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Abstract

This paper characterizes optimal commitment policy in the New Keynesian model using a recursive formulation of the central bank's infinite horizon optimization problem in which promised inflation and output gap—as opposed to lagged Lagrange multipliers—act as pseudo-state variables. Our recursive formulation is motivated by Kydland and Prescott (1980). Using three well known variants of the model—one featuring inflation bias, one featuring stabilization bias, and one featuring a lower bound constraint on nominal interest rates—we show that the proposed formulation sheds new light on the nature of the intertemporal trade-off facing the central bank.

JEL: E32, E52, E61, E62, E63

Keywords: Commitment, Inflation Bias, Optimal Policy, Ramsey Plans, Stabilization Bias, Zero Lower Bound.

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1 Introduction

Optimal commitment policy is a widely adopted approach among economists and policymakers to studying the question of how to best conduct monetary policy. For example, at the Federal Reserve, the results of optimal commitment policy analysis from the FRB/US model have for some time been regularly presented to the Federal Open Market Committee to help inform its policy decisions (Brayton, Laubach, and Reifschneider (2014)). Most recently, in many advanced economies where the policy rate was constrained at the effective lower bound (ELB), the insights from the optimal commitment policy in a stylized New Keynesian model have played a key role in the inquiry on how long the policy rate should be kept at the ELB (Bullard (2013), Evans (2013), Kocherlakota (2011), Plosser (2013), Woodford (2012)). Accordingly, a deep understanding of optimal commitment policy is as relevant as ever.

In this paper, we contribute to a better understanding of optimal commitment policy in the New Keynesian model—a workhorse model for analyzing monetary policy—by characterizing it using a novel recursive method. Our method uses promised values of inflation and output as pseudo-state variables in the spirit of Kydland and Prescott (1980) instead of lagged Lagrange multipliers as in the standard method of Marcet and Marimon (2016). We describe our recursive approach—which we will refer to as the promised value approach—in three variants of the New Keynesian model that have been widely studied in the literature: the model with inflation bias, the model with stabilization bias, and the model with an ELB constraint. In each model, we define the infinite-horizon problem of the Ramsey planner, provide the recursive formulations of the Ramsey planner's problem via the promised value approach, and describe the tradeoff facing the central bank in determining the optimal commitment policy.

The idea of using promised values as pseudo-state variables to recursify the infinite-horizon problem of the Ramsey planner was first suggested by Kydland and Prescott (1980) in the context of an optimal capital taxation problem. Later, Chang (1998) and Phelan and Stacchetti (2001) formally described, as an intermediate step toward characterizing sustainable policies, the recursive formulation of the Ramsey planner's problem using promised marginal utility in models with money and with fiscal policy, respectively. However, because their focus was on characterizing sustainable policies, they did not solve for the Ramsey policy. To our knowledge, we are the first to formulate and solve the Ramsey policy using the promised value approach.[1]

Our aim is not to argue that readers should use the promised value approach instead of the Lagrange multiplier approach. Rather, our aim is to show that the promised value approach can be a useful analytical tool to supplement the analysis based on the standard

[1]The only exception is a recent lecture note by Sargent and Stachurski (2018) which characterizes the Ramsey policy in the linear-quadratic version of the model of Cagan (1956) using the promised value approach. Note that, while the Lagrange multiplier approach is almost always used in solving the Ramsey problems in business cycle models, the promised value approach is extensively used in the literature of dynamic contract.

Lagrange multiplier approach. Both approaches should be able to find the same allocation; we indeed find that both approaches reliably compute the optimal commitment policies in the New Keynesian model. However, the Ramsey policies are often history-dependent in complex ways, and it is not always straightforward for researchers to understand the trade-off facing the central bank. Accordingly, it is useful for researchers to have an alternative way to analyze the Ramsey policy, as it may provide new insights on the optimal commitment policy.

One difficulty associated with the promised value approach is that it requires researchers to compute the set of feasible promised values (see discussion in Marcet and Marimon (2016)). We find that the extent to which this computation poses a challenge depends on the model. For the model with inflation bias and the model with stabilization bias, the promised rate of inflation is the only pseudo state variable, and we analytically show that the set of feasible *promised* inflation rates are identical to the set of feasible *actual* inflation rates—which is a primitive of the models—under nonrestrictive conditions. For the model with the ELB constraint, the set of feasible promised inflation-output pairs cannot be found analytically, and one needs a computationally nontrivial method described in Chang (1998) and Phelan and Stacchetti (2001), among others, to find the set. In our numerical example, we find that the set is large and does not represent a binding constraint for the control variables in the Bellman equation. Thus, if one wants to casually use the promised value approach, abstracting from the task of characterizing the set of feasible promises is unlikely to be harmful.

Part of our contribution is pedagogical. As discussed above, the idea of solving the Ramsey policy using the promised value approach has been around for a few decades. Yet, researchers almost always use the Lagrange multiplier approach to solve the Ramsey policy. Our detailed description of how to adopt the promised value approach to a well-known optimal policy problem will be useful to other researchers who would like to adopt this approach to other interesting optimal policy problems.

In addition to Chang (1998) and Phelan and Stacchetti (2001) who recursified the Ramsey planner's problem using promised values, our paper is closely related to the large literature on optimal policy in New Keynesian models. Optimal commitment policies in the model with inflation bias and in the model with stabilization bias have been studied by many, including Clarida, Gali, and Gertler (1999), Galí (2015), and Woodford (2003). Optimal commitment policy in the model with the ELB constraint has been studied by Eggertsson and Woodford (2003), Jung, Teranishi, and Watanabe (2005), Adam and Billi (2006), and Nakov (2008), among others. All of these papers—often implicitly but sometimes explicitly—rely on the method of Marcet and Marimon (2016) and use the lagged Lagrange multipliers as pseudo-state variables to characterize optimal commitment policies; our contribution is to provide an alternative method to characterize them.

Our paper is also related to Waki, Dennis, and Fujiwara (2018) who study a mechanism-design problem under private information in a New Keynesian model. The recursive characterization of their mechanism-design problem features promised inflation as a pseudo state variable, as in our paper, and the limiting full-information version of their model corresponds to the standard New Keynesian model with stabilization bias we consider in Section 4. Our paper is different from their work because we (i) examine the intertemporal trade-off facing the central bank under the promised-value approach, (ii) contrast it with the trade-off under the Lagrange multiplier approach, and (iii) consider two other versions of the New Keynesian model—one with inflation bias and the other with ELB—that are commonly used in the literature.

The rest of the paper is organized as follows. Sections 2, 3, and 4 study the model with inflation bias, the model with stabilization bias, and the model with the ELB constraint, respectively. In each section, we first present the infinite-horizon problem of the Ramsey planner and describe how the infinite-horizon problem is made recursive under the promised value approach. We then discuss the dynamics of the Ramsey equilibrium, describe the key trade-off the central bank faces, and contrast the promised value approach with the standard Lagrange approach. Section 5 concludes.

2 Model with inflation bias

Our first model is the one with inflation bias, which is a version of the standard New Keynesian model in which the inefficiency associated with monopolistic competition in the product market is not offset by a production subsidy. As the model is standard, we refer interested readers to Woodford (2003) and Galí (2015) for more detailed descriptions. The economy starts at time one. The model is loglinearized around its deterministic steady state. Its private sector equilibrium conditions at time t are given by

$$\sigma y_t = \sigma y_{t+1} + \pi_{t+1} - r_t + r^*, \tag{1}$$

$$\pi_t = \kappa y_t + \beta \pi_{t+1}, \tag{2}$$

where y_t, π_t, and r_t are the output gap, inflation, and the policy rate, respectively. σ, κ, β are the inverse intertemporal elasticity of substitution, the slope of the Phillips curve, and the time discount rate, respectively. r^* is the long-run natural rate of interest. Equations (1) and (2) are referred to as the Euler equation and the Phillips curve, respectively. In the model with inflation bias, we abstract from the ELB constraint on r_t. With this abstraction, the Euler equation does not constrain the allocations the central bank can choose; it merely pins down the policy rate given the sequence of inflation and output. This abstraction is a common practice in the literature.

We assume that $y_t \in \mathbb{K}_Y$ and $\pi_t \in \mathbb{K}_\Pi$ where $\mathbb{K}_Y$ and $\mathbb{K}_\Pi$ are closed intervals on the real line, $\mathbb{R}$. For any variable x, let us denote $\{x_t\}_{t=1}^{\infty}$ by a bold font $\boldsymbol{x}$. We say $(\boldsymbol{y}, \boldsymbol{\pi})$ (that is, $\{y_t, \pi_t\}_{t=1}^{\infty}$) is a competitive outcome if equation (2) is satisfied for all $t \geq 1$, and use **CE** to denote the set of all competitive outcomes.

The sequence of values, $\{V_t\}_{t=1}^{\infty}$, associated with a competitive outcome, $\{y_t,\ \pi_t\}_{t=1}^{\infty}$, is given by

$$V_t = \sum_{k=t}^{\infty} \beta^{k-t} u(y_k, \pi_k),$$

where $u(\cdot, \cdot)$, the payoff function, is given by

$$u(y, \pi) = -\frac{1}{2}[\pi^2 + \lambda(y - y^*)^2]. \tag{3}$$

This quadratic payoff function can be derived as the second-order approximation to the household welfare.[2] The presence of y^* in this objective function captures the inefficiency associated with monopolistic competition in the product market. The problem of the Ramsey planner is to choose a competitive outcome that maximizes the time-one value as follows:

$$V_{ram,1} = \max_{(\boldsymbol{y},\boldsymbol{\pi})\in\mathbf{CE}} V_1. \tag{4}$$

The Ramsey outcome is defined as the solution to this optimization problem and is denoted by $\{y_{ram,t},\ \pi_{ram,t}\}_{t=1}^{\infty}$. The value sequence associated with the Ramsey outcome is denoted by $\{V_{ram,t}\}_{t=1}^{\infty}$.

2.1 Promised value approach

Under the promised value approach, the infinite-horizon optimization problem of the Ramsey planner given by equation (4) is divided into two steps. In the first step, the following constrained infinite-horizon Ramsey problem is formulated:

$$w^*(\eta) = \max_{(\boldsymbol{y},\boldsymbol{\pi})\in\Gamma(\eta)} -\frac{1}{2}\sum_{t=1}^{\infty} \beta^{t-1}\left[\pi_t^2 + \lambda\,(y_t - y^*)^2\right],$$

where $\Gamma(\eta)$ is the set of competitive outcomes in which the initial inflation, π_1, is η. This set is formally defined in Appendix A. In the second step, the Ramsey planner chooses the initial inflation promise, η, that maximizes $w^*(\eta)$. That is,

$$V_{ram,1} = \max_{\eta\in\Omega} \quad w^*(\eta), \tag{5}$$

where Ω is the set of time-one inflation rates consistent with the existence of a competitive outcome. This set is formally defined and computed analytically in Appendix A.

By the standard dynamic programming argument, it can be shown that $w^*(\eta)$ satisfies

[2] See, for example, Galí (2015) for the derivation.

the following functional equation:

$$w(\eta) = \max_{y \in \mathbb{K}_Y, \pi \in \mathbb{K}_\Pi, \eta' \in \Omega} \; u(y, \pi) + \beta w(\eta') \tag{6}$$

subject to

$$\pi = \eta$$
$$\pi = \kappa y + \beta \eta',$$

where η is the promised rate of inflation for the current period from the previous period, and η' is the promised rate of inflation for tomorrow. Conversely, if a bounded function, $w : \Omega \to \mathbb{R}$, satisfies this functional equation, then $w = w^*$.[3]

Let $\{w_{PV}(\cdot),\, y_{PV}(\cdot),\, \pi_{PV}(\cdot),\, \eta'_{PV}(\cdot)\}$ be the value and policy functions associated with this Bellman equation. The Ramsey value sequence and the Ramsey outcome are obtained by iterating over these functions with the time-one inflation rate set to the argmax of $w^*(\eta)$ in equation (5).

2.2 Lagrange multiplier approach

It is useful to contrast the recursive formulation of the promised value approach with that of the more standard Lagrange multiplier approach of Marcet and Marimon (2016). In the Lagrange multiplier approach, a saddle-point functional equation is used to recursify the infinite-horizon optimization problem of the Ramsey planner.[4] In the model with inflation bias, it is given by

$$W(\phi) = \min_{\phi'} \max_{y \in \mathbb{K}_Y, \pi \in \mathbb{K}_\Pi} \; f(y, \pi, \phi, \phi') + \beta W(\phi'), \tag{7}$$

where $f(\cdot)$, the modified payoff function, is given by

$$f(y, \pi, \phi, \phi') = u(y, \pi) + \phi'(\pi - \kappa y) - \phi \pi.$$

Let $\{y_{LM}(\cdot),\, \pi_{LM}(\cdot),\, \phi'_{LM}(\cdot)\}$ be the policy functions associated with this saddle-point functional equation. One can find the Ramsey outcome by iterating over these policy functions with the initial Lagrange multiplier set to zero.

[3] The proof is closely related to the proof of the Bellman optimality principle. See Chang (1998).

[4] In many papers, authors casually refer to the theory of Marcet and Marimon (2016) to justify the recursive characterization of the Ramsey policy with lagged Lagrange multipliers, and the saddle-point function equation is rarely explicitly formulated. For examples of papers explicitly formulating the saddle-point functional equation associated with the infinite-horizon optimization problem of the Ramsey planner in the context of sticky-price models, see Khan, King, and Wolman (2003), Adam and Billi (2006), Svensson (2010), and Nakata (2016).

2.3 Analysis of optimal policy

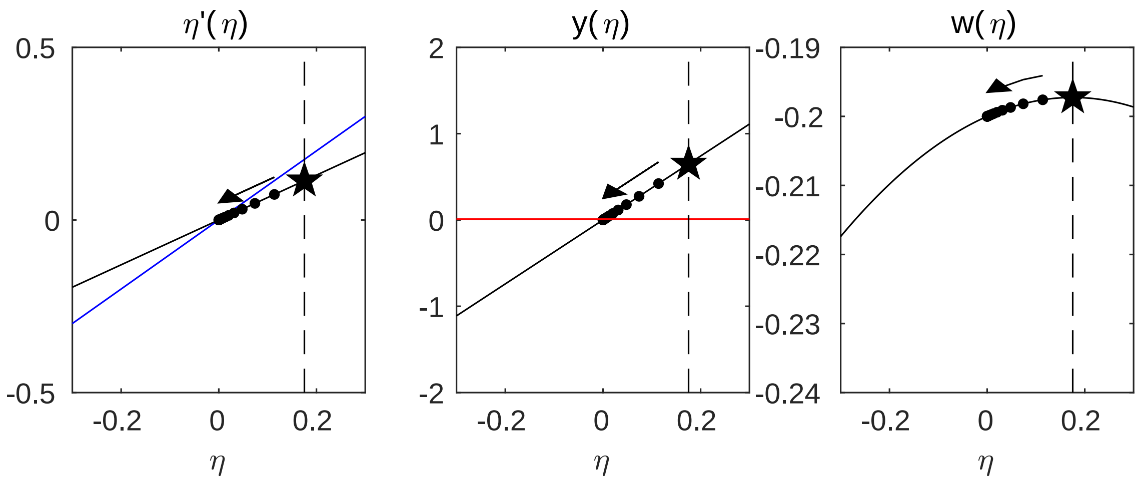
For both the Bellman equation of the promised value approach and the saddle-point functional equation of the Lagrange multiplier approach, the payoff function is quadratic and the constraints are linear. This linear-quadratic structure allows us to solve the model analytically for both approaches.[5] However, to describe how the promised value approach works in a transparent way, we use a numerical example in the main text and relegate the closed-form solutions to Appendix D. The parameter values used in the numerical example are from Woodford (2003) and are shown in Table 1.

Table 1: Parameters
—Model with Inflation Bias—

β	y^*	λ	κ
0.9925	0.01	0.003	0.024

Figure 1 shows the policy functions for the promised rate of inflation in the next period and the output gap in the current period as well as the value function associated with the Bellman equation.

Figure 1: Policy Functions from the Promised Value Approach
—Model with Inflation Bias—



Note: η is the rate of inflation that was promised in the previous period and needs to be delivered in the current period. η' is the promised rate of inflation for the next period. These rates are expressed in annualized percent. w is the value associated with the Bellman equation (equation (6)).

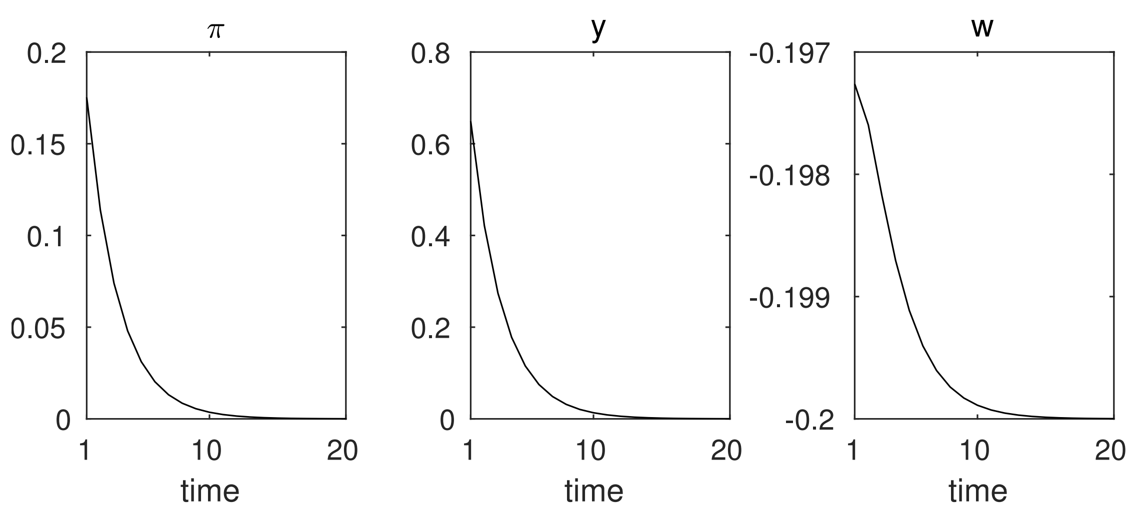
In the promised value approach, the initial inflation rate is given by the argmax of the value function associated with the Bellman equation—shown in the right panel of Figure 1.

[5] If the upper and lower bounds of the two closed intervals, $\mathbb{K}_Y$ and $\mathbb{K}_\Pi$, are binding constraints, the problem is not linear-quadratic. We confirmed that they are not binding constraints in our model.

According to the panel, the initial inflation rate—indicated by the dashed vertical line—is slightly below 0.2 percent. Once the initial inflation rate is determined, the dynamics of the economy are sequentially pinned down by the policy functions linking the promised rate of inflation in the current period (η) to the promised rate of inflation in the next period (η') and output in the current period (y), shown in the left and middle panels, respectively. For example, the time-two inflation rate is determined by the policy function for the promised rate of inflation evaluated at the initial inflation rate and is shown by the pentagram in the left-panel. The time-one output is determined by the policy function for output evaluated at the initial rate of inflation and is shown by the pentagram in the middle-panel. The black dots in the policy functions trace the dynamics of the economy afterward.

The implied dynamics of the economy are shown in Figure 2. The central bank has an incentive to generate a positive inflation rate at time one, which is associated with a level of output gap that is above zero but below y^*. Inflation and output converge eventually to zero, a well-known feature of the optimal commitment policy in this model.[6]

Figure 2: Dynamics
—Model with Inflation Bias—



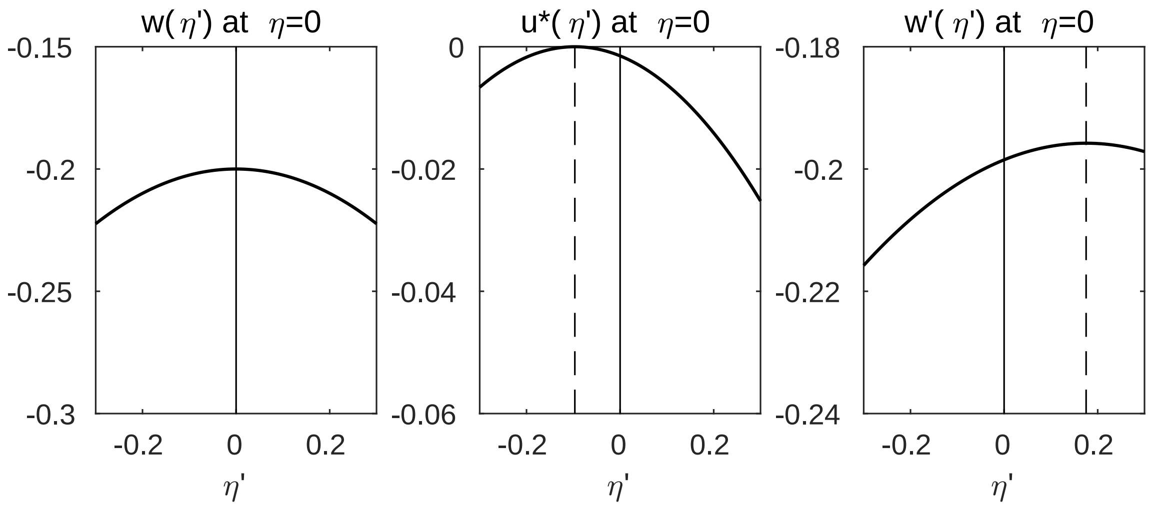
Note: The rate of inflation is expressed in annualized percent. The output gap is expressed in percent.

To understand the trade-off associated with the Bellman equation (equation (6)), we show in Figure 3 the objective function to be maximized and its two subcomponents—today's payoff, $u(\cdot, \cdot)$, and the discounted continuation value, $\beta w(\cdot)$—at $t = 20$ when the economy has essentially converged to its steady state of zero inflation so that $\eta = 0$. Note that two arguments for the payoff function, inflation and output, are functions of η and η'. Thus, the payoff function $u(\pi, y)$ can be transformed to an indirect payoff function, $u^*(\eta, \eta')$.

The fact that inflation is zero at the steady state is captured by the fact that the objective

[6] In Appendix G, we contrast the Ramsey equilibrium to the Markov perfect equilibrium and the value-maximizing pair of inflation and output.

Figure 3: Trade-off under the Promised Value Approach
—Model with Inflation Bias—



Note: η is the rate of inflation that was promised in the previous period and needs to be delivered in the current period. η' is the promised rate of inflation for the next period. These rates are expressed in annualized percent.

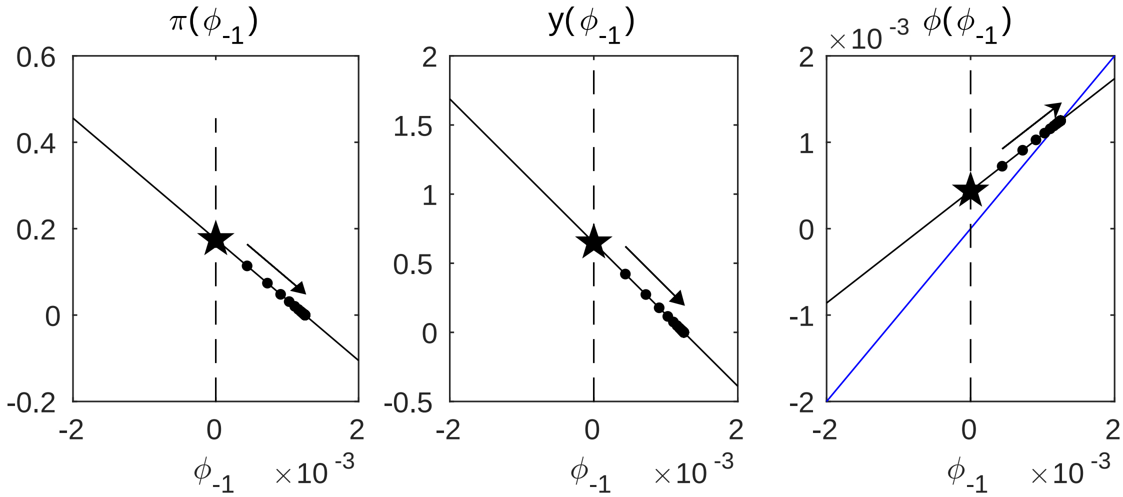
function evaluated at $\eta = 0$, shown by the left panel, is maximized at $\eta' = 0$. The optimality of promising zero inflation in the next period when the promised inflation rate for the current period is zero reflects two competing forces. The first force is how the promised inflation rate affects today's payoff. Given that the central bank needs to deliver zero inflation today, the lower the promised inflation rate is for next period, the higher the output today has to be in order to satisfy the Phillips curve.[7] Because of the presence of y^* in the payoff function, a higher output (a lower inflation) means a higher payoff as long as output is below y^*. Thus, the central bank has an incentive to promise some deflation next period, as captured by the middle panel of Figure 3 which shows that today's utility is maximized at $\eta' < 0$.

The second force is how the promised inflation affects the discounted continuation value. As shown in the right panel of Figure 3, a higher promised inflation rate is associated with a higher continuation value up to a certain point, as a higher future inflation is associated with a higher future level of output that is closer to y^*. The optimality of promising a zero inflation rate reflects these two competing effects of adjusting the inflation promise on the today's payoff and on the discounted continuation value.

We will close the section by examining the policy functions from the standard Lagrange multiplier approach. Figure 4 shows the policy functions for inflation, output, and the Lagrange multiplier associated with the saddle-point functional equation (7). Unlike in the promised value approach, these functions are functions of the lagged Lagrange multiplier, ϕ_{-1}. Time-one allocations are given by the policy functions evaluated at the initial lagged

[7]To see this, set $\eta = 0$ in the two constraints in the Bellman equation (6).

Figure 4: Policy Functions from the Lagrange-Multiplier Approach
—Model with Inflation Bias—



Note: ϕ_{-1} is the lagged Lagrange multiplier, whereas ϕ is the Lagrange multplier in the current period. The rate of inflation is expressed in annualized percent. The output gap is expressed in percent.

Lagrange multiplier of zero and are indicated by the pentagram. The black dots trace the dynamics of inflation, output, and the Lagrange multiplier after the first period. According to the right panel, the Lagrange multiplier eventually converges to a positive value. As the Lagrange multiplier converges, inflation and output also converge to zero, as shown in the left and middle panels, respectively. The dynamics of inflation and output derived from the Lagrange multiplier approach are of course identical to those implied by the promised value approach shown in Figure 2. In Appendix D, we provide analytical proof for their equivalence.

3 Model with stabilization bias

Our second model is the model with stabilization bias. The private sector equilibrium conditions in this model at time t are given by

$$\sigma y_t(s^t) = \sigma E_t y_{t+1}(s^{t+1}) + E_t \pi_{t+1}(s^{t+1}) - r_t(s^t) + r^*,$$
$$\pi_t(s^t) = \kappa y_t(s^t) + \beta E_t \pi_{t+1}(s^{t+1}) + s_t.$$

The key difference between this model and the model in the previous section is that in this model, there is a cost-push shock, denoted by s_t, that additively enters into the Phillips curve. The cost-push shock follows an N-state Markov process and its possible values are given by the set, $\mathbb{S} := \{e_1, e_2, ..., e_N\}$. The probability of moving from state i to state j is denoted by $p(e_j|e_i)$. s^t denotes the history of shocks up to time t. That is, $s^t := \{s_h\}_{h=1}^t$. Because there is uncertainty, the allocations are state-contingent and depend on s^t.

As in the model with inflation bias and consistent with common practice in the literature on stabilization bias, we abstract from the ELB constraint on the policy rate, which in turn allows us to abstract from the Euler equation. We assume that $y_t \in \mathbb{K}_Y$ and $\pi_t \in \mathbb{K}_\Pi$, where $\mathbb{K}_Y$ and $\mathbb{K}_\Pi$ are closed intervals on the real line, $\mathbb{R}$. For any variable x, let us denote its state-contingent sequence $\{x_t(s^t)\}_{t=1}^{\infty}$ by $\boldsymbol{x}$ (bold font) and its state-contingent sequence with the time-one state $s_1 = s$ by $\boldsymbol{x}(s)$. We say $(\boldsymbol{y}, \boldsymbol{\pi})$ is a competitive outcome if the Philips curve is satisfied for all $t \geq 1$. We use **CE** to denote the set of all competitive outcomes and use $\mathbf{CE}(s)$ to denote the set of competitive outcomes in which the initial state s_1 is s.

The sequence of values $\{V_t(s^t)\}_{t=1}^{\infty}$ associated with a competitive outcome is given by

$$V_t(s^t) = \sum_{k=t}^{\infty} \beta^{k-t} \sum_{s^k|s^t} \mu(s^k|s^t) u(y_k(s^k), \pi_k(s^k)),$$

where $\mu(s^k|s^t)$ is the conditional probability of observing s^k after observing s^t. The payoff function, $u(\cdot,\cdot)$, is given by

$$u(y, \pi) = -\frac{1}{2}[\pi^2 + \lambda y^2]. \tag{8}$$

The Ramsey problem is to choose the state-contingent sequences of inflation and output to maximize the time-one value for each $s \in \mathbb{S}$. That is,

$$V_{ram,1}(s) = \max_{(\boldsymbol{y}(s), \boldsymbol{\pi}(s)) \in \mathbf{CE}(s)} V_1(s).$$

The Ramsey outcome is defined as the state-contingent sequences of inflation and output that solve this optimization problem and is denoted by $\{y_{ram,t}(s^t),\, \pi_{ram,t}(s^t)\}_{t=1}^{\infty}$. The value sequence associated with the Ramsey outcome is denoted by $\{V_{ram,t}(s^t)\}_{t=1}^{\infty}$.

3.1 Promised value approach

As in the model with inflation bias, the infinite-horizon optimization problem of the Ramsey planner is divided into two stages. In the first stage, the constrained Ramsey problem is formulated as follows:

$$w^*(\eta, s) = \max_{(\boldsymbol{y}(s), \boldsymbol{\pi}(s)) \in \Gamma(\eta, s)} \quad -\frac{1}{2} \sum_{t=1}^{\infty} \beta^{t-1} \sum_{s^t | s_1 = s} \mu(s^t | s_1 = s) \left[\pi_t^2 + \lambda y_t^2\right],$$

where $\Gamma(\eta, s)$ is the set of competitive outcomes with the initial state $s_1 = s$ in which the initial inflation is η. This set is formally defined in Appendix B. In the second stage, the Ramsey planner chooses the initial inflation to maximize $w^*(\eta, s)$:

$$V_{ram,1}(s) = \max_{\eta \in \Omega(s)} \quad w^*(\eta, s), \tag{9}$$

where $\Omega(s)$ is the set of time-one inflation rates consistent with the existence of a competitive outcome with the initial state $s_1 = s$. This set is formally defined and computed analytically in Appendix B. The Bellman equation associated with the first-stage constrained Ramsey problem is given by

$$w(\eta_i, e_i) = \max_{y \in \mathbb{K}_Y, \pi \in \mathbb{K}_\Pi, \{\eta'_j \in \Omega_j\}_{j=1}^N} u(y, \pi) + \beta \sum_{j=1}^{N} p(e_j|e_i) w(\eta'_j, e_j) \tag{10}$$

subject to

$$\pi = \eta_i$$
$$\pi = \kappa y + \beta \sum_{j=1}^{N} p(e_j|e_i)\eta'_j + e_i,$$

where we are now explicit about the specifics of the shock (recall that $\mathbb{S} := \{e_1, e_2, ..., e_N\}$). Note that the control variables include η'_j for each $j \in \{1, 2, ..., N\}$.

Let $\{w_{PV}(\cdot),\ y_{PV}(\cdot),\ \pi_{PV}(\cdot),\ \{\eta'_{PV,j}(\cdot)\}_{t=1}^N\}$ be the value and policy function associated with the Bellman equation. Note that there are N promised inflation rates that have to be chosen.[8] The Ramsey value sequence and the Ramsey outcome can be obtained by iterating over these functions with the time-one inflation set to the argmax of $w^*(\eta, s)$ in equation (9).

3.2 Lagrange multiplier approach

The saddle-point functional equation associated with the Ramsey planner's problem above is given by

$$W(\phi, e_i) = \min_{\phi'} \max_{y \in \mathbb{K}_Y, \pi \in \mathbb{K}_\Pi} f(y, \pi, \phi, \phi', e_i) + \beta \sum_{j=1}^{N} p(e_j|e_i) W'(\phi', e_j)$$

where $f(\cdot)$, the modified payoff function, is given by

$$f(y, \pi, \phi, \phi', e_i) = u(y, \pi) + \phi'(\pi - \kappa y - e_i) - \phi\pi.$$

Let $\{y_{LM}(\cdot),\ \pi_{LM}(\cdot),\ \phi'_{LM}(\cdot)\}$ be the policy functions associated with this saddle-point functional equation. One can find the Ramsey outcome by iterating over these policy functions with the initial Lagrange multiplier set to zero.

3.3 Analysis of optimal policy

Given the linear-quadratic structure of the model, the solutions to the Bellman equation from the promised value approach and the saddle-point functional equation can be obtained

[8] As a result, the larger the number of exogenous states is, the larger the number of policy functions to solve for is. However, because an increase in the number of exogenous states does not affect the state space, it does not necessarily lead to increased computational burden.

analytically.[9] However, we will again use a numerical example to illustrate the mechanics of the promised value approach in a transparent way; the analytical results are provided in Appendix E. To make the exposition as transparent as possible, we will assume that (i) there are only two states (high and normal), (ii) the economy starts in the high state, (iii) the economy will move to the normal state with certainty in period 2, and (iv) the normal state is absorbing. In the remainder of this section, we will use the notation e_h and e_n, instead of e_1 and e_2, to refer to the high and normal states, respectively. Parameter values are shown in Table 2. The values for the parameter governing the private sector behavior are the same as in the previous section.

Table 2: Parameters and Transition Probabilities
—Model with Stabilization Bias—

β	λ	κ	e_h	e_n	$p(e_h\|e_h)$	$p(e_n\|e_h)$	$p(e_h\|e_n)$	$p(e_n\|e_n)$
0.9925	0.003	0.024	0.001	0	0	1	0	1

Figure 5 shows the policy functions for the promised inflation rate in the next period and output in the current period as well as the value function associated with the Bellman equation of the promised value approach. The top and bottom panels are for the high state and the normal state, respectively.

The initial inflation rate is given by the argmax of the value function from the high state, shown by the top-right panel of the figure. The initial inflation—indicated by the dashed vertical line—is about 0.25 percent. Once the time-one inflation rate is determined, the time-two inflation rate (π_2) and the time-one output (y_1) are determined by the high-state policy functions shown in the top-left and top-middle panels, respectively. Subsequent sequences of inflation and output—shown by the black dots—are determined by the normal-state policy functions shown in the bottom panels.

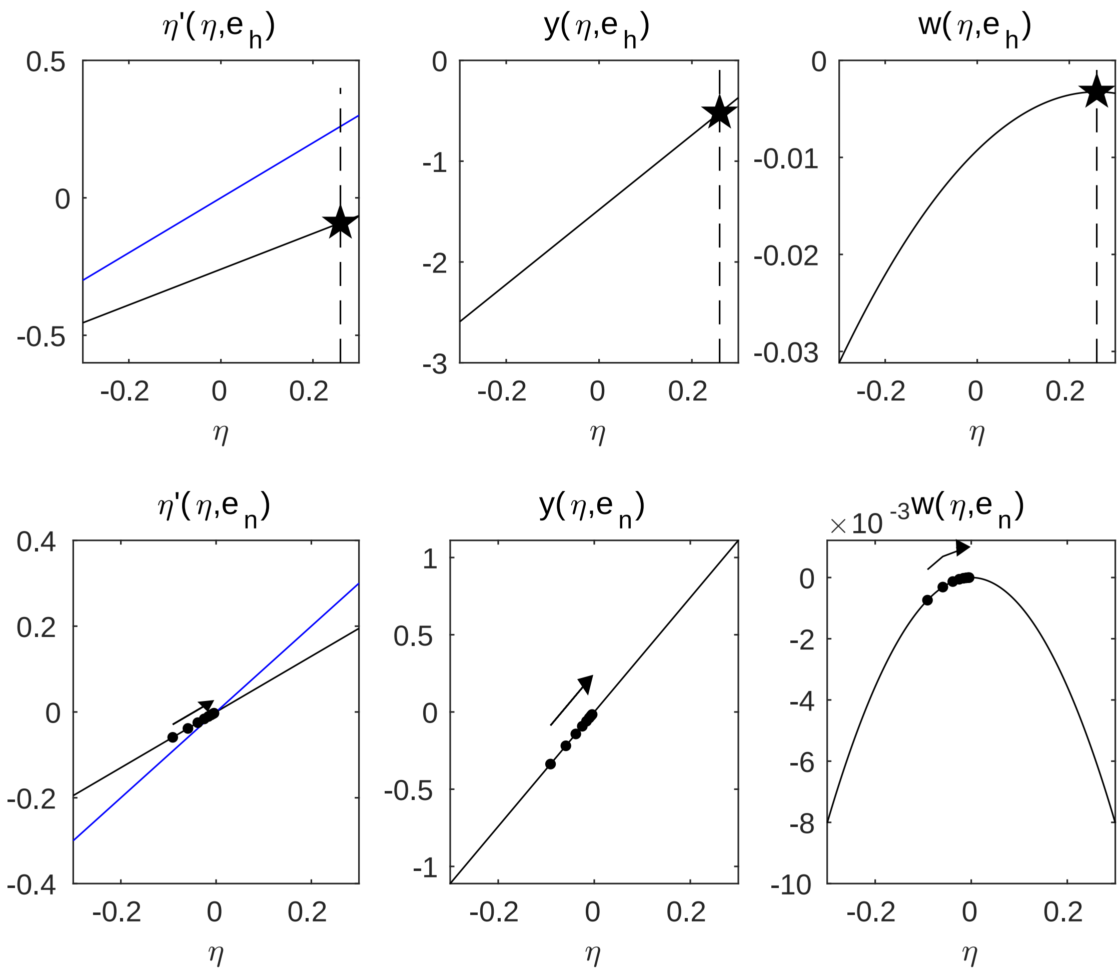
Figure 6 shows the implied dynamics of inflation, output, and the value. A well-known feature of the optimal commitment policy in the model with stabilization bias is that, in the initial period, the central bank promises to undershoot its inflation target once the shock disappears. Relative to the equilibrium under the Markov perfect policy—shown by the dashed lines—in which the central bank does not have a commitment technology, such promise of undershooting improves the trade-off between inflation and output stabilization at $t = 1$ through expectations when the economy is buffeted by the cost-push shock, allowing the central bank to achieve a higher period-one value.[10] The undershooting of inflation and output will fade gradually, and inflation and output will eventually converge to zero.

To understand the trade-off the central bank faces in choosing to create deflation in

[9] As in the model with inflation bias, we confirm that the upper and lower bounds implied by the closed intervals on choice variables are not binding in equilibrium.

[10] Appendix H formulates the optimization of the discretionary central bank and solves for the Markov perfect policy.

Figure 5: Policy Functions from the Promised Value Approach
—Model with Stabilization Bias—

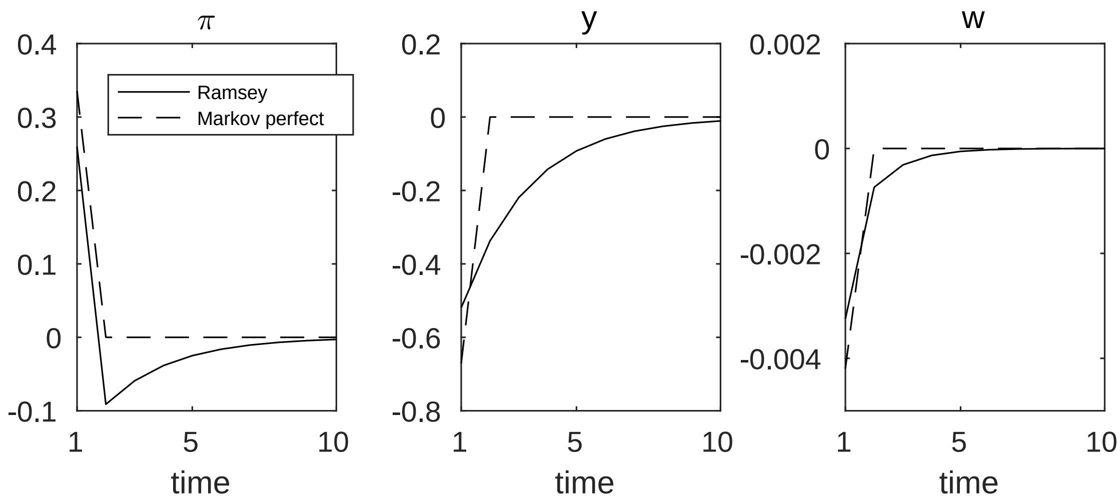


Note: η is the rate of inflation that was promised in the previous period and needs to be delivered in the current period. η' is the promised rate of inflation for the next period. These rates are expressed in annualized percent. w is the value associated with the Bellman equation (equation (10)).

the second period, we show in Figure 7 the objective function associated with the Bellman equation—shown in the left-panel—and its two subcomponents—shown in the middle and right panels—at time one when the cost-push shock is present. Consistent with Figures 5 and 6, the value of the objective function, $w(\cdot)$, is maximized at $\eta' < 0$. To understand why some deflation is optimal, we need to examine how η' affects today's payoff as well as the discounted continuation value, shown in the middle and right panels of Figure 7, respectively.

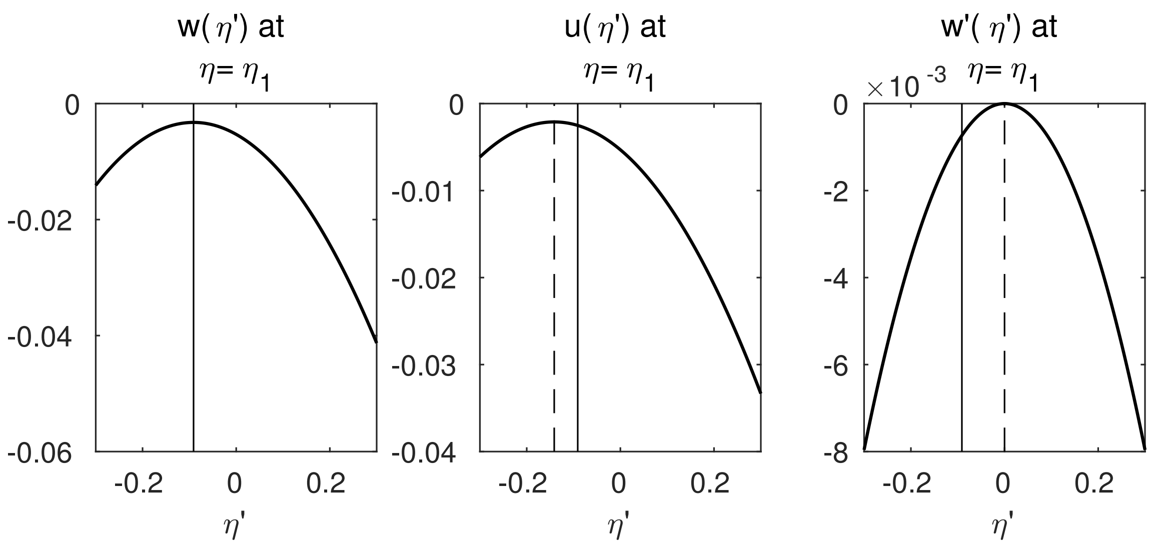
On the one hand, because the inflation rate in the current period has been chosen in the previous period, today's payoff is maximized when today's output is zero. Conditional on the initial promised inflation rate of $\eta = \eta_1 = 0.26/400$ and the cost-push shock of $e = e_h = 0.1/100$, the Phillips curve implies that zero output today is achieved only by

Figure 6: Dynamics
—Model with Stabilization Bias—



Note: The rate of inflation is expressed in annualized percent. The output gap is expressed in percent.

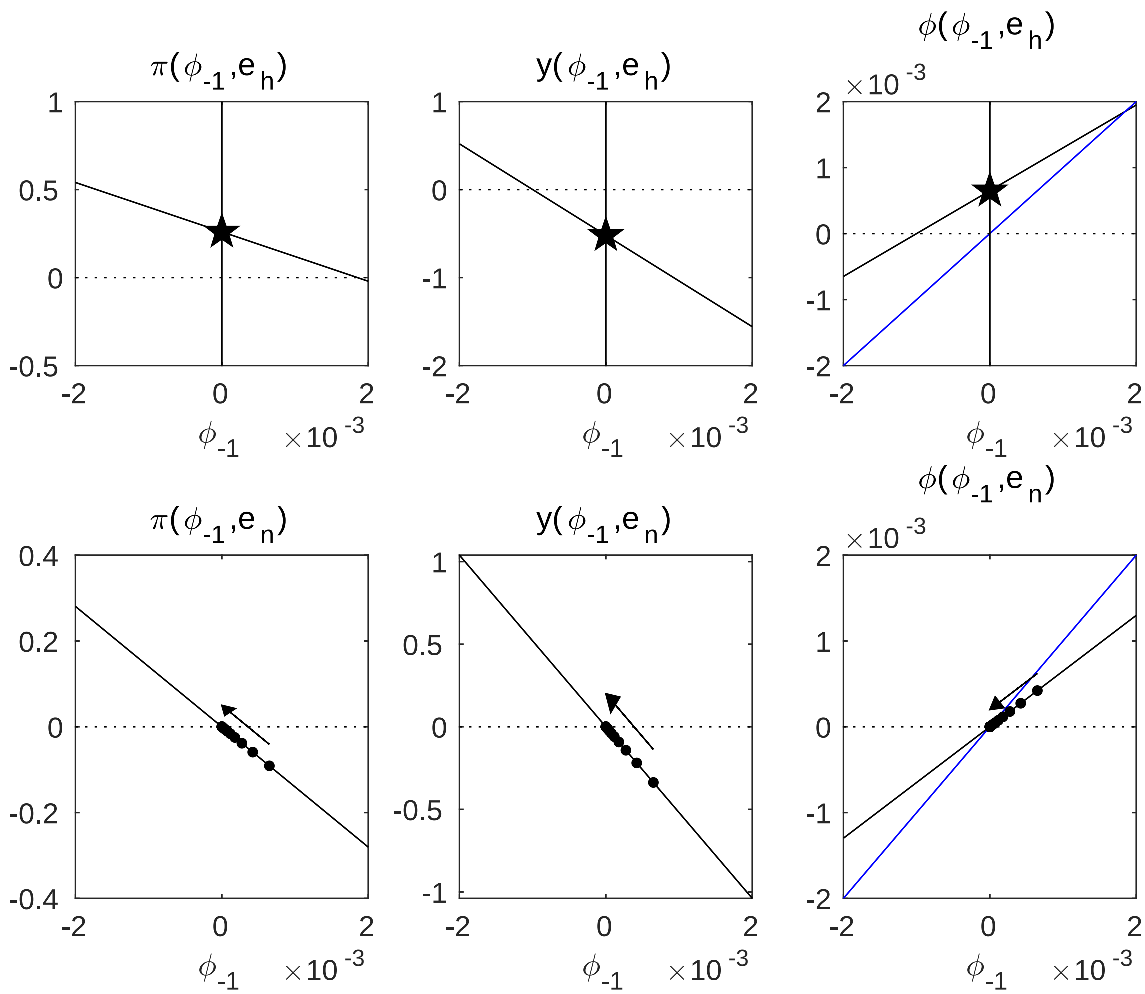
Figure 7: Trade-off under the Promised Value Approach
—Model with Stabilization Bias—



Note: η is the rate of inflation that was promised in the previous period and needs to be delivered in the current period. η' is the promised rate of inflation for the next period. These rates are expressed in annualized percent.

promising some deflation for the next period, as indicated by the dashed vertical line in the middle panel of Figure 7. On the other hand, the discounted continuation value is maximized if the central bank chooses to promise zero inflation for the next period ($\eta' = 0$), as shown by the dashed vertical line in the right panel of Figure 7. The promised inflation rate of zero maximizes the discounted continuation value because it is associated with fully stabilized paths of inflation and output in the future and thus with the highest possible continuation value of zero. The optimal rate of promised inflation—indicated by the solid vertical line—balances these two forces. All told, the overall objective function—shown in the left-panel of Figure 7—is maximized at $\eta' < 0$.

Figure 8: Policy Functions from the Lagrange-Multiplier Approach
—Model with Stabilization Bias—



Note: ϕ_{-1} is the lagged Lagrange multiplier, whereas ϕ is the Lagrange multiplier in the current period. The rate of inflation is expressed in annualized percent. The output gap is expressed in percent.

Finally, Figure 8 shows the policy functions for inflation, output, and the Lagrange multiplier associated with the saddle-point functional equation (3.2) of the Lagrange multiplier approach. The top and bottom panels are for the high state and the normal state, respectively. The time-one inflation, output, and Lagrange multiplier are indicated by the pentagrams in the top panels, which are the high-state policy functions evaluated at the initial lagged Lagrange multiplier of zero. Thereafter, the dynamics of the economy are governed by the normal-state policy functions shown in the bottom panels—because the cost-push shock is assumed to disappear after the first period—and are traced by the black dots. In Appendix E, we analytically verify the equivalence of the dynamics of the economy obtained from the promised value and Lagrange multiplier approaches.

4 Model with the ELB

Our final model features the ELB constraint on nominal interest rates and a natural rate shock. The private sector equilibrium conditions at time t are given by

$$\sigma y_t(s^t) = \sigma E_t y_{t+1}(s^{t+1}) + E_t \pi_{t+1}(s^{t+1}) - r_t(s^t) + r^* + s_t \tag{11}$$

$$\pi_t(s^t) = \kappa y_t(s^t) + \beta E_t \pi_{t+1}(s^{t+1}) \tag{12}$$

where s_t is a natural rate shock following a N-state Markov process. We assume that $y_t \in \mathbb{K}_Y$ and $\pi_t \in \mathbb{K}_\Pi$ where $\mathbb{K}_Y$ and $\mathbb{K}_\Pi$ are closed intervals on the real line, $\mathbb{R}$. We introduce the ELB constraint on the policy rate by imposing that $r_t \in \mathbb{K}_R := [r_{ELB}, r_{max}]$, where r_{ELB} is the ELB constraint on the policy rate.

Possible values of the natural rate shock are given by the set, $\mathbb{S} := \{\delta_1, \delta_2, ..., \delta_n\}$. The probability of moving from state i to state j is denoted by $p(\delta_j|\delta_i)$. s^t denotes the history of shocks up to time t. That is, $s^t := \{s_h\}_{h=1}^t$. Because there is uncertainty, the allocations are state-contingent and depend on s^t.

For any variable x, let us denote its state-contingent sequence $\{x_t(s^t)\}_{t=1}^\infty$ by a bold font $\boldsymbol{x}$ and its state-contingent sequence with the time-one state $s_1 = s$ by $\boldsymbol{x}(s)$. We say $(\boldsymbol{y}, \boldsymbol{\pi}, \boldsymbol{r})$ is a competitive outcome if equations (11) and (12) are satisfied for all $t \geq 1$. We use **CE** to denote the set of all competitive outcomes and use **CE**(s) to denote the set of competitive outcomes in which the initial state s_1 is s.

The value sequence, $\{V_t(s^t)\}_{t=1}^\infty$ associated with a competitive outcome is given by

$$V_t(s^t) = \sum_{k=t}^{\infty} \beta^{k-t} \sum_{s^k|s^t} \mu(s^k|s^t) u(y_k(s^k), \pi_k(s^k)),$$

where $\mu(s^k|s^t)$ is the conditional probability of observing s^k after observing s^t. The payoff function, $u(\cdot,\cdot)$, is given by equation (8) from the previous section. The Ramsey planner's problem is to choose the state-contingent sequences of inflation and output to maximize the time-one value for each $s \in \mathbb{S}$. That is,

$$V_{ram,1}(s) = \max_{(\boldsymbol{y}(s), \boldsymbol{\pi}(s), \boldsymbol{r}(s)) \in \mathbf{CE}(s)} V_1(s). \tag{13}$$

The Ramsey outcome is defined by the solution to this problem and is denoted by $\{y_{ram,t}(s^t), \pi_{ram,t}(s^t), r_{ram,t}(s^t)\}_{t=1}^\infty$. The value sequence associated with the Ramsey outcome is denoted by $\{V_{ram,t}(s^t)\}_{t=1}^\infty$.

4.1 Promised value approach

As in the previous two models, the infinite-horizon optimization problem of the Ramsey planner is divided into two stages. In the first stage, the constrained Ramsey problem is formulated as follows:

$$w^*(\eta_1, \eta_2, s) = \max_{(\boldsymbol{y}(s), \boldsymbol{\pi}(s), \boldsymbol{r}(s)) \in \Gamma(\eta_1, \eta_2, s)} -\frac{1}{2} \sum_{t=1}^{\infty} \beta^{t-1} \sum_{s^t | s_1 = s} \mu(s^t | s_1 = s) \left[\pi_t(s^t)^2 + \lambda y_t(s^t)^2 \right],$$

where $\Gamma(\eta_1, \eta_2, s)$ is the set of competitive outcomes with the initial state $s_1 = s$ in which the initial output and inflation are η_1 and η_2, respectively. This set is more formally defined in Appendix C. In the second stage, the Ramsey planner chooses the initial inflation and output promises that maximize $w^*(\eta_1, \eta_2, s)$:

$$w^r(s) = \max_{(\eta_1, \eta_2) \in \Omega(s)} \quad w^*(\eta_1, \eta_2, s), \tag{14}$$

where $\Omega(s)$ is the set of pairs of time-one inflation rates and output gaps consistent with the existence of a competitive outcome with the initial state $s_1 = s$. This set is formally defined and computed numerically in Appendix C. The Bellman equation associated with the first-stage constrained Ramsey problem is given by

$$w(\eta_{1,i}, \eta_{2,i}, \delta_i) = \max_{y \in \mathbb{K}_Y, \pi \in \mathbb{K}_\Pi, r \in \mathbb{K}_R, \{(\eta'_{1,j}, \eta'_{2,j}) \in \Omega_j\}_{j=1}^N} \quad u(y, \pi) + \beta \sum_{j=1}^{n} p(\delta_j | \delta_i) w(\eta'_{1,j}, \eta'_{2,j}, \delta_j) \tag{15}$$

subject to

$$\begin{aligned} y &= \eta_{1,i} \\ \pi &= \eta_{2,i} \\ \sigma y &= \sum_{j=1}^{N} p(\delta_j | \delta_i) [\sigma \eta'_{1,j} + \eta'_{2,j}] - r + r^* + \delta_i \\ \pi &= \kappa y + \beta \sum_{j=1}^{N} p(\delta_j | \delta_i) \eta'_{2,j}, \end{aligned}$$

where we are now explicit about the specifics of the shock (recall that $\mathbb{S} := \{\delta_1, \delta_2, ..., \delta_n\}$). Let $\{w_{PV}(\cdot), y_{PV}(\cdot), \pi_{PV}(\cdot), r_{PV}(\cdot), \{\eta'_{1,j,PV}(\cdot), \eta'_{2,j,PV}(\cdot)\}_{j=1}^N\}$ be the value and policy functions associated with this Bellman equation. Note that there are N promises for both inflation and output that have to be chosen. The Ramsey value sequence and the Ramsey outcome can be obtained by iterating over the policy functions for inflation, output, and the policy rate found in the first step with time-one output and inflation set to the argmax of $w^*(\eta_1, \eta_2, s)$ in equation (14).

4.2 Lagrange multiplier approach

The saddle-point functional equation associated with the infinite-horizon optimization problem of the Ramsey planner above (equation (13)) is given by

$$W(\phi_1, \phi_2, \delta_i) = \min_{\phi_1', \phi_2'} \max_{y \in \mathbb{K}_Y, \pi \in \mathbb{K}_\Pi, r \in \mathbb{K}_R} f(y, \pi, r, \phi_1, \phi_1', \phi_2, \phi_2', \delta_i) + \beta \sum_{j=1}^{N} p(\delta_j | \delta_i) W(\phi_1', \phi_2', \delta_j), \tag{16}$$

where $f(\cdot)$, the modified payoff function, is given by

$$\begin{aligned} & f(y, \pi, r, \phi_1, \phi_1', \phi_2, \phi_2', \delta_i) \\ =& u(y, \pi) + \phi_1'(r - r^* + \sigma y - \delta_i) - \frac{\phi_1}{\beta}(\sigma y + \pi) + \phi_2'(\pi - \kappa y) - \phi_2 \pi. \end{aligned}$$

Let $\{y_{LM}(\cdot), \pi_{LM}(\cdot), r_{LM}(\cdot), \phi_{1,LM}'(\cdot), \phi_{2,LM}'(\cdot)\}$ be the policy functions associated with this saddle-point functional equation. As in the previous two models, one can find the Ramsey value and outcome by iterating over these policy functions with the initial Lagrange multiplier set to zero.

4.3 Analysis of optimal policy

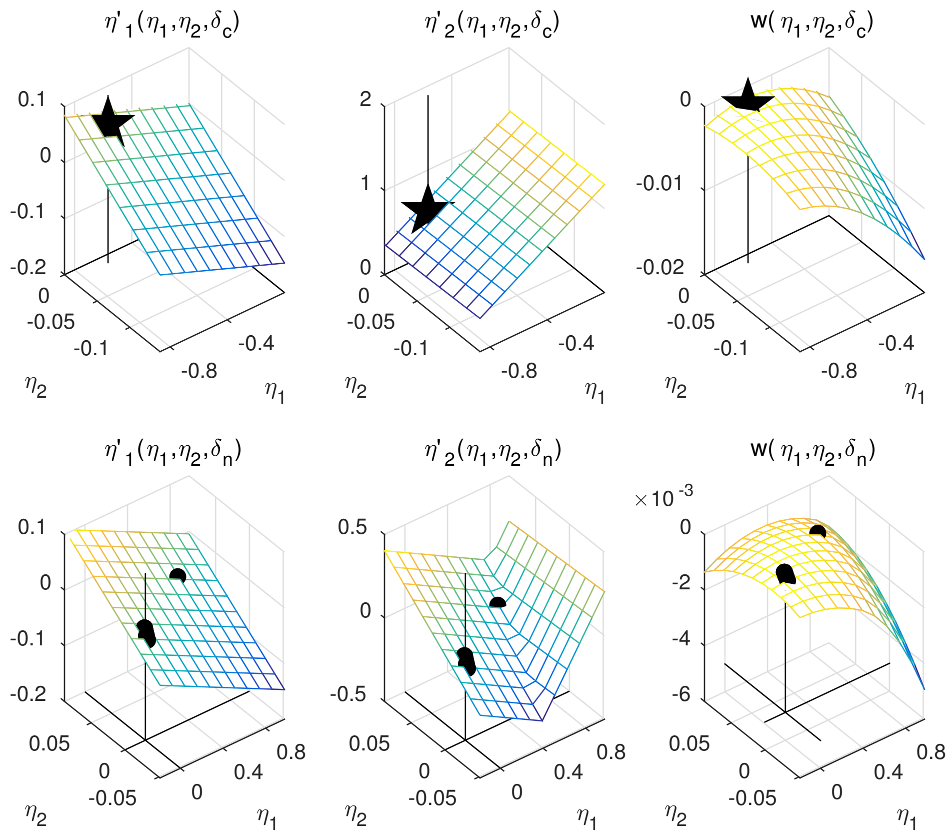
Unlike the first two models, the model with the ELB constraint cannot be solved analytically under either approach. Thus, we solve the model numerically. The solution methods are standard and their details are described in Appendix F. As in the model with stabilization bias, we will simplify the shock structure in order to describe the mechanics of the promised value approach in a transparent way. In particular, we assume that (i) there are only two states (crisis and normal), (ii) the economy starts in the crisis state, (iii) the economy will move to the normal state with certainty in the second period, and (iv) the normal state is absorbing. In the remainder of the section, we will use the notation δ_c and δ_n, instead of δ_1 and δ_2, to refer to the crisis and normal states, respectively. Parameter values are shown in Table 3. The values for the parameters governing the private sector behavior are the same as those in the previous sections.

Table 3: Parameters and Transition Probabilities
—Model with ELB—

β	λ	κ	r_{ELB}	δ_c	δ_n	$p(\delta_c\|\delta_c)$	$p(\delta_n\|\delta_c)$	$p(\delta_c\|\delta_n)$	$p(\delta_n\|\delta_n)$
0.9925	0.003	0.024	0	-0.02	0	0	1	0	1

Figure 9 shows the policy functions for the promised inflation and output in the next period as well as the value function associated with the Bellman equation from the promised value approach, while Figure 10 shows the dynamics of inflation, output, the policy rate, and the value implied by these functions.

Figure 9: Policy Functions from the Promised Value Approach
—Model with ELB—

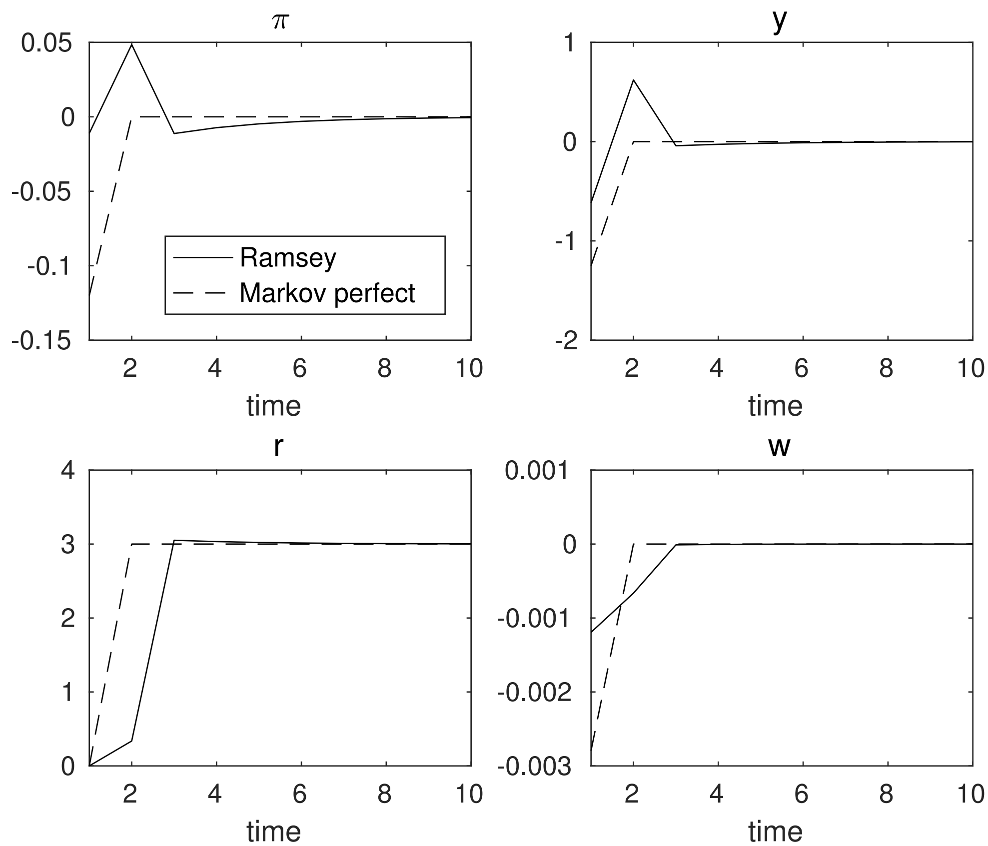


Note: η_1 and η_2 are the output gap and the rate of inflation, respectively, that were promised in the previous period and need to be delivered in the current period. η_1' and η_2' are the promised output gap and the promised rate of inflation, respectively, for the next period. w is the value associated with the Bellman equation (equation (15)). The rate of inflation is expressed in annualized percent. The output gap is expressed in percent.

The pair of the initial inflation rate and output is given by the argmax of the crisis-state value function—shown in the top-right panel of Figure 9—and is indicated by the solid vertical line. The initial inflation rate and output are about minus 0.01 percent and minus 0.6 percent, respectively. Once the initial inflation rate and output are determined, the dynamics of the economy are governed by the normal-state policy functions linking the promised inflation rate and output today to the promised inflation rate and output next period, shown by the bottom panels. The dots in the policy functions trace the dynamics of the economy. The economy's dynamics are shown in Figure 10.

The key feature of the optimal commitment policy in the model with the ELB constraint is that in the initial period, the central bank promises to overshooting inflation and output once the crisis shock disappears in the second period—a feature well-known in the literature (Eggertsson and Woodford (2003); Jung, Teranishi, and Watanabe (2005); Adam and Billi

Figure 10: Dynamics
—Model with ELB—



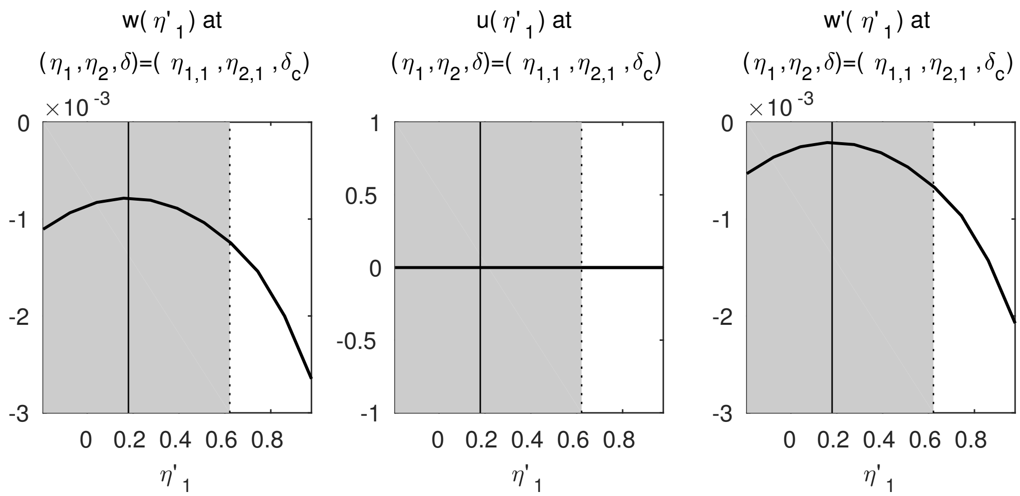
Note: The rate of inflation and the policy rate are expressed in annualized percent. The output gap is expressed in percent.

(2006)). The overshooting commitment mitigates the declines in inflation and output at the ELB via expectations. After the second period, inflation and output gradually approach to their steady state values of zero. Note that, under the optimal discretionary policy—shown by the dashed lines—there is no overshooting in the aftermath of the crisis shock, and the declines in inflation and output are larger during the crisis than under the optimal commitment policy.[11]

Figure 11 shows the trade-off associated with the Bellman equation, given by equation 15, in the first period when the economy is in the crisis state today but is expected to return to the normal state in the next period. Given the initial inflation rate and output the central bank has to deliver today ($\eta_{1,c}$ and $\eta_{2,c}$), the Phillips curve pins down the promised inflation in the next period. Thus, the only control variable available for the central bank to adjust is the promised output for the next period, η_1'. The left panel shows how the overall objective

[11]We formulate the optimization problem of the discretionary central bank and solve for the Markov perfect equilibrium in Appendix I.

Figure 11: Trade-off under the Promised Value Approach (Time-One Value)
—Model with ELB—



Note: η_1' and η_2' are the promised output gap and the promised rate of inflation for the next period. Grey shades indicate the range of η_1' consistent with negative nominal interest rates.

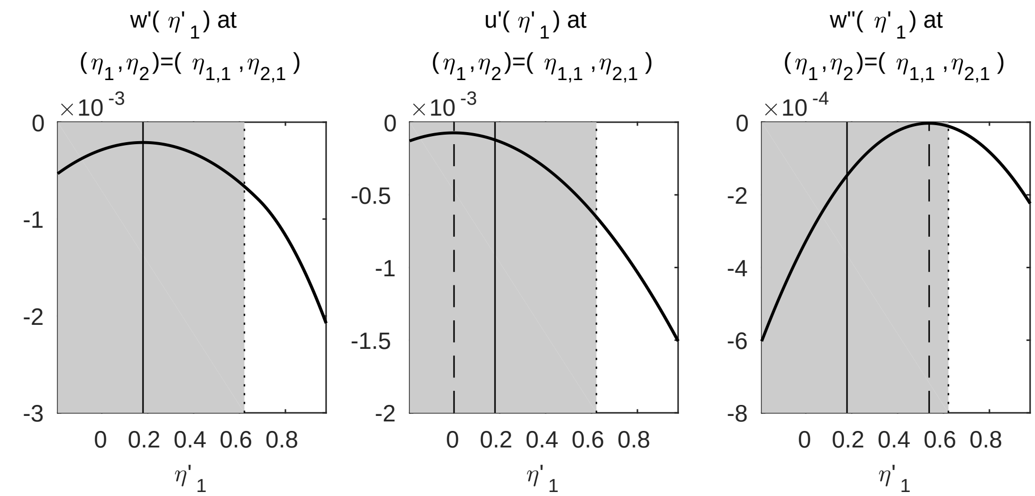
function varies with the promised output, whereas the middle and right panels show how the two subcomponents of the overall objective function, today's payoff and the discounted continuation value, vary with the promised output.

As shown in the middle panel, today's payoff is constant, as it depends only on the current-period inflation rate and output that were promised in the previous period and thus does not depend on the promised output for the next period. Thus, what maximizes the discounted continuation value—shown in the right panel—also maximizes the overall objective function—shown in the left panel. As indicated by the solid vertical line in that panel, the discounted continuation value is maximized at around $\eta_1' = 0.2$, meaning that it is optimal to promise an output overshoot.

The optimality of a positive time-two output for the discounted continuation value reflects the following two competing forces. On the one hand, because the time-two inflation is given—it is implied by the time-one inflation and output, as discussed above—promising the time-two inflation rate of zero maximizes the time-two payoff, as shown in the middle panel of Figure 12. On the other hand, because the time-two inflation rate is positive, a promise of zero time-two output means that the time-three inflation has to be positive and even slightly higher than the time-two inflation because of the time-two Phillips curve constraint.[12] By promising a higher output for time two, the central bank ensures that the time-three inflation is closer to zero, which is desirable because it is associated with a higher value, as shown by the right panel of Figure 12. The optimality of promising a positive time-two output is the outcome of the intertemporal trade-off between these two forces.

[12]With $y_2 = 0$, the time-two Phillips curve implies that $\pi_3 = \pi_2/\beta$.

Figure 12: Trade-off under the Promised Value Approach (Time-Two Value)
—Model with ELB—

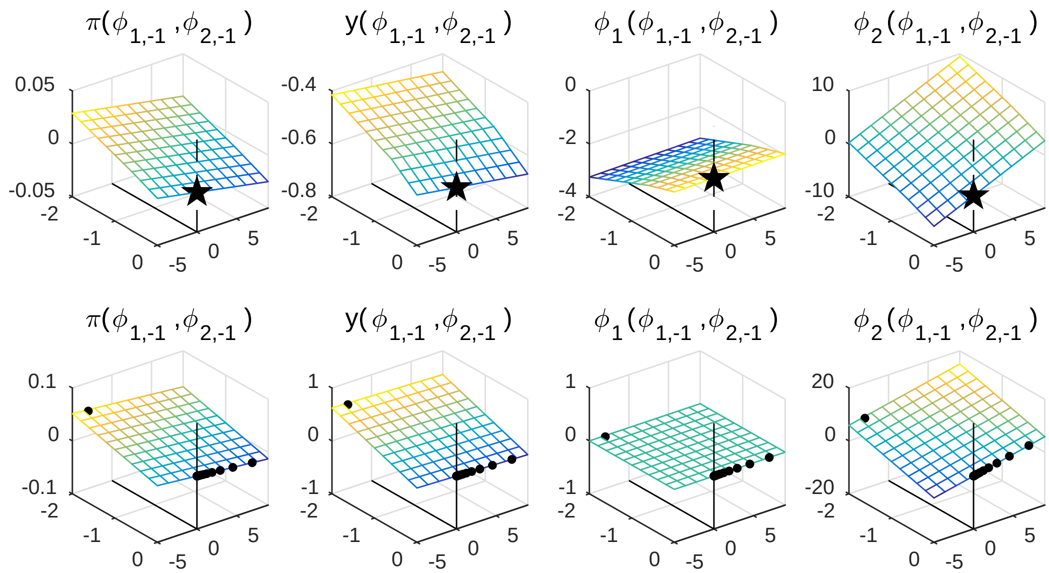


Note: η_1' and η_2' are the promised output gap and the promised rate of inflation, respectively, for the next period. Grey shades indicate the range of η_1' consistent with negative nominal interest rates.

While $\eta_1' = 0.2$ maximizes the time-one value, this is not the level of output the central bank ends up promising for $t = 2$ because of the ELB constraint on the policy rate. The ELB constraint on the policy rate puts a lower bound on the promised output the central bank can choose due to the Euler equation; given today's output and the rate of inflation in the next period, the policy rate needs to be sufficiently low in order to support a low level of output in the next period. In Figure 12, any promised output below the dashed vertical line is associated with a negative policy rate in the current period. The maximum is attained when the promised output is at its lower bound and the policy rate is zero.

Turning to the Lagrange multiplier approach, we show in Figure 13 the policy functions and value function associated with the saddle-point functional equation (16). The time-one inflation, output, and Lagrange multipliers are given by the crisis-state policy functions—shown in the top panels—evaluated at $(\phi_{1,-1}, \phi_{2,-1}) = (0, 0)$ and are indicated by the pentagrams. Thereafter, the dynamics of the economy are determined by the normal-state policy functions shown in the bottom panels and are traced by the black dots. The dynamics of inflation and output derived from the Lagrange multiplier approach are identical to those implied by the promised value approach, up to the accuracy of the numerical methods used. In Appendix F, we contrast the dynamics obtained from the promised value and Lagrange multiplier approaches and show that the differences are of a magnitude in line with the numerical errors associated with the global solution methods.

Figure 13: Policy Functions from the Lagrange-Multiplier Approach
—Model with ELB—



Note: $\phi_{1,-1}$ and $\phi_{2,-1}$ are the lagged Lagrange multipliers, whereas ϕ_1 and ϕ_2 are the Lagrange multipliers in the current period. The rate of inflation is expressed in annualized percent. The output gap is expressed in percent.

5 Conclusion

In this paper, we characterized optimal commitment policies in three well-known versions of the New Keynesian model using a novel recursive approach—which we called the promised value approach—inspired by Kydland and Prescott (1980). Under the promised value approach, promised inflation and output act as pseudo state variables, as opposed to the lagged Lagrange multipliers under the standard approach of Marcet and Marimon (2016). The Bellman equation from the promised value approach sheds new light on the trade-off facing the central bank and provides fresh perspectives on optimal commitment policies. The promised value approach can serve as a useful analytical tool for those economists interested in analyzing optimal monetary policy.

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Technical Appendix for Online Publication

This technical appendix is organized as follows:

- Sections A, B, and C describe the technical details of the promised value approach for the three models considered in the paper.
- Sections D and E present analytical results for the model with inflation bias and the model with stabilization bias, respectively.
- Section F describes the global solution methods used to solve the model with the effective lower bound (ELB) constraint and presents the accuracy of the solution.
- Section G presents a few additional results for the model with inflation bias.
- Sections H and I characterize Markov perfect equilibria in models with stabilization bias and with the ELB constraint on nominal interest rates, respectively.

A Details of the promised value approach for the model with inflation bias

Notations closely follow Chang (1998). For any variable, x_t, let $\boldsymbol{x} \equiv \{x_t\}_{t=1}^{\infty}$. $(\boldsymbol{y}, \boldsymbol{\pi})$ is said to be a competitive outcome if, for all $t \geq 1$, $y_t \in \mathbb{K}_Y$, $\pi_t \in \mathbb{K}_\Pi$, and

$$\pi_t = \kappa y_t + \beta \pi_{t+1}.$$

Let **CE** denote the set of all competitive outcomes. That is,

$$\mathbf{CE} \equiv \{(\boldsymbol{y}, \boldsymbol{\pi}) \, | (\boldsymbol{y}, \boldsymbol{\pi}) \, \text{is a competitive outcome}\} .$$

The sequence of values, $\{V_t\}_{t=1}^{\infty}$, associated with a competitive outcome, $\{y_t, \, \pi_t\}_{t=1}^{\infty}$, is given by

$$V_t = \sum_{k=t}^{\infty} \beta^{k-t} u(y_k, \pi_k)$$

where $u(\cdot, \cdot)$, the payoff function, is given by

$$u(y, \pi) = -\frac{1}{2}[\pi^2 + \lambda(y - y^*)^2]$$

The Ramsey problem is to choose a competitive outcome that maximizes the time-one value:

$$V_{ram,1} = \max_{(\boldsymbol{y},\boldsymbol{\pi}) \in \mathbf{CE}} V_1. \tag{17}$$

A.1 Recursive formulation

A recursive treatment of the Ramsey problem entails the use of *promised* inflation made in period t given by $\eta_{t+1} = \pi_{t+1}$. Hence, η_{t+1} is period $t + 1$'s inflation rate that is promised by the equilibrium in period t.

Let Ω denote the set of all possible initial inflation *promises* that are consistent with a competitive outcome. That is,

$$\Omega \equiv \{\eta \in \mathbb{R} | \eta = \pi_1 \text{ for some } (\boldsymbol{y}, \boldsymbol{\pi}) \in \mathbf{CE}\}.$$

Let $\Gamma(\eta)$ be the set of all possible competitive outcomes whose initial promise is given by η. That is,

$$\Gamma(\eta) \equiv \{(\boldsymbol{y}, \boldsymbol{\pi}) \in \mathbf{CE} | \pi_1 = \eta\}.$$

Under the promised value approach, the problem of the Ramsey planner is divided into two steps. In the first step, the following constrained problem is formulated for all $\eta \in \Omega$.

$$w^*(\eta) = \max_{(\boldsymbol{y},\boldsymbol{\pi}) \in \Gamma(\eta)} -\frac{1}{2} \sum_{t=1}^{\infty} \beta^{t-1} \left[\pi_t^2 + \lambda (y_t - y^*)^2\right]$$

In the second step, the Ramsey planner chooses the initial inflation promise, η, that maximize $w^*(\eta)$.

$$V_{ram,1} = \max_{\eta \in \Omega} \quad w^*(\eta).$$

By the standard dynamic programing argument, it can be shown that $w^*(\eta)$ satisfies the following functional equation:

$$w(\eta) = \max_{y,\pi,\eta'} \quad -\frac{1}{2}[\pi^2 + \lambda (y - y^*)^2] + \beta w(\eta')$$

$$\text{such that } (y, \pi, \eta') \in \mathbb{K}_y \times \mathbb{K}_\pi \times \Omega,$$

$$\pi = \eta,$$

and

$$\pi = \kappa y + \beta \eta'.$$

Conversely, it can be also shown that, if a bounded function, $w : \Omega \to \mathbb{R}$, satisfies the above functional equation, then $w = w^*$.

Since $\mathbb{K}_\Pi$ and $\mathbb{K}_Y$ are primitives of the model and are known, the object of interest becomes Ω. To find Ω, we define an operator, $\mathbb{B}$, as follows. For $Q \in \mathbb{R}$, let

$$\mathbb{B}(Q) = \left\{\eta \in \mathbb{R} \mid \exists (y, \pi, \eta') \in \mathbb{K}_Y \times \mathbb{K}_\Pi \times Q, \text{ where } \pi = \eta \text{ and } \pi = \kappa y + \beta \eta'\right\}.$$ [13]

It can be shown that (i) $Q \subseteq \mathbb{B}(Q) \Rightarrow \mathbb{B}(Q) \subseteq \Omega$ (a.k.a *self generation*), (ii) $\Omega = \mathbb{B}(\Omega)$ (a.k.a. *factorization*), and (iii) letting $Q_0 = [\underline{\eta}, \overline{\eta}]$ and $Q_n = \mathbb{B}(Q_{n-1})$, $Q_n \supseteq Q_{n+1}$ and $\Omega = \cap_{n=0}^{\infty} Q_n$. As a result, in order to find Ω, one can start from a closed interval, apply the operator until it converges. The converged set is Ω. See Chang (1998) for the proof.

A.2 Computing Ω with the $\mathbb{B}$ operator

We can show that, if $\mathbb{K}_Y$ is sufficiently large, $\Omega = \mathbb{K}_\Pi$.

Proof: Let $Q_0 := \mathbb{K}_\Pi$. From the properties of $\mathbb{B}$-operator described above, we can prove $\Omega = \mathbb{K}_\Pi$ by showing $Q_0 \subseteq \mathbb{B}(Q_0)$. Let $\mathbb{K}_\Pi := [\pi_{min}, \pi_{max}]$. Take $\mathbb{K}_Y := [y_{min}, y_{max}]$ with

[13] See Appendix A.2

$y_{min} \leq (1-\beta)\pi_{min}/\kappa$ and $y_{max} \geq (1-\beta)\pi_{max}/\kappa$. Take $\eta \in Q_0$. We want to show that $\eta \in \mathbb{B}(Q_0)$. Consider $y = (1-\beta)\eta/\kappa$, $\pi = \eta$, and $\eta' = \eta$. By construction, $y_{min} \leq y \leq y_{max}$, $\pi \in Q_0 := \mathbb{K}_\Pi$, and $\eta' \in Q_0$. That is, $(y, \pi, \eta') \in \mathbb{K}_Y \times \mathbb{K}_\Pi \times Q_0$ Thus, $\eta \in \mathbb{B}(Q_0)$.

B Details of the promised value approach for the model with stabilization bias

Let s_t denote the exogenous shock of the model at time t and let s^t denote the history of shocks up to time t.

For any variable, x, with range $\mathbb{X}$, let us denote its corresponding state-contingent sequence by $\boldsymbol{x} \equiv \left\{x_t\left(\delta^t\right)\right\}_{t=1}^{\infty}$. That is, $\boldsymbol{x}$ is a sequence of functions mapping a history of states into $\mathbb{X}$:

$$x_1 : \mathbb{S} \to \mathbb{X}$$

and

$$x_t : \mathbb{S}^t \to \mathbb{X}.$$

For any variable, x, with range $\mathbb{X}$, $\boldsymbol{x}(s)$ represents a state-contingent sequence with $s_1 = s$ defined by a sequence of functions mapping a history of states with $s_1 = s$ into $\mathbb{X}$:

$$x_1 : s \to \mathbb{X}$$

and

$$x_t : \mathbb{S}^t \to \mathbb{X}.$$

A state-contingent sequence of inflation and output, $(\boldsymbol{y}, \boldsymbol{\pi})$, is said to be a competitive outcome if, $\forall t \geq 1$ and $\forall s^t \in \mathbb{S}^t$, $y_t(s^t) \in \mathbb{K}_Y$, $\pi_t(s^t) \in \mathbb{K}_\Pi$, and

$$\pi_t(s^t) = \kappa y_t(s^t) + \beta \sum_{s^{t+1}|s^t} \mu(s^{t+1}|s^t)\pi_{t+1}(s^{t+1}) + s_t.$$

For each $s \in \mathbb{S}$, let **CE** denote the set of all competitive outcomes. That is,

$$\mathbf{CE} \equiv \{(\boldsymbol{y}, \boldsymbol{\pi}) \mid (\boldsymbol{y}, \boldsymbol{\pi}) \, \text{is a competitive outcome}\}.$$

For each $s \in \mathbb{S}$, let $\mathbf{CE}(s)$ denote the set of all competitive outcomes with $s_1 = s$. That is,

$$\mathbf{CE}(s) \equiv \{(\boldsymbol{y}(s), \boldsymbol{\pi}(s)) \mid (\boldsymbol{y}(s), \boldsymbol{\pi}(s)) \, \text{is a competitive outcome with} \, s_1 = s\}.$$

The Ramsey planner's problem is to choose the competitive outcome that maximizes the time-one value:

$$V_{ram,1}(s) = \max_{(\boldsymbol{y}(s), \boldsymbol{\pi}(s)) \in \mathbf{CE}(s)} V_1(s_1)$$

B.1 Recursive formulation

A recursive treatment of the Ramsey problem entails the use of state-contingent *promised* value(s) made in period t given by $\eta_{t+1}(u_{t+1}|u_t) = \pi_{t+1}(s_{t+1}|s_t)$ given $s_t \in \mathbb{S}$ and $s_{t+1} \in \mathbb{S}$. Hence, $\eta_{t+1}(s_{t+1}|s_t)$ is period $t+1$'s inflation rate, for some state $s_{t+1} \in \mathbb{S}$, that is promised by the equilibrium in period t for a given $s_t \in \mathbb{S}$. From now on, we will denote these promised variables by $\eta_{t+1}^{s'}$, $s' \in \mathbb{S}$, $s_{t+1} = s'$.

For any $s \in \mathbb{S}$, denote the set of all possible initial inflation *promises* consistent with a competitive outcome by $\Omega(s)$. That is,

$$\Omega(s) \equiv \{\eta \in \mathbb{R} \mid \eta = \pi_1 \text{ for some } (\boldsymbol{y}(s), \boldsymbol{\pi}(s)) \in \mathbf{CE}(s)\}.$$

For any $s \in \mathbb{S}$, denote the set of all possible competitive outcomes whose initial promise is η by $\Gamma(\eta, s)$. That is,

$$\Gamma(\eta, s) \equiv \{(\boldsymbol{y}(s), \boldsymbol{\pi}(s)) \in \mathbf{CE}(s) \mid \pi_1 = \eta\}.$$

For a given $s \in \mathbb{S}$, the recursive formulation takes two steps. In the first step, the constrained Ramsey problem is formulated.

$$w^*(\eta, s) = \max_{(\boldsymbol{y}(s), \boldsymbol{\pi}(s)) \in \Gamma(\eta, s)} -\frac{1}{2} \sum_{t=1}^{\infty} \beta^{t-1} \sum_{s^t | s_1 = s} \mu(s^t | s_1 = s) \left[\pi_t(s^t)^2 + \lambda y_t(s^t)^2\right]$$

In the second step, the Ramsey planner chooses the initial promise to maximize $w^*(\eta, s)$.

$$V_{ram,1}(s) = \max_{\eta \in \Omega(s)} \quad w^*(\eta, s).$$

By the standard dynamic programming argument, it can be shown that, for a given $s \in \mathbb{S}$, $w^*(\eta, s)$ satisfies the functional equation:

$$w(\eta, s) = \max_{y, \pi, \{\eta^{s'}\}_{s' \in \mathbb{S}}} \quad -\frac{1}{2}[\pi^2 + \lambda y^2] + \beta \sum_{s' \in \mathbb{S}} p(s'|s) w(\left\{\eta^{s'}\right\}, s' \in \mathbb{S})$$

$$\text{such that } \left(y, \pi, \left\{\eta^{s'}\right\}_{s' \in \mathbb{S}}\right) \in \mathbb{K}_Y \times \mathbb{K}_\Pi \times \{\Omega(s')\}_{s' \in \mathbb{S}},$$

$$\pi = \eta,$$

and

$$\pi = \kappa y + \beta \sum_{s' \in \mathbb{S}} p(s'|s) \eta^{s'} + s.$$

Conversely, it can be also shown that, if a bounded function, $w : \Omega(u) \times \mathbb{U} \to \mathbb{R}$, satisfies the above functional equation, then $w = w^*$.

Since $\mathbb{K}_\pi$ and $\mathbb{K}_y$ are already defined, the objects of interest become $\Omega(s)$, $s \in \mathbb{S}$. To find $\Omega(s)$ for a given $s \in \mathbb{S}$, we define an operator, $\mathbb{B}$, as follows. For $Q(s) \in \mathbb{R}$ and $s \in \mathbb{S}$, let

$$\mathbb{B}(Q)(s) = \Big\{\eta \mid \exists \left(y, \pi, \left\{\eta^{s'}\right\}_{s' \in \mathbb{S}}\right) \in \mathbb{K}_\pi \times \mathbb{K}_y \times \left\{Q(s')\right\}_{s' \in \mathbb{S}},$$
$$\text{where } \pi = \eta \text{ and } \pi = \kappa y + \beta \sum_{s' \in \mathbb{S}} p(s'|s) \eta^{s'} + s\Big\}.$$

It can be shown that, for a given $s \in \mathbb{S}$, (i) $Q(s) \subseteq \mathbb{B}(Q)(s) \Rightarrow \mathbb{B}(Q)(s) \subseteq \Omega(s)$ (a.k.a *self generation*), (ii) $\Omega(s) = \mathbb{B}(\Omega)(s)$ (a.k.a. *factorization*), and (iii) letting $Q_0(s) = \left[\underline{\eta}, \overline{\eta}\right]$ and $Q_n(s) = \mathbb{B}(Q_{n-1})(s)$, $Q_n(s) \supseteq Q_{n+1}(s)$ and $\Omega(s) = \cap_{n=0}^{\infty} Q_n(s)$.

B.2 Computing $\Omega(s)$ with the $\mathbb{B}$ operator

We can show that, if $\mathbb{K}_Y$ is sufficiently large, $\Omega(e_h) = \Omega(e_n) = \mathbb{K}_\Pi$.

Proof: Let $Q_0(e_h) := \mathbb{K}_\Pi$ and $Q_0(e_n) := \mathbb{K}_\Pi$. From the properties of $\mathbb{B}$-operator described above, we can prove $\Omega(e_h) = \Omega(e_n) = \mathbb{K}_\Pi$ by showing $Q_0(e_h) \subseteq \mathbb{B}(Q_0)(e_h)$ and $Q_0(e_n) \subseteq \mathbb{B}(Q_0)(e_n)$. Let $\mathbb{K}_\Pi := [\pi_{min}, \pi_{max}]$. Take $\mathbb{K}_Y := [y_{min}, y_{max}]$ with $y_{min} \leq [(1-\beta)\pi_{min} - e_h]/\kappa$ and $y_{max} \geq [(1-\beta)\pi_{max} - e_n]/\kappa$.

Take $\eta \in Q_0(e_h)$. We want to show that $\eta \in \mathbb{B}(Q_0)(e_h)$. Consider $y = [(1-\beta)\pi - e_h]/\kappa$, $\pi = \eta$, $\eta'_h = \eta$, and $\eta'_n = \eta$. Note that the lowest possible y (when $\eta = \pi_{min}$) is given by $y = [(1-\beta)\pi_{min} - e_h]/\kappa \geq y_{min}$ and the highest possible y (when $\eta = \pi_{max}$) is given by $y = [(1-\beta)\pi_{max} - e_h]/\kappa \leq [(1-\beta)\pi_{max} - e_n]/\kappa = y_{max}$. Thus, $y \in \mathbb{K}_Y$. $\pi \in Q_0 := \mathbb{K}_\Pi$, $\eta'_h \in Q_0(e_h)$, $\eta'_n \in Q_0(e_h) := Q_0(e_n)$. That is, $(y, \pi, \eta'_h, \eta'_n) \in \mathbb{K}_Y \times \mathbb{K}_\Pi \times Q_0(e_h) \times Q_0(e_n)$ Thus, $\eta \in \mathbb{B}(Q_0)(e_h)$.

Take $\eta \in Q_0(e_n)$. We want to show that $\eta \in \mathbb{B}(Q_0)(e_n)$. Consider $y = [(1-\beta)\pi - e_n]/\kappa$, $\pi = \eta$, $\eta'_h = \eta$, and $\eta'_n = \eta$. Note that the lowest possible y (when $\eta = \pi_{min}$) is given by $y = [(1-\beta)\pi_{min} - e_n]/\kappa \geq [(1-\beta)\pi_{min} - e_h]/\kappa \geq y_{min}$ and the highest possible y (when $\eta = \pi_{max}$) is given by $y = [(1-\beta)\pi_{max} - e_n]/\kappa \leq y_{max}$. Thus, $y \in \mathbb{K}_Y$. $\pi \in Q_0 := \mathbb{K}_\Pi$, $\eta'_h \in Q_0(e_h)$, $\eta'_n \in Q_0(e_h) := Q_0(e_n)$. That is, $(y, \pi, \eta'_h, \eta'_n) \in \mathbb{K}_Y \times \mathbb{K}_\Pi \times Q_0(e_h) \times Q_0(e_n)$ Thus, $\eta \in \mathbb{B}(Q_0)(e_n)$.

C Details of the promised value approach for the model with ELB

Notations, s_t, s^t, $\boldsymbol{x}$, $\boldsymbol{x}(s)$, are the same as in the model with stabilization bias.

A state-contingent sequence of inflation, output, and the policy rate, $(\boldsymbol{y}, \boldsymbol{\pi}, \boldsymbol{r})$, is said to be a competitive outcome if, $\forall t \geq 1$ and $\forall s^t \in \mathbb{S}^t$, $y_t(s^t) \in \mathbb{K}_Y$, $\pi_t(s^t) \in \mathbb{K}_\Pi$, $r_t(s^t) \in \mathbb{K}_R$, and

$$\begin{aligned} \sigma y_t(s^t) &= \sigma E_t y_{t+1}(s^{t+1}) + E_t \pi_{t+1}(s^{t+1}) - r_t(s^t) + r^* + s_t \\ \pi_t(s^t) &= \kappa y_t(s^t) + \beta E_t \pi_{t+1}(s^{t+1}) \\ r_t &\geq r_{ELB} \end{aligned}$$

Let **CE** denote the set of all competitive outcomes. That is,

$$\mathbf{CE} \equiv \{(\boldsymbol{y}, \boldsymbol{\pi}, \boldsymbol{r}) \mid (\boldsymbol{y}(s), \boldsymbol{\pi}(s), \boldsymbol{r}(s)) \text{ is a competitive outcome}\}.$$

For each $s \in \mathbb{S}$, let $\mathbf{CE}(s)$ denote the set of all competitive outcomes with $s_1 = s$. That is,

$$\mathbf{CE}(s) \equiv \{(\boldsymbol{y}(s), \boldsymbol{\pi}(s), \boldsymbol{r}(s)) \mid (\boldsymbol{y}(s), \boldsymbol{\pi}(s), \boldsymbol{r}(s)) \text{ is a competitive outcome with } s_1 = s\}.$$

The Ramsey planner's problem is to chooses the competitive outcome that maximizes the time-one value:

$$V_{ram,1}(s) = \max_{(\boldsymbol{y}(s), \boldsymbol{\pi}(s), \boldsymbol{r}(s)) \in \mathbf{CE}(s)} V_1(s)$$

C.1 Recursive formulation

A recursive treatment of the Ramsey problem entails the use of state-contingent *promised* value(s) made in period t given by $\eta_{1,t+1}(s_{t+1}|s_t) = \pi_{t+1}(s_{t+1}|s_t)$ and $\eta_{2,t+1}(s_{t+1}|s_t) = y_{t+1}(s_{t+1}|s_t)$ given $s_t \in \mathbb{S}$ and $s_{t+1} \in \mathbb{S}$. Hence, $\eta_{1,t+1}(s_{t+1}|s_t)$ and $\eta_{2,t+1}(s_{t+1}|s_t)$ are period $t+1$'s inflation rate and consumption levels, respectively, for some state $s_{t+1} \in \mathbb{S}$, that is promised by the equilibrium in period t for a given $s_t \in \mathbb{S}$. From now on, we will denote these promised variables by $\eta_{1,t+1}^{s'}$ and $\eta_{2,t+1}^{s'}$, $s' \in \mathbb{S}$, $s_{t+1} = s'$.

For any $s \in \mathbb{S}$, let $\Omega(s)$ denote the set of all possible pairs of initial inflation and output *promises* consistent with the existence of a competitive outcome. That is,

$$\Omega(s) \equiv \{(\eta_1, \eta_2) \in \mathbb{R}^2 \mid \eta_1 = \pi_1 \text{ and } \eta_2 = y_1 \text{ for some } (\boldsymbol{y}(s), \boldsymbol{\pi}(s), \boldsymbol{r}(s)) \in \mathbf{CE}(s)\}.$$

For any $s \in \mathbb{S}$, let $\Gamma(\eta_1, \eta_2, s)$ denote the set of all possible competitive outcomes whose initial promise pair is given by (η_1, η_2). That is,

$$\Gamma(\eta_1, \eta_2, s) \equiv \{(\boldsymbol{y}(s), \boldsymbol{\pi}(s), \boldsymbol{r}(s)) \in \mathbf{CE}(s) \mid y_1 = \eta_1, \text{ and } \pi_1 = \eta_2\}.$$

For a given $s \in \mathbb{S}$, the recursive formulation takes two steps. In the first step, the constrained Ramsey problem is formulated:

$$w^*(\eta_1, \eta_2, s) = \max_{(\boldsymbol{y}(s), \boldsymbol{\pi}(s), \boldsymbol{r}(s)) \in \Gamma(\eta_1, \eta_2, s)} -\frac{1}{2} \sum_{t=1}^{\infty} \beta^{t-1} \sum_{s^t | s_1 = s} \mu(s^t | s_1 = s) \left[\pi_t(s^t)^2 + \lambda y_t(s^t)^2\right]$$

In the second step, the Ramsey planner chooses the initial inflation and output promises that maximize $w^*(\eta_1, \eta_2, s)$.

$$V_{ram,1}(s) = \max_{(\eta_1, \eta_2) \in \Omega(s)} w^*(\eta_1, \eta_2, s).$$

By the standard dynamic programming argument, it can be shown that, for any $s \in \mathbb{S}$, $w^*(\eta_1, \eta_2, s)$ satisfies the functional equation:

$$w(\eta_1, \eta_2, s) = \max_{y, \pi, r, \left\{(\eta_1^{s'}, \eta_2^{s'})\right\}_{s' \in \mathbb{S}}} -\frac{1}{2}[\pi^2 + \lambda y^2] + \beta \sum_{s' \in \mathbb{S}} p(s'|s) w(\left\{\eta_1^{s'}, \eta_2^{s'}\right\}, s' \in \mathbb{S})$$

$$\text{such that } \left(y, \pi, r, \left\{(\eta_1^{s'}, \eta_2^{s'})\right\}_{s' \in \mathbb{S}}\right) \in \mathbb{K}_Y \times \mathbb{K}_\Pi \times \mathbb{K}_R \times \{\Omega(s')\}_{s' \in \mathbb{S}},$$

$$y = \eta_1,$$

$$\pi = \eta_2,$$

$$r = \frac{1}{\sigma} \sum_{s' \in \mathbb{S}} p(s'|s) \eta_1^{s'} + \sum_{s' \in \mathbb{S}} p(s'|s) \eta_2^{s'} - \frac{1}{\sigma} y + s,$$

and

$$\pi = \kappa y + \beta \sum_{s' \in \mathbb{S}} p(s'|s) \eta_2^{s'}.$$

Conversely, if a bounded function, $w : \Omega(s) \times \mathbb{S} \to \mathbb{R}$, satisfies the above functional equation, then $w = w^*$.

Since $\{0, \mathbb{R}^+\}$, $\mathbb{K}_\pi$ and $\mathbb{K}_y$ are already defined, the objects of interest become $\Omega(s)$, $s \in \mathbb{S}$. To find $\Omega(s)$ for a given $s \in \mathbb{S}$, we define an operator, $\mathbb{B}$, as follows. For $Q(s) \in \mathbb{R}^2$ and $s \in \mathbb{S}$, let

$$\mathbb{B}\left(Q(s)\right) = \Big\{(\eta_1,\eta_2) \mid \exists \left(y,\pi,r,\left\{\eta_{1s'},\eta_2^{s'}\right\}_{s'\in\mathbb{S}}\right) \in \mathbb{K}_Y\times\mathbb{K}_\Pi\times\mathbb{K}_R\times\left\{Q(s')\right\}_{s'\in\mathbb{S}},$$
$$\text{where } \; y=\eta_1, \pi=\eta_2, r=\frac{1}{\sigma}\sum_{s'\in\mathbb{S}}p(s'|s)\eta_2^{s'}+\sum_{s'\in\mathbb{S}}p(s'|s)\eta_1^{s'}-\frac{1}{\sigma}y+s,$$
$$\text{and } \; \pi=\kappa y+\beta\sum_{s'\in\mathbb{S}}p(s'|s)\eta_1^{s'}\Big\}.$$

It can be shown that, for any $s \in \mathbb{S}$, (i) $Q(s) \subseteq \mathbb{B}\left(Q\right)(s) \Rightarrow \mathbb{B}\left(Q\right)(s) \subseteq \Omega(s)$ (a.k.a *self generation*), (ii) $\Omega(s) = \mathbb{B}\left(\Omega\right)(s)$ (a.k.a. *factorization*), and (iii) letting $Q_0(s) = \left[\underline{\eta}_1, \overline{\eta}_1\right] \times \left[\underline{\eta}_2, \overline{\eta}_2\right]$ and $Q_n(s) = \mathbb{B}\left(Q_{n-1}\right)(s)$, $Q_n(s) \supseteq Q_{n+1}(s)$ and $\Omega(s) = \cap_{n=0}^{\infty} Q_n(s)$.

C.2 Computing $\Omega(s)$ with the operator $\mathbb{B}$

The obvious guess for $\Omega(\delta_c)$ and $\Omega(\delta_n)$ is $\mathbb{K}_Y \times \mathbb{K}_\Pi$. However, we can show that, with $Q_0(\delta_c) := \mathbb{K}_Y \times \mathbb{K}_\Pi$ and $Q_0(\delta_n) := \mathbb{K}_Y \times \mathbb{K}_\Pi$, $\mathbb{B}\left(Q_0\right)(\delta_c) \subset Q_0(\delta_c)$ and $\mathbb{B}\left(Q_0\right)(\delta_n) \subset Q_0(\delta_n)$.

<u>Proof</u>: Let $Q_0(\delta_c) := \mathbb{K}_Y \times \mathbb{K}_\Pi$ and $Q_0(\delta_n) := \mathbb{K}_Y \times \mathbb{K}_\Pi$. Take $(\eta_1, \eta_2) = (y_{max}, \pi_{min}) \in Q_0(\delta_n)$. We want to show that $(\eta_1, \eta_2) \notin \mathbb{B}\left(Q_0\right)(\delta_n)$. Let $y = \eta_1$ and $\pi = \eta_2$. Note that, in order to satisfy the Phillips curve, $\eta_2(\delta_n) = (\eta_2 - \kappa\eta_1)/\beta$. Note that $\eta_2(\delta_n) = (\eta_2 - \kappa\eta_1)/\beta = (\pi_{min} - \kappa y_{max})/\beta < \pi_{min} - \kappa y_{max}/\beta \leq \pi_{min}$. The second-to-last inequality follows from the fact that $\pi_{min} < 0$. The last inequality follows from the fact that $y_{max} > 0$.[14] Thus, $(\eta_1, \eta_2) \notin \mathbb{B}\left(Q_0\right)(\delta_n)$. Similarly, we can prove that $(\eta_1, \eta_2) \notin \mathbb{B}\left(Q_0\right)(\delta_c)$.

Since $\mathbb{B}\left(Q_0\right)(\delta_c) \subset Q_0(\delta_c) := \mathbb{K}_Y \times \mathbb{K}_\Pi$ and $\mathbb{B}\left(Q_0\right)(\delta_n) \subset Q_0(\delta_n) := \mathbb{K}_Y \times \mathbb{K}_\Pi$, $\Omega(\delta_c) \neq \mathbb{K}_Y \times \mathbb{K}_\Pi$ and $\Omega(\delta_n) \neq \mathbb{K}_Y \times \mathbb{K}_\Pi$. Thus, we have to apply the operator $\mathbb{B}$ repeatedly until it converges to find $\Omega(\delta_c)$ and $\Omega(\delta_n)$.

Analytically characterizing the sequence of $\{Q_j(\delta_c), Q_j(\delta_n)\}_{j=0}^{\infty}$ seems daunting, if not infeasible. Thus, we will use a numerical method similar to Feng, Miao, Peralta-Alva, and Santos (2014) in order to numerically compute $\{Q_j(\delta_c), Q_j(\delta_n)\}_{j=0}^{\infty}$ and their convergent sets, $\Omega(\delta_c)$ and $\Omega(\delta_n)$.

C.2.1 Setup

Let $\mathbb{A} = \mathbb{K}_Y \times \mathbb{K}_\Pi \times \mathbb{K}_R$ be known as the action space. Due to the following for each $s \in \mathbb{S}$.
(i) Make an initial guess for $\Omega(\delta)$, i.e. $\hat{Q}^0(\delta) = \mathbb{K}_Y \times \mathbb{K}_\Pi$.
(ii) Create an object, $\hat{Q}_{grid}(\delta)$, by discretizing each dimension, η, of $\hat{Q}(\delta)$ into N_η equidistant points. This results in $(N_{\eta_1} - 1) \times (N_{\eta_2} - 1)$ rectangles each denoted by Ξ_i—$i = 1, ..., (N_{\eta_1} - 1) \times (N_{\eta_2} - 1)$—which in turn yields a position (j_{η_1}, j_{η_2}), $j_{\eta_1} = 1, ..., N_{\eta_1} - 1$ and $j_{\eta_2} = 1, ..., N_{\eta_2} - 1$, in the discretized state space, $\hat{Q}_{grid}(\delta)$.
(iii) Let $\mathbb{G}_\delta^0\left(\hat{Q}_{grid}(\delta)\right) = \left\{\mathbb{I}_{0,1}\left(\Xi_1\right), ..., \mathbb{I}_{0,1}\left(\Xi_{\left(N_{\eta_1}-1\right)\times\left(N_{\eta_2}-1\right)}\right)\right\}$ be a vector of indicator functions indicating the inclusion of each rectangle, Ξ_i, where a value of 1 indicates inclusion

[14] Since we want to allow (y, π) to take the value of $(0, 0)$, both $\mathbb{K}_Y$ and $\mathbb{K}_\Pi$ have to cover 0. In other words, π_{min} and y_{min} has to be strictly negative and π_{max} and y_{max} has to be strictly positive.

and 0 does not.

(iv) Set $\mathbb{G}_{\delta}^{0}\left(\hat{Q}_{grid}(\delta)\right) = \underline{\underline{\mathbf{1}}}$ (a vector of ones).

(v) Let $\hat{Q}_{grid}^{0}(\delta) = \left\{\Xi_i \in \hat{Q}_{grid} | \mathbb{G}_{\delta}^{0}\left(\Xi_i\right) = 1\right\}$.

C.2.2 Brute-force search algorithm

At each iteration, $n \geq 0$, and for each $s \in \mathbb{S}$, do the following:

(i) Given $\hat{Q}_{grid}^{n}(\delta)$ and $\mathbb{G}_{\delta}^{n}\left(\hat{Q}_{grid}(\delta)\right)$, we want to update $\hat{Q}_{grid}^{n+1}(\delta)$ and $\mathbb{G}_{\delta}^{n+1}\left(\hat{Q}_{grid}(\delta)\right)$.

(ii) For each $\Xi_i \in \hat{Q}_{grid}^{n}(\delta)$, make a judicious selection of points to test.[15] If for at least one point, $(\eta_1, \eta_2) \in \Xi_i$, $\exists \left(y, \pi, r, \left\{\eta_1^{\delta'}, \eta_2^{\delta'}\right\}_{\delta' \in \mathbb{D}}\right) \in \mathbb{A} \times \left\{\hat{Q}_{grid}^{n}(\delta')\right\}_{\delta' \in \mathbb{D}}$ such that $y = \eta_1$, $\pi = \eta_2$, $r = \frac{1}{\sigma} \sum_{\delta' \in \mathbb{D}} p(\delta'|\delta) \eta_1^{\delta'} + \sum_{\delta' \in \mathbb{D}} p(\delta'|\delta) \eta_2^{\delta'} - \frac{1}{\sigma} y + \delta$, and $\pi = \kappa y + \beta \sum_{\delta' \in \mathbb{D}} p(\delta'|\delta) \eta_2^{\delta'}$, set $\mathbb{G}_{\delta}^{n+1}\left(\Xi_i\right) = 1$. Otherwise, set $\mathbb{G}_{\delta}^{n+1}\left(\Xi_i\right) = 0$.

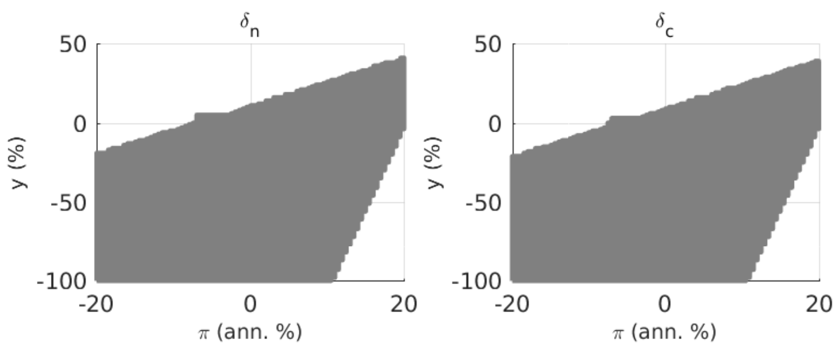
(iii) Update $\hat{Q}_{grid}^{n+1}(\delta) = \left\{\Xi_i \in \hat{Q}_{grid} | \mathbb{G}_{\delta}^{n+1}\left(\Xi_i\right) = 1\right\}$.

(iv) If $\hat{Q}_{grid}^{n}(\delta) = \hat{Q}_{grid}^{n+1}(\delta)$ $\forall \delta \in \mathbb{D}$, stop the algorithm and set $\hat{\Omega}(\delta) = \hat{Q}_{grid}^{n+1}(\delta)$, $s \in \mathbb{D}$. Otherwise, repeat the algorithm.

C.2.3 Results

Figure 14 shows the set of feasible pairs of initial inflation and output promises, $\Omega(\delta)$.

Figure 14: The set of feasible pairs of initial (y, π)-promises



According to the figure, for both high (crisis) and low (normal) states, (i) combinations of a very high output and a very low output—northwest corner—and (ii) combinations of a very low output and very high inflation—southeast corner—are not feasible. This makes sense because Phillips curve constraint requires that, all else equal, inflation today has to be lower when output is lower.

The set of feasible pairs of initial inflation and output promises is large. In particular, for both states, a wide range of areas around the steady state of $(\pi = 0, y = 0)$ is feasible. Thus, the boundary of this set does not pose any binding constraints on the optimization problem associated with the Bellman equation.

[15] We use a total of nine points: the vertices, the midpoints between the vertices, and the point in the middle of the rectangle.

D Analytical results for the model with inflation bias

In this section, we first provide the analytical solutions to the saddle-point functional equation associated with the Lagrange multiplier approach and the Bellman equation associated with the promised value approach for the model with inflation bias (section D.1 and D.2). We then prove that the allocations obtained from the Lagrange multiplier approach are identical to those obtained from the promised value approach (section D.3).

D.1 Lagrange multiplier approach

Guess that the solution to the saddle-point functional equation takes the following form:

$$\pi = \alpha_{0,\pi} + \alpha_{1,\pi}\phi_{-1}, \tag{18}$$
$$\phi = \alpha_{0,\phi} + \alpha_{1,\phi}\phi_{-1}, \tag{19}$$
$$y = \alpha_{0,y} + \alpha_{1,y}\phi_{-1}. \tag{20}$$

We would like to find $(\alpha_{0,\pi}, \alpha_{1,\pi}, \alpha_{0,\phi}, \alpha_{1,\phi})$ such that the following FONCs associated with the saddle-point functional equation are satisfied:

$$\phi = \pi + \phi_{-1}, \tag{21}$$
$$0 = -\lambda(y - y^*) - \kappa\phi, \tag{22}$$
$$\pi = \kappa y + \beta\pi'. \tag{23}$$

Substituting (18) and (19) into (21), we obtain

$$\alpha_{0,\phi} + \alpha_{1,\phi}\phi_{-1} = \alpha_{0,\pi} + \alpha_{1,\pi}\phi_{-1} + \phi_{-1}.$$

For this equation to hold for any ϕ_{-1}, the following two equations must hold:

$$\alpha_{0,\phi} = \alpha_{0,\pi}, \tag{24}$$
$$\alpha_{1,\phi} = 1 + \alpha_{1,\pi}. \tag{25}$$

Substituting (21), (18), (19), and (20) into (22), we obtain

$$\begin{aligned} 0 &= -\lambda\left(\alpha_{0,y} + \alpha_{1,y}\phi_{-1} - y^*\right) - \kappa(\pi + \phi_{-1}), \\ &= -\lambda\left(\alpha_{0,y} + \alpha_{1,y}\phi_{-1} - y^*\right) - \kappa(\alpha_{0,\pi} + \alpha_{1,\pi}\phi_{-1} + \phi_{-1}), \\ &= -\lambda\alpha_{0,y} + \lambda y^* - \kappa\alpha_{0,\pi} - \lambda\alpha_{1,y}\phi_{-1} - \kappa(1 + \alpha_{1,\pi})\phi_{-1}. \end{aligned}$$

For this equation to hold for any ϕ_{-1}, the following two equations must hold:

$$\alpha_{1,y} = -\lambda^{-1}\kappa(1 + \alpha_{1,\pi}), \tag{26}$$
$$\alpha_{0,y} = y^* - \kappa\lambda^{-1}\alpha_{0,\pi}. \tag{27}$$

Substituting (18)-(20) into (23), we obtain

$$\begin{aligned} \alpha_{0,\pi} + \alpha_{1,\pi}\phi_{-1} &= \kappa(\alpha_{0,y} + \alpha_{1,y}\phi_{-1}) + \beta(\alpha_{0,\pi} + \alpha_{1,\pi}\phi), \\ &= \kappa(\alpha_{0,y} + \alpha_{1,y}\phi_{-1}) + \beta\alpha_{0,\pi} + \beta\alpha_{1,\pi}(\alpha_{0,\phi} + \alpha_{1,\phi}\phi_{-1}), \\ &= \kappa\alpha_{0,y} + \kappa\alpha_{1,y}\phi_{-1} + \beta\alpha_{0,\pi} + \beta\alpha_{1,\pi}\alpha_{0,\phi} + \beta\alpha_{1,\pi}\alpha_{1,\phi}\phi_{-1}. \end{aligned}$$

For this equation to hold for any ϕ_{-1}, the following two equations must hold:

$$\alpha_{0,\pi} = \kappa\alpha_{0,y} + \beta(\alpha_{0,\pi} + \alpha_{1,\pi}\alpha_{0,\phi}), \tag{28}$$

$$\alpha_{1,\pi} = \kappa\alpha_{1,y} + \beta\alpha_{1,\pi}\alpha_{1,\phi}. \tag{29}$$

Substituting (24) and (27) into (28),

$$\alpha_{0,\pi} = \kappa y^* - \kappa^2\lambda^{-1}\alpha_{0,\pi} + \beta\alpha_{0,\pi}(1 + \alpha_{1,\pi}),$$

$$\implies \quad \alpha_{0,\pi} = \frac{\kappa y^*}{1 + \kappa^2\lambda^{-1} - \beta(1 + \alpha_{1,\pi})}.$$

Substituting (26) into (29),

$$\alpha_{1,\pi} = -\lambda^{-1}\kappa^2(1 + \alpha_{1,\pi}) + \beta\alpha_{1,\pi}(1 + \alpha_{1,\pi}),$$

$$\implies \quad \beta\alpha_{1,\pi}^2 - \left(1 - \beta + \lambda^{-1}\kappa^2\right)\alpha_{1,\pi} - \lambda^{-1}\kappa^2 = 0. \tag{30}$$

Substituting (25) into (30) and arranging it,

$$\beta\alpha_{1,\phi}^2 - \left(1 + \beta + \lambda^{-1}\kappa^2\right)\alpha_{1,\phi} + 1 = 0.$$

The solution to this quadratic function is

$$\frac{\left(1 + \beta + \lambda^{-1}\kappa^2\right) \pm \sqrt{\left(1 + \beta + \lambda^{-1}\kappa^2\right)^2 - 4\beta}}{2\beta}.$$

Since $1 + \beta + \lambda^{-1}\kappa^2 > 0$,

$$\alpha_{1,\phi} = \frac{\left(1 + \beta + \lambda^{-1}\kappa^2\right) - \sqrt{\left(1 + \beta + \lambda^{-1}\kappa^2\right)^2 - 4\beta}}{2\beta}.$$

Summary of coefficients

$$\alpha_{1,\phi} = \frac{\left(1 + \beta + \lambda^{-1}\kappa^2\right) - \sqrt{\left(1 + \beta + \lambda^{-1}\kappa^2\right)^2 - 4\beta}}{2\beta}$$

$$\alpha_{0,\pi} = \frac{\kappa y^*}{1 + \kappa^2\lambda^{-1} - \beta\alpha_{1,\phi}}$$

$$\alpha_{0,\phi} = \alpha_{0,\pi}$$

$$\alpha_{0,y} = y^* - \kappa\lambda^{-1}\alpha_{0,\pi}$$

$$\alpha_{1,\pi} = \alpha_{1,\phi} - 1$$

$$\alpha_{1,y} = -\lambda^{-1}\kappa(1 + \alpha_{1,\pi})$$

D.2 Promised value approach

Guess that the solution to the Bellman equation takes the following form:

$$\eta' = a_\eta\eta, \tag{31}$$

$$y = a_y\eta. \tag{32}$$

We would like to find (a_η, α_y) such that the following FONCs associated with the Bellman equation are satisfied:

$$-\lambda(y - y^*) - \kappa\omega = 0,$$
$$\beta\frac{\partial W(\eta')}{\partial \eta'} - \omega\beta = 0,$$
$$\eta = \kappa y + \beta\eta'.$$

where ω is the Lagrange multiplier on the Phillips curve. The envelope condition associated with the Bellman equation is

$$\frac{\partial W(\eta)}{\partial \eta} = -\eta + \omega.$$

From above four equations, we have

$$0 = \frac{\lambda}{\kappa}y - \eta' - \frac{\lambda}{\kappa}y', \tag{33}$$
$$\eta = \kappa y + \beta\eta'. \tag{34}$$

Substituting (31) and (32) into (34), we obtain

$$a_y = \frac{1 - \beta a_\eta}{\kappa}. \tag{35}$$

Substituting (31) and (32) into (33), we obtain

$$\begin{aligned} 0 &= \frac{\lambda}{\kappa}a_y\eta - a_\eta\eta - \frac{\lambda}{\kappa}a_y\eta', \\ &= \frac{\lambda}{\kappa}a_y\eta - a_\eta\eta - \frac{\lambda}{\kappa}a_y a_\eta\eta, \\ &= \frac{\lambda}{\kappa}a_y - a_\eta - \frac{\lambda}{\kappa}a_y a_\eta. \end{aligned} \tag{36}$$

Finally, substituting (35) into (36), we obtain

$$\begin{aligned} 0 &= \frac{\lambda}{\kappa}\left(\frac{1 - \beta a_\eta}{\kappa}\right) - a_\eta - \frac{\lambda}{\kappa}\left(\frac{1 - \beta a_\eta}{\kappa}\right)a_\eta, \\ &= \lambda(1 - \beta a_\eta) - \kappa^2 a_\eta - a_\eta\lambda(1 - \beta a_\eta), \\ &= \lambda - \lambda\beta a_\eta - \kappa^2 a_\eta - a_\eta\lambda + a_\eta^2\lambda\beta, \\ &= \beta\lambda a_\eta^2 - \left[(1 + \beta)\lambda + \kappa^2\right]a_\eta + \lambda, \\ &= \beta a_\eta^2 - \left(1 + \beta + \lambda^{-1}\kappa^2\right)a_\eta + 1. \end{aligned}$$

The solution to this quadratic function is

$$a_\eta = \frac{\left(1 + \beta + \lambda^{-1}\kappa^2\right) \pm \sqrt{(1 + \beta + \lambda^{-1}\kappa^2)^2 - 4\beta}}{2\beta}.$$

Since $1 + \beta + \lambda^{-1}\kappa^2 > 0$,

$$a_\eta = \frac{\left(1 + \beta + \lambda^{-1}\kappa^2\right) - \sqrt{\left(1 + \beta + \lambda^{-1}\kappa^2\right)^2 - 4\beta}}{2\beta}.$$

Summary of coefficients

$$a_y = \frac{1 - \beta a_\eta}{\kappa}$$

$$a_\eta = \frac{\left(1 + \beta + \lambda^{-1}\kappa^2\right) - \sqrt{\left(1 + \beta + \lambda^{-1}\kappa^2\right)^2 - 4\beta}}{2\beta}$$

In the promised value approach, one needs to know the value function associated with the Bellman equation to find the initial inflation. Guess that the value function takes the following form:

$$W_{PV}(\eta) = \mu_{PV,0} + \mu_{PV,1}\eta + \frac{1}{2}\mu_{PV,2}\eta^2. \tag{37}$$

The value function satisfies

$$W_{PV}(\eta) = -\frac{1}{2}\left[\eta^2 + \lambda(y - y^*)^2\right] + \beta W_{PV}(\eta'). \tag{38}$$

Substituting (31), (32), and (37) into (38), we obtain

$$\begin{aligned}
W_{PV}(\eta) &= -\frac{1}{2}\left[\eta^2 + \lambda(y - y^*)^2\right] + \beta\left[\mu_{PV,0} + \mu_{PV,1}\eta' + \frac{1}{2}\mu_{PV,2}{\eta'}^2\right], \\
&= -\frac{1}{2}\eta^2 - \frac{1}{2}\lambda\left[a_y^2\eta^2 - 2a_y\eta y^* + [y^*]^2\right] + \beta\mu_{PV,0} + \beta\mu_{PV,1}a_\eta\eta + \frac{1}{2}\beta\mu_{PV,2}a_\eta^2\eta^2, \\
&= -\frac{1}{2}\eta^2 - \frac{1}{2}\lambda a_y^2\eta^2 + \lambda a_y y^*\eta - \frac{1}{2}\lambda[y^*]^2 + \beta\mu_{PV,0} + \beta\mu_{PV,1}a_\eta\eta + \frac{1}{2}\beta\mu_{PV,2}a_\eta^2\eta^2, \\
&= \beta\mu_{PV,0} - \frac{1}{2}\lambda[y^*]^2 + \left(\lambda a_y y^* + \beta\mu_{PV,1}a_\eta\right)\eta + \frac{1}{2}\left[\beta\mu_{PV,2}a_\eta^2 - (1 + \lambda a_y^2)\right]\eta^2.
\end{aligned}$$

Comparing the constant terms and coefficients on η and η^2, we obtain

$$\mu_{PV,0} = -\frac{\lambda[y^*]^2}{2(1 - \beta)},$$

$$\mu_{PV,1} = \frac{\lambda a_y y^*}{1 - \beta a_\eta},$$

$$\mu_{PV,2} = -\frac{1 + \lambda a_y^2}{1 - \beta a_\eta^2}.$$

The initial inflation is given by

$$\pi_1 = \text{argmax}_{\text{J}}\ W_{PV}$$

$$
\begin{aligned}
\implies \quad & \frac{\partial W_{PV}}{\partial \eta} = 0 \\
\implies \quad & \pi_1 = \eta \\
& = -\frac{\mu_{PV,1}}{\mu_{PV,2}} \\
& = \frac{\lambda a_y y^*}{1-\beta a_\eta} \frac{1-\beta a_\eta^2}{1+\lambda a_y^2}.
\end{aligned}
$$

D.3 Equivalence

We now prove that the allocations under the Lagrange multiplier and the promised value approaches are identical. We do so in two steps. First, we show that the initial inflation and output implied by the two approaches are identical. We then show that if the initial output is the same, the allocations from $t = 2$ on are identical.

D.3.1 Equivalence of time-one allocations

We will first show $\pi_{lm,1} = \pi_{pv,1}$ and then $y_{lm,1} = y_{pv,1}$.
In the promised value approach, the time-1 inflation is given by

$$
\begin{aligned}
\pi_{pv,1} &= \frac{\lambda \alpha_y y^*}{1-\beta \alpha_\eta} \frac{1-\beta \alpha_\eta^2}{1+\lambda \alpha_y^2}, \\
&= \frac{\lambda y^*}{\kappa} \frac{1-\beta \alpha_\eta^2}{1+\lambda \alpha_y^2},
\end{aligned}
$$

with

$$
\alpha_y = \frac{1-\beta\alpha_\eta}{\kappa}.
$$

In Lagrange multiplier approach, the time-1 inflation is given by

$$
\pi_{lm,1} = \frac{\kappa y^*}{1+\kappa^2\lambda^{-1} - \beta\alpha_\eta}.
$$

$$
\begin{aligned}
& \pi_{pv,1} = \pi_{lm,1} \\
\iff \quad & \frac{\lambda}{\kappa}(1-\beta\alpha_\eta^2)(1+\kappa^2\lambda^{-1}-\beta\alpha_\eta) = \kappa(1+\lambda\alpha_y^2), \\
\iff \quad & \frac{\lambda}{\kappa}(1-\beta\alpha_\eta^2)(1+\kappa^2\lambda^{-1}-\beta\alpha_\eta) = \kappa\left(1+\lambda\left[\frac{1-\beta\alpha_\eta}{\kappa}\right]^2\right), \\
\iff \quad & \kappa\lambda(1-\beta\alpha_\eta^2)(1+\kappa^2\lambda^{-1}-\beta\alpha_\eta) = \kappa\left(\kappa^2+\lambda\left[1-\beta\alpha_\eta\right]^2\right), \\
\iff \quad & \lambda(1-\beta\alpha_\eta^2)(1+\kappa^2\lambda^{-1}-\beta\alpha_\eta) = \kappa^2+\lambda[1-\beta\alpha_\eta]^2, \\
\iff \quad & (1-\beta\alpha_\eta^2)(\lambda+\kappa^2-\beta\lambda\alpha_\eta) = \kappa^2+\lambda[1-\beta\alpha_\eta]^2, \\
\iff \quad & \lambda+\kappa^2-\beta\lambda\alpha_\eta-\beta\lambda\alpha_\eta^2-\beta\kappa^2\alpha_\eta^2+\beta^2\lambda\alpha_\eta^3 = \kappa^2+\lambda-2\lambda\beta\alpha_\eta+\lambda\beta^2\alpha_\eta^2, \\
\iff \quad & -\beta\lambda\alpha_\eta-\beta\lambda\alpha_\eta^2-\beta\kappa^2\alpha_\eta^2+\beta^2\lambda\alpha_\eta^3 = -2\lambda\beta\alpha_\eta+\lambda\beta^2\alpha_\eta^2, \\
\iff \quad & -1-\alpha_\eta-\kappa^2\frac{1}{\lambda}\alpha_\eta+\beta\alpha_\eta^2 = -2+\beta\alpha_\eta, \\
\iff \quad & \beta\alpha_\eta^2-\left(1+\beta+\frac{\kappa^2}{\lambda}\right)\alpha_\eta+1 = 0.
\end{aligned}
$$

Note that α_η was constructed so that the last equality holds. Thus, the last equality holds and $\pi_{lm,1} = \pi_{pv,1}$.

Now, we will show $y_{lm,1} = y_{pv,1}$. In the promised value approach, the time-1 output is given by

$$
\begin{aligned}
y_{pv,1} &= \frac{1-\beta\alpha_\eta}{\kappa}\pi_{pv,1} \\
&= \frac{1-\beta\alpha_\eta}{\kappa}\frac{\kappa y^*}{1+\lambda^{-1}\kappa^2-\beta\alpha_\eta} \\
&= \frac{1-\beta\alpha_\eta}{1+\lambda^{-1}\kappa^2-\beta\alpha_\eta}y^* \\
&= [1-\frac{\lambda^{-1}\kappa^2}{1+\lambda^{-1}\kappa^2-\beta\alpha_\eta}]y^*
\end{aligned}
$$

In the Lagrange multiplier approach, the time-1 output is given by

$$
\begin{aligned}
y_{lm,1} &= y^* - \frac{\kappa^2}{\lambda}\frac{\kappa y^*}{1+\kappa^2\lambda^{-1}-\beta\alpha_\eta} \\
&= [1-\frac{\lambda^{-1}\kappa^2}{1+\lambda^{-1}\kappa^2-\beta\alpha_\eta}]y^*
\end{aligned}
$$

Thus, $y_{lm,1} = y_{pv,1}$.

D.3.2 Equivalence of allocations from $t = 2$ on

To show the equivalence of allocations from $t = 2$ on, we first express the allocation at t as a function of output at $t-1$ under the two approaches. We then show that the function is identical under the two approaches. Since we have already shown that time-one output is the same across the two approaches, it follows that the allocations from $t = 2$ on are the same across them.

Let the mapping from output in the previous period to today's allocations under the Lagrange multiplier approach be given by:

$$\pi_{lm,t} = \gamma_\pi y_{lm,t-1},$$
$$y_{lm,t} = \gamma_y y_{lm,t-1},$$

for $t = 2, 3, ...$ Using $0 = -\lambda(y_{lm,t} - y^*) - \kappa\phi_{lm,t}$, we have

$$\begin{aligned}\pi_{lm,t} &= \alpha_{0,\pi} + \alpha_{1,\pi}\phi_{lm,t} \\ &= \alpha_{0,\pi} + \frac{\lambda}{\kappa}\alpha_{1,\pi}y^* - \frac{\lambda}{\kappa}\alpha_{1,\pi}y_{lm,t-1} \\ &= -\frac{\lambda}{\kappa}\alpha_{1,\pi}y_{lm,t-1},\end{aligned}$$

and

$$\begin{aligned}y_{lm,t} &= \alpha_{0,\pi} + \alpha_{1,\pi}\phi_{lm,t} \\ &= \alpha_{0,y} + \frac{\lambda}{\kappa}\alpha_{1,y}y^* - \frac{\lambda}{\kappa}\alpha_{1,y}y_{lm,t-1} \\ &= -\frac{\lambda}{\kappa}\alpha_{1,y}y_{lm,t-1},\end{aligned}$$

for $t = 1, 2, ...$ Note that $\alpha_{0,\pi} = -\frac{\lambda}{\kappa}\alpha_{1,\pi}y^*$ and $\alpha_{0,y} = -\frac{\lambda}{\kappa}\alpha_{1,y}y^*$ are implied by

$$\beta\alpha_{1,\phi}^2 - \left(1 + \beta + \lambda^{-1}\kappa^2\right)\alpha_{1,\phi} + 1 = 0.$$

Thus,

$$\begin{aligned}\gamma_\pi &= -\frac{\lambda}{\kappa}\alpha_{1,\pi} \\ \gamma_y &= -\frac{\lambda}{\kappa}\alpha_{1,y}.\end{aligned}$$

Now, let the mapping from output in the previous period to today's allocations under the promised value approach be given by:

$$\pi_{pv,t} = c_\pi y_{pv,t-1},$$

$$y_{pv,t} = c_y y_{pv,t-1},$$

for $t = 2, 3, ...$ Using the solution from the promised value approach,

$$\begin{aligned}\pi_{pv,t} &= a_\eta \pi_{pv,t-1} \\ &= a_\eta a_y^{-1} y_{pv,t-1},\end{aligned}$$

$$\begin{aligned}y_{pv,t} &= a_y \pi_{pv,t} \\ &= a_y a_\eta a_y^{-1} y_{pv,t-1} \\ &= a_\eta y_{pv,t-1}.\end{aligned}$$

Thus,

$$c_\pi = a_\eta a_y^{-1}$$
$$c_y = a_\eta.$$

We want to show $\gamma_y = c_y$ and $\gamma_\pi = c_\pi$, which imply $y_{lm,t} = y_{pv,t}$ and $\pi_{lm,t} = \pi_{pv,t}$ for $t = 2, 3, ...$ given that $y_{lm,1} = y_{pv,1}$. Let us first show

$$\gamma_y = c_y.$$

$$\begin{aligned}
\gamma_y &= -\frac{\lambda}{\kappa}\alpha_{1,y}, \\
&= -\frac{\lambda}{\kappa}\left(-\frac{\kappa}{\lambda}\alpha_{1,\phi}\right), \\
&= \alpha_{1,\phi}, \\
&= \frac{1+\beta+\lambda^{-1}\kappa^2 - \sqrt{(1+\beta+\lambda^{-1}\kappa^2)^2 - 4\beta}}{2\beta}, \\
&= \frac{(1+\beta)\lambda+\kappa^2 - \sqrt{[(1+\beta)\lambda+\kappa^2]^2 - 4\beta\lambda^2}}{2\beta\lambda}, \\
&= a_\eta = c_y.
\end{aligned}$$

Next, let us show

$$\gamma_\pi = a_\eta a_y^{-1}.$$

We have

$$\gamma_\pi = -\frac{\lambda}{\kappa}(a_\eta - 1),$$

and

$$a_\eta a_y^{-1} = \frac{\kappa a_\eta}{1-\beta a_\eta}.$$

$$\begin{aligned}
& & \frac{\kappa a_\eta}{1-\beta a_\eta} &= -\frac{\lambda}{\kappa}(a_\eta - 1) \\
&\iff & \kappa^2 a_\eta &= -\lambda(a_\eta - 1)(1-\beta a_\eta) \\
&\iff & \kappa^2 a_\eta &= -\lambda(a_\eta - 1 - \beta a_\eta^2 + \beta a_\eta) \\
&\iff & \kappa^2 a_\eta &= -\lambda a_\eta + \lambda + \beta\lambda a_\eta^2 - \beta\lambda a_\eta \\
&\iff & & \beta\lambda a_\eta^2 - (\lambda+\beta\lambda+\kappa^2)\alpha_\eta + \lambda = 0 \\
&\iff & & \beta a_\eta^2 - (1+\beta+\frac{\kappa^2}{\lambda})a_\eta + 1 = 0
\end{aligned}$$

Note that a_η was constructed so that the last equality holds. Thus, the last equality holds by construction.

E Analytical results for the model with stabilization bias

In this section, we first provide the analytical solutions to the saddle-point functional equation associated with the Lagrange multiplier approach and the Bellman equation associated with the promised value approach for the model with stabilization bias (section E.1 and E.2). We then prove that the allocations obtained from the Lagrange multiplier approach are identical to those obtained from the promised value approach (section E.3).

E.1 Lagrange multiplier approach

Given our assumption that the cost-push shock disappears after $t = 1$, the solution of the saddle-point functional equation in this model is identical to that in the model with inflation bias from $t = 2$ on, and is given by:

$$\begin{aligned}
\alpha_{1,\pi}(2) &= \alpha_{1,\phi}(2) - 1 \\
\alpha_{1,y}(2) &= -\kappa\lambda^{-1}(1 + \alpha_{1,\pi}(2)) \\
\alpha_{1,\phi}(2) &= \frac{1 + \beta + \kappa^2\lambda^{-1} - \sqrt{(1 + \beta + \kappa^2\lambda^{-1})^2 - 4\beta}}{2\beta}
\end{aligned}$$

Turning our attention to $t = 1$ when the cost-push shock is present, the FONCs are given by

$$\phi(s_1) = \pi(s_1) + \phi_{-1}, \tag{39}$$

$$0 = -\lambda y(s_1) - \kappa\phi(s_1), \tag{40}$$

$$\pi(s_1) = \kappa y(s_1) + \beta\pi(\phi(s_1), e_2) + e_1. \tag{41}$$

Guess that the solution takes the following form:

$$\pi(s_1) = \alpha_{0,\pi}(1) + \alpha_{1,\pi}(1)\phi_{-1}, \tag{42}$$

$$\phi(s_1) = \alpha_{0,\phi}(1) + \alpha_{1,\phi}(1)\phi_{-1}, \tag{43}$$

$$y(s_1) = \alpha_{0,y}(1) + \alpha_{1,y}(1)\phi_{-1}, \tag{44}$$

Note that

$$\begin{aligned}
\pi(\phi(s_1), e_2) &= \alpha_{1,\pi}(2)\phi(s_1) \\
&= \alpha_{1,\pi}(2)(\alpha_{0,\phi}(1) + \alpha_{1,\phi}(1)\phi_{-1}) \\
&= \alpha_{1,\pi}(2)\alpha_{0,\phi}(1) + \alpha_{1,\pi}(2)\alpha_{1,\phi}(1)\phi_{-1}.
\end{aligned} \tag{45}$$

Substituting (42) and (43) into (39),

$$\alpha_{0,\phi}(1) + \alpha_{1,\phi}(1)\phi_{-1} = \alpha_{0,\pi}(1) + (1 + \alpha_{1,\pi}(1))\phi_{-1}.$$

Therefore, we have

$$\alpha_{0,\phi}(1) = \alpha_{0,\pi}(1), \tag{46}$$

$$\alpha_{1,\phi}(1) = 1 + \alpha_{1,\pi}(1). \tag{47}$$

Substituting (39), (42), and (44) into (40),

$$0 = -\lambda(\alpha_{0,y}(1) + \alpha_{1,y}(1)\phi_{-1}) - \kappa(\alpha_{0,\pi}(1) + \alpha_{1,\pi}(1)\phi_{-1} + \phi_{-1}).$$

Therefore, we have

$$\alpha_{0,y}(1) = -\kappa\lambda^{-1}\alpha_{0,\pi}(1), \tag{48}$$
$$\alpha_{1,y}(1) = -\lambda^{-1}\kappa(1 + \alpha_{1,\pi}(1)). \tag{49}$$

Substituting (42), (44), and (45) into (41),

$$\alpha_{0,\pi}(1) + \alpha_{1,\pi}(1)\phi_{-1} = \kappa(\alpha_{0,y}(1) + \alpha_{1,y}(1)\phi_{-1}) + \beta(\alpha_{1,\pi}(2)\alpha_{0,\phi}(1) + \alpha_{1,\pi}(2)\alpha_{1,\phi}(1)\phi_{-1}) + e_1.$$

Therefore, we have

$$\alpha_{0,\pi}(1) = \kappa\alpha_{0,y}(1) + \beta(\alpha_{1,\pi}(2)\alpha_{0,\phi}(1)) + e_1, \tag{50}$$
$$\alpha_{1,\pi}(1) = \kappa\alpha_{1,y}(1) + \beta\alpha_{1,\pi}(2)\alpha_{1,\phi}(1). \tag{51}$$

Substituting (46) and (48) into (50), we have

$$\alpha_{0,\pi}(1) = -\kappa^2\lambda^{-1}\alpha_{0,\pi}(1) + \beta\alpha_{1,\pi}(2)\alpha_{0,\pi}(1) + e_1,$$
$$\implies \quad \alpha_{0,\pi}(1) = \frac{e_1}{1 - \beta\alpha_{1,\pi}(2) + \kappa^2\lambda^{-1}}. \tag{52}$$

Substituting (47) and (49) into (51), we have

$$\alpha_{1,\pi}(1) = -\kappa^2\lambda^{-1}(1 + \alpha_{1,\pi}(1)) + \beta\alpha_{1,\pi}(2)(1 + \alpha_{1,\pi}(1)),$$
$$\implies \quad \alpha_{1,\pi}(1) = \frac{\beta\alpha_{1,\pi}(2) - \kappa^2\lambda^{-1}}{1 + \kappa^2\lambda^{-1} - \beta\alpha_{1,\pi}(2)}. \tag{53}$$

Summary of coefficients for $t = 1$

$$\begin{aligned}
\alpha_{0,\phi}(1) &= \alpha_{0,\pi}(1) \\
\alpha_{1,\phi}(1) &= 1 + \alpha_{1,\pi}(1) \\
\alpha_{0,y}(1) &= -\kappa\lambda^{-1}\alpha_{0,\pi}(1) \\
\alpha_{1,y}(1) &= -\lambda^{-1}\kappa(1 + \alpha_{1,\pi}(1)) \\
\alpha_{0,\pi}(1) &= \frac{e_1}{1 - \beta\alpha_{1,\pi}(2) + \kappa^2\lambda^{-1}} \\
\alpha_{1,\pi}(1) &= \frac{\beta\alpha_{1,\pi}(2) - \kappa^2\lambda^{-1}}{1 + \kappa^2\lambda^{-1} - \beta\alpha_{1,\pi}(2)}
\end{aligned}$$

E.2 Promised value approach

As with the Lagrange multiplier approach, given our assumption that the shock disappears after $t = 1$, the solution to the relevant Bellman equation in this model is the same as that in the model with inflation bias, and is given by:

$$
\begin{aligned}
a_{1,y}(2) &= \frac{1-\beta a_{1,\eta}(2)}{\kappa} \\
a_{1,\eta}(2) &= \frac{1+\beta+\lambda^{-1}\kappa^2-\sqrt{[(1+\beta)+\lambda^{-1}\kappa^2]^2-4\beta}}{2\beta} \\
\mu_{pv,0}(2) &= 0 \\
\mu_{pv,1}(2) &= 0 \\
\mu_{pv,2}(2) &= -\frac{1+\lambda a_{1,y}(2)^2}{1-\beta a_{1,\eta}(2)^2}
\end{aligned}
$$

Turning our attention to $t = 1$ when the cost-push shock is present, the FONCs are given by

$$\lambda y(s_1) - \lambda y(\eta'(s_1), e_2) - \kappa\eta'(s_1) = 0, \tag{54}$$

$$\eta = \kappa y(s_1) + \beta\eta'(s_1) + e_1. \tag{55}$$

Guess that the solution takes the following form:

$$\eta'(s_1) = a_{0,\eta}(1) + a_{1,\eta}(1)\eta, \tag{56}$$
$$y(s_1) = a_{0,y}(1) + a_{1,y}(1)\eta, \tag{57}$$
$$y(\eta'(s_1), e_2) = a_{1,y}(2)(a_{0,\eta}(1) + a_{1,\eta}(1)\eta). \tag{58}$$

Substituting (56)-(58) into (54),

$$\lambda(a_{0,y}(1) + a_{1,y}(1)\eta) - \lambda a_{1,y}(2)(a_{0,\eta} + a_{1,\eta}(1)\eta) - \kappa(a_{0,\eta}(1) + a_{1,\eta}(1)\eta) = 0.$$

Therefore, we have

$$\lambda a_{1,y}(1) - \lambda a_{1,y}(2)a_{1,\eta}(1) - \kappa a_{1,\eta}(1) = 0, \tag{59}$$
$$\lambda a_{0,y}(1) - \lambda a_{1,y}(2)a_{0,\eta}(1) - \kappa a_{0,\eta}(1) = 0. \tag{60}$$

Substituting (56) and (57) into (55),

$$\eta = \kappa(a_{0,y}(1) + a_{1,y}(1)\eta) + \beta(a_{0,\eta}(1) + a_{1,\eta}(1)\eta) + e_1.$$

Therefore, we have

$$1 = \kappa a_{1,y}(1) + \beta a_{1,\eta}(1), \tag{61}$$
$$0 = \kappa a_{0,y}(1) + \beta a_{0,\eta}(1) + e_1. \tag{62}$$

From (61),

$$a_{1,y}(1) = \frac{1-\beta a_{1,\eta}(1)}{\kappa}. \tag{63}$$

Substituting (63) into (59),

$$\lambda\left(\frac{1-\beta a_{1,\eta}(1)}{\kappa}\right) - \lambda a_{1,y}(2)a_{1,\eta}(2) - \kappa a_{1,\eta}(1) = 0,$$

$$\lambda - \lambda\beta a_{1,\eta}(1) - \lambda\kappa a_{1,y}(2)a_{1,\eta}(1) - \kappa^2 a_{1,\eta}(1) = 0,$$
$$\lambda = (\lambda\beta + \lambda\kappa a_{1,y}(2) + \kappa^2)a_{1,\eta}(1),$$

$$\implies a_{1,\eta}(1) = \frac{\lambda}{\lambda\beta + \kappa^2 + \lambda\kappa a_{1,y}(2)}.$$

Furthermore,

$$\begin{aligned}
a_{1,\eta}(1) &= \frac{\lambda}{\kappa^2 + \lambda\beta + \lambda\kappa a_{1,y}(2)} \\
&= \frac{\lambda}{\kappa^2 + \lambda\beta + \lambda\kappa\frac{1-\beta a_{1,\eta}(2)}{\kappa}} \\
&= \frac{\lambda}{\kappa^2 + \lambda\beta + \lambda - \lambda\beta a_{1,\eta}(2)} \\
&= \frac{\lambda}{\lambda(1+\beta+\lambda^{-1}\kappa^2) - \lambda\beta\frac{1+\beta+\lambda^{-1}\kappa^2 - \sqrt{[(1+\beta)+\lambda^{-1}\kappa^2]^2 - 4\beta}}{2\beta}} \\
&= \frac{1}{1+\beta+\lambda^{-1}\kappa^2 - \frac{1+\beta+\lambda^{-1}\kappa^2 - \sqrt{[(1+\beta)+\lambda^{-1}\kappa^2]^2 - 4\beta}}{2}} \\
&= \frac{1}{\frac{1+\beta+\lambda^{-1}\kappa^2 + \sqrt{[(1+\beta)+\lambda^{-1}\kappa^2]^2 - 4\beta}}{2}} \\
&= \frac{2}{1+\beta+\lambda^{-1}\kappa^2 + \sqrt{[(1+\beta)+\lambda^{-1}\kappa^2]^2 - 4\beta}} \\
&= \frac{2}{1+\beta+\lambda^{-1}\kappa^2 + \sqrt{[(1+\beta)+\lambda^{-1}\kappa^2]^2 - 4\beta}}\frac{1+\beta+\lambda^{-1}\kappa^2 - \sqrt{[(1+\beta)+\lambda^{-1}\kappa^2]^2 - 4\beta}}{1+\beta+\lambda^{-1}\kappa^2 - \sqrt{[(1+\beta)+\lambda^{-1}\kappa^2]^2 - 4\beta}} \\
&= \frac{1+\beta+\lambda^{-1}\kappa^2 - \sqrt{[(1+\beta)+\lambda^{-1}\kappa^2]^2}}{2\beta} \\
&= a_{1,\eta}(2)
\end{aligned}$$

Note that $a_{1,\eta}(1) = a_{1,\eta}(2)$ implies $a_{1,y}(1) = a_{1,y}(2)$
From (60),

$$a_{0,y}(1) = \frac{\kappa}{\lambda}a_{0,\eta}(1) + a_{1,y}(2)a_{0,\eta}(1). \tag{64}$$

Substituting (64) into (62),

$$0 = \frac{\kappa^2}{\lambda}a_{0,\eta}(1) + \kappa a_{1,y}(2)a_{0,\eta}(1) + \beta a_{0,\eta}(1) + e_1,$$

$$0 = \kappa^2 a_{0,\eta}(1) + \lambda\kappa a_{1,y}(2)a_{0,\eta}(1) + \beta\lambda a_{0,\eta}(1) + e_1,$$
$$(\kappa^2 + \lambda\kappa a_{1,y}(2) + \beta\lambda)a_{0,\eta}(1) = -\lambda e_1,$$
$$\implies a_{0,\eta}(1) = -\frac{\lambda e_1}{\kappa^2 + \lambda\beta + \lambda\kappa a_{1,y}(2)}.$$

Summary of coefficients

$$
\begin{aligned}
a_{0,y}(1) &= \frac{\kappa}{\lambda} a_{0,\eta}(1) + a_{1,y}(2) a_{0,\eta}(1) \\
a_{1,y}(1) &= a_{1,y}(2) \\
a_{0,\eta}(1) &= -\frac{\lambda e_1}{\kappa^2 + \lambda\beta + \lambda\kappa a_{1,y}(2)} \\
a_{1,\eta}(1) &= a_{1,\eta}(2)
\end{aligned}
$$

Now, we solve for the value function at $t = 1$.

$$
W_{pv}(s_1) = -\frac{1}{2}[\eta^2 + \lambda y(s_1)^2] + \beta W_{pv}(\eta'(s_1), e_2). \tag{65}
$$

Guess that the solution takes the following form:

$$
W_{pv}(s_1) = \mu_{pv,0}(1) + \mu_{pv,1}(1)\eta + \frac{1}{2}\mu_{pv,2}(1)\eta^2.
$$

Then,

$$
W_{pv}(\eta'(s_1), e_2) = \mu_{pv,0}(2) + \mu_{pv,1}(2)(a_{0,\eta}(1) + a_{1,\eta}(1)\eta) + \frac{1}{2}\mu_{pv,2}(2)(a_{0,\eta}(1) + a_{1,\eta}(1)\eta)^2. \tag{66}
$$

Since $\mu_{pv,0}(2) = \mu_{pv,1}(2) = 0$, (66) can be rewritten as follows:

$$
W_{pv}(\eta'(s_1, e_2)) = \frac{1}{2}\mu_{pv,2}(2)(a_{0,\eta}(1) + a_{1,\eta}(1)\eta)^2. \tag{67}
$$

Substituting (57) and (67) into (65), we obtain

$$
\begin{aligned}
W_{pv}(s_1) &= -\frac{1}{2}[\eta^2 + \lambda(a_{0,y}(1) + a_{1,y}(1)\eta)^2] + \frac{1}{2}\beta\mu_{pv,,2}(2)(a_{0,\eta}(1) + a_{1,\eta}(1)\eta)^2, \\
&= -\frac{1}{2}(\eta^2 + \lambda a_{0,y}(1)^2 + 2\lambda a_{0,y}(1)a_{1,y}(1)\eta + \lambda a_{1,y}(1)^2\eta^2) \\
&\quad + \frac{1}{2}\beta\mu_{pv,2}(2)(a_{0,\eta}(1)^2 + 2a_{0,\eta}(1)a_{1,\eta}(1)\eta + a_{1,\eta}(1)^2\eta^2), \\
&= -\frac{\lambda}{2}a_{0,y}(1)^2 + \frac{1}{2}\beta\mu_{pv,2}(2)a_{0,\eta}(1)^2 \\
&\quad - \lambda a_{0,y}(1)a_{1,y}(1)\eta + \beta\mu_{pv,2}(2)a_{0,\eta}(1)a_{1,\eta}(1)\eta \\
&\quad - \frac{1}{2}\eta^2 - \frac{\lambda}{2}a_{1,y}(1)^2\eta^2 + \frac{1}{2}\beta\mu_{pv,2}(2)a_{1,\eta}(1)^2\eta^2.
\end{aligned}
$$

Comparing the constant terms and coefficients on η and η^2, we obtain

$$
\begin{aligned}
\mu_{pv,0}(1) &= -\frac{\lambda}{2}a_{0,y}(1)^2 + \frac{1}{2}\beta\mu_{pv,2}(2)a_{0,\eta}(1)^2, \\
\mu_{pv,1}(1) &= \beta\mu_{pv,2}(2)a_{0,\eta}(1)a_{1,\eta}(1) - \lambda a_{0,y}(1)a_{1,y}(1), \\
\mu_{pv,2}(1) &= \beta\mu_{pv,2}(2)a_{1,\eta}(2)^2 - 1 - \lambda a_{1,y}(1)^2.
\end{aligned}
$$

E.3 Equivalence

We now prove that the allocations under the Lagrange multiplier and the promised value approaches are identical in the model with stabilization bias. Since the model's solution is the same across two approaches from $t = 2$ on if time-one output is the same, it is sufficient to show that the initial inflation and output implied by the two approaches are identical (that is, $\pi_{lm,1} = \pi_{pv,1}$ and $y_{lm,1} = y_{pv,1}$).

We will first show $\pi_{lm,1} = \pi_{pv,1}$ and then $y_{lm,1} = y_{pv,1}$.

In the promised value approach, time-one inflation is given by

$$\begin{aligned}
\pi_{pv,1} &= -\frac{\mu_{pv,1}(1)}{\mu_{pv,2}(1)} \\
&= -\frac{\beta\mu_{pv,2}(2)a_{0,\eta}(1)a_{1,\eta}(1) - \lambda a_{0,y}(1)a_{1,y}(1)}{\beta\mu_{pv,2}(2)a_{1,\eta}(1)^2 - 1 - \lambda a_{1,y}(1)^2} \\
&= -\frac{\beta\mu_{pv,2}(2)(-a_{1,\eta}(1)e_1)a_{1,\eta}(1) - \lambda[-(\frac{\kappa}{\lambda} + a_{1,y}(2))a_{1,\eta}e_1]a_{1,y}(1)}{\beta\mu_{pv,2}(2)a_{1,\eta}(1)^2 - 1 - \lambda a_{1,y}(1)^2} \\
&= -e_1\frac{-\beta\mu_{pv,2}(2)a_{1,\eta}(1)^2 + \lambda(\frac{\kappa}{\lambda} + a_{1,y}(2))a_{1,\eta}a_{1,y}(1)}{\beta\mu_{pv,2}(2)a_{1,\eta}(1)^2 - 1 - \lambda a_{1,y}(1)^2}
\end{aligned}$$

In the Lagrange multiplier approach, time-one inflation is given by

$$\begin{aligned}
\pi_{lm,1} &= \alpha_{0,\pi}(1) \\
&= \frac{e_1}{1 - \beta a_{1,\pi}(2) + \lambda^{-1}\kappa^2} \\
&= \frac{e_1}{1 - \beta(a_{1,\phi}(2) - 1) + \lambda^{-1}\kappa^2} \\
&= \frac{e_1}{1 + \beta + \lambda^{-1}\kappa^2 - \beta a_{1,\phi}(2)} \\
&= \frac{e_1}{1 + \beta + \lambda^{-1}\kappa^2 - \beta a_{1,\eta}(2)} \\
&= \frac{a_{1,\eta}(2)e_1}{[1 + \beta + \lambda^{-1}\kappa^2 - \beta a_{1,\eta}(2)]a_{1,\eta}(2)} \\
&= a_{1,\eta}(2)e_1
\end{aligned}$$

where the last inequality follows from the definition of $a_{1,\eta}(2)$. Putting things together, showing $\pi_{lm,1} = \pi_{pv,1}$ amounts to showing

$$a_{1,\eta}(2) = -\frac{-\beta\mu_{pv,2}(2)a_{1,\eta}(1)^2 + \lambda(\frac{\kappa}{\lambda} + a_{1,y}(2))a_{1,\eta}a_{1,y}(1)}{\beta\mu_{pv,2}(2)a_{1,\eta}(1)^2 - 1 - \lambda a_{1,y}(1)^2}$$

$\Longrightarrow$

$$\beta\mu_{pv,2}(2)a_{1,\eta}(1)^2a_{1,\eta}(2) - a_{1,\eta}(2) - \lambda a_{1,y}(1)^2a_{1,\eta}(2) = \beta\mu_{pv,2}(2)a_{1,\eta}(1)^2 - \lambda(\frac{\kappa}{\lambda} + a_{1,y}(2))a_{1,\eta}a_{1,y}(1)$$

$\Longrightarrow$

$$-a_{1,\eta}(2) - \lambda a_{1,y}(1)^2a_{1,\eta}(2) = \beta\mu_{pv,2}(2)a_{1,\eta}(1)^2(1 - a_{1,\eta}(2)) - \lambda(\frac{\kappa}{\lambda} + a_{1,y}(2))a_{1,\eta}a_{1,y}(1)$$

$\Longrightarrow$

$$-a_{1,\eta}(2) = \beta\mu_{pv,2}(2)a_{1,\eta}(1)^2(1 - a_{1,\eta}(2)) - \kappa a_{1,\eta}a_{1,y}(1)$$

$\Longrightarrow$

$$-a_{1,\eta}(2) = -\beta\frac{1 + \lambda a_{1,y}(2)^2}{1 - \beta a_{1,\eta}(2)^2}a_{1,\eta}(1)^2(1 - a_{1,\eta}(2)) - \kappa a_{1,\eta}a_{1,y}(1)$$

Multiplying both sides by $1 - \beta a_{1,\eta}(2)^2$, we obtain

$$-a_{1,\eta}(2)(1 - \beta a_{1,\eta}(2)^2) = -\beta(1 + \lambda a_{1,y}(2)^2)a_{1,\eta}(1)^2(1 - a_{1,\eta}(2)) - \kappa a_{1,\eta}a_{1,y}(1)(1 - \beta a_{1,\eta}(2)^2)$$

Dividing both sides by $a_{1,\eta}(2)$, we obtain

$$-(1 - \beta a_{1,\eta}(2)^2) = -\beta(1 + \lambda a_{1,y}(2)^2)a_{1,\eta}(1)(1 - a_{1,\eta}(2)) - \kappa a_{1,y}(1)(1 - \beta a_{1,\eta}(2)^2)$$

$\Longrightarrow$

$$-(1 - \beta a_{1,\eta}(2)^2) = -\beta(1 + \lambda a_{1,y}(2)^2)a_{1,\eta}(1)(1 - a_{1,\eta}(2)) - \kappa\frac{1 - \beta a_{1,\eta}(1)}{\kappa}(1 - \beta a_{1,\eta}(2)^2)$$

$\Longrightarrow$

$$-(1 - \beta a_{1,\eta}(2)^2) = -\beta(1 + \lambda a_{1,y}(2)^2)a_{1,\eta}(1)(1 - a_{1,\eta}(2)) - (1 - \beta a_{1,\eta}(1))(1 - \beta a_{1,\eta}(2)^2)$$

$\Longrightarrow$

$$-\beta a_{1,\eta}(1)(1 - \beta a_{1,\eta}(2)^2) = -\beta(1 + \lambda a_{1,y}(2)^2)a_{1,\eta}(1)(1 - a_{1,\eta}(2))$$

Dividing both sides by $-\beta a_{1,\eta}(1)$, we obtain

$$\begin{aligned}
1 - \beta a_{1,\eta}(2)^2 &= (1 + \lambda a_{1,y}(2)^2)(1 - a_{1,\eta}(2)) \\
&= (1 + \lambda[\frac{1 - \beta a_{1,\eta}(2)}{\kappa}]^2)(1 - a_{1,\eta}(2)) \\
&= (1 + \frac{\lambda}{\kappa^2}[1 - 2\beta a_{1,\eta}(2) + \beta^2 a_{1,\eta}(2)^2])(1 - a_{1,\eta}(2)) \\
&= (1 + \frac{\lambda}{\kappa^2}[1 - 2\beta a_{1,\eta}(2) + \beta(1 + \beta + \lambda^{-1}\kappa^2)a_{1,\eta}(2) - \beta])(1 - a_{1,\eta}(2)) \\
&= (1 + \frac{\lambda(1 - \beta)}{\kappa^2} + \frac{\lambda\beta}{\kappa^2}(-1 + \beta + \lambda^{-1}\kappa^2)a_{1,\eta}(2))(1 - a_{1,\eta}(2)) \\
&= 1 + \frac{\lambda(1 - \beta)}{\kappa^2} + \frac{\lambda\beta}{\kappa^2}(-1 + \beta + \lambda^{-1}\kappa^2)a_{1,\eta}(2) \\
&\quad - (1 + \frac{\lambda(1 - \beta)}{\kappa^2})a_{1,\eta}(2) - \frac{\lambda\beta}{\kappa^2}(-1 + \beta + \lambda^{-1}\kappa^2)a_{1,\eta}(2)^2
\end{aligned}$$

Multiplying both sides by κ^2, we obtain

$$\begin{aligned}
\kappa^2 - \beta\kappa^2 a_{1,\eta}(2)^2 &= \kappa^2 + \lambda(1 - \beta) + \lambda\beta(-1 + \beta + \lambda^{-1}\kappa^2)a_{1,\eta}(2) \\
&\quad - (\kappa^2 + \lambda(1 - \beta))a_{1,\eta}(2) - \lambda\beta(-1 + \beta + \lambda^{-1}\kappa^2)a_{1,\eta}(2)^2 \\
&= \kappa^2 + \lambda(1 - \beta) + (\lambda\beta^2 + \kappa^2\beta - \kappa^2 - \lambda)a_{1,\eta}(2) \\
&\quad + (\lambda\beta - \lambda\beta^2 - \beta\kappa^2)a_{1,\eta}(2)^2
\end{aligned}$$

Subtracting $\kappa^2 - \beta\kappa^2 a_{1,\eta}(2)^2$ from both sides, we obtain

$$0 = \lambda(1-\beta) + [\lambda(\beta-1)(\beta+1) + \kappa^2(1-\beta)]a_{1,\eta}(2) + \lambda\beta(1-\beta)a_{1,\eta}(2)^2$$

Dividing both sides by $\lambda(1-\beta)$, we obtain

$$0 = -(1+\beta+\frac{\kappa^2}{\lambda})a_{1,\eta}(2) + \beta a_{1,\eta}(2)^2$$

This quadratic equation holds by the definition of $a_{1,\eta}(2)$. Thus, $\pi_{lm,1} = \pi_{pv,1}$.

Now, we will show $y_{lm,1} = y_{pv,1}$. In the promised value approach, the time-1 output is given by

$$\begin{aligned} y_{pv,1} &= a_{0,y} + a_{1,y}\pi_{pv,1} \\ &= -[\frac{\kappa}{\lambda} + a_{1,y}(2)]a_{1,\eta}(1)e_1 + a_{1,y}(1)\pi_{pv,1} \\ &= -[\frac{\kappa}{\lambda} + a_{1,y}(2)]\pi_{pv,1} + a_{1,y}(2)\pi_{pv,1} \\ &= -\frac{\kappa}{\lambda}\pi_{pv,1} \end{aligned}$$

In the Lagrange multiplier approach, the time-1 output is given by

$$\begin{aligned} y_{lm,1} &= \alpha_{0,y}(1) \\ &= -\frac{\kappa}{\lambda}\alpha_{0,\pi} \\ &= -\frac{\kappa}{\lambda}\pi_{lm,1} \end{aligned}$$

Thus, $y_{lm,1} = y_{pv,1}$.

F Global solution methods and their accuracy for the model with ELB

F.1 Lagrange multiplier approach

F.1.1 Marcet and Marimon's recursive formulation

Marcet and Marimon (2016) recursify the Ramsey problem using the Lagrange multipliers as pseudo state variables:

$$\begin{aligned} W(\phi_1, \phi_2, \delta_i) = \min_{\phi_1', \phi_2'} \max_{y,\pi,r\geq 0} \Big[&-\frac{\lambda}{2}y^2 - \frac{1}{2}\pi^2 \\ &+ \phi_1'\left(r + \sigma y + \delta_i\right) - \beta^{-1}\phi_1\left(\sigma y + \pi\right) \\ &+ \phi_2'\left(\pi - \kappa y\right) - \phi_2\pi + \beta\sum_{j=1}^{N} p(\delta_j|\delta_i)W(\phi_1', \phi_2', \delta_j) \Big]. \end{aligned}$$

The first-order necessary conditions (FONCs) are given by

$$\begin{aligned}
\partial y :& -\lambda y + \sigma\phi_1' - \sigma\beta^{-1}\phi_1 - \kappa\phi_2' = 0\\
\partial \pi :& -\pi - \beta^{-1}\phi_1 + \phi_2' - \phi_2 = 0\\
\partial \phi_1' :& r + \sigma y - \delta_i - \sum_{j=1}^{N} p(\delta_j|\delta_i) W_1(\phi_1', \phi_2', \delta_j) = 0\\
\partial \phi_2' :& \pi - \kappa y - \beta\sum_{j=1}^{N} p(\delta_j|\delta_i) W_2(\phi_1', \phi_2', \delta_j) = 0
\end{aligned}$$

where $W_1(\phi_1', \phi_2', \delta_j) = \partial W(\phi_1', \phi_2', \delta_j)/\partial\phi_1'$ and $W_2(\phi_1', \phi_2', \delta_j) = \partial W(\phi_1', \phi_2', \delta_j)/\partial\phi_2'$. The following Karush-Kuhn-Tucker conditions (KKTCs) must be satisfied as well

$$\phi_1' r = 0, \quad \phi_1' \leq 0, \quad \text{and} \quad r \geq 0.$$

The intitial conditions are such that $\phi_1 = 0$ and $\phi_2 = 0$.

F.1.2 Time iteration method

We explicitly consider a vector of policy functions $\varsigma(\xi_i) = [y(\xi_i), \pi(\xi_i), r(\xi_i), \phi_1'(\xi_i), \phi_2'(\xi_i)]'$ as functions of the state variables $\xi_i = (\phi_1, \phi_2, \delta_i)$ for $i = 1, ..., N$. Using the envelope theorem (i.e., $W_1(\phi_1, \phi_2, \delta_i) = \sigma y(\xi_i) + \pi(\xi_i)$ and $W_2(\phi_1, \phi_2, \delta_i) = \pi(\xi_i) - \kappa y(\xi_i)$), we have the following a system of functional equations

$$\begin{aligned}
e_{\text{LM},1}(\xi_i) \equiv& -\lambda y(\xi_i) + \sigma\phi_1'(\xi_i) - \frac{\sigma}{\beta}\phi_1 - \kappa\phi_2'(\xi_i) = 0,\\
e_{\text{LM},2}(\xi_i) \equiv& -\pi(\xi_i) - \beta^{-1}\phi_1 + \phi_2'(\xi_i) - \phi_2 = 0,\\
e_{\text{LM},3}(\xi_i) \equiv& r(\xi_i) + \sigma y(\xi_i) - \delta_j - \sum_{j=1}^{N} p(\delta_j|\delta_i)\left[\sigma y(\phi_1'(\xi_i), \phi_2'(\xi_i), \delta_j) + \pi(\phi_1'(\xi_i), \phi_2'(\xi_i), \delta_j)\right] = 0,\\
e_{\text{LM},4}(\xi_i) \equiv& \pi - \kappa y - \beta\sum_{j=1}^{N} p(\delta_j|\delta_i)\left[\kappa y(\phi_1'(\xi_i), \phi_2'(\xi_i), \delta_j) + \pi(\phi_1'(\xi_i), \phi_2'(\xi_i), \delta_j)\right] = 0.
\end{aligned}$$

Algorithm The time iteration method takes the following steps:

1. Make an initial guess for the policy function $\varsigma^{(0)}(\xi_i)$ for $i = 1, ..., N$.
2. For $k = 1, 2, ...$ (k is an index for the number of iteration), given the policy function

previously obtained $\varsigma^{(k-1)}(\xi_i)$ for each i, solve

$$\begin{aligned}
&-\lambda y+\sigma\phi_1'-\frac{\sigma}{\beta}\phi_1-\kappa\phi_2'=0\\
&-\pi-\beta^{-1}\phi_1+\phi_2'-\phi_2=0\\
&r+\sigma y-\delta_i\\
&-\sum_{j=1}^{N}p(\delta_j|\delta_i)\left[\sigma y^{(k-1)}(\phi_1',\phi_2',\delta_j)+\pi^{(k-1)}(\phi_1',\phi_2',\delta_j)\right]=0\\
&\pi-\kappa y-\beta\sum_{j=1}^{N}p(\delta_j|\delta_i)\left[\kappa y^{(k-1)}(\phi_1',\phi_2',\delta_j)+\pi^{(k-1)}(\phi_1',\phi_2',\delta_j)\right]=0
\end{aligned}$$

for $(y,\pi,r,\phi_1',\phi_2')$.

3. Update the policy function by setting $y=y^{(k)}(\xi_i)$, $\pi=\pi^{(k)}(\xi_i)$, $r=r^{(k)}(\xi_i)$, $\phi_1'=\phi_1^{'(k)}(\xi_i)$, $\phi_2'=\phi_2^{'(k)}(\xi_i)$ for $i=1,...,N$.

4. Repeat 2-3 until $\left\|\varsigma^{(k)}(\xi_i)-\varsigma^{(k-1)}(\xi_i)\right\|$ is small enough.

We use the following indicator function approach as in Gust, Herbst, López-Salido, and Smith (2017), Nakata (2017), and Hirose and Sunakawa (2017). That is, for $\varsigma\in\{y,\pi,\phi_1',\phi_2'\}$,

$$\varsigma(\xi_i)=\mathbb{I}_{r_{\text{NZLB}}(\xi_i)\geq 0}\varsigma_{\text{NZLB}}(\xi_i)+\left(1-\mathbb{I}_{r_{\text{NZLB}}(\xi_i)\geq 0}\right)\varsigma_{\text{ZLB}}(\xi_i)$$

where $\varsigma_{\text{NZLB}}(\xi_i)$ is the policy function assuming that ZLB *always* does not bind and $\varsigma_{\text{ZLB}}(\xi_i)$ is the policy function assuming that ZLB *always* binds. $\mathbb{I}_{r_{\text{NZLB}}(\xi_i)\geq 0}$ is the indicator function that takes the value of one when $r_{\text{NZLB}}(\xi_i)\geq 0$, otherwise takes the value of zero. Then, in Steps 2 and 3, the problem becomes finding a pair of policy functions, $(\varsigma_{\text{NZLB}}(\xi_i),\varsigma_{\text{ZLB}}(\xi_i))$, as follows (we denote a tuple of variables to be solved $(y_{\text{NZLB}},\pi_{\text{NZLB}},\phi_{1,\text{NZLB}}',\phi_{2,\text{NZLB}}',r_{\text{NZLB}})$ in the non-ZLB regime and $(y_{\text{ZLB}},\pi_{\text{ZLB}},\phi_{1,\text{ZLB}}',\phi_{2,\text{ZLB}}',r_{\text{ZLB}})$ in the ZLB regime): (i) When we assume that ZLB does not bind, given the values of ϕ_1, ϕ_2 and $\phi_{1,\text{NZLB}}'=0$, solve

$$\begin{aligned}
&\phi_{2,\text{NZLB}}'=\kappa^{-1}\left(-\lambda y_{\text{NZLB}}-\sigma\beta^{-1}\phi_1\right)\\
&\pi_{\text{NZLB}}=\phi_{2,\text{NZLB}}'-\phi_2-\beta^{-1}\phi_1\\
&\pi_{\text{NZLB}}-\kappa y_{\text{NZLB}}-\beta\sum_{j=1}^{N}p(\delta_j|\delta_i)\left[\kappa y^{(k-1)}(0,\phi_{2,\text{NZLB}}',\delta_j)+\pi^{(k-1)}(0,\phi_{2,\text{NZLB}}',\delta_j)\right]=0
\end{aligned}$$

for $(y_{\text{NZLB}},\pi_{\text{NZLB}},\phi_{2,\text{NZLB}}')$. Then we have

$$r_{\text{NZLB}}=-\sigma y_{\text{NZLB}}+\delta_i+\sum_{j=1}^{N}p(\delta_j|\delta_i)\left[\sigma y^{(k-1)}(0,\phi_{2,\text{NZLB}}',\delta_j)+\pi^{(k-1)}(0,\phi_{2,\text{NZLB}}',\delta_j)\right]$$

and set $y_{\text{NZLB}} = y^{(k)}_{\text{NZLB}}(\xi_i)$, $\pi_{\text{NZLB}} = \pi^{(k)}_{\text{NZLB}}(\xi_i)$, and $r_{\text{NZLB}} = r^{(k)}_{\text{NZLB}}(\xi_i)$. (ii) When we assume that ZLB binds, given the values of ϕ_1, ϕ_2 and $r_{\text{ZLB}} = 0$, solve

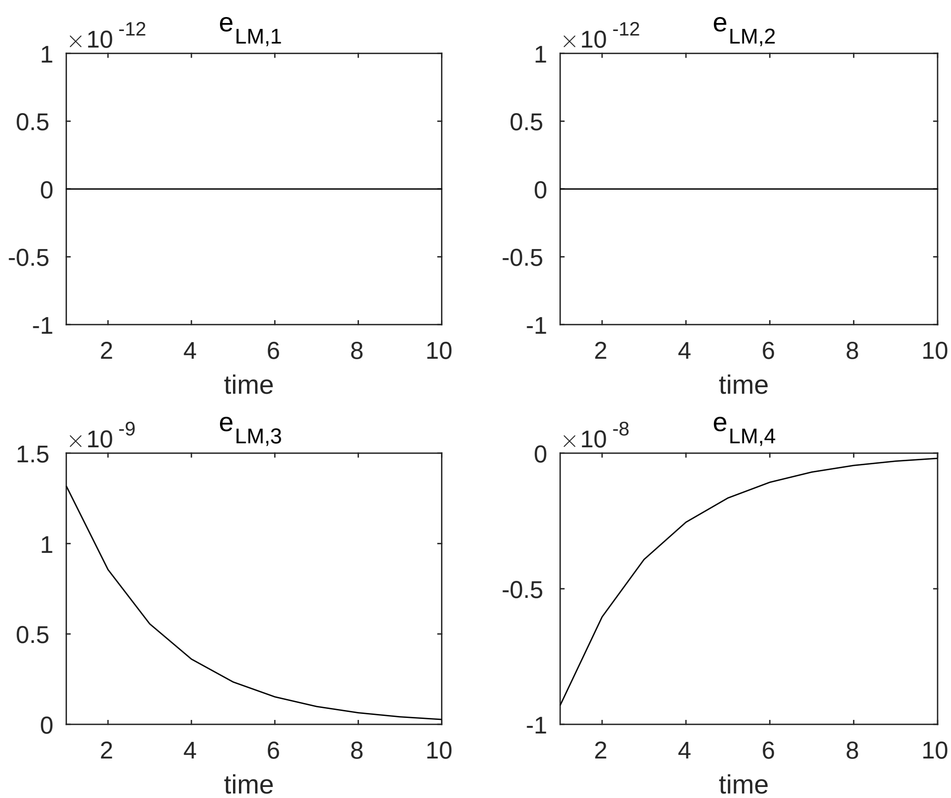
$$\phi'_{2,\text{ZLB}} = \pi + \phi_2 + \beta^{-1}\phi_1$$
$$\phi'_{1,\text{ZLB}} = \sigma^{-1}\kappa\phi'_{2,\text{ZLB}} + \sigma^{-1}\lambda y_{\text{ZLB}} + \beta^{-1}\phi_1$$
$$\pi_{\text{ZLB}} - \kappa y_{\text{ZLB}} - \sum_{j=1}^{N} p(\delta_j|\delta_i)\left[\kappa y^{(k-1)}(\phi'_{1,\text{ZLB}}, \phi'_{2,\text{ZLB}}, \delta_j) + \pi^{(k-1)}(\phi'_{1,\text{ZLB}}, \phi'_{2,\text{ZLB}}, \delta_j)\right] = 0$$
$$-\sigma y_{\text{ZLB}} + \delta_i + \beta\sum_{j=1}^{N} p(\delta_j|\delta_i)\left[\sigma y^{(k-1)}(\phi'_{1,\text{ZLB}}, \phi'_{2,\text{ZLB}}, \delta_j) + \pi^{(k-1)}(\phi'_{1,\text{ZLB}}, \phi'_{2,\text{ZLB}}, \delta_j)\right] = 0$$

for $(y_{\text{ZLB}}, \pi_{\text{ZLB}}, \phi'_{1,\text{ZLB}}, \phi'_{2,\text{ZLB}})$ and set $y_{\text{ZLB}} = y^{(k)}_{\text{ZLB}}(\xi_i)$ and $\pi_{\text{ZLB}} = \pi^{(k)}_{\text{ZLB}}(\xi_i)$.

When we solve the problem on a computer, we discretize a rectangle of the state space of (ϕ_1, ϕ_2). We use 21 points for each state variable. We set $\phi_1 \in [-0.002, 0]$ and $\phi_2 \in [-0.005, 0.009]$, and divide the state space by evenly spaced grid points. We use piecewise-linear functions to approximate the policy functions off the grid points.

Figure 15 shows the impulse response of the residual functions $(e_{\text{LM},1}, e_{\text{LM},2}, e_{\text{LM},3}, e_{\text{LM},4})$. Note that $(e_{\text{LM},1}, e_{\text{LM},2})$ (the FONCs) hold with equality (up to the machine precision), as we use these equations to substitute variables other than the ones we solve for with the other equations.

Figure 15: Euler errors: LM approach.



F.2 Promised-value approach

F.2.1 Recursive formulation

We substitute out $\eta_1 = \pi$, $\eta'_{1,j} = \pi'_j$, $\eta_2 = y$, and $\eta'_{2,j} = y'_j$. For each $\delta_i \in \mathbb{D}$, the problem for the optimal commitment policy planner can be written as

$$w\left(y_i, \pi_i, \delta_i\right) = \max_{y,\pi,r,\{\pi'_j,y'_j\}} -\frac{\lambda}{2}y^2 - \frac{1}{2}\pi^2 + \beta\sum_{j=1}^{N} p(\delta_j|\delta_i) w\left(y'_j, \pi'_j, \delta_j\right)$$

subject to

$$\begin{aligned}
y &= y_i \\
\pi &= \pi_i \\
\sigma y &= \sum_{j=1}^{N} p(\delta_j|\delta_i)\left[\sigma y'_j + \pi'_j\right] - r + \delta_i \\
\pi &= \kappa y + \beta\sum_{j=1}^{N} p(\delta_j|\delta_i)\pi'_j
\end{aligned}$$

Let the Lagrange multipliers on the constraints ω_1 and ω_2. The FONCs are given as follows:

$$\begin{aligned}
\partial y'_j :& -\sigma\omega_1 + \beta w_1\left(y'_j, \pi'_j, \delta_j\right) = 0 \\
\partial \pi'_j :& -\omega_1 - \beta\omega_2 + \beta w_2\left(y'_j, \pi'_j, \delta_j\right) = 0 \\
\partial \omega_1 :& \sigma y + r - \delta_i - \sum_{j=1}^{N} p(\delta_j|\delta_i)\left[\sigma y'_j + \pi'_j\right] = 0 \\
\partial \omega_2 :& \pi - \kappa y - \beta\sum_{j=1}^{N} p(\delta_j|\delta_i)\pi'_j = 0
\end{aligned}$$

for $j = 1, ..., N$, where $w_1(y'_j, \pi'_j, \delta_j) = \partial w(y'_j, \pi'_j, \delta_j)/\partial y'_j$ and $w_2(y'_j, \pi'_j, \delta_j) = \partial w(y'_j, \pi'_j, \delta_j)/\partial \pi'_j$. Note that these equations hold at each state j due to the state-contingent promises $\left\{y'_j, \pi'_j\right\}_{j=1}^{N}$. By using the envelope theorem and noting $y = y_i$ and $\pi = \pi_i$,

$$\begin{aligned}
w_1\left(y_i, \pi_i, \delta_i\right) &= -\lambda y_i + \sigma\omega_1 - \kappa\omega_2 \\
w_2\left(y_i, \pi_i, \delta_i\right) &= -\pi_i + \omega_2
\end{aligned}$$

for $i = 1, ..., N$. Then we have

$$\begin{aligned}
&-\lambda y'_j - \beta^{-1}\sigma\omega_1 + \sigma\omega'_1 - \kappa\omega'_2 = 0 \\
&-\pi'_j - \beta^{-1}\omega_1 + \omega'_2 - \omega_2 = 0
\end{aligned}$$

for $j = 1, ..., N$. As an important reminder, these equations yield a total of $2N$ FONCs since there are N states. The following KKTCs must be satisfied as well

$$\omega_1 r = 0, \quad \omega_1 \leq 0, \quad \text{and} \quad r \geq 0.$$

F.2.2 Time iteration method with simulated grid

We explicitly consider a vector of policy functions $\tilde{\varsigma}(\tilde{\xi}_i) = [\{y'_j(\tilde{\xi}_i), \pi'_j(\tilde{\xi}_i)\}_{j=1}^N, r(\tilde{\xi}_i), \omega_1(\tilde{\xi}_i), \omega_2(\tilde{\xi}_i)]'$ as functions of the state variables $\tilde{\xi}_i = (y, \pi, \delta_i)$ for $i = 1, ..., N$. Then we have the following a system of functional equations

$$
\begin{aligned}
e_{\text{PV},1,j}(\tilde{\xi}_i) \equiv & -\lambda y'_j(\tilde{\xi}_i) + \sigma\omega_1(y'_j(\tilde{\xi}_i), \pi'_j(\tilde{\xi}_i), \delta_j) - \sigma\beta^{-1}\omega_1(\tilde{\xi}_i) - \kappa\omega_2(y'_j(\tilde{\xi}_i), \pi'_j(\tilde{\xi}_i), \delta_j) = 0, \\
& \text{for } j = 1, ..., N, \\
e_{\text{PV},2,j}(\tilde{\xi}_i) \equiv & -\pi'_j(\tilde{\xi}_i) - \beta^{-1}\omega_1(\tilde{\xi}_i) + \omega_2(y'_j(\tilde{\xi}_i), \pi'_j(\tilde{\xi}_i), \delta_j) - \omega_2(\tilde{\xi}_i) = 0, \text{ for } j = 1, ..., N, \\
e_{\text{PV},3}(\tilde{\xi}_i) \equiv & r(\tilde{\xi}_i) + \sigma y - \delta_j - \sum_{j=1}^{N} p(\delta_j|\delta_i)\left[\sigma y'_j(\tilde{\xi}_i) + \pi'_j(\tilde{\xi}_i)\right] = 0, \\
e_{\text{PV},4}(\tilde{\xi}_i) \equiv & \pi - \kappa y - \beta\sum_{j=1}^{N} p(\delta_j|\delta_i)\pi'_j(\tilde{\xi}_i) = 0.
\end{aligned}
$$

We solve for the Ramsey equilibrium in a way that captures the full range of plausible values for y and π. Given the set of parameter values and the shock process we assume, some pairs (y, π) in a rectangle state space (as in Section F.1) may not be *plausible* in the Ramsey equilibrium. This makes solving for the policy and value functions with the rectangle state space impossible. In order to circumvent this problem, we adapt the approach of Maliar and Maliar (2015) (hereafter MM). That is, we solve for the policy functions on simulated grid points based on ergodic distribution of $\{y_t, \pi_t, \delta_t\}$, which are presumably included in the distribution of plausible promised pairs.

EDS algorithm As in MM, we merge the simulation-based sparse grid and the time iteration method by the following steps:

1. Initialization:
   (a) Choose initial values $\tilde{\xi}_0 = (y_0, \pi_0, \delta_0)$ and simulation length, T.
   (b) Draw a sequence of $\{\delta_t\}_{t=1}^T$ where $\delta_t \in \mathbb{D}$ and fix the sequence throughout the iterations.
   (c) Choose approximating policy functions $\tilde{\varsigma}(\tilde{\xi}_i; \boldsymbol{\theta})$ and make an initial guess of $\boldsymbol{\theta}$, where $\boldsymbol{\theta}$ is a vector of coefficients on a polynomial.
2. Construction of an EDS grid
   (a) Given $\{\delta_t\}_{t=1}^T$, use $\tilde{\varsigma}(\tilde{\xi}_i; \boldsymbol{\theta})$ to simulate $\{y_t, \pi_t\}_{t=1}^T$.
   (b) Construct an EDS grid $\Gamma(\delta_i) \equiv \{y_m, \pi_m; \delta_i\}_{m=1}^{M_i}$ for each $i = 1, ..., N$.
3. Computation of a solution on EDS grid, $\tilde{\varsigma}(\tilde{\xi}_i; \boldsymbol{\theta})$, using the time iteration method
   (a) Make an initial guess for the policy function $\tilde{\varsigma}^{(0)}$.
   (b) For $k = 1, 2, ...$ (k is an index for the number of iteration), given the policy function

previously obtained $\tilde{\varsigma}^{(k-1)}$, solve

$$
\begin{aligned}
&-\lambda y_j' + \sigma\omega_1^{(k-1)}(y_j', \pi_j', \delta_j; \boldsymbol{\theta}) - \sigma\beta^{-1}\omega_1 - \kappa\omega_2^{(k-1)}(y_j', \pi_j', \delta_j; \boldsymbol{\theta}) = 0, \text{ for } j = 1, ..., N, \\
&-\pi_j' - \beta^{-1}\omega_1 + \omega_2^{(k-1)}(y_j', \pi_j', \delta_j; \boldsymbol{\theta}) - \omega_2 = 0, \text{ for } j = 1, ..., N, \\
&r + \sigma y - \delta_i - \sum_{j=1}^{N} p(\delta_j|\delta_i)\left[\sigma y_j' + \pi_j'\right] = 0, \\
&\pi - \kappa y - \beta\sum_{j=1}^{N} p(\delta_j|\delta_i)\pi_j' = 0,
\end{aligned}
$$

for $(\{y_j', \pi_j'\}_{j=1}^{N}, r, \omega_1, \omega_2)$.

(c) Update the policy function by setting $y_j' = y_j'^{(k)}(\tilde{\xi}_i)$ for $j = 1, ..., N$, $\pi_j' = \pi_j'^{(k)}(\tilde{\xi}_i)$ for $j = 1, ..., N$, $r = r^{(k)}(\tilde{\xi}_i)$, $\omega_1 = \omega_1^{(k)}(\tilde{\xi}_i)$, $\omega_2 = \omega_2^{(k)}(\tilde{\xi}_i)$.

(d) Repeat 2-3 until $\left\|\tilde{\varsigma}^{(k)} - \tilde{\varsigma}^{(k-1)}\right\|$ is small enough.

4. Repeat 2-3 until convergence of the EDS grid.

In Step 2, we construct an EDS grid $\Gamma(\delta_i)$ indexed by $\delta_i \in \{\delta_n, \delta_c\}$ (we assume $N = 2$ and $\delta_i \in \{\delta_n, \delta_c\}$ hereafter) from an essentially ergodic set $\{y_t, \pi_t, \delta_t\}_{t=1}^{T}$. Given the policy function $\tilde{\varsigma}(s_i; \boldsymbol{\theta})$ and the sequence of $\{\delta_t\}_{t=1}^{T}$, we first simulate the economy to obtain an essentially ergodic set. As we assume the normal state is absorbing, i.e., $p(\delta_c|\delta_c) = p(\delta_c|\delta_n) = 0$, in order to obtain samples in the crisis state, $\{\delta_t\}_{t=1}^{T}$ is not necessarily consistent with the true stochastic process. In other words, the ergodic set we obtain here is *quasi-ergodic* (see Figure 18).

In constructing an EDS grid from the ergodic set, we do the following two step procedure (see MM for more details):

1. Selecting points within an essentially ergodic set (called Algorithm $\mathcal{A}^{\eta}$ in MM)

2. Constructing a uniformly spaced set of points that covers the essentially ergodic set (called Algorithm P^{ϵ} in MM)

There are two important parameters in this two step procedure: The interval of sampling, ι, and the threshold of density, ϵ. We set these parameters depending on the exogenous state variable δ_i. We set $(\iota_n, \iota_c) = (5, 1)$ and $(\epsilon_n, \epsilon_c) = (0.001, 0.000001)$, considering the ergodic set has fewer number of samples in the crisis state. The number of grid points is set to $M_n = M_c = 40$.

In Step 3, as in the LM approach, we use the following indicator function approach. That is, for $\tilde{\varsigma} \in \{y_n', \pi_n', y_c', \pi_c', \omega_1, \omega_2\}$,

$$
\tilde{\varsigma}(\tilde{\xi}_i) = \mathbb{I}_{r_{\text{NZLB}}(\tilde{\xi}_i)\geq 0}\tilde{\varsigma}_{\text{NZLB}}(\tilde{\xi}_i) + \left(1 - \mathbb{I}_{r_{\text{NZLB}}(\tilde{\xi}_i)\geq 0}\right)\tilde{\varsigma}_{\text{ZLB}}(\tilde{\xi}_i).
$$

Then, in Steps 3(b) and 3(c), the problem becomes finding a pair of policy functions, $\left(\tilde{\varsigma}_{\text{NZLB}}(\tilde{\xi}_i), \tilde{\varsigma}_{\text{ZLB}}(\tilde{\xi}_i)\right)$, as follows (we denote a tuple of variables ($y_{n,\text{NZLB}}'$, $\pi_{n,\text{NZLB}}'$, $y_{c,\text{NZLB}}'$, $\pi_{c,\text{NZLB}}'$, $\omega_{1,\text{NZLB}}$, $\omega_{2,\text{NZLB}}$, r_{NZLB}) in the non-ZLB regime and ($y_{n,\text{ZLB}}'$, $\pi_{n,\text{ZLB}}'$, $y_{c,\text{ZLB}}'$, $\pi_{c,\text{ZLB}}'$, $\omega_{1,\text{ZLB}}$, $\omega_{2,\text{ZLB}}$, r_{ZLB}) in the ZLB regime): (i) When we assume that ZLB does not bind, given the values of y, π and $\omega_{1,\mathrm{NZLB}} = 0$, we solve

$$
\begin{aligned}
&-\lambda y'_{n,\mathrm{NZLB}} + \sigma\omega_1^{(k-1)}(y'_{n,\mathrm{NZLB}}, \pi'_{n,\mathrm{NZLB}}, \delta_n; \boldsymbol{\theta}) - \sigma\beta^{-1}\omega_1 - \kappa\omega_2^{(k-1)}(y'_{n,\mathrm{NZLB}}, \pi'_{n,\mathrm{NZLB}}, \delta_n; \boldsymbol{\theta}) = 0,\\
&-\lambda y'_{c,\mathrm{NZLB}} + \sigma\omega_1^{(k-1)}(y'_{c,\mathrm{NZLB}}, \pi'_{c,\mathrm{NZLB}}, \delta_c; \boldsymbol{\theta}) - \sigma\beta^{-1}\omega_1 - \kappa\omega_2^{(k-1)}(y'_{c,\mathrm{NZLB}}, \pi'_{c,\mathrm{NZLB}}, \delta_c; \boldsymbol{\theta}) = 0,\\
&-\pi'_{n,\mathrm{NZLB}} + \omega_2^{(k-1)}(y'_{n,\mathrm{NZLB}}, \pi'_{n,\mathrm{NZLB}}, \delta_n; \boldsymbol{\theta}) - \omega_{2,\mathrm{NZLB}} = 0,\\
&-\pi'_{c,\mathrm{NZLB}} + \omega_2^{(k-1)}(y'_{c,\mathrm{NZLB}}, \pi'_{c,\mathrm{NZLB}}, \delta_c; \boldsymbol{\theta}) - \omega_{2,\mathrm{NZLB}} = 0,\\
&\pi - \kappa y - \beta\left[p(\delta_n|\delta_i)\pi'_{n,\mathrm{NZLB}} + p(\delta_c|\delta_i)\pi'_{c,\mathrm{NZLB}}\right] = 0,
\end{aligned}
$$

for $(y'_{n,\mathrm{NZLB}}, \pi'_{n,\mathrm{NZLB}}, y'_{c,\mathrm{NZLB}}, \pi'_{c,\mathrm{NZLB}}, \omega_{2,\mathrm{NZLB}})$ and

$$
r_{\mathrm{NZLB}} = -\sigma y + \delta_i + p(\delta_n|\delta_i)\left[\sigma y'_{n,\mathrm{NZLB}} + \pi'_{n,\mathrm{NZLB}}\right] + p(\delta_c|\delta_i)\left[\sigma y'_{c,\mathrm{NZLB}} + \pi'_{c,\mathrm{NZLB}}\right]
$$

and set $0 = \omega^{(k)}_{1,\mathrm{NZLB}}(\tilde{\xi}_i)$, $\omega_{2,\mathrm{NZLB}} = \omega^{(k)}_{2,\mathrm{NZLB}}(\tilde{\xi}_i)$, and $r_{\mathrm{NZLB}} = r^{(k)}_{\mathrm{NZLB}}(\tilde{\xi}_i)$. (ii) When we assume that ZLB binds, given the values of y, π and $r_{\mathrm{ZLB}} = 0$, we solve

$$
\begin{aligned}
&-\lambda y'_{n,\mathrm{ZLB}} + \sigma\omega_1^{(k-1)}(y'_{n,\mathrm{ZLB}}, \pi'_{n,\mathrm{ZLB}}, \delta_n; \boldsymbol{\theta}) - \sigma\beta^{-1}\omega_{1,\mathrm{ZLB}} - \kappa\omega_2^{(k-1)}(y'_{n,\mathrm{ZLB}}, \pi'_{n,\mathrm{ZLB}}, \delta_n; \boldsymbol{\theta}) = 0,\\
&-\lambda y'_{c,\mathrm{ZLB}} + \sigma\omega_1^{(k-1)}(y'_{c,\mathrm{ZLB}}, \pi'_{c,\mathrm{ZLB}}, \delta_c; \boldsymbol{\theta}) - \sigma\beta^{-1}\omega_{1,\mathrm{ZLB}} - \kappa\omega_2^{(k-1)}(y'_{c,\mathrm{ZLB}}, \pi'_{c,\mathrm{ZLB}}, \delta_c; \boldsymbol{\theta}) = 0,\\
&-\pi'_{n,\mathrm{ZLB}} + \omega_2^{(k-1)}(y'_{n,\mathrm{ZLB}}, \pi'_{n,\mathrm{ZLB}}, \delta_n; \boldsymbol{\theta}) - \omega_{2,\mathrm{ZLB}} = 0,\\
&-\pi'_{c,\mathrm{ZLB}} + \omega_2^{(k-1)}(y'_{c,\mathrm{ZLB}}, \pi'_{c,\mathrm{ZLB}}, \delta_c; \boldsymbol{\theta}) - \omega_{2,\mathrm{ZLB}} = 0,\\
&-\sigma y + \delta_i + p(\delta_n|\delta_i)\left[\sigma y'_{c,\mathrm{ZLB}} + \pi'_{n,\mathrm{ZLB}}\right] + p(\delta_c|\delta_i)\left[\sigma y'_{c,\mathrm{ZLB}} + \pi'_{c,\mathrm{ZLB}}\right] = 0,\\
&\pi - \kappa y - \beta\left[p(\delta_n|\delta_i)\pi'_{n,\mathrm{ZLB}} + p(\delta_c|\delta_i)\pi'_{c,\mathrm{ZLB}}\right] = 0,
\end{aligned}
$$

for $(y'_{n,\mathrm{ZLB}}, \pi'_{n,\mathrm{ZLB}}, y'_{c,\mathrm{ZLB}}, \pi'_{c,\mathrm{ZLB}}, \omega_1, \omega_2)$ and set $\omega_1 = \omega^{(k)}_{1,\mathrm{ZLB}}(\tilde{\xi}_i)$ and $\omega_2 = \omega^{(k)}_{2,\mathrm{ZLB}}(\tilde{\xi}_i)$.

We use second-order polynomials to approximate the policy functions off the grid points. That is, we fit a second-order polynomial,

$$
\tilde{\varsigma}(\tilde{\xi}_i; \boldsymbol{\theta}) = \theta_{i,(0,0)} + \theta_{i,(1,0)} y_m + \theta_{i,(0,1)} \pi_m + \theta_{i,(2,0)} y_m^2 + \theta_{i,(1,1)} y_m \pi_m + \theta_{i,(0,2)} \pi_m^2
$$

for each variable $\left(y'_{n,l}, \pi'_{n,l}, y'_{c,l}, \pi'_{c,l}, \omega_{1,l}, \omega_{2,l}, r_l\right)_{l\in\{\mathrm{NZLB},\mathrm{ZLB}\}}$ on the grid points $\Gamma(\delta_i) = \{y_m, \pi_m; \delta_i\}_{m=1}^{M_i}$. When we fit polynomials, we use the LS-SVD algorithm by following Judd, Maliar, and Maliar (2011) to avoid potential multicollinearlity problems.

Figure 16 shows the impulse response of the residual functions ($e_{\mathrm{PV},1,n}$, $e_{\mathrm{PV},1,c}$, $e_{\mathrm{PV},2,n}$, $e_{\mathrm{PV},2,c}$, $e_{\mathrm{PV},3}$, $e_{\mathrm{PV},4}$). Note that $(e_{\mathrm{PV},3}, e_{\mathrm{PV},4})$ (the consumption Euler equation and NKPC) hold with equality (up to machine precision), as we use these equations to substitute variables other than the ones we solve for with the other equations.

Figure 16: Euler errors: PV approach.

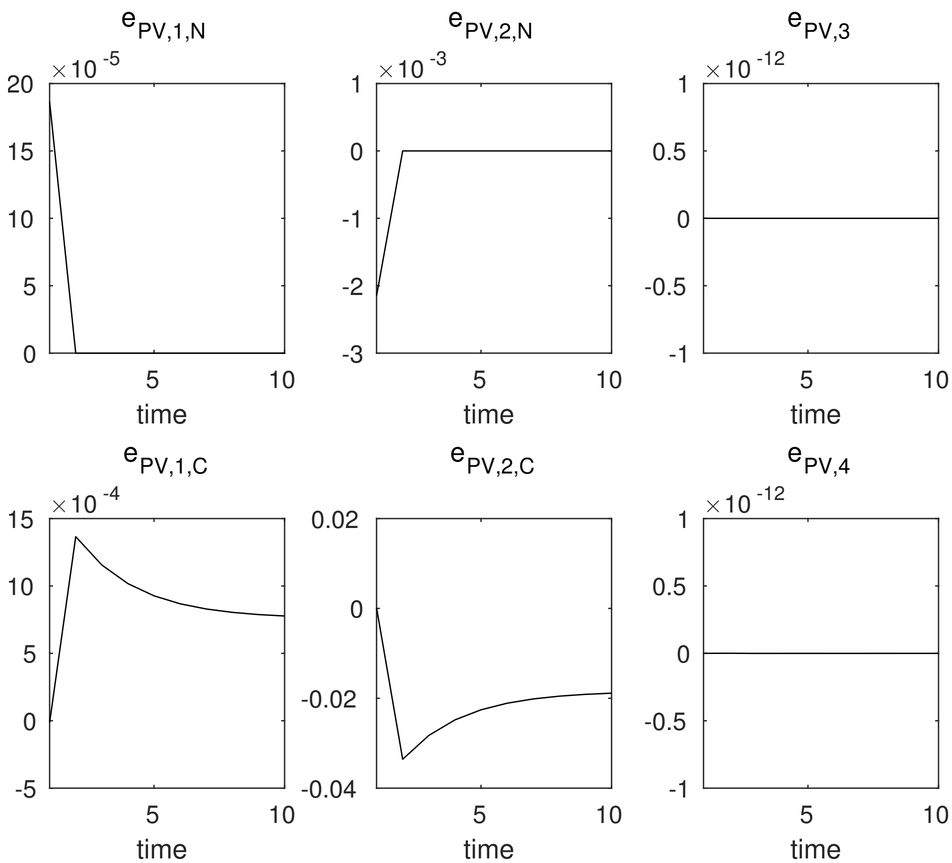
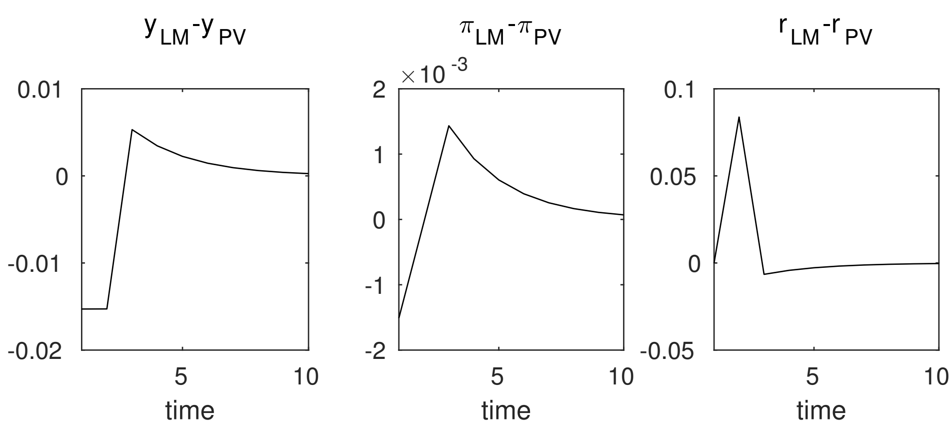


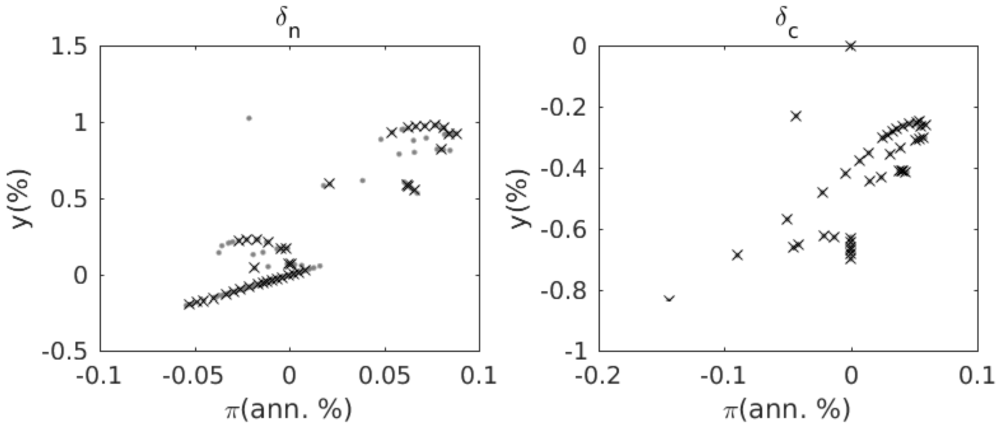
Figure 17 shows the difference of the impulse response of inflation, output and the policy rate under LM approach and PV approach.

Figure 17: Difference in dynamics between LM and PV approaches.



Note: The rate of inflation and the policy rate are expressed in annualized percent. The output gap is expressed in percent.

Figure 18: Quasi-ergodic distribution and EDS grid of (y, π)



G Additional results for the model with inflation bias

An interesting feature of the Ramsey outcome in the model with inflation bias is that inflation and the output gap eventually converge to zero. To better appreciate this feature, it is useful to contrast this convergent point with two time-invariant pairs of inflation and the output gap. The first pair is the one that prevails in the Markov perfect equilibrium. The second pair is the one that maximizes the time-one value. The analysis of Markov perfect equilibrium in the model with inflation bias has been studied by many. The value-maximizing pair of constant inflation and output is studied in Wolman (2001) in a sticky-price model with inflation bias.[16]

G.1 Markov perfect policy in the model with inflation bias

The problem of the discretionary central bank is to choose $\{y_t, \pi_t\}$, taking as given the future value, V_{t+1}, and inflation, π_{t+1}:

$$V_t = \max_{y_t, \pi_t} \; -\frac{1}{2}[\pi_t^2 + \lambda(y_t - y^*)^2] + \beta V_{t+1}$$

subject to the Phillips curve constraint. The Markov perfect equilibrium in the model with inflation bias is given by a set of time-invariant value and policy functions that solve this

[16] Wolman (2001) analyzes a fully nonlinear model and did not provide analytical results, whereas we study a semi-loglinear model that permits analytical results.

Bellman equation and is denoted by $\{V_{MP}, y_{MP}, \pi_{MP}, r_{MP}\}$. They are given by

$$
\begin{aligned}
y_{MP} &= \frac{(1-\beta)\lambda^2}{\kappa^2\lambda + (1-\beta)\lambda^2} y^* \\
\pi_{MP} &= \frac{\lambda\kappa}{\kappa^2 + (1-\beta)\lambda} y^* \\
V_{MP} &= \frac{1}{1-\beta} u(y_{MP}, \pi_{MP})
\end{aligned}
$$

<u>Proof</u>: Let ϕ be the Lagrange multiplier on the Phillips curve constraint. Then,

$$
\begin{aligned}
\frac{\partial V}{\partial y} &: 0 = -\lambda(y - y^*) - \kappa\phi \\
\frac{\partial V}{\partial y} &: 0 = -\pi + \phi.
\end{aligned}
$$

These equations imply

$$y = y^* - \frac{\kappa}{\lambda}\pi$$

Putting this into the Phillips curve,

$$\pi = \kappa(y^* - \frac{\kappa}{\lambda}\pi) + \beta\pi'.$$

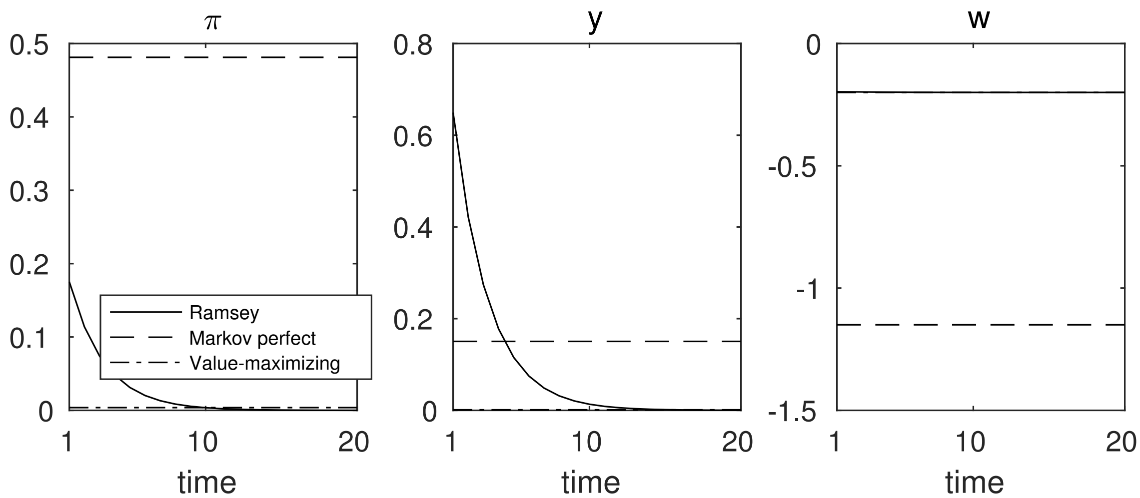
In equilibrium, $\pi = \pi'$, which we can call π_{MP}. Thus,

$$
\begin{aligned}
& \pi_{MP} = \kappa(y^* - \frac{\kappa}{\lambda}\pi_{MP}) + \beta\pi_{MP} \\
\iff \quad & (1 + \frac{\kappa^2}{\lambda} - \beta)\pi_{MP} = \kappa y^* \\
\iff \quad & \frac{\kappa^2 + (1-\beta)\lambda}{\lambda}\pi_{MP} = \kappa y^* \\
\iff \quad & \pi_{MP} = \frac{\lambda\kappa}{\kappa^2 + (1-\beta)\lambda} y^*.
\end{aligned}
$$

With this π_{MP},

$$
\begin{aligned}
y_{MP} &= y^* - \frac{\kappa}{\lambda}\pi_{MP} \\
&= y^* - \frac{\kappa}{\lambda}\frac{\lambda\kappa}{\kappa^2 + (1-\beta)\lambda} y^* \\
&= y^* - \frac{\lambda\kappa^2}{\kappa^2\lambda + (1-\beta)\lambda^2} y^* \\
&= (1 - \frac{\lambda\kappa^2}{\kappa^2\lambda + (1-\beta)\lambda^2}) y^* \\
&= \frac{(1-\beta)\lambda^2}{\kappa^2\lambda + (1-\beta)\lambda^2} y^*
\end{aligned}
$$

Figure 19: Dynamics
—Model with Inflation Bias—



Note: The rate of inflation is expressed in annualized percent. The output gap is expressed in percent.

The pair of output and inflation consistent with the Markov perfect equilibrium is shown by the dashed lines in Figure 19. As is well known in the literature, the discretionary central bank that takes the expected inflation as given will try to increase output by raising inflation today. A higher inflation today in turn worsens the inflation-output trade-off for the central bank in the previous period. In equilibrium, the economy ends up with positive inflation and output that is positive, but below y^*.

G.2 A value-maximizing pair of constant inflation and output gap

The value-maximizing pair of constant inflation and output gap, denoted by (π_{VM}, y_{VM}), is the pair of constant inflation and output gap that maximize the time-one value. That is,

$$(\pi_{VM}, y_{VM}) := \text{argmax}_{y,\pi} \;\; V_1$$

where the optimization is subject to

$$\pi = \kappa y + \beta \pi.$$

It is straightforward to show that

$$\pi_{VM} = \frac{\kappa\lambda(1-\beta)}{\kappa + \lambda(1-\beta)^2} y^*,$$
$$y_{VM} = y^* - \kappa \frac{\pi_{VM}}{\lambda(1-\beta)},$$
$$V_{VM} = u(\pi_{VM}, y_{VM})$$

Proof: Let ϕ be the Phillips curve constraint. Then,

$$\frac{\partial V}{\partial y} : 0 = -\lambda(y - y^*) - \kappa\phi$$
$$\frac{\partial V}{\partial y} : 0 = -\pi + \phi(1 - \beta)$$

The second equation means

$$\phi = \frac{\pi}{1-\beta}$$

$$\begin{aligned} & 0 = -\lambda(y - y^*) - \kappa\phi \\ \iff \quad & 0 = -\lambda(y - y^*) - \kappa\frac{\pi}{1-\beta} \\ \iff \quad & \lambda(y - y^*) = -\kappa\frac{\pi}{1-\beta} \\ \iff \quad & y = y^* - \kappa\frac{\pi}{\lambda(1-\beta)} \end{aligned}$$

$$\begin{aligned} & (1-\beta)\pi = \kappa y \\ \iff \quad & (1-\beta)\pi = \kappa = \kappa y^* - \kappa^2\frac{\pi}{\lambda(1-\beta)} \\ \iff \quad & [(1-\beta) + \frac{\kappa}{\lambda(1-\beta)}]\pi = \kappa y^* \\ \iff \quad & \frac{\kappa + \lambda(1-\beta)^2}{\lambda(1-\beta)}\pi = \kappa y^* \\ \iff \quad & \pi = \frac{\kappa\lambda(1-\beta)}{\kappa + \lambda(1-\beta)^2}y^* \end{aligned}$$

The value-maximizing pair of output and inflation is shown by the dash-dotted lines in Figure 19. Because of the presence of $y^* > 0$, the value maximizing pair features positive inflation and a positive output gap, as in the Markov perfect equilibrium. However, the magnitudes are much smaller than under the Markov perfect equilibrium.

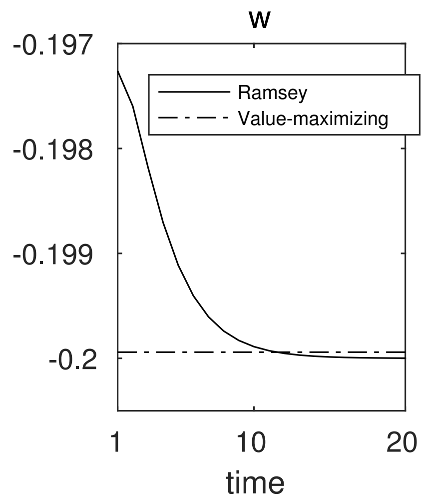
As shown in Figure 20, the time-one value associated with the value-maximizing pair is by construction lower than the time-one value under the Ramsey equilibrium. However, once inflation and output converge under the Ramsey equilibrium, the Ramsey value is lower than the value associated with the value-maximizing pair.

H Markov perfect policy in the model with stabilization bias

In the discussion of the optimal commitment policy for the model with stabilization bias, we contrasted the allocations under the optimal commitment policy to those under the Markov perfect policy to describe the benefit of commitment. In this section, we formulate the problem of the discretionary central bank and solve for the Markov perfect equilibrium.

For each $s_t \in S$, the problem of the discretionary central bank is to choose $\{y_t, \pi_t\}$, taking as given the future value function, V_{t+1}, and the future policy function for inflation, π_{t+1}.

Figure 20: Dynamics
—Model with Inflation Bias—



Note: The rate of inflation is expressed in annualized percent. The output gap is expressed in percent.

That is,

$$V_t(s_t) = \max_{y_t, \pi_t} \quad -\frac{1}{2}[\pi_t^2 + \lambda y_t^2] + \beta E_t V_{t+1}(s_{t+1}) \tag{68}$$

subject to the Phillips curve constraint. The Markov perfect equilibrium in the model with stabilization bias is given by a set of time-invariant value and policy functions that solves this Bellman equation and is denoted by $\{V_{MP}(\cdot),\, y_{MP}(\cdot),\, \pi_{MP}(\cdot),\, r_{MP}(\cdot)\}$. For the simple shock case considered in the main text, the solution can be found analytically. For the normal state, we have

$$y_{MP}(e_n) = 0$$
$$\pi_{MP}(e_n) = 0$$
$$V_{MP}(e_n) = 0$$

because the normal state is an absorbing state. For the high state when the cost-push shock hits the economy, we have

$$y_{MP}(e_h) = -\frac{\kappa e_h}{\lambda + \kappa^2}$$
$$\pi_{MP}(e_h) = \frac{e_h}{1 + \frac{\kappa^2}{\lambda}}$$
$$V_{MP}(e_h) = u(y_{MP}(e_h), \pi_{MP}(e_h))$$

I Markov perfect policy in the model with the ELB

In the discussion of the optimal commitment policy for the model with the ELB, we contrasted the allocations under the optimal commitment policy to those under the Markov perfect policy to describe the benefit of commitment. In this section, we formulate the problem of the discretionary central bank and solve for the Markov perfect equilibrium.

The problem of the discretionary central bank is to choose $\{y_t, \pi_t\}$, taking as given the future value (V_{t+1}) and inflation π_{t+1}:

$$V_t(s_t) = \max_{y_t, \pi_t} \; -\frac{1}{2}[\pi_t^2 + \lambda y_t^2] + \beta E_t V_{t+1}(s_{t+1})$$

subject to the Euler equation and Phillips curve constraints. The Markov perfect equilibrium in the model with the ELB is given by a set of time-invariant value and policy functions that solves this Bellman equation and is denoted by $\{V_{MP}(\cdot),\, y_{MP}(\cdot),\, \pi_{MP}(\cdot),\, r_{MP}(\cdot)\}$.

For the two-state shock case considered in the main text, the Markov Perfect equilibrium can be characterized analytically. For the normal state, we have

$$\begin{aligned} y_{MP}(\delta_n) &= 0 \\ \pi_{MP}(\delta_n) &= 0 \\ r_{MP}(\delta_n) &= r^* \\ V_{MP}(\delta_n) &= 0 \end{aligned}$$

because of the absorbing state assumption. For the crisis state, we have

$$\begin{aligned} y_{MP}(\delta_c) &= \frac{r^* + s_t}{\sigma} \\ \pi_{MP}(\delta_c) &= \kappa \frac{r^* + s_t}{\sigma} \\ r_{MP}(\delta_c) &= 0 \\ V_{MP}(\delta_c) &= u(y_{MP}(\delta_c), \pi_{MP}(\delta_c)). \end{aligned}$$