1} = 0.10$, $\sigma_{e,2} = 0.10$, $\sigma_{e,3} = 0.10$, $x_1 = 0.10$, $x_2 = 0.10$, $b_1 = -\mu_{1,1} = 0.10$, $b_2 = -\mu_{2,1} = 0.10$, $m_1 = -\frac{\mu_{1,0}}{\mu_{1,1}} = 0.20$, $m_2 = -\frac{\mu_{2,0}}{\mu_{2,1}} = 0.20$. $\sigma_1 = -0.10$, $\sigma_2 = 0.10$, $\tilde{\lambda}_1 = -0.10$, $\tilde{\lambda}_2 = 0.10$, $t = 0$, $T = 30$, in (106)-(109), where we rewrite SDEs of X_1 and X_2 in (108) as mean-reverting processes

$$\begin{aligned} dX_{j,t} &= (\mu_{j,1} X_{j,t} + \mu_{j,0})dt + \sigma_j \sqrt{X_{j,t}} dB_{j,t} \\ &= b_j(m_j - X_{j,t})dt + \sigma_j \sqrt{X_{j,t}} dB_{j,t}, \\ X_{j,0} &= x_j > 0, \ j = 1, 2. \end{aligned} \tag{131}$$

Particularly, we note that $\sigma_{e,1}\tilde{\lambda}_1 < 0$, $\sigma_{e,2}\tilde{\lambda}_2 > 0$ in (121). Hereafter, we call X_1, which is driven by the market risk source B_1 the agent is conservative about, as the conservative factor, and X_2, which is driven by the market risk source B_2 the agent is aggressive about, as the aggressive factor.

Firstly, Table 1 shows that if the mean-reverting level m_2 for the aggressive factor X_2 in (131) shifts from 0.20 to 1.00, the long-end yields move higher, since the higher mean-reverting level makes the future short interest rate r higher as observed in (121), which leads to a yield curve steepening effect.

	Spot	2y	5y	10y	20y	30y
Mean-reversion level m_2 for X_2: 0.20	0.38%	0.37%	0.37%	0.37%	0.36%	0.36%
Mean-reversion level m_2 for X_2: 1.00	0.38%	0.42%	0.46%	0.51%	0.55%	0.58%
Change in zero yields	0.00%	0.05%	0.09%	0.14%	0.19%	0.22%

Table 1: Change in the zero yield curve when the mean-reverting level m_2 for X_2 shifts

Next, Table 2 shows that if the parameter on the degree of aggressiveness $\tilde{\lambda}_2$ in (109) increases from 0.10 to 1.00, the whole yield curve moves up, since the change in $\tilde{\lambda}_2$ increases the effect of the aggressive factor X_2 on rising the equilibrium short rate r as in (121). Moreover, as Tables 3 indicate, if $\tilde{\lambda}_2$ shifts from 0.10 to 1.00, the steepening effect by the change in the mean-reverting level m_2 intensify.

	Spot	2y	5y	10y	20y	30y
Degree of aggressiveness $\tilde{\lambda}_2 : 0.1$	0.38%	0.37%	0.37%	0.37%	0.36%	0.36%
Degree of aggressiveness $\tilde{\lambda}_2 : 1.0$	1.80%	1.91%	2.07%	2.30%	2.67%	2.98%
Change in zero yields	1.42%	1.54%	1.70%	1.93%	2.31%	2.62%

Table 2: Change in the yield curve when $\tilde{\lambda}_2$ shifts from 0.10 to 1.00

	Spot	2y	5y	10y	20y	30y
Mean-reversion level m_2 for X_2: 0.20	1.80%	1.91%	2.07%	2.30%	2.67%	2.98%
Mean-reversion level m_2 for X_2: 1.00	1.80%	2.40%	3.05%	3.86%	5.06%	5.99%
Change in zero yields	0.00%	0.49%	0.98%	1.56%	2.39%	3.02%

Table 3: Change in the yield curve when the mean-reverting level m_2 for X_2 shifts ($\tilde{\lambda}_2 = 1.00$)

Then, we can interpret the effects of the shifts in the parameters $\tilde{\lambda}_2$ and m_2 as follows. (i) ((ii)) An increase (a decrease) in $\tilde{\lambda}_2$, which implies that the agent has more (less) aggressive economic views as observed in (114), makes the equilibrium interest rate higher (lower). (a) ((b)) An increase (a decrease) in m_2, which implies that the aggressive factor becomes larger (smaller) as time passes as in (131), makes the future equilibrium interest rate higher (lower).

Thus, four patterns of the yield curve movements are described by shifts in the aggressiveness parameters as follows. If we assume an upward sloping yield curve as a default curve shape, (i)-(a): the whole yield curve moves up (bond prices decline = bear) while the curve steepens (bear-steepening). (i)-(b): The whole yield curve moves up while the curve flattens (bear-flattening). (ii)-(a): The whole yield curve moves down (bond prices increase = bull) while the curve steepens (bull-steepening). (ii)-(b): The whole yield curve moves down while the curve flattens (bull-flattening). We remark that the same (opposite) effects are observed by the opposite (same) changes in the parameters of the conservative factor X_1. Net effect of those two factors results in the movements of yield curves observed in reality.

For example, Figure 1 shows changes in the yield curve of Japanese government bond (JGB) from 2009/1/5 to 2019/9/30, in which the data are obtained from the website of Ministry of Finance, Japan [16]. As observed in Figure 1, the yield curve moved considerably lower and flattened from 2009 to 2019. Such bull-flattening of a yield curve can be explained by (ii)-(b), that is, the market's aggressiveness in the economic view shrinks and an aggressive factor decreases as time passes. It can also be explained that the market's conservativeness in the economic view grows and a conservative factor increases as time passes, in terms of the conservativeness. Estimation of parameters and identification of the two factors are one of our future research topics.

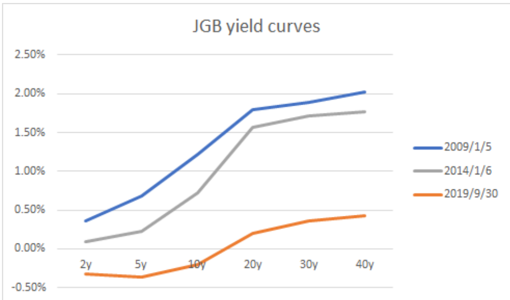

Figure 1: Changes in JGB yield curve from 2009/1/5 to 2019/9/30

As we observed, for central banks, who aims to control the long-end yields lower, it could try to control the uncertainties related factors by effectively making public statements to flatten the yield curve. Similarly, the hedge funds could trade on the yield curve movements by estimating the factors in the model from market data. For example, if they predict that the yield curve flattens, which means that long-end interest rates could go lower compared to the short-end interest rates, then they can buy the long-end bonds and sell the short-end bonds to earn profits.

8 Stochastic endowment volatility

In the following, we consider the cases in which $sgn(Z_1^{\lambda_1^*,\lambda_2^*})$ and $sgn(Z_2^{\lambda_1^*,\lambda_2^*})$ in the expressions of λ_1^* and λ_2^* in (68) are not determined by the comparison theorems as in Section 6, but are obtained by solving BSDE (72) explicitly. Particularly, in Sections 8.1-8.3, we consider three cases, the standard power, the standard log, and the stochastic differential log-utility under random volatility endowment process. Also, we present the expressions of the short rate r. In Section 8.4, we present the explicit expression of the continuously compounded zero yield $Y(t,t+\tau)$ for the standard power and the standard log-utility.

Firstly, we restate SDEs (65), (66) and BSDE (69) in Section 5 in a more general setting under $P^{\lambda_1^*,\lambda_2^*}$. Let us assume the endowment process, which is equivalent to the consumption process in equilibrium, and the state-variable process as follows:

$$\frac{de_t}{e_t} = \mu_e^* dt + \sigma_e(x_t,t)\cdot dB_t^{\lambda_1^*,\lambda_2^*}, \tag{132}$$

$$dx_t = \mu_x^* dt + \sigma_x(x_t,t)\cdot dB_t^{\lambda_1^*,\lambda_2^*}, \tag{133}$$

where $B^{\lambda_1^*,\lambda_2^*}\in\mathcal{R}^d$, $\sigma_e(x,t)\in\mathcal{R}^d$, $\sigma_x(x,t)\in\mathcal{R}^{l\times d}$ $(3\le d\le l)$ and with $\lambda^*\in\mathcal{R}^d$,

$$\mu_e^* = \mu_e(x_t,t) + \lambda^*\cdot\sigma_e(x_t,t), \tag{134}$$

$$\mu_x^* = \mu_x(x_t,t) + \lambda^*\cdot\sigma_x(x_t,t). \tag{135}$$

In particular, as in (68), the stochastic process of λ^* is given as follows: for $j=1,2$,

$$\lambda_{j,t}^* = (-1)^j|\bar{\lambda}_j(x_t,t)|sgn(\sigma_V^j(t)), \tag{136}$$

and $\lambda_{j,t}^*\equiv 0$ for $j=3,\cdots,d$.

Here, σ_V^j is the j-th element of $\sigma_V\in\mathcal{R}^d$, the volatility of the following SDU:

$$dV_t = -f(e_t,V_t)dt + \sigma_V(t)\cdot dB_t^{\lambda_1^*,\lambda_2^*},\ V_T=\xi. \tag{137}$$

where $f:\mathcal{R}_+\times\mathcal{R}\to\mathcal{R}$ be an aggregator satisfying conditions (i)-(iii) in Section 5, and ξ is a bounded $\mathcal{F}_T$-measurable random variable.

Since the forward SDEs (132) and (133) include $\sigma_V^j,\ j=1,2$, we note that the system consisting of (132)-(137) is considered to be coupled FBSDEs.

In the subsequent subsections, we solve coupled FBSDEs (132)-(137) for the standard power and the standard/stochastic differential log-utility in (94) in the following way. We first suppose $sgn(\sigma_V^j(t)) = (-1)^j,\ j=1,2$, which indicates $\lambda_j^* = |\bar{\lambda}_j(x_t,t)|,\ j=1,2$, and make coupled FBSDEs (132)-(137) separated into forward SDEs and a BSDE, since

$$\mu_e^* = \mu_e^*(x,t);\ \mu_x^* = \mu_x^*(x,t) \tag{138}$$

by (134) and (135). Then, we confirm that $sgn(\sigma_V^j(t)) = +1,\ j=1,2$ by explicitly solving BSDE (137) under certain conditions. If these conditions are met, we observe that x and (V,σ_V) also satisfy the original coupled FBSDEs (132)-(137).

Remark 7. *In the following subsections 8.1-8.3, without any modifications, the boundedness on $f(e,0)$ and ξ is not necessarily satisfied. In the same way as in Remark 4, we can consider bounded modifications of e and X in SDEs (132) and (133), in particular so that X does not take values in a neighborhood of 0. Then, the boundedness on $f(e,0)$ and ξ is satisfied. Also, we can consider bounded modifications of $f(e,0)$ and ξ as functionals of e.*

We also remark that even without those modifications, the FBSDEs with f and ξ in Sections 8.1-8.3 are explicitly solved and the expressions of the equilibrium interest rate are obtained. If we further restrict Λ in (3) so that BSDE (24) for (λ_1,λ_2) has a unique solution, *and* $\left\{\sum_{j=1}^d \int_0^t \bar{Z}_{j,s} dB_{j,s}^{\lambda_1,\lambda_2^*}\right\}_{0\le t\le T}$ *in (53) becomes a* $P^{\lambda_1,\lambda_2^*}$*-martingale for any* λ_1 *in* (λ_1,λ_2) *in the set, then the optimality of* $(\lambda_1^*,\lambda_2^*)$ *for the sup-inf/inf-sup problem (11)/(12) is guaranteed as in the proof of Theorem 1. For instance, in the standard log-utility in Section 8.2, if we restrict the set of* λ *to* λ *of the form* $\lambda_j = \hat{\lambda}_j\sqrt{X_j},\ j=1,2,\ \hat{\lambda}_1<0\ \hat{\lambda}_2>0$*, then the BSDE (24) for* (λ_1,λ_2) *is solvable. In this case, the martingale property* $\left\{\sum_{j=1}^d \int_0^t \bar{Z}_{j,s} dB_{j,s}^{\lambda_1,\lambda_2^*}\right\}_{0\le t\le T}$ *in (53) holds and the result of Theorem 1 follows (see Section 8.2 for details).*

8.1 Standard power utility case with random endowment volatility

Firstly, we consider the standard power utility with the aggregator f in (94), which is defined as

$$f(e_t,V_t) = \frac{\beta}{\rho}(e_t^\rho - 1) - \beta V_t. \tag{139}$$

Then, BSDE (137) becomes

$$dV_t = -\left[\frac{\beta}{\rho}(e_t^\rho - 1) - \beta V_t\right]dt + \sigma_V \cdot dB_t^{\lambda_1^*,\lambda_2^*},\ V_T = \frac{e_T^\rho - 1}{\rho}. \tag{140}$$

Let us further suppose a functional form of V as

$$V_t = \frac{A(x_t,t)e_t^\rho - 1}{\rho}, \tag{141}$$

where $A : \mathcal{R}^l \times [0,T] \to \mathcal{R}$ and $A \in \mathcal{C}^{2,1}$.

Then, by applying Ito's formula to (141) and comparing the diffusion and the drift term with (140), we have

$$\sigma_V = \partial_x A(x,t)\cdot\sigma_x(x,t)\frac{e^\rho}{\rho} + A(x,t)e^\rho\sigma_e(x,t) \tag{142}$$

$$= A(x,t)e^\rho\left[\frac{1}{\rho}\frac{\partial_x A(x,t)}{A(x,t)}\sigma_x(x,t) + \sigma_e(x,t)\right] \tag{143}$$

with $\partial_x A(x,t) \in \mathcal{R}^l$ and $\sigma_x(x,t) \in \mathcal{R}^{l\times d}$, and

$$\begin{aligned}&\beta + \partial_t A(x,t) - A(x,t)\left[\beta + \frac{\rho(1-\rho)}{2}|\sigma_e(x,t)|^2 - \rho\mu_e^*(x,t)\right]\\&+\sum_{i=1}^l \partial_{x_i}A(x,t)\left[\mu_{x_i}^*(x,t) + \rho\sum_{k=1}^d \sigma_e^k(x,t)\sigma_{x_i}^k(x,t)\right] + \frac{1}{2}\sum_{i,j=1}^l \partial_{x_ix_j}^2 A(x,t)\sum_{k=1}^d \sigma_{x_i}^k(x,t)\sigma_{x_j}^k(x,t) = 0,\\&A_T = 1.\end{aligned} \tag{144}$$

By Feynman-Kac formula, $A(x,t)$ is expressed as

$$A(x,t) = \mathbf{E}_t^A\left[e^{-\int_t^T r_s^A ds} + \beta\int_t^T e^{-\int_t^u r_s^A ds}du\right] > 0\ (\beta > 0), \tag{145}$$

where

$$r_t^A = r^A(x_t, t) = \beta + \frac{\rho(1-\rho)}{2}|\sigma_e(x_t, t)|^2 - \rho\mu_e^*(x_t, t), \tag{146}$$

with

$$\mu_e^* = \mu_e(x, t) + \lambda^* \cdot \sigma_e(x, t). \tag{147}$$

The conditional expectation $\mathbf{E}_t^A[\cdot]$ is taken under a probability measure P^A with the following SDE for x:

$$dx_t = \mu^A(x_t, t)dt + \sigma_x(x_t, t)dB_t^A, \tag{148}$$

where B^A is a d-dimensional Brownian motion under P^A and

$$\mu^A(x_t, t) = \left[\mu_{x_i}^*(x_t, t) + \rho \sum_{k=1}^{d} \sigma_e^k(x_t, t)\sigma_{x_i}^k(x_t, t)\right]_{i=1,\cdots,l}. \tag{149}$$

Then,

$$\partial_{x_i} A(x, t) = -\mathbf{E}_t^A \left[\left(\int_t^T \partial_{x_i} r_s^A ds\right) e^{-\int_t^T r_s^A ds} + \beta \int_t^T \left(\int_t^u \partial_{x_i} r_s^A ds\right) e^{-\int_t^u r_s^A ds} du\right]. \tag{150}$$

Hence, since $\beta > 0$, the sign of $\partial_{x_i} r_s^A$ determines the sign of $\partial_{x_i} A(x, t)$.

Since $A(x, t) > 0$ and $e > 0$,

- When $\rho \in (0, 1)$ and $k = 1, 2$,
  if $\sigma_e^k(x, t) > 0$ and

  $$\sum_{i=1}^{l} \partial_{x_i} A(x, t)\sigma_{x,i}^k(x, t) > 0, \tag{151}$$

  then $\sigma_V^k > 0$.

- When $\rho < 0$ and $k = 1, 2$,
  if $\sigma_e^k(x, t) > 0$, and

  $$\sum_{i=1}^{l} \partial_{x_i} A(x, t)\sigma_{x,i}^k(x, t) < 0, \tag{152}$$

  then $\sigma_V^k > 0$.

This indicates that if the above conditions are satisfied, then $sgn(\sigma_V^j) = (-1)^j,\ j = 1, 2$, and thus (x, V, σ_V), which is obtained by explicitly solving the separated forward SDEs and BSDE with the predetermined values on $sgn(\sigma_V^j)$, is also a solution of the original coupled FBSDEs (132)-(137).

By applying Ito's formula to (309) and comparing the drift term with (308), we obtain the short interest rate r as

$$r = -f_v - \frac{\mathcal{L}f_e}{f_e} \tag{153}$$

where $\mathcal{L}f_e$ denotes the drift part of f_e, that is

$$\frac{\mathcal{L}f_e}{f_e} = \frac{f_{ee}e\mu_e - f_{ev}f + f_{eev}e\sigma_e\sigma_V + \frac{1}{2}f_{eee}e^2|\sigma_e|^2 + \frac{1}{2}f_{evv}|\sigma_V|^2}{f_e}. \tag{154}$$

Then, we have

$$r = \beta + (1-\rho)\mu_e^*(x,t) - \frac{1}{2}(\rho-1)(\rho-2)|\sigma_e(x,t)|^2. \tag{155}$$

The market price of risk θ is given as the minus of the coefficient of Brownian motion in $\frac{df_e}{f_e}$:

$$\theta = (1-\rho)\sigma_e(x,t). \tag{156}$$

Example 3. (Random endowment volatility with a square-root state-variable process) If e and x in (132)-(135) are stochastic processes with coefficients

$$\mu_{x_1}^*(x_t,t) = (\tilde{\mu}_{x_1,1} + \bar{\lambda}_{1,t}\tilde{\sigma}_{x,1})x_{1,t} + \mu_{x_1,0}, \tag{157}$$
$$\mu_{x_2}^*(x_t,t) = (\tilde{\mu}_{x_2,1} + \bar{\lambda}_{2,t}\tilde{\sigma}_{x,2})x_{2,t} + \mu_{x_2,0}, \tag{158}$$
$$\mu_{x_3}^*(x_t,t) = \tilde{\mu}_{x_3,1}x_{3,t} + \mu_{x_3,0}, \tag{159}$$

$$\mu_e^* = (\tilde{\mu}_{e,1} + \bar{\lambda}_{1,t}\tilde{\sigma}_{e,1})x_1 + (\tilde{\mu}_{e,2} + \bar{\lambda}_{2,t}\tilde{\sigma}_{e,2})x_2 + \tilde{\mu}_{e,3}x_3 + \tilde{\mu}_{e,0}, \tag{160}$$

$$\sigma_{x,i}^k(x,t) = \tilde{\sigma}_{x,i}\sqrt{x_{i,t}},\ i = 1,2,\ i = k, \tag{161}$$
$$\sigma_{x,3}^k(x,t) = \tilde{\sigma}_{x,3},\ i = 3,\ i = k, \tag{162}$$
$$\sigma_{x,i}^k(x,t) = 0,\ i = 1,2,3,\ i \neq k, \tag{163}$$

$$\sigma_e^1(x,t) = \tilde{\sigma}_{e,1}\sqrt{x_{1,t}}, \tag{164}$$
$$\sigma_e^2(x,t) = \tilde{\sigma}_{e,2}\sqrt{x_{2,t}}, \tag{165}$$
$$\sigma_e^3(x,t) = \tilde{\sigma}_{e,3}, \tag{166}$$

with $d = l = 3$, then

$$sgn\left(\partial_{x_k}A(x,t)\sigma_{x,i}^k(x,t)\right) = sgn\left(\left[-\frac{\rho(1-\rho)}{2}\tilde{\sigma}_{e,k}^2 + \rho(\tilde{\mu}_{e,k} + \bar{\lambda}_{k,s}\tilde{\sigma}_{e,k})\right]\tilde{\sigma}_{x,k}\right) \tag{167}$$

holds in (151) and (152) (see Appendix D for details).

Particularly, when $\bar{\lambda}_{j,t} \equiv \bar{\lambda}_j,\ j = 1,2$, the expressions of the short interest rate r and the market price of risk θ in (155) and (156) become

$$\begin{aligned} r &= \beta + (1-\rho)[(\tilde{\mu}_{e,1} + \bar{\lambda}_1\tilde{\sigma}_{e,1})x_{1,t} + (\tilde{\mu}_{e,2} + \bar{\lambda}_2\tilde{\sigma}_{e,2})x_{2,t} + \tilde{\mu}_{e,3}x_{3,t} + \tilde{\mu}_{e,0}] \\ &\quad - \frac{1}{2}(\rho-1)(\rho-2)[\tilde{\sigma}^2_{e,1}x_{1,t} + \tilde{\sigma}^2_{e,2}x_{2,t} + \tilde{\sigma}^2_{e,3}], \end{aligned} \tag{168}$$

and

$$\theta_{j,t} = \begin{cases} (1-\rho)\tilde{\sigma}_{e,j}\sqrt{x_{j,t}},\ j = 1,2, \\ (1-\rho)\tilde{\sigma}_{e,j},\ j = 3. \end{cases} \tag{169}$$

8.2 Standard log-utility case with random endowment volatility

Next, we consider the standard log-utility with the aggregator f in (94), which is defined as

$$f(e_t, V_t) = \beta(\log e_t - V_t), \tag{170}$$

with the terminal condition $\xi = \log e_T$. Particularly, we consider the case of the random volatility endowment and state-variable processes with coefficients (157)-(166) of SDEs (132)-(135) as in Example 3 with $\bar{\lambda}_{j,t} \equiv \bar{\lambda}_j,\ j = 1,2$.

In a similar way as in Section 8.1, we first suppose a functional form of V as

$$V_t = m_1(t)x_{1,t} + m_2(t)x_{2,t} + m_3(t)x_{3,t} + n(t) + \log e_t, \tag{171}$$

where $m_i,\ i = 1,2,3,\ n : [0,T] \to \mathcal{R}$ are differentiable.

Then, applying Ito's formula to (171) and comparing the drift and the diffusion term with (137), we observe the following.

$m_i(t), i = 1,2,3$, and $n(t)$ are solutions of linear ODEs

$$\begin{cases} -\beta m_i(t) + \dot{m}_i(t) + \bar{\mu}_{e,i} - \frac{1}{2}\tilde{\sigma}^2_{e,i} + m_i(t)\bar{\mu}_{x_i,1} = 0,\ m_i(T) = 0,\ i = 1,2, \\ -\beta m_i(t) + \dot{m}_i(t) + \bar{\mu}_{e,i} + m_i(t)\bar{\mu}_{x_i,1} = 0,\ m_i(T) = 0,\ i = 3, \end{cases} \tag{172}$$

and

$$-\beta n(t) + \dot{n}(t) + \mu_{e,0} - \frac{1}{2}\tilde{\sigma}^2_{e,3} + \sum_{i=1}^{3} m_i(t)\mu_{x_i,0} = 0,\ n(T) = 0, \tag{173}$$

where

$$\bar{\mu}_{e,i} = \begin{cases} \mu_{e,i} + \bar{\lambda}_i\tilde{\sigma}_{e,i},\ i = 1,2, \\ \mu_{e,i},\ i = 3, \end{cases}$$

$$\bar{\mu}_{x_i,1} = \begin{cases} \mu_{x_i,1} + \bar{\lambda}_i\tilde{\sigma}_{x,i},\ i = 1,2, \\ \mu_{x_i,1},\ i = 3. \end{cases}$$

Also, we have

$$\sigma_V^j = \sqrt{x_{j,t}}(\tilde{\sigma}_{x,j} m_j(t) + \tilde{\sigma}_{e,j}), \ j = 1, 2. \tag{174}$$

We note that ODEs (172) and (173) are solved as

$$m_i(t) = \begin{cases} \left(\bar{\mu}_{e,i} - \frac{1}{2}\tilde{\sigma}_{e,i}^2\right) \left\{ \frac{1}{-(\beta - \bar{\mu}_{x_i,1})} (e^{-(\beta - \bar{\mu}_{x_i,1})(T-t)} - 1) \right\}, \ i = 1, 2, \\ \bar{\mu}_{e,i} \left\{ \frac{1}{-(\beta - \bar{\mu}_{x_i,1})} (e^{-(\beta - \bar{\mu}_{x_i,1})(T-t)} - 1) \right\}, \ i = 3, \end{cases} \tag{175}$$

and

$$n(t) = \int_t^T \left(\mu_{e,0} - \frac{1}{2}\tilde{\sigma}_{e,3} + \sum_{i=1}^{3} m_i(s)\bar{\mu}_{x_i,0} \right) e^{-\beta(s-t)} ds. \tag{176}$$

Hence, by (174), for $k = 1, 2$, if

$$\tilde{\sigma}_{x,k} m_k(t) + \tilde{\sigma}_{e,k} > 0, \tag{177}$$

then $\sigma_V^k > 0$.

Thus, if this condition is satisfied, $sgn(\sigma_V^k), \ k = 1, 2$ agree with the ones predetermined and x and (V, σ_V) satisfy the original coupled FBSDEs (132)-(137).

In the same way as in (155) and (156) in Section 8.1, the short rate r and the market price of risk θ are given by

$$\begin{aligned} r &= \beta + \mu_e^*(x, t) - |\sigma_e(x, t)|^2 \\ &= \beta + [(\tilde{\mu}_{e,1} + \bar{\lambda}_1 \tilde{\sigma}_{e,1}) x_{1,t} + (\tilde{\mu}_{e,2} + \bar{\lambda}_2 \tilde{\sigma}_{e,2}) x_{2,t} + \tilde{\mu}_{e,3} x_{3,t} + \tilde{\mu}_{e,0}] \\ &\quad - [\tilde{\sigma}_{e,1}^2 x_{1,t} + \tilde{\sigma}_{e,2}^2 x_{2,t} + \tilde{\sigma}_{e,3}^2], \end{aligned} \tag{178}$$

and

$$\theta_{j,t} = \sigma_e^j(x, t) = \begin{cases} \tilde{\sigma}_{e,j} \sqrt{x_{j,t}}, \ j = 1, 2, \\ \tilde{\sigma}_{e,j}, \ j = 3. \end{cases} \tag{179}$$

8.2.1 Optimality of $(\lambda_1^*, \lambda_2^*)$ in sup-inf/inf-sup problem

As in Remark 7, in the case of standard log-utility in (170) with the random volatility endowment and the state-variable process, if we restrict $\lambda \in \Lambda$ in the sup-inf/inf-sup problem (11)/(12) to λ of the form $\lambda_j = \hat{\lambda}_j \sqrt{x_j}, \ j = 1, 2, \ \bar{\lambda}_1 < 0, \ \bar{\lambda}_2 > 0, \ |\hat{\lambda}_j| \leq |\bar{\lambda}_j| \ (j = 1, 2)$, BSDE (24) for (λ_1, λ_2) is solved in the same manner without any further modifications, and it is proved that the optimality of $(\lambda_1^*, \lambda_2^*)$ on the sup-inf/inf-sup problem (11)/(12) holds.

Let us assume the endowment process, which is equivalent to the consumption process in equilibrium, and the state-variable process as follows:

$$\frac{de_t}{e_t} = \mu_e dt + \sigma_e(x, t) \cdot dB_t, \tag{180}$$

$$dx_t = \mu_x dt + \sigma_x(x_t, t) dB_t, \tag{181}$$

where $B^{\lambda_1,\lambda_2} \in \mathcal{R}^3$, $\sigma_e(x,t) \in \mathcal{R}^3$, $\sigma_x(x,t) \in \mathcal{R}^{3\times3}$ with the coefficients

$$\mu_e = \tilde{\mu}_{e,1}x_1 + \tilde{\mu}_{e,2}x_2 + \tilde{\mu}_{e,3}x_3 + \tilde{\mu}_{e,0}, \tag{182}$$

$$\begin{cases} \mu_{x_1}(x_t,t) = \tilde{\mu}_{x_1,1}x_{1,t} + \mu_{x_1,0}, \\ \mu_{x_2}(x_t,t) = \tilde{\mu}_{x_2,1}x_{2,t} + \mu_{x_2,0}, \\ \mu_{x_3}(x_t,t) = \tilde{\mu}_{x_3,1}x_{3,t} + \mu_{x_3,0}, \end{cases} \tag{183}$$

$$\begin{cases} \sigma_e^1(x,t) = \tilde{\sigma}_{e,1}\sqrt{x_{1,t}}, \\ \sigma_e^2(x,t) = \tilde{\sigma}_{e,2}\sqrt{x_{2,t}}, \\ \sigma_e^3(x,t) = \tilde{\sigma}_{e,3}, \end{cases} \tag{184}$$

$$\begin{cases} \sigma_{x,i}^k(x,t) = \tilde{\sigma}_{x,i}\sqrt{x_{i,t}}, \ i = 1,2, \ i = k, \\ \sigma_{x,3}^k(x,t) = \tilde{\sigma}_{x,3}, \ i = 3, \ i = k, \\ \sigma_{x,i}^k(x,t) = 0, \ i = 1,2,3, \ i \neq k, \end{cases} \tag{185}$$

By Girsanov's theorem, under P^{λ_1,λ_2}, SDEs (252) and (253) are rewritten as

$$\frac{de_t}{e_t} = \mu_e^\lambda dt + \sigma_e(x,t) \cdot dB_t^{\lambda_1,\lambda_2}, \tag{186}$$

$$dx_t = \mu_x^\lambda dt + \sigma_x(x_t,t)dB_t^{\lambda_1,\lambda_2}, \tag{187}$$

where

$$\mu_e^\lambda = \mu_e(x,t) + \lambda \cdot \sigma_e(x,t), \tag{188}$$

$$\mu_x^\lambda = \mu_x(x,t) + \sigma_x(x,t)\lambda, \tag{189}$$

$$\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} \hat{\lambda}_1\sqrt{x_1} \\ \hat{\lambda}_2\sqrt{x_2} \\ 0 \end{pmatrix}. \tag{190}$$

In detail, μ_e^λ and μ_x^λ in (188) and (189) are expressed as

$$\mu_e^\lambda = (\tilde{\mu}_{e,1} + \hat{\lambda}_1\tilde{\sigma}_{e,1})x_1 + (\tilde{\mu}_{e,2} + \hat{\lambda}_2\tilde{\sigma}_{e,2})x_2 + \tilde{\mu}_{e,3}x_3 + \tilde{\mu}_{e,0}, \tag{191}$$

$$\begin{cases} \mu_{x_1}^\lambda(x_t,t) = (\tilde{\mu}_{x_1,1} + \hat{\lambda}_1\tilde{\sigma}_{x,1})x_{1,t} + \mu_{x_1,0}, \\ \mu_{x_2}^\lambda(x_t,t) = (\tilde{\mu}_{x_2,1} + \hat{\lambda}_2\tilde{\sigma}_{x,2})x_{2,t} + \mu_{x_2,0}, \\ \mu_{x_3}^\lambda(x_t,t) = \tilde{\mu}_{x_3,1}x_{3,t} + \mu_{x_3,0}. \end{cases} \tag{192}$$

Let us define $(V^{\lambda_1,\lambda_2}, \sigma_V^{\lambda_1,\lambda_2})$ as

$$\begin{cases} V_t^{\lambda_1,\lambda_2} = m_1^{\lambda_1,\lambda_2}(t)x_{1,t} + m_2^{\lambda_1,\lambda_2}(t)x_{2,t} + m_3^{\lambda_1,\lambda_2}(t)x_{3,t} + n(t)^{\lambda_1,\lambda_2} + \log e_t, \\ \sigma_V^{\lambda_1,\lambda_2,j} = \sqrt{x_{j,t}}(\tilde{\sigma}_{x,j} m_j^{\lambda_1,\lambda_2}(t) + \tilde{\sigma}_{e,j}),\ j = 1, 2, \end{cases} \tag{193}$$

where

$$m_i^{\lambda_1,\lambda_2}(t) = \begin{cases} \left(\bar{\mu}_{e,i} - \frac{1}{2}\tilde{\sigma}_{e,i}^2\right)\left\{\frac{1}{-(\beta - \bar{\mu}_{x_i,1})}(e^{-(\beta - \bar{\mu}_{x_i,1})(T-t)} - 1)\right\},\ i = 1, 2, \\ \bar{\mu}_{e,i}\left\{\frac{1}{-(\beta - \bar{\mu}_{x_i,1})}(e^{-(\beta - \bar{\mu}_{x_i,1})(T-t)} - 1)\right\},\ i = 3, \end{cases} \tag{194}$$

$$n(t)^{\lambda_1,\lambda_2} = \int_t^T \left(\mu_{e,0} - \frac{1}{2}\tilde{\sigma}_{e,3} + \sum_{i=1}^3 m_i^{\lambda_1,\lambda_2}(s)\bar{\mu}_{x_i,0}\right) e^{-\beta(s-t)} ds, \tag{195}$$

with

$$\bar{\mu}_{e,i} = \begin{cases} \mu_{e,i} + \hat{\lambda}_i \tilde{\sigma}_{e,i},\ i = 1, 2, \\ \mu_{e,i},\ i = 3, \end{cases}$$

$$\bar{\mu}_{x_i,1} = \begin{cases} \mu_{x_i,1} + \hat{\lambda}_i \tilde{\sigma}_{x,i},\ i = 1, 2, \\ \mu_{x_i,1},\ i = 3. \end{cases}$$

Note that $m_i^{\lambda_1,\lambda_2}(t)$ and $n^{\lambda_1,\lambda_2}(t)$ satisfy ODEs

$$\begin{cases} -\beta m_i^{\lambda_1,\lambda_2}(t) + \dot{m}_i^{\lambda_1,\lambda_2}(t) + \bar{\mu}_{e,i} - \frac{1}{2}\tilde{\sigma}_{e,i}^2 + m_i^{\lambda_1,\lambda_2}(t)\bar{\mu}_{x_i,1} = 0,\ m_i^{\lambda_1,\lambda_2}(T) = 0,\ i = 1, 2, \\ -\beta m_i^{\lambda_1,\lambda_2}(t) + \dot{m}_i^{\lambda_1,\lambda_2}(t) + \bar{\mu}_{e,i} + m_i^{\lambda_1,\lambda_2}(t)\bar{\mu}_{x_i,1} = 0,\ m_i^{\lambda_1,\lambda_2}(T) = 0,\ i = 3, \end{cases} \tag{196}$$

and

$$-\beta n^{\lambda_1,\lambda_2}(t) + \dot{n}^{\lambda_1,\lambda_2}(t) + \mu_{e,0} - \frac{1}{2}\tilde{\sigma}_{e,3}^2 + \sum_{i=1}^3 m_i^{\lambda_1,\lambda_2}(t)\mu_{x_i,0} = 0,\ n(T) = 0. \tag{197}$$

Particularly when $\hat{\lambda}_j = \bar{\lambda}_j,\ j = 1, 2$, we write (λ_1, λ_2) as $(\lambda_1^*, \lambda_2^*)$, that is,

$$\begin{aligned} \lambda_{1,t}^* &= \bar{\lambda}_1\sqrt{x_{1,t}}, \\ \lambda_{2,t}^* &= \bar{\lambda}_2\sqrt{x_{2,t}}. \end{aligned} \tag{198}$$

The next proposition shows that $(V^{\lambda_1,\lambda_2}, \sigma_{V^{\lambda_1,\lambda_2}})$ is a solution of a BSDE under P^{λ_1,λ_2} and a stochastic Lipschitz BSDE under P.

Proposition 4. *$(V^{\lambda_1,\lambda_2}, \sigma_{V^{\lambda_1,\lambda_2}})$ is a solution of a BSDE*

$$\begin{aligned} dV_t^{\lambda_1,\lambda_2} &= -f(c_t, V_t^{\lambda_1,\lambda_2})dt + \sigma_{V^{\lambda_1,\lambda_2}} dB_t^{\lambda_1,\lambda_2} \\ &= -\left[f(c_t, V_t^{\lambda_1,\lambda_2}) + \sum_{j=1}^2 \hat{\lambda}_j\sqrt{x_{j,t}}\sigma_{V^{\lambda_1,\lambda_2},j}\right]dt + \sigma_{V^{\lambda_1,\lambda_2}} dB_t,\ V_T^{\lambda_1,\lambda_2} = \log c_T. \end{aligned} \tag{199}$$

Particularly, when $\hat{\lambda}_j = \bar{\lambda}_j$, $j = 1, 2$, $(V^{\lambda_1,\lambda_2}, \sigma_{V^{\lambda_1,\lambda_2}})$ is a solution of a BSDE

$$dV_t^{\lambda_1^*,\lambda_2^*} = -\left[f(c_t, V_t^{\lambda_1^*,\lambda_2^*}) + \sum_{j=1}^{2} \bar{\lambda}_j \sqrt{x_{j,t}} \sigma_{V^{\lambda_1^*,\lambda_2^*},j}\right] dt + \sigma_{V^{\lambda_1^*,\lambda_2^*}} dB_t, \ V_T^{\lambda_1^*,\lambda_2^*} = \log c_T. \quad (200)$$

Proof. By applying Ito's formula to V^{λ_1,λ_2} in (193) and using (196) and (197), we obtain (199). □

We further assume the condition

$$\begin{cases} \tilde{\sigma}_{x,k} m_k^{\lambda_1,\lambda_2}(t) + \tilde{\sigma}_{e,k} > 0, \ k = 1, \\ \tilde{\sigma}_{x,k} m_k^{\lambda_1,\lambda_2}(t) + \tilde{\sigma}_{e,k} > 0, \ k = 2. \end{cases} \quad (201)$$

Then, $\sigma_V^{\lambda_1,\lambda_2,1} > 0$ and $\sigma_V^{\lambda_1,\lambda_2,2} > 0$ by (193), and in particular,

$$\begin{aligned} \bar{\lambda}_1 \sqrt{x_{1,t}} \sigma_{V^{\lambda_1^*,\lambda_2^*},1} &= -\bar{\lambda}_1 \sqrt{x_{1,t}} |\sigma_{V^{\lambda_1^*,\lambda_2^*},1}| \leq \hat{\lambda}_1 \sqrt{x_{1,t}} \sigma_{V^{\lambda_1^*,\lambda_2^*},1}, \\ \bar{\lambda}_2 \sqrt{x_{2,t}} \sigma_{V^{\lambda_1^*,\lambda_2^*},2} &= \bar{\lambda}_2 \sqrt{x_{2,t}} |\sigma_{V^{\lambda_1^*,\lambda_2^*},2}| \geq \hat{\lambda}_2 \sqrt{x_{2,t}} \sigma_{V^{\lambda_1^*,\lambda_2^*},2}, \end{aligned} \quad (202)$$

for all $|\hat{\lambda}_j| \leq |\bar{\lambda}_j|, j = 1, 2$, in (200).

Theorem 2. *Let $\lambda \in \Lambda$ in the sup-inf/inf-sup problem (11)/(12) be λ of the form $\lambda_j = \hat{\lambda}_j x_j$, $j = 1, 2$, $|\bar{\lambda}_j| \geq |\hat{\lambda}_j| > 0$. Suppose that c and x are solutions of SDEs (252) and (253), respectively, with their coefficients (183) - (184). We also suppose f in (170) and assume that the condition (201) is satisfied.*

Then, $(\lambda_1^*, \lambda_2^*)$ attains the sup-inf in the problem (11), as well as the inf-sup in the problem (12).

Proof. We show that $(\lambda_1^*, \lambda_2^*)$ is a saddle point. In particular, we show

$$V_t^{\lambda_1^*,\lambda_2^*} - V_t^{\lambda_1,\lambda_2^*} \leq 0. \quad (203)$$

We first note that under $P^{\lambda_1^*,\lambda_2^*}$, SDE (200) becomes

$$dV_t^{\lambda_1^*,\lambda_2^*} = -\left[f(c_t, V_t^{\lambda_1^*,\lambda_2^*}) + (\bar{\lambda}_1 - \hat{\lambda}_1) x_{j,t} \sigma_{V^{\lambda_1^*,\lambda_2^*},1}\right] dt + \sigma_{V^{\lambda_1^*,\lambda_2^*}} dB_t^{\lambda_1,\lambda_2^*}, \ V_T^{\lambda_1^*,\lambda_2^*} = \log c_T, \quad (204)$$

Let

$$\bar{V}_t = V_t^{\lambda_1^*,\lambda_2^*} - V_t^{\lambda_1,\lambda_2^*}. \quad (205)$$

Then, by Ito's formula, we have

$$\begin{aligned} d\bar{V}_t &= -\left[f(c_t, V_t^{\lambda_1^*,\lambda_2^*}) - f(c_t, V_t^{\lambda_1,\lambda_2^*}) + (\bar{\lambda}_1 - \hat{\lambda}_1) x_{1,t} \sigma_{V^{\lambda_1^*,\lambda_2^*},1}\right] dt \\ &\quad + (\sigma_{V^{\lambda_1^*,\lambda_2^*}} - \sigma_{V^{\lambda_1,\lambda_2^*}}) dB_t^{\lambda_1,\lambda_2^*} \\ &= +\beta \bar{V}_t dt - (\bar{\lambda}_1 - \hat{\lambda}_1) x_{1,t} \sigma_{V^{\lambda_1^*,\lambda_2^*},1} dt + (\sigma_{V^{\lambda_1^*,\lambda_2^*}} - \sigma_{V^{\lambda_1,\lambda_2^*}}) dB_t^{\lambda_1,\lambda_2^*}, \ \bar{V}_T = 0, \end{aligned} \quad (206)$$

and

$$d(e^{-\beta t}\bar{V}_t) = -(\bar{\lambda}_1 - \hat{\lambda}_1)x_{1,t}\sigma_{V^{\lambda_1^*,\lambda_2^*},1}e^{-\beta t}dt + (\sigma_{V^{\lambda_1^*,\lambda_2^*}} - \sigma_{V^{\lambda_1,\lambda_2^*}})e^{-\beta t}dB_t^{\lambda_1,\lambda_2^*}. \tag{207}$$

Thus,

$$-e^{-\beta t}\bar{V}_t = \int_t^T -(\bar{\lambda}_1 - \hat{\lambda}_1)x_{1,s}\sigma_{V^{\lambda_1^*,\lambda_2^*},1}e^{-\beta s}ds + \int_t^T (\sigma_{V^{\lambda_1^*,\lambda_2^*}} - \sigma_{V^{\lambda_1,\lambda_2^*}})e^{-\beta s}dB_s^{\lambda_1,\lambda_2^*}, \tag{208}$$

and by taking the conditional expectation $E_t^{\lambda_1,\lambda_2^*}[\cdot]$ in both sides of (288), we have

$$-e^{-\beta t}\bar{V}_t = E_t^{\lambda_1,\lambda_2^*}\left[\int_t^T -(\bar{\lambda}_1 - \hat{\lambda}_1)x_{1,s}\sigma_{V^{\lambda_1^*,\lambda_2^*},1}e^{-\beta s}ds\right] \geq 0. \tag{209}$$

Hence, $\bar{V}_t \leq 0$. Here, we used the fact that

$$\left\{\int_0^t (\sigma_{V^{\lambda_1^*,\lambda_2^*},s} - \sigma_{V^{\lambda_1,\lambda_2^*},s})e^{-\beta s}dB_s^{\lambda_1,\lambda_2^*}\right\}_{0\leq t\leq T} \tag{210}$$

is a martingale under $P^{\lambda_1,\lambda_2^*}$, which is proved in the following lemma.

Lemma 1.

$$\left\{\int_0^t (\sigma_{V^{\lambda_1^*,\lambda_2^*},s} - \sigma_{V^{\lambda_1,\lambda_2^*},s})e^{-\beta s}dB_s^{\lambda_1,\lambda_2^*}\right\}_{0\leq t\leq T} \tag{211}$$

is a $P^{\lambda_1,\lambda_2^*}$-martingale.

Proof. As in the proof of Theorem 1 in Section 4, by (202) and (288), it suffices to show that $E^{\lambda_1,\lambda_2^*}\left[\sup_{0\leq t\leq T}|V_t^{\lambda_1^*,\lambda_2^*}|\right]$ and $E^{\lambda_1,\lambda_2^*}\left[\sup_{0\leq t\leq T}|V_t^{\lambda_1,\lambda_2^*}|\right] < \infty$.

Firstly, $E^{\lambda_1,\lambda_2^*}[\sup_{0\leq t\leq T}|V_t^{\lambda_1,\lambda_2^*}|] < \infty$ is proved as follows. Under $P^{\lambda_1,\lambda_2^*}$, SDEs (252) and (253) become

$$\frac{de_t}{e_t} = \hat{\mu}_e^* dt + \sigma_e(x_t,t)\cdot dB_t^{\lambda_1,\lambda_2^*}, \tag{212}$$

$$dx_t = \hat{\mu}_x^* dt + \sigma_x(x_t,t)\cdot dB_t^{\lambda_1,\lambda_2^*}, \tag{213}$$

where

$$\hat{\mu}_{x_1}^*(x_t,t) = (\tilde{\mu}_{x_1,1} + \hat{\lambda}_1\tilde{\sigma}_{x_1})x_{1,t} + \mu_{x_1,0}, \tag{214}$$

$$\hat{\mu}_{x_2}^*(x_t,t) = (\tilde{\mu}_{x_2,1} + \bar{\lambda}_2\tilde{\sigma}_{x_2})x_{2,t} + \mu_{x_2,0}, \tag{215}$$

$$\hat{\mu}_{x_3}^*(x_t,t) = \tilde{\mu}_{x_3,1}x_{3,t} + \mu_{x_3,0}, \tag{216}$$

$$\hat{\mu}_e^* = (\tilde{\mu}_{e,1} + \hat{\lambda}_1\tilde{\sigma}_{e,1})x_1 + (\tilde{\mu}_{e,2} + \bar{\lambda}_2\tilde{\sigma}_{e,2})x_2 + \tilde{\mu}_{e,3}x_3 + \tilde{\mu}_{e,0}, \tag{217}$$

$$\sigma^k_{x,i}(x,t) = \tilde{\sigma}_{x,i}\sqrt{x_{i,t}},\ i = 1,2,\ i = k, \tag{218}$$
$$\sigma^k_{x,3}(x,t) = \tilde{\sigma}_{x,3},\ i = 3,\ i = k, \tag{219}$$
$$\sigma^k_{x,i}(x,t) = 0,\ i = 1,2,3,\ i \neq k, \tag{220}$$

$$\sigma^1_e(x,t) = \tilde{\sigma}_{e,1}\sqrt{x_{1,t}}, \tag{221}$$
$$\sigma^2_e(x,t) = \tilde{\sigma}_{e,2}\sqrt{x_{2,t}}, \tag{222}$$
$$\sigma^3_e(x,t) = \tilde{\sigma}_{e,3}. \tag{223}$$

By (193), we have

$$\sup_{0\le t\le T}|V^{\lambda_1^*,\lambda_2^*}_t| = \sum_{i=1}^{3}\sup_{0\le t\le T}|m_i^{\lambda_1^*,\lambda_2^*}(t)|\sup_{0\le t\le T}|x_{i,t}| + \sup_{0\le t\le T}|n^{\lambda_1^*,\lambda_2^*}(t)| + \sup_{0\le t\le T}|\log e_t|. \tag{224}$$

Since $\log e_t$ is written as

$$\log e_t = \log e_0 + \int_0^t (\sum_{i=1}^{3}\alpha_i x_{i,s} + \beta)ds + \int_0^t \left(\tilde{\sigma}_{e,1}\sqrt{x_{1,t}}dB^{\lambda_1,\lambda_2^*}_{1,s} + \tilde{\sigma}_{e,2}\sqrt{x_{2,t}}dB^{\lambda_1,\lambda_2^*}_{2,s} + \tilde{\sigma}_{e,3}dB^{\lambda_1,\lambda_2^*}_{3,s}\right), \tag{225}$$

where $\alpha_i, i = 1,2,3, \beta \in \mathcal{R}$, we have

$$\begin{aligned}\sup_{0\le t\le T}|\log e_t| \le \log e_0 + \left(\beta + \sum_{i=1}^{3}|\alpha_i|\sup_{0\le t\le T}|x_{i,s}|\right)T + \sum_{i=1}^{2}|\tilde{\sigma}_{e,i}|\sup_{0\le t\le T}\left|\int_0^t \sqrt{x_{i,s}}dB^{\lambda_1,\lambda_2^*}_{i,s}\right| \\ + |\tilde{\sigma}_{e,3}|\sup_{0\le t\le T}|B^{\lambda_1,\lambda_2^*}_{3,t}|.\end{aligned} \tag{226}$$

By Burkholder's inequality, we have

$$\begin{aligned}E^{\lambda_1,\lambda_2^*}\left[\sup_{0\le t\le T}\left|\int_0^t \sqrt{x_{i,s}}dB^{\lambda_1,\lambda_2^*}_s\right|\right] \le KE^{\lambda_1,\lambda_2^*}\left[\left(\int_0^t |x_{i,s}|ds\right)^{\frac{1}{2}}\right] \\ \le K\left(E^{\lambda_1,\lambda_2^*}\left[\left(\int_0^t |x_{i,s}|ds\right)\right]\right)^{\frac{1}{2}} \le KT\left(E^{\lambda_1,\lambda_2^*}\left[\sup_{0\le t\le T}|x_{i,t}|\right]\right)^{\frac{1}{2}},\ i = 1,2.\end{aligned} \tag{227}$$

Since $E^{\lambda_1,\lambda_2^*}[\sup_{0\le t\le T}|x_{i,t}|] < \infty,\ i = 1,2,3$, and $E^{\lambda_1,\lambda_2^*}[\sup_{0\le t\le T}|B^{\lambda_1,\lambda_2^*}_{3,t}|] < \infty$ (for instance, see Problem 5.3.15 in Karatzas and Shreve [12]), we have $E^{\lambda_1,\lambda_2^*}[\sup_{0\le t\le T}|\log e_t|] < \infty$. Hence, $E^{\lambda_1,\lambda_2^*}[\sup_{0\le t\le T}|V^{\lambda_1^*,\lambda_2^*}_t|] < \infty$. $E^{\lambda_1,\lambda^*}\left[\sup_{0\le t\le T}|V^{\lambda_1,\lambda_2^*}_t|\right] < \infty$ is also proved in the same manner. $\square$

$V^{\lambda_1^*,\lambda_2^*}_t - V^{\lambda_1^*,\lambda_2}_t \ge 0$. is also proved in the same manner. Hence, $(\lambda_1^*, \lambda_2^*)$ is a saddle point and the optimality of $(\lambda_1^*, \lambda_2^*)$ on the sup-inf/inf-sup problem (11)/(12) holds. $\square$

8.3 Stochastic differential log-utility case with random endowment volatility

Finally, we consider the case of the stochastic differential log-utility f in (94), which is defined as

$$f(e_t, V_t) = \beta(1 + \alpha V_t)\left[\log e_t - \frac{\log(1 + \alpha V_t)}{\alpha}\right], \tag{228}$$

with the terminal condition $\xi = \frac{e_T^\alpha - 1}{\alpha}$ in BSDE (137). As in Section 8.2, we consider the case of the random volatility endowment and state-variable processes with coefficients (157)-(166) and $\bar{\lambda}_{j,t} \equiv \bar{\lambda}_j,\ j = 1, 2$ of their SDEs (132)-(135). Also, for the equilibrium short rate and the term structure of interest rates without fundamental uncertainties under the stochastic differential log-utility, see Nakamura et al. [18].

We suppose

$$V_t = \frac{A(x_t, t)e_t^\alpha - 1}{\alpha}, \tag{229}$$

where

$$A(x_t, t) = \exp(\alpha\{m_1(t)x_1 + m_2(t)x_2 + m_3(t)x_3 + n(t)\}),$$

and $m_i, i = 1, 2, 3, n : [0, T] \to \mathcal{R}$ are differentiable.

Then, by applying Ito's formula to (229) and comparing the drift and the diffusion term with (137), we observe the following.

$m_1(t), m_2(t), m_3(t), n(t)$ are obtained by solving Riccati equations

$$-\beta m_i(t) + \dot{m}_i(t) + \bar{\mu}_{e,i} - \frac{1}{2}(1 - \alpha)\tilde{\sigma}_{e,i}^2 + m_i(t)\bar{\mu}_{x_i,1} + \alpha m_i(t)\tilde{\sigma}_{e,i}\tilde{\sigma}_{x,i} + \alpha m_i^2(t)\tilde{\sigma}_{x,i}^2 = 0,$$
$$m_i(T) = 0,\ i = 1, 2, 3, \tag{230}$$

and

$$-\beta n(t) + \dot{n}(t) + m_1(t)\mu_{x_1,0} + m_2(t)\mu_{x_2,0} + m_3(t)\mu_{x_3,0} = 0,\ n(T) = 0, \tag{231}$$

where

$$\begin{aligned}
\bar{\mu}_{e,1} &= \mu_{e,1} + \bar{\lambda}_1\tilde{\sigma}_{e,1},\\
\bar{\mu}_{e,2} &= \mu_{e,2} + \bar{\lambda}_2\tilde{\sigma}_{e,2},\\
\bar{\mu}_{e,3} &= \mu_{e,3},\\
\bar{\mu}_{x_1,1} &= \mu_{x_1,1} + \bar{\lambda}_1\tilde{\sigma}_{x,1},\\
\bar{\mu}_{x_2,1} &= \mu_{x_2,1} + \bar{\lambda}_2\tilde{\sigma}_{x,2},\\
\bar{\mu}_{x_3,1} &= \mu_{x_3,1}.
\end{aligned} \tag{232}$$

Next, let

$$\begin{aligned}
A_i &= \alpha\tilde{\sigma}_{x,i}^2,\\
B_i &= (\alpha\tilde{\sigma}_{e,i}\tilde{\sigma}_{x,i} + \bar{\mu}_{x_i,1} - \beta),\\
C_i &= \bar{\mu}_{e,i} - \frac{1}{2}(1 - \alpha)\tilde{\sigma}_{e,i}^2,
\end{aligned} \tag{233}$$

and assume that $B_i^2 - 4A_iC_i \geq 0$, and set γ_i as a solution of

$$A_i\gamma_i^2 + B_i\gamma_i + C_i = 0. \tag{234}$$

Then, the solution of Riccati equations (230) is obtained as

$$m_i(t) = \frac{1}{\frac{-A_i\gamma_i - B_i}{\gamma_i(2A_i\gamma_i+B_i)e^{-(2A_i\gamma_i+B_i)(T-t)} - \frac{A_i}{2A_i\gamma_i+B_i}}} + \gamma_i. \tag{235}$$

Moreover, we have

$$\sigma_V^i(t) = e_t^\alpha A\sqrt{x_{i,t}}(\tilde{\sigma}_{x,i}m_i(t) + \tilde{\sigma}_{e,i}). \tag{236}$$

Hence, when $0 < \alpha < 1$, $sgn(\sigma_V^1) = +1$ as long as $m_1(t)\tilde{\sigma}_{x,1} + \sigma_{e,1} > 0$ and $sgn(\sigma_V^2) = +1$ as long as $m_2(t)\tilde{\sigma}_{x,2}+\sigma_{e,2} > 0$. Also, when $\alpha < 0$, $sgn(\sigma_V^1) = +1$ as long as $m_1(t)\tilde{\sigma}_{x,1}+\sigma_{e,1} < 0$ and $sgn(\sigma_V^2) = +1$ as long as $m_2(t)\tilde{\sigma}_{x,2} + \sigma_{e,2} < 0$.

If these conditions are satisfied, the signs of $\sigma_V^j,\ j = 1, 2$ agree with the predetermined ones, and x and (V, σ_V) also satisfy the original coupled FBSDEs (132)-(137).

In the same way as in (155) and (156) in Section 8.1, the short rate r and the market price of risk θ are given by

$$\begin{aligned} r &= \beta + \mu_e^*(x,t) - |\sigma_e(x,t)|^2 + \alpha\sigma_e(x,t)[\sigma_e(x,t) + \sigma_x(x,t)m(t)] \\ &= +\beta + ((\tilde{\mu}_{e,1} + \bar{\lambda}_{1,t}\tilde{\sigma}_{e,1})x_{1,t} + (\tilde{\mu}_{e,2} + \bar{\lambda}_{2,t}\tilde{\sigma}_{e,2})x_{2,t} + \tilde{\mu}_{e,3}x_{3,t} + \tilde{\mu}_{e,0}) \\ &\quad - (1-\alpha)(\tilde{\sigma}_{e,1}^2 x_{1,t} + \tilde{\sigma}_{e,2}^2 x_{2,t} + \tilde{\sigma}_{e,3}^2) \\ &\quad + \alpha\{m_1(t)\tilde{\sigma}_{e,1}\tilde{\sigma}_{x,1}x_{1,t} + m_2(t)\tilde{\sigma}_{e,2}\tilde{\sigma}_{x,2}x_{2,t} + m_3(t)\tilde{\sigma}_{e,3}\tilde{\sigma}_{x,3}\}, \end{aligned} \tag{237}$$

where

$$\sigma_e(x,t) + \sigma_x(x,t)m(t) = \left(\sigma_e^k(x,t) + \sum_{i=1}^{3} m_i(t)\sigma_{x_i}^k(x,t)\right)_{k=1,2,3}, \tag{238}$$

and

$$\begin{aligned} \theta_t &= \sigma_e(x,t) - \alpha[\sigma_e(x,t) + \sigma_x(x,t)m(t)] \\ &= (1-\alpha)\sigma_e(x,t) - \alpha\sigma_x(x,t)m(t) \\ &= \begin{cases} \{(1-\alpha)\tilde{\sigma}_{e,j} - \alpha\tilde{\sigma}_{x,j}m_j(t)\}\sqrt{x_{j,t}},\ j = 1, 2, \\ (1-\alpha)\tilde{\sigma}_{e,j} - \alpha\tilde{\sigma}_{x,j}m_j(t),\ j = 3. \end{cases} \end{aligned} \tag{239}$$

8.4 Term structure of interest rates for the random endowment process

As we observed in (168),(178) and (237) in Sections 8.1-8.3, in the case of the square-root state-variable process and the random volatility endowment process with coefficients (157)-(166) of SDEs (132)-(135) with $\bar{\lambda}_{j,t} \equiv \bar{\lambda}_j,\ j = 1, 2$, r becomes a linear functional with respect to x. Moreover, under the risk-neutral measure Q^{λ^*} in (315), x also remains a square-root process, which is observed by the expressions of the market price of risk $\theta = -\sigma^\pi$ calculated as (169),(179), and (239). In particular, when f is the standard power or the standard log-utility as in Sections 8.1 and 8.2, we can calculate the zero yield $Y(t, t+\tau)$ explicitly as follows. We remark that in the case of the stochastic differential log-utility in Section 8.3, since the coefficients of the SDE for x are time-dependent, we calculate the zero yield $Y(t, t+\tau)$ by Monte Carlo simulations.

To obtain a time-t zero yield with maturity T under a square-root model, we evaluate the following zero-coupon bond price:

$$P(t,T) = E^{Q^{\lambda^*}}[e^{-\int_t^T r_u du}|\mathcal{F}_t] = E^{Q^{\lambda^*}}[e^{\int_t^T x_{1,u}du}|\mathcal{F}_t]E^{Q^{\lambda^*}}[e^{-\int_t^T x_{2,u}du}|\mathcal{F}_t], \tag{240}$$

where

$$\begin{aligned} r_t &= x_{2,t} - x_{1,t}, & (241)\\ dx_{j,t} &= b_j(\theta_j - x_{j,t})dt + \sigma_j\sqrt{x_{j,t}}dB_{j,t}^{Q^{\lambda^*}}\ ;\ x_{j,0} > 0, & (242)\\ &= (a_j - b_j x_{j,t})dt + \sigma_j\sqrt{x_{j,t}}dB_{j,t}^{Q^{\lambda^*}}\ ,\ j = 1,2, & (243) \end{aligned}$$

with independent Brownian motions $B_j^{Q^{\lambda^*}}$ under a risk-neutral probability, and with constants $\sigma_1 < 0$, $\sigma_2 > 0$, $\theta_j > 0$, $b_j > 0$, $a_j = b_j\theta_j$,$j = 1,2$, such that $b_1^2 > 2\sigma_1^2$. Then, we obtain with $\tau = T - t$, $\gamma_1 = \frac{\sqrt{b_1^2 - 2\sigma_1^2}}{2}$ $(b_1^2 > 2\sigma_1^2)$, $\gamma_2 = \frac{\sqrt{b_2^2+2\sigma_2^2}}{2}$,

$$P_1(t,T) = E^{Q^{\lambda^*}}[e^{\int_t^T x_{1,u}du}|\mathcal{F}_t] = e^{x_{1,t}B_1(t,T)+A_1(t,T)}, \tag{244}$$

$$P_2(t,T) = E^{Q^{\lambda^*}}[e^{-\int_t^T x_{2,u}du}|\mathcal{F}_t] = e^{-x_{2,t}B_2(t,T)-A_2(t,T)}, \tag{245}$$

$$B_j(t,T) = \frac{\sinh(\gamma_j\tau)}{\gamma_j\cosh(\gamma_j\tau) + \frac{b_j}{2}\sinh(\gamma_j\tau)}\ (j = 1,2), \tag{246}$$

$$A_1(t,T) = \frac{2a_1}{\sigma_1^2}\log\left\{\frac{\gamma_1\exp(\frac{b_1\tau}{2})}{\gamma_1\cosh(\gamma_1\tau) + \frac{b_1}{2}\sinh(\gamma_1\tau)}\right\}, \tag{247}$$

$$A_2(t,T) = \frac{-2a_2}{\sigma_2^2}\log\left\{\frac{\gamma_2\exp(\frac{b_2\tau}{2})}{\gamma_2\cosh(\gamma_2\tau) + \frac{b_2}{2}\sinh(\gamma_2\tau)}\right\}, \tag{248}$$

where $\sinh(y) = \frac{e^y - e^{-y}}{2}$ and $\cosh(y) = \frac{e^y+e^{-y}}{2}$.

Thus, a time-t zero yield with term τ is given by

$$Y(t,t+\tau) = \frac{-1}{\tau}\log P(t,T) = \frac{1}{\tau}\left[-\{x_{1,t}B_1(t,T) + A_1(t,T)\} + \{x_{2,t}B_2(t,T) + A_2(t,T)\}\right]. \tag{249}$$

Remark 8. *If $r = \alpha_2 x_2 - \alpha_1 x_1$, $\alpha_j > 0$, $j = 1,2$, we define $\hat{x}_j = \alpha_j x_j$*

$$d\hat{x}_{j,t} = (\hat{a}_j - b_j\hat{x}_{j,t})dt + \hat{\sigma}_j\sqrt{\hat{x}_{j,t}}dB_t^{Q^{\lambda^*}}\ ;\ \hat{a}_j = \alpha_j x_j;\ \hat{\sigma}_j = \sqrt{\alpha_j}\sigma_j, \tag{250}$$

and then, the same formula as above is applied.

8.5 Gaussian quadratic-Gaussian interest rate model

This section explains a Gaussian quadratic-Gaussian interest rate model in which the state-variable process is Gaussian and the equilibrium short rate is expressed as a quadratic function of the state-variable process. Particularly, we consider a case of the standard log-utility.

8.5.1 Optimality of $(\lambda_1^*, \lambda_2^*)$ in sup-inf/inf-sup problem

Firstly, let us specify the consumption (endowment) process as follows:

$$\frac{dc_t}{c_t} = \mu_c dt + \sigma_c(x,t) \cdot dB_t, \tag{251}$$

$$\tag{252}$$

$$dx_t = \mu_x dt + \sigma_x(x_t,t) dB_t, \tag{253}$$

with $B^{\lambda_1^*,\lambda_2^*} \in \mathcal{R}^d$, $\sigma_c(x,t) \in \mathcal{R}^d$, $\sigma_x(x,t) \in \mathcal{R}^{l \times d}$ $(d = l = 3)$, where

$$\mu_c(x,t) = \mu_0 + l_1(t)x_1 + l_2(t)x_2 + \mu_1 x_1^2 + \mu_2 x_2^2 + \mu_3 x_3, \tag{254}$$

$$\sigma_c^j(x,t) = \sigma_{c,j} x_j, (j = 1,2),\ \sigma_{c,j}, (j = 3), \tag{255}$$

$$\mu_{x,j}(x,t) = a_j - b_j x_j, \tag{256}$$

$$\sigma_{x,j}(x,t) = \begin{pmatrix} \sigma_{x,1} & 0 & 0 \\ 0 & \sigma_{x,2} & 0 \\ 0 & 0 & \sigma_{x,3} \end{pmatrix}. \tag{257}$$

Next, let us define for given $\bar{\lambda}_1 < 0$, $\bar{\lambda}_2 > 0$, $\lambda_{1,t} = \hat{\lambda}_1 x_{1,t}$, $\lambda_{2,t} = \hat{\lambda}_2 x_{2,t}$, $|\hat{\lambda}_1| \leq |\bar{\lambda}_1|$, $|\hat{\lambda}_2| \leq \bar{\lambda}_2$. Since $x_{1,t}, x_{2,t}$ are Gaussian process under P, a probability measure P^{λ_1,λ_2} in (2) is well defined. Also, let

$$\lambda_{1,t}^* = \bar{\lambda}_1 x_{1,t}, \tag{258}$$

$$\lambda_{2,t}^* = \bar{\lambda}_2 x_{2,t}, \tag{259}$$

$$\tag{260}$$

and then, $P^{\lambda_1^*,\lambda_2^*}$ is well defined by (2), too.

By Girsanov's theorem, B^{λ_1,λ_2} in (13) is a Brownian motion under P^{λ_1,λ_2}. (252) and (253) are rewritten as

$$\frac{dc_t}{c_t} = \mu_c^\lambda dt + \sigma_c(x,t) \cdot dB_t^{\lambda_1,\lambda_2}, \tag{261}$$

$$dx_t = \mu_x^\lambda dt + \sigma_x(x_t,t) dB_t^{\lambda_1,\lambda_2}, \tag{262}$$

where

$$\mu_c^\lambda = \mu_c(x,t) + \lambda \cdot \sigma_c(x,t), \tag{263}$$

$$\mu_x^\lambda = \mu_x(x,t) + \sigma_x(x,t)\lambda. \tag{264}$$

$$\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} \hat{\lambda}_1 x_1 \\ \hat{\lambda}_2 x_2 \\ 0 \end{pmatrix}. \tag{265}$$

Now, we specify the function f as the following standard log-utility:

$$f(c, V) = \beta(\log c - V), \ \beta > 0. \tag{266}$$

Let us define $(V^{\lambda_1,\lambda_2}, \sigma_{V^{\lambda_1,\lambda_2}})$ as

$$\begin{cases} V_t^{\lambda_1,\lambda_2} & = m_1^{\hat{\lambda}_1,\hat{\lambda}_2}(t) x_{1,t}^2 + m_2^{\hat{\lambda}_1,\hat{\lambda}_2}(t) x_{2,t}^2 + m_3(t) x_{3,t} \\ & \quad + n^{\hat{\lambda}_1,\hat{\lambda}_2}(t) + \log c_t, \\ \sigma_{V^{\hat{\lambda}_1,\hat{\lambda}_2},j} & = \begin{cases} x_{j,t}(\sigma_{c,j} + 2m_j^{\hat{\lambda}_1,\hat{\lambda}_2}(t)\sigma_{x,j}), \ j = 1,2, \\ \sigma_{c,j} + m_j^{\hat{\lambda}_1,\hat{\lambda}_2}(t)\sigma_{x,j}, \ j = 3. \end{cases} \end{cases} \tag{267}$$

where

$$m_j^{\hat{\lambda}_1,\hat{\lambda}_2}(t) = \left(\mu_j^{\hat{\lambda}_1,\hat{\lambda}_2} - \frac{1}{2}\sigma_{c,j}^2\right) \int_t^T e^{-(\beta + 2b_j^{\hat{\lambda}_1,\hat{\lambda}_2})(s-t)} ds, \ j = 1,2, \tag{268}$$

$$m_3(t) = \mu_3 \int_t^T e^{-(\beta + b_3)(s-t)} ds, \tag{269}$$

$$n^{\hat{\lambda}_1,\hat{\lambda}_2}(t) = \int_t^T \left(\mu_0 + \sum_{j=1}^2 m_i^{\hat{\lambda}_1,\hat{\lambda}_2}(s)\sigma_{x,j}^2 + m_3 a_3 - \frac{1}{2}\sigma_{c,3}^2\right) e^{-\beta(s-t)} ds, \tag{270}$$

and

$$l_j(t) = -2m_j^{\hat{\lambda}_1,\hat{\lambda}_2}(t) a_j, \ j = 1,2, \tag{271}$$

with

$$\begin{aligned} \mu_j^{\hat{\lambda}_1,\hat{\lambda}_2} &= \mu_j + \hat{\lambda}_j \sigma_{c,j}, \ j = 1,2, \\ b_j^{\hat{\lambda}_1,\hat{\lambda}_2} &= b_j - \hat{\lambda}_j \sigma_{x,j}, \ j = 1,2. \end{aligned} \tag{272}$$

Note that $m_j^{\hat{\lambda}_1,\hat{\lambda}_2}(t), \ j = 1,2, m_3(t)$, and $n^{\hat{\lambda}_1,\hat{\lambda}_2}(t)$ in (268)-(270) satisfy a system of ODEs below:

$$\dot{m}_j^{\hat{\lambda}_1,\hat{\lambda}_2}(t) - m_j^{\hat{\lambda}_1,\hat{\lambda}_2}(t)(\beta + 2b_j^{\hat{\lambda}_1,\hat{\lambda}_2}) + \mu_j^{\hat{\lambda}_1,\hat{\lambda}_2} - \frac{1}{2}\sigma_{c,j}^2 = 0; \ m_j^{\hat{\lambda}_1,\hat{\lambda}_2}(T) = 0, \ j = 1,2, \tag{273}$$

$$\dot{m}_3(t) - m_3(t)(\beta + b_3) + \mu_3 = 0; \ m_3(T) = 0, \tag{274}$$

and

$$\dot{n}^{\hat{\lambda}_1,\hat{\lambda}_2}(t) - \beta n^{\hat{\lambda}_1,\hat{\lambda}_2}(t) + \mu_0 + \sum_{j=1}^2 m_j^{\hat{\lambda}_1,\hat{\lambda}_2}(t)\sigma_{x,j}^2 + m_3(t) a_3 - \frac{1}{2}\sigma_{c,3}^2 = 0; \ n^{\hat{\lambda}_1,\hat{\lambda}_2}(T) = 0. \tag{275}$$

The next proposition shows that $(V^{\lambda_1,\lambda_2}, \sigma_{V^{\lambda_1,\lambda_2}})$ in (267) satisfies two BSDEs:

Proposition 5. *$(V^{\lambda_1,\lambda_2}, \sigma_{V^{\lambda_1,\lambda_2}})$ in (267) is a solution of BSDE:*

$$dV_t^{\lambda_1,\lambda_2} = -f(c_t, V_t^{\lambda_1,\lambda_2})dt + \sigma_{V^{\lambda_1,\lambda_2}} dB_t^{\lambda_1,\lambda_2}, \; V_T^{\lambda_1,\lambda_2} = \log c_T. \tag{276}$$

Moreover, $(V^{\lambda_1,\lambda_2}, \sigma_{V^{\lambda_1,\lambda_2}})$ satisfies a stochastic Lipschitz BSDE:

$$dV_t^{\lambda_1,\lambda_2} = -\left[f(c_t, V_t^{\lambda_1,\lambda_2}) + \sum_{j=1}^{2} \hat{\lambda}_j x_{j,t} \sigma_{V^{\lambda_1,\lambda_2},j}\right] dt + \sigma_{V^{\lambda_1,\lambda_2}} dB_t, \; V_T^{\lambda_1,\lambda_2} = \log c_T. \tag{277}$$

Particularly, when $\hat{\lambda}_1 = \bar{\lambda}_1, \hat{\lambda}_2 = \bar{\lambda}_2$, $(V^{\lambda_1^*,\lambda_2^*}, \sigma_{V^{\lambda_1^*,\lambda_2^*}})$ defined as

$$\begin{cases} V_t^{\lambda_1^*,\lambda_2^*} & = m_1^{\bar{\lambda}_1,\bar{\lambda}_2}(t)x_{1,t}^2 + m_2^{\bar{\lambda}_1,\bar{\lambda}_2}(t)x_{2,t}^2 + m_3(t)x_{3,t} \\ & \quad + n^{\bar{\lambda}_1,\bar{\lambda}_2}(t) + \log c_t, \\ \sigma_{V^{\lambda_1^*,\lambda_2^*},j} & = \begin{cases} x_{j,t}(\sigma_c^j + 2m_j^{\bar{\lambda}_1,\bar{\lambda}_2}(t)\sigma_{x,j}), \; j = 1, 2, \\ \sigma_c^j + m_j^{\bar{\lambda}_1,\bar{\lambda}_2}(t)\sigma_{x,j}, \; j = 3, \end{cases} \end{cases} \tag{278}$$

is a solution of BSDE

$$dV_t^{\lambda_1^*,\lambda_2^*} = -\left[f(c_t, V_t^{\lambda_1^*,\lambda_2^*}) + \sum_{j=1}^{2} \bar{\lambda}_j x_{j,t} \sigma_{V^{\lambda_1^*,\lambda_2^*},j}\right] dt + \sigma_{V^{\lambda_1^*,\lambda_2^*}} dB_t, \; V_T^{\lambda_1^*,\lambda_2^*} = \log c_T. \tag{279}$$

Remark 9. *This implies that $V_t^{\lambda_1,\lambda_2}$ satisfies (9).*

Proof. By applying Ito's formula to $V_t^{\lambda_1,\lambda_2}$ in (267) and using (274) and (275), we obtain BSDEs (276) and (277). □

Moreover, if conditions,

$$\begin{aligned} \sigma_c^j + 2m_j^{\bar{\lambda}_1,\bar{\lambda}_2}(t)\sigma_{x,j} > 0, \; j = 1, \\ \sigma_c^j + 2m_j^{\bar{\lambda}_1,\bar{\lambda}_2}(t)\sigma_{x,j} > 0, \; j = 2 \end{aligned} \tag{280}$$

are satisfied, then the equation for $\sigma_{V^{\lambda_1^*,\lambda_2^*},j}$ in (278) implies that

$$sgn(\sigma_{V^{\lambda_1^*,\lambda_2^*},j}) = sgn(x_{j,t}). \tag{281}$$

Then,

$$\bar{\lambda}_j x_{j,t} \sigma_{V^{\lambda_1^*,\lambda_2^*},j} = \begin{cases} \bar{\lambda}_1 |x_{1,t}||\sigma_{V^{\lambda_1^*,\lambda_2^*},1}| \leq \hat{\lambda}_1 x_{1,t} \sigma_{V^{\lambda_1^*,\lambda_2^*},1} \;, j = 1 \\ +\bar{\lambda}_2 |x_{2,t}||\sigma_{V^{\lambda_1^*,\lambda_2^*},2}| \geq \hat{\lambda}_2 x_{2,t} \sigma_{V^{\lambda_1^*,\lambda_2^*},2} \;, j = 2 \end{cases} \tag{282}$$

holds in stochastic Lipschitz BSDE (279).

The next theorem shows the optimality of $V^{\lambda_1^*,\lambda_2^*}$ on the sup-inf/inf-sup problem.

Theorem 3. *Let $\lambda \in \Lambda$ in the sup-inf/inf-sup problem (11)/(12) be of the form $\lambda_j = \hat{\lambda}_j x_j$, $j = 1, 2$, $\bar{\lambda}_j \geq |\hat{\lambda}_j| > 0$, and assume f in the problem as (266). Suppose that the condition (280) is satisfied. Suppose also that c and x are solutions of SDEs (261) and (262), respectively, with their coefficients (254) - (257). Then, $(\lambda_1^*, \lambda_2^*)$ attains the sup-inf in problem the (11), as well as the inf-sup in the problem (12).*

Proof.

We show that $(\lambda_1^*, \lambda_2^*)$ is a saddle point. Particularly, we show

$$V_t^{\lambda_1^*,\lambda_2^*} - V_t^{\lambda_1,\lambda_2^*} \leq 0. \tag{283}$$

Set

$$\bar{V}_t = V_t^{\lambda_1^*,\lambda_2^*} - V_t^{\lambda_1,\lambda_2^*}. \tag{284}$$

Since SDE (279) is rewritten as

$$dV_t^{\lambda_1^*,\lambda_2^*} = -\left[f(c_t, V_t^{\lambda_1^*,\lambda_2^*}) + (\bar{\lambda}_1 - \hat{\lambda}_1)x_{1,t}\sigma_{V^{\lambda_1^*,\lambda_2^*},1}\right]dt + \sigma_{V^{\lambda_1^*,\lambda_2^*}}dB_t^{\lambda_1,\lambda_2^*},\ V_T^{\lambda_1^*,\lambda_2^*} = \log c_T, \tag{285}$$

we have

$$\begin{aligned} d\bar{V}_t = &-\left[f(c_t, V_t^{\lambda_1^*,\lambda_2^*}) - f(c_t, V_t^{\lambda_1,\lambda_2^*}) + (\bar{\lambda}_1 - \hat{\lambda}_1)x_{1,t}\sigma_{V^{\lambda_1^*,\lambda_2^*},1}\right]dt \\ &+ (\sigma_{V^{\lambda_1^*,\lambda_2^*}} - \sigma_{V^{\lambda_1,\lambda_2^*}})dB_t^{\lambda_1,\lambda_2^*} \\ = &+\beta\bar{V}_t dt - (\bar{\lambda}_1 - \hat{\lambda}_1)x_{1,t}\sigma_{V^{\lambda_1^*,\lambda_2^*},1}dt + (\sigma_{V^{\lambda_1^*,\lambda_2^*}} - \sigma_{V^{\lambda_1,\lambda_2^*}})dB_t^{\lambda_1,\lambda_2^*},\ \bar{V}_T = 0. \end{aligned} \tag{286}$$

By Ito's formula, we have

$$d(e^{-\beta t}\bar{V}_t) = -(\bar{\lambda}_1 - \hat{\lambda}_1)x_{1,t}\sigma_{V^{\lambda_1^*,\lambda_2^*},1}e^{-\beta t}dt + (\sigma_{V^{\lambda_1^*,\lambda_2^*}} - \sigma_{V^{\lambda_1,\lambda_2^*}})e^{-\beta t}dB_t^{\lambda_1,\lambda_2^*}, \tag{287}$$

and thus

$$-e^{-\beta t}\bar{V}_t = \int_t^T -(\bar{\lambda}_1 - \hat{\lambda}_1)x_{1,s}\sigma_{V^{\lambda_1^*,\lambda_2^*},1}e^{-\beta s}ds + \int_t^T (\sigma_{V^{\lambda_1^*,\lambda_2^*}} - \sigma_{V^{\lambda_1,\lambda_2^*}})e^{-\beta s}dB_s^{\lambda_1,\lambda_2^*}. \tag{288}$$

Taking the conditional expectation $E_t^{\lambda_1,\lambda_2^*}[\cdot]$ in both sides of (288), we have

$$-e^{-\beta t}\bar{V}_t = E_t^{\lambda_1,\lambda_2^*}\left[\int_t^T -(\bar{\lambda}_1 - \hat{\lambda}_1)x_{1,s}\sigma_{V^{\lambda_1^*,\lambda_2^*},1}e^{-\beta s}ds\right] \geq 0. \tag{289}$$

Thus, by (282), we have $\bar{V}_t \leq 0$.

Here, we used the fact that

$$\left\{\int_0^t (\sigma_{V^{\lambda_1^*,\lambda_2^*},s} - \sigma_{V^{\lambda_1,\lambda_2^*},s})e^{-\beta s}dB_s^{\lambda_1,\lambda_2^*}\right\}_{0\leq t\leq T} \tag{290}$$

is a martingale under $P^{\lambda_1,\lambda_2^*}$, which is proved in the following Lemma.

Lemma 2. $\{\int_0^t(\sigma_{V^{\lambda_1^*,\lambda_2^*},s} - \sigma_{V^{\lambda_1,\lambda_2^*},s})e^{-\beta s}dB_s^{\lambda_1,\lambda_2^*}\}_{0\leq t\leq T}$ *is a martingale under* $P^{\lambda_1,\lambda_2^*}$.

Proof.
To confirm the martingale property, it suffices to show that

$$E^{\lambda_1,\lambda^*}\left[\sup_{0\le t\le T}|V_t^{\lambda_1^*,\lambda_2^*}|\right], E^{\lambda_1,\lambda^*}\left[\sup_{0\le t\le T}|V_t^{\lambda_1,\lambda_2^*}|\right] < \infty. \tag{291}$$

Firstly, $E^{\lambda_1,\lambda_2^*}[\sup_{0\le t\le T}|V_t^{\lambda_1^*,\lambda_2^*}|] < \infty$ is proved as follows.
Under $P^{\lambda_1,\lambda_2^*}$, SDEs become

$$\frac{dc_t}{c_t} = \hat{\mu}_c^* dt + \sigma_c(x_t,t)\cdot dB_t^{\lambda_1,\lambda_2^*}, \tag{292}$$

$$dx_t = \hat{\mu}_x^* dt + \sigma_x(x_t,t)\cdot dB_t^{\lambda_1,\lambda_2^*}, \tag{293}$$

where

$$\hat{\mu}_{x_1}^*(x_t,t) = a_1 - (b_1 - \hat{\lambda}_1\sigma_{x_1})x_{1,t}, \tag{294}$$

$$\hat{\mu}_{x_2}^*(x_t,t) = a_2 - (b_2 - \bar{\lambda}_2\sigma_{x_2})x_{2,t}, \tag{295}$$

$$\hat{\mu}_{x_3}^*(x_t,t) = a_3 - b_3 x_{3,t}, \tag{296}$$

$$\hat{\mu}_c^* = (\mu_{c,1} + \hat{\lambda}_1\sigma_{c,1})x_{1,t}^2 + (\mu_{c,2} + \bar{\lambda}_2\sigma_{c,2})x_{2,t}^2 + \mu_{c,3}x_{3,t} + l_1(t)x_{1,t} + l_2(t)x_{2,t} + \mu_{e,0}, \tag{297}$$

$$\sigma_{x,i}^k(x,t) = \sigma_{x,i},\ i=1,2,3,\ i=k, \tag{298}$$

$$\sigma_{x,i}^k(x,t) = 0,\ i=1,2,3,\ i\neq k, \tag{299}$$

$$\sigma_{c,1}^1(x,t) = \sigma_{c,1}x_{1,t}, \tag{300}$$

$$\sigma_{c,2}^2(x,t) = \sigma_{c,2}x_{2,t}, \tag{301}$$

$$\sigma_{c,3}^3(x,t) = \sigma_{c,3}, \tag{302}$$

.

$$\begin{aligned}\sup_{0\le t\le T}|V_t^{\lambda_1^*,\lambda_2^*}| \le& \sum_{i=1}^{2}\sup_{0\le t\le T}|m_i(t)|\sup_{0\le t\le T}|x_{i,t}|^2 + \sup_{0\le t\le T}|m_3(t)|\sup_{0\le t\le T}|x_{3,t}| \\ &+ \sup_{0\le t\le T}|n(t)| + \sup_{0\le t\le T}|\log e_t|.\end{aligned} \tag{303}$$

Since $\log e_t$ is written as

$$\begin{aligned}\log e_t =& \log e_0 + \int_0^t (\sum_{i=1}^{3}\alpha_i x_{i,s}^2 + \beta_i x_{i,s} + \gamma)ds \\ &+ \int_0^t \left(\sigma_{c,1}x_{1,t}dB_{1,s}^{\lambda_1,\lambda_2^*} + \sigma_{c,2}x_{2,t}dB_{2,s}^{\lambda_1,\lambda_2^*} + \sigma_{c,3}x_{3,t}dB_{3,s}^{\lambda_1,\lambda_3^*}\right),\end{aligned} \tag{304}$$

where $\alpha_i, \beta_i, i = 1, 2, 3, \gamma \in \mathcal{R}$, we have

$$\sup_{0\le t\le T} |\log e_t| \le \log e_0 + \left(\gamma + \sum_{i=1}^{3}\left(|\alpha_i| \sup_{0\le t\le T} |x_{i,s}|^2 + |\beta_i| \sup_{0\le t\le T} |x_{i,s}|\right)\right) T$$
$$+ \sum_{i=1}^{2} |\sigma_{c,i}| \sup_{0\le t\le T}\left|\int_0^t x_{i,s} dB_{i,s}^{\lambda_1,\lambda_2^*}\right| + |\sigma_{c,3}| \sup_{0\le t\le T}\left|B_{3,t}^{\lambda_1,\lambda_2^*}\right|. \tag{305}$$

By Burkholder's inequality, we have

$$E^{\lambda_1,\lambda_2^*}\left[\sup_{0\le t\le T}\left|\int_0^t x_{i,s} dB_s\right|\right] \le K E^{\lambda_1,\lambda_2^*}\left[\left(\int_0^t |x_{i,s}|^2 ds\right)^{\frac{1}{2}}\right]$$
$$\le K\left(E^{\lambda_1,\lambda_2^*}\left[\left(\int_0^t |x_{i,s}|^2 ds\right)\right]\right)^{\frac{1}{2}} \le KT\left(E^{\lambda_1,\lambda_2^*}\left[\sup_{0\le t\le T} |x_{i,t}|^2\right]\right)^{\frac{1}{2}}, \ i = 1, 2. \tag{306}$$

Since $E^{\lambda_1,\lambda_2^*}[\sup_{0\le t\le T} |x_{i,t}|^2] < \infty$, $i = 1, 2$ and $E^{\lambda_1,\lambda_2^*}[\sup_{0\le t\le T} |B_{3,t}^{\lambda_1,\lambda_2^*}|] < \infty$ (for instance, see Problem 5.3.15 in Karatzas and Shreve), we have $E^{\lambda_1,\lambda_2^*}[\sup_{0\le t\le T} |\log e_t|] < \infty$. Hence, $E^{\lambda_1,\lambda_2^*}[\sup_{0\le t\le T} |V_t^{\lambda_1^*,\lambda_2^*}|] < \infty$. $E^{\lambda_1,\lambda^*}\left[\sup_{0\le t\le T} |V_t^{\lambda_1,\lambda_2^*}|\right] < \infty$ is also proved in the same manner. □

In the same manner, we can prove that

$$V_t^{\lambda_1^*,\lambda_2^*} - V_t^{\lambda_1^*,\lambda_2} \ge 0. \tag{307}$$

Thus, $(\lambda_1^*, \lambda_2^*)$ is a saddle point of $J(\lambda_1, \lambda_2)$, $(\lambda_1, \lambda_2) \in \Lambda$. □

8.5.2 Equilibrium interest rate (Three Factor Gaussian Quadratic-Gaussian interest rate model)

Let π be a state-price density process satisfying a SDE

$$\frac{d\pi_t}{\pi_t} = -r_t dt + \sigma_t^\pi \cdot dB_t^{\lambda_1^*,\lambda_2^*}, \ \pi_0 = 1, \tag{308}$$

where r is a risk-free interest rate and $-\sigma^\pi$ is called a market price of risk in equilibrium. We denote the market price of risk $-\sigma^\pi$ by θ.

The state-price density process π is given by

$$\pi_t = \exp\left(\int_0^t f_y(c_s, V_s^{\lambda_1^*,\lambda_2^*}) ds\right) f_c(c_t, V_t^{\lambda_1^*,\lambda_2^*}). \tag{309}$$

in equilibrium where the consumption process c is equivalent to the endowment. Here, subscripts y and c of f describe the partial derivatives of f with respect to those variables.

By applying Ito's formula to (309) and compare its drift and diffusion terms with (308), we obtain

$$r = \beta + \mu_c^*(x, t) - |\sigma_c(x, t)|^2$$
$$= \beta + \left\{\mu_0 + l_1(t)x_{1,t} + l_2(t)x_{2,t} + \sum_{i=1}^{2}(\mu_i + \bar{\lambda}_i\sigma_{c,i})x_{i,t}^2 + \mu_3 x_{3,t}\right\} - \left\{\sum_{i=1}^{2}\sigma_{c,i}^2 x_{i,t}^2 + \sigma_{c,3}^2\right\}, \tag{310}$$

and

$$\theta = \sigma_c(x,t) = \begin{pmatrix} \sigma_{c,1}x_1 \\ \sigma_{c,2}x_2 \\ \sigma_{c,3} \end{pmatrix}. \tag{311}$$

Let D be a cumulative dividend process which is RCLL (right-continuous with left limits) and $\mathcal{R}$-valued $\{\mathcal{F}_t\}$-progressively measurable process.

It is well known that under a complete market assumption, a risky asset with the dividend stream D is priced as

$$S_t = E^{P^{\lambda_1^*,\lambda_2^*}}\left[\int_t^T \frac{\pi_s}{\pi_t} dD_s \middle| \mathcal{F}_t\right]. \tag{312}$$

Particularly, the zero-coupon bond price $P(t,T)$ becomes

$$\begin{aligned} P(t,T) &= E^{P^{\lambda_1^*,\lambda_2^*}}\left[\frac{\pi_T}{\pi_t}\middle| \mathcal{F}_t\right] \\ &= E^P\left[\frac{\pi_T \mathcal{Z}_T^{\lambda_1^*,\lambda_2^*}}{\pi_t \mathcal{Z}_t^{\lambda_1^*,\lambda_2^*}}\middle| \mathcal{F}_t\right]. \end{aligned} \tag{313}$$

We note that $P(t,T)$ in (313) is also rewritten as

$$P(t,T) = E^{Q^{\lambda^*}}\left[\exp\left(-\int_t^T r_s ds\right)\middle| \mathcal{F}_t\right], \tag{314}$$

where Q^{λ^*} is a risk-neutral measure with respect to $P^{\lambda_1^*,\lambda_2^*}$:

$$\begin{aligned} Q^{\lambda^*}(A) &= E^{\lambda_1^*,\lambda_2^*}[\mathcal{Z}_T^{Q^{\lambda^*}} 1_A];\ A \in \mathcal{F}_T, \\ \mathcal{Z}_T^{Q^{\lambda^*}} &= \exp\left(-\frac{1}{2}\int_0^T |\sigma_s^\pi|^2 ds + \int_0^T \sigma_s^\pi \cdot dB_s^{\lambda_1^*,\lambda_2^*}\right). \end{aligned} \tag{315}$$

Thus, under Q^{λ^*},

$$dB_{i,t}^{Q^{\lambda^*}} = \begin{cases} dB_{i,t}^{\lambda_1^*,\lambda_2^*} + \sigma_{c,i}x_i dt,\ (i=1,2) \\ dB_{i,t}^{\lambda_1^*,\lambda_2^*} + \sigma_{c,i}dt,\ i=3. \end{cases} \tag{316}$$

is a Brownian motion, so, for $i = 1, 2$,

$$\begin{aligned} dx_{i,t} &= (a_i - b_i^* x_{i,t})dt + \sigma_{x,i} dB_{i,t}^{\lambda_1^*,\lambda_2^*} = (a_i - b_i^* x_{i,t})dt + \sigma_{x,i}(dB_{i,t}^{Q^{\lambda^*}} - \sigma_{c,i}x_{i,t}dt) \\ &= (a_i - (b_i^* + \sigma_{x,i}\sigma_{c,i})x_{i,t})dt + \sigma_{x,i} dB_{i,t}^{Q^{\lambda^*}}, \end{aligned} \tag{317}$$

where $b_1^* + \sigma_{x,1}\sigma_{c,1} < b_1^*$ due to $\sigma_{c,1} > 0$ and $\sigma_{x,1} < 0$, and $b_2^* + \sigma_{x,2}\sigma_{c,2} > b_2^*$ due to $\sigma_{c,2} > 0$ and $\sigma_{x,2} > 0$.

Similarly, for $i = 3$,

$$dx_{3,t} = ((a_3 - \sigma_{x,3}\sigma_{c,3}) - b_3 x_{3,t})dt + \sigma_{x,3} dB_{3,t}^{Q^{\lambda^*}}. \tag{318}$$

Consequently, evaluation of (314) with (310) under SDEs (317) and (318) provides an equilibrium yield curve in a three-factor Gaussian Quadratic-Gaussian interest rate model.

9 Conclusion

In this study, we have proposed a novel asset pricing model incorporating fundamental uncertainties by choice of a probability measure. Particularly, the model takes into account both aggressive (positive) and conservative (cautious) attitudes of the agent towards different market risk sources by a sup-inf/inf-sup problem on the agent's utility, which reduces to solving FBSDEs. Moreover, we have applied the result of the sup-inf/inf-sup problem to asset pricing, in particular, term structures of interest rates. Furthermore, we have presented numerical examples of the term structure of interest rates with market sentiment. Such term structure models are important in yield curve trading of hedge funds as well as central banks' monetary policy making in the global low interest environments, in which the yield curves are controlled by central banks and less affected by economic factors, but driven mainly by sentiment of market participants.

Firstly, we have formulated a sup-inf/inf-sup problem of a representative agent's utility that describes aggressive (positive)/conservative (cautious) attitudes toward risks of Brownian motions and shown that the optimal solution is obtained by solving the associated BSDE. Moreover, we have presented expressions of the equilibrium interest rate and the term structure of interest rates under the probability measure obtained through the sup-inf/inf-sup problem. Furthermore, we have provided explicit expressions of the equilibrium interest rate by solving the system of FBSDEs by two approaches. The first approach is by comparison theorems with which the signs of the volatilities of the BSDE are uniquely determined and thus, the system of FBSDEs is reduced to a combination of a solvable BSDE and forward SDEs. The second approach is to predetermine those signs and confirm them by explicitly solving the separated BSDE. Finally, we have provided numerical examples on the term structure of interest rates under fundamental uncertainties, which give implications on yield curve trading by hedge funds and yield curve controls for monetary policies by central banks.

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A Proof of Proposition 1

In this section, we prove Proposition 1 with modifications of the arguments in the proofs of Theorem 9.20 in Cohen and Elliott [4] and Theorem I-3 in Hamadene and Lepeltier [8].

Hereafter, we suppress the superscript λ_1, λ_2 of Y and Z.

Let $\phi : [0,T] \times \mathcal{R}^l \times \mathcal{C}([0,T] \to \mathcal{R}^d) \times \mathcal{R} \times \mathcal{R}^d \to \mathcal{R}$ be

$$\phi(t,x,\omega,y,z) = g(t,\omega,x,y) + \lambda_{1,t}(\omega)z_1 + \lambda_{2,t}(\omega)z_2. \tag{319}$$

Then,

$$\begin{aligned} dY_t &= -\bigg(g(s,B,X_s,Y_s) + \lambda_{1,t}Z_{1,t} + \lambda_{2,t}Z_{2,t}\bigg)dt + \sum_{j=1}^{d} Z_{j,t}dB_{j,t},\ Y_T = 0 \\ &= -\phi(t,X_t,B,Y_t,Z_t)dt + Z_t dB_t,\ Y_T = \xi. \end{aligned} \tag{320}$$

$$
\begin{aligned}
|\phi(t,x,\omega,y,z)-\phi(t,x,\omega,y',z')| &= |\lambda_{1,t}(\omega)(z_1-z_1')+\lambda_{2,t}(\omega)(z_2-z_2')|+L|y-y'| \\
&\le \|\lambda(\omega)\|_t|z-z'|+L|y-y'|, \qquad (321)
\end{aligned}
$$

where $\lambda(\omega)=(\lambda_1(\omega),\lambda_2(\omega))$ and $\|\lambda(\omega)\|_t=\sup_{0\le s\le t}|\lambda_s(\omega)|$.

Let

$$
\phi^{n,m}(t,x,\omega,y,z)=\phi(t,x,\omega,y,z)1_{\{\|\lambda(\omega)\|_t\le n\}}1_{\{\phi(t,x,y,\omega,z)\ge 0\}}+\phi(t,x,\omega,y,z)1_{\{\|\lambda(\omega)\|_t\le m\}}1_{\{\phi(t,x,\omega,y,z)<0\}}. \qquad (322)
$$

Then, we have

$$
|\phi^{n,m}(t,x,\omega,y,z)-\phi^{n,m}(t,x,\omega,y',z')|\le(n+m)|z-z'|+L|y-y'|. \qquad (323)
$$

Hence, $\phi^{n,m}(t,x,\omega,y,z)$ satisfies the uniform Lipschitz condition and by Theorem 6.2.1 in Pham [21] , there exists a unique solution $(Y^{n,m},Z^{n,m})$ for a BSDE

$$
dY_t^{n,m}=-\phi^{n,m}(t,X_t,B,Y_t^{n,m},Z_t^{n,m})dt+Z_t^{n,m}dB_t,\ Y_T^{n,m}=\xi, \qquad (324)
$$

such that

$$
E\left[\int_0^T (Y_s^{n,m})^2+|Z_s^{n,m}|^2ds\right]<\infty. \qquad (325)
$$

Namely,

$$
\begin{aligned}
Y_t^{n,m} &= \xi+\int_t^T \phi^{n,m}(s,X_s,B,Y_s^{n,m},Z_s^{n,m})ds-\int_t^T Z_s^{n,m}dB_s \\
&= \xi+\int_t^T \phi^{n,m}(s,X_s,B,0,0)ds \\
&\quad+\left(\int_t^T \phi^{n,m}(s,X_s,B,Y_s^{n,m},Z_s^{n,m})ds-\int_t^T \phi^{n,m}(s,X_s,B,0,Z_s^{n,m})ds\right) \\
&\quad+\left(\int_t^T \phi^{n,m}(s,X_s,B,0,Z_s^{n,m})ds-\int_t^T \phi^{n,m}(s,X_s,B,0,0)ds\right) \\
&\quad-\int_t^T Z_s^{n,m}dB_s \\
&= \xi+\int_t^T \phi^{n,m}(s,X_s,B,0,0)ds+\int_t^T a_s^{n,m}Y_s^{n,m}ds-\int_t^T Z_s^{n,m}dB_s^{n,m}, \qquad (326)
\end{aligned}
$$

where

$$
a_s^{n,m}=\frac{\phi^{n,m}(s,X_s,B,Y_s^{n,m},Z_s^{n,m})-\phi^{n,m}(s,X_s,B,0,Z_s^{n,m})}{Y_s^{n,m}}1_{\{Y_s^{n,m}\neq 0\}}, \qquad (327)
$$

$$
dB_{j,s}^{n,m}=dB_{j,s}-\frac{\phi^{n,m}(s,X_s,B,0,Z_s^{n,m})-\phi^{n,m}(s,X_s,B,0,0)}{|Z_s^{n,m}|^2}Z_{j,s}^{n,m}1_{\{Z_s^{n,m}\neq 0\}}ds. \qquad (328)
$$

Note that by (323), we have

$$|a_s^{n,m}| \leq L. \tag{329}$$

Let

$$\theta_{j,s}^{n,m} = \frac{\phi^{n,m}(s, X_s, B, 0, Z_s^{n,m}) - \phi^{n,m}(s, X_s, B, 0, 0)}{|Z_s^{n,m}|^2} Z_{j,s}^{n,m} 1_{\{Z_s^{n,m} \neq 0\}}, \ j = 1, \ldots, d. \tag{330}$$

Then,

$$|\theta_s^{n,m}| = \frac{|\phi^{n,m}(s, X_s, B, 0, Z_s^{n,m}) - \phi^{n,m}(s, X_s, B, 0, 0)|}{|Z_s^{n,m}|} 1_{\{Z_s^{n,m} \neq 0\}}. \tag{331}$$

Since

$$|\phi^{n,m}(t, x, \omega, y, z) - \phi^{n,m}(t, x, \omega, y, z')| \leq (n+m)|z - z'|, \tag{332}$$

$$|\theta_s^{n,m}| \leq \frac{(n+m)|Z_s^{n,m}|}{|Z_s^{n,m}|} = n + m. \tag{333}$$

By Girsanov's theorem, $P^{n,m}$ defined by

$$P^{n,m}(A) = E\left[\exp\left(-\frac{1}{2}\int_0^T |\theta_s^{n,m}|^2 ds + \int_0^T \theta_s^{n,m} dB_s\right) 1_A\right], \ A \in \mathcal{F} \tag{334}$$

is a probability measure, and $B^{n,m}$ is a Brownian motion under $P^{n,m}$.

By Ito's formula, we have

$$\begin{aligned} d(e^{\int_0^t a_s^{n,m} ds} Y_t^{n,m}) &= a_t^{n,m} e^{\int_0^t a_s^{n,m} ds} Y_t^{n,m} dt + e^{\int_0^t a_s^{n,m} ds} dY_t^{n,m} \\ &= -e^{\int_0^t a_s^{n,m} ds} \phi^{n,m}(t, X_t, B, 0, 0) dt + e^{\int_0^t a_s^{n,m} ds} Z_t^{n,m} dB_t^{n,m}, \end{aligned} \tag{335}$$

$$e^{\int_0^T a_s^{n,m} ds} \xi - e^{\int_0^t a_s^{n,m} ds} Y_t^{n,m} = -\int_t^T e^{\int_0^s a_u^{n,m} du} \phi^{n,m}(s, X_s, B, 0, 0) ds + \int_t^T e^{\int_0^s a_u^{n,m} du} Z_s^{n,m} dB_s^{n,m}, \tag{336}$$

and

$$Y_t^{n,m} = e^{\int_t^T a_s^{n,m} ds} \xi + \int_t^T e^{\int_t^s a_u^{n,m} du} \phi^{n,m}(s, X_s, B, 0, 0) ds - \int_t^T e^{\int_t^s a_u^{n,m} du} Z_s^{n,m} dB_s^{n,m}. \tag{337}$$

Then, due to the fact that

$$\left\{\int_t^v e^{\int_t^s a_u^{n,m} du} Z_s^{n,m} dB_s^{n,m}\right\}_{0 \leq v \leq T} \tag{338}$$

is a $P^{n,m}$- martingale for each $0 \leq t \leq T$, we have

$$Y_t^{n,m} = E^{n,m}\left[e^{\int_t^T a_s^{n,m} ds} \xi + \int_t^T e^{\int_t^s a_u^{n,m} du} \phi^{n,m}(s, X_s, B, 0, 0) ds \middle| \mathcal{F}_t\right]. \tag{339}$$

Let K_1 and K_2 be positive constants by which $|\xi|$ and $|\phi(s, x, \omega, 0, 0)|$ are bounded, respectively. Since $|\phi^{n,m}(s, X_s, B, 0, 0)| \leq |\phi(s, X_s, B, 0, 0)| \leq K_2$, we have

$$|Y_t^{n,m}| \leq e^{LT}(K_1 + T \sup_{0 \leq s \leq T} |\phi(s, x, \omega, 0, 0)|) \leq e^{LT}(K_1 + TK_2). \tag{340}$$

Here, we used (329) in the first inequality. Thus, $Y^{n,m}$ is uniformly bounded with respect to t, ω, n, m.

A.1

The fact that

$$\left\{ \int_t^v e^{\int_t^s a_u^{n,m} du} Z_s^{n,m} dB_s^{n,m} \right\}_{t \le v \le T} \tag{341}$$

is a $P^{n,m}$- martingale is proved as follows.

We first show

$$E^{n,m} \left[\left\{ \int_0^T (Z_s^{n,m})^2 ds \right\}^{1/2} \right] < \infty. \tag{342}$$

Let

$$\mathcal{L}_t^{n,m} = \exp\left(-\frac{1}{2} \int_0^t |\theta_s^{n,m}|^2 ds + \int_0^t \theta_s^{n,m} dB_s \right). \tag{343}$$

Then,

$$\begin{aligned} E^{n,m} \left[\left\{ \int_0^t (Z_s^{n,m})^2 ds \right\}^{1/2} \right] &= E \left[\mathcal{L}_T^{n,m} \left\{ \int_0^t (Z_s^{n,m})^2 ds \right\}^{1/2} \right] \\ &\le E \left[\mathcal{L}_T^{n,m2} \right]^{1/2} E \left[\int_0^t (Z_s^{n,m})^2 ds \right]^{1/2}. \end{aligned} \tag{344}$$

Since $\mathcal{L}_t^{n,m}$ satisfies

$$d\mathcal{L}_t^{n,m} = \theta_t^{n,m} \mathcal{L}_t^{n,m} dB_t, \ \mathcal{L}_0^{n,m} = 1, \tag{345}$$

where $\theta_t^{n,m}$ is bounded, and thus $E[\mathcal{L}_T^{n,m2}] < \infty$.

Hence, noting that $E\left[\int_0^t (Z_s^{n,m})^2 ds\right] < \infty$ in (344), we have

$$E^{n,m} \left[\left\{ \int_0^T (Z_s^{n,m})^2 ds \right\}^{1/2} \right] < \infty. \tag{346}$$

By the BDG inequality,

$$\begin{aligned} E^{n,m} \left[\sup_{t \le v \le T} \left| \int_t^v e^{\int_t^s a_u^{n,m} du} Z_s^{n,m} dB_s^{n,m} \right| \right] &\le K E^{n,m} \left[\left\{ \int_t^T e^{\int_t^s 2a_u^{n,m} du} (Z_s^{n,m})^2 ds \right\}^{1/2} \right] \\ &\le K e^{LT} E^{n,m} \left[\left\{ \int_0^T (Z_s^{n,m})^2 ds \right\}^{1/2} \right] < \infty. \end{aligned} \tag{347}$$

Hence, $\left\{ \int_t^v e^{\int_t^s a_u^{n,m} du} Z_s^{n,m} dB_s^{n,m} \right\}_{t \le v \le T}$ is a uniformly integrable local martingale and thus a martingale.

A.2

Since $Y^{n,m}$ is uniformly bounded, increasing with respect to n and decreasing with respect to m, we can define Y_t as

$$Y_t = \lim_{m\to\infty}\lim_{n\to\infty} Y_t^{n,m},\ P\text{-a.s.} \tag{348}$$

Here, the monotonicity of $Y^{n,m}$ with respect to n, m follows from a comparison theorem (e.g. see Theorem 6.2.2. in Pham [21]) applied to BSDE (324) with a monotone driver which is increasing with regard to n and decreasing with respect to m. For each $p \geq 1$, we can choose a subsequence such that

$$\lim_{k\to\infty} E\left[\int_0^T |Y_s^{n(k),k} - Y_s|^p ds\right] = 0. \tag{349}$$

In fact, we choose a subsequence $n(k)$ as follows.

Let $Y^{\infty,m} = \lim_{n\to\infty} Y^{n,m}$. Note that by the bounded convergence theorem, for all $j \geq 1$,

$$\lim_{n\to\infty} \|Y^{n,j} - Y^{\infty,j}\|_{L^p([0,T]\times\Omega)} = 0. \tag{350}$$

Let

$$n(1) = \min\{n \in \mathbf{N} | \|Y^{n,1} - Y^{\infty,1}\|_{L^p([0,T]\times\Omega)} \leq \|Y^{\infty,1} - Y\|_{L^p([0,T]\times\Omega)}\}. \tag{351}$$

For $j \geq 2$,

$$n(j) = \min\{n \geq n(j-1) | \|Y^{n,j} - Y^{\infty,j}\|_{L^p([0,T]\times\Omega)} \leq \|Y^{\infty,j} - Y\|_{L^p([0,T]\times\Omega)}\}. \tag{352}$$

Then, by the dominated convergence theorem, for all $\epsilon > 0$, there exists N such that, for all $k > N$, $\|Y^{\infty,k} - Y\|_{L^p([0,T]\times\Omega)} < \frac{1}{2}\epsilon$.

Moreover, by (351) and (352),

$$\|Y^{n(k),k} - Y^{\infty,k}\|_{L^p([0,T]\times\Omega)} < \frac{1}{2}\epsilon. \tag{353}$$

Hence,

$$\begin{aligned}\|Y^{n(k),k} - Y\|_{L^p([0,T]\times\Omega)} &\leq \|Y^{n(k),k} - Y^{\infty,k}\|_{L^p([0,T]\times\Omega)} + \|Y^{\infty,k} - Y\|_{L^p([0,T]\times\Omega)} \\ &< \epsilon. \end{aligned} \tag{354}$$

A.3

Then, $\{Z_t^{n(k),k}\}_{k\in\mathbf{N}}$ is a Cauchy sequence in the space $\left\{ Z \middle| E\left[\int_0^T Z_s^2 ds\right] < \infty \right\}$, by the following discussion.

Since by Ito's formula,

$$d(Y_t^{n,m} - Y_t^{n',m'})^2 = 2(Y_t^{n,m} - Y_t^{n',m'})d(Y_t^{n,m} - Y_t^{n',m'}) + d\left\langle Y^{n,m} - Y^{n',m'}\right\rangle_t, \tag{355}$$

we have

$$
\begin{aligned}
&-(Y_t^{n,m}-Y_t^{n',m'})^2\\
&=\int_t^T 2(Y_s^{n,m}-Y_s^{n',m'})(-\phi^{n,m}(s,X_s,B,Y_s^{n,m},Z_s^{n,m})+\phi^{n',m'}(s,X_s,B,Y_s^{n',m'},Z_s^{n',m'}))ds\\
&\quad+\int_t^T 2(Y_s^{n,m}-Y_s^{n',m'})(Z_s^{n,m}-Z_s^{n',m'})dB_s+\int_t^T (Z_s^{n,m}-Z_s^{n',m'})^2 ds. \qquad (356)
\end{aligned}
$$

Note that $\{\int_0^t 2(Y_s^{n,m}-Y_s^{n',m'})(Z_s^{n,m}-Z_s^{n',m'})dB_s\}_{0\le t\le T}$ is a P-martingale, since $Y^{n,m}-Y^{n',m'}$ is uniformly bounded with respect to ω,n,m,n',m' and $E[\int_0^T(Z^{n,m}-Z^{n',m'})_s^2ds]<\infty$. Taking the expectation with respect to P, we have

$$
\begin{aligned}
&E[(Y_t^{n,m}-Y_t^{n',m'})^2]+E\left[\int_t^T (Z_s^{n,m}-Z_s^{n',m'})^2ds\right]\\
&=E\left[\int_t^T 2(Y_s^{n,m}-Y_s^{n',m'})(\phi^{n,m}(s,X_s,B,Z_s^{n,m})-\phi^{n',m'}(s,X_s,B,Z_s^{n',m'}))ds\right] \qquad (357)
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&E\left[\int_t^T (Z^{n,m}-Z^{n',m'})^2ds\right]\\
&\le E\left[\int_t^T 2(Y^{n,m}-Y^{n',m'})(\phi^{n,m}(s,X_s,B,Y^{n,m},Z_s^{n,m})-\phi^{n',m'}(s,X_s,B,Y^{n',m'},Z_s^{n',m'}))ds\right]\\
&\le E\left[\int_t^T 2|Y^{n,m}-Y^{n',m'}||\phi^{n,m}(s,X_s,B,Y^{n,m},Z_s^{n,m})-\phi^{n',m'}(s,X_s,B,Y^{n',m'},Z_s^{n',m'})|ds\right]\\
&\le E\left[\int_t^T \|\lambda(B)\|_s 2|Y^{n,m}-Y^{n',m'}|(|Z_s^{n,m}|+|Z_s^{n',m'}|)ds\right]\\
&\quad+E\left[\int_t^T 2L|Y^{n,m}-Y^{n',m'}|(|Y_s^{n,m}|+|Y_s^{n',m'}|)ds\right]\\
&\quad+E\left[4\int_t^T |\phi(s,X_s,B,0,0)||Y^{n,m}-Y^{n',m'}|ds\right]\\
&\le E\left[\int_t^T (|Z_s^{n,m}|+|Z_s^{n',m'}|)^2ds\right]^{1/2} E\left[\int_t^T 16\|\lambda(B)\|_s^4ds\right]^{1/4} E\left[\int_t^T |Y^{n,m}-Y^{n',m'}|^4ds\right]^{1/4}\\
&\quad+E\left[\int_t^T 2L|Y^{n,m}-Y^{n',m'}|(|Y_s^{n,m}|+|Y_s^{n',m'}|)ds\right]\\
&\quad+E\left[4\int_t^T |\phi(s,X_s,B,0,0)||Y^{n,m}-Y^{n',m'}|ds\right]. \qquad (358)
\end{aligned}
$$

By the assumption on the integrability on $\lambda(B)$ in (18), and the fact that $E\left[\int_t^T|Z_s^{n,m}|^2ds\right]$ and $|Y^{n,m}|$ are uniformly bounded with respect to n,m, taking the subsequence $n(k)$ for $p=4$, we observe that $\{Z^{n(k),k}\}_{k\in\mathbf{N}}$ is a Cauchy sequence in $L^2([0,T]\times\Omega)$. We define Z as the limit of $\{Z^{n(k),k}\}_{k\in\mathbf{N}}$ in $L^2([0,T]\times\Omega)$.

A.4

The fact that

$$E\left[\int_t^T |Z_s^{n,m}|^2 ds\right] \tag{359}$$

is uniformly bounded with respect to n, m is proved as follows.

$$\xi^2 - (Y_t^{n,m})^2 = \int_t^T -2Y_s^{n,m}\phi^{n,m}(s, X_s, B, Y_s^{n,m}, Z_s^{n,m})ds + \int_t^T (Z_s^{n,m})^2 ds + \int_t^T 2(Y_s^{n,m})Z_s^{n,m} dB_s. \tag{360}$$

Then,

$$\int_t^T (Z_s^{n,m})^2 ds \leq \xi^2 + \int_t^T 2Y_s^{n,m}\phi^{n,m}(s, X_s, B, Y_s^{n,m}, Z_s^{n,m})ds - \int_t^T 2(Y_s^{n,m})Z_s^{n,m} dB_s. \tag{361}$$

Taking the expectation with respect to P, we have

$$\begin{aligned}
E\left[\int_t^T (Z_s^{n,m})^2 ds\right] &\leq E[\xi^2] + E\left[\int_t^T 2Y_s^{n,m}\phi^{n,m}(s, X_s, B, Y_s^{n,m}, Z_s^{n,m})ds\right] \\
&\leq E[\xi^2] + E\left[\int_t^T 2|Y_s^{n,m}||\phi^{n,m}(s, X_s, B, 0, 0)|ds\right] + E\left[\int_t^T 2|Y_s^{n,m}|\|\lambda(B)\|_s|Z_s^{n,m}|ds\right] \\
&\quad + E\left[\int_t^T 2L|Y_s^{n,m}|^2 ds\right] \\
&\leq E[\xi^2] + E\left[\int_t^T 2|Y_s^{n,m}||\phi^{n,m}(s, X_s, B, 0, 0)|ds\right] + 2E\left[\int_t^T |Y_s^{n,m}|^2\|\lambda(B)\|_s^2 ds\right] \\
&\quad + E\left[\int_t^T 2L|Y_s^{n,m}|^2 ds\right] + \frac{1}{2}E\left[\int_t^T |Z_s^{n,m}|^2 ds\right].
\end{aligned} \tag{362}$$

Thus,

$$\begin{aligned}
\frac{1}{2}E\left[\int_t^T (Z_s^{n,m})^2 ds\right] &\leq E[\xi^2] + E\left[\int_t^T 2|Y_s^{n,m}||\phi^{n,m}(s, X_s, B, 0, 0)|ds\right] + 2E\left[\int_t^T |Y_s^{n,m}|^2\|\lambda(B)\|_s^2 ds\right] \\
&\quad + E\left[\int_t^T 2L|Y_s^{n,m}|^2 ds\right].
\end{aligned} \tag{363}$$

Since $Y_s^{n,m}$ is uniformly bounded with respect to n, m and $\phi^{n,m}(s, X_s, B, 0, 0)$ is bounded, the right-hand side does not depend on n, m.

A.5

We observe that (Y_t, Z_t) is a solution of BSDE (14) by the following discussion.

We consider the limit $k \to \infty$ in $L^1(\Omega)$ in both sides of

$$Y_t^{n(k),k} = \xi + \int_t^T \phi^{n(k),k}(s, X_s, B, Y_s^{n(k),k}, Z_s^{n(k),k})ds - \int_t^T Z_s^{n(k),k} dB_s. \tag{364}$$

Firstly, by the dominated convergence theorem, we have

$$\lim_{k\to\infty} E[|Y_t^{n(k),k} - Y_t|] = 0. \tag{365}$$

Secondly,

$$\lim_{k\to\infty} E\left[\left|\int_t^T Z_s^{n(k),k} dB_s - \int_t^T Z_s dB_s\right|\right] = 0, \tag{366}$$

since

$$\lim_{k\to\infty} E\left[\left|\int_t^T Z_s^{n(k),k} dB_s - \int_t^T Z_s dB_s\right|^2 ds\right] = \lim_{k\to\infty} E\left[\int_t^T |Z_s^{n(k),k} - Z_s|^2 ds\right] = 0. \tag{367}$$

Finally,

$$\lim_{k\to\infty} E\left[\left|\int_t^T \phi^{n(k),k}(s, X_s, B, Y_s^{n(k),k}, Z_s^{n(k),k})ds - \int_t^T \phi(s, X_s, B, Y_s, Z_s)ds\right|\right] = 0. \tag{368}$$

This is proved as follows.

$$\begin{aligned}
&\left|\int_t^T \phi^{n(k),k}(s, X_s, B, Y_s^{n(k),k}, Z_s^{n(k),k})ds - \int_t^T \phi(s, X_s, B, Y_s, Z_s)ds\right| \\
&\leq \left|\int_t^T \phi^{n(k),k}(s, X_s, B, Y_s^{n(k),k}, Z_s^{n(k),k})ds - \int_t^T \phi^{n(k),k}(s, X_s, B, Y_s, Z_s)ds\right| \\
&\quad + \left|\int_t^T \phi^{n(k),k}(s, X_s, B, Y_s, Z_s)ds - \int_t^T \phi(s, X_s, B, Y_s, Z_s)ds\right|.
\end{aligned} \tag{369}$$

Then,

$$\begin{aligned}
&E\left[\left|\int_t^T \phi^{n(k),k}(s, X_s, B, Y_s^{n(k),k}, Z_s^{n(k),k})ds - \int_t^T \phi^{n(k),k}(s, X_s, B, Y_s, Z_s)ds\right|\right] \\
&\leq E\left[\int_t^T \|\lambda(B)\|_s |Z_s^{n(k),k} - Z_s| ds\right] + E\left[\int_t^T L|Y_s^{n(k),k} - Y_s| ds\right] \\
&\leq E\left[\int_t^T \|\lambda(B)\|_s^2 ds\right]^{1/2} E\left[\int_t^T |Z_s^{n(k),k} - Z_s|^2 ds\right]^{1/2} + E\left[\int_t^T L|Y_s^{n(k),k} - Y_s| ds\right] \\
&\to 0 \ (k \to \infty).
\end{aligned} \tag{370}$$

Next, by (322), we have

$$|\phi^{n(k),k}(s, X_s, B, Y_s, Z_s) - \phi(s, X_s, B, Y_s, Z_s)| \leq 2|\phi(s, X_s, B, Y_s, Z_s)|. \tag{371}$$

Also,

$$E\left[\int_t^T 2|\phi(s,X_s,B,Y_s,Z_s)|ds\right] \le 2E\left[\int_t^T \|\lambda(B)\|_s^2 ds\right]^{1/2} E\left[\int_t^T |Z_s|^2 ds\right]^{1/2} + 2E\left[\int_t^T L|Y_s|ds\right] + 2E\left[\int_t^T |\phi(s,X_s,B,0,0)|ds\right] < \infty. \tag{372}$$

Hence, by the dominated convergence theorem, we have

$$\begin{aligned} &E\left[\left|\int_t^T \phi^{n(k),k}(s,X_s,B,Y_s,Z_s)ds - \int_t^T \phi(s,X_s,B,Y_s,Z_s)ds\right|\right] \\ &\le E\left[\int_0^T \left|\phi^{n(k),k}(s,X_s,B,Y_s,Z_s)ds - \phi(s,X_s,B,Y_s,Z_s)\right| ds\right] \to 0 \ (k\to\infty). \end{aligned} \tag{373}$$

Here, we used

$$\lim_{k\to\infty} \phi^{n(k),k}(s,X_s,B,Y_s,Z_s) = \phi(s,X_s,B,Y_s,Z_s), \ \ \mu \times P\text{-a.e.}, \tag{374}$$

where μ is Lebesgue measure on $[0,T]$. This follows from (322).

Since

$$\begin{aligned} &E\left[\left|Y_t - \left(\xi + \int_t^T \phi(s,X_s,B,Y_s,Z_s)ds - \int_t^T Z_s dB_s\right)\right|\right] \\ &= E\left[\left|\left(Y_t^{n(k),k} - \left(\int_t^T \xi + \phi^{n(k),k}(s,X_s,B,Y_s^{n(k),k},Z_s^{n(k),k})ds - \int_t^T Z_s^{n(k),k} dB_s\right)\right)\right.\right. \\ &\quad \left.\left. - \left(Y_t - \left(\xi + \int_t^T \phi(s,X_s,B,Y_s,Z_s)ds - \int_t^T Z_s dB_s\right)\right)\right|\right] \\ &\le E[|Y_t^{n(k),k} - Y_t|] \\ &\quad + E\left[\left|\left(\int_t^T \phi^{n(k),k}(s,X_s,B,Y_s^{n(k),k},Z_s^{n(k),k})ds - \int_t^T Z_s^{n(k),k} dB_s\right)\right.\right. \\ &\quad\quad \left.\left. - \left(\int_t^T \phi(s,X_s,B,Y_s,Z_s)ds - \int_t^T Z_s dB_s\right)\right|\right] \to 0 \ (k\to\infty), \end{aligned} \tag{375}$$

we have

$$Y_t = \xi + \int_t^T \phi(s,X_s,B,Y_s,Z_s)ds - \int_t^T Z_s dB_s, \ \ P\text{-a.s.} \tag{376}$$

Since Y is continuous P-almost surely as we shall observe in next subsection,

$$Y_t = \xi + \int_t^T \phi(s,X_s,B,Y_s,Z_s)ds - \int_t^T Z_s dB_s, \ \ 0 \le \forall t \le T, \ \ P\text{-a.s} \tag{377}$$

holds.

A.6

The fact that Y is continuous P-almost surely is proved by the following discussion.

If we show

$$\sup_{0\le s\le T}|Y_s^{n(o(p(k))),o(p(k))}-Y_s|\to 0\ (k\to\infty), \tag{378}$$

for some subsequence $\{n(o(p(k)))\}_{k\in\mathbf{N}}$ of $\{n(k)\}_{k\in\mathbf{N}}$, then it follows that Y is continuous, since Y is a uniform convergence limit of continuous processes $\{Y^{n(o(p(k))),o(p(k))}\}_{k\in\mathbf{N}}$.

By (364), we have

$$\begin{aligned}|Y_t^{n(k),k}-Y_t^{n(l),l}|\le&\left|\int_t^T\phi^{n(k),k}(s,X_s,B,Y_s^{n(k),k},Z_s^{n(k),k})-\phi^{n(l),l}(s,X_s,B,Y_s^{n(l),l},Z_s^{n(l),l})ds\right|\\&+\left|\int_t^T(Z_s^{n(k),k}-Z_s^{n(l),l})dB_s\right|.\end{aligned}\tag{379}$$

Since $(Y_t^{n(l),l},\int_t^T\phi^{n(l),l}(s,X_s,B,Z_s^{n(l),l})ds,\int_t^T Z_s^{n(l),l}dB_s)$ converges to $(Y_t,\int_t^T\phi(s,X_s,B,Z_s)ds,\int_t^T Z_s dB_s)$ in $L^1(\Omega)$, which follows from (365),(366), and (373), there exists a subsequence $\{o(l)\}_{l\in\mathbf{N}}$ of $\{l\}_{l\in\mathbf{N}}$ such that the convergence holds in P-almost surely and in particular,

$$\int_0^T\left|\phi^{n(o(k)),o(k)}(s,X_s,B,Y_s^{n(o(k)),o(k)},Z_s^{n(o(k)),o(k)})-\phi(s,X_s,B,Y_s,Z_s)\right|ds\to 0\ (k\to\infty). \tag{380}$$

Then, we have

$$\begin{aligned}|Y_t^{n(o(k)),o(k)}-Y_t|\le&\left|\int_t^T\phi^{n(o(k)),o(k)}(s,X_s,B,Y_s^{n(o(k)),o(k)},Z_s^{n(o(k)),o(k)})-\phi(s,X_s,B,Y_s,Z_s)ds\right|\\&+\left|\int_t^T(Z_s^{n(o(k)),o(k)}-Z_s)dB_s\right|\\\le&\int_t^T\left|\phi^{n(o(k)),o(k)}(s,X_s,B,Y_s^{n(o(k)),o(k)},Z_s^{n(o(k)),o(k)})-\phi(s,X_s,B,Y_s,Z_s)\right|ds\\&+\left|\int_t^T(Z_s^{n(o(k)),o(k)}-Z_s)dB_s\right|.\end{aligned}\tag{381}$$

By taking $\sup_{0\le t\le T}$ in both sides, we have

$$\begin{aligned}&\sup_{0\le t\le T}|Y_t^{n(o(k)),o(k)}-Y_t|\\&\le\int_0^T\left|\phi^{n(o(k)),o(k)}(s,X_s,B,Y_s^{n(o(k)),o(k)},Z_s^{n(o(k)),o(k)})-\phi(s,X_s,B,Y_s,Z_s)\right|ds\\&\quad+\sup_{0\le t\le T}\left|\int_t^T(Z_s^{n(o(k)),o(k)}-Z_s)dB_s\right|.\end{aligned}\tag{382}$$

Finally, to prove (378), it suffices to show that there exists a subsequence $\{n(o(p(k)))\}_{k\in\mathbf{N}}$ of $\{n(o(k))\}_{k\in\mathbf{N}}$ such that

$$\lim_{k\to\infty}\sup_{0\le t\le T}\left|\int_t^T (Z_s^{n(o(p(k))),o(p(k))} - Z_s)dB_s\right| = 0, \tag{383}$$

since (380) holds.

(383) is proved as follows. By Burkholder's inequality,

$$E\left[\sup_{0\le t\le T}\left|\int_t^T (Z_s^{n(o(p(k))),o(p(k))} - Z_s)dB_s\right|^2\right] \le KE\left[\int_0^T \left|(Z_s^{n(o(p(k))),o(p(k))} - Z_s)\right|^2 ds\right] \to 0\ (k\to\infty). \tag{384}$$

Hence, (383) follows.

A.7

The uniqueness of (Y,Z) holds by the following discussion. Let (Y,Z) and (Y',Z') be solutions of the BSDE (14).

By Ito's formula, we have

$$d(Y_t - Y_t')^2 = 2(Y_t - Y_t')d(Y_t - Y_t') + d\left\langle Y - Y'\right\rangle_t, \tag{385}$$

where

$$\begin{aligned} d(Y_t - Y_t') &= -(g(t,B,X_t,Y_t) - g(t,B,X_t,Y_t') + \lambda_{1,t}(Z_{1,t} - Z_{1,t}') + \lambda_{2,t}(Z_{2,t} - Z_{2,t}'))dt \\ &\quad + \sum_{j=1}^d (Z_{j,t} - Z_{j,t}')dB_{j,t}, \\ &= -\frac{g(t,B,X_t,Y_t) - g(t,B,X_t,Y_t')}{Y_t - Y_t'} 1_{\{Y_t - Y_t' \neq 0\}}(Y_t - Y_t')dt + \sum_{j=1}^d (Z_{j,t} - Z_{j,t}')d\tilde{B}_{j,t}, \\ &= -b_t(Y_t - Y_t')dt + \sum_{j=1}^d (Z_{j,t} - Z_{j,t}')d\tilde{B}_{j,t}, \end{aligned} \tag{386}$$

and

$$d\left\langle Y - Y'\right\rangle_t = \sum_{j=1}^d (Z_{j,t} - Z_{j,t}')^2 dt. \tag{387}$$

Here, we set

$$d\tilde{B}_{j,t} = dB_{j,t} - \lambda_{j,t}dt,\ j = 1,2, \tag{388}$$

where $\tilde{B}$ is a $\tilde{P}$-Brownian motion because of (17) and Girsanov's theorem if we define

$$\tilde{P}(A) = E\left[\exp\left(-\frac{1}{2}\sum_{j=1}^2\int_0^T \lambda_{j,s}^2 ds + \sum_{j=1}^2\int_0^T \lambda_{j,s}dB_{j,s}\right)1_A\right],\ A\in\mathcal{F}, \tag{389}$$

and

$$b_t = \frac{g(t,B,X_t,Y_t) - g(t,B,X_t,Y_t')}{Y_t - Y_t'} 1_{\{Y_t - Y_t' \neq 0\}}. \tag{390}$$

Let $\bar{Y}_t = (Y_t - Y_t')e^{\int_0^t b_u du}$, $\bar{Z}_t = (Z_t - Z_t')e^{\int_0^t b_u du}$.

Then,

$$-\bar{Y}_t^2 = \sum_{j=1}^d \int_t^T 2\bar{Y}_t \bar{Z}_{j,s} d\tilde{B}_{j,s} + \int_t^T \sum_{j=1}^d \bar{Z}_{j,s}^2 ds, \tag{391}$$

and taking conditional expectation with respect to $\tilde{P}$ and the filtration $\mathcal{F}_t$, we have

$$\tilde{E}\left[\bar{Y}_t^2 + \int_t^T \sum_{j=1}^d \bar{Z}_{j,s}^2 ds \middle| \mathcal{F}_t\right] = 0. \tag{392}$$

We used the fact that $\{\sum_{j=1}^d \int_0^t 2\bar{Y}_s \bar{Z}_{j,s} d\tilde{B}_{j,s}\}_{0\le t\le T}$ is a $\tilde{P}$-martingale, which can be shown by a localization argument as follows.

Noting that

$$\sum_{j=1}^d \int_0^t 2\bar{Y}_s \bar{Z}_{j,s} d\tilde{B}_{j,s} = \bar{Y}_t^2 - \bar{Y}_0^2 - \int_0^t \sum_{j=1}^d \bar{Z}_{j,s}^2 ds, \tag{393}$$

and $\{\sum_{j=1}^d \int_0^t 2\bar{Y}_s \bar{Z}_{j,s} d\tilde{B}_{j,s}\}_{0\le t\le T}$ is a $\tilde{P}$-local martingale, we can take an increasing sequence of stopping times $\{\tau_n\}_{n\in\mathbf{N}}$ such that $\tau_n = T$ for sufficiently large n, $\tilde{P}$-almost surely, the $\tilde{P}$-local martingale $\{\sum_{j=1}^d \int_0^{t\wedge\tau_n} 2\bar{Y}_s \bar{Z}_{j,s} d\tilde{B}_{j,s}\}_{0\le 0\le T}$ is a $\tilde{P}$-martingale for all $n \in \mathbf{N}$.

Thus, we observe that for all $0 \le t_1 \le t_2 \le T$,

$$\begin{aligned}\lim_{n\to\infty} \tilde{E}\left[\sum_{j=1}^d \int_0^{t_2\wedge\tau_n} 2\bar{Y}_s \bar{Z}_{j,s} d\tilde{B}_{j,s} \middle| \mathcal{F}_{t_1}\right] &= \lim_{n\to\infty} \sum_{j=1}^d \int_0^{t_1\wedge\tau_n} 2\bar{Y}_s \bar{Z}_{j,s} d\tilde{B}_{j,s} \\ &= \sum_{j=1}^d \int_0^{t_1} 2\bar{Y}_s \bar{Z}_{j,s} d\tilde{B}_{j,s},\end{aligned} \tag{394}$$

and

$$\begin{aligned}\lim_{n\to\infty} \tilde{E}\left[\sum_{j=1}^d \int_0^{t_2\wedge\tau_n} 2\bar{Y}_s \bar{Z}_{j,s} d\tilde{B}_{j,s} \middle| \mathcal{F}_{t_1}\right] &= \lim_{n\to\infty} \tilde{E}\left[\bar{Y}_{t_2\wedge\tau_n}^2 - \bar{Y}_0^2 - \int_0^{t_2\wedge\tau_n} \sum_{j=1}^d \bar{Z}_{j,s}^2 ds, \middle| \mathcal{F}_{t_1}\right] \\ &= \tilde{E}\left[\bar{Y}_{t_2}^2 - \bar{Y}_0^2 - \int_0^{t_2} \sum_{j=1}^d \bar{Z}_{j,s}^2 ds, \middle| \mathcal{F}_{t_1}\right] \\ &= \tilde{E}\left[\sum_{j=1}^d \int_0^{t_2} 2\bar{Y}_s \bar{Z}_{j,s} d\tilde{B}_{j,s} \middle| \mathcal{F}_{t_1}\right].\end{aligned} \tag{395}$$

In the second equality in (395), we used the dominated convergence theorem and the monotone convergence theorem, since $\{\bar{Y}_t^2 - \bar{Y}_0^2\}_{0\le t\le T}$ is uniformly bounded with respect to ω and t and $\{-\int_0^t \sum_{j=1}^d \bar{Z}_{j,s}^2 ds\}_{0\le t\le T}$ is an decreasing process.

Hence,

$$\tilde{E}\left[\sum_{j=1}^d \int_0^{t_2} 2\bar{Y}_s \bar{Z}_{j,s} d\tilde{B}_{j,s} \middle| \mathcal{F}_{t_1}\right] = \sum_{j=1}^d \int_0^{t_1} 2\bar{Y}_s \bar{Z}_{j,s} d\tilde{B}_{j,s}. \tag{396}$$

Therefore, noting that $e^{\int_0^t b_u du} > 0,\ 0 \le \forall t \le T$, we have

$$Y_t = Y'_t,\ 0 \le \forall t \le T, P\text{-a.s.} \tag{397}$$

$$Z_{j,t} = Z'_{j,t},\ \mu \times P\text{-a.e.} \tag{398}$$

□

B Proof of Proposition 2

Proposition 2 is proved in the same manner as Proposition 1 with the following modifications.

In the proof of existence of a solution, instead of (319) and (322), we set

$$\phi(t,x,\omega,y,z) = g(t,\omega,x,y) - |\bar{\lambda}_1(t,x)||z_1| + |\bar{\lambda}_2(t,x)||z_2| \tag{399}$$

and

$$\begin{aligned}\phi^{n,m}(t,x,\omega,y,z) &= \phi(t,x,\omega,y,z)1_{\{\|\bar{\lambda}(x)\|_t \le n\}}1_{\{\phi(t,x,\omega,y,z)\ge 0\}} \\ &\quad + \phi(t,x,\omega,y,z)1_{\{\|\bar{\lambda}(x)\|_t \le m\}}1_{\{\phi(t,x,\omega,y,z)< 0\}},\end{aligned} \tag{400}$$

respectively.

Then, noting that

$$\begin{aligned}&|\phi(t,x,\omega,y,z) - \phi(t,x,\omega,y',z')| \\ &\le L|y-y'| + |\bar{\lambda}_{1,t}(x)|||z_1| - |z'_1|| + |\bar{\lambda}_{2,t}(x)|||z_2| - |z'_2|| \\ &\le L|y-y'| + |\bar{\lambda}_{1,t}(x)||z_1 - z'_1| + |\bar{\lambda}_{2,t}(x)||z_2 - z'_2| \\ &\le L|y-y'| + |\bar{\lambda}_t(x)||z-z'| \\ &\le L|y-y'| + \|\bar{\lambda}(x)\|_t|z-z'|,\end{aligned} \tag{401}$$

we have

$$\begin{aligned}&|\phi^{n,m}(t,x,\omega,y,z) - \phi^{n,m}(t,x,\omega,y',z')| \\ &\le |\phi(t,x,\omega,y,z) - \phi(t,x,\omega,y',z')|1_{\{\|\bar{\lambda}(x)\|_t \le n\}}1_{\{\phi(t,x,\omega,y,z)\ge 0\}} \\ &\quad + |\phi(t,x,\omega,y,z) - \phi(t,x,\omega,y',z')|1_{\{\|\bar{\lambda}(x)\|_t \le m\}}1_{\{\phi(t,x,\omega,y,z)< 0\}} \\ &\le L|y-y'| + (n+m)|z-z'|,\end{aligned} \tag{402}$$

which corresponds to (323).

For uniqueness of the solution, let $(Y,Z),(Y',Z')$ be solutions of BSDE (23). Then, instead of (386), we have

$$\begin{aligned} d(Y_t - Y'_t) &= -\frac{g(t,B,X_t,Y_t) - g(t,B,X_t,Y'_t)}{Y_t - Y'_t} 1_{\{Y_t - Y'_t \neq 0\}}(Y_t - Y'_t)dt \\ &\quad + |\bar{\lambda}_{1,t}(X_t)| \frac{|Z_{1,t}| - |Z'_{1,t}|}{Z_{1,t} - Z'_{1,t}} 1_{\{Z_{1,t} - Z'_{1,t} \neq 0\}}(Z_{1,t} - Z'_{1,t})dt \\ &\quad - |\bar{\lambda}_{2,t}(X_t)| \frac{|Z_{2,t}| - |Z'_{2,t}|}{Z_{2,t} - Z'_{2,t}} 1_{\{Z_{2,t} - Z'_{2,t} \neq 0\}}(Z_{2,t} - Z'_{2,t})dt \\ &\quad + \sum_{j=1}^{d} (Z_{j,t} - Z'_{j,t})dB_{j,t}, \ Y_T - Y'_T = 0. \end{aligned} \tag{403}$$

Setting $\bar{Y}_t = Y_t - Y'_t$, $\bar{Z}_{j,t} = Z_{j,t} - Z'_{j,t}$, $j = 1, \ldots, d$, $b_t = \frac{g(t,B,X_t,Y_t) - g(t,B,X_t,Y'_t)}{Y_t - Y'_t} 1_{\{Y_t - Y'_t \neq 0\}}$, $c_{1,t} = -|\bar{\lambda}_{1,t}(X_t)| \frac{|Z_{1,t}| - |Z'_{1,t}|}{Z_{1,t} - Z'_{1,t}} 1_{\{Z_{1,t} - Z'_{1,t} \neq 0\}}$, $c_{2,t} = |\bar{\lambda}_{2,t}(X_t)| \frac{|Z_{2,t}| - |Z'_{2,t}|}{Z_{2,t} - Z'_{2,t}} 1_{\{Z_{2,t} - Z'_{2,t} \neq 0\}}$, we have

$$d\bar{Y}_t = -b_t \bar{Y}_t dt + \bar{Z}_{1,t}(dB_{1,t} - c_{1,t}dt) + \bar{Z}_{2,t}(dB_{2,t} - c_{2,t}dt) + \sum_{j=3}^{d} \bar{Z}_{j,t} dB_{j,t}, \ \bar{Y}_T = 0. \tag{404}$$

Since

$$\begin{aligned} |c_{1,t}| &= |\bar{\lambda}_{1,t}(X_t)| \frac{||Z_{1,t}| - |Z'_{1,t}||}{|Z_{1,t} - Z'_{1,t}|} \\ &\leq |\bar{\lambda}_{1,t}(X_t)|, \\ |c_{2,t}| &= |\bar{\lambda}_{2,t}(X_t)| \frac{||Z_{2,t}| - |Z'_{2,t}||}{|Z_{2,t} - Z'_{2,t}|} \\ &\leq |\bar{\lambda}_{2,t}(X_t)|, \end{aligned} \tag{405}$$

if a weak version of Novikov's condition (21) is satisfied for $\bar{\lambda}_1$ and $\bar{\lambda}_2$, then the probability measure P^{c_1,c_2}

$$P^{c_1,c_2}(A) = E\left[\exp\left(-\frac{1}{2}\sum_{j=1}^{2}\int_0^T c_{j,s}^2 ds + \sum_{j=1}^{2}\int_0^T c_{j,s} dB_{j,s}\right) 1_A\right], \ A \in \mathcal{F}, \tag{406}$$

is well-defined and Girsanov's theorem is applied.

Then,

$$d\bar{Y}_t = -b_t \bar{Y}_t dt + \sum_{j=1}^{d} \bar{Z}_{j,t} dB_{j,t}^{c_1,c_2}, \ \bar{Y}_T = 0, \tag{407}$$

where $B^{c_1,c_2} = (B_1^{c_1,c_2}, \ldots, B_d^{c_1,c_2})$ define by

$$\begin{aligned} B_{1,t}^{c_1,c_2} &= B_{1,t} - \int_0^t c_{1,s} ds, \\ B_{2,t}^{c_1,c_2} &= B_{2,t} - \int_0^t c_{2,s} ds, \\ B_{j,t}^{c_1,c_2} &= B_{j,t} \ (3 \leq j \leq d). \end{aligned} \tag{408}$$

is a d-dimensional Brownian motion under P^{c_1,c_2}.

The rest of the proof is the same as the one for Proposition 1. $\square$

C Equilibrium interest rate without fundamental uncertainties

In this section, we show that BSDE (97) with the stochastic differential power utility f in (94)

$$\begin{aligned} dV_t &= -\left[\beta\frac{e_t^\rho}{\rho}(1+\alpha V_t)^{(\alpha-\rho)/\alpha} - \frac{\beta}{\rho}(1+\alpha V_t)\right]dt + \sigma_{v,t}(1+\alpha V_t)dB_t, \\ V_T &= \frac{e_T^\alpha - 1}{\alpha}, \end{aligned} \tag{409}$$

that is

$$dV_t = -f(e_t, V_t)dt + \sigma_{V,t}dB_t \tag{410}$$

where

$$f(e_t, V_t) = \beta\frac{e_t^\rho}{\rho}(1+\alpha V_t)^{(\alpha-\rho)/\alpha} - \frac{\beta}{\rho}(1+\alpha V_t), \tag{411}$$

$$\sigma_{V,t} = \sigma_{v,t}(1+\alpha V_t), \tag{412}$$

is explicitly solved and r is obtained as in (101).

Remark 10. *The stochastic differential power utility in (411) is derived from the standard version of stochastic differential power utility (continuous version of recursive power utility) that includes the volatility of the BSDE in the aggregator (driver), as follows.*

$$\frac{dv_t}{v_t} = -\left[\beta\frac{(c_t/v_t)^\rho - 1}{\rho} - \frac{\gamma}{2}|\sigma_{v,t}|^2\right]dt + \sigma_{v,t}dB_t, \tag{413}$$

$$v_T = c_T \tag{414}$$

with $\beta > 0$, $\rho = 1-\delta < 1$, $\gamma > 0$.

For $\rho = 0$ ($\delta = 1$),

$$\frac{dv_t}{v_t} = -\left[\beta\log(c_t/v_t) - \frac{\gamma}{2}|\sigma_{v,t}|^2\right]dt + \sigma_{v,t}dB_t, \tag{415}$$

$$v_T = c_T \tag{416}$$

Let

$$V = \phi_A(v) := \frac{v^\alpha - 1}{\alpha} \tag{417}$$

with $\alpha = 1-\gamma < 1$. Then, $v_t = (1+\alpha V_t)^{1/\alpha}$, $\phi_A'(v) = v^{\alpha-1}$ and $\phi_A''(v) = (\alpha-1)v^{\alpha-2}$.

Applying Ito's formula, we have

$$dV_t = -\left[\beta\frac{c^\rho}{\rho}(1+\alpha V_t)^{(\alpha-\rho)/\alpha} - \frac{\beta}{\rho}(1+\alpha V_t)\right]dt + \sigma_{v,t}(1+\alpha V_t)dB_t, \tag{418}$$

$$V_T = \frac{c_T^\alpha - 1}{\alpha}. \tag{419}$$

Then, we have the following:
For $\beta > 0$, $\rho < 1(\rho \neq 0)$ and $\alpha < 1(\alpha \neq 0)$ (stochastic differential power utility),

$$dV_t = -f(c_t, V_t)dt + \sigma_{V,t}dB_t \tag{420}$$

where

$$f(c_t, V_t) = \beta\frac{c_t^\rho}{\rho}(1+\alpha V_t)^{(\alpha-\rho)/\alpha} - \frac{\beta}{\rho}(1+\alpha V_t), \tag{421}$$

$$\sigma_{V,t} = \sigma_{v,t}(1+\alpha V_t). \tag{422}$$

For $\rho = 0$ with $\beta > 0$ and $\alpha < 1(\alpha \neq 0)$ (stochastic differential log-utility),

$$f(c_t, V_t) = \beta(1+\alpha V_t)\left[\log c_t - \frac{\log(1+\alpha V_t)}{\alpha}\right], \tag{423}$$

$$\sigma_{V,t} = \sigma_{v,t}(1+\alpha V_t). \tag{424}$$

For $\rho = \alpha \neq 0$ with $\beta > 0$ (standard power utility),

$$f(c_t, V_t) = \frac{\beta}{\rho}(c_t^\rho - 1) - \beta V_t \ (\rho = \alpha \neq 0, \tag{425}$$

$$\sigma_{V,t} = \sigma_{v,t}(1+\alpha V_t). \tag{426}$$

For $\rho = \alpha = 0$ with $\beta > 0$ (standard log-utility),

$$f(c_t, V_t) = \beta\left[\log c_t - V_t\right], \tag{427}$$

$$\sigma_{V,t} = \sigma_{v,t}(1+\alpha V_t). \tag{428}$$

Let us suppose that the endowment process is given as

$$\frac{de_t}{e_t} = \mu_e dt + \sigma_e dB_t \tag{429}$$

with a constant μ_e and a constant vector σ_e.

Let us assume $V_t = V(e_t, t) = \frac{A(t)e_t^\alpha - 1}{\alpha} \in C^{2,1}$ and since $\int_0^t f(e_s, V_s)dt + V(e_t, t)$ is a martingale, Ito's formula implies

$$f + \partial_t V + \mu_{e,t}e_t\partial_e V + \frac{1}{2}\sigma_{e,t}^2 e_t^2 \partial_e^2 V = 0,$$

$$V_T = \frac{e_T^\alpha - 1}{\alpha}. \tag{430}$$

Then, by (430), we obtain an ODE

$$A'(t) = pA(t) + qA^{1-\frac{\rho}{\alpha}}(t),\ A(T) = 1, \tag{431}$$

with

$$p = \alpha(\beta/\rho - \mu_e - (\alpha - 1)\sigma_e^2/2),\ q = -\alpha\beta/\rho. \tag{432}$$

Let $u(t) = A(t)^{\rho/\alpha}$, and then

$$u'(t) = au(t) + b,\ u(T) = 1, \tag{433}$$

with $a = p\rho/\alpha = \beta - \rho(\mu_e + (\alpha - 1)\sigma_e^2/2)$ and $b = q\rho/\alpha = -\beta$.

Hence, we obtain

$$\begin{aligned} u(t) &= e^{-a(T-t)}u(T) + \frac{\beta}{a}(1 - e^{-a(T-t)}) \\ &= e^{-a(T-t)} + \frac{\beta}{a}(1 - e^{-a(T-t)}), \end{aligned} \tag{434}$$

with $a = \beta - \rho(\mu_e + (\alpha - 1)\sigma_e^2/2)$, and $u(t) > 0$ for $t \in [0, T]$ since $\beta > 0$.

Thus,

$$V_t = \frac{A(t)e_t^\alpha - 1}{\alpha}, \tag{435}$$

with $A(t) = u(t)^{\alpha/\rho}$.

We also obtain $\sigma_V(t) = A(t)\sigma_e e_t^\alpha = (1 + \alpha V)\sigma_e$, and $\sigma_v = \sigma_e$.

By applying Ito's formula to (309), the interest rate r in (308) is given by

$$r_t = \beta + (1 - \rho)\mu_e - \frac{|\sigma_e|^2}{2}(1 - \alpha)(2 - \rho). \tag{436}$$

with $1 - \alpha > 0$, and $\sigma_\pi = -\lambda = -(1 - \alpha)\sigma_{e,t}$. ($\lambda$ denotes the market price of risk.)

For $\rho = 0$ ($\delta = 1$),

$$r_t = \beta + \mu_e - \sigma_e^2(1 - \alpha) \tag{437}$$

with $1 - \alpha > 0$, and $\sigma_\pi = -\lambda = -(1 - \alpha)\sigma_{e,t}$.

D $A(x, t)$ for determination of $sgn(\sigma_V)$ in Example 3

$A(x, t)$ in Example 3 in Section 8.1 is explicitly calculated and concrete conditions that determine the signs of $\sigma_V^k,\ k = 1, 2$ are obtained as follows.

$$r_t^A = r^A(x_t, t) = \beta + \frac{\rho(1 - \rho)}{2}|\sigma_e(x_t, t)|^2 - \rho\mu_e^*(x_t, t) \tag{438}$$

$$= \beta + \frac{\rho(1 - \rho)}{2}[\tilde{\sigma}_{e,1}^2 x_{1,t} + \tilde{\sigma}_{e,2}^2 x_{2,t} + \tilde{\sigma}_{e,3}^2] \tag{439}$$

$$-\rho\left[(\tilde{\mu}_{e,1} + \bar{\lambda}_{1,t}\tilde{\sigma}_{e,1})x_1 + (\tilde{\mu}_{e,2} + \bar{\lambda}_{2,t}\tilde{\sigma}_{e,2})x_2 + \tilde{\mu}_{e,3}x_3 + \tilde{\mu}_{e,0}\right] \tag{440}$$

$$r_t^A(x,t) = \sum_{i=1}^{2}\left[\frac{\rho(1-\rho)}{2}\sigma_{e,i}^2 - \rho(\tilde{\mu}_{e,i} + \bar{\lambda}_{i,t}\tilde{\sigma}_{e,i})\right]x_i - \rho\tilde{\mu}_{e,3}x_3 + \left[\beta + \frac{\rho(1-\rho)}{2}\tilde{\sigma}_{e,3}^2 - \rho\tilde{\mu}_{e,0}\right] \tag{441}$$

$$r_t^A = \gamma_0 + \sum_{i=1}^{3}\gamma_i x_{i,t}, \tag{442}$$

with

$$\gamma_0 = \beta + \frac{\rho(1-\rho)}{2}\tilde{\sigma}_{e,3}^2 - \rho\tilde{\mu}_{e,0}, \tag{443}$$

$$\gamma_i = \frac{\rho(1-\rho)}{2}\sigma_{e,i}^2 - \rho(\tilde{\mu}_{e,i} + \bar{\lambda}_{i,t}\tilde{\sigma}_{e,i}),\ (i=1,2), \tag{444}$$

$$\gamma_3 = -\rho\tilde{\mu}_{e,3}. \tag{445}$$

$$\begin{aligned}
&\beta + \partial_t A(x,t) - A(x,t)\left[\gamma_0 + \sum_{i=1}^{3}\gamma_i x_{i,t}\right] \\
&+ \sum_{i=1}^{3}\partial_{x_i}A(x,t)\mu_i^A(x_t,t) + \frac{1}{2}\left[\partial_{x_1^2}^2 A(x,t)\tilde{\sigma}_{x,1}^2 x_{1,t} + \partial_{x_2^2}^2 A(x,t)\tilde{\sigma}_{x,2}^2 x_{2,t} + \partial_{x_3^2}^2 A(x,t)\tilde{\sigma}_{x,3}^2\right] = 0, \\
&A_T = 1.
\end{aligned} \tag{446}$$

By Feynman-Kac formula,

$$A(x,t) = \mathbf{E}_t^A\left[e^{-\int_t^T r_s^A ds} + \beta\int_t^T e^{-\int_t^u r_s^A ds}du\right]. \tag{447}$$

By independence among B_i^A $(i=1,2,3)$,

$$E_t^A[e^{-\int_t^u r_s^A ds}] = e^{-\gamma_0(u-t)}\Pi_{i=1}^3 E_t^A[e^{-\int_t^u \gamma_{i,s}x_{i,s}ds}]. \tag{448}$$

For $i=1,2$,

$$\mu_i^A(x_t,t) = \mu_{x_i}^*(x_t,t) + \rho\sum_{k=1}^{d}\sigma_e^k(x_t,t)\sigma_{x_i}^k(x_t,t) \tag{449}$$

$$= (\tilde{\mu}_{x_i,1} + \bar{\lambda}_{i,t}\tilde{\sigma}_{x_i})x_{i,t} + \mu_{x_i,0} + \rho\tilde{\sigma}_{e,i}\tilde{\sigma}_{x,i}x_{i,t} \tag{450}$$

$$= (\tilde{\mu}_{x_i,1} + \bar{\lambda}_{i,t}\tilde{\sigma}_{x_i} + \rho\tilde{\sigma}_{e,i}\tilde{\sigma}_{x,i})x_{i,t} + \mu_{x_i,0}. \tag{451}$$

For $i=3$,

$$\mu_i^A(x_t,t) = \tilde{\mu}_{x_3,1}x_{3,t} + \mu_{x_3,0}. \tag{452}$$

$$dx_{1,t} = (a_1 - b_1 x_{1,t})dt + \tilde{\sigma}_{x,1}\sqrt{x_{1,t}}dB^A_{1,t},\ x_{1,0} > 0, \tag{453}$$

with $a_1 = \mu_{x_1,0} \geq 0$ and $b_1 = -(\tilde{\mu}_{x_1,1} + \bar{\lambda}_{1,t}\tilde{\sigma}_{x_1} + \rho\tilde{\sigma}_{e,1}\tilde{\sigma}_{x,1})$.

Let $\hat{x}_{1,t} := |\gamma_1| x_{1,t}$.

$$d\hat{x}_{1,t} = (\hat{a}_1 - b_1 \hat{x}_{1,t})dt + \hat{\sigma}_{x,1}\sqrt{\hat{x}_{1,t}}dB^A_{1,t},\ \hat{x}_{1,0} > 0 \tag{454}$$

with $\hat{a}_1 = |\gamma_1|\mu_{x_1,0} \geq 0$ and $\hat{\sigma}_{x,1} = \sqrt{|\gamma_1|}\tilde{\sigma}_{x,1}$.

We note that there exits the unique strong solution $\hat{x}_{1,t} \geq 0$ for all $t \geq 0$.

$$dx_{2,t} = (a_2 - b_2 x_{2,t})dt + \tilde{\sigma}_{x,2}\sqrt{x_{2,t}}dB^A_{2,t},\ x_{2,0} > 0 \tag{455}$$

with $a_2 = \mu_{x_2,0} \geq 0$ and $b_2 = -(\tilde{\mu}_{x_2,1} + \bar{\lambda}_{2,t}\tilde{\sigma}_{x_2} + \rho\tilde{\sigma}_{e,2}\tilde{\sigma}_{x,2})$.

Let $\hat{x}_{2,t} := |\gamma_2| x_{2,t}$.

$$d\hat{x}_{2,t} = (\hat{a}_2 - b_2 \hat{x}_{2,t})dt + \hat{\sigma}_{x,2}\sqrt{\hat{x}_{2,t}}dB^A_{2,t},\ \hat{x}_{2,0} > 0 \tag{456}$$

with $\hat{a}_2 = |\gamma_2|\mu_{x_2,0} \geq 0$, and $\hat{\sigma}_{x,2} = \sqrt{|\gamma_2|}\tilde{\sigma}_{x,2}$.

Again, there exits the unique strong solution $\hat{x}_{2,t} \geq 0$ for all $t \geq 0$.

$$dx_{3,t} = (a_3 - b_3 x_{3,t})dt + \tilde{\sigma}_{x,3}dB^A_{3,t} \tag{457}$$

with $a_3 = \mu_{x_3,0}$ and $b_3 = -\tilde{\mu}_{x_3,1}$.

Let $\hat{x}_{3,t} := \gamma_3 x_{3,t}$.

$$d\hat{x}_{3,t} = (\hat{a}_3 - b_3 \hat{x}_{3,t})dt + \hat{\sigma}_{x,3}dB^A_{3,t}, \tag{458}$$
$$(\hat{a}_3 = \gamma_3\mu_{x_3,0},\ \hat{\sigma}_{x,3} = \gamma_3\tilde{\sigma}_{x,3}). \tag{459}$$

$$A(x,t) = \mathbf{E}^A_t\left[e^{-\int_t^T r^A_s ds}\right] + \beta\int_t^T \mathbf{E}^A_t\left[e^{-\int_t^u r^A_s ds}\right]du \tag{460}$$

$$\tag{461}$$

with

$$E^A_t[e^{-\int_t^u r^A_s ds}] = e^{-\gamma_0(u-t)}\Pi^3_{i=1}A_i(t,u). \tag{462}$$

For $i = 1, 2$, let

$$A_i(t,u) = E^A_t[e^{-\int_t^u \gamma_{i,s}x_{i,s}ds}] = E^A_t[e^{-\int_t^u \hat{x}_{i,s}ds}],\ (\gamma_i > 0), \tag{463}$$
$$A_i(t,u) = E^A_t[e^{-\int_t^u \gamma_{i,s}x_{i,s}ds}] = E^A_t[e^{\int_t^u \hat{x}_{i,s}ds}],\ (\gamma_i < 0). \tag{464}$$

For $i = 1, 2$, with constants, $a_i \geq 0(\mu_{x_i,0} \geq 0)$, b_i, σ_i, $i = 1, 2$ such that $b_i^2 > 2\sigma_i^2$ when $\gamma_i < 0$.

Then, we obtain with $\tau = u - t$, $h_i = \frac{\sqrt{b_i^2+2\sigma_i^2}}{2}$ ($\gamma_i > 0$) and $h_i = \frac{\sqrt{b_i^2-2\sigma_i^2}}{2}$ ($\gamma_i < 0$, $b_i^2 > 2\sigma_i^2$),

$$A_i(t,u) = e^{-\hat{x}_{i,t}\beta_i(\tau)-\rho_i(\tau)},\ (\gamma_i > 0) \tag{465}$$
$$\beta_i(\tau) = \frac{\sinh(h_i\tau)}{h_i\cosh(h_i\tau)+\frac{b_i}{2}\sinh(h_i\tau)}, \tag{466}$$
$$\rho_i(\tau) = \frac{-2\hat{a}_i}{\hat{\sigma}_i^2}\log\left\{\frac{h_i\exp(\frac{b_i\tau}{2})}{h_i\cosh(h_i\tau)+\frac{b_i}{2}\sinh(h_i\tau)}\right\}, \tag{467}$$

$$A_i(t,u) = e^{\hat{x}_{i,t}\beta_i(\tau)+\rho_i(\tau)},\ (\gamma_i < 0) \tag{468}$$
$$\beta_i(\tau) = \frac{\sinh(h_i\tau)}{h_i\cosh(h_i\tau)+\frac{b_i}{2}\sinh(h_i\tau)}, \tag{469}$$
$$\rho_i(\tau) = \frac{2\hat{a}_i}{\hat{\sigma}_i^2}\log\left\{\frac{h_i\exp(\frac{b_i\tau}{2})}{h_i\cosh(h_i\tau)+\frac{b_i}{2}\sinh(h_i\tau)}\right\}, \tag{470}$$

where $\sinh(y) = \frac{e^y-e^{-y}}{2}$ and $\cosh(y) = \frac{e^y+e^{-y}}{2}$.
For $i = 3$ with $b_3 = 0$,

$$A_3(t,u) = \exp\left(-\hat{x}_{3,t}\tau - \frac{\hat{a}_3}{2}\tau^2 + \frac{\hat{\sigma}_3^2}{6}\tau^3.\right) \tag{471}$$

For $i = 3$ with $b_3 \neq 0$,

$$A_3(t,u) = \exp\left(-\hat{x}_{3,t}\beta_3(\tau) - \rho_3(\tau)\right), \tag{472}$$

where

$$\beta_3(\tau) = \frac{1-e^{-b_3\tau}}{b_3}, \tag{473}$$
$$\rho_3(\tau) = -\frac{1}{2}\left(\frac{\hat{\sigma}_3}{b_3}\right)^2\left[\tau - 2\beta_3(\tau) + \frac{1-e^{-2b_3\tau}}{2b_3}\right] + \frac{\hat{a}_3}{b_3}\left[\tau - \beta_3(\tau)\right]. \tag{474}$$

$$\partial_{x_1} r_t^A = \frac{\rho(1-\rho)}{2}\tilde{\sigma}_{e,1}^2 - \rho(\tilde{\mu}_{e,1} + \bar{\lambda}_{1,t}\tilde{\sigma}_{e,1}), \tag{475}$$
$$\partial_{x_2} r_t^A = \frac{\rho(1-\rho)}{2}\tilde{\sigma}_{e,2}^2 - \rho(\tilde{\mu}_{e,2} + \bar{\lambda}_{2,t}\tilde{\sigma}_{e,2}), \tag{476}$$
$$\partial_{x_3} r_t^A = -\rho\tilde{\mu}_{e,3} \tag{477}$$

For $k = 1, 2$,

$$\sum_{i=1}^{l}\partial_{x_i}A(x,t)\sigma_{x,i}^k(x,t) = \partial_{x_k}A(x,t)\sigma_{x,k}^k(x,t) = \partial_{x_k}A(x,t)\tilde{\sigma}_{x,k}\sqrt{x_{k,t}}, \tag{478}$$

where

$$\partial_{x_k} A(x,t) = -\mathbf{E}_t^A \left[\int_t^T \left[\frac{\rho(1-\rho)}{2} \tilde{\sigma}_{e,k}^2 - \rho(\tilde{\mu}_{e,k} + \bar{\lambda}_{k,s} \tilde{\sigma}_{e,k}) \right] ds \; e^{-\int_t^T r_s^A ds} \right. \tag{479}$$

$$\left. +\beta \int_t^T \int_t^u \left[\frac{\rho(1-\rho)}{2} \tilde{\sigma}_{e,k}^2 - \rho(\tilde{\mu}_{e,k} + \bar{\lambda}_{k,s} \tilde{\sigma}_{e,k}) \right] ds \; e^{-\int_t^u r_s^A ds} du \right]. \tag{480}$$

Thus, the sign of

$$\left[-\frac{\rho(1-\rho)}{2} \tilde{\sigma}_{e,k}^2 + \rho(\tilde{\mu}_{e,k} + \bar{\lambda}_{k,s} \tilde{\sigma}_{e,k}) \right] \tilde{\sigma}_{x,k} \tag{481}$$

determines the sign of $\partial_{x_k} A(x,t) \tilde{\sigma}_{x,k}$.