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Supplementary file for ”Sup-inf/inf-sup problem on choice of a probability measure by FBSDE approach”

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***Abstract*—This paper presents a problem on model uncertainties in stochastic control, in which an agent assumes a best case scenario on one risk and at the same time a worst case scenario on another risk. Particularly, the agent maximizes its view on a Brownian motion, simultaneously minimizing its view on another Brownian motion in choice of a probability measure. This selection method of a probability measure generalizes an approach to model uncertainties in which one considers the worst case scenarios for the views on Brownian motions, such as in the robust control. Specifically, we newly formulate and solve this problem based on a backward stochastic differential equation (BSDE) approach as a sup-inf (resp., inf-sup) optimal control problem on choice of a probability measure with the control domains dependent on stochastic processes. Concretely, we show that under certain conditions, the sup-inf and inf-sup problems are equivalent and these are solved by finding a solution of a BSDE with a stochastic Lipschitz driver. Then, we investigate two cases in which the optimal probability measure is explicitly obtained. The expression of the optimal probability measure includes signs of the diffusion terms of the value process, which are hard to determine in general. In these cases, we show two methods of determining the signs: the first one is by comparison theorems, and the second one is to predetermine the signs a priori and confirm them afterwards by explicitly solving the corresponding equations.**

***Index Terms*—Backward stochastic differential equations, Stochastic control, Application to finance.**

I. INTRODUCTION

In this paper, we present a sup-inf (resp., inf-sup) problem with respect to choice of a probability measure, arising from a motivation for asset pricing with market sentiments in finance. Specifically, in the choice of a probability measure, the agent maximizes its expectation on utility with respect to its view on a Brownian motion, simultaneously minimizing the expectation with respect to its view on another Brownian motion.

In a model where risks are expressed by Brownian motions (a multi-dimensional Brownian motion), there may exist a fundamental uncertainty, which is an uncertainty about a risk of each Brownian motion and is represented by a stochastic process λ_j for the j-th risk (Brownian motion B_j). When there is a fundamental uncertainty about the j-th risk, we only know the true j-th risk is one of $\{B_j^\lambda; \lambda_j \in \Lambda_j\}$ with $B_{j,t}^\lambda := B_{j,t} - \int_0^t \lambda_{j,s} ds$, $0 \leq t < \infty$ given some set Λ_j. In contrast, if $\lambda_j \equiv 0$ (i.e. $B_j^\lambda = B_j$), there is no fundamental uncertainty about the j-th risk. Here, B_j^λ is a Brownian motion under a new probability measure induced by $(\lambda_j)_j$ through *a change of a probability measure*. (See the details for the following sections.) Further, in order to model the best and worst case scenarios towards fundamental uncertainties, we consider a sup-inf (resp., inf-sup) problem on a single agent's objective (e.g. utility), who optimally chooses its probability measure through minimizing the objective with respect to the fundamental uncertainty (λ_1) for the first Brownian motion (B_1) while maximizing the objective with respect to the fundamental uncertainty (λ_2) for the second Brownian motion (B_2). Here, the agent assumes the worst case scenario on a risk of one Brownian motion say B_1 and the best case scenario on a risk of a different Brownian motion B_2, while there are no fundamental uncertainties about the risks associated with the other Brownian motions.

This selection method of a probability measure generalizes an approach to model uncertainties in which one selects the worst case scenarios for uncertainties of stochastic systems, such as in the robust control. Petersen et al. [19] introduce relative entropy constraints to stochastic uncertain systems, where the worst case scenario in the choice of a probability measure is taken into account. Hansen and Sargent [12] consider worst scenarios of an agent towards risks. Chen and Epstein [4] also consider a stochastic control problem, in which an agent is uncertain about risks and takes the worst case on the Brownian motions in its choice of a probability measure.

Particularly, we take a forward-backward stochastic differential equation (FBSDE) approach to solve the sup-inf (resp., inf-sup) problem. In the problem, the control domains are unbounded and dependent on stochastic processes, and its corresponding backward stochastic differential equation (BSDE) has a stochastic Lipschitz driver. We emphasize that our study is to express a single agent's pessimistic and optimistic views on risks as a sup-inf (resp.,inf-sup) control problem in contrast to a two-person zero sum stochastic differential game in which two persons maximize their objective functions of opposite signs. Moreover, the work is different from two-person zero sum games in that we do not necessarily require the objective values in the sup-inf and inf-sup problems coincide. Whether we should consider the sup-inf or the inf-sup problem depends on which side of the views (pessimistic or optimistic) we put more emphasis on. In other words, if we aim to put more emphasis on viewing risks pessimistically (optimistically), we

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consider a sup-inf (inf-sup) problem since the value for the sup-inf problem is less than or equal to that of the inf-sup problem as long as the set of admissible controls is expressed as a direct product of two control sets. In this study, we provide some conditions in which those two problems are solved at the same time and the objective values coincide.

For related literature on two-person zero-sum stochastic differential games, Hamadene and Lepeltier [11], Hamadene [10], Buckdahn and Li [2] and Buckdahn et al. [3] investigate the games with a recursive or non-recursive driver with bounded control domains, for instance. Bayraktar and Yao [1] and Cosso [8] deal with the cases of unbounded control domains, but the driver of the corresponding BSDE is not a stochastic Lipschitz type. For the drivers with a linear quadratic type, Hamadene [9] and Yu [26] consider linear quadratic drivers for zero-sum stochastic differential games and Wang and Yu [24] and Wang et al. [25] for non-zero-sum stochastic differential games, for example. Possamaï et al. [22] and Pham and Zhang [21] work on path-dependent stochastic differential games in weak formulation.

The motivation of our study is as follows. In financial markets, not only worst scenarios but also best scenarios on some market risks are reflected in asset prices. In particular, the term structure of interest rates in the recent global low interest environments, especially when the market is controlled by authorities such as central banks and governments, are driven by those optimistic and pessimistic sentiments of the market. In order to express effects of those sentiments in asset prices, we model the market participants' best and worst case scenarios towards uncertainties by a sup-inf (resp., inf-sup) problem on an agent's utility by choice of a probability measure.

For example, B_1 and B_2 can be taken as Brownian motions associated with global and domestic specific risks, respectively. In such a case, the market, which is deemed to be the agent, has a pessimistic sentiment about taking global risks (e.g. news on fiscal conditions of foreign countries), while it does an optimistic one on domestic specific risks (e.g. domestic business conditions) and hence is willing to take those risks aggressively. The effectiveness of an interest rate model with such sentiment factors in a period including the global financial crisis is shown empirically by a text mining approach (see Nishimura et al. [18], for instance). This paper provides a theoretical foundation for the asset pricing model with market sentiments.

This paper is organized as follows. Section II introduces a sup-inf (resp., inf-sup) problem with respect to fundamental uncertainties. Section III presents solution methods for the sup-inf (resp., inf-sup) problem. Finally, Section IV concludes. Appendices show the proofs of propositions in the main text.

II. Sup-inf (resp., inf-sup) problem with respect to fundamental uncertainties

In this section, we consider a sup-inf (resp., inf-sup) problem with respect to uncertainties on Brownian motions that express worst and best scenarios of a single agent towards risks. Hereafter, we call the uncertainties on Brownian motions as fundamental uncertainties. Particularly, we obtain a probability measure of the agent, who assumes the worst case on a certain risk, while the best case on another risk, by solving the sup-inf (resp., inf-sup) problem through a BSDE approach.

A. Motivating example

We would like to consider an objective function (e.g. expected profit and loss) determined by a state variable X_0 for a single agent who has different views on risks driving the state variable. In particular, as a simple example, we consider a case in which the agent is cautious about global risks expressed by a Brownian motion B_1 and aggressive about domestic specific risks represented by another Brownian motion B_2, respectively. Particularly, let $Y_0^{\lambda_1,\lambda_2}$ be an objective value at time 0 dependent on the state variable X_0 under the probability measure of the agent P^{λ_1,λ_2}, who has an uncertainty λ_1 on B_1 and an uncertainty λ_2 on B_2 taking values in Λ_1 and Λ_2, respectively. Here, $\Lambda_j,\ j=1,2$, are sets of progressively measurable processes satisfying

$$-\bar{\lambda}_j \leq \lambda_{j,t} \leq \bar{\lambda}_j, \tag{1}$$

where $\bar{\lambda}_j,\ j=1,2$ are positive constants.

$$Y_0^{\lambda_1,\lambda_2} = E\bigg[\mathcal{Z}_T(\lambda)\left(\frac{X_{0,T}^2}{2} + \int_0^T \frac{1}{2}X_{0,s}^2 ds\right)\bigg], \tag{2}$$

with X_0, X_1, and X_2 satisfying

$$\frac{dX_{0,t}}{X_{0,t}} = \mu_{x_0}dt + \sigma_{x_0,1}dB_{1,t} + \sigma_{x_0,2}dB_{2,t},\ x_0>0, \tag{3}$$

where $\sigma_{x_0,1}, \sigma_{x_0,2} > 0$, μ_{x_0} is a constant, and

$$\mathcal{Z}_t(\lambda) := \exp\left\{\sum_{j=1}^{2}\int_0^t \lambda_{j,s}dB_{j,s} - \sum_{j=1}^{2}\frac{1}{2}\int_0^t \lambda_{j,s}^2 ds\right\} \tag{4}$$

is a martingale that defines a probability measure P^{λ_1,λ_2} by

$$P^{\lambda_1,\lambda_2}(A) := E[\mathcal{Z}_T(\lambda)1_A];\ A\in\mathcal{F}_T. \tag{5}$$

By Girsanov's theorem, (2) and (3) are rewritten as

$$Y_0^{\lambda_1,\lambda_2} = E^{P^{\lambda_1,\lambda_2}}\left[\frac{X_{0,T}^2}{2} + \int_0^T \frac{1}{2}X_{0,s}^2 ds\right], \tag{6}$$

$$\begin{aligned}\frac{dX_{0,t}}{X_{0,t}} &= (\mu_{x_0} + \sigma_{x_0,1}\lambda_{1,t} + \sigma_{x_0,2}\lambda_{2,t})dt \\ &\quad + \sigma_{x_0,1}dB_{1,t}^{\lambda_1,\lambda_2} + \sigma_{x_0,2}dB_{2,t}^{\lambda_1,\lambda_2},\ x_0>0.\end{aligned} \tag{7}$$

Thus, we observe that λ_1 and λ_2 appear in the drift term of X_0 under P^{λ_1,λ_2}. Since $\sigma_{x_0,1}, \sigma_{x_0,2} > 0$, an increase in λ_1 and λ_2 indicates an increase in X_0 and $Y_0^{\lambda_1,\lambda_2}$. We aim to express the agent's pessimistic view on risks B_1 and optimistic view on risks B_2 by minimization and maximization of the objective value on λ_1 and λ_2, respectively, namely by solving the following sup-inf and inf-sup problems,

$$\sup_{\lambda_2\in\Lambda_2}\inf_{\lambda_1\in\Lambda_1} Y_0^{\lambda_1,\lambda_2},\ \inf_{\lambda_1\in\Lambda_1}\sup_{\lambda_2\in\Lambda_2} Y_0^{\lambda_1,\lambda_2}. \tag{8}$$

These problems are solved in Example 2 in Section III-A. In this particular example, the sup-inf and inf-sup problems in (8) are solved as $\lambda_1^* = -\bar{\lambda}_1$, $\lambda_2^* = \bar{\lambda}_2$.

In the following sections, we formulate an agent's pessimistic and optimistic views about the Brownian motions by a sup-inf (resp. inf-sup) problem in a general setting, in which $\bar{\lambda}_1$ and $\bar{\lambda}_2$, the bounds of λ_1 and λ_2, are stochastic processes.

Remark 1. *We may consider the problem formulation in another way which corresponds to strong formulation in two-person zero-sum stochastic differential games as follows. For a fixed probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{0\le t\le T}, P)$ equipped with a two-dimensional standard Brownian motion $B = (B_1, B_2)$, where $\{\mathcal{F}\}_{0\le t\le T}$ is a natural filtration generated by B, we define*

$$\begin{aligned}\frac{dX_{0,t}}{X_{0,t}} &= (\mu_{x_0} + \sigma_{x_0,1}\lambda_{1,t} + \sigma_{x_0,2}\lambda_{2,t})dt\\ &\quad + \sigma_{x_0,1}dB_{1,t} + \sigma_{x_0,2}dB_{2,t},\ x_0 > 0,\end{aligned}$$

$$Y_0^{\lambda_1,\lambda_2} = E\left[\frac{X_{0,T}^2}{2} + \int_0^T \frac{1}{2}X_{0,s}^2 ds\right].$$

Then, for $\lambda_j \in \Lambda_j$ $(j = 1, 2)$ as in (1), we consider the sup-inf and inf-sup problems (8). These problems express the agent's views on the expected return of X_0, the drift part of the above SDE, in which the agent is conservative about λ_1 and aggressive about λ_2. However, our formulation considers the sup-inf and inf-sup with respect to P^{λ_1,λ_2} determining the distribution of X_0, which corresponds to weak formulation of two-person zero-sum stochastic differential games.

B. Problem formulation

Firstly, we suppose that a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, P)$ and a d-dimensional Brownian motion $B = (B_1, \dots, B_d)$ $(d \ge 2)$ are given, where $\{\mathcal{F}_t\}_{0\le t\le T}$ is the augmentation of the natural filtration generated by B, and we call P the physical measure, hereafter. Next, for a $\mathcal{R}^2$-valued $\{\mathcal{F}_t\}$-progressively measurable process $\lambda = (\lambda_1, \lambda_2)$, satisfying that $\mathcal{Z}_t(\lambda)$ set by

$$\mathcal{Z}_t(\lambda) := \exp\left\{\sum_{j=1}^{2}\int_0^t \lambda_{j,s}dB_{j,s} - \sum_{j=1}^{2}\frac{1}{2}\int_0^t \lambda_{j,s}^2 ds\right\} \quad (9)$$

is a martingale, we define a probability measure P^{λ_1,λ_2} by

$$P^{\lambda_1,\lambda_2}(A) := E[\mathcal{Z}_T(\lambda)1_A];\ A \in \mathcal{F}_T. \quad (10)$$

Here, λ_1 and λ_2 stand for uncertainties about the risks associated with Brownian motions B_1 and B_2, respectively.

Let $\mathcal{R}^l$-valued stochastic process X be a state variable process satisfying a stochastic differential equation (SDE)

$$dX_t = \mu_x(X_t)dt + \sum_{j=1}^{d}\sigma_{x,j}(X_t)dB_{j,t}, \quad (11)$$

where $\mu_x, \sigma_{x,j} : \mathcal{R}^l \to \mathcal{R}^l,\ j = 1, 2, \dots, d,\ l \ge d$, $rank(\sigma_{x,1}\sigma_{x,2}\dots\sigma_{x,d}) = d$. Hereafter, we assume that X is exogenously given and SDE (11) has a unique strong solution.

Then, the agent supposes the worst (best) case on Brownian motion $B_1 (B_2)$ and implements optimization with respect to λ_j $(j = 1, 2)$, that is, it minimizes (maximizes) its utility with respect to $\lambda_1 (\lambda_2)$. In contrast, the agent has no uncertainties for risks represented by Brownian motions B_j, $j = 3, \cdots, d$, so that we have $\lambda_j \equiv 0$. Then, $B_{1,t}^{\lambda_1,\lambda_2} = B_{1,t} - \int_0^t \lambda_{1,s}ds$, $B_{2,t}^{\lambda_1,\lambda_2} = B_{2,t} - \int_0^t \lambda_{2,s}ds$ and $B_j^{\lambda_1,\lambda_2} = B_j$ for $j = 3, \cdots, d$ are Brownian motions under the probability measure P^{λ_1,λ_2} generated by a martingale $\mathcal{Z}(\lambda)$ with $\lambda = (\lambda_1, \lambda_2, 0, \cdots, 0)$.

Let us define the agent's stochastic differential utility (SDU, continuous-time version of recursive utility. For an introduction of SDU, see Section 1.3 in Ma and Yong [15] for instance) Y^{λ_1,λ_2} as follows: with an aggregator $g : [0, T] \times \mathcal{C}([0, T] \to \mathcal{R}^d) \times \mathcal{R}^l \times \mathcal{R} \to \mathcal{R}$,

$$\begin{aligned}Y_t^{\lambda_1,\lambda_2} &= E\left[\frac{\mathcal{Z}_T(\lambda)}{\mathcal{Z}_t(\lambda)}\left(\xi + \int_t^T g(s, B, X_s, Y_s^{\lambda_1,\lambda_2})ds\right)\middle|\mathcal{F}_t\right],\\ &= E^{P^{\lambda_1,\lambda_2}}\left[\xi + \int_t^T g(s, B, X_s, Y_s^{\lambda_1,\lambda_2})ds\middle|\mathcal{F}_t\right], \quad (12)\end{aligned}$$

where ξ is a bounded $\mathcal{F}_T$-measurable random variable (for example, a standard utility $g(t, \omega, x, y) = u(x) - \beta y$ is well known) and X satisfies the following SDE under probability measure P^{λ_1,λ_2}

$$dX_t = \left(\mu_x(X_t) + \sum_{j=1}^{d}\sigma_{x,j}(X_t)\lambda_{j,t}\right)dt + \sum_{j=1}^{d}\sigma_{x,j}(X_t)dB_{j,t}^{\lambda_1,\lambda_2}. \quad (13)$$

Remark 2. *For example, a well-known standard utility*

$$Y_t^{\lambda_1,\lambda_2} = E^{P^{\lambda_1,\lambda_2}}\left[\xi + \int_t^T e^{-\beta(s-t)}u(X_s)ds\middle|\mathcal{F}_t\right]$$

is expressed as a special case of (12), where $u : \mathcal{R}^d \to \mathcal{R}$ and $\beta > 0$, that is,

$$Y_t^{\lambda_1,\lambda_2} = E^{P^{\lambda_1,\lambda_2}}\left[\xi + \int_t^T (u(X_s) - \beta Y_s^{\lambda_1,\lambda_2})ds\middle|\mathcal{F}_t\right].$$

Next, let us set $J(\lambda_1, \lambda_2)$ as

$$J(\lambda_1, \lambda_2) = Y_0^{\lambda_1,\lambda_2},\ (\lambda_1, \lambda_2) \in \Lambda, \quad (14)$$

where $\Lambda = \Lambda_1 \times \Lambda_2$,

$$\Lambda_j = \{\lambda_j; |\lambda_{j,t}| \le |\bar{\lambda}_{j,t}|,\ 0 \le t \le T\},\ j = 1, 2, \quad (15)$$

and $\bar{\lambda}_j,\ j = 1, 2$ are $\mathcal{R}$-valued $\{\mathcal{F}_t\}$-progressively measurable processes satisfying a weak version of Novikov's condition (e.g. Corollary 3.5.14 in Karatzas and Shreve [14]); there exists a partition of $[0, T]$, $0 = t_0 < t_1 < \cdots < t_N = T$, such that

$$E\left[\exp\left(\sum_{j=1}^{2}\frac{1}{2}\int_{t_{n-1}}^{t_n}\bar{\lambda}_{j,s}^2 ds\right)\right] < \infty,$$
$$\text{for all } 1 \le n \le N. \quad (16)$$

Remark 3. *The weak Novikov's condition (16) guarantees that for all $\lambda = (\lambda_1, \lambda_2)$ with $|\lambda_{j,t}| \le |\bar{\lambda}_{j,t}|,\ 0 \le t \le T,\ j = 1, 2$, $\{\mathcal{Z}_t(\lambda)\}_{0\le t\le T}$ is a martingale. Moreover, by Girsanov's theorem, P^{λ_1,λ_2} in (10) is well-defined as a probability measure* *and B^{λ_1,λ_2} in (18) below is a d-dimensional Brownian motion under P^{λ_1,λ_2} for all $\lambda \in \Lambda = \Lambda_1 \times \Lambda_2$. Some general criteria that guarantee the exponential local martingale to be a martingale are known (for instance, see Chikvinidze [5], [6]). However, a set of (λ_1, λ_2) that satisfies such a criterion does not take a direct product form as $\Lambda_1 \times \Lambda_2$, thus we use the weak Novikov's condition (16) in this paper.*

Then, we consider the following sup-inf and inf-sup problems:

$$\sup_{\lambda_2\in\Lambda_2}\inf_{\lambda_1\in\Lambda_1} J(\lambda_1,\lambda_2),\ \inf_{\lambda_1\in\Lambda_1}\sup_{\lambda_2\in\Lambda_2} J(\lambda_1,\lambda_2) \qquad (17)$$

where the agent takes the worst case scenario on the uncertainty over B_1 and minimizes the SDU $J(\lambda_1, \lambda_2)$ with respect to λ_1, while it supposes the best case scenario on the uncertainty over B_2 and maximizes $J(\lambda_1, \lambda_2)$ with respect to λ_2.

We note that λ_1 and λ_2 represent deviation of the agent's probability measure P^{λ_1,λ_2} from the physical measure P. Specifically, the relation between $B_j,\ j = 1, 2$, under the physical measure P and $B_j^{\lambda_1,\lambda_2}$ under the probability measure P^{λ_1,λ_2} is $dB_{j,t} = dB_{j,t}^{\lambda_1,\lambda_2} + \lambda_{j,t}dt$. Taking the conditional expectations under P^{λ_1,λ_2} with respect to $\mathcal{F}_t$ in the both sides , we obtain $E^{P^{\lambda_1,\lambda_2}}[dB_{j,t}|\mathcal{F}_t] = \lambda_{j,t}dt$, which implies that under the probability measure P^{λ_1,λ_2}, $dB_{j,t}$ is expected as $\lambda_{j,t}dt$.

Thus, the sup-inf (resp., inf-sup) problem in (17) is considered to be an optimization to determine the views on Brownian motions B_1 and B_2 so that the utility is minimized with respect to λ_1 for given λ_2, and maximized with respect to λ_2 for given λ_1. In other words, the agent takes the worst case scenario for the uncertainty λ_1 on B_1, and at the same time, the best case scenario for the uncertainty λ_2 on B_2.

C. BSDEs for the model with fundamental uncertainties

Next, we introduce BSDEs associated with the sup-inf (resp., inf-sup) problem. Particularly, we provide existence and uniqueness of solutions of the BSDEs under certain conditions in Propositions 1 and 2.

First, for a $\mathcal{R}^2$-valued $\{\mathcal{F}_t\}$-progressively measurable process (λ_1, λ_2) satisfying that $\mathcal{Z}^{\lambda_1,\lambda_2}$ is a P-martingale, by Girsanov's theorem, we can define a d-dimensional Brownian motion under P^{λ_1,λ_2}, $B^{\lambda_1,\lambda_2} = (B_1^{\lambda_1,\lambda_2}, \dots, B_d^{\lambda_1,\lambda_2})$, by

$$\begin{aligned} B_{j,t}^{\lambda_1,\lambda_2} &= B_{j,t} - \int_0^t \lambda_{j,s}ds,\ j = 1, 2,\\ B_{j,t}^{\lambda_1,\lambda_2} &= B_{j,t}\ (3 \le j \le d). \end{aligned} \qquad (18)$$

Then, under a certain condition, Y^{λ_1,λ_2} in (12) is characterized as a unique solution of the following BSDE as a consequence of the martingale representation theorem and the Girsanov transformation:

$$\begin{aligned} dY_t^{\lambda_1,\lambda_2} &= -g(t, B, X_t, Y_t^{\lambda_1,\lambda_2})dt + \sum_{j=1}^d Z_{j,t}^{\lambda_1,\lambda_2} dB_{j,t}^{\lambda_1,\lambda_2}\\ &= -\left(g(t, B, X_t, Y_t^{\lambda_1,\lambda_2}) + \lambda_{1,t}Z_{1,t}^{\lambda_1,\lambda_2} + \lambda_{2,t}Z_{2,t}^{\lambda_1,\lambda_2}\right)dt\\ &\quad + \sum_{j=1}^d Z_{j,t}^{\lambda_1,\lambda_2} dB_{j,t},\ Y_T^{\lambda_1,\lambda_2} = \xi. \end{aligned} \qquad (19)$$

In detail, by taking a conditional expectation under P^{λ_1,λ_2} in both sides of the first equality in the integral form of (19), we obtain (12) if the Itô integral is a martingale.

In the following, we show two propositions on existence and uniqueness of a BSDE, which will be used in Theorem 1 in Section II-D. The next proposition presents conditions under which BSDE (19) with stochastic Lipschitz coefficients λ_1 and λ_2 has a unique solution.

Proposition 1. *Suppose that SDE (11) has a unique strong solution and $g : [0, T] \times \mathcal{C}([0, T] \to \mathcal{R}^d) \times \mathcal{R}^l \times \mathcal{R} \to \mathcal{R}$ satisfies the following conditions: (i) $g(t, \omega, x, 0)$ is bounded. (ii) There exists a constant $L > 0$ such that*

$$\begin{gathered} |g(t, \omega, x, y) - g(t, \omega, x, y')| \le L|y - y'|,\\ \forall y, y' \in \mathcal{R}, x \in \mathcal{R}^l, \omega \in \mathcal{C}([0, T] \to \mathcal{R}^d), t \in [0, T]. \end{gathered}$$

Suppose also that the exponential local martingale $\mathcal{Z}(\lambda)$ in (9) is a martingale and

$$E\left[\sup_{0\le s\le T}|\lambda_s|^4\right] < \infty. \qquad (20)$$

Then, BSDE

$$\begin{aligned} dY_t^{\lambda_1,\lambda_2} &= -\Big(g(t, B, X_t, Y_t^{\lambda_1,\lambda_2}) + \lambda_{1,t}Z_{1,t}^{\lambda_1,\lambda_2}\\ &\quad + \lambda_{2,t}Z_{2,t}^{\lambda_1,\lambda_2}\Big)dt + \sum_{j=1}^d Z_{j,t}^{\lambda_1,\lambda_2}dB_{j,t},\ Y_T^{\lambda_1,\lambda_2} = \xi, \end{aligned} \qquad (21)$$

has a unique solution $(Y^{\lambda_1,\lambda_2}, Z^{\lambda_1,\lambda_2})$ such that $E\left[\int_0^T |Z_s^{\lambda_1,\lambda_2}|^2 ds\right] < \infty$ and Y^{λ_1,λ_2} is uniformly bounded with respect to $(t, \omega) \in [0, T] \times \mathcal{C}([0, T] \to \mathcal{R}^d)$.

Proof. See Appendix A.

In addition, the next proposition provides conditions under which existence and uniqueness of a solution for a BSDE, containing the absolute values of the diffusion terms $|Z_1^{\lambda_1^*,\lambda_2^*}|$ and $|Z_2^{\lambda_1^*,\lambda_2^*}|$ in the driver, holds.

Proposition 2. *Suppose that SDE (11) has a unique strong solution and $g : [0, T] \times \mathcal{C}([0, T] \to \mathcal{R}^d) \times \mathcal{R}^l \times \mathcal{R} \to \mathcal{R}$ satisfies the following conditions: (i) $g(t, \omega, x, 0)$ is bounded. (ii) There exists a constant $L > 0$ such that*

$$\begin{gathered} |g(t, \omega, x, y) - g(t, \omega, x, y')| \le L|y - y'|,\\ \forall y, y' \in \mathcal{R}, x \in \mathcal{R}^l, \omega \in \mathcal{C}([0, T] \to \mathcal{R}^d), t \in [0, T]. \end{gathered}$$

Also, suppose that a weak version of Novikov's condition (16) holds and

$$E\left[\sup_{0\le s\le T}|\bar{\lambda}_s|^4\right] < \infty. \qquad (22)$$

Then BSDE

$$\begin{aligned} dY_t^{\lambda_1^*,\lambda_2^*} &= -\Big(g(t, B, X_t, Y_t^{\lambda_1^*,\lambda_2^*}) - |\bar{\lambda}_{1,t}||Z_{1,t}^{\lambda_1^*,\lambda_2^*}|\\ &\quad + |\bar{\lambda}_{2,t}||Z_{2,t}^{\lambda_1^*,\lambda_2^*}|)\Big)dt + \sum_{j=1}^d Z_{j,t}^{\lambda_1^*,\lambda_2^*}dB_{j,t},\ Y_T^{\lambda_1^*,\lambda_2^*} = \xi, \end{aligned} \qquad (23)$$

has a unique solution $(Y^{\lambda_1^*,\lambda_2^*}, Z^{\lambda_1^*,\lambda_2^*})$ *such that* $E\left[\int_0^T |Z_s^{\lambda_1^*,\lambda_2^*}|^2 ds\right] < \infty$ *and* $Y^{\lambda_1^*,\lambda_2^*}$ *is uniformly bounded with respect to* $(t,\omega) \in [0,T] \times \mathcal{C}([0,T] \to \mathcal{R}^d)$.

Proof. See Appendix B.

Example 1. (Square-root process)

Suppose that $|\lambda_j| \le \bar{\lambda}_{j,t} = \tilde{\lambda}_{j,t}\sqrt{X_{j,t}},\ j = 1,2$ for some bounded deterministic functions $\tilde{\lambda}_{j,t} > 0$, and X_t is a $\mathcal{R}^2$-valued square root process following an SDE:

$$dX_{j,t} = (a_{j,t} - b_{j,t}X_{j,t})dt + \sigma_{x,j,t}\sqrt{X_{j,t}}dB_{j,t},\\ X_{j,0} = x_j > 0,\ j = 1,2,$$

where $a_{j,t}, b_{j,t}, \sigma_{x,j,t} : [0,T] \to \mathcal{R}$ are bounded functions with $a_{j,t}, b_{j,t} > 0,\ \sigma_{x,j,t} > c_0$ for some $c_0 > 0$, $0 \le t \le T,\ j = 1,2$, and B_1 and B_2 are independent. Then, a weak version of Novikov's condition (16) is satisfied and exponential local martingale (9) is a martingale, which follows from Theorem 3.2 in Shirakawa [23]. Moreover, the moment conditions (20) and (22) are satisfied by Eq. 5.3.17 in Problem 5.3.15 in Karatzas and Shreve [14]. Notice that existence and uniqueness of a strong solution of the SDE follow from Theorems 4.1.1, 4.2.3, and 4.2.4 in Ikeda and Watanabe [13] and Proposition 5.2.13 in Karatzas and Shreve [14].

D. *Solution of the sup-inf (resp., inf-sup) problem*

Then, we show that under certain conditions, the sup-inf and inf-sup problems (17) are equivalent and these are solved by finding a solution of BSDE (23), which is summarized in the following theorem.

Theorem 1. *Suppose that assumptions in Proposition 2 hold. Let* $(Y^{\lambda_1^*,\lambda_2^*}, Z^{\lambda_1^*,\lambda_2^*})$ *be a unique solution of BSDE (23) and*

$$\lambda_{j,t}^* = (-1)^j |\bar{\lambda}_{j,t}| sgn(Z_{j,t}^{\lambda_1^*,\lambda_2^*}),\ (j = 1,2). \tag{24}$$

Then, $(\lambda_1^*, \lambda_2^*)$ *attains the sup-inf and inf-sup in problems (17).*

Proof. In the following, we show that

$$Y_0^{\lambda_1^*,\lambda_2^*} = \sup_{\lambda_2 \in \Lambda_2} \inf_{\lambda_1 \in \Lambda_1} Y_0^{\lambda_1,\lambda_2} = \inf_{\lambda_1 \in \Lambda_1} \sup_{\lambda_2 \in \Lambda_2} Y_0^{\lambda_1,\lambda_2}.$$

To prove this, it suffices to show that $(\lambda_1^*, \lambda_2^*)$ is a saddle point of $J(\lambda_1, \lambda_2)$, meaning that

$$J(\lambda_1^*, \lambda_2) \le J(\lambda_1^*, \lambda_2^*) \le J(\lambda_1, \lambda_2^*), \forall \lambda_1 \in \Lambda_1, \lambda_2 \in \Lambda_2. \tag{25}$$

Next, we show the second inequality in (25),

$$J(\lambda_1^*, \lambda_2^*) - J(\lambda_1, \lambda_2^*) = Y_0^{\lambda_1^*,\lambda_2^*} - Y_0^{\lambda_1,\lambda_2^*} \le 0.$$

Note that BSDE (23) is rewritten as

$$\begin{aligned} &dY_t^{\lambda_1^*,\lambda_2^*} \\ &= -\left(g(t,B,X_t,Y_t^{\lambda_1^*,\lambda_2^*}) + \lambda_{1,t}^* Z_{1,t}^{\lambda_1^*,\lambda_2^*} + \lambda_{2,t}^* Z_{2,t}^{\lambda_1^*,\lambda_2^*}\right)dt \\ &\quad + \sum_{j=1}^d Z_{j,t}^{\lambda_1^*,\lambda_2^*} dB_{j,t}, Y_T^{\lambda_1^*,\lambda_2^*} = \xi. \end{aligned} \tag{26}$$

Here, uniqueness and existence of a solution of BSDE (26) are guaranteed by Proposition 2.

Also note that $Y_t^{\lambda_1,\lambda_2^*}$ is a unique solution of a BSDE

$$\begin{aligned} &dY_t^{\lambda_1,\lambda_2^*} \\ &= -\left(g(t,B,X_t,Y_t^{\lambda_1,\lambda_2^*}) + \lambda_{1,t} Z_{1,t}^{\lambda_1,\lambda_2^*} + \lambda_{2,t}^* Z_{2,t}^{\lambda_1,\lambda_2^*}\right)dt \\ &\quad + \sum_{j=1}^d Z_{j,t}^{\lambda_1,\lambda_2^*} dB_{j,t},\ Y_T^{\lambda_1,\lambda_2^*} = \xi, \end{aligned} \tag{27}$$

in which existence and uniqueness of a solution are guaranteed by Proposition 1 as in the following discussion; for any $(\lambda_1, \lambda_2^*) \in \Lambda$, $\mathcal{Z}^{\lambda_1,\lambda_2^*}$ is a P-martingale since a weak version of Novikov's condition holds for (λ_1, λ_2^*). Similarly, the condition (20) in Proposition 1 holds for any $(\lambda_1, \lambda_2^*) \in \Lambda$ due to the assumption (22) on the moment of $(\bar{\lambda}_{1,t}, \bar{\lambda}_{2,t})$.

Then, by (26) and (27), we have

$$\begin{aligned} &d(Y_t^{\lambda_1^*,\lambda_2^*} - Y_t^{\lambda_1,\lambda_2^*}) \\ &= -b_t(Y_t^{\lambda_1^*,\lambda_2^*} - Y_t^{\lambda_1,\lambda_2^*})dt - (\lambda_{1,t}^* - \lambda_{1,t})Z_{1,t}^{\lambda_1^*,\lambda_2^*}dt \\ &+ \sum_{j=1}^d (Z_{j,t}^{\lambda_1^*,\lambda_2^*} - Z_{j,t}^{\lambda_1,\lambda_2^*})dB_{j,t}^{\lambda_1,\lambda_2^*}, \end{aligned}$$

where $b_t = -\frac{g(t,B,X_t,Y_t^{\lambda_1^*,\lambda_2^*}) - g(t,B,X_t,Y_t^{\lambda_1,\lambda_2^*})}{Y_t^{\lambda_1^*,\lambda_2^*} - Y_t^{\lambda_1,\lambda_2^*}}$ $\times 1_{\{Y_t^{\lambda_1^*,\lambda_2^*} - Y_t^{\lambda_1,\lambda_2^*} \neq 0\}}$, and $B^{\lambda_1,\lambda_2^*} = (B_1^{\lambda_1,\lambda_2^*}, \ldots, B_d^{\lambda_1,\lambda_2^*})$ is a d-dimensional Brownian motion under $P^{\lambda_1,\lambda_2^*}$ defined as in (18).

Set $\bar{Y}_t = e^{\int_0^t b_u du}(Y_t^{\lambda_1^*,\lambda_2^*} - Y_t^{\lambda_1,\lambda_2^*})$ and $\bar{Z}_{j,t} = e^{\int_0^t b_u du}(Z_{j,t}^{\lambda_1^*,\lambda_2^*} - Z_{j,t}^{\lambda_1,\lambda_2^*}),\ j = 1,\ldots,d.$

Then, we have

$$d\bar{Y}_t = -(\lambda_{1,t}^* - \lambda_{1,t})Z_{1,t}^{\lambda_1^*,\lambda_2^*} e^{\int_0^t b_u du} dt + \sum_{j=1}^d \bar{Z}_{j,t} dB_{j,t}^{\lambda_1,\lambda_2^*},$$

and thus

$$\bar{Y}_0 = \int_0^T (\lambda_{1,s}^* - \lambda_{1,s})Z_1^{\lambda_1^*,\lambda_2^*} e^{\int_0^s b_u du} ds - \sum_{j=1}^d \int_0^T \bar{Z}_{j,s} dB_{j,s}^{\lambda_1,\lambda_2^*}. \tag{28}$$

Next, we note that $\left\{\sum_{j=1}^d \int_0^t \bar{Z}_{j,s} dB_{j,s}^{\lambda_1,\lambda_2^*}\right\}_{0 \le t \le T}$ is a $P^{\lambda_1,\lambda_2^*}$-martingale. This follows from the fact that $\bar{Y}_t$ is uniformly bounded and $\int_0^t (\lambda_{1,s}^* - \lambda_{1,s})Z_{1,s}^{\lambda_1^*,\lambda_2^*} e^{\int_0^s b_u du} ds$ is a negative decreasing process, which is due to the following inequality

$$\begin{aligned} \lambda_{1,s}^* Z_{1,s}^{\lambda_1^*,\lambda_2^*} &= -|\bar{\lambda}_{1,s}||Z_{1,s}^{\lambda_1^*,\lambda_2^*}| \le -|\lambda_{1,s}||Z_{1,s}^{\lambda_1^*,\lambda_2^*}| \\ &= -|\lambda_{1,s} Z_{1,s}^{\lambda_1^*,\lambda_2^*}| \le \lambda_{1,s} Z_{1,s}^{\lambda_1^*,\lambda_2^*}. \end{aligned}$$

Thus, we have $E^{\lambda_1,\lambda_2^*}\left[\sum_{j=1}^d \int_0^{t_2} \bar{Z}_{j,s} dB_{j,s}^{\lambda_1,\lambda_2^*} \middle| \mathcal{F}_{t_1}\right] = \sum_{j=1}^d \int_0^{t_1} \bar{Z}_{j,s} dB_{j,s}^{\lambda_1,\lambda_2^*}$.

Taking the expectation with respect to $P^{\lambda_1,\lambda_2^*}$ in both sides in (28), we have

$$\bar{Y}_0 = E^{\lambda_1,\lambda_2^*}\left[\int_0^T (\lambda_{1,s}^* - \lambda_{1,s})Z_{1,s}^{\lambda_1^*,\lambda_2^*} e^{\int_0^s b_u du} ds\right].$$

Hence, $\bar{Y}_0 \leq 0$ and

$$Y_0^{\lambda_1^*,\lambda_2^*} \leq Y_0^{\lambda_1,\lambda_2^*}.$$

The first inequality in (25) also follows in the same manner. Therefore, $(\lambda_1^*, \lambda_2^*)$ is a saddle point of $J(\lambda_1, \lambda_2)$. □

Remark 4. *In general, as in the strong formulation of two person zero-sum stochastic differential games, results of sup-inf and inf-sup problems are not equivalent (for example, see Example 9.1.1 in Zhang [27]). Compared with the strong formulation of stochastic differential games, in which the state variables include the controls in their SDEs under a common probability measure, in our case, the state variable X does not include the control processes λ_1 or λ_2 in its SDE (11) under P. As in the proof of Theorem 1, while X does not include λ_1 or λ_2 in its SDE (11), BSDE (19) for Y^{λ_1,λ_2} includes those in its driver, then, by comparison in the BSDE (19) that includes both λ_1 and λ_2 in the driver under P, we obtain λ_1^* and λ_2^* in (24) that solve the sup-inf and inf-sup problems in (17).*

III. Solution methods

As we observed in (24) of Theorem 1 in Section II-D, $(\lambda_1^*, \lambda_2^*)$ that defines the probability measure of the agent $P^{\lambda_1^*,\lambda_2^*}$ includes signs of $Z_j^{\lambda_1^*,\lambda_2^*},\ j = 1, 2$, in its expression. Thus, in order to obtain $P^{\lambda_1^*,\lambda_2^*}$, we need to solve BSDE (23) for $sgn(Z_j^{\lambda_1^*,\lambda_2^*}), j = 1, 2$. In this section, we present two methods to solve for $sgn(Z_j^{\lambda_1^*,\lambda_2^*}), j = 1, 2$ in the system of equations consisting of forward SDE (11) and BSDE (23) in concrete cases. Firstly, in Section III-A, we investigate a case in which $sgn(Z_1^{\lambda_1^*,\lambda_2^*})$ and $sgn(Z_2^{\lambda_1^*,\lambda_2^*})$ in (24) are determined by comparison theorems. Secondly, in Section III-B, we show a case in which we predetermine these signs a priori and confirm them afterwards by explicitly solving the equations.

Let $\mathcal{R}^l$-valued stochastic process X be a state-variable process satisfying

$$dX_t = \mu_x(X_t)dt + \sum_{j=1}^{d} \sigma_{x,j}(X_t)dB_{j,t}, \tag{29}$$

and X_0 be a $\mathcal{R}_+$-valued stochastic process satisfying an SDE

$$\frac{dX_{0,t}}{X_{0,t}} = \mu_{x_0}(X_t)dt + \sum_{j=1}^{d} \sigma_{x_0,j}(X_t)dB_{j,t},\ x_0 > 0 \tag{30}$$

where $\mu_x, \sigma_{x,j} : \mathcal{R}^l \to \mathcal{R}^l$, $\sigma_{x_0,j}, \mu_{x_0} : \mathcal{R}^l \to \mathcal{R}$, $j = 1, 2, \ldots, d$, and $\mathcal{R}_+$ is the set of positive real numbers. This type of modeling frequently appears in finance (e.g. [18]), and corresponds to the case in which $\mathcal{R}^l$-valued process X in (11) in Section II is replaced with $\mathcal{R}^{l+1}$-valued process (X_0, X).

First of all, we consider the corresponding sup-inf (resp., inf-sup) problem in (17) replacing g in (12) with f, which does not depend on $B \in \mathcal{C}([0,T] \to \mathcal{R}^d)$. Here, $f : [0,T] \times \mathcal{R}_+ \times \mathcal{R}^l \times \mathcal{R} \to \mathcal{R}$ satisfies the following conditions: (i) $f(t, x_0, x, 0)$ is bounded, (ii) There exists a constant $L > 0$ such that $|f(t, x_0, x, y) - f(t, x_0, x, y')| \leq L|y - y'|, \forall y, y' \in \mathcal{R},\ x_0 \in \mathcal{R}_+,\ x \in \mathcal{R}^l,\ t \in [0,T]$, and (iii) f is continuously differentiable with respect to x_0 and y. Moreover, we assume that a weak version of Novikov's condition (16) and the moment condition (22) hold for $\{\bar{\lambda}_{j,t}\}_{0\leq t\leq T},\ j = 1, 2$. Then, by Theorem 1, $(\lambda_1^*, \lambda_2^*)$ that attains the sup-inf and inf-sup in problems (17) is expressed as

$$\lambda_{1,t}^* = -|\bar{\lambda}_{1,t}| sgn(Z_{1,t}^{\lambda_1^*,\lambda_2^*}), \lambda_{2,t}^* = |\bar{\lambda}_{2,t}| sgn(Z_{2,t}^{\lambda_1^*,\lambda_2^*}), \tag{31}$$

where $Z^{\lambda_1^*,\lambda_2^*}$ is from a unique solution $(Y^{\lambda_1^*,\lambda_2^*}, Z^{\lambda_1^*,\lambda_2^*})$ of a BSDE:

$$dY_t^{\lambda_1^*,\lambda_2^*} = -\Big(f(t, X_{0,t}, X_t, Y_t^{\lambda_1^*,\lambda_2^*}) - |\bar{\lambda}_{1,t}||Z_{1,t}^{\lambda_1^*,\lambda_2^*}| + |\bar{\lambda}_{2,t}||Z_{2,t}^{\lambda_1^*,\lambda_2^*}|)\Big)dt + \sum_{j=1}^{d} Z_{j,t}^{\lambda_1^*,\lambda_2^*} dB_{j,t}, Y_T^{\lambda_1^*,\lambda_2^*} = h(X_{0,T}), \tag{32}$$

where $h : \mathcal{R} \to \mathcal{R}$ is a bounded function.

A. The case in which the diffusion terms of X_0 on B_1 and B_2 only depend on X_0 (Method by comparison theorems)

Firstly, we provide the case in which the signs of $Z_1^{\lambda_1^*,\lambda_2^*}$ and $Z_2^{\lambda_1^*,\lambda_2^*}$ in the equations (29), (30) and (32) are determined by comparison theorems.

As a specific case of (29) and (30), we consider the following SDEs for the state-variable process X and X_0. Here, the first two diffusion terms of X_0 only depend on itself. Let $d = l = 3$. We assume that under the physical measure P, X_1, X_2, X_3 and X_0 are unique strong solutions of SDEs

$$\begin{cases} dX_{j,t} = \mu_j(X_{j,t})dt + \sigma_j(X_{j,t})dB_{j,t},\ X_{j,0} = x_j, \\ j = 1, 2, 3, \\ \frac{dX_{0,t}}{X_{0,t}} = \mu_{x_0}(X_{1,t}, X_{2,t}, X_{3,t})dt + \sigma_{x_0,1}dB_{1,t} \\ \quad +\sigma_{x_0,2}dB_{2,t} + \sigma_{x_0,3}(X_{3,t})dB_{3,t},\ X_{0,0} = x_0, \end{cases} \tag{33}$$

where $\mu_j, \sigma_j : \mathcal{R} \to \mathcal{R},\ j = 1, 2, 3,\ \mu_{x_0} : \mathcal{R}^3 \to \mathcal{R}$ with $\int_0^T |\mu_{x_0}(X_{1,t}, X_{2,t}, X_{3,t})|dt < \infty,\ x_0 > 0,\ \sigma_{x_0,1}, \sigma_{x_0,2} \in \mathcal{R}$, and $\sigma_{x_0,3} : \mathcal{R} \to \mathcal{R}$. Let $f : [0,T] \times \mathcal{R}_+ \times \mathcal{R}^3 \times \mathcal{R} \to \mathcal{R}$ be an aggregator satisfying conditions (i)-(iii) in Section III.

Then, the next proposition shows that $sgn(Z_1^{\lambda_1^*,\lambda_2^*})$ and $sgn(Z_2^{\lambda_1^*,\lambda_2^*})$ in (31) are uniquely determined under certain conditions.

Proposition 3. *Let $v : [0,\infty) \times \mathcal{R}^3 \times \mathcal{R}^+ \to \mathcal{R}$ be a value function defined by*

$$v(t, \mathbf{x}, x_0) = Y_t^{t,\mathbf{x},x_0},\ \mathbf{x} = (x_1, x_2, x_3) \tag{34}$$

where $(Y_s^{t,\mathbf{x},x_0}, Z_s^{t,\mathbf{x},x_0})$ is the solution of a BSDE

$$\begin{aligned} &dY_s^{t,\mathbf{x},x_0} \\ &= -(f(s, X_{0,s}^{t,x_0}, X_{1,s}^{t,x_1}, X_{2,s}^{t,x_2}, X_{3,s}^{t,x_3}, Y_s^{t,\mathbf{x},x_0}) \\ &\quad -|\bar{\lambda}_1(s, X_{1,s}^{t,x_1})||Z_{1,s}^{t,\mathbf{x},x_0}| + |\bar{\lambda}_2(s, X_{2,s}^{t,x_2})||Z_{2,s}^{t,\mathbf{x},x_0}|)ds \\ &\quad + \sum_{j=1}^{3} Z_{j,s}^{t,\mathbf{x},x_0} dB_{j,s}, \\ &Y_T^{t,\mathbf{x},x_0} = h(X_{0,T}^{t,x_0}),\ t \leq s \leq T, \end{aligned} \tag{35}$$

and $X_{j,s}^{t,x_j},\ j = 1, 2, 3$, and $X_{0,s}^{t,x_0}$ $(t \leq s \leq T)$ are unique strong solutions of SDEs (33) with the initial conditions replaced with $X_{j,t}^{t,x_j} = x_j$ and $X_{0,t}^{t,x_0} = x_0$ at $s = t$.

Here, $h : \mathcal{R} \to \mathcal{R}$ is a bounded increasing function. Let $\bar{\lambda}_j(s, x_j),\ j = 1, 2 : [0, T] \times \mathcal{R} \to \mathcal{R}$ be measurable functions and we assume that a weak version of Novikov's condition (16) and the moment condition (22) hold for $\bar{\lambda}_{j,t} = \bar{\lambda}_j(t, X_{j,t}), j = 1, 2$. Assume that f is increasing with respect to x_0, decreasing with respect to x_1, and increasing with respect to x_2, $\mu_{x_0}(x_1, x_2, x_3)$ is decreasing with respect to x_1 and increasing with respect to x_2, and $|\bar{\lambda}_j(s, x_j)|,\ j = 1, 2$, are increasing with respect to x_j. Also, assume that $\sigma_{x_0,1}, \sigma_{x_0,2} > 0$, $\sigma_1(X_{1,t}) < 0, \sigma_2(X_{2,t}) > 0$ in (33). Suppose that $v(t, \mathbf{x}, x_0)$ is continuously differentiable with respect to t and twice continuously differentiable with respect to x_1, x_2, x_3 and x_0.

Then, we have

$$sgn(Z_{j,t}^{\lambda_1^*,\lambda_2^*}) = +1\ (j = 1, 2), \tag{36}$$

for $(Z_1^{\lambda_1^*,\lambda_2^*}, Z_2^{\lambda_1^*,\lambda_2^*})$ in (32).

Proof. By a comparison theorem on SDEs (see Proposition 5.2.18 in Karatzas and Shreve [14], for example), when x_1 increases, $X_{1,s}^{t,x_1}$ increases. Since μ_{x_0} is decreasing with respect to the first variable, $X_{0,s}^{t,x_0}$ also decreases. As a result, by a slight modification of the proof of Theorem 1, a comparison theorem on a stochastic Lipschitz BSDE holds and it follows that $Y_t^{t,\mathbf{x},x_0}$ decreases. In detail, when x_1 increases, $X_{0,s}^{t,x_0}$ decreases, and consequently, the driver $f(s, X_{0,s}^{t,x_0}, X_{1,s}^{t,x_1}, X_{2,s}^{t,x_2}, X_{3,s}^{t,x_3}, y) - |\bar{\lambda}_{1,s}(X_{1,s}^{t,x_1})||z_1| + |\bar{\lambda}_{2,s}(X_{2,s}^{t,x_2})||z_2|$ in (35) decreases for all $s \in [0, T]$, $y \in \mathcal{R}$, $(z_1, z_2) \in \mathcal{R}^2$, since $f_{x_0} > 0$ and $-|\bar{\lambda}_{1,s}(X_{1,s}^{t,x_1})|$ also decreases.

Similarly, when x_2 increases, both $X_{2,s}^{t,x_2}$ and $X_{0,s}^{t,x_0}$ increase, and then $Y_t^{t,\mathbf{x},x_0}$ increases. When x_0 increases, $X_{0,s}^{t,x_0}$ increases, and then $Y_t^{t,\mathbf{x},x_0}$ increases.

Thus, we have

$$\begin{aligned}
&\partial_{x_1} v(t, \mathbf{x}, x_0) \le 0,\\
&\partial_{x_2} v(t, \mathbf{x}, x_0) \ge 0,\\
&\partial_{x_0} v(t, \mathbf{x}, x_0) \ge 0,
\end{aligned} \tag{37}$$

since $v(t, \mathbf{x}, x_0)$ is differentiable with respect to x_1, x_2 and x_0.

Also, by applying Ito's formula to $v(t, \mathbf{X}_t, X_{0,t})$, $\mathbf{X}_t = (X_{1,t}, X_{2,t}, X_{3,t})$, and comparing the result with (35), we have

$$Z_{j,t}^{\lambda_1^*,\lambda_2^*} = \begin{cases} \sigma_j(X_{j,t})\partial_{x_j} v(t, \mathbf{X}_t, X_{0,t}) \\ +\sigma_{x_0,j} X_{0,t}\partial_{x_0} v(t, \mathbf{X}_t, X_{0,t}),\ j = 1, 2, \\ \sigma_j(X_{3,t})\partial_{x_3} v(t, \mathbf{X}_t, X_{0,t}) \\ +\sigma_{x_0,3}(X_{0,t})X_{0,t}\partial_{x_0} v(t, \mathbf{X}_t, X_{0,t}),\ j = 3. \end{cases}$$

By (37) and $\sigma_{x_0,1}, \sigma_{x_0,2} > 0$, $\sigma_1(X_{1,t}) < 0, \sigma_2(X_{2,t}) > 0$,

$$Z_{j,t}^{\lambda_1^*,\lambda_2^*} \ge 0,\ j = 1, 2.$$

Thus,

$$sgn(Z_{1,t}^{\lambda_1^*,\lambda_2^*}) = sgn(Z_{2,t}^{\lambda_1^*,\lambda_2^*}) = +1.$$

□

Example 2 (continuation of the motivating example in Section II-A). First, we consider a quadratic objective function case in which f in (32) is given by $f(x_0, y) = \frac{1}{2}x_0^2$, with X_0 satisfying

$$\frac{dX_{0,t}}{X_{0,t}} = \mu_{x_0} dt + \sigma_{x_0,1} dB_{1,t} + \sigma_{x_0,2} dB_{2,t},\ x_0 > 0, \tag{38}$$

where $\sigma_{x_0,1}, \sigma_{x_0,2} > 0$ and μ_{x_0} is a constant. We also assume $\bar{\lambda}_j(t, X_t) = \tilde{\lambda}_j,\ j = 1, 2,\ \tilde{\lambda}_1, \tilde{\lambda}_2 > 0$.

We remark that although the boundedness assumptions in Propositions 1 and 2 and Theorem 1 are not satisfied in this quadratic objective function case, the results of the propositions and the theorem hold. In detail, since λ is bounded as in (1), BSDE (32) is a Lipschitz driver case. Then, a theorem for the existence and uniqueness of a solution for a BSDE with a Lipschitz driver (e.g. Theorem 6.2.1 in Pham [20]) can be applied and the conclusions of Propositions 1 and 2 hold. Moreover, by a comparison theorem for a BSDE with a uniform Lipschitz driver (e.g. Theorem 6.2.2 in Pham [20]), the conclusion of Theorem 1 holds.

We first note that in this example, the assumptions on σ_{x_0} and $\bar{\lambda}$ in Proposition 3 are satisfied. If the continuous differentiablity of v is satisfied, then by Proposition 3, we obtain $sgn(Z_j^{\lambda_1^*,\lambda_2^*}) = +1,\ j = 1, 2$.

In fact, we can confirm the continuous differentiability of v as follows. Let us consider a BSDE

$$dY_t^{\lambda_1^*,\lambda_2^*} = -\frac{1}{2}X_{0,t}^2 dt + Z_t^{\lambda_1^*,\lambda_2^*} dB_t^{\lambda_1^*,\lambda_2^*},\ Y_T^{\lambda_1^*,\lambda_2^*} = \frac{X_{0,T}^2}{2}, \tag{39}$$

$$\begin{aligned}\frac{dX_{0,t}}{X_{0,t}} &= (\mu_{x_0,t} - \sigma_{x_0,1}\tilde{\lambda}_1 + \sigma_{x_0,2}\tilde{\lambda}_2)dt \\ &\quad + \sigma_{x_0,1} dB_{1,t}^{\lambda_1^*,\lambda_2^*} + \sigma_{x_0,2} dB_{2,t}^{\lambda_1^*,\lambda_2^*},\ x_0 > 0,\end{aligned} \tag{40}$$

where $dB_{j,t}^{\lambda_1^*,\lambda_2^*} = dB_{j,t} - (-1)^j\tilde{\lambda}_j dt,\ j = 1, 2$, which corresponds to BSDE (32) under $P^{\lambda_1^*,\lambda_2^*}$ when $sgn(Z_j^{\lambda_1^*,\lambda_2^*}) = +1,\ j = 1, 2$.

We suppose a solution for BSDE (39) of the form

$$Y_t^{\lambda_1^*,\lambda_2^*} = \frac{A(t)X_{0,t}^2}{2}. \tag{41}$$

By applying Ito's formula to (41), we have

$$\begin{aligned}\dot{A}(t) + 2A(t)(\mu_{x_0,t} - \sigma_{x_0,1}\tilde{\lambda}_1 + \sigma_{x_0,2}\tilde{\lambda}_2) + 2|\sigma_{x_0}|^2 = -1,\\ A(T) = 1,\end{aligned} \tag{42}$$

which is solved as $A(t) = e^{-\int_0^t b_s ds}\left(1 + \int_t^T (2|\sigma_{x_0}|^2 + 1)e^{\int_0^v b_s ds} dv\right) > 0$, with $b_t = 2(\mu_{x_0,t} - \sigma_{x_0,1}\tilde{\lambda}_1 + \sigma_{x_0,2}\tilde{\lambda}_2)$, and

$$Z_t^{\lambda_1^*,\lambda_2^*} = A(t)X_{0,t}^2\sigma_{x_0}. \tag{43}$$

Hence, $(Y^{\lambda_1^*,\lambda_2^*}, Z^{\lambda_1^*,\lambda_2^*})$ given by (41) and (43) satisfies BSDE (40).

Then, it follows that $(Y_t^{\lambda_1^*,\lambda_2^*}, Z_t^{\lambda_1^*,\lambda_2^*})$ also satisfies BSDE (32). Thus, by (41), $v(t, x_0) = A(t)x_0^2$, which indicates the continuous differentiability of v.

Remark 5. *In this particular example, the proof of Proposition 3 is described as follows. By applying Ito's formula to $v(t, X_{0,t})$ and focusing on the diffusion terms, we have*

$Z_{j,t}^{\lambda_1^*,\lambda_2^*} = \sigma_{x_0,j} X_{0,t}\partial_{x_0} v(t, X_{0,t})$. *For* $v(t,x_0) = Y_t^{t,x_0}$, *by a comparison theorem for SDEs (e.g. Proposition 5.2.18 in Karatzas and Shreve [14]), if* x_0 *increases* $X_{0,s}^{t,x_0}$, $t \le s \le T$ *increases. Then, since* f *and* h *are increasing with respect to* $X_{0,s}^{t,x_0}$, *by a comparison theorem for BSDEs (e.g. Theorem 6.2.2 in Pham [20]),* Y_s^{t,x_0} *also increases for* $t \le s \le T$. *Thus* $\partial_{x_0} v(t,x_0) \ge 0$, *and since* $\sigma_{x_0,j} > 0$ *and* $X_{0,t} > 0$, *we obtain* $sgn(Z_j^{\lambda_1^*,\lambda_2^*}) = +1,\ j = 1,2$.

Remark 6. *Alternatively, if some Lipschitz modification is made on the quadratic objective function and its terminal condition, then Assumption 5.0.1 in Zhang [27] is satisfied and by Problem 5.7.4 in Zhang [27], the continuously differentiability of* v *is guaranteed. Thus, we also have* $sgn(Z_j^{\lambda_j^*,\lambda_2^*}) = +1,\ j = 1,2$ *in that case.*

Example 3.

Let $\rho < 1$ with $\rho \ne 0$, and $\beta > 0$. We consider the case of a standard power utility

$$f(x_0, y) = \frac{\beta}{\rho}(x_0^\rho - 1) - \beta y, \tag{44}$$

with X_0 satisfying

$$\frac{dX_{0,t}}{X_{0,t}} = \mu_{x_0} dt + \sigma_{x_0,1} dB_{1,t} + \sigma_{x_0,2} dB_{2,t},\ x_0 > 0, \tag{45}$$

where $\sigma_{x_0,1}, \sigma_{x_0,2} > 0$, and state-variable processes X_1 and X_2 satisfying

$$dX_{j,t} = (\mu_{j,1} X_{j,t} + \mu_{j,0})dt + \sigma_j \sqrt{X_{j,t}} dB_{j,t}, \\ X_{j,0} = x_j > 0,\ j = 1,2, \tag{46}$$

where $\mu_{1,0}, \mu_{2,0} > 0$, $\mu_{1,1}, \mu_{2,1} < 0$, $\sigma_1 < 0, \sigma_2 > 0$. Here, X_0 typically represents a consumption process.

We also assume $h(X_{0,T}) = \frac{X_{0,T}^\rho - 1}{\rho}$ for the terminal condition, and

$$\bar\lambda_1(t, X_1) = \tilde\lambda_1 \sqrt{X_{1,t}},\ \bar\lambda_2(t, X_2) = \tilde\lambda_2 \sqrt{X_{2,t}},$$

where $\tilde\lambda_1 < 0$, $\tilde\lambda_2 > 0$.

Then, we note that $f_{x_0} > 0$ and $|\bar\lambda_j(s, x_j)|$ is increasing with respect to x_j. With some necessary modifications on X_0 and the state-variable process X in (45) and (46) or the aggregator f in (44) and the terminal condition $h(X_{0,T})$ as in Remark 7, by applying Proposition 3, we obtain $sgn(Z_j^{\lambda_1^*,\lambda_2^*}) = +1,\ j = 1,2$, and by (31), we have $\lambda_{1,t}^* = \tilde\lambda_1 \sqrt{X_{1,t}},\ \lambda_{2,t}^* = \tilde\lambda_2 \sqrt{X_{2,t}}$.

Remark 7. *In the above example, without any modifications, the boundedness on* $f(x_0, 0)$ *and* $h(X_{0,T})$, *and the continuous differentiability of the value function* v, *which are assumptions in Proposition 3, are not necessarily satisfied. One possible adjustment is that we consider bounded modifications of* X_0 *and* X *in SDEs (45) and (46), in particular so that* X *does not take values in a neighborhood of* 0. *Then, the boundedness on* $f(x_0, 0)$ *and* $h(X_{0,T})$, *as well as a uniform Lipschitz condition on the driver of BSDE (32), follows, and by Lemma 5.2.3 in Zhang [27], the continuous differentiability of* v *is obtained. Another approach is that we consider bounded modifications of* $f(x_0, 0)$ *in (44) and* $h(X_{0,T})$ *as functionals of* x_0, *and assume existence of a classical solution of PDE (A.14) without the jump component in Theorem A.9.22 in Cohen and Elliott [7], which also yields the continuous differentiability of* v.

B. The case in which the diffusion terms of X_0 *include both* X_0 *and* X

Next, we consider the case in which $sgn(Z_1^{\lambda_1^*,\lambda_2^*})$ and $sgn(Z_2^{\lambda_1^*,\lambda_2^*})$ in the expressions of λ_1^* and λ_2^* in (31) are obtained by solving the equations (29), (30) and (32) explicitly with the diffusion terms of X_0 and X including both X_0 and X.

Firstly, we rewrite SDEs (29) and (30) in Section III under $P^{\lambda_1^*,\lambda_2^*}$ by Girsanov's theorem as follows.

$$dX_t = \mu_{x,t}^* dt + \sigma_x(X_t) dB_t^{\lambda_1^*,\lambda_2^*}, \tag{47}$$

$$\frac{dX_{0,t}}{X_{0,t}} = \mu_{x_0,t}^* dt + \sigma_{x_0}(X_t) \cdot dB_t^{\lambda_1^*,\lambda_2^*}, \tag{48}$$

where $B^{\lambda_1^*,\lambda_2^*} \in \mathcal{R}^d$, $\sigma_x(x) \in \mathcal{R}^{l\times d}$ $(2 \le d \le l)$, $\sigma_{x_0}(x) \in \mathcal{R}^d$,

$$\mu_{x,t}^* = \mu_x(X_t) + \sigma_x(X_t)\lambda_t^*, \tag{49}$$

$$\mu_{x_0,t}^* = \mu_{x_0}(X_t) + \lambda_t^* \cdot \sigma_{x_0}(X_t), \tag{50}$$

with the stochastic process λ^* in (31), which is

$$\lambda_{j,t}^* = (-1)^j |\bar\lambda_{j,t}| sgn(Z_{j,t}^{\lambda_1^*,\lambda_2^*}) \tag{51}$$

for $j = 1,2$, and $\lambda_{j,t}^* \equiv 0$ for $j = 3, \cdots, d$.

Here, $Z^{\lambda_1^*,\lambda_2^*} \in \mathcal{R}^d$ satisfies BSDE (32) under $P^{\lambda_1^*,\lambda_2^*}$

$$dY_t^{\lambda_1^*,\lambda_2^*} = -f(t, X_{0,t}, X_t, Y_t^{\lambda_1^*,\lambda_2^*})dt + Z_t^{\lambda_1^*,\lambda_2^*} \cdot dB_t^{\lambda_1^*,\lambda_2^*}, \\ Y_T^{\lambda_1^*,\lambda_2^*} = h(X_{0,T}). \tag{52}$$

Then, we solve the equations (47)-(52) in the following way. We first suppose $sgn(Z_{j,t}^{\lambda_1^*,\lambda_2^*}) = +1,\ j = 1,2$, which indicates $\lambda_j^* = (-1)^j |\bar\lambda_{j,t}|,\ j = 1,2$ by (51), and separate FBSDEs (47)-(52) into forward SDEs and a BSDE. We confirm that $sgn(Z_{j,t}^{\lambda_1^*,\lambda_2^*}) = +1,\ j = 1,2$, by explicitly solving BSDE (52) under certain conditions. If these conditions are met, we observe that X and $(Y^{\lambda_1^*,\lambda_2^*}, Z^{\lambda_1^*,\lambda_2^*})$ also satisfy the equations (47)-(52).

Example 4. Next, we consider the case of a stochastic differential log-utility as in equation (2.9) in Nakamura et al. [16], whose aggregator f does not depend on t or x and is defined as

$$f(x_0, y) = \beta(1 + \alpha y)\left[\log x_0 - \frac{\log(1 + \alpha y)}{\alpha}\right], \tag{53}$$

with the terminal condition $h(X_{0,T}) = \frac{X_{0,T}^\alpha - 1}{\alpha}$ in BSDE (32). Note that this is a generalization of a standard log-utility whose aggregator is given by $f(x_0, y) = \beta(\log x_0 - y)$ with $h(X_{0,T}) = \log X_{0,T}$. This aggregator for the standard log-utility is obtained by sending α to 0 in (53) and $h(X_{0,T})$.

The next proposition shows that the equations (47)-(52), in which the diffusion terms of X_0 include X, are explicitly solved and $sgn(Z_j^{\lambda_1^*,\lambda_2^*}),\ j = 1,2$, are determined.

For equations (47)-(52), we assume $d = l = 3$ and the following coefficients. We denote $(1, j)$ component of $\sigma_{x_0}(x)$ by $\sigma^j_{x_0}(x)$ and (i, k) component of $\sigma_x(x)$ by $\sigma^k_{x,i}(x)$.

$$\begin{aligned}\mu_{x_0}(x) = \tilde{\mu}_{x_0,1}x_1 + \tilde{\mu}_{x_0,2}x_2 + \tilde{\mu}_{x_0,3}x_3 \\ +\tilde{\mu}_{x_0,0},\end{aligned}$$

$$\begin{aligned}&\sigma^j_{x_0}(x) = \tilde{\sigma}_{x_0,j}\sqrt{x_j},\ j = 1, 2,\\ &\sigma^3_{x_0}(x) = \tilde{\sigma}_{x_0,3},\end{aligned}$$

$$\begin{aligned}&\mu_{x_j}(x) = \tilde{\mu}_{x_j,1}x_j + \tilde{\mu}_{x_j,0},\ j = 1, 2,\\ &\mu_{x_3}(x) = \tilde{\mu}_{x_3,1}x_3 + \tilde{\mu}_{x_3,0},\end{aligned}$$

$$\begin{aligned}&\sigma^k_{x,i}(x) = \tilde{\sigma}_{x,i}\sqrt{x_i},\ i = 1, 2,\ i = k,\\ &\sigma^k_{x,3}(x) = \tilde{\sigma}_{x,3},\ k = 3,\\ &\sigma^k_{x,i}(x) = 0,\ i = 1, 2, 3,\ i \neq k,\end{aligned}$$

$$\bar{\lambda}_j(t, X_t) = \tilde{\lambda}_j\sqrt{X_{j,t}},\ j = 1, 2,$$

where $\tilde{\mu}_{x_0,0}, \tilde{\mu}_{x_0,j}, \tilde{\sigma}_j, \tilde{\mu}_{x_j,i}, \tilde{\sigma}_{x,j} \in \mathcal{R},\ i = 0, 1,\ j = 1, 2, 3,\ \tilde{\lambda}_1 \leq 0$, and $\tilde{\lambda}_2 \geq 0$.

Also, we assume

$$B_i^2 - 4A_iC_i \geq 0,\ i = 1, 2, 3, \tag{54}$$

where

$$\begin{aligned}A_i &= \alpha\tilde{\sigma}^2_{x,i},\\ B_i &= (\alpha\tilde{\sigma}_{x_0,i}\tilde{\sigma}_{x,i} + \bar{\mu}_{x_i,1} - \beta),\\ C_i &= \bar{\mu}_{x_0,i} - \frac{1}{2}(1-\alpha)\tilde{\sigma}^2_{x_0,i},\end{aligned}$$

with

$$\begin{aligned}\bar{\mu}_{x_0,i} &= \tilde{\mu}_{x_0,i} + \tilde{\lambda}_i\tilde{\sigma}_{x_0,i},\ i = 1, 2,\\ \bar{\mu}_{x_0,3} &= \tilde{\mu}_{x_0,3},\\ \bar{\mu}_{x_i,1} &= \tilde{\mu}_{x_i,1} + \tilde{\lambda}_i\tilde{\sigma}_{x,i},\ i = 1, 2,\\ \bar{\mu}_{x_3,1} &= \tilde{\mu}_{x_3,1}.\end{aligned}$$

Let

$$m_i(t) = \frac{1}{\frac{-A_i\gamma_i - B_i}{\gamma_i(2A_i\gamma_i + B_i)e^{-(2A_i\gamma_i+B_i)(T-t)} - \frac{A_i}{2A_i\gamma_i+B_i}}} + \gamma_i, \tag{55}$$

where $\gamma_i = \frac{-B_i + \sqrt{B_i^2 - 4A_iC_i}}{2A_i}$, $i = 1, 2, 3$.

Proposition 4. *Suppose that conditions*

$$\tilde{\sigma}_{x,j}m_j(t) + \tilde{\sigma}_{x_0,j} > 0,\ j = 1, 2, \tag{56}$$

hold.

Then,

(i) $sgn(Z_j^{\lambda_1^*,\lambda_2^*}) = +1,\ j = 1, 2,$ *holds.*

(ii) In particular, $(Y^{\lambda_1^*,\lambda_2^*}, Z^{\lambda_1^*,\lambda_2^*})$ *of the form*

$$Y_t^{\lambda_1^*,\lambda_2^*} = \frac{A(X_t, t)X^\alpha_{0,t} - 1}{\alpha}, \tag{57}$$

$$Z_{j,t}^{\lambda_1^*,\lambda_2^*} = \begin{cases} X^\alpha_{0,t}A(X_t, t)\sqrt{X_{j,t}}(\tilde{\sigma}_{x,j}m_j(t) + \tilde{\sigma}_{x_0,j}), j = 1, 2,\\ X^\alpha_{0,t}A(X_t, t)(\tilde{\sigma}_{x,j}m_j(t) + \tilde{\sigma}_{x_0,j}), j = 3,\end{cases} \tag{58}$$

satisfies FBSDEs (47)-(52), where

$$A(x, t) = \exp(\alpha\{m_1(t)x_1 + m_2(t)x_2 + m_3(t)x_3 + n(t)\}),$$

$$n(t) = \int_t^T e^{-\beta(s-t)}\sum_{j=1}^3 m_j(s)\tilde{\mu}_{x_j,0}ds, \tag{59}$$

and (X, X_0) *is a unique strong solution of (47) and (48) with* $sgn(Z_j^{\lambda_1^*,\lambda_2^*}) = +1,\ j = 1, 2$ *being substituted.*

Proof. First, let X be a unique strong solution of forward SDEs (47)-(49) with supposing $sgn(Z_j^{\lambda_1^*,\lambda_2^*}) = +1,\ j = 1, 2,$ in λ^* in (51).

By applying Ito's formula to (57) and comparing the drift and the diffusion term with (52), we observe that $(Y^{\lambda_1^*,\lambda_2^*}, Z^{\lambda_1^*,\lambda_2^*})$ given by (57) and (58) satisfies BSDE (52), if Riccati equations

$$\begin{aligned}-\beta m_i(t) + \dot{m}_i(t) + \bar{\mu}_{x_0,i}\\ -\frac{1}{2}(1-\alpha)\tilde{\sigma}^2_{x_0,i} + m_i(t)\bar{\mu}_{x_i,1} + \alpha m_i(t)\tilde{\sigma}_{x_0,i}\tilde{\sigma}_{x,i}\\ +\alpha m_i^2(t)\tilde{\sigma}^2_{x,i} = 0,\\ m_i(T) = 0,\end{aligned} \tag{60}$$

and ODE

$$\begin{aligned}-\beta n(t) + \dot{n}(t)\\ +m_1(t)\tilde{\mu}_{x_1,0} + m_2(t)\tilde{\mu}_{x_2,0} + m_3(t)\tilde{\mu}_{x_3,0} = 0,\\ n(T) = 0,\end{aligned} \tag{61}$$

hold. In fact, $m_i(t),\ i = 1, 2, 3,$ in (55) and $n(t)$ in (59) are solutions of (60) and (61), respectively.

Then, by condition (56), $sgn(Z_j^{\lambda_1^*,\lambda_2^*}) = +1,\ j = 1, 2,$ and thus X, originally defined as a unique strong solution of forward SDEs (47)-(51) supposing $sgn(Z_j^{\lambda_1^*,\lambda_2^*}) = +1,\ j = 1, 2$ in λ^* in (51), satisfies forward SDEs (47)-(51). □

Remark 8. *In this case, without any modifications, the boundedness on* $f(x_0, 0)$ *and* $h(X_{0,T})$ *is not necessarily satisfied. However, we can consider bounded modifications of* X_0 *and* X *in SDEs (47) and (48), in particular so that* X *does not take values in a neighborhood of* 0*. Then, the boundedness on* $f(x_0, 0)$ *and* $h(X_{0,T})$ *is satisfied. Also, we can consider bounded modifications of* $f(x_0, 0)$ *and* $h(X_{0,T})$ *as functionals of* X_0*.*

IV. Concluding remarks

In this study, we have presented a sup-inf (resp., inf-sup) problem on choice of a probability measure in which a single agent assumes a best case scenario on one risk at the same time a worst case scenario on another risk. This selection method of a probability measure generalizes the approach to model uncertainties in which one considers the worst scenario on the views of Brownian motions, such as in the robust control. Besides, this sup-inf (resp., inf-sup) problem has unbounded control domains dependent on stochastic processes and is solved via a BSDE with a stochastic Lipschitz driver.

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APPENDIX A
PROOF OF PROPOSITION 1

In this section, we prove Proposition 1 with modifications of the arguments in the proofs of Theorem 9.20 in Cohen and Elliott [7] and Theorem I-3 in Hamadene and Lepeltier [11].

Let $\phi : [0,T] \times \mathcal{R}^l \times \mathcal{C}([0,T] \to \mathcal{R}^d) \times \mathcal{R} \times \mathcal{R}^d \to \mathcal{R}$ be

$$\phi(t,x,\omega,y,z) = g(t,\omega,x,y) + \lambda_{1,t}(\omega)z_1 + \lambda_{2,t}(\omega)z_2. \quad (62)$$

Then, (19) is rewritten as

$$\begin{aligned} &dY_t^{\lambda_1,\lambda_2} \\ &= -\bigg(g(s,B,X_s^{\lambda_1,\lambda_2},Y_s^{\lambda_1,\lambda_2}) + \lambda_{1,t}Z_{1,t}^{\lambda_1,\lambda_2} \\ &\quad + \lambda_{2,t}Z_{2,t}^{\lambda_1,\lambda_2}\bigg)dt + \sum_{j=1}^{d} Z_{j,t}^{\lambda_1,\lambda_2}dB_{j,t}, \\ &= -\phi(t,X_t,B,Y_t^{\lambda_1,\lambda_2},Z_t^{\lambda_1,\lambda_2})dt + Z_t^{\lambda_1,\lambda_2}dB_t^{\lambda_1,\lambda_2}, \\ &Y_T^{\lambda_1,\lambda_2} = \xi. \end{aligned} \quad (63)$$

Hereafter, we suppress the superscript λ_1,λ_2 of Y^{λ_1,λ_2} and Z^{λ_1,λ_2}.

By condition (ii), we have

$$\begin{aligned} &|\phi(t,x,\omega,y,z) - \phi(t,x,\omega,y',z')| \\ &= |\lambda_{1,t}(\omega)(z_1 - z_1') + \lambda_{2,t}(\omega)(z_2 - z_2')| + L|y-y'| \\ &\le \|\lambda(\omega)\|_t|z-z'| + L|y-y'|, \end{aligned} \quad (64)$$

where $\lambda(\omega) = (\lambda_1(\omega),\lambda_2(\omega))$ and $\|\lambda(\omega)\|_t = \sup_{0\le s\le t}|\lambda_s(\omega)|$.

Set

$$\begin{aligned} &\phi^{n,m}(t,x,\omega,y,z) \\ &= \phi(t,x,\omega,y,z)1_{\{\|\lambda(\omega)\|_t\le n\}}1_{\{\phi(t,x,y,\omega,z)\ge 0\}} \\ &+\phi(t,x,\omega,y,z)1_{\{\|\lambda(\omega)\|_t\le m\}}1_{\{\phi(t,x,\omega,y,z)<0\}}. \end{aligned} \quad (65)$$

Then, we have

$$\begin{aligned} &|\phi^{n,m}(t,x,\omega,y,z) - \phi^{n,m}(t,x,\omega,y',z')| \\ &\le (n+m)|z-z'| + L|y-y'|. \end{aligned} \quad (66)$$

Since $\phi^{n,m}(t,x,\omega,y,z)$ satisfies the uniform Lipschitz condition and by Theorem 6.2.1 in Pham [20] , there exists a unique solution $(Y^{n,m},Z^{n,m})$ for a BSDE

$$\begin{aligned} dY_t^{n,m} = -\phi^{n,m}(t,X_t,B,Y_t^{n,m},Z_t^{n,m})dt + Z_t^{n,m}dB_t, \\ Y_T^{n,m} = \xi, \end{aligned} \quad (67)$$

such that

$$E\left[\int_0^T (Y_s^{n,m})^2 + |Z_s^{n,m}|^2 ds\right] < \infty, \quad (68)$$

and by the boundedness of ξ and condition (i), it follows that $Y^{n,m}$ is uniformly bounded with respect to t,ω,n,m.

Also, by applying comparison theorem (e.g. see Theorem 6.2.2. in Pham [20]) to BSDE (63), it follows that $Y^{n,m}$ is increasing with respect to n and decreasing with respect to m. Then, we define Y_t as

$$Y_t = \lim_{m\to\infty}\lim_{n\to\infty} Y_t^{n,m}, \ P\text{-a.s.} \quad (69)$$

For each $p \geq 1$, we take a subsequence $\{n(k)\}_{k\in\mathbf{N}}$, such that

$$\lim_{k\to\infty} E\left[\int_0^T |Y_s^{n(k),k} - Y_s|^p ds\right] = 0. \tag{70}$$

By the assumption on the integrability on λ in (20) and the uniform boundedness of $|Y^{n,m}|$, taking the subsequence $n(k)$ for $p = 4$, it follows that $\{Z^{n(k),k}\}_{k\in\mathbf{N}}$ is a Cauchy sequence in $L^2([0,T]\times\Omega)$. We define Z as the limit of $\{Z^{n(k),k}\}_{k\in\mathbf{N}}$ in $L^2([0,T]\times\Omega)$.

Then, by taking the limit of $k\to\infty$ in the integral form of (67) in which (n,m) is replaced by $(n(k),k)$, we observe that (Y,Z) is a solution of BSDE (19).

The uniqueness of (Y,Z) holds by the following discussion. Let (Y,Z) and (Y',Z') be solutions of the BSDE (19).

By Ito's formula, we have

$$d(Y_t - Y_t')^2 = 2(Y_t - Y_t')d(Y_t - Y_t') + d\left\langle Y - Y'\right\rangle_t, \tag{71}$$

where

$$\begin{aligned}
&d(Y_t - Y_t') \\
&= -(g(t,B,X_t,Y_t) - g(t,B,X_t,Y_t') \\
&\quad + \lambda_{1,t}(Z_{1,t} - Z'_{1,t}) + \lambda_{2,t}(Z_{2,t} - Z'_{2,t}))dt \\
&\quad + \sum_{j=1}^d (Z_{j,t} - Z'_{j,t})dB_{j,t}, \\
&= -\frac{g(t,B,X_t,Y_t) - g(t,B,X_t,Y_t')}{Y_t - Y_t'} 1_{\{Y_t - Y_t' \neq 0\}}(Y_t - Y_t')dt \\
&\quad + \sum_{j=1}^d (Z_{j,t} - Z'_{j,t})d\tilde{B}_{j,t}, \\
&= -b_t(Y_t - Y_t')dt + \sum_{j=1}^d (Z_{j,t} - Z'_{j,t})d\tilde{B}_{j,t},
\end{aligned} \tag{72}$$

and

$$d\left\langle Y - Y'\right\rangle_t = \sum_{j=1}^d (Z_{j,t} - Z'_{j,t})^2 dt. \tag{73}$$

Here, we set

$$b_t = \frac{g(t,B,X_t,Y_t) - g(t,B,X_t,Y_t')}{Y_t - Y_t'} 1_{\{Y_t - Y_t' \neq 0\}}, \tag{74}$$

and

$$d\tilde{B}_{j,t} = dB_{j,t} - \lambda_{j,t}dt, \ j = 1,2, \tag{75}$$

where $\tilde{B}$ is a $\tilde{P}$-Brownian motion because of (9) and Girsanov's theorem if we define

$$\begin{aligned}
&\tilde{P}(A) \\
&= E\left[\exp\left(-\frac{1}{2}\sum_{j=1}^2\int_0^T \lambda_{j,s}^2 ds + \sum_{j=1}^2\int_0^T \lambda_{j,s}dB_{j,s}\right)1_A\right], \\
&A \in \mathcal{F}.
\end{aligned} \tag{76}$$

Let $\bar{Y}_t = (Y_t - Y_t')e^{\int_0^t b_u du}$, $\bar{Z}_t = (Z_t - Z_t')e^{\int_0^t b_u du}$.

Then,

$$-\bar{Y}_t^2 = \sum_{j=1}^d \int_t^T 2\bar{Y}_t\bar{Z}_{j,s}d\tilde{B}_{j,s} + \int_t^T \sum_{j=1}^d \bar{Z}_{j,s}^2 ds. \tag{77}$$

By taking conditional expectation with respect to $\tilde{P}$ and the filtration $\mathcal{F}_t$, we have

$$\tilde{E}\left[\bar{Y}_t^2 + \int_t^T \sum_{j=1}^d \bar{Z}_{j,s}^2 ds \middle| \mathcal{F}_t\right] = 0. \tag{78}$$

Here, we used the fact that $\{\sum_{j=1}^d \int_0^t 2\bar{Y}_s\bar{Z}_{j,s}d\tilde{B}_{j,s}\}_{0\leq t\leq T}$ is a $\tilde{P}$-martingale, which can be shown by a localization argument. □

Appendix B
Proof of Proposition 2

Proposition 2 is proved in the same manner as Proposition 1 with the following modifications.

In the proof of existence of a solution, instead of (62) and (65), we set

$$\phi(t,x,\omega,y,z) = g(t,\omega,x,y) - |\bar{\lambda}_1(t,x)||z_1| + |\bar{\lambda}_2(t,x)||z_2| \tag{79}$$

and

$$\begin{aligned}
&\phi^{n,m}(t,x,\omega,y,z) \\
&= \phi(t,x,\omega,y,z)1_{\{\|\bar{\lambda}(x)\|_t\leq n\}}1_{\{\phi(t,x,\omega,y,z)\geq 0\}} \\
&+ \phi(t,x,\omega,y,z)1_{\{\|\bar{\lambda}(x)\|_t\leq m\}}1_{\{\phi(t,x,\omega,y,z)< 0\}},
\end{aligned} \tag{80}$$

respectively.

Then, noting that

$$\begin{aligned}
&|\phi(t,x,\omega,y,z) - \phi(t,x,\omega,y',z')| \\
&\leq L|y-y'| + |\bar{\lambda}_{1,t}(x)|||z_1| - |z_1'|| + |\bar{\lambda}_{2,t}(x)|||z_2| - |z_2'|| \\
&\leq L|y-y'| + |\bar{\lambda}_{1,t}(x)||z_1 - z_1'| + |\bar{\lambda}_{2,t}(x)||z_2 - z_2'| \\
&\leq L|y-y'| + |\bar{\lambda}_t(x)||z-z'| \\
&\leq L|y-y'| + \|\bar{\lambda}(x)\|_t|z-z'|,
\end{aligned} \tag{81}$$

we have

$$\begin{aligned}
&|\phi^{n,m}(t,x,\omega,y,z) - \phi^{n,m}(t,x,\omega,y',z')| \\
&\leq |\phi(t,x,\omega,y,z) - \phi(t,x,\omega,y',z')|1_{\{\|\bar{\lambda}(x)\|_t\leq n\}} \\
&\quad 1_{\{\phi(t,x,\omega,y,z)\geq 0\}} \\
&+ |\phi(t,x,\omega,y,z) - \phi(t,x,\omega,y',z')| \\
&\quad 1_{\{\|\bar{\lambda}(x)\|_t\leq m\}}1_{\{\phi(t,x,\omega,y,z)<0\}} \\
&\leq L|y-y'| + (n+m)|z-z'|,
\end{aligned} \tag{82}$$

which corresponds to (66).

For uniqueness of the solution, let $(Y,Z),(Y',Z')$ be solutions of BSDE (23).

Then, instead of (72), we have

$$
\begin{aligned}
&d(Y_t - Y_t') \\
&= -\frac{g(t,B,X_t,Y_t) - g(t,B,X_t,Y_t')}{Y_t - Y_t'} 1_{\{Y_t - Y_t' \neq 0\}} (Y_t - Y_t')dt \\
&\quad + |\bar{\lambda}_{1,t}(X_t)| \frac{|Z_{1,t}| - |Z_{1,t}'|}{Z_{1,t} - Z_{1,t}'} 1_{\{Z_{1,t} - Z_{1,t}' \neq 0\}} (Z_{1,t} - Z_{1,t}')dt \\
&\quad - |\bar{\lambda}_{2,t}(X_t)| \frac{|Z_{2,t}| - |Z_{2,t}'|}{Z_{2,t} - Z_{2,t}'} 1_{\{Z_{2,t} - Z_{2,t}' \neq 0\}} (Z_{2,t} - Z_{2,t}')dt \\
&\quad + \sum_{j=1}^{d} (Z_{j,t} - Z_{j,t}')dB_{j,t}, \ Y_T - Y_T' = 0. \qquad (83)
\end{aligned}
$$

Setting $\bar{Y}_t = Y_t - Y_t'$, $\bar{Z}_{j,t} = Z_{j,t} - Z_{j,t}'$, $j = 1, \ldots, d$, $b_t = \frac{g(t,B,X_t,Y_t) - g(t,B,X_t,Y_t')}{Y_t - Y_t'} 1_{\{Y_t - Y_t' \neq 0\}}$, $c_{1,t} = -|\bar{\lambda}_{1,t}(X_t)| \frac{|Z_{1,t}| - |Z_{1,t}'|}{Z_{1,t} - Z_{1,t}'} 1_{\{Z_{1,t} - Z_{1,t}' \neq 0\}}$, $c_{2,t} = |\bar{\lambda}_{2,t}(X_t)| \frac{|Z_{2,t}| - |Z_{2,t}'|}{Z_{2,t} - Z_{2,t}'} 1_{\{Z_{2,t} - Z_{2,t}' \neq 0\}}$, we have

$$
\begin{aligned}
d\bar{Y}_t = -b_t \bar{Y}_t dt + \bar{Z}_{1,t}(dB_{1,t} - c_{1,t}dt) + \bar{Z}_{2,t}(dB_{2,t} - c_{2,t}dt) \\
+ \sum_{j=3}^{d} \bar{Z}_{j,t} dB_{j,t}, \ \bar{Y}_T = 0. \qquad (84)
\end{aligned}
$$

Since

$$
\begin{aligned}
|c_{1,t}| &= |\bar{\lambda}_{1,t}(X_t)| \frac{||Z_{1,t}| - |Z_{1,t}'||}{|Z_{1,t} - Z_{1,t}'|} \\
&\leq |\bar{\lambda}_{1,t}(X_t)|, \\
|c_{2,t}| &= |\bar{\lambda}_{2,t}(X_t)| \frac{||Z_{2,t}| - |Z_{2,t}'||}{|Z_{2,t} - Z_{2,t}'|} \\
&\leq |\bar{\lambda}_{2,t}(X_t)|, \qquad (85)
\end{aligned}
$$

by a weak version of Novikov's condition (16) for $\bar{\lambda}_1$ and $\bar{\lambda}_2$, the probability measure P^{c_1,c_2}

$$
\begin{aligned}
&P^{c_1,c_2}(A) \\
&= E\left[\exp\left(-\frac{1}{2}\sum_{j=1}^{2}\int_0^T c_{j,s}^2 ds + \sum_{j=1}^{2}\int_0^T c_{j,s} dB_{j,s}\right) 1_A\right], \\
&A \in \mathcal{F}, \qquad (86)
\end{aligned}
$$

is well-defined and Girsanov's theorem is applied.

Then,

$$
d\bar{Y}_t = -b_t \bar{Y}_t dt + \sum_{j=1}^{d} \bar{Z}_{j,t} dB_{j,t}^{c_1,c_2}, \ \bar{Y}_T = 0, \qquad (87)
$$

where $B^{c_1,c_2} = (B_1^{c_1,c_2}, \ldots, B_d^{c_1,c_2})$ define by

$$
\begin{aligned}
B_{1,t}^{c_1,c_2} &= B_{1,t} - \int_0^t c_{1,s} ds, \\
B_{2,t}^{c_1,c_2} &= B_{2,t} - \int_0^t c_{2,s} ds, \\
B_{j,t}^{c_1,c_2} &= B_{j,t} \ (3 \leq j \leq d). \qquad (88)
\end{aligned}
$$

is a d-dimensional Brownian motion under P^{c_1,c_2}.

The rest of the proof is the same as the one for Proposition 1. $\square$