CARF Working Paper

CARF-F-510

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March 29, 2021

CARF is presently supported by The Dai-ichi Life Insurance Company, Limited, Nomura Holdings, Inc., Sumitomo Mitsui Banking Corporation, Mizuho Financial Group, Inc., MUFG Bank, Ltd., The Norinchukin Bank and The University of Tokyo Edge Capital Partners Co., Ltd. This financial support enables us to issue CARF Working Papers.

Repricing Avalanches*

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Abstract

We present a menu-cost pricing model with a large but finite number n of firms. A firm's nominal price increase lowers other firms' relative prices, thereby inducing further nominal price increases. The distribution of these *repricing avalanches* converges as $n \to \infty$ to a mixture of Generalized Poisson Distributions (GPD) with an index of dispersion $= \frac{1}{(1-\theta)^2}$, where θ is determined by the equilibrium of the continuous limit. We calibrate the model to the U.S. experience during 1988–2005 and obtain a θ surprisingly close to unity. Simulations show that a GPD fits well the distribution of avalanches but that, once we account for the dynamics, the multiplier effect derived from a firm adjusting prices by paying menu costs is even larger. We also show that the model can account for the positive relationship between inflation level and volatility that was observed in 1988–2005 in the U.S.

*We thank Fernando Alvarez, Jean-Philippe Bouchaud, Mike Golosov, Lars Hansen, Simon Mongey, Harald Uhlig, Mike Woodford, and seminar participants at Columbia's Economics Faculty Lunch Seminar, the BUMP working group at The University of Chicago, the CFM Chair of EconophysiX at École Polytechnique, Bank of Japan, Hitotsubashi University, SED, SAET, Econometric Society World Congress, and the Money & Banking Workshop at The University of Chicago for comments. We also thank Haaris Mateen for excellent research assistance. Part of this research was done during visits by JAS to the Center for Advanced Research in Finance at the University of Tokyo and by MN to Columbia University. MN acknowledges financial support from KAKENHI 18H00831. Portions of this paper were previously circulated under the titles "Self-organization of inflation volatility" and "Menu costs and the volatility of inflation."

1 Introduction

Monthly aggregate price changes exhibit chronic fluctuations but the aggregate shocks that drive these fluctuations are elusive. Macroeconomic models often add stochastic macro shocks such as technology shocks or monetary policy shocks to produce aggregate fluctuations but empirical counterparts to these macro-level shocks have been identified largely as residuals. Direct evidences for exogenous shocks are usually found at more disaggregated levels.

In this paper, we show that a state-dependent pricing model, with a large but finite number of firms, is capable of generating large fluctuations in the number of firms that adjust prices in response to an idiosyncratic (Calvo) shock to the cost of price adjustment of a single firm. These fluctuations, in turn, cause fluctuations in aggregate price changes even in the absence of aggregate shocks.

In the model, firms choose prices considering competitors' prices and costs incurred when adjusting prices. A firm's upward repricing reduces all competitors' relative prices, inducing some of these competitors to also reprice. Thus pricing behavior exhibits complementarity in equilibrium. We first treat a stationary equilibrium of a model in a continuum of firms, as it is commonly done in the literature, and show that the strength of this propagation depends on a parameter θ, the ratio of the measure of firms that increase prices as a result of an increase in prices by a set of firms of measure m, as $m \to 0$. However, this complementarity does not generate any volatility of inflation, since in a stationary equilibrium the fraction of firms that adjust prices is constant.[1]

We will argue however that the continuum model is a bad approximation to examine the distribution of price changes when a large but finite number of firms are present. To avoid the proliferation of state variables in the model with a finite number of firms as the number of firms becomes large, we will assume that agents use the policy functions for the continuum model.[2] With a finite number of firms, the complementarity in price-changing creates *repricing avalanches*. We show that if we consider an i.i.d. sample of n firms from the stationary distribution of the continuum model, conditional on the (normalized) relative price charged by the firm that receives a Calvo shock, the distribution of the number of firms that change prices converges, as $n \to \infty$, to a *Generalized Poisson Distribution*[3], henceforth GPD, with an index of dispersion (Variance/Mean) that equals $\frac{1}{(1-\theta)^2}$. Under stronger parametric assumptions that are verified in our calibrated example, this convergence holds in $\mathcal{L}^1$, and

[1] For example, Caplin and Spulber [22], Dotsey et al. [29], and Golosov and Lucas [35] present a state-dependent pricing model with a continuum of firms similar to ours. The models in these papers share the same feature of complementarity of price setting in equilibrium. However, complementarity was not the main issue studied in their papers.

[2] This is in the same spirit as the strategy used by Krusell and Smith [42], or in set-ups similar to ours, by Caballero and Engel [20] and Alvarez and Lippi [4].

[3] A discrete value random variable has a GPD with parameters θ_0 and $\theta > 0$ if $P_x = \theta_0(\theta_0+\theta x)^{x-1}e^{-\theta_0-\theta x}/x!$ (*e.g.* Consul and Famoye [26, Chapter 9]).

as a consequence the index of dispersion of the limit GPD is a lower bound to the index of dispersion of the distribution of price change avalanches when n is large. Hence as $\theta \to 1$ the index of dispersion of the distribution of the number of firms that reprice $\to \infty$.

We calibrate our model so that the stationary equilibrium for the continuum model reproduces the average inflation observed in the US in 1988–2005, the observed volatility of inflation at this average inflation and targets from Nakamura and Steinsson [48] and Golosov and Lucas [35]. To accommodate price decreases, we introduce a generalization of the model that allows for productivity shocks, and possibly different menu costs to increase or decrease prices.[4] Using the theoretical formula for θ, we show that for the calibrated values of the parameters we obtain a θ surprisingly close to 1, even though inflation rates are modest. In addition, simulations of histories, using our calibrated values, produce a distribution of total daily avalanches that is well fitted by a GPD.[5] However the $\hat{\theta}$ implied by the fitted GPD is substantially larger than the θ obtained when we fit the stationary distribution to the target data, or equivalently, the index of dispersion implied by the equilibrium distribution of avalanches obtained from the calibration is substantially smaller than the index of dispersion in the distribution of simulated histories of avalanches. We argue that, in the simulations of the dynamics, the distribution of relative prices fluctuates around the stationary distribution and that this fluctuation of the distribution of relative prices generates the excess dispersion. Our simulation also shows that, using the calibrated values, we are able to reasonably fit the observed time series relationship between average inflation and volatility of inflation in the U.S., and in particular, the observed positive association between level and volatility of inflation. First observed by Okun [55], this positive correlation has been confirmed repeatedly, *e.g.*, by Judson and Orphanides [37] or Dincer and Eichengreen [28]. In addition, the positive association between volatility and trend inflation is used to justify central banks' choice of low inflation targets (see, *e.g.*, Fischer [30], Briault [17], and Billi and Kahn [13]).

The rest of the paper is organized as follows. Section 2 reviews the related literature. Section 3 builds a tractable state-dependent pricing model with linear production and a continuum of firms, and solves for the pricing policy and the stationary distribution of relative prices. We characterize rigorously the pricing policy and the stationary distribution using verification theorems from the theory of optimal impulse control (*e.g.* Øksendal and Sulem [54]). We then show that, for each constant rate of inflation, there exists a unique real-wage level that produces a candidate stationary equilibrium, and that this candidate is an actual equilibrium, provided menu costs are not excessive or that firms have a sufficiently large cost of quitting, or that the inflation rate is high enough.[6] Thus we establish, in particular,

[4] In the online appendix we show the existence and uniqueness of a stationary equilibrium and characterize the equilibrium stationary distribution for this generalized model.

[5] This helps to explain the good fit by a GPD for several distributions of daily price changes reported by the billion prices project (Cavallo and Rigobon [24]) that is documented in Leal et al. [43].

[6] Of course there is no reason to expect that menu costs are independent of the rate of inflation. As the rate of inflation becomes larger, economies often adopt strategies to minimize menu costs such as indexation

the existence of a unique stationary equilibrium for a state-dependent pricing model with endogenous real wage and without appealing to quadratic approximations.

We also derive several comparative statics results and, in particular, we show that for inflation rates that are sufficiently large, the equilibrium real wage rate declines when inflation increases. Thus our model provides a theoretical justification for the negative relation between real wage and inflation in high inflation economies that is documented by Braumann [16]. Section 3 also contains the construction of the contagion parameter θ. In Section 4 we consider a model with a finite number n of firms, where agents use the policy functions for the continuum model. We characterize the limit distribution of avalanches and establish the convergence results. In addition, since data on price changes are typically not available continuously, we provide a formula to decompose the dispersion of the number of avalanches in a finite interval into a component associated to the fluctuations in the number of Calvo shocks and another component associated to the fluctuations on the size of the avalanches that follow a Calvo shock. This decomposition allows one to evaluate the importance of fluctuations on the size of avalanches in determining the number of price changes at the available data frequency. Section 5 contains the calibrated example. Section 6 concludes.

2 Related Literature

A recent literature demonstrates that the micro shocks may produce macro shocks in an economy in which some firms or industries are very large. Gabaix [32], an early contributor, dubbed this mechanism the "granular hypothesis." Acemoglu et al. [1] show that input-output relationships can produce industries that have a very large influence in the overall economy. The granular hypothesis implies that the origin of aggregate shocks should be identifiable as a set of micro shocks on large firms or key industries.[7] Nonetheless, estimates by Gabaix [32] or Carvalho and Grassi [23] leave the majority of aggregate shocks still unexplained.

Central banks in developed economies have commonly targeted annual inflation rates around 2%; even researchers who advocate higher inflation targets do not suggest inflation rates above 4% (*e.g.* Ball [10]), because the higher inflation rate would bring high social costs. In addition to the redistribution effects of inflation, greater dispersion of relative prices and greater volatility or uncertainty of inflation rates are commonly cited as cost of high inflation rates (*e.g.* Fischer and Modigliani [31]). The connection between the relative price dispersion and inflation is well established and its welfare implications are extensively discussed (see, *e.g.*, Golosov and Lucas [35]; Burstein and Hellwig [18]; Nakamura and Steinsson [49]). In

or dollarization (see *e.g.* Shiller [60]).

[7] In a recent paper, Moran and Bouchaud [47] examine the effect of replacing the Cobb-Douglas production function in [1] by a CES with "small" degree of substitutability, while not assuming the existence of "key industries." They show that near critical parameter values these networks generate power-tailed size distributions of firms and that small idiosyncratic shocks cause aggregate output losses that obey a power law distribution.

contrast, the positive association between inflation level and volatility has been less explored in state-dependent pricing models.[8]

The impact of trend inflation and its implication for volatility have been investigated mostly in time-dependent pricing models (see Ascari and Sbordone [6] for a survey). However, the lack of microfoundation for repricing frequency in the time-dependent models has been a serious limitation for policy analysis. The state-dependent pricing model, which provides a microfoundation, has been beyond analytical analysis. The model loses tractability quickly, unless one employs linear-quadratic approximation or exogenous real wage.

Our state-dependent pricing model is similar to Dotsey et al. [29], Golosov and Lucas [35], and Gertler and Leahy [34], which rely heavily on numerical results. These models, as well as other state-dependent pricing models (Sheshinski and Weiss [59]; Caplin and Spulber [22]; Bénabou and Konieczny [12]; Caplin and Leahy [21]; Danziger [27]; Ahlin and Shintani [2]), and the broader literature on (S,s) economies (Caballero and Engel [19]; Khan and Thomas [39]; Stokey [63]), use a continuum-of-firms setup and do not examine the stochastic nature of firms' interactions that we emphasize. Proposition 7 in Alvarez and Lippi [4] justifies the simplifying assumption of Caballero and Engel [20] of using steady state decision rules in the presence of stochastic deviations from the steady state, an assumption that is similar in nature to our assumption that, in the economy with n firms, agents use the policy function of the continuum model.

The state-dependent pricing models motivated empirical analyses on the intensive and extensive margins of price adjustments with micro-level data including Klenow and Kryvtsov [41] and Nakamura and Steinsson [48]. Midrigan [45] extends the menu-costs model to a multiproduct setting in which repricing exhibits economies of scope, and shows that this extension predicts larger real effects of monetary policy shocks than Golosov and Lucas [35].[9] Gagnon [33] documents that a higher level of inflation is associated with a higher frequency of price increases—a key mechanism to generate large endogenous price fluctuations in our model when the number of firms is finite. Alvarez et al. [3] review and extend several comparative statics results in a state dependent model in the Golosov-Lucas style, and confront them to Argentinian data. In particular, they report that the variance of inflation during high inflation periods is explained more by the extensive margin than the intensive margin, which is consistent with our model's prediction.

[8] Earlier work has focused on the possibility that monetary authority chooses the combination of high-level, high-volatility inflation (see Ball [8] and references therein). A contrasting New Keynesian view was provided by Ball et al. [9]. In their model, a high level of inflation induces frequent repricing and more flexible aggregate prices, and exogenous demand shocks generate large volatility of aggregate price and small volatility of output. Hence high inflation would have a positive impact on social welfare.

[9] Although much of the literature on menu costs assumes, as we do, the presence of competitive or monopolistically competitive sectors, models with other forms of strategic interactions that generate complementarity have also been developed and tested (*e.g.* Slade [62]; Nakamura and Zerom [50]; Neiman [51]; Kano [38]; Mongey [46]).

3 Model with a continuum of firms

In this section we exposit a dynamic continuous-time general equilibrium model with a continuum of firms that will underlie our analysis of the case with a finite number of firms. We consider a continuum of intermediate goods indexed by $i \in [0, 1]$ and a competitive sector of final goods producers with constant returns to scale technology such that aggregate output of final goods is:

$$Y_t = \left(\int_0^1 (y_t^i)^{(\eta-1)/\eta} di \right)^{\eta/(\eta-1)}, \tag{1}$$

where y_t^i is the amount of intermediate good i employed by the final goods production sector and $\eta > 1$ is the substitution elasticity. Given prices P for the final good and P^i for intermediate good i, $i \in [0, 1]$, the demand of the final goods sector for good i satisfies:[10]

$$y_t^i = (p_t^i)^{-\eta} Y_t, \tag{2}$$

where $p^i = P^i/P$. Competition and constant returns to scale in the final-goods industry insures that

$$1 = \int_0^1 (p_t^i)^{1-\eta} di. \tag{3}$$

Alternatively we may think each good i as a final good with a nominal price P_t^i, Y_t as the consumption-basket and P_t as the CPI. Equation (3) then defines the CPI.

Each firm i produces output with a linear technology with labor being the single input,[11]

$$y_t^i = l_t^i. \tag{4}$$

Firm i fixes a price P_t^i at time t and must satisfy demand, that is hire a labor force:

$$l_t^i = (p_t^i)^{-\eta} Y_t. \tag{5}$$

The net income that firm i obtains from production when it sells output at price p^i, wages are W and final good output is Y is thus, measured in consumption-goods equivalent,

$$Z(p^i, w, Y) := p^i y^i - w l^i = ((p^i)^{1-\eta} - w(p^i)^{-\eta})Y, \tag{6}$$

where $w = W/P$.

[10] Formally this demand equation needs only to be satisfied for a set of full measure. We will not explicitly distinguish between equalities or inequalities that hold for almost all i versus for all i.

[11] In Section 5 and the online appendix we consider a model in which each firm i has two possible levels of productivity t, $a_t^i \in \{a^L, a^H\}$, with $a^L < a^H$.

The specification of demand in equation (2), widely used in this literature, implies, as shown in equation (6), that a firm's profits are fully determined by its own price and productivity, and aggregate variables. In particular, a firm is equally affected by the price choice of any other firm.

Inflation is measured by the price of the final good that is:

$$\pi := \frac{dP_t}{P_t}. \tag{7}$$

If P_t^i is constant, the relative price of firm i would fall at the rate $-\pi$, or $dp_t^i = -\pi p_t^i dt$.

Price changes require payment of a menu cost which at t equals $\delta^+ Y_t$ units of the final consumption good if the price change is positive, and $\delta^- Y_t$ if the price change is negative. The proportionality of menu costs to final-good output allows us to reduce the number of state variables in the firm's optimization problem. In a pioneering paper in this literature, Sheshinski and Weiss [59] refer to "costs associated with the transmission of price information to the consumers and with the decision process itself," which suggests $\delta^+ = \delta^-$. The earlier literature on menu costs emphasizes the costs of transmitting price information, while some of the more recent literature on price adjustments (*e.g.* Mankiw and Reis [44]; Woodford [65]; Alvarez et al. [5]) concentrates on the cost of acquiring information for a price-change decision. In all cases, buyers are presumed to be fully informed on all prices, thus ignoring search motivations of buyers, and to have a purely cognitive reaction to price changes, represented by a static demand function. Rotemberg [57] argues that consumers also have emotional reactions to price changes. In fact, in Blinder et al. [15] survey 21% of US executives polled answered that they did infrequent price changes because "price changes would antagonize customers" and only 14% mentioned "costs associated with changes in prices." Once one considers menu costs in a less literal manner and includes negative reaction from buyers, it is natural to consider also the case when $\delta^+ > \delta^-$. Thus we will assume $\delta^+ \geq \delta^-$. In fact, as we will show later, when $\pi > 0$, δ^- has a minor role in determining the stationary equilibrium in the model of a continuum of firms. We introduce the possibility of asymmetry in menu costs because it will play a role in the case of a finite but large number of firms.

Firms are also subject to a Calvo shock that is determined by a Poisson process with rate μ. When a Poisson time arrives, the firm is allowed to change prices at a zero-cost. We assume that the Poisson shocks are independent across firms.[12]

The representative household has utility function,

$$E\left[\int_0^\infty e^{-\rho t}\left(U(c_t, N_t) + \iota\left(\frac{M_t}{P_t}\right)\right)dt\right], \tag{8}$$

[12] Although this independence is generally assumed in economics, there is no mathematically sound way to construct this continuum of independent processes. However, as Sun [64] showed, one can rigorously construct a continuum of processes such that both the Fubini property and an exact law of large numbers that eliminates idiosyncratic randomness hold. This is all that is needed in what follows.

where c_t is the amount consumed at t, N_t the amount of labor supplied by the household and M_t is the nominal balance held by the household. We assume that the utility function is strongly concave on c and convex in N, that U satisfies Inada conditions and that $\partial^2 U(c,N)/\partial c \partial N \leq 0$ to ensure the existence and uniqueness of a steady state equilibrium. In addition, ι is increasing and strictly concave. The representative household owns all firms and receives a process $\mathcal{D}_t$ with $\mathcal{D}_0 = 0$ corresponding to the cumulative profits generated by the set of intermediate-goods firms. For simplicity we assume that firms' shares are non-tradable but it will become clear that one could calculate prices for shares that would yield no trading in equilibrium. Thus writing $m_t = M_t/P_t$, the household's budget constraint is:

$$dm_t = w_t N_t dt + d\mathcal{D}_t + (R_t - \pi_t) m_t dt + d\mathcal{T}_t - c_t dt \tag{9}$$

where R_t is the nominal interest rate paid on money holdings and $\mathcal{T}_t$ are cumulative real lump-sum transfers given by the monetary authorities to households.

3.1 Stationary equilibrium: definition

In a stationary equilibrium, prices grow at a constant rate $\pi \geq 0$,[13] which equals the rate of growth of the money supply M_t. The monetary authorities also choose a constant nominal rate of interest on cash balances R and make the real lump-sum transfers necessary to balance its budget constraint, that is:

$$d\mathcal{T}_t = (\pi - R) m dt. \tag{10}$$

This equation also guarantees that $d\mathcal{T}_t/dt$ is well defined. In addition, in a stationary equilibrium, the distribution of relative prices is stationary and all real quantities—output $\bar{Y}$, consumption $\bar{c}$, real wages $\bar{w}$, employment $\bar{N}$, and the fraction of firms lowering (raising) prices and paying menu costs between t and $t + dt$, $\lambda^- dt$ ($\lambda^+ dt$) are constant. Hence aggregate dividends are also a constant. Thus consumer's optimization problem yields the HJB equation:

$$\rho \mathcal{V}(m) = \sup_{\{c,N\}} \left[U(c,N) + \iota(m) + \left(wN + D + (R - \pi)m + \frac{d\mathcal{T}_t}{dt} - c \right) \mathcal{V}'(m) \right]. \tag{11}$$

Thus if U satisfies Inada conditions,[14]

$$\mathcal{V}'(m) = U_c(c,N) = -\frac{U_N(c,N)}{w}$$

and in a stationary equilibrium,

$$\rho \mathcal{V}'(\bar{m}) = \iota'(\bar{m}) + (R - \pi)\mathcal{V}'(\bar{m})$$

[13] The case where $\pi < 0$ can be examined symmetrically.

[14] If U is separable and linear in N as in Golosov and Lucas [35], Inada conditions with respect to c suffice.

which requires $\rho > R - \pi$. In particular, in a stationary equilibrium

$$\iota'(\bar{m}) = (\rho - R + \pi)U_c(\bar{c}, \bar{N}) = (R - \rho - \pi)\frac{U_N(\bar{c}, \bar{N})}{\bar{w}}.$$

In a stationary equilibrium where the rate of inflation is π and $\bar{c} > 0$, if F denotes the stationary probability distribution of prices, we must have:

$$\pi = \frac{dP_t}{P_t} \qquad \text{(stationary inflation)} \tag{12}$$

$$1 = E^F[p^{1-\eta}] \qquad \text{(zero-profit final good)} \tag{13}$$

$$\bar{N} = E^F[p^{-\eta}\bar{Y}] \qquad \text{(labor market clearing condition)} \tag{14}$$

$$\bar{c} = \bar{Y}(1 - \delta^+\lambda^+ - \delta^-\lambda^-) \qquad \text{(consumption equals output minus menu costs)} \tag{15}$$

$$\bar{w} = -\frac{u_2(\bar{c}, \bar{N})}{u_1(\bar{c}, \bar{N})}. \qquad \text{(intra-temporal optimal choice)} \tag{16}$$

In addition, if firms optimize the present value of profits, taking $\bar{w}$, $\bar{Y}$ and the discount rate ρ as fixed, and the initial joint distribution of firms' prices is F, this distribution is preserved and λ^+ (λ^-) firms increase (decrease) prices and pay menu costs per unit of time.

3.2 Stationary equilibrium: firm's optimization problem

Stationarity of equilibrium and the Poisson character of the Calvo shocks guarantee that the problem a firm faces is stationary. We will examine a symmetric equilibrium where every firm makes the same choices conditional on their current price. The firm must decide a stopping time T' for paying the menu costs of a price change, conditional on not having a Calvo shock before T', and the new relative price level. It must also decide on the price it will charge if it receives a Calvo shock. Given the stationarity of the problem and the fixed cost of repricing, the optimal price to which a firm will move, p', should be independent of the reason for repricing or the price just before the repricing. In a stationary equilibrium any firm that charges prices p has net real income from production, normalized by final goods output $\bar{Y}$:

$$z(p) := \frac{Z(p, \bar{w}, \bar{Y})}{\bar{Y}} = p^{1-\eta} - \bar{w}p^{-\eta}.$$

The function $z(\cdot)$ has a unique maximum at $\hat{p} := \bar{w}(\eta/(\eta - 1))$ and it is increasing for $p < \hat{p}$ and decreasing for $p > \hat{p}$. Furthermore $z(\hat{p}) > 0$.

Write $v(p) = V(p)/\bar{Y}$, where $V(p)$ is the expected maximum discounted profits for a firm that charges p at time zero.

Suppose that in a stationary equilibrium at some random time τ a Calvo shock arrives. Then the firm should choose $p_\tau = \arg\max v(p)$. Suppose now that at a random time T' the firm decides to pay menu costs. If the firm chooses to increase price then it faces no restriction and would always choose $p_{T'} = \arg\max v(p)$, independent of the current price. On the other hand if it chooses to lower prices it would choose $p_{T'} = \arg\max_{\{p \le p_{T'-}\}} v(p)$. Since $\delta^- > 0$ the optimal choice of $p_{T'}$, if it exists, is interior in this stationary equilibrium. Thus

$$v(p) = \sup_{p', T', p'' \le p_{T'-}} E_0 \Bigg\{ \int_0^{T' \wedge \tau} e^{-\rho t} z(p_t) dt + \mathbf{1}_{\tau < T'} e^{-\rho \tau} v(p') \\ + \mathbf{1}_{T' < \tau} e^{-\rho T'} \max\{v(p') - \delta^+, v(p'') - \delta^-\} \Bigg\}. \tag{17}$$

This is an *impulse control* problem as in *e.g.* Øksendal and Sulem [54]. By slightly abusing notation we will refer to v as the value function. We will rigorously prove the existence of a stationary equilibrium by first showing that conditional on the real wage rate w we can construct the value function $v(p)$, the associated optimal policy, a unique stationary distribution $F(p)$ and the associated constant rate of firms that pay menu costs in order to raise prices, $\lambda^+(w)$. We will also show that no firms with prices in the support of F pay menu costs to lower prices, that is $\lambda^-(w) = 0$. Proposition 2 provides sufficient conditions on the pair (π, δ^+) so that there exists a unique wage rate w such that $v(p) \ge 0$ in the support of F, and (w, F, λ, c, N, Y) solves (13)–(16), that is, a stationary equilibrium obtains.

A standard argument (*e.g.* Shiryaev [61, Theorem 21, p.91]) shows that we can restrict ourselves to feedback controls, that is stopping times T' such that $T' = 0$ if and only if $p_0 \in \mathcal{S} := \mathcal{S}^1 \cup \mathcal{S}^2$, where $\mathcal{S}^1 := \{p \in (0, \infty) : v(p) = \sup_{p'} v(p') - \delta^+\}$ and $\mathcal{S}^2 = \{p \in (0, \infty) : v(p) = \sup_{p' \le p} v(p') - \delta^-\}$. Taking limits in the dynamic programming equation (17) we obtain:

$$z(p) - \rho v(p) - \pi p v'(p) + \mu[\sup_{p'} v(p') - v(p)] \le 0. \tag{18}$$

The left-hand side includes the profit at time zero and the expected drift of $e^{-\rho t} v(p_t, a)$ taking into consideration the fact that, if a Calvo shock occurs, the value would jump to $\sup_{p'} v(p')$, and the rate μ associated to the Poisson process of Calvo shocks. In addition, since stopping immediately is always a choice $v(p) \ge \max\{v(p') - \delta^+, v(p'') - \delta^-\}$. Notice that if $p \notin \mathcal{S}$ then necessarily

$$z(p) - \rho v(p) - \pi p v'(p) + \mu[\sup_{p'} v(p') - v(p)] = 0, \tag{19}$$

so the function v satisfies the *quasi-variational inequality*:

$$\max \left\{ \begin{array}{c} z(p) - \rho v(p) - \pi p v'(p) + \mu[\sup_{p'} v(p') - v(p)]; \\ \max\{\sup_{p'} v(p) - \delta^+, \sup_{p'' \le p} v(p'') - \delta^-\} - v(p) \end{array} \right\} = 0. \tag{20}$$

To characterize the value function, we start by showing that there exists a unique solution satisfying appropriate boundary conditions to equation (19), and construct a candidate value function using this solution. We then apply the procedure in Øksendal and Sulem [54, Chapter 9],[15] to show that the candidate is actually the value function.

Lemma 1. *Suppose that $R(\cdot)$ solves*

$$z(p) - \rho R(p) - \pi p R'(p) + \mu \left(\sup_{p' \geq \hat{p}} R(p') - R(p) \right) = 0. \tag{21}$$

Then:

(a) For each $\delta^+ > 0$ there exists at most one pair $(\underline{p}, p^*)$, with $\underline{p} \leq \hat{p} \leq p^*$ with $p^* > \underline{p}$ that solves,

$$R'(\underline{p}) = R'(p^*) = 0 \tag{22}$$

$$R(\underline{p}) = R(p^*) - \delta^+. \tag{23}$$

(b) If $\underline{p}$ (p^*) exists it is a global minimum of $R(p)$ in $(0, \hat{p})$ (maximum in $(\hat{p}, \infty)$ resp.).

(c) Consequently, if p^* exists,

$$R(p^*) = \frac{z(p^*)}{\rho}, \tag{24}$$

and if $\underline{p}$ and p^* exist,

$$\delta^+ = R(p^*) - R(\underline{p}) = \frac{z(p^*) - z(\underline{p})}{\mu + \rho}. \tag{25}$$

Proof. First note that if R solves (21) then:

$$z'(p) - (\pi + \mu + \rho) R'(p) - \pi p R''(p) = 0. \tag{26}$$

Thus whenever $R'(p) = 0$ and $p < \hat{p}$, $R''(p) > 0$. Hence there is at most one critical point in $(0, \hat{p})$ and it is a global minimum in $(0, \hat{p})$. Similarly for $p \in (\hat{p}, \infty)$ there is at most one critical point and it is a global maximum in $(\hat{p}, \infty)$. (c) is obvious. □

The proof of the next proposition, which characterizes the value and policy functions, is in the appendix A.1.

Proposition 1. *Fix $w > 0$ and $\delta^+ > 0$. Then*

(a) There exist a unique function $R(p)$ and points $\underline{p}$ and p^* (which depend on (w, δ^+)) that satisfy equations (21)–(23).

[15] A major difference between our set up and the one in [54] is the absence of a diffusion term, which allows for less smoothness of the candidate value function.

(b) The following equation holds:

$$w = \frac{\eta - 1}{\eta} \frac{\varphi(p^*/\underline{p}, (\mu+\rho)/\pi + 1 - \eta)}{\varphi(p^*/\underline{p}, (\mu+\rho)/\pi - \eta)} \underline{p} \tag{27}$$

with

$$\varphi(q, x) := (q^x - 1)/x.$$

(c) $v(p)$ given by:

$$v(p) = \begin{cases} R(p^*) - \delta^+ & \text{for } p < \underline{p} \\ R(p) & \text{for } p \in [\underline{p}, p^*] \\ \max\{R(p), R(p^*) - \delta^-\} & \text{for } p > p^* \end{cases}$$

is the value function.

(d) The optimal policy is: If $p_{t\text{-}} \in \mathcal{S}^1 = \{p : v(p) = R(p^*) - \delta^+\}$, then pay the menu cost (δ^+) and set $p_t = p^*$, and if $p_{t\text{-}} \in \mathcal{S}^2 = \{p : v(p) = R(p^*) - \delta^-\}$, then pay the menu cost (δ^-) and set $p_t = p^*$. Otherwise, unless you receive a Calvo shock, do nothing. If a Calvo shock arrives, move to p^*.

The characterization of the optimal policy in Proposition 1 implies that $[\underline{p}, p^*]$ is an absorbing set. In fact if:

$$p^\dagger := \inf\{p : p \in \mathcal{S}^2\},$$

and $p_0^i \leq \underline{p}$ or $p_0^i \geq p^\dagger$ then the firm immediately pays the applicable menu cost and moves to $p = p^*$. If $p_0^i \in (p^*, p^\dagger)$ then the firm waits for the price to drift to p^*. Furthermore, if $p_t^i \in (\underline{p}, p^*]$ then:

$$dp_t^i = -\pi p_t^i dt + \mu(p^* - p_t^i)dt + \mathcal{M}_t,$$

where $\mathcal{M}_t$ is a martingale.

In addition, if $h(p, t)$ denotes the time t density of prices for firms, then h satisfies the *forward equation*:

$$\frac{\partial}{\partial t} h(p, t) = \frac{\partial}{\partial p}[\pi p h(p, t)] - \mu h(p, t). \tag{28}$$

Corollary 1. *(a) In a stationary equilibrium, the distribution of firm's prices has support $[\underline{p}, p^*]$.*

(b) The stationary distribution of prices has density:

$$f(p) = \frac{p^{\mu/\pi - 1}}{\underline{p}^{\mu/\pi} \varphi(p^*/\underline{p}, \mu/\pi)}. \tag{29}$$

(c) *In the stationary equilibrium firms choose to pay menu costs only when increasing prices ($\lambda = \lambda^+$). The rate of firms paying menu costs, λ, satisfies:*

$$\lambda = \frac{\mu}{(p^*/\underline{p})^{\mu/\pi} - 1}. \tag{30}$$

Proof. Equation (29) follows from (28) and Item (a). Moreover, we obtain (30) from $\lambda = f(\underline{p})\pi\underline{p} = \pi/\varphi(p^*/\underline{p}, \mu/\pi) = \mu/((p^*/\underline{p})^{\mu/\pi} - 1)$. □

Notice that the stationary distribution does not depend on δ^- and neither does the value function for p's in the support of the stationary distribution. The parameter δ^- only affects the value function $v(p)$ for $p > p^*$. This is because in a stationary equilibrium, the rate of change of the price of the final good is always a constant $\pi \geq 0$.

Remark 1. *We have assumed that firms do not choose to exit the market at any p. This assumption can be justified if we assume that firms must pay an exit cost $c_e > \delta^+$. Note that for a given inflation rate π and any $\epsilon > 0$, a firm can choose to move to a price p^ϵ high enough so that expected discounted profits starting at p^ϵ exceed $-\epsilon$. Choosing $\epsilon < c_e - \delta^+$ this strategy yields higher profits than exiting.*

3.3 The equilibrium wage rate

In the previous section we showed that given w we can find a vector $(\underline{p}_w, p^*_w, f_w(\cdot), \lambda_w)$ that characterizes the behavior of the intermediate goods firms. However, a necessary condition for equilibrium is that the average price of intermediate goods must be one, that is, equation (13) must hold. In this section, we show that given the parameters of the model $(\eta, \rho, \mu, \delta^+)$ and a rate of inflation π there is exactly one w such that (13) holds. In addition, we show that given this candidate equilibrium value of w we can choose uniquely a vector (m, c, N, Y) such that $(w, \underline{p}_w, p^*_w, f_w, \lambda_w, m, c, N, Y)$ solve equations (13)–(16) that characterize a stationary equilibrium. In what follows, it is useful to introduce the notation:

$$q := \frac{p^*}{\underline{p}} > 1.$$

Thus q denotes the proportional increase in relative price when a firm pays the menu cost. With this notation, we may rewrite the zero-profit condition for the competitive firms, equation (13), as:

$$1 = \int p^{1-\eta} f(p)dp = \frac{\varphi(q, \mu/\pi + 1 - \eta)}{\varphi(q, \mu/\pi)} \underline{p}^{1-\eta} \tag{31}$$

and rewrite the value-matching condition (25) as:

$$(\rho + \mu)\delta^+ = \left((q^{1-\eta} - 1) - \frac{w}{\underline{p}}(q^{-\eta} - 1) \right) \underline{p}^{1-\eta}. \tag{32}$$

The inverse of the "minimum markup" $\underline{p}/w$ satisfies (27), that is,

$$\frac{w}{\underline{p}} = \frac{\eta - 1}{\eta} \frac{\varphi(q, (\mu + \rho)/\pi + 1 - \eta)}{\varphi(q, (\mu + \rho)/\pi - \eta)}. \tag{33}$$

Thus, equations (31)–(33) determine three unknowns $(w, \underline{p}, q)$. By substituting (33) into (32), and substituting out $\underline{p}^{1-\eta}$ from (31), we obtain:

$$\frac{\eta - 1}{(\mu + \rho)\delta^+} = \frac{\varphi(q, \mu/\pi + 1 - \eta)}{\varphi(q, \mu/\pi)\varphi(q, 1 - \eta)} \left(\frac{\varphi(q, (\rho + \mu)/\pi + 1 - \eta)}{\varphi(q, (\rho + \mu)/\pi - \eta)} \frac{\varphi(q, -\eta)}{\varphi(q, 1 - \eta)} - 1 \right)^{-1}. \tag{34}$$

Lemma 2. *There exists a unique positive vector $(q, \underline{p}, w) \in \mathbf{R}^3$ with $q > 1$ that solves (31)–(34). If λ given by (30) satisfies $\lambda\delta^+ < 1$, then there exists a unique positive vector $(\bar{c}, \bar{N}, \bar{Y})$ satisfying equations (14)–(16).*

Proof: See Appendix A.3.

The next lemma, an application of Fubini's Theorem, will be useful to produce sufficient conditions for $\lambda\delta^+ < 1$ to hold.

Lemma 3. *If $(q, \underline{p}, w)$, with $q > 1$ solves (31)–(34) and*

$$E^F v(p) > 0,$$

then $\lambda\delta^+ < 1$.

Proof: See Appendix A.4.

The following proposition states sufficient conditions for $\lambda\delta^+ < 1$ to hold, and hence for existence of a (stationary) equilibrium.

Proposition 2. *(a) For any $\bar{\pi} > 0$ there exists a $\bar{\delta}$ such that for any $\pi \leq \bar{\pi}$ and $\delta^+ \leq \bar{\delta}$, $v(p) \geq 0$. As a consequence, $\lambda\delta^+ < 1$ and thus there exists a stationary equilibrium $(\underline{p}, p^*, w, \lambda, c, N, Y)$.*

(b) For any δ^+, if π is large enough then $\lambda\delta^+ < 1$, and thus there exists a stationary equilibrium $(\underline{p}, p^*, w, \lambda, c, N, Y)$.

Proof: See Appendix A.6.

Proposition 2 establishes that an equilibrium exists provided the rate of inflation and the cost of price-changing are not too large. It also establishes that given any fixed cost of price adjustment, if inflation is high enough an equilibrium exists. Unfortunately the Proposition does not establish existence of an equilibrium for an arbitrary pair (δ^+, π).

In this model, the real wage is affected by the inflation rate through optimal markup behavior, even though the marginal product of labor is constant due to the linear production technology. We can show that the stationary equilibrium satisfies the following properties:

Proposition 3. *(a)* $\log q$ *grows asymptotically linearly in* π. *In particular, the optimal stopping time satisfies* $T^* = (\log q)/\pi \to (\log(1 + \delta^+\eta\mu))/\mu$ *as* $\pi \to \infty$. *Moreover,* $\lambda = \mu/(e^{\mu T^*} - 1) \to 1/(\delta^+\eta)$ *as* $\pi \to \infty$.

(b) p^* *increases unboundedly as* π *increases.*

(c) w *decreases as* π *increases for sufficiently large* π.

Proof: See Appendix A.7.

The repricing size increases in π as in Sheshinski and Weiss [59], and the intuition is clear: a higher inflation would cause more frequent repricing, inducing a firm to adjust its price by a greater intensive margin in order to gain more time until the next repricing. Item (a) establishes the sharper result that the size of repricing is asymptotically linear on inflation. Item (b) implies that the target price is increasing in the inflation π. Combined with the next claim, (c), this implies that, in the range of high inflation rates, a higher inflation induces firms to choose a higher markup when they move to p^*. Item (c) states that, when the inflation rate is sufficiently high, the real wage decreases as inflation increases. When the inflation is higher, the relative price dispersion and the resulting inefficiency loss in the production sector are larger. Thus, in an equilibrium of the good and labor markets, as the inflation rate increases, the real wage decreases. To our knowledge, this is a novel result in a literature where most analytical results are obtained with an exogenous real wage or using quadratic approximations.

Next, we consider the case when menu costs fall as inflation increases. For constants $\delta(0) > 0$ and $\sigma_\delta > 0$, let $\delta^+(\pi) = \delta_0 e^{-\sigma_\delta \pi}$. Proposition 4 states the existence of an equilibrium for any $\pi \geq 0$ when $\delta(0)$ is small and $\delta^+(\pi)$ decreases fast enough.

Proposition 4. *There exists a* $\bar{\delta}^+$ *such that for every* $\delta(0) < \bar{\delta}^+$ *there exists a* σ_0 *such that, for* $\sigma_\delta \geq \sigma_0$, *there exists a stationary equilibrium for the parameters* $(\delta^+(\pi), \pi)$ *for* $\pi \geq 0$.

Proof: See Appendix A.8.

3.4 A measure of complementarity

In this section we introduce measures of the complementarity of price-adjustments across firms. As we discuss below, these measures of complementarity play an important role in determining moments of aggregate fluctuations in an economy with a finite number of firms.

It is convenient to define a relative log price chosen by firms in the support of the stationary distribution,

$$s(p) := \frac{\log p - \log \underline{p}}{\log p^* - \log \underline{p}} \in [0, 1].$$

Conversely, we may define

$$p(s) := q^s \underline{p}.$$

Using (29) one obtains the stationary density of s,

$$g(s) = \frac{\log p^* - \log \underline{p}}{\varphi(p^*/\underline{p}, \mu/\pi)} \left(\frac{p^*}{\underline{p}}\right)^{\frac{\mu s}{\pi}} = \frac{\log q}{\varphi(q, \mu/\pi)} q^{(\mu/\pi)s}. \tag{35}$$

For $0 < s < 1$ and for each $\nu \leq \frac{1-s}{\log q}$, let

$$m_s(\nu) := \int_s^{s+\nu/\log q} g(\tau) d\tau,$$

that is, $m_s(\nu)$ is the measure, under the stationary distribution, of firms that charge prices between $p(s)$ and $e^\nu p(s)$.

Suppose we perform the following thought experiment. All firms i with $s \leq s^i \leq s + \nu/\log q$ reprice (set $p = p^*$). This increases the price level P to a new level P' that can be computed using the zero-profit condition (13) as:

$$\frac{P'}{P} = \left(\int_0^{p(s)} p^{1-\eta} f(p) dp + (p^*)^{1-\eta} m_s(\nu) + \int_{p(s)e^\nu}^{p^*} p^{1-\eta} f(p) dp\right)^{1/(1-\eta)} > 1.$$

Let

$$\nu'_s(\nu) = \log(P'/P).$$

If firms expect that the equilibrium stationary dynamics prevails, and ν is small enough, all firms with original prices p such that $\log p \leq \log \underline{p} + \nu'_s(\nu)$ would then raise their price to p^*. Thus if ν is small enough the measure of firms that respond to the first round of repricing is:

$$m'_s(\nu) := \int_0^{\nu'_s(\nu)/\log q} g(\tau) d\tau.$$

Let

$$\theta_0(s) := \lim_{\nu \to 0} \frac{m'_s(\nu)}{m_s(\nu)}.$$

For each s we can think of $\theta_0(s)$, as the density per unit of time of the number of firms that adjust their prices if the firm at s changes its price, presumably because this firm received a Calvo shock. In Appendix A.9 we show that

$$\theta_0(s) = \frac{q^{1-\eta} - q^{(1-\eta)s}}{(1-\eta)\varphi(q, 1-\eta+\mu/\pi)}, \tag{36}$$

which decreases with s. In Appendix A.9 we show that the average $\theta_0(s)$, θ_0, satisfies:

$$\theta_0 := \int_0^1 \theta_0(s) g(s) ds = \frac{\varphi(p^*, 1-\eta)}{\varphi(q, \mu/\pi)}. \tag{37}$$

Of particular interest is

$$\theta := \lim_{s\to 0} \theta_0(s) = \frac{\varphi(q, 1-\eta)}{\varphi(q, 1-\eta+\mu/\pi)}. \tag{38}$$

Whereas a price change for a firm with $s > 0$ must be the result of a Calvo shock, in the stationary equilibrium there is a constant positive density of firms repricing, because they reach $s = 0$ and choose to pay menu costs. The parameter θ is similar to the rate of reproduction in epidemiological models and is a measure of complementarity of menu-cost payments that we observe in equilibrium. It follows from equation (31) that $\theta < 1$, since, by Lemma 4(b) in Appendix A.2, $\varphi(q, x)$ is increasing on x if $q > 1$. Thus the payment of menu costs has a (finite) multiplier effect

$$\frac{1}{1-\theta}.$$

In Appendix A.9 we show that

$$\pi = \frac{\mu}{1-\theta}\varphi(p^*, 1-\eta), \tag{39}$$

which relates the rate of inflation to the multiplier effect. Equation (39) shows that $\theta \to 1$ as $\pi \to \infty$, since $\varphi(p^*, 1-\eta) < 1/(\eta - 1)$. It also implies $1/(1-\theta) > (\eta-1)\pi/\mu$, that is, the multiplier effect is bounded below by a linearly increasing function of π. In Appendix A.10 we also show that $\lim_{\pi\to 0} \theta(\pi) = 0$ and that there exists a π' such that $\theta(\pi)$ is non-decreasing for $\pi \geq \pi'$.

Finally, combining (30), (37), and (39) yields

$$\lambda = \frac{\mu\theta_0}{1-\theta}. \tag{40}$$

This equation shows that the measure of firms that pay menu costs at each instant is the product of three terms: the rate of Calvo shocks μ, the direct effect of the Calvo-hit firms on the firms at the extensive margin θ_0, and the multiplier effect $1/(1-\theta)$. In particular, the complementarity measure θ produces a multiplier effect on the constant rate of menu-cost payment λ. In the next section, we argue that complementarity generates not only the mean multiplier effect but also *volatility* in aggregate price when n is large but finite.

4 The model with a finite number of firms

4.1 Convergence to the continuum model

We now consider a model with a finite number of firms n, that approximates the continuum model as $n \to \infty$. To accomplish this, we will choose each firm $i = 1, 2, \ldots, n$ to have measure

$1/n$ and consider $\tilde{y}^i$ the *intensity* of output of firm i, that is $\tilde{y}^i = ny^i$, where y^i is the output of firm i. Similarly, the intensity of labor input is denoted by $\tilde{l}^i$. We assume that output of the final good is given by:

$$Y^{(n)} = \left(\sum_{i=1}^{n} (\tilde{y}^i)^{(\eta-1)/\eta}/n \right)^{\eta/(\eta-1)}. \tag{41}$$

If $\tilde{y}(k)$ is a function of bounded variation and we set $\tilde{y}^i = \tilde{y}(j)$ for some $j \in [\frac{i-1}{n}, \frac{i}{n})$, $i = 1, 2, \ldots, n$ then as $n \to \infty$,

$$Y^{(n)} \to \left(\int_0^1 \tilde{y}(k)^{(\eta-1)/\eta} dk \right)^{\eta/(\eta-1)},$$

the postulated production of final goods in the model with a continuum of firms, when final goods producers use a total amount $\tilde{y}(k)$ of the input produced by firm k.

From now on, we drop the superscript (n) except if needed for clarity. Equation (41) can be rewritten as:

$$Y = \left(\sum_{i=1}^{n} (y^i)^{(\eta-1)/\eta} n^{-1/\eta} \right)^{\eta/(\eta-1)}. \tag{42}$$

This aggregator function with the normalization factor $n^{-1/\eta}$ is used by, for example, Blanchard and Kiyotaki [14].

A final good producer's nominal profit is

$$\max_{Y, \{y^i\}} PY - \sum_{i=1}^{n} P^i y^i. \tag{43}$$

The first-order condition with respect to y^i yields a factor demand function

$$y^i = \left(\frac{P^i}{P} \right)^{-\eta} Y/n = (p^i)^{-\eta} Y/n \tag{44}$$

where $p^i := P^i/P$ denotes the relative price of good i. Aggregating across i, we obtain the competitive price of final goods

$$P = \left(\sum_{i=1}^{n} (P^i)^{1-\eta}/n \right)^{1/(1-\eta)}. \tag{45}$$

Again, the intermediate goods are produced using a linear technology $y^i = l^i$. Households choose aggregate labor supply N. Then, the market clearing condition for labor is:

$$N = \sum_{i=1}^{n} l^i = \sum_{i=1}^{n} y^i = Y \sum_{i=1}^{n} \frac{(p^i)^{-\eta}}{n}. \tag{46}$$

In the model with a finite number of firms and continuous time, the presence of menu costs would have no effect on aggregate consumption, except at a set of times of measure 0, which would not affect consumer's welfare given criteria (8). To remedy this, we first consider a variant of the continuum model in which firms pay flow menu costs during an interval of time of length ξ and show that decision problem of firms and equilibrium outcomes are exactly the same as in the model where menu costs are paid immediately, provided we scale menu costs appropriately. In the finite n approximation to this variant of the model, consumers' welfare is affected by the payment of menu costs. Suppose that if a firm increases (decreases) prices at a time t, without experiencing a Calvo shock, it must pay a flow of menu costs $\delta_f^+ Y_t$ ($\delta_f^- Y_t$) during the interval $[t, t+\xi]$. Notice that if $\delta^+ Y = \int_0^\xi e^{-\rho t} \delta_f^+ Y \, dt$ and similarly $\delta^- Y = \int_0^\xi e^{-\rho t} \delta_f^- Y \, dt$ hold, and $Y_t \equiv Y$, then the firm's decision problem is identical to the case where the firm pays menu costs $\delta^+ Y_t$ or $\delta^- Y_t$ at time t, provided $\delta_f^\cdot = \frac{\rho \delta^\cdot}{1-e^{-\rho \xi}}$. Firm i's dividend $Z(p)$ is reduced by the menu cost $\delta_f^+ Y$ for a period of length ξ after a price increase in the absence of a Calvo shock. At the stationary equilibrium, firms hit $\underline{p}$ at rate $\lambda = \lambda^+$. Thus, the measure of firms paying menu costs for price increases is at any time is equal to $\lambda \xi$. In the stationary equilibrium, aggregate consumption satisfies the natural modification of (15),

$$c = Y - \lambda \xi \delta_f^+ Y.$$

For the model with a finite number of firms, when a firm adjusts price at a time t, in the absence of a Calvo shock, it would pay for a period ξ an amount that equals $\delta_f^+ Y_t / n$ of the final good if it increased price and $\delta_f^- Y_t / n$ if it decreased price. If Λ_t^+ (Λ_t^-) denotes the fraction of firms paying menu costs at time t as a result of price increases (decreases) then, in the model with a finite number of firms,

$$c_t = Y_t - \left(\Lambda_t^+ \delta_f^+ + \Lambda_t^- \delta_f^- \right) Y_t. \tag{47}$$

With this convention, the system of equations (44)–(47) naturally provide a discrete approximation to the equations for the corresponding system of equations in the continuum model.

4.2 Equilibrium with a finite number of firms

When n is finite a firm's optimal decision would depend on the cross section of current prices. Households' demand for money would depend on their expected inflation, which also would be a function of the n-dimensional vector of current prices. We adopt here simplifying behavioral assumptions that greatly economize on the number of state variables.

Assumption 1. *(a) Consumption-good sector: price P_t of the final good guarantees zero-profits at t.*

(b) Intermediate goods producers: all firms expect future inflation to equal π, the real wage w to stay at the stationary equilibrium value in the continuum model, and use the discount rate ρ. Each firm observes current P_t and sets P_t^i using the policy function for the continuum model, in particular, reprices to $P_t^i = qP_{t\text{-}}^i$ if $P_{t\text{-}}^i/P_t \leq \underline{p}$ and reprices to $P_t^i = (p^*/p^\dagger)P_{t\text{-}}^i$ if $P_{t\text{-}}^i/P_t \geq p^\dagger$.

(c) Households: expect future inflation to equal π and the real wage w to stay at the stationary equilibrium value in the continuum model.

(d) Monetary authority: accommodates price level changes,

$$d\mathcal{T}_t = \left(1 - \frac{P_{t\text{-}}}{P_t} - Rdt\right) m,$$

where m is the real-money balance held by households.

Since there are only a finite number of firms, Assumption 1 implies that households and firms do not display fully rational expectations. In Section 5 we will show that, in an example with calibrated parameter values using Nakamura and Steinsson [48] and Golosov and Lucas [35], the impact of these expectational errors are minor. This numerical result agrees with theoretical results in Alvarez and Lippi [4, Proposition 7(ii)] for the Golosov-Lucas model.

4.3 Complementarity of repricing at the extensive margin

When n is finite, the impact of the price change by a firm that receives a Calvo shock at time t may cause other firms to pay menu costs and change prices. In turn, these price-changes may induce other firms to change prices. This *repricing avalanche* will continue until the change in prices by a group of firms induces no other firm to change prices. The size of this avalanche will determine the number of firms that reprice at t.[16] In this section we show that the function $\theta_0(s)$, especially the reproduction number θ, plays a crucial role in determining the distribution of the size of repricing avalanches in the model with a large finite number of firms. In particular, we show that the dispersion index of the distribution of avalanche sizes is bounded below by $\frac{1}{(1-\theta)^2}$.

[16] Notice that one could examine self-fulfilling equilibria where the number of firms that reprice at t could be larger than the size of the avalanche as we define it. In this sense, we are studying the minimal number of firms that change prices at t, as a result of a Calvo shock.

Suppose the aggregate good price is $P_{t\text{-}}$ and that a single firm i changes its (relative) price to p^* at t. Then $\Delta \log P_t^i := \log p^* - \log p_{t\text{-}}^i$. The price change by firm i increases the final good price P_t, which in turn decreases all relative prices. To compute the impact of firm i's price-change on P_t, recall that

$$\Delta\left((P_t^i)^{1-\eta}\right) = \frac{(P_t^i)^{1-\eta} - (P_{t\text{-}}^i)^{1-\eta}}{n} = \frac{(P_{t\text{-}}^i)^{1-\eta}}{n}(e^{(1-\eta)\Delta \log P_t^i} - 1)$$

and $P_t = \left(\sum_{i=1}^n e^{(1-\eta)\log P_t^i}/n\right)^{1/(1-\eta)}$. Thus,

$$\Delta \log P_t = \frac{\Delta(P_{t\text{-}}^{1-\eta})}{(1-\eta)P_{t\text{-}}^{1-\eta}} - \epsilon_P(s_{t\text{-}}^i) = \frac{1}{n}\left(\frac{P_{t\text{-}}^i}{P_{t\text{-}}}\right)^{1-\eta} \frac{e^{(1-\eta)\Delta \log P_t^i} - 1}{1-\eta} - \epsilon_P(s_{t\text{-}}^i),$$

where $\epsilon_P(s_{t\text{-}}^i) = (\Delta P_t)^2/\tilde{P}$ for some $\tilde{P} \in [P_{t\text{-}}, P_{t\text{-}} + \Delta P_t]$ and $\Delta P_t = O(1/n)$.[17] Hence,

$$\Delta \log P_t = \frac{(p_{t\text{-}}^i)^{1-\eta}}{n} \frac{e^{(1-\eta)\Delta \log P_t^i} - 1}{1-\eta} - \epsilon_P(s_{t\text{-}}^i), \quad \text{where} \tag{48}$$

$$\epsilon_P(s_{t\text{-}}^i) = O(n^{-2}). \tag{49}$$

To analyze the avalanche initiated by a Calvo shock, it is again helpful to work with the normalized price $s_{t\text{-}}^i = (\log p_{t\text{-}}^i - \log \underline{p})/\log q$ that has stationary distribution given in (35). We fix a time t that, to simplify notation, we temporarily omit, and choose s^1 using the distribution $g(s)$ and assume that firm 1 receives a Calvo shock. In addition, we draw independently $n-1$ other firms and thus $(s^\ell)_{\ell=2}^n$ is distributed as $g^{n-1}(s)$. Equation (48) implies that s^j for $j > 1$ decreases by

$$\epsilon_0^n = \frac{1}{\log p^* - \log \underline{p}}\left[\frac{1}{n}\frac{(p^*)^{1-\eta} - (p_{t\text{-}}^1)^{1-\eta}}{1-\eta} - \epsilon_P(s^1)\right]. \tag{50}$$

Set $m_0^n = \#M_0^n$, where $M_0^n := \{j > 1 : s^j \leq \epsilon_0^n\}$ is the set of firms that choose to reprice because firm 1 repriced. Write G for the cumulative distribution with density g. Then, m_0^n conditional on s^1 follows a binomial distribution with population $n-1$ and probability $\kappa_0^n := G(\epsilon_0^n)$.

Notice that s^ℓ for $\ell \notin M_0^n$, $\ell > 1$, decreases by $\epsilon_0^n + \sum_{j \in M_0^n} \tilde{\epsilon}_1^j$, with

$$\tilde{\epsilon}_1^j := \epsilon_1^n - \epsilon_P'(s^j)$$

[17] Throughout this paper, we write $y(n) = O(x(n))$ if and only if there exists a positive real number M and a real number n_o such that $|y(n)| \leq Mx(n)$ for all $n \geq n_o$. Also, $x(n) \sim y(n)$ means $\lim_{n\to\infty} x(n)/y(n) = 1$.

where

$$\epsilon_1^n := \frac{1}{n}\frac{\underline{p}^{1-\eta}}{\log p^* - \log \underline{p}}\frac{(p^*/\underline{p})^{1-\eta}-1}{1-\eta} = \frac{\epsilon_1^1}{n} \geq \epsilon_0^n \tag{51}$$

$$\epsilon_P'(s^j) := \frac{1}{\log p^* - \log \underline{p}}\left[\frac{\underline{p}^{1-\eta}}{n}\frac{(p^*/\underline{p})^{(1-\eta)s^j}-1}{1-\eta} + \epsilon_P(s^j)\right] > 0,$$

where $\epsilon_P'(s^j) = O(n^{-2})$ and positive since $0 < s^j < \epsilon_0^n = O(n^{-1})$. Furthermore, applying (31) and (35) to (51) we obtain,

$$\epsilon_1^1 g(0) = \theta.$$

The set,

$$M_1^n := \{\ell > 1 : s^\ell \leq \epsilon_0^n + \sum_{j\in M_0^n} \tilde{\epsilon}_1^j\} \setminus M_0^n,$$

where $\setminus$ denotes the set difference, is the set of firms that reprice because the Calvo shock led firm 1 and firms in M_0^n to reprice. The random variable $m_1^n = \# M_1^n$ is distributed as a binomial with population $n-1-m_0^n$ and probability:

$$\kappa_1^n := G\left(\epsilon_0^n + \sum_{j\in M_0^n} \tilde{\epsilon}_1^j\right) - G\left(\epsilon_0^n\right).$$

In turn, the price-changes of these m_1^n firms cause an additional set of firms to change prices and we write,

$$M_2^n := \{\ell > 1 : s^\ell \leq \epsilon_0^n + \sum_{j\in M_0^n\cup M_1^n} \tilde{\epsilon}_1^j\} \setminus (M_0^n \cup M_1^n),$$

for the set of firms reacting to the price changes by the firms in M_1^n, $m_2^n = \# M_2^n$ etc...

Conditional on $(m_k^n)_{k=0}^u$, m_{u+1}^n follows a binomial with population $n - 1 - \sum_{k=0}^u m_k^n$ and probability κ_u^n where,

$$\kappa_u^n := G\left(\epsilon_0^n + \sum_{j\in\bigcup_{k=0}^u M_k^n} \tilde{\epsilon}_1^j\right) - G\left(\epsilon_0^n + \sum_{j\in\bigcup_{k=0}^{u-1} M_k^n} \tilde{\epsilon}_1^j\right). \tag{52}$$

The total size of the avalanche initiated by a Calvo shock at s is given by,

$$L^n = \sum_{u=0}^{\infty} m_u^n.$$

Notice that if $m_U^n = 0$ then (52) implies that $m_u^n = 0$ for each $u > U$, that is the avalanche stops whenever $m_U^n = 0$. These L^n firms' repricing size is close to $\log p^* - \log \underline{p}$. Suppose all the

$n-1$ firms reprice after a firm reprices due to a Calvo shock. These repricings increase $\log P_t$ by at most $\log q$, which decreases s^i by at most 1. This maximum increase has probability 0, since it requires that the Calvo-hit firm has $s^1 = 0$. Hence, $L^n < n$ holds with probability 1.

It is well known that the distribution of properly normalized binomials converge to the distribution of a Poisson as the number of observations $n \to \infty$. Using (51) we obtain,

$$\epsilon_0^n + \sum_{j \in \bigcup_{k=0}^{u} M_k^n} \tilde{\epsilon}_1^j = \epsilon_0^n + \frac{\epsilon_1^1}{n} \sum_{k=0}^{u} m_k^n - \sum_{j \in \bigcup_{k=0}^{u} M_k^n} \epsilon'_P(s^j). \tag{53}$$

Fixing the history of m_k^n for $k \leq u$ as $m_k^n = m_k$, since ϵ_0^n and ϵ_1^n are $O(n^{-1})$, (52) implies that $\lim_{n\to\infty} n\kappa_u^n = m_u \epsilon_1^1 g(0) = \theta m_u$. Hence,

$$\lim_{n\to\infty} E[m_{u+1}^n \mid (m_k^n)_{k=0}^u = (m_k)_{k=0}^u] = \lim_{n\to\infty} (n - 1 - \sum_{k=0}^{u} m_k)\kappa_u^n = \theta m_u.$$

Using (36), (50), and $E[m_0 \mid s^1] = (n-1)\kappa_0^n = (n-1)G(\epsilon_0^n)$ we obtain:

$$\lim_{n\to\infty} E[m_0^n \mid s^1] = \frac{g(0)}{\log q} \frac{(p^*)^{1-\eta} - (p^1)^{1-\eta}}{1-\eta} = \theta_0(s^1).$$

This suggests that conditional on s^1, the random variable L^n approaches (in distribution) as $n \to \infty$ a random variable L where

$$L := \sum_{u=0}^{\infty} m_u,$$

and m_u follows a branching process where the number of children per parent follows a Poisson distribution with mean θ and m_0 follows a Poisson with mean $\theta_0(s^1)$, what we establish formally in the next Proposition.

From now on we fix $s^1 = s$ and let $\mathcal{Q}^n$ ($\mathcal{Q}$) denote the probability distribution of L^n (L) conditional on s. Recall that for distributions $\mathcal{Q}_1$ and $\mathcal{Q}_2$ with support in Z_+ *the total variation distance* satisfies:

$$d_\tau(\mathcal{Q}_1, \mathcal{Q}_2) := 2 \sup_{A \subset Z_+} |\mathcal{Q}_1(A) - \mathcal{Q}_2(A)| = 2 \sum_r |\mathcal{Q}_1(\{r\}) - \mathcal{Q}_2(\{r\})|.$$

In Appendix A.11, we establish:

Proposition 5.

$$\lim_{n\to\infty} d_\tau(\mathcal{Q}^n, \mathcal{Q}) = 0.$$

Since $L^n(s) \to L(s)$ in total variation, we can construct random variables X^n and X in [0,1] with the natural filtration, such that X^n has the distribution of $L^n(s)$ and X has the distribution of $L(s)$, with $X^n \to X$ a.s.[18] Thus Fatou's Lemma, applied to X^n guarantees:

[18] *E.g.* set $X^n(y) = \Pr\{L^n(s) \leq y\}$ and $X(y) = \Pr\{L(s) \leq y\}$.

Corollary 2.
$$\liminf_{n\to\infty} E[(L^n)^p|s] \geq E[(L)^p|s].$$

Stronger assumptions can establish that $E[L^n|s] \to E[L|s]$. The assumption used below is especially interesting, because it is verified for parameter values calibrated in Section 5.

Proposition 6. *Write* $\Theta := (q^{\mu/\pi} - 1)\theta$. *If either* $\Theta \leq 1$ *or* $\theta e^{\Theta-1}/\Theta < 1$ *then the set* $\{L^n : n \in Z_+\}$ *is uniformly integrable and as a consequence:*

(a) $E[L^n|s] \to E[L|s]$.

(b) $\liminf_{n\to\infty} \frac{Var[L^n|s]}{E[L^n|s]} \geq \frac{Var[L|s]}{E[L|s]}$.

Proof. Uniform integrability and hence Item (a) is shown by Lemma 9 in Appendix A.12. Item (b) is a consequence of Corollary 2. □

We next describe the distribution of L and thus the asymptotic distribution of L^n.

Proposition 7. *(a) L conditional on m_0 follows the Borel-Tanner distribution:*

$$\Pr(L = \ell \mid m_0) = \frac{m_0}{\ell} \frac{e^{-\theta\ell}(\theta\ell)^{\ell-m_0}}{(\ell - m_0)!} \tag{54}$$

for $\ell = m_0, m_0+1, m_0+2, \ldots$, where m_0 follows a mixture of Poisson distributions with mean $\theta_0(s)$ and s is distributed as $g(s)$.

(b) L conditional on the Calvo-hit firm being at s follows the Generalized Poisson Distribution (GPD):

$$\Pr(L = \ell \mid s) = \frac{\theta_0(s)}{\ell!} e^{-(\theta\ell+\theta_0(s))}(\theta\ell + \theta_0(s))^{\ell-1} \tag{55}$$

for $\ell = 0, 1, \ldots$, where s is distributed as $g(s)$. The conditional mean and variance of L are $\theta_0(s)/(1-\theta)$ and $\theta_0(s)/(1-\theta)^3$, respectively.

(c) The tail of both (54) and (55) is proportional to $e^{-(\theta-1-\log\theta)\ell}\ell^{-1.5}$ as $\ell \to \infty$.

(d) The unconditional distribution of L follows a general Lagrangian probability distribution. The unconditional mean μ_L and variance σ_L^2 of L are, respectively,

$$\mu_L = \frac{\theta_0}{1-\theta}$$
$$\sigma_L^2 = \frac{\theta_0}{(1-\theta)^3} + \frac{\sigma_{\theta_0}^2}{(1-\theta)^2},$$

where $\sigma^2_{\theta_0}$ denotes the variance of $\theta_0(s)$ under the stationary distribution $g(s)$,

$$\sigma^2_{\theta_0} = \frac{1}{(1-\eta)^2\varphi^2(q,\mu/\pi)}\left(\frac{\varphi(q,2(1-\eta)+\mu/\pi)\varphi(q,\mu/\pi)}{\varphi^2(q,1-\eta+\mu/\pi)} - 1\right).$$

In particular, the index of dispersion of L is bounded below by

$$\frac{1}{(1-\theta)^2}$$

and thus arbitrarily large as $\theta \to 1$.

Proof. (a) The sum L conditional on m_0 of a Poisson branching process is known to follow the Borel-Tanner distribution (54) (see Kingman [40]; Nirei [52, 53]). (b) Equation (55) is obtained by combining (54) with the fact that m_0 conditional on s follows a Poisson distribution with mean $\theta_0(s)$ (see Consul and Shoukri [25]). The mean and variance follow from the formulae for the moments of a GPD (see *e.g.* Consul and Famoye [26, Equation (9.9), p.166]). (c) from Stirling's formula applied to $(\ell - m_0)!$ in (54) and $\ell!$ in (55). (d) See Consul and Famoye [26, p.111] for the Lagrangian distribution property and Appendix A.13 for the derivation of μ_L and σ^2_L. □

Item (c) can be sharpened when L is conditional on $m_0 = 1$.

Corollary 3. *The distribution of L satisfies:*

$$\Pr(L = \ell \mid m_0 = 1) = \frac{1}{\sqrt{2\pi}e^{1-\theta}} e^{-(1-\theta-\log\theta)\ell}\ell^{-1.5} + O(e^{-(1-\theta-\log\theta)\ell}\ell^{-2.5}), \qquad \text{for } \ell \to \infty, \tag{56}$$

which has a power-law tail when $\theta = 1$.

Proof. A direct application of Harris [36, Theorem 13.1, p.32]. Note that π in expression (56) denotes the ratio of circumference to diameter of a circle instead of the inflation rate. □

The implications of Proposition 7 are three-fold. First, it shows that the limit of the unconditional mean of L^n as $n \to \infty$ is bounded below by $\theta_0/(1-\theta) = \lambda/\mu$, obtained from (30), (39), and Corollary 2, with equality under the assumptions of Proposition 6. This is expected—since each firm draws a Calvo event at rate μ, on average, the number of firms that draw a Calvo event in a short time horizon dt is $n\mu dt$. Hence, the average fraction of firms that adjust within dt is $(\lambda/\mu)(n\mu dt)/n = \lambda dt$. Thus in the $n \to \infty$ limit, the average fraction of firms that pay menu costs in a small interval dt coincides with the rate of firms that pay menu costs in the continuum economy. Furthermore, $\lambda/\mu = \theta_0/(1-\theta)$ decomposes the avalanche into the direct effect of a Calvo-hit firm and the additional effects due to menu-cost paying firms, since θ_0 is the multiplier of a Calvo-hit firm on the fraction of firms that adjust, and θ is the multiplier associated to repricing by a menu-cost paying firm.

As shown in Section 3.4, in the continuum case, when a firm pays menu costs and reprices, it induces θ firms to reprice, leading to a total number of repricing firms that equals $(1-\theta)^{-1}$. A second implication of Proposition 7 is that even in the limit as $n \to \infty$ the multiplier effect is stochastic with a dispersion index that is bounded below by $(1-\theta)^{-2}$, which diverges as $\theta \to 1$. We show in Appendix A.10 that θ converges to 1 as $\pi \to \infty$ and there exists a π' such that θ is non-decreasing for $\pi \geq \pi'$. Furthermore, we show in Section 3.4 that $(1-\theta)^{-2}$ dominates an increasing function of π^2. More importantly, we show in Section 5, for parameters calibrated from U.S. data, $\theta \sim 1$ even for very modest levels of π.

Thirdly, Corollary 3 (equation 56) states that the number of repricing firms, conditional on a single firm reacting to the Calvo-hit firm, approaches a power-law distribution with exponential truncation. The multiplier effect caused by the complementarity of repricing has a distribution that exhibits a fat right tail, up to the exponential truncation point determined by θ. When θ reaches 1, the truncation point is infinite, implying that the entire tail is characterized by a power-law distribution. Using this limit law, we obtain

$$\lim_{n\to\infty} n^{0.5} E\left[\left(\frac{L^n}{n}\right)^2\right] = \lim_{n\to\infty} n^{0.5} \sum_{\ell=0}^{n} \left(\frac{\ell}{n}\right)^2 \Pr(L^n = \ell) \approx \lim_{n\to\infty} n^{0.5} b_o \int_1^n \frac{\ell^{0.5}}{n^2} d\ell = \frac{b_o}{1.5},$$

that is, $E[(L^n/n)^2] \sim n^{-0.5}$ when $\theta = 1$. Similarly, we obtain that $(E[L^n/n])^2 \sim n^{-1}$, which implies $V(L^n/n) \sim n^{-0.5}$ for the case of $\theta = 1$.

For finite n, the aggregate price $\log P_t$ follows a compound Poisson process; a Calvo shock arrives at intensity rate $n\mu$ and the price jump size $\log P_t - \log P_{t\text{-}}$ is a stationary random variable. The price jump consists of a repricing by firm i that receives a Calvo shock and the subsequent repricing by L_t^n firms who pay menu costs. By aggregating (48) across those repricing firms and using (31), we obtain

$$\log P_t - \log P_{t\text{-}} = \underbrace{\frac{(p^*)^{1-\eta} - (p^i_{t\text{-}})^{1-\eta}}{(1-\eta)n}}_{\text{Calvo-hit firm } i} + \underbrace{\frac{\varphi(q, \mu/\pi)\varphi(q, 1-\eta)}{\varphi(q, 1-\eta+\mu/\pi)} \frac{L_t^n}{n}}_{\text{Menu-cost paying firms}} - O(n^{-2}). \tag{57}$$

Note that the intensive margin effect of menu-cost firms, $h(\pi) := \frac{\varphi(q,\mu/\pi)\varphi(q,1-\eta)}{\varphi(q,1-\eta+\mu/\pi)}$, is positive for each π. Moreover, the extensive margin effect of menu-cost firms, L_t^n, has a positive covariance with the effect of the Calvo-hit firm, because the latter is decreasing in s^i and $E[L_t^n|s^i] = \theta_0(s^i)/(1-\theta)$, with $\theta_0(s^i)$ decreasing. Hence, a lower bound on the behavior of $V(\log P_t - \log P_{t\text{-}})$ as a function of h can be obtained from equation (57). In addition, as shown in Appendix A.14 that for large π, the variance of $h(\pi)L_t^n$ and the variance of the effect of the Calvo-hit firm are increasing in π.

4.4 Finite interval of time

In actual data there are at least two complications. First it is unlikely that s, the relative price of the firm that undergoes a Calvo shock, is observed. Second one can only observe the sum of avalanches over a sequence of finite non-overlapping intervals.

For definiteness, assume that in the model a unit interval of time is one year and suppose one observes data in a sequence of non-overlapping intervals of length T. The independence of Calvo shocks insures that if $\hat{\mu} = \mu n/T$, the count of total number of Calvo shocks K in an interval of length T satisfies:

$$\Pr(K = k) = \frac{(\hat{\mu})^k}{k!} e^{-\bar{\mu}}.$$

In an interval of length T the total count of price changes Z^n would correspond to a sum of avalanches and the Calvo shocks. Thus if an interval has k Calvo shocks:

$$Z^n = k + \sum_{i=1}^{k} L_i^n,$$

where L_i^n denotes the avalanche caused by the i-th Calvo shock. Let $S = s^1, s^2 \dots$ be an infinite sequence of Calvo shocks drawn independently using the distribution g.

We can decompose the uncertainty of Z^n as follows. One source is k, the number of Calvo shocks, and another the location of each s^i and the distribution of avalanches conditional on a Calvo shock at s^i. Since the distributions of s^i and L_i^n are endogenous, this decomposes the variability of Z^n into an endogenous and an exogenous portion.

Write $E[L^n] = \int_0^1 E[L^n(s)]g(s)ds$, the expected value of the unconditional distribution of L^n. Similarly, let $V[L^n]$ denote the variance of the unconditional distribution of L^n. Since the number of Calvo shocks is independent of the realization of s,

$$E[Z^n] = E[E[Z^n|k]] = E\left[\sum_{i=j+1}^{j+k} (E[L^n(s^i)] + 1)\right] = \hat{\mu}(E[L^n] + 1).$$

Furthermore,

$$\begin{aligned} V[Z^n] &= E[V[Z^n|k]] + V[E[Z^n|k]] = E[V[Z^n|k]] + V\left[\sum_{i=j+1}^{j+k} (E[L^n(s^i)] + 1)\right] \\ &= E[V[Z^n|k]] + V[k(E[L^n] + 1)] = E[V[Z^n|k]] + \hat{\mu}(E[L^n] + 1)^2. \end{aligned}$$

Hence,

$$\frac{V[Z^n]}{E[Z^n]} = \frac{E[V[Z^n|k]]}{E[Z^n]} + \frac{E[Z^n]}{\hat{\mu}}. \tag{58}$$

Equation (58) decomposes the dispersion of Z into a term that reflects the stochastic nature of the avalanches and a portion caused by variations on the number of realized Calvo shocks. Notice that, since the LHS and last term on the RHS can be estimated/calibrated from data at the observable frequency, we can calculate the portion that the stationary distribution of stochastic avalanches contributes to the dispersion of Z^n.[19]

Assuming that the density g is flat at $s = 0$, we can write, if an interval contains avalanches $j+1, \dots, j+k$:

$$Z^n = k + \sum_{i=j+1}^{j+k} L^n(s^i),$$

where $L^n(s^i)$ are independent random variables with a GPD with parameters $\theta_0(s^i)$ and θ. Thus, conditional on k, Z^n is the sum of k and a random variable that is a GPD with parameters $\tilde{\theta}_0 = \sum_{i=j+1}^{j+k} \theta_0(s^i)$ and θ (Consul and Famoye [26, Theorem 9.1]). When k is relatively small, the distribution of Z^n, except in the range of small total avalanches, is not much affected by k. Since in a GPD the parameter θ dictates the tail behavior, it is expected that the dispersion index is not much affected by the uncertainty on the number and location of Calvo shocks. Furthermore the independence of avalanches implies that:

$$E[V[Z^n|k]] = E\left[\sum_{i=1}^{k} V[L^n(s^i)]\right] = \hat{\mu} V[L^n].$$

Hence, equation (58) reduces to:

$$\frac{V[Z^n]}{E[Z^n]} = \frac{V[L^n]}{E(L^n)+1} + \frac{E[Z^n]}{\hat{\mu}}.$$

5 Numerical simulation

5.1 Productivity shocks, calibration, and inflation volatility

In this section, we investigate the model quantitatively within a realistic range of long-run inflation rates. We will introduce a generalization of the model that accommodates the possibility of price decreases. We will assume that firm i produces output with a linear technology with labor being the single input,

$$y_t^i = a_t^i l_t^i,$$

and productivity at time t, $a_t^i \in \{a^L, a^H\}$, with $a^L \le a^H$. By choosing the units in which we measure labor, we may assume $\frac{a^L + a^H}{2} = 1$. At the time of a Calvo shock, firms may

[19]See Section 5 for a calibration of μ, and thus $\hat{\mu}$, using the equilibrium equations of the model. Using the calibration we estimate the exogenous uncertainty to account for only 3.15% of the total dispersion of daily price changes.

also experience a productivity shock. We assume that, conditional on a Calvo shock, the probability of a productivity change is a constant $0 < \zeta < 1$. In the online appendix we show the existence and uniqueness of a stationary equilibrium and characterize the equilibrium stationary distribution, including the values of the $\underline{p}_a$ and p_a^*, the minimum and maximum relative prices charged by firms with productivity a in equilibrium, and thus the ratios $q_a = p_a^*/\underline{p}_a$, $a \in \{a^L, a^H\}$.[20] Although the existence and uniqueness results fully generalize, we do not establish in the online appendix the full set of comparative statics and full characterization of the distribution of avalanches. Nonetheless, as we show in subsection 5.2, the distribution of avalanches is well fitted by a GPD.

Remark 2. *Although firms are assumed to use the policy functions of the continuum, when n is finite, in contrast to the continuum case, firms may pay menu costs to adjust prices downward. Suppose $p^*_{a^L} > p^*_{a^H}$ and that a firm with productivity a^L and price close to $p^*_{a^L}$ suffers a Calvo shock and experiences a change in productivity at t. That firm would now optimally charge $p^*_{a^H}$ and the price of the final good would fall by Assumption 1(a). Firms with productivity a^L (a^H) and prices $p_{t\text{-}}$ close to $p^*_{a^L}$ ($p^*_{a^H}$), may now choose to cut prices if δ^- is small enough.*[21]

We calibrate the model parameters $(\mu, \delta^+, \delta^-)$ so that the implied stationary distribution matches some key empirical patterns reported by Nakamura and Steinsson [48] for U.S. CPI micro data during the period 1988–2005—a median size of repricing of 7.3% (the median increase of non-sale price), a median frequency of non-sale price change of 9–12% per month, and that a third of non-sale price changes are price decreases. This implies that the net frequency of price increases is 3–4% per month, one-third of the total frequency of repricing.

U.S. CPI increased at 3% on average during their sample period. Thus we set $\pi = 0.03$. The elasticity of substitution η is set at 3, so that labor's share is two-thirds of value added. The time discount rate ρ is set to the long-term real interest rate 0.02.

We set the parameter values related to productivity shocks to match Golosov and Lucas [35], who choose for log productivity an Ornstein-Uhlenbeck process with stationary variance 0.01 and auto-covariance $0.01e^{-0.55}/(2 * 0.55)$. Matching these two moments, we set $(a^L, a^H) = (0.9, 1.1)$ and $\mu\zeta = (1 - e^{-0.55})/2$.

The average positive repricing size and frequency correspond to $\sum_a \log q_a/2$ and $\mu(1 - \zeta/2) + \lambda^+$, respectively, evaluated at $\pi = 0.03$. Thus, we choose μ, δ^+ and ζ so that the computed $\sum_a \log q_a/2$ and $\mu(1 - \zeta/2) + \lambda^+$ match the target repricing size and frequency with the restriction that $\mu\zeta = (1 - e^{-0.55})/2$. We obtain calibrated parameter values by minimizing the sum of log-deviation from the two target moments. This results in $\mu = 0.2672$, $\delta^+ = 0.0021$ and $\zeta = 0.7916$.

[20] In the online appendix we also allow the probability of the productivity shock to depend on the productivity of the firm just prior to the Calvo shock.

[21] However for a fixed δ^-, the probability that this occurs goes to 0 as $n \to \infty$.

We set the menu cost for price cuts as $\delta^- = n^{-2.2}$, so that the fraction of firms that lower prices is roughly a half of the number of firms that raise prices, as Nakamura and Steinsson's estimate. In a positive inflation environment with the calibrated size of positive productivity shock a^H/a^L, the average number of price cuts in equilibrium is small unless δ^- is very small. As a consequence of small δ^-, our model does not match the median size of price cut (10.5%) that Nakamura and Steinsson reported. With this gap, inflation in our model would be overestimated if the pricing frequency of 9–12% was used directly. Therefore, we focus on the frequency of *net* price increases. Taking a mid-point of the estimated range, we set 3.5% as a target for monthly (positive) repricing frequency.

Finally, we set $n = 30000$ to match inflation volatility observed in the sample period of Nakamura and Steinsson [48], when the average monthly standard deviation of inflation rates (year-on-year price increase) was 0.55% while the average inflation rate was 3%.

For these parameter values, we compute stationary equilibrium prices and policy functions for different levels of π, and use these to compute θ, which determines the volatility of extensive margins of price adjustments. As can be seen in Figure 1, θ approaches 1 quickly for a low range of π under the parameter values calibrated to the U.S. economy. At $\pi = 0.03$, our model produces $\theta = 0.7270$ and $\theta_0 = 0.3023$. Also, the slope of the stationary distribution $g(s)$ at the repricing threshold $s = 0$ is $g'(0) = \sum_a (1/2)(\mu/\pi)(\log q_a)^2/\varphi(q_a, \mu/\pi) = 0.4592$.

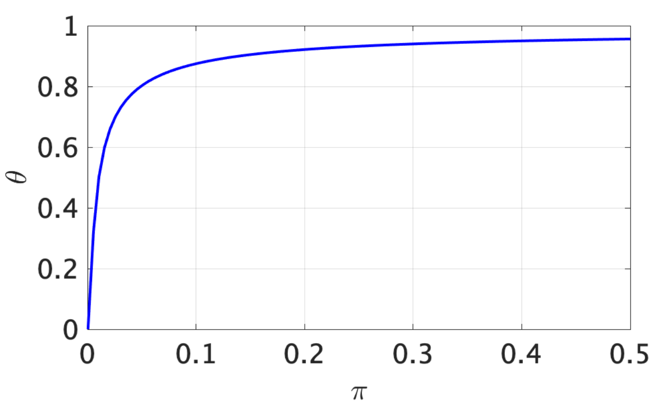

Figure 1: Complementarity measure θ for various levels of π. We numerically compute the stationary equilibrium and obtain θ using Equation (38).

We conduct Monte Carlo simulations, drawing Calvo and productivity shocks for up to $n\mu \times 8000$ rounds for each inflation level π and generate aggregate-price time-series P_t. We plot the standard deviation of the annual aggregate price increase $\log P_{t+1} - \log P_t$ against π. The result is shown in Figure 2. Notice that inflation volatility increases with inflation level, and the slope of the relationship in the calibration matches the experience in the U.S.

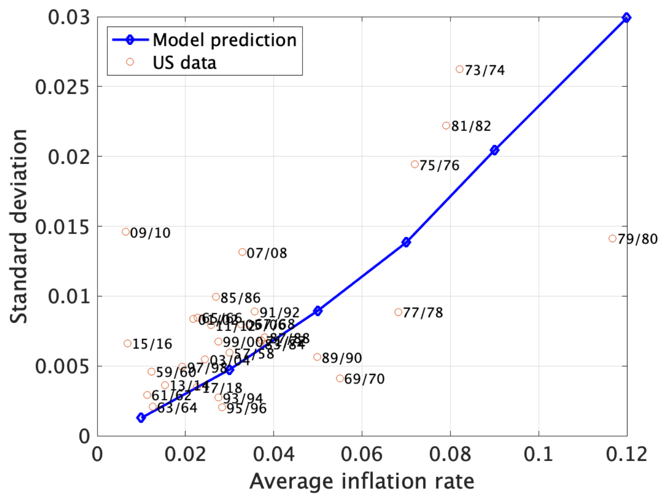

Figure 2: Average annual inflation rates and standard deviations of inflation rates in the model simulations and the U.S. data. The line shows the simulation result. For the U.S. time series, we use IMF-IFS dataset for monthly CPI for 1955M1-2018M12. We compute the year-on-year inflation rates at monthly frequency, and compute the average and standard deviation of the inflation rates in two-year windows. The starting year of the two-year window is $\{1955, 1957, \ldots, 2017\}$. The scatter plot of the average and the standard deviation is shown in the figure.

5.2 Distribution of simulated avalanches L_t

We focus on avalanches of price increases, neglecting avalanches of price decreases. We write L for the number of firms that reprice, excluding the Calvo-hit firm. The distribution of simulated L is in fact well approximated by a fitted GPD (see Figure 3). Here we employ moment-estimators for θ_0 and θ, exploiting the properties of the GPD: $\sqrt{E(L)^3/V(L)} = \theta_0$ and $V(L)/E(L) = 1/(1-\theta)^2$. The point estimates for the parameters are $\hat{\theta}_0 = 0.1758$ and $\hat{\theta} = 0.9061$. The root mean square error (normalized by the absolute mean) of the fitting of the logarithm of the counter-cumulative distribution function is 2.99%.[22]

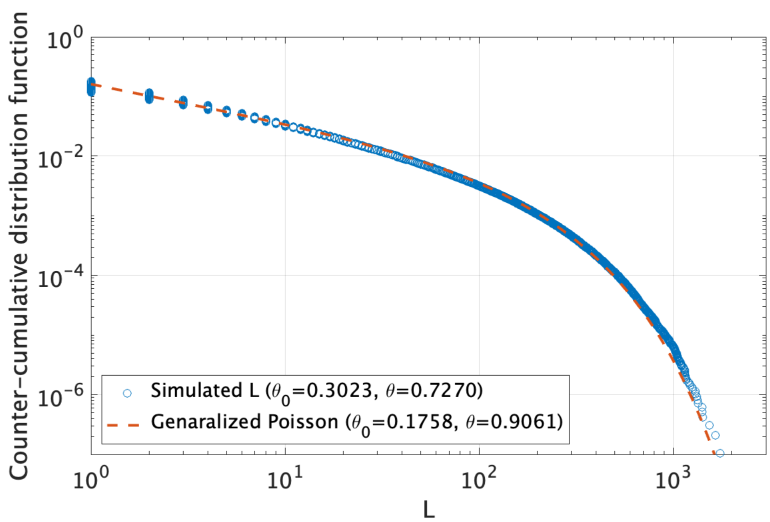

Figure 3: Counter-cumulative distribution of simulated L and a fitting by a Generalized Poisson distribution

While a GPD fits the simulated distribution well, the estimated key multiplier $\hat{\theta}$ is substantially larger than the θ of the data-generating model. There are four potential sources for the gap. First, the moment estimator is subject to an estimation error and bias. However, assuming the parameters of the GPD are θ_0 and θ of the data-generating model, we can use the formulae in Consul and Famoye [26, p.170] to compute standard error of the moment estimator (.0003) and a bias (-.0000), up to n^{-2}. These cannot account for the gap $\hat{\theta} - \theta$. Second, our theory predicts that L follows a Lagrangian distribution, which is obtained by a mixture of GPDs. Thus, $1/(1-\theta)^2$ is only a lower bound for the dispersion index (see Proposition

[22] The NRMSE equals $\sqrt{\sum_{i=0}^{m}(\hat{y}(i) - y(i))^2/(m+1)}/|\sum_{i=0}^{m} y(i)/(m+1)|$, where $y(i)$ is the log relative frequency of avalanches greater than or equal to i in the simulation, $\hat{y}(i) := \log \Pr(L \geq i)$ for the GPD obtained from the moment estimators, and m is the maximum avalanche in the sample.

7(d)). Third, the simulations allow for productivity shocks. To quantify the effect of these two departures from a GPD, we generalize the formula for the dispersion index in Proposition 7 to the two-state productivity economy (Proposition B.3 in the online appendix). The dispersion index using the calibrated parameters in the generalized formula in Proposition B.3 is $V(L)/E(L) = 13.8108$, while the value of θ used to generate the simulations implies a lower bound $1/(1-\theta)^2 = 13.4216$. Thus the value of θ obtained from the calibration amounts to 97% of the predicted dispersion index that accounts for these two departures from a GPD; neither the mixture of GPDs stemming from the random draws of the firm s that receives a Calvo shocks nor the presence of productivity shocks explain much of the excess dispersion and hence the substantially larger estimated multiplier of the simulated L.

The fourth possible reason is the dynamics of the distribution of relative prices. When fitting L, we are assuming that the underlying θ is constant over time. However, in an economy with finite n, the relative price distribution evolves over time, which affects the number of firms near the repricing threshold $s = 0$ and generates shifts in the multiplier θ.[23] The higher dispersion index obtained in the simulations suggests that, once we account for the dynamics, the multiplier effect of a firm's decision to pay menu costs is higher than what would obtain in the stationary equilibrium. We explore this interpretation further in subsection 5.3.

The simulated series L also allow for quantification of the exogenous shocks' contributions to the dispersion index in a finite-time interval, as discussed in Section 4.4. We set the relevant interval of time to a day, and construct the daily number of positive repricing, Z. Variations on Z are partially driven by the (exogenous) variation on the number of Calvo shocks and by the (endogenous) variations on the size of realized avalanches. We compute equation (58) using daily simulated data and obtain a daily dispersion index $V(Z)/E(Z) = 91.14$, whereas the contributions of the exogenous shock is $E(Z)/\hat{\mu} = 2.87$.[24] This decomposition shows that the exogenous shocks only account for 3.15% of the total dispersion index. The simulated distribution of Z is also well-fitted by a GPD (even though it follows a more general Lagrangian distribution in principle), with moment estimates $\hat{\theta}_0^Z = 3.9924$ and $\hat{\theta}^Z = 0.8953$. Recall that the sum of independent random variables, each with GPD with the same θ, has a GPD with the same θ (Consul and Famoye [26, Theorem 9.1]). Thus our numerical estimates indicate that the dependence of successive avalanches does not affect much the value of $\hat{\theta}$.

[23] In the simulations, conditional on an avalanche $L_i > 0$, the expected value of the next avalanche is 4.15 while the unconditional expectation $E(L) = 1.87$. This failure of independence is reflected in the presence of a positive autocorrelation (.0223) between avalanches.

[24] We used the two-productivity model of Section 5 and collected only positive price changes. The mean daily Calvo/productivity shocks, $\hat{\mu}$, is computed as $(n/T)\mu(1-\zeta/2)$, that is, the number of Calvo/productivity shocks that led to positive price changes in a period of length $1/T$.

5.3 Dynamics of price distribution and inflation volatility

The aggregate price volatility would be quite different if price-changes were independent across firms. Suppose that firms adjust only when they draw Calvo shocks. The number of firms that receive Calvo shocks in time interval dt follows a Poisson distribution with mean $n\mu dt$. The index of dispersion for the number of Calvo-hit firms is equal to one.

With Calvo shocks alone, the variance of the fraction of firms that change prices in dt is $\mu dt/n$. That is, the variance of aggregate fluctuations decline as n^{-1}, as predicted by the central limit theorem. Because our avalanches are stochastic and the distribution displays a right fat tail, the convergence of aggregate volatility to 0 is slowed down to the extent that we observe substantial aggregate fluctuations even for $n = 30000$.

Compared to the case of independent price changes, (57) exhibits two departures. The first is the *selection effect* (Caplin and Spulber [22]; Golosov and Lucas [35]), that is, the size of the price change of firms paying menu costs, $\log q$, is greater than that of the typical Calvo-hit firm. The second departure is the volatility of L^n, a result of the *synchronization* of individual price changes. These two effects act as complements to produce variations in the aggregate price level; since all L^n firms adjust their price by $\log q$ (the selection effect), the synchronization effect generates non-trivial variations on inflation.

The power-law tailed distribution for avalanches of simultaneously repricing firms is reminiscent of the self-organized criticality model of inventories in Bak et al. [7] and Scheinkman and Woodford [58]. In their model, the configuration of agents' states (inventory profile) converges to the critical point of pairwise correlation of actions, at which the power-law distribution of simultaneous actions emerges, resulting in non-trivial aggregate fluctuations arising from micro-level shocks. In our model, the relevant configuration is the profile of relative prices. When a large number of firms are near the repricing threshold, small shocks cause large avalanches of repricing, leading to a decrease in the number of firms near the threshold. In contrast, if a small number of firms are located near the threshold, only small-sized avalanches occur, and the number of firms near the threshold gradually rises. In any case, the relative price distribution converges to the stationary distribution, where the complementarity of repricing is given by θ. Therefore, when θ is close to 1, the relative prices converge to the point at which substantial fluctuations of the number of repricing firms emerge.

This mechanism of self-organized criticality may be used to understand the excess dispersion of simulated L_t that we noted in Section 5.2. Write G_t for the distribution of normalized prices that prevails at time t. When $G_t(s)$ is small for $s \sim 0$, only small avalanches occur, and as a result, more firms accumulate near $s = 0$, leading to higher $G_t(s)$. When $G_t(s)$ is large for $s \sim 0$, large avalanches tend to occur and $G_t(s)$ for $s \sim 0$ goes down. These shifts in G_t generate fluctuations in the estimated θ. Since the dispersion index $1/(1-\theta)^2$ is a strongly convex function of θ, fluctuations in θ increase the average dispersion index.

The following numerical exercise supports this interpretation. We estimate $\hat{\theta}$ for subsamples of the entire series of L, each with 500 consecutive avalanches. The histogram of $\hat{\theta}$ shown in Figure 4 indicates wide variation relative to the standard error produced by the formula in Consul and Famoye [26, p.170], 0.0404, suggesting that the effective multiplier θ varies during the simulated history.

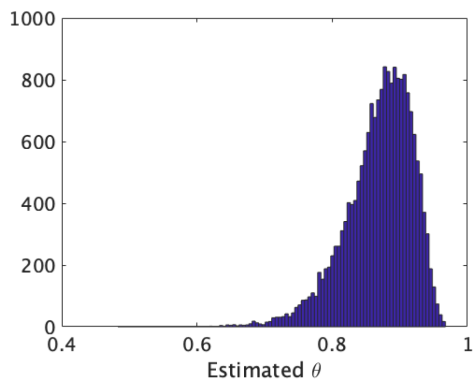

Figure 4: Histogram of $\hat{\theta}$. Each $\hat{\theta}$ is estimated using the dispersion index for a subsample of L with sample size 500. The standard error of the estimator $\hat{\theta}$, computed using the formula in Consul and Famoye [26, p.170], is 0.0404.

5.4 Welfare loss of households following behavioral assumptions

In the simulation, we compute the statistic

$$\Upsilon := \frac{1}{\sum_{i=1}^{n}(p_t^i)^{-\eta}/(a^i n)},$$

which should converge to the stationary output of the final good per worker, $\bar{Y}/\bar{N}$, as $n \to \infty$. Each time a Calvo shock occurs in the simulation, we update the relative price vector (p_t) and compute the statistics, and take monthly averages. For values of π between 1% and 12%, the standard deviation of monthly $\log \Upsilon$ varies between .0016 and .0017. Thus the standard deviation of $\log \Upsilon$ is very small for all π in this range. This insures that the deviation of the cross-sectional price distribution from the stationary distribution is small, and the deviation of the output per worker from the stationary output is small, when compared to the volatility of aggregate price changes shown in Figure 2.

Finally, we compute the volatility of consumption caused by the variation in menu-cost payments. Since $c/Y = 1 - \delta^+\lambda^+ - \delta^-\lambda^-$, the standard deviation of $\log c$ given Y is constant and can be directly computed. For the benchmark case $\pi = 0.03$, the monthly standard deviation of $\log c$ is 0.002%. Thus the utility losses caused to households, because firms follow the behavioral assumptions we postulate, are very small.

6 Conclusion

In this paper we studied a model where menu costs determinate the distribution of price-changes. The main departure from the previous theoretical literature is that we consider a large finite number of firms instead of a continuum. When the number of firms is finite, a firm's price increase necessarily leads to a decrease in other firms' relative prices. Thus, repricing behavior exhibits complementarity across firms, which generates repricing avalanches. The strength of this complementarity when n is large can be summarized by a quantity $0 < \theta < 1$ that is a property of the equilibrium of the continuum approximation.

Analytical results show that as $n \to \infty$ the distribution of the number of firms that reprice following a single firm's Calvo-shock approaches a mixture of Generalized Poisson Distributions. The index of dispersion (variance divided by mean) of this limit distribution is bounded below by $\frac{1}{(1-\theta)^2}$. We also show that as $\pi \to \infty$, $\theta \to 1$. The intuition is that when trend inflation is high, firms' relative prices quickly drift away from the target to which they adjust when they reprice, causing a relatively higher density of firms located near the repricing threshold. Hence, there is a higher probability for a firm's repricing to cause another firm's repricing. A sufficiently high long-run inflation rate causes high volatility in short-run inflation rates. Furthermore we show that for $\theta \sim 1$ the fluctuations of the avalanche size L substantially slow down the convergence, as $n \to \infty$, of the variance of the price level.

We calibrate the model to the U.S. experience during 1988–2005. Simulations show that a GPD fits well the distribution of avalanches but that the dispersion index under the calibrated stationary distribution substantially underestimates the dispersion index of avalanches obtained in the simulations. We argue that this occurs because in a model with a large but finite number of firms the distribution of relative prices fluctuates around the stationary distribution. Our simulation thus suggests that once we account for the dynamics, the multiplier effect derived from a firm adjusting prices by paying menu costs is larger than the one that obtains in the stationary equilibrium.

Our simulations also show that, even in a low inflation range, one observes a positive association between the level and volatility of inflation as it has been the case for the U.S. in 1988–2005.

Appendix

A Proofs

A.1 Proof of Proposition 1

If we write $S(p) = R'(p)$ and differentiate both sides of (21) we obtain,

$$S'(p) = \frac{z'(p)}{\pi p} - \left(\frac{\pi + \mu + \rho}{\pi p}\right) S(p). \tag{59}$$

Set

$$A(r) := \int_1^r -\frac{\pi + \mu + \rho}{\pi s} ds = -\frac{\pi + \mu + \rho}{\pi} \log(r).$$

Then the general solution to (59) is:

$$S(\sigma, p) = \sigma e^{A(p)} + e^{A(p)} \int_{\hat{p}}^{p} e^{-A(r)} \frac{z'(r)}{\pi r} dr = p^{-\frac{\pi+\mu+\rho}{\pi}} \left[\sigma + \int_{\hat{p}}^{p} r^{\frac{\mu+\rho}{\pi}} \frac{z'(r)}{\pi} dr\right].$$

If

$$\Gamma(p) := \int_{\hat{p}}^{p} r^{\frac{\mu+\rho}{\pi}} \frac{z'(r)}{\pi} dr,$$

$S(\sigma, p) = 0$ if and only if

$$\sigma + \Gamma(p) = 0. \tag{60}$$

Since $\Gamma'(p) := p^{\frac{\mu+\rho}{\pi}} z'(p)/\pi$ is strictly positive (negative) for $p < \hat{p}$ ($p > \hat{p}$), $\Gamma(p)$ achieves a maximum of zero at $\hat{p}$. In addition, $\Gamma''(\hat{p}) := \hat{p}^{\frac{\mu+\rho}{\pi}} z''(\hat{p})/\pi < 0$. Thus there exists $\epsilon > 0$ such that for $\sigma < \epsilon$, there are exactly two solutions $p_1(\sigma) < \hat{p} < p_2(\sigma)$ to $S(\sigma, p) = 0$ and $\lim_{\sigma \searrow 0} p_i(\sigma) = \hat{p}$, $i = 1, 2$. Moreover since

$$\frac{\partial p_i}{\partial \sigma} = -\frac{1}{\Gamma'(p_i)} \tag{61}$$

and $\Gamma'(p_i(\sigma)) \neq 0$, unless $p_i = \hat{p}$, we can prolong the function $p_1(\sigma)$ until a $\sigma_{\max}$ such that $p_1(\sigma_{\max}) = 0$ ($\sigma_{\max} = \infty$ is not ruled out) and, since for any σ, $-1/\Gamma'(p_2(\sigma))$ is uniformly bounded above, we may also prolong $p_2(\sigma)$ until $\sigma_{\max}$. Furthermore if $\sigma' > \sigma$, $p_1(\sigma') < p_1(\sigma) < \hat{p} < p_2(\sigma) < p_2(\sigma')$.

If $\Psi(\sigma) := z(p_2(\sigma)) - z(p_1(\sigma))$, then Ψ achieves a minimum of zero at $\sigma = 0$. In addition, using equation (61), we obtain

$$\Psi'(\sigma) = \pi \left[p_1^{-\frac{\mu+\rho}{\pi}} - p_2^{-\frac{\mu+\rho}{\pi}}\right] > 0,$$

provided $\sigma > 0$. Since $z(0) = -\infty$, this delivers the existence and uniqueness of $\sigma(\delta^+) < \sigma_{\max}$ such that the value matching condition

$$R(\sigma, p_2(\sigma)) - R(\sigma, p_1(\sigma)) = \delta^+ \tag{62}$$

holds at $\sigma = \sigma(\delta^+)$ for any $\delta^+ \geq 0$.

Set $p^* = p_2(\sigma(\delta^+))$ and $\underline{p} = p_1(\sigma(\delta^+))$ and set $R(\sigma(\delta^+), \cdot)$ using (21) and (24). In addition, (60) applied to $\underline{p}$ and p^*, implies

$$\int_{\underline{p}}^{p^*} r^{\frac{\mu+\rho}{\pi}} \frac{z'(r)}{\pi} dr = 0,$$

which after integration yields (27).

Following the procedure in Øksendal and Sulem [54, Chapter 9],[25] one can in fact verify that since $R(\sigma(\delta^+), p)$ satisfies (21)–(23) then

$$v(p) = \begin{cases} R(\sigma(\delta^+), p) & \text{for } \underline{p} \leq p \leq p^* \\ R(\sigma(\delta^+), p^*) - \delta^+ & \text{for } p < \underline{p} \\ \max\{R(\sigma(\delta^+), p), R(\sigma(\delta^+), p^*) - \delta^-\} & \text{for } p > p^* \end{cases}$$

is the value function and the associated optimal policy is given by (d) in the statement of the Proposition.

A.2 Properties of $\varphi(q, x)$

We summarize properties of the function $\varphi(q, x)$ in the following lemma. Here $\varphi_q(q, x)$ and $\varphi_x(q, x)$ denote the partial derivatives of $\varphi(q, x)$.

Lemma 4. *For $q > 1$, define $\varphi(q, x) := (q^x - 1)/x$ for $x \neq 0$ and $\varphi(q, 0) := \log q$. Then, $\varphi(q, x)$ satisfies the following properties.*

(a) $\varphi(q, x) > 0$, $\varphi_q(q, x) > 0$, *and* $\varphi(q, -x) = \varphi(q, x)/q^x = \varphi(q, x)/(\varphi_q(q, x)q)$

(b) $\log \varphi(q, x)$ *is strictly increasing and convex in* x,

$$\frac{\varphi_x}{\varphi}(q, x) = \frac{\log q}{1 - q^{-x}} - \frac{1}{x} > 0,$$

$$\frac{\partial}{\partial x}\left(\frac{\varphi_x}{\varphi}\right)(q, x) = \frac{1}{x^2}\left(1 - \frac{q^x (\log q^x)^2}{(q^x - 1)^2}\right) \geq 0,$$

and $(\partial/\partial x)(\varphi_x/\varphi)(q, x) \leq 1/x^2$.

[25] A major difference between our set up and the one in [54] is the absence of a diffusion term, which allows for less smoothness of the candidate value function.

(c) If and only if $x > 0$, for any y

$$\frac{\partial}{\partial q}\frac{q^x\varphi(q,y)}{\varphi(q,x+y)} = \frac{\partial}{\partial q}\frac{\varphi(q,-y)}{\varphi(q,-x-y)} > 0$$

(d) For any x and y,

$$\frac{\partial}{\partial q}\frac{\varphi(q,x)\varphi(q,y)}{\varphi(q,x+y)} > 0.$$

(e) Define a function $\phi(y) := (y\log y - y + 1)/(y-1)^2$ for $y > 0$. Then,

$$\frac{\varphi_x}{\varphi^2}(q,x) = \frac{\partial}{\partial x}\left(\frac{-1}{\varphi}\right)(q,x) = \phi(q^x)$$

is strictly positive, continuous and decreasing in x.

Proof.

(a) $\varphi(q,x) = (q^x-1)/x > 0$ for $q > 1$ and $x \neq 0$, and $\varphi(q,0) = \log q > 0$ for $q > 1$. Also, $\varphi_q(q,x) = q^{x-1} > 0$. Furthermore, $\varphi(q,-x)q^x = (1-q^x)/(-x) = \varphi(q,x)$.

(b) $\varphi_x(q,x) = \frac{\partial}{\partial x}\frac{q^x-1}{x} = \frac{1}{x^2}\left[q^x\log(q^x) - (q^x-1)\right]$. Since $y\log y - (y-1) > 0$ for $y > 0$ and $y \neq 1$, $\varphi_x(q,x)$ is strictly positive for $x \neq 0$. For $x = 0$, l'Hôpital's rule yields $\lim_{x\to 0}\varphi_x(q,x) = (\log q)^2/2 > 0$. Hence $\varphi_x(q,x) > 0$ for any x. Moreover,

$$\frac{\varphi_x}{\varphi}(q,x) = \frac{\partial}{\partial x}\log\frac{q^x-1}{x} = \frac{x}{q^x-1}\frac{(q^x\log q)x-(q^x-1)}{x^2} = \frac{\log q}{1-q^{-x}} - \frac{1}{x}$$

and

$$\frac{\partial}{\partial x}\left(\frac{\varphi_x}{\varphi}\right)(q,x) = -\frac{q^{-x}(\log q)^2}{(1-q^{-x})^2} + \frac{1}{x^2} = \frac{1}{x^2}\left(1 - \frac{q^x(\log q^x)^2}{(q^x-1)^2}\right).$$

Let $h(y) := (y-1)^2 - y(\log y)^2$ for $y > 0$. Note that $h'(y) = 2(y-1) - 2\log y - (\log y)^2$ and $h(1) = h'(1) = 0$. Also, $h''(y) = (2/y)(y-1-\log y)$ is positive for $y > 0$. Thus, $h'(y)$ is increasing in y, implying that $h'(y) < 0$ for $0 < y < 1$ and $h'(y) > 0$ for $y > 1$. Hence, $h(y)$ achieves a minimum 0 at $y = 1$ and $h(y) > 0$ for $y \neq 1$. This leads to $1 \geq y(\log y)^2/(y-1)^2$ for $y > 0$ with equality holding at $y = 1$. Thus, $(\partial^2/\partial x^2)\log\varphi(q,x) \geq 0$.

(c) Using $\varphi(q,-x) = \varphi(q,x)/q^x$ from Lemma 4(a), we have

$$\frac{q^x\varphi(q,y)}{\varphi(q,x+y)} = \frac{\varphi(q,y)}{q^y}\frac{q^{x+y}}{\varphi(q,x+y)} = \frac{\varphi(q,-y)}{\varphi(q,-x-y)}.$$

Thus, the derivative with respect to q is

$$\begin{aligned}\frac{\partial}{\partial q}\frac{\varphi(q,-y)}{\varphi(q,-x-y)} &= \frac{1}{\varphi^2(q,-x-y)}\left[q^{-y-1}\varphi(q,-x-y) - \varphi(q,-y)q^{-x-y-1}\right] \\ &= \frac{q^{-x-2y-1}}{\varphi^2(q,-x-y)}\left[\varphi(q,x+y) - \varphi(q,y)\right].\end{aligned}$$

By Lemma 4(b), $\varphi(q,x)$ is increasing in x. Hence, the last expression has the same sign as x.

(d) First we consider the case $x \neq 0$. By Lemma 4(c), we know that $\frac{\partial}{\partial q}\frac{q^x\varphi(q,y)}{\varphi(q,x+y)}$ has the same sign as x. Write

$$\frac{\partial}{\partial q}\frac{\varphi(q,x)\varphi(q,y)}{\varphi(q,x+y)} = \frac{1}{x}\left[\frac{\partial}{\partial q}\frac{q^x\varphi(q,y)}{\varphi(q,x+y)} - \frac{\partial}{\partial q}\frac{\varphi(q,y)}{\varphi(q,x+y)}\right].$$

The first term inside the square brackets has the same sign as x and so does the second term (including the negative sign) by Lemma 4(c). Hence, the entire expression is strictly positive. If $x = 0$, we have $\frac{\partial}{\partial q}\frac{\varphi(q,0)\varphi(q,y)}{\varphi(q,y)} = 1/q > 0$.

(e) Note that $y\log y > y - 1$ for $y \neq 1$. By l'Hôpital's rule,

$$\phi(1) = \lim_{y\to 1}\frac{\log y}{2(y-1)} = \lim_{y\to 1}\frac{1}{2y} = 1/2.$$

Thus, $\phi(y)$ is continuous and $\phi(y) > 0$. Moreover,

$$\phi'(y) = \frac{(y-1)^2\log y - 2(y-1)(y\log y - y + 1)}{(y-1)^4} = \frac{2(y-1) - (y+1)\log y}{(y-1)^3}. \tag{63}$$

Note that a function $\psi(y) := 2(y-1) - (y+1)\log y$ satisfies $\psi(1) = 0$ and $\psi'(y) = 1 - 1/y - \log y \leq 0$ with equality holding at $y = 1$. Hence $\psi(y)$ is positive for $0 < y < 1$ and negative for $y > 1$. Thus $\phi'(y) = \psi(y)/(y-1)^3$ is negative for $0 < 1 < y$ or $y > 1$, and $\phi(y)$ is a decreasing function for $y > 0$. Hence, $(\partial/\partial x)(-1/\varphi)(q,x) = \phi(q^x)$ is a decreasing function in x for $q > 1$. □

A.3 Proof of Lemma 2

In the next section A.3.1 we show that (34) has a unique solution q. Then equations (31)–(32) determine a unique $\underline{p}$ and equilibrium wage rate w. Equation (30) determines λ and if $\lambda < 1/\delta^+$, equations (14)–(15) deliver $c > 0$ and $N > 0$ as linear functions of Y. The assumption that Inada conditions hold, $\partial^2 U(c,N)/\partial c^2 < 0$ and $\partial^2 U(c,N)/\partial c \partial N \leq 0$ guarantees that there exists exactly one level of Y that satisfies equation (16).[26]

A.3.1 Unique solution of Equation (34)

We define functions $A(q,\pi)$ and $B(q,\pi)$ as

$$A(q,\pi) := \frac{\varphi(q,(\rho+\mu)/\pi+1-\eta)}{\varphi(q,(\rho+\mu)/\pi-\eta)}\frac{\varphi(q,-\eta)}{\varphi(q,1-\eta)}$$
$$B(q,\pi) := \frac{\varphi(q,\mu/\pi+1-\eta)}{\varphi(q,\mu/\pi)\varphi(q,1-\eta)}$$

and rewrite (34) as

$$1 = \frac{\delta^+(\rho+\mu)}{\eta-1}\frac{B(q,\pi)}{A(q,\pi)-1}. \tag{64}$$

First we investigate $B(q,\pi)$. We have $\partial B/\partial q < 0$ for any π by Lemma 4(d). Also, $\lim_{q\to 1} B(q,\pi) = \infty$ for any π, since, by l'Hôpital's rule, any ratio $\varphi(q,x)/\varphi(q,y)$ converges to 1 as $q \to 1$ and since $\lim_{q\to 1}\varphi(q,\mu/\pi) = 0$. Finally, $\lim_{q\to\infty} B(q,\pi) = 0$, because for any $x, y > 0$, $\lim_{q\to\infty}\varphi(q,y-x)/\varphi(q,y) = 0$ and $\lim_{q\to\infty}\varphi(q,-x) = 1/x$.

Next we investigate $A(q,\pi)$. Using Lemma 4(a), we have

$$A(q,\pi) = \frac{\varphi(q,-\eta)}{\varphi(q,1-\eta)}\frac{\varphi(q,1-\eta+\frac{\rho+\mu}{\pi})}{\varphi(q,-\eta+\frac{\rho+\mu}{\pi})} = \frac{\varphi(q,\eta)}{\varphi(q,\eta-1)}\frac{\varphi(q,\eta-1-\frac{\rho+\mu}{\pi})}{\varphi(q,\eta-\frac{\rho+\mu}{\pi})}. \tag{65}$$

Also using Lemma 4(a) we have $\partial \log\varphi(q,x)/\partial q = 1/(q\varphi(q,-x))$. Hence,

$$\begin{aligned}\frac{\partial A/\partial q}{A} &= \frac{1}{q}\left(\frac{1}{\varphi(q,-\eta)} - \frac{1}{\varphi(q,1-\eta)} + \frac{1}{\varphi(q,1+\frac{\rho+\mu}{\pi}-\eta)} - \frac{1}{\varphi(q,\frac{\rho+\mu}{\pi}-\eta)}\right) \\ &= \frac{1}{q}\left(\int_{-\eta}^{\min\{1-\eta,\frac{\rho+\mu}{\pi}-\eta\}}\frac{\partial}{\partial x}\left(\frac{-1}{\varphi}\right)(q,x)dx - \int_{\max\{1-\eta,\frac{\rho+\mu}{\pi}-\eta\}}^{1+\frac{\rho+\mu}{\pi}-\eta}\frac{\partial}{\partial x}\left(\frac{-1}{\varphi}\right)(q,x)dx\right).\end{aligned} \tag{66}$$

By Lemma 4(e), the integrand $(\partial/\partial x)(-1/\varphi)(q,x)$ in (66) is a positive-valued, decreasing function. Thus $\partial A/\partial q$ is strictly positive, since $-\eta < \min\{1-\eta,(\rho+\mu)/\pi-\eta\} \leq \max\{1-\eta,(\rho+\mu)/\pi-\eta\} < 1+(\rho+\mu)/\pi-\eta$. Moreover, $\lim_{q\to 1} A(q,\pi) = 1$ for any π.

[26] Again, if U is separable and linear in N, Inada conditions with respect to c suffice.

In sum, we have $\partial A/\partial q > 0$, $\lim_{q\to 1} A(q,\pi) = 1$, $\lim_{q\to\infty} A(q,\pi) < \infty$, $\partial B/\partial q < 0$, $\lim_{q\to 1} B(q,\pi) = \infty$, and $\lim_{q\to\infty} B(q,\pi) = 0$. These properties guarantee that the right-hand side of (64) decreases monotonically from $+\infty$ to 0 as q increases from 1 to $+\infty$, which insures the existence and uniqueness of solutions to (64).

Finally, we verify that the solution to (31)–(34) generates inflation π. Note that, at any instance t, $P_t^{1-\eta}$ is increased by the price adjustments of firms hit by Calvo shocks and price adjustments of firms that pay menu costs. We compute these two effects on the growth rate $(dP_t^{1-\eta}/dt)/P_t^{1-\eta}$. The effect due to Calvo shocks is written as $\mu\left((p^*)^{1-\eta} - \int p^{1-\eta} f(p)dp\right)$. By zero-profit condition (13) we have $1 = \int p^{1-\eta} f(p)dp$. Thus the above expression is reduced to

$$\mu\left((p^*)^{1-\eta} - 1\right). \tag{67}$$

The effect due to firms paying menu cost is

$$\lambda\left((p^*)^{1-\eta} - \underline{p}^{1-\eta}\right). \tag{68}$$

Summing (67) and (68), and using $\lambda/\mu = 1/(q^{\mu/\pi} - 1)$ from (30), we have

$$\frac{dP_t^{1-\eta}/dt}{P_t^{1-\eta}} = \mu\left((p^*)^{1-\eta} - 1\right) + \lambda\left((p^*)^{1-\eta} - \underline{p}^{1-\eta}\right) = \mu\left(\frac{(q^{\mu/\pi+1-\eta} - 1)\underline{p}^{1-\eta}}{q^{\mu/\pi} - 1} - 1\right).$$

Using $\underline{p}^{1-\eta} = \varphi(q,\mu/\pi)/\varphi(q,\mu/\pi + 1 - \eta)$ from (31), we obtain $\frac{dP_t^{1-\eta}/dt}{P_t^{1-\eta}} = \pi(1-\eta)$. Hence, we obtain the desired result $(dP_t/dt)/P_t = \pi$.

A.4 Non-negative average $v(p)$ implies $\lambda\delta^+ < 1$

Proof of Lemma 3. Let $(\Omega, \mathcal{F}, Q)$ be the probability space that defines the Poisson processes. Then,

$$v(p) = E^Q\left[\int_0^\infty e^{-\rho t}\left(z(p_t(\omega), t) - \delta^+ \mathbf{1}_{\{t: p_t(\omega) = \underline{p}\}}\right) dt \mid p_0 = p\right].$$

Averaging over the stationary distribution of prices F, we obtain:

$$E^F v(p) = E^F E^Q\left[\int_0^\infty e^{-\rho t}\left(z(p_t(\omega)) - \delta^+ \mathbf{1}_{\{t: p_t(\omega) = \underline{p}\}}\right) dt \mid p_0 = p\right].$$

Since $z(p_t) - \delta^+$ is bounded below by $\min\{z(p^*) - \delta^+; z(\underline{p}) - \delta^+\}$, we can apply Tonelli's theorem to obtain

$$\begin{aligned} E^F v(p) &= E^Q \int_0^\infty E^F\left[e^{-\rho t}\left(z(p_t(\omega)) - \delta^+ \mathbf{1}_{\{t: p_t(\omega) = \underline{p}\}}\right)\right] dt \\ &= \frac{1}{\rho}\left(E^F z(p) - \lambda\delta^+\right). \end{aligned}$$

Since w is necessarily positive, (13) implies that $E^F z(p) < 1$. Thus if $E^F v(p) > 0$,

$$0 < E^F \rho v(p) = E^F z(p) - \lambda\delta^+ < 1 - \lambda\delta^+.$$

A.5 Comparative statics

We establish some comparative statics with respect to π and δ as well as the limit property of q when $\delta^+ \to 0$ or $\pi \to 0$.

A.5.1 $dq/d\pi > 0$ and $dp^*/d\pi > 0$ for any $\pi > 0$

Lemma 4(b) implies that for any $y > x$,

$$\frac{\partial}{\partial u}\left(\frac{\varphi(q, y+u)}{\varphi(q, x+u)}\right) > 0. \tag{69}$$

Using (69) we obtain

$$\frac{\partial A(q,\pi)}{\partial \pi} < 0 \quad \text{and} \quad \frac{\partial B(q,\pi)}{\partial \pi} > 0.$$

Also, as shown in A.3.1, we have $\partial A/\partial q > 0$ and $\partial B/\partial q < 0$. Hence, (64) implies that $dq/d\pi > 0$ for $\pi > 0$.

From (31), we have

$$(p^*)^{\eta-1} = \frac{q^{\eta-1}\varphi(q, 1-\eta+\mu/\pi)}{\varphi(q, \mu/\pi)} = \frac{\varphi(q, \eta-1-\mu/\pi)}{\varphi(q, -\mu/\pi)}. \tag{70}$$

Equation (70) establishes that $p^*(\pi) = p^*(q(\pi), \pi)$. From Lemma 4(c) it follows that $\partial p^*/\partial q > 0$, and using (69) we conclude that $\partial p^*/\partial \pi > 0$. Since $dq/d\pi > 0$, we obtain the claim $dp^*/d\pi > 0$.

A.5.2 $dq/d\delta^+ > 0$, $dp^*/d\delta^+ > 0$, and $d(w/p^*)/d\delta^+ < 0$

Equation (64) is written as $A(q,\pi) - 1 = B(q,\pi)\delta^+(\rho+\mu)/(\eta-1)$. Then,

$$\frac{dq}{d\delta^+} = \frac{B(q,\pi)(\rho+\mu)/(\eta-1)}{(\partial A/\partial q) - (\partial B/\partial q)\delta^+(\rho+\mu)/(\eta-1)} > 0,$$

where the inequality obtains from $\partial A/\partial q > 0$ and $\partial B/\partial q < 0$. Moreover, since $dp^*/dq > 0$, we obtain $dp^*/d\delta^+ > 0$.

From (27), we have

$$\frac{w}{p^*} = \frac{\eta-1}{\eta}\frac{\varphi(q, 1-\eta+\frac{\rho+\mu}{\pi})}{q\varphi(q, -\eta+\frac{\rho+\mu}{\pi})} = \frac{\eta-1}{\eta}\frac{\varphi(q, \eta-1-\frac{\rho+\mu}{\pi})}{\varphi(q, \eta-\frac{\rho+\mu}{\pi})}. \tag{71}$$

From the above equation, we may write $w/p^* = (w/p^*)(q, \pi)$ and Lemma 4(c) guarantees that

$$\frac{\partial(w/p^*)}{\partial q} < 0.$$

Using $dq/d\delta^+ > 0$, we obtain that $d(w/p^*)/d\delta^+ < 0$.

A.5.3 Continuity of $q(\delta^+, \pi)$ at $\delta^+ = 0$

When $\delta^+ = 0$, it is optimal for firms to adjust price immediately at any time regardless of π. This implies $q = 1$ and $p^* = \hat{p}$. With all the firms pricing at $\hat{p}$, price aggregation implies $\hat{p}^{1-\eta} = 1$. Hence, $w = (\eta - 1)/\eta$ and $p^* = \hat{p} = 1$ hold.

As δ^+ decreases to 0, the right-hand side of (64) decreases to 0 for fixed q. This implies that the solution q of (64) decreases as δ^+ decreases, since $B(q, \pi)/(A(q, \pi) - 1)$ is a decreasing function in q. Moreover, the solution q converges to 1 as $\delta^+ \to 0$, since $\lim_{q\to 1} A(q, \pi) = 1$. Hence, $q(\delta^+, \pi)$ is continuous in $\delta^+ \geq 0$ and $q(0, \pi) = 1$ for any π.

A.5.4 Continuity of $q(\delta^+, \pi)$ at $\pi = 0$

When there is no inflation, the relative price does not move, unless a firm chooses to change it. Once a firm adjusts its price, it would opt to change the price again only when it receives a productivity shock. Thus, the relative price of a firm is $p^*(0) = \hat{p}(0) = w(0)\eta/(\eta - 1)$. Hence, the zero-profit condition (13) implies $1 = E^F[p^{1-\eta}] = (p^*)^{1-\eta} = (w\eta/(\eta - 1))^{1-\eta}$.

The firm pays menu costs and reprices if $p < \underline{p}(0)$ where $\underline{p}(0)$ is determined by the value matching condition (32). Using the results above, (32) is modified as

$$\frac{(\rho + \mu)\delta^+}{\eta - 1} = \varphi(q(0), \eta) - \varphi(q(0), \eta - 1). \tag{72}$$

Equation (72) shows that $q(0) > 1$, since function $h(q) := ((\eta - 1)/\eta)q^\eta - q^{\eta-1} + 1/\eta - (\rho + \mu)\delta^+(a/\tilde{a})^{1-\eta}$ is increasing in $q \geq 1$ toward infinity and $h(1) < 0$.

Now we show that $\lim_{\pi\to 0} q(\pi) = q(0)$. First, notice that for any fixed $q > 1$, using l'Hôpital's rule:

$$\lim_{y\to\infty} \frac{q^x \varphi(q, y)}{\varphi(q, x + y)} = \lim_{y\to\infty} \frac{q^x(q^y - 1)}{q^{x+y} - 1} \lim_{y\to\infty} \frac{x + y}{y} = \lim_{y\to\infty} \frac{q^x q^y \log q}{q^{x+y} \log q} = 1.$$

In addition, Lemma 4(c) states that, for $q > 1$, if $x > 0$ then $q^x\varphi(q, y)/\varphi(q, x + y)$ increases with q, and if $x < 0$, $q^x\varphi(q, y)/\varphi(q, x + y)$ decreases with q.

Let $\bar{q}$ be an upper bound of $q(\pi)$ in $\pi \in (0, \bar{\pi})$. For $\pi \in (0, \bar{\pi})$, since $\bar{q} > q(\pi) > 1$,

$$\lim_{q\to 1} \frac{\varphi(q, 1 - \eta + \mu/\pi)}{q^{1-\eta}\varphi(q, \mu/\pi)} \leq \frac{\varphi(q(\pi), 1 - \eta + \mu/\pi)}{q(\pi)^{1-\eta}\varphi(q(\pi), \mu/\pi)} < \frac{\varphi(\bar{q}, 1 - \eta + \mu/\pi)}{\bar{q}^{1-\eta}\varphi(\bar{q}, \mu/\pi)}.$$

By l'Hôpital's rule, the leftmost side is equal to 1 for any π. Thus, taking the limit $\pi \to 0$ for all the terms, we obtain

$$\lim_{\pi \to 0} \frac{\varphi(q(\pi), 1-\eta+\mu/\pi)}{q(\pi)^{1-\eta}\varphi(q(\pi), \mu/\pi)} = 1$$

and if $q_o := \lim_{\pi\to 0} q(\pi)$ then

$$\lim_{\pi \to 0} \frac{\varphi(q(\pi), 1-\eta+\mu/\pi)}{\varphi(q(\pi), \mu/\pi)} = q_o^{1-\eta}.$$

Similarly,

$$\lim_{\pi \to 0} \frac{\varphi(q(\pi), 1-\eta+(\mu+\rho)/\pi)}{\varphi(q(\pi), -\eta+(\mu+\rho)/\pi)} = q_o.$$

Applying these limits to (31) and (33), we obtain $\lim_{\pi\to 0} w(\pi) = (\eta-1)/\eta$ and $\lim_{\pi\to 0} \underline{p}(\pi) = q_o$. Substituting these into (32) yields

$$\frac{(\rho+\mu)\delta^+}{\eta-1} = \varphi(q_o, \eta) - \varphi(q_o, \eta-1).$$

This equation is identical to (72). Hence, $q(\pi)$ is continuous at $\pi = 0$.

A.6 Existence of equilibrium when δ^+ is constant

Proof of Proposition 2. Since for any p, $v(p) \geq v(p^*(\delta^+,\pi)) - \delta^+ = \frac{z(p^*(\delta^+,\pi))}{\rho} - \delta^+$ if

$$z(p^*(\delta^+,\pi), w(\delta^+,\pi)) = p^*(\delta^+,\pi)^{1-\eta}\left(1 - \frac{w(\delta^+,\pi)}{p^*(\delta^+,\pi)}\right) \geq \rho\delta^+,$$

$v(p) > 0$. Also since $p^*(\delta^+,\pi) \geq \hat{p}(\delta^+,\pi)$, we have $w(\delta^+,\pi)/p^*(\delta^+,\pi) \leq w(\delta^+,\pi)/\hat{p}(\delta^+,\pi) = (\eta-1)/\eta$. Hence,

$$z(p^*(\delta^+,\pi), w(\delta^+,\pi)) \geq \frac{1}{\eta}(p^*(\delta^+,\pi))^{1-\eta}.$$

In addition, as we proved in A.5.1, $p^*(\delta^+,\pi)$ is an increasing function of π for each δ^+. Furthermore, $p^*(\delta^+,\pi) \leq q(\delta^+,\pi)$, since $q(\delta^+,\pi)\underline{p}(\delta^+,\pi) = p^*(\delta^+,\pi)$ and from (31), $\underline{p}(\delta^+,\pi) \leq 1$. Thus for any δ^+ and any $\pi \leq \bar{\pi}$,

$$z(p^*(\delta^+,\pi), w(\delta^+,\pi)) \geq \frac{1}{\eta}(p^*(\delta^+,\pi))^{1-\eta} \geq \frac{1}{\eta}(p^*(\delta^+,\bar{\pi}))^{1-\eta} \geq \frac{1}{\eta}(q(\delta^+,\bar{\pi}))^{1-\eta}.$$

Since q is continuous and $\lim_{\delta^+\to 0} q(\delta^+,\bar{\pi}) = 1$ (see A.5.3), there exists $\delta_1 > 0$ such that,

$$(q(\delta^+,\bar{\pi}))^{1-\eta} > \frac{1}{2}$$

for $\delta^+ \in [0, \delta_1]$. Hence if $\bar{\delta} = \min\{\frac{1}{2\eta\rho}, \delta_1\}$,

$$z(p^*(\delta^+, \pi), w(\delta^+, \pi)) \geq \frac{1}{\eta}(q(\delta^+, \bar{\pi}))^{1-\eta} > \rho\bar{\delta} \geq \rho\delta^+ \tag{73}$$

for each $\pi \in [0, \bar{\pi}]$ and $\delta^+ \in [0, \bar{\delta}]$. This together with Lemma 3 establishes Item (a).

Furthermore, Proposition 3(a) showed that for each δ^+,

$$\lim_{\pi \to \infty} (\log q(\pi, \delta^+))/\pi = (\log(1 + \delta^+ \eta\mu))/\mu,$$

and thus,

$$\lim_{\pi \to \infty} q(\pi, \delta^+)^{\frac{\mu}{\pi}} = e^{(\log(1+\delta^+\eta\mu))} = 1 + \delta^+ \eta\mu.$$

Hence,

$$\delta^+ \lim_{\pi \to \infty} \lambda(\pi, \delta^+) = \delta^+ \lim_{pi \to \infty} \frac{\mu}{q(\pi, \delta^+)^{\mu/\pi} - 1} = \frac{1}{\eta} < 1,$$

establishing Item (b).

A.7 Proof of Proposition 3

Proof of Item (a). First, we show that $\log q(\pi) \to \infty$ as $\pi \to \infty$. Suppose to the contrary that there exists a sequence $\pi_n \to \infty$ such that $\log q_n = q(\pi_n)$ is bounded above. Then $A(q_n, \pi_n)$ converges to 1 as $\pi_n \to \infty$. Hence, (64) implies that $B(q_n, \pi_n)$ must converge to 0 as $\pi_n \to \infty$. The numerator of B, $\varphi(q, 1-\eta+\mu/\pi)$, is strictly positive and increasing in q for $q > 1$. Therefore, the denominator must tend to infinity. However, $\varphi(q, 1-\eta)$ is bounded. Thus, $\varphi(q_n, \mu/\pi_n)$ must tend to infinity. For a fixed $\log q$, we have $\lim_{\pi \to \infty}(e^{\mu(\log q)/\pi} - 1)/(\mu/\pi) = \lim_{\pi \to \infty}(e^{\mu(\log q)/\pi}\mu(\log q)/(-\pi^2))/(\mu/(-\pi^2)) = \log q$. Since $\log q_n$ is bounded by our hypothesis, this contradicts the divergence of $\varphi(q_n, \mu/\pi_n)$. Hence, $\log q(\pi) \to \infty$ as $\pi \to \infty$.

Next, we show asymptotic relations for $A(q, \pi)$ and $B(q, \pi)$. Function $\varphi(q, x) = (q^x - 1)/x$ for fixed $q > 1$ is analytic in region $x < 0$, and so is $\log \varphi(q, x)$. Thus, a Taylor series expansion of $\log \varphi(q, x)$ around $x = -\eta$ yields

$$\log \varphi(q, -\eta + u) - \log \varphi(q, -\eta) = u \frac{\varphi_x}{\varphi}(q, -\eta) + \frac{\partial}{\partial x} \frac{\varphi_x}{\varphi}(q, x)\bigg|_{x=x_1} \frac{u^2}{2}$$

for $|u| < 1$ where $x_1 \in [-\eta, -\eta + u]$. Similar expansion around $x = 1 - \eta$ yields, for some $x_2 \in [1-\eta, 1-\eta+u]$,

$$\log \varphi(q, 1-\eta+u) - \log \varphi(q, 1-\eta) = u \frac{\varphi_x}{\varphi}(q, 1-\eta) + \frac{\partial}{\partial x} \frac{\varphi_x}{\varphi}(q, x)\bigg|_{x=x_2} \frac{u^2}{2}.$$

Note that, from Lemma 4(b), the coefficients of u^2 terms in the above two equations are uniformly bounded for $q \in [1, \infty]$. Moreover, Lemma 4(b) gives

$$\frac{\varphi_x}{\varphi}(q, 1-\eta) - \frac{\varphi_x}{\varphi}(q, -\eta) = \int_{-\eta}^{1-\eta} \frac{1}{x^2} \left(1 - \frac{q^x (\log q^x)^2}{(q^x - 1)^2}\right) dx. \tag{74}$$

Using notation $u := (\rho + \mu)/\pi$, we obtain the first-order Taylor expansion of $\log A$ around $u = 0$ as

$$\begin{aligned}\log A(q, \pi) &= \log \varphi(q, 1-\eta+u) - \log \varphi(q, 1-\eta) - \log \varphi(q, -\eta+u) + \log \varphi(q, -\eta) \\ &= \left(\frac{\varphi_x}{\varphi}(q, 1-\eta) - \frac{\varphi_x}{\varphi}(q, -\eta)\right) u + O(u^2).\end{aligned}$$

Combining with (74), we obtain

$$\pi \log A(q, \pi) = (\rho + \mu) \int_{-\eta}^{1-\eta} \frac{1}{x^2} \left(1 - \frac{q^x (\log q^x)^2}{(q^x - 1)^2}\right) dx + O(u).$$

We have previously shown that $\lim_{\pi\to\infty} q(\pi) = \infty$. Thus, $\lim_{\pi\to\infty} q^x(\pi)(\log q^x(\pi))^2/(q^x(\pi)-1)^2 = 0$ for $x < 0$. Also note $\int_{-\eta}^{1-\eta} 1/x^2 dx = 1/(\eta(\eta-1))$. Thus, $\lim_{\pi\to\infty} \pi \log A(q(\pi), \pi) = (\rho + \mu)/(\eta(\eta - 1))$. This implies $\lim_{\pi\to\infty} A(q(\pi), \pi) = 1$. Thus, $\lim_{\pi\to\infty}(A(q(\pi), \pi) - 1)/\log A(q(\pi), \pi) = 1$ by l'Hôpital's rule. Combining with the above result yields

$$\pi(A(q(\pi), \pi) - 1) \to \frac{\rho + \mu}{\eta(\eta - 1)} \quad \text{as } \pi \to \infty. \tag{75}$$

Moreover, a Taylor series expansion of $\varphi(q, 1-\eta+\mu/\pi)$ around $x = 1-\eta$ yields, for some $x_3 \in [1-\eta, 1-\eta+\mu/\pi]$,

$$\varphi(q, 1-\eta+\mu/\pi) = \varphi(q, 1-\eta) + \frac{x_3 q^{x_3} \log(q) - (q^{x_3} - 1)}{x_3^2} \frac{\mu}{\pi}.$$

Since $x_3 < 0$ for sufficiently large π such that $1 - \eta + \mu/\pi < 0$, the final term tends to 0 as $\pi \to \infty$ and $q(\pi) \to \infty$. Thus we have,

$$\varphi(q(\pi), \mu/\pi) B(q(\pi), \pi) = \frac{\varphi(q(\pi), 1-\eta+\mu/\pi)}{\varphi(q(\pi), 1-\eta)} \to 1 \quad \text{as } \pi \to \infty. \tag{76}$$

Substituting (75) and (76) into (64), we obtain $\lim_{\pi\to\infty}(\log q(\pi))/\pi = (\log(1 + \delta^+\eta\mu))/\mu$. Finally, applying the above limit to (30), $\lambda(\pi) = \mu/(e^{\mu(\log q(\pi))/\pi} - 1)$, we obtain $\lim_{\pi\to\infty} \lambda(\pi) = 1/(\delta^+\eta)$. □

Proof of Item (b). Recall that (70) states $(p^*)^{\eta-1} = \frac{\varphi(q,1-\eta+\mu/\pi)}{\varphi(q,\mu/\pi)} q^{\eta-1}$. First, as $\pi \to \infty$ and $q(\pi) \to \infty$, $\varphi(q(\pi), 1-\eta+\mu/\pi)$ converges to a positive constant. Second, we know from Item (a) that $\log q$ grows linearly in π for large π. This implies that, for large π, $q^{\eta-1} = e^{(\eta-1)\log q}$ grows exponentially in π, while $\varphi(q, \mu/\pi) = (\pi/\mu)(e^{\mu(\log q)/\pi} - 1)$ grows only linearly in π. Therefore, the numerator dominates the denominator for large π. Thus p^* grows unboundedly as π increases. $\square$

In order to prove Item (c), we need the following lemma.

Lemma 5.

$$\frac{d \log q}{d\pi} \sim \frac{\log q}{\pi}.$$

Proof. We evaluate the asymptotic behavior of

$$\frac{d \log q}{d\pi} = \frac{-1}{q} \frac{\partial A/\partial \pi - \delta_o \partial B/\partial \pi}{\partial A/\partial q - \delta_o \partial B/\partial q},$$

where $\delta_o := \delta(\rho+\mu)/(\eta-1)$. First we investigate $\partial A/\partial q$ in the denominator. Using (66) and notation $u = (\rho+\mu)/\pi$, we have for sufficiently large π such that $u < 1$,

$$\begin{aligned}
\frac{\partial A}{\partial q} &= \frac{A}{q}\left[\int_{-\eta}^{\min\{1-\eta,u-\eta\}} \frac{\partial}{\partial x}\left(\frac{-1}{\varphi}\right)(q,x)dx - \int_{\max\{1-\eta,u-\eta\}}^{1+u-\eta} \frac{\partial}{\partial x}\left(\frac{-1}{\varphi}\right)(q,x)dx\right] \\
&= \frac{Au}{q}\left[\frac{\partial}{\partial x}\left(\frac{-1}{\varphi}\right)(q,x_4) - \frac{\partial}{\partial x}\left(\frac{-1}{\varphi}\right)(q,x_5)\right] \\
&= \frac{Au}{q}\int_{x_5}^{x_4} \frac{\partial^2}{\partial x^2}\left(\frac{-1}{\varphi}\right)(q,x)dx = \frac{Au}{q}\int_{x_4}^{x_5} \frac{\partial^2}{\partial x^2}\left(\frac{1}{\varphi}\right)(q,x)dx
\end{aligned}$$

where $x_4 \in [-\eta, u-\eta]$ and $x_5 \in [1-\eta, 1+u-\eta]$ and $x_4 < x_5 < 0$ for large π.

From Lemma 4(e) and equation (63), we have

$$\frac{\partial^2}{\partial x^2}\left(\frac{-1}{\varphi}\right)(q,x) = \frac{\partial}{\partial x}\frac{q^x \log(q^x) - q^x + 1}{(q^x-1)^2} = \frac{1}{(q^x-1)^2}\left(2 - \frac{q^x+1}{q^x-1}\log(q^x)\right) q^x \log q.$$

Substituting this into the equation above, we obtain

$$\begin{aligned}
\frac{\partial A}{\partial q} &= \frac{Au}{q}\left[\frac{-q^{x_6}\log q}{(q^{x_6}-1)^2}\left(2 - \frac{q^{x_6}+1}{q^{x_6}-1}\log(q^{x_6})\right)\right] \quad \text{for some } x_6 \in [x_4, x_5] \\
&= \frac{(\log q)^2}{\pi q^{1-x_6}} \frac{-x_6 A(\rho+\mu)}{(1-q^{x_6})^2}\left(\frac{2}{x_6 \log q} + \frac{1+q^{x_6}}{1-q^{x_6}}\right) \\
&= \frac{(\log q)^2}{\pi q^{1-x_6}}(-x_6 A(\rho+\mu))(1 - O(1/\log q))
\end{aligned}$$

where the last equality uses the fact that q^{x_6} tends to 0 as a power function with exponent $x_6 < 0$, and hence it is dominated by $1/\log q$ for large π.

We also obtain

$$
\begin{aligned}
-\frac{\partial B}{\partial q} &= -\frac{\partial \log B}{\partial q} B = \left(\frac{1}{\varphi(q, -\mu/\pi)} + \frac{1}{\varphi(q, \eta - 1)} - \frac{1}{\varphi(q, \eta - 1 - \mu/\pi)} \right) \frac{B}{q} \\
&= \left(\frac{1}{\varphi(q, -\mu/\pi)} + \int_{\eta-1-\mu/\pi}^{\eta-1} \frac{\partial}{\partial x} \left(\frac{1}{\varphi} \right) (q, x) dx \right) \frac{B}{q} \\
&= \left(\frac{1}{\varphi(q, -\mu/\pi)} - \frac{\mu}{\pi} \frac{\varphi_x}{\varphi^2}(q, x_7) \right) \frac{B}{q} \quad \text{where } x_7 \in [\eta - 1 - \mu/\pi, \eta - 1].
\end{aligned}
$$

By Lemma 4(e), we have $(\varphi_x/\varphi^2)(q, x_7) = O(q^{-x_7} \log q)$, which is dominated by $1/\varphi(q, -\mu/\pi)$ for $x_7 > 0$. Thus we obtain that $\pi^2 q(\partial A/\partial q - \delta_o \partial B/\partial q)$ is equal to

$$
\frac{\pi (\log q)^2}{q^{-x_6}} (-x_6 A(\rho + \mu)) \left(1 - O\left(\frac{1}{\log q} \right) \right) + \left(\frac{\pi^2 \delta_o}{\varphi(q, -\mu/\pi)} - O\left(\frac{\pi \log q}{q^{x_7}} \right) \right) B.
$$

Since q grows asymptotically exponentially as $\pi \to \infty$, both q^{-x_6} and q^{x_7} (with $x_6 < 0$ and $x_7 > 0$) grow exponentially. From (75) and (76) we have $\lim_{\pi\to\infty} A(q(\pi), \pi) = 1$ and $B(q(\pi), \pi) \sim_{\pi\to\infty} 1/\varphi(q(\pi), \mu/\pi)$. From Item (a) we have $\lim_{\pi\to\infty} \varphi(q(\pi), \mu/\pi)/\pi = \delta^+\eta$, which implies that $(\log q(\pi))/\pi$ converges to a positive constant as well as that $\upsilon := \lim_{\pi\to\infty} \varphi(q(\pi), -\mu/\pi)/\pi$ is a positive constant. Collecting these results, we obtain

$$
\lim_{\pi\to\infty} \pi^2 q \left(\frac{\partial A}{\partial q} - \delta_o \frac{\partial B}{\partial q} \right) = \frac{\delta_o}{\delta^+ \eta \upsilon}, \qquad \text{where } \upsilon = \lim_{\pi\to\infty} \frac{\varphi(q(\pi), -\mu/\pi)}{\pi}.
$$

Next, we turn to the numerator $-\partial A/\partial \pi + \delta_o \partial B/\partial \pi$. Using Lemma 4(b), we calculate $\partial A/\partial \pi$ as

$$
\begin{aligned}
\frac{\partial A/\partial \pi}{A} &= \frac{\partial \log A}{\partial \pi} = \left(\frac{\varphi_x}{\varphi}(q, \eta - 1 - u) - \frac{\varphi_x}{\varphi}(q, \eta - u) \right) \frac{d(-u)}{d\pi} \\
&= -\frac{\rho + \mu}{\pi^2} \int_{\eta-1-u}^{\eta-u} \frac{1}{x^2} \left(1 - \frac{q^x (\log q^x)^2}{(q^x - 1)^2} \right) dx. \qquad (77)
\end{aligned}
$$

Thus, $\lim_{\pi\to\infty} \pi^2 \partial A/\partial \pi = -(\rho + \mu) \int_{\eta-1}^{\eta} x^{-2} dx = -(\rho + \mu)/(\eta(\eta - 1))$.

Using Lemma 4(b), we have

$$
\begin{aligned}
\frac{\partial B}{\partial \pi} &= \left(\frac{\varphi_x}{\varphi}(q, 1 - \eta + \mu/\pi) - \frac{\varphi_x}{\varphi}(q, \mu/\pi) \right) \left(\frac{-\mu}{\pi^2} \right) B(q(\pi), \pi) \qquad (78) \\
&= \left(\frac{\log q}{1 - q^{\eta-1-\mu/\pi}} + \frac{1}{\eta - 1 - \mu/\pi} - \frac{\log q}{1 - q^{-\mu/\pi}} + \frac{\pi}{\mu} \right) \left(\frac{-\mu}{\pi^2} \right) B(q(\pi), \pi).
\end{aligned}
$$

Rearranging terms, we obtain

$$\begin{aligned}\frac{\partial B}{\partial \pi} &= \left(-O\left(\frac{\log q}{\pi q^{\eta-1}}\right) + \frac{1}{\pi(\eta-1)/\mu - 1} - \frac{\log q}{\varphi(q,-\mu/\pi)} + 1\right)\frac{B(q(\pi),\pi)}{-\pi} \\ &= \left(\frac{\log q}{\varphi(q,-\mu/\pi)} - 1 - O(\pi^{-1})\right)\frac{B(q(\pi),\pi)}{\pi}.\end{aligned}$$

Using (76) and $\pi/\varphi(q,\mu/\pi) \sim 1/(\delta^+\eta)$ from Item (a), we have

$$\lim_{\pi\to\infty} \pi^2 \frac{\partial B}{\partial \pi} = \left(\frac{1}{\upsilon}\lim_{\pi\to\infty}\frac{\log q}{\pi} - 1\right)\frac{1}{\delta^+\eta}.$$

Collecting the results above, we obtain

$$\lim_{\pi\to\infty}\frac{\pi^2(-\partial A/\partial\pi + \delta_o \partial B/\partial \pi)}{\pi^2 q(\partial A/\partial q - \delta_o \partial B/\partial q)} = \left(\frac{\rho+\mu}{\eta(\eta-1)} + \left(\frac{1}{\upsilon}\lim_{\pi\to\infty}\frac{\log q}{\pi} - 1\right)\frac{\delta_o}{\delta^+\eta}\right)\frac{\delta^+\eta\upsilon}{\delta_o} = \lim_{\pi\to\infty}\frac{\log q}{\pi}$$

where we used $\delta_o = \delta^+(\rho+\mu)/(\eta-1)$. Hence, we obtain the desired result: $d\log q/d\pi \sim (\log q)/\pi$. □

Proof of Item (c). The real wage w is determined by (31) and (33) as

$$w = \frac{\eta-1}{\eta}\frac{\varphi(q,1-\eta+(\rho+\mu)/\pi)}{\varphi(q,-\eta+(\rho+\mu)/\pi)}\left(\frac{\varphi(q,1-\eta+\mu/\pi)}{\varphi(q,\mu/\pi)}\right)^{1/(\eta-1)}. \tag{79}$$

Note that $\varphi(q(\pi),\mu/\pi)$ tends to infinity as $\pi\to\infty$ and all the other terms with φ are bounded for $q>1$. Thus, w converges to 0 as π increases.

Rewrite the right-hand side of (79) using $u = (\rho+\mu)/\pi$ as

$$\frac{\eta-1}{\eta(\eta-1-\mu/\pi)^{1/(\eta-1)}}\frac{\eta-u}{\eta-1-u}\frac{(1-q^{1-\eta+u})(1-q^{1-\eta+\mu/\pi})^{1/(\eta-1)}}{(1-q^{-\eta+u})\varphi(q,\mu/\pi)^{1/(\eta-1)}}. \tag{80}$$

The first two fractions are monotonically decreasing in π. We focus on the third fraction. We have

$$\frac{d}{d\pi}\left(1-q^{1-\eta+\mu/\pi}\right) = -q^{1-\eta+\mu/\pi}\left((1-\eta+\mu/\pi)\frac{d\log q}{d\pi} - \frac{\mu\log q}{\pi^2}\right).$$

By Lemma 5(a), we have $d(\log q)/d\pi \sim (\log q)/\pi \sim (\log(1+\mu\delta\eta))/\mu$. Hence, for any small $\epsilon>0$ there exists π_o such that for all $\pi>\pi_o$, $|d(\log q)/d\pi - (\log q)/\pi| < \epsilon$, $|d(\log q)/d\pi - (\log(1+\mu\delta\eta))/\mu| < \epsilon$, and $|(\log q)/\pi - (\log(1+\mu\delta\eta))/\mu| < \epsilon$ hold. Hence, we have

$$\frac{d}{d\pi}\left(1-q^{1-\eta+\mu/\pi}\right) < \frac{1}{q^{\eta-1-\mu/\pi}}\left[(\eta-1)\left(\frac{\log(1+\mu\delta\eta)}{\mu}+\epsilon\right)+\frac{\epsilon\mu}{\pi}\right].$$

Since q asymptotically grows exponentially in π, the left-hand side is bounded by a function exponentially decreasing to 0. The same analysis holds true for functions $(1-q^{1-\eta+\mu/\pi})^{1/(\eta-1)}$, $1-q^{1-\eta+u}$, and $1-q^{-\eta+u}$. Since all of these functions are bounded above by 1 and bounded below by positive constants for sufficiently large π, the logarithms of these functions also have derivatives exponentially decreasing in π for large π.

Next, we examine the derivative of $\log\varphi(q,\mu/\pi)$. For $\pi>\pi_o$ we have the following inequality:

$$\begin{aligned}\frac{d\log\varphi(q,\mu/\pi)}{d\pi} &= \frac{1}{\pi}+\frac{\mu}{\pi(1-e^{-\mu(\log q)/\pi})}\left(\frac{d\log q}{d\pi}-\frac{\log q}{\pi}\right)\\ &> \frac{1}{\pi}\left(1-\frac{\mu\epsilon}{1-e^{\mu\epsilon}/(1+\mu\delta\eta)}\right).\end{aligned}$$

Thus, the left-hand side is bounded from below by a function that declines as $1/\pi$.

Combining the results, the derivative of the logarithm of the third fraction of (80),

$$\frac{d}{d\pi}\left(\log\frac{(1-q^{1-\eta+u})(1-q^{1-\eta+\mu/\pi})^{1/(\eta-1)}}{(1-q^{-\eta+u})\varphi(q,\mu/\pi)^{1/(\eta-1)}}\right),$$

consists of three terms that are bounded by exponentially declining functions and one term, with negative sign, which is bounded below by a function declining as $1/\pi$. Thus, the negative term dominates the other terms for large π. Hence, the third fraction is a decreasing function in π.

Since all fractions in (80) are decreasing in π for large π, we obtain that w is asymptotically decreasing in π. □

A.8 Existence of equilibrium when $\delta(\pi)$ is a decreasing function

Proof of Proposition 4. Choose

$$\bar{\pi}>\max\left\{\rho+\mu,\ \frac{\rho+\mu}{\eta-1}\right\}.$$

Let $\bar{\delta}>0$ be chosen as in Proposition 2. For $\delta_0\le\bar{\delta}$, and $\sigma_\delta>0$, set

$$q(\pi):=q(\delta(0),\pi).$$

Then $(q(\pi))^{1-\eta}\le(q(\delta(\pi),\pi))^{1-\eta}$. Moreover, from (73),

$$(q(\pi))^{1-\eta}>\rho\delta(0)\ge\rho\delta(\pi)$$

for $\pi \leq \bar{\pi}$. To establish that this inequality continues to hold for $\pi > \bar{\pi}$, it suffices to show that $d(q^{1-\eta})/d\pi > \rho\delta'(\pi)$ if $(q(\pi))^{1-\eta} = \rho\delta(\pi)$. This is equivalent to show

$$\frac{d\log q}{d\pi} < \frac{1}{1-\eta}\frac{\delta'(\pi)}{\delta(\pi)} = \frac{\sigma_\delta}{\eta-1}.$$

If $\delta_o := \delta(0)(\rho+\mu)/(\eta-1)$,

$$\frac{d\log q}{d\pi} = \frac{1}{q}\frac{-\partial A/\partial\pi + \delta_o\partial B/\partial\pi}{\partial A/\partial q - \delta_o\partial B/\partial q} \leq \frac{(-\partial A/\partial\pi + \delta_o\partial B/\partial\pi)\pi/A}{(\partial A/\partial q)q\pi/A} \tag{81}$$

where the inequality holds because $\partial B/\partial q < 0$. To establish our result it suffices to find a uniform upper bound for the right-hand side of (81) which does not depend on σ_δ.

First, we obtain $-(\partial A/\partial\pi)\pi/A$ from (77) as

$$-\pi\frac{\partial\log A}{\partial\pi} = \frac{\rho+\mu}{\pi}\int_{\eta-1-u}^{\eta-u}\frac{1}{x^2}\left(1-\frac{q^x(\log q^x)^2}{(q^x-1)^2}\right)dx$$

where $u = (\rho+\mu)/\pi$. Lemma 4(b) implies that the integrand is positive and bounded above by $1/x^2$. Moreover, using $\eta-u > \eta-1-u > 0$ we obtain

$$-\pi\frac{\partial\log A}{\partial\pi} \leq \frac{\rho+\mu}{\pi}\int_{\eta-1-u}^{\eta-u}\frac{1}{x^2}dx \leq \frac{\rho+\mu}{\pi}\frac{1}{(\eta-1-u)^2}.$$

Since we have set $\bar{\pi} > (\rho+\mu)/(\eta-1)$, $\eta-1-u$ is uniformly bounded below by a positive value for $\pi > \bar{\pi}$. Hence, $-(\partial A/\partial\pi)\pi/A$ is uniformly bounded above for $\pi > \bar{\pi}$, and this bound is independent of σ_δ.

Second, using (66) we obtain

$$\begin{aligned}\frac{\partial\log A}{\partial q} &= \frac{1}{q}\left(\int_{-\eta}^{u-\eta}\frac{\partial}{\partial x}\left(\frac{-1}{\varphi}\right)(q,x)dx - \int_{1-\eta}^{1+u-\eta}\frac{\partial}{\partial x}\left(\frac{-1}{\varphi}\right)(q,x)dx\right)\\ &\geq \frac{u}{q}\left(\frac{\partial}{\partial x}\left(\frac{-1}{\varphi}\right)(q,u-\eta) - \frac{\partial}{\partial x}\left(\frac{-1}{\varphi}\right)(q,1-\eta)\right)\\ &= \frac{u}{q}\left(\phi(q^{u-\eta}) - \phi(q^{1-\eta})\right)\end{aligned}$$

where the inequality holds from the fact that $(\partial/\partial x)(-1/\varphi)(q,x)$ is a positive-valued, decreasing function of x by Lemma 4(e). Since we chose $\bar{\pi} > \rho+\mu$, we have $u < 1$ for any $\pi > \bar{\pi}$. With $u-\eta < 1-\eta < 0$, the mean value theorem implies that there exists $y^* \in [q^{u-\eta}, q^{1-\eta}]$ such that

$$\phi(q^{u-\eta}) - \phi(q^{1-\eta}) = (u-1)\phi'(y^*).$$

Note that $y^* \le (q(\pi))^{1-\eta} \le (q(0))^{1-\eta} < 1$ where the last inequality was obtained in A.5.4. Thus we have $\phi'(y^*) < 0$ by (63) in the proof of Lemma 4(e). Also, $1-u > 1-(\rho+\mu)/\bar{\pi} > 0$. Hence, we obtain a positive lower bound

$$\frac{q\pi}{A}\frac{\partial A}{\partial q} \ge (\rho+\mu)(1-(\rho+\mu)/\bar{\pi})(-\phi'(y^*)).$$

Finally, we have from (78)

$$\pi\frac{\partial B}{\partial \pi} = \left[\frac{\varphi_x}{\varphi}(q,\mu/\pi) - \frac{\varphi_x}{\varphi}(q,1-\eta+\mu/\pi)\right]\frac{\mu}{\pi}B(q,\pi).$$

Since $(\varphi_x/\varphi)(q,x) = (\log q)/(1-q^{-x}) - 1/x > 0$, we obtain

$$\pi\frac{\partial B}{\partial \pi} < \frac{\varphi_x}{\varphi}(q,\mu/\pi)\frac{\mu}{\pi}B(q,\pi) = \left(\frac{\log q^{\mu/\pi}}{1-q^{-\mu/\pi}} - 1\right)B(q,\pi).$$

The function $h(y) := (\log y^{-1})/(1-y)$ is a positive, decreasing function of $y < 1$. Thus, $h(e^{-\mu(\log q(\pi))/\pi})$ is uniformly bounded on $\pi > \bar{\pi}$ if $(\log q(\pi))/\pi$ is uniformly bounded on $\pi > \bar{\pi}$. Proposition 3(a) established that $\lim_{\pi\to\infty}(\log q(\pi))/\pi = (\log(1+\mu\eta\delta(0)))/\mu < \infty$. Moreover, $(\log q(\bar{\pi}))/\bar{\pi} < \infty$ and $(\log q(\pi))/\pi$ is continuous in π. Hence we obtain a uniform bound for $(\log q(\pi))/\pi$ on $\pi > \bar{\pi}$.

It suffices thus to find a uniform bound for $B(q(\pi),\pi)$ at π such that $(q(\pi))^{1-\eta} = \rho\delta(\pi)$. From (73) we have

$$(q(\bar{\pi}))^{1-\eta} > \eta\rho\delta(0) > \rho\delta(\pi) = (q(\pi))^{1-\eta},$$

which implies that $q(\bar{\pi}) < q(\pi)$. Since $\partial B/\partial q < 0$ it suffices to uniformly bound $B(q(\bar{\pi}),\pi)$ from above. Moreover, since $\partial B/\partial \pi > 0$, the uniform upper bound is given by $\lim_{\pi\to\infty} B(q(\bar{\pi}),\pi)$ which is finite because $q(\bar{\pi}) > 1$, and again is independent of σ_δ.

Finally, for any q and π,

$$\frac{1}{A} = \frac{\varphi(q,1-\eta)\varphi(q,-\eta+(\rho+\mu)/\pi)}{\varphi(q,-\eta)\varphi(q,1-\eta+(\rho+\mu)/\pi)} \le 1.$$

Therefore, if evaluated at $q(\pi)^{1-\eta} = \rho\delta(\pi)$, $\delta_o(\partial B/\partial \pi)\pi/A$ is uniformly bounded above for $\pi > \bar{\pi}$, and the bound is independent of σ_δ.

Hence it suffices to choose $\sigma_\delta = (\eta-1)K$ where K is the bound we obtained for the right-hand side of (81).

A.9 Derivation of $\theta_0(s)$, θ_0 and θ in Section 3.4

We want to evaluate

$$\theta_0(s) := \lim_{\nu\to 0}\frac{m_s'(\nu)}{m_s(\nu)} = \lim_{\nu\to 0}\frac{dm_s'(\nu)/d\nu}{dm_s(\nu)/d\nu}.$$

By differentiation we obtain

$$\frac{dm_s(\nu)}{d\nu} = \frac{d}{d\nu}\int_s^{s+\nu/\log q} g(\tau)d\tau = \frac{g(s+\nu/\log q)}{\log q}$$
$$\frac{dm_s'(\nu)}{d\nu} = \frac{d}{d\nu}\int_0^{\nu_s'(\nu)/\log q} g(\tau)d\tau = \frac{d\nu_s'}{d\nu}\frac{g(\nu_s'(\nu)/\log q)}{\log q}.$$

By differentiation, we also obtain

$$\frac{d\nu_s'(\nu)}{d\nu} = \frac{d\log(P'/P)}{d\nu} = \frac{1}{1-\eta}\left((p^*)^{1-\eta}\frac{dm_s(\nu)}{d\nu} - (p\cdot p^{1-\eta}f(p))\big|_{p=p(s)e^{\nu}}\right).$$

Using

$$f(p) = \frac{p^{\mu/\pi-1}}{\underline{p}^{\mu/\pi}\varphi(q,\mu/\pi)}, \quad g(s) = \frac{(\log q)q^{s\mu/\pi}}{\varphi(q,\mu/\pi)}, \quad \text{and} \quad \underline{p}^{1-\eta} = \frac{\varphi(q,\mu/\pi)}{\varphi(q,\mu/\pi+1-\eta)},$$

we obtain

$$\lim_{\nu\to 0}\frac{d\nu_s'(\nu)}{d\nu} = \frac{1}{1-\eta}\left((p^*)^{1-\eta}\frac{q^{s\mu/\pi}}{\varphi(q,\mu/\pi)} - \frac{p(s)^{\mu/\pi+1-\eta}}{\underline{p}^{\mu/\pi}\varphi(q,\mu/\pi)}\right) = \frac{q^{s\mu/\pi}\underline{p}^{1-\eta}(q^{1-\eta}-q^{s(1-\eta)})}{(1-\eta)\varphi(q,\mu/\pi)}$$
$$= \frac{q^{s\mu/\pi}(q^{1-\eta}-q^{s(1-\eta)})}{(1-\eta)\varphi(q,\mu/\pi+1-\eta)}.$$

Thus we obtain (36) as

$$\theta_0(s) = \lim_{\nu\to 0}\frac{dm_s'(\nu)/d\nu}{dm_s(\nu)/d\nu} = \frac{g(0)}{g(s)}\lim_{\nu\to 0}\frac{d\nu_s'}{d\nu} = \frac{q^{1-\eta}-q^{s(1-\eta)}}{(1-\eta)\varphi(q,\mu/\pi+1-\eta)}.$$

From this expression we observe that $\theta := \lim_{s\to 0}\theta_0(s) = \varphi(q,1-\eta)/\varphi(q,\mu/\pi+1-\eta)$.

Equation (37) is obtained by taking the expected value of the above expression,

$$\theta_0 = E^{G_s}[\theta_0(s)] = \frac{q^{1-\eta}-\varphi(q,\mu/\pi+1-\eta)/\varphi(q,\mu/\pi)}{(1-\eta)\varphi(q,\mu/\pi+1-\eta)} = \frac{\varphi(p^*,1-\eta)}{\varphi(q,\mu/\pi)},$$

where we used $p^* = q\underline{p}$.

Finally, we derive (39). In A.3.1, we showed that the repricing of Calvo-hit firms (67) and the repricing of menu-cost paying firms (68) constitute inflation:

$$(1-\eta)\pi = \mu((p^*)^{1-\eta}-1) + \lambda((p^*)^{1-\eta}-\underline{p}^{1-\eta}).$$

This is rewritten as

$$\pi = \mu\varphi(p^*,1-\eta) + \frac{\varphi(q,1-\eta)}{\varphi(q,\mu/\pi+1-\eta)}\pi = \mu\varphi(p^*,1-\eta) + \theta\pi.$$

Hence, we obtain (39), $\pi = \mu\varphi(q^*,1-\eta)/(1-\theta)$.

A.10 Asymptotic behavior of θ

Proposition 8. $\lim_{\pi\to 0}\theta = 0$*, and* θ *is increasing in* π *for sufficiently large* π*.*

Proof. From (38), we have $\theta(\pi) = \frac{\varphi(q(\pi),1-\eta)}{\varphi(q(\pi),1-\eta+\mu/\pi)}$. Since $q(0) > 0$, we obtain

$$\lim_{\pi\to 0}\theta(\pi) = \frac{\varphi(q(0), 1-\eta)}{\lim_{\pi\to 0}\varphi(q(\pi), 1-\eta+\mu/\pi)} = 0.$$

Applying (70),

$$\begin{aligned}(p^*(\pi))^{1-\eta}\pi &= \frac{\varphi(q(\pi), -\mu/\pi)}{\varphi(q(\pi), \eta-1-\mu/\pi)}\pi = \frac{q(\pi)^{-\mu/\pi}-1}{q(\pi)^{\eta-1-\mu/\pi}-1}\frac{\pi(\eta-1)-\mu}{-\mu}\pi \\ &= \frac{e^{-\mu\log(q(\pi))/\pi}-1}{q(\pi)^{\eta-1}e^{-\mu\log(q(\pi))/\pi}-1}\frac{\pi(\eta-1)-\mu}{-\mu}\pi.\end{aligned}$$

Since $q(\pi)$ increases exponentially in π and $\log(q(\pi))/\pi$ converges to a constant by Proposition 3, there exist constants $a > 0$ and $A > 0$ such that the last expression is bounded by $Ae^{-a\pi}$. By differentiating the logarithm of (70), we obtain,

$$\begin{aligned}(\eta-1)\frac{d\log p^*}{d\pi} &= \left(\frac{\varphi_x}{\varphi}(q, \eta-1-\mu/\pi) - \frac{\varphi_x}{\varphi}(q, -\mu/\pi)\right)\frac{\mu}{\pi^2} \\ &\quad + \left(\frac{1}{\varphi(q, -\eta+1+\mu/\pi)} - \frac{1}{\varphi(q, \mu/\pi)}\right)\frac{d\log q}{d\pi}.\end{aligned}$$

By Lemma 4(b), the first term on the RHS is positive and $O(\pi^{-2})$. The second term is smaller than $(d\log q/d\pi)/\varphi(q(\pi), 1-\eta)$, which is bounded for sufficiently large π. Furthermore,

$$\frac{d}{d\pi}\frac{\varphi(p^*, 1-\eta)}{\pi} = \frac{\varphi(p^*, 1-\eta)}{\pi^2}\left(\frac{(p^*)^{1-\eta}\pi}{\varphi(p^*, 1-\eta)}\frac{d\log p^*}{d\pi} - 1\right).$$

We have shown that $(p^*(\pi))^{1-\eta}\pi$ is dominated by $Ae^{-a\pi}$ for some pair (A, a) of positive constants and that $d\log p^*/d\pi$ is bounded for large π. Also, since q grows exponentially in π, by (70) p^* grows arbitrarily large for large π. Thus, for large π, $\varphi(p^*(\pi), 1-\eta)$ is positive and $1/\varphi(p^*(\pi), 1-\eta)$ is bounded. Hence, for sufficiently large π the above expression is negative. Combined with $1-\theta = \varphi(p^*, 1-\eta)/(\pi/\mu)$ from (39), this establishes that there exists a π' such that for $\pi > \pi'$, θ is increasing in π. □

A.11 Proof of Proposition 5

In what follows, we write $\mathcal{B}(n, p)$ for a binomial with population n and probability p and $\mathcal{P}(\lambda)$ for a Poisson with mean λ. Conditional on s, m_0 follows $\mathcal{P}(\theta_0(s))$ and m_0^n follows

$\mathcal{B}(n-1, G(\epsilon_0^n))$. Consider a third random variable $\tilde{m}_0^n$ distributed as $\mathcal{B}(n-1, \theta_0(s)/(n-1))$ and

$$\tilde{L}^n = \sum_{u=0}^{\infty} \tilde{m}_u^n$$

where $\tilde{m}_{u+1}^n$ conditional on $(\tilde{m}_k^n)_{k=0}^u$ and s is distributed as

$$\mathcal{B}\left(n-1-\sum_{k\le u} \tilde{m}_k^n,\ \frac{\theta \tilde{m}_u^n}{n-1-\sum_{k\le u} \tilde{m}_k^n}\right),$$

and write $\tilde{\mathcal{Q}}^n$ for the associated distribution of $\tilde{L}^n$.

Lemma 6. *For any $r \in Z_+$,*

$$\lim_{n\to\infty} |\tilde{\mathcal{Q}}^n(\{r\}) - \mathcal{Q}(\{r\})| = 0$$

Proof. First notice that if $\tilde{L}^n = L = r$ then $\tilde{L}^n = \sum_{u\le r} \tilde{m}_u^n$ and $L = \sum_{u\le r} m_u$. In addition if $J^r := \{\bar{j} = (j_0, \ldots, j_r) : \sum_{i\le r} j_i = r\}$ then $\Pr(\tilde{L}^n = r)$ is written as

$$\begin{aligned}
&\sum_{\bar{j}\in J^r} \Pr\left(\tilde{m}_r = r - \sum_{u=0}^{r-1} j_u \mid (\tilde{m}_u)_{u=0}^{r-1} = (j_u)_{u=0}^{r-1}\right) \\
&\quad \cdot \Pr\left(\tilde{m}_{r-1} = j_{r-1} \mid (\tilde{m}_u)_{u=0}^{r-2} = (j_u)_{u=0}^{r-2}\right) \cdots \Pr(\tilde{m}_1 = j_1 \mid \tilde{m}_0 = j_0) \cdot \Pr(\tilde{m}_0 = j_0) \\
&= \sum_{\bar{j}\in J^r} \Pr\left[\mathcal{B}\left(n-1-\sum_{u=0}^{r-1} j_u,\ \frac{\theta j_{r-1}}{n-1-\sum_{u=0}^{r-1} j_u}\right) = j_r\right] \\
&\quad \cdots \Pr\left[\mathcal{B}\left(n-1-j_0,\ \frac{\theta j_0}{n-1-j_0}\right) = j_1\right] \cdot \Pr\left[\mathcal{B}\left(n-1,\ \frac{\theta_0(s)}{n-1}\right) = j_0\right]. \qquad (82)
\end{aligned}$$

Moreover for each $\bar{j} \in J^r$, $n > r+1$,

$$d_\tau\left(\mathcal{B}\left(n-1-\sum_{i\le u} j_i, \frac{\theta j_u}{n-1-\sum_{i\le u} j_i}\right), \mathcal{P}(\theta j_u)\right) < \frac{\theta j_u}{n-1-\sum_{i\le u} j_i} \le \frac{\theta r}{n-r},$$

for $u > 0$ and

$$d_\tau\left(\mathcal{B}\left(n-1, \frac{\theta_0(s)}{n-1}\right), \mathcal{P}(\theta_0(s))\right) < \frac{\theta_0(s)}{n-1} \le \frac{\theta}{n-1}$$

(see *e.g.* Barbour and Hall [11, Equation (1.1)]). Since $\#J^r \le 2^r - 1$, for each $\epsilon > 0$ we may choose an n large enough so that $|\tilde{\mathcal{Q}}^n(\{r\}) - \mathcal{Q}(\{r\})| < \epsilon$. □

Lemma 7. *For each* $\bar{j} \in J^r$

$$\left| \kappa_i^n - \frac{\theta j_i}{n - 1 - \sum_{\ell \le i} j_\ell} \right| \le O(n^{-2}), \text{ for each } i = 0, \cdots, r.$$

Moreover,

$$\left| G(\epsilon_0^n) - \frac{\theta_0(s)}{n-1} \right| \le O(n^{-2}).$$

Proof. Note that

$$\kappa_i^n = G\left(\epsilon_0^n + \sum_{\ell \in \bigcup_{k=1}^{i} M_k^n} \tilde{\epsilon}_1^\ell \right) - G\left(\epsilon_0^n + \sum_{\ell \in \bigcup_{k=1}^{i-1} M_k^n} \tilde{\epsilon}_1^\ell \right) = g(x_i^n(\bar{j})) \sum_{\ell \in M_i^n} \tilde{\epsilon}_1^\ell, \tag{83}$$

for some $x_i^n(\bar{j}) \in \left[\epsilon_0^n + \sum_{\ell \in \bigcup_{k=1}^{i-1} M_k^n} \tilde{\epsilon}_1^\ell,\ \epsilon_0^n + \sum_{\ell \in \bigcup_{k=1}^{i} M_k^n} \tilde{\epsilon}_1^\ell\right]$. Moreover, using (53) and the fact that $\epsilon_1^1 = \theta/g(0)$, we obtain

$$\sum_{\ell \in M_i^n} \tilde{\epsilon}_1^\ell = j_i \epsilon_1^n - O(n^{-2}) = \frac{\theta j_i}{ng(0)} - O(n^{-2}) \le \frac{\theta r}{ng(0)} - O(n^{-2}). \tag{84}$$

Hence,

$$\begin{aligned} \left| \kappa_i^n - \frac{\theta j_i}{n - 1 - \sum_{\ell \le i} j_\ell} \right| &= \left| \left(\frac{\theta j_i}{ng(0)} - O(n^{-2}) \right) g(x_i^n(\bar{j})) - \frac{\theta j_i}{n - 1 - \sum_{\ell \le i} j_\ell} \right| \\ &= \frac{\theta j_i}{n} \left| \frac{g(x_i^n(\bar{j}))}{g(0)} - g(x^n(\bar{j}))O(n^{-1}) - \frac{n}{n - 1 - \sum_{\ell \le i} j_\ell} \right| \\ &\le O(n^{-2}) + \frac{\theta r}{n} \left[\left| \frac{g(x_i^n(\bar{j}))}{g(0)} - 1 \right| + \left| \frac{n}{n - 1 - \sum_{\ell \le i} j_\ell} - 1 \right| \right]. \end{aligned}$$

Note that $\epsilon_0^n \le \epsilon_1^n = \theta/(ng(0))$ by (51). Since $x_i^n(\bar{j}) \le (1 + \sum_{\ell=0}^{i} j_\ell)\theta/(ng(0)) \le (1 + r)\theta/(ng(0))$ and since g is increasing,

$$0 \le g(x^n(\bar{j})) \le g\left(\frac{(1+r)\theta}{ng(0)} \right).$$

In addition $g(x)/g(0) = q^{(\mu/\pi)x}$. Hence,

$$\left| \kappa_i^n - \frac{\theta j_i}{n - 1 - \sum_{\ell \le i} j_\ell} \right| \le O(n^{-2}).$$

Finally, we evaluate $|G(\epsilon_0^n) - \theta_0(s)/(n-1)|$. Using (31), (36) and (50), we obtain

$$G(\epsilon_0^n) = \frac{q^{\epsilon_0^n} - 1}{\varphi(q, \mu/\pi)} = \frac{1}{\varphi(q, \mu/\pi)} \left[e^{\frac{1}{n}\theta_0(s)\varphi(q,\mu/\pi) - \epsilon_P(s)} - 1 \right].$$

Note that for any $x > 0$ there exists $y \in (0, x)$ such that $|e^x - 1 - x| = y^2/2 < x^2/2$. Hence,

$$\begin{aligned} \left| G(\epsilon_0^n) - \frac{\theta_0(s)}{n-1} \right| &= \frac{1}{\varphi(q, \mu/\pi)} \left| e^{\frac{1}{n}\theta_0(s)\varphi(q,\mu/\pi) - \epsilon_P(s)} - 1 - \frac{\theta_0(s)\varphi(q, \mu/\pi)}{n-1} \right| \\ &\leq \frac{1}{\varphi(q, \mu/\pi)} \left[\frac{\left(\frac{1}{n}\theta_0(s)\varphi(q, \mu/\pi) - \epsilon_P(s)\right)^2}{2} + \frac{\theta_0(s)\varphi(q, \mu/\pi)}{n(n-1)} + \epsilon_P(s) \right]. \end{aligned}$$

Using (49), we obtain that $|G(\epsilon_0^n) - \theta_0(s)/(n-1)| \leq O(n^{-2})$. □

Lemma 8. *For any $r \in Z_+$,*

$$\lim_{n\to\infty} |\tilde{\mathcal{Q}}^n(\{r\}) - \mathcal{Q}^n(\{r\})| = 0.$$

Proof. Roos [56, inequality (15)] for the case $s = 0$ states that for each (ℓ, p, p') with $p, p' \in (0, 1)$,

$$d_\tau\left(\mathcal{B}(\ell, p), \mathcal{B}(\ell, p')\right) \leq \frac{\sqrt{e}}{2} \frac{\chi}{(1-\chi)^2},$$

where

$$\chi = |p' - p| \sqrt{\frac{2 + \ell}{2p'(1 - p')}}.$$

If $\ell = n - 1 - \sum_{k=0}^{i} j_k$, $p = \kappa_i^n$ and $p' = \frac{\theta j_i}{n-1-\sum_{k=0}^{i} j_k}$, then $j_i \geq 1$ if and only if $p > 0$ and $p' > 0$. Thus we may choose $\chi = 0$ when $j_i = 0$. In addition $p < 1$ and $p' < 1$ provided $n \geq r$, which is necessary for $L^n = \tilde{L}^n = r$.

By Lemma 7 $|p' - p| = O(n^{-2})$ and since the term under the square root is of order n^2 we obtain that $\chi = O(n^{-1})$ for each $j_i \leq r$.

The remaining of the proof is exactly as in Lemma 6 noting that when $j_i = 0$ both binomials are degenerate. □

Proof of Proposition. The result follows since Lemmas 6 and 8 establish that for each r, $\lim_{n\to\infty} |\mathcal{Q}^n(\{r\}) - \mathcal{Q}(\{r\})| = 0$ and because pointwise convergence in distribution insures convergence in Total Variation for distributions with support in Z_+. □

A.12 Lemma for proof of Proposition 6

Lemma 9. *Write* $\Theta := (q^{\mu/\pi} - 1)\theta$. *If either* $\Theta \leq 1$ *or* $\theta e^{\Theta-1}/\Theta < 1$ *then there exists* K *such that* $E[(L^n)^2 \mid s] \leq K$.

Proof. All expectations used in this proof are conditional on s, and, to simplify notation, we omit the dependence on s until the very end. Conditional on $m_k^n = j_k$ for $k \leq u$, the distribution of m_{u+1}^n is a binomial with population $n - 1 - \sum_{k\leq u} j_k$ and probability κ_u^n. Let $r_u^n := (1 + \sum_{k\leq u} j_k)/n \leq 1$. The conditional mean and variance of m_{u+1}^n are respectively $n(1-r_u^n)\kappa_u^n$ and $n(1-r_u^n)\kappa_u^n(1-\kappa_u^n)$. Since G is convex, using (83) and (84), we obtain

$$\kappa_u^n \leq \frac{\theta j_u}{n} \frac{g\left(\theta r_u^n/g(0)\right)}{g(0)} = \frac{\theta j_u}{n} q^{(\mu/\pi)(\theta r_u^n/g(0))} = \frac{\theta j_u}{n} e^{(q^{\mu/\pi}-1)\theta r_u^n} = \frac{\theta j_u}{n} e^{\Theta r_u^n}.$$

If $F(r) := (1-r)e^{\Theta r}$, for $0 \leq r \leq 1$, then

$$E[m_{u+1} \mid (m_k)_{k=0}^u = (j_k)_{k=0}^u] = n(1-r_u^n)\kappa_u^n \leq (1-r_u^n)\theta j_u e^{\Theta r_u^n} = \theta F(r_u^n) j_u.$$

If $\Theta \leq 1$, then F is non-increasing, since $F'(r) = e^{\Theta r}((1-r)\Theta - 1) \leq 0$. Hence $F(r) \leq F(0) = 1$. If $\Theta > 1$, then $F(r)$ attains a maximum at $r^* = 1 - 1/\Theta$, because $F'(r^*) = 0$, $F''(r^*) < 0$, and $F'(r) < 0$ for $r > r^*$. In this case, for any $0 \leq r \leq 1$ $F(r)$ is bounded by $F(r^*) = e^{\Theta-1}/\Theta$ and thus by hypothesis $\theta F(r) < 1$. Hence there exists $\vartheta < 1$ such that $\theta F(r_u^n) \leq \vartheta$, and furthermore

$$E[m_{u+1}^n \mid (m_k^n)_{k=0}^u = (j_k)_{k=0}^u] \leq \theta F(r_u^n) j_u \leq \vartheta j_u. \tag{85}$$

Moreover,

$$E[(L^n)^2] = E\left[\left(\sum_{u=0}^n m_u^n\right)^2\right] \leq 2E\left[\sum_{u=0}^n \sum_{k=0}^{n-u} m_u^n m_{u+k}^n\right].$$

By the Law of Iterated Expectations and (85), we have

$$\begin{aligned} E[m_u^n m_{u+k}^n] &= E[m_u^n E[m_{u+k}^n \mid m_u^n]] = E[m_u^n E[\cdots E[m_{u+k}^n \mid m_u^n, \ldots, m_{u+k-1}^n] \cdots \mid m_u^n]] \\ &\leq \vartheta^k E[(m_u^n)^2]. \end{aligned}$$

Hence,

$$E[(L^n)^2] \leq 2E\left[\sum_{u=0}^n \sum_{k=0}^{n-u} m_u^n m_{u+k}^n\right] \leq 2\sum_{u=0}^n \sum_{k=0}^{n-u} \vartheta^k E[(m_u^n)^2] \leq \frac{2}{1-\vartheta} \sum_{u=0}^n E[(m_u^n)^2].$$

Using the conditional mean and variance of m_u^n and (85) we obtain

$$\begin{aligned} E[(m_u^n)^2 \mid m_{u-1}^n = j_{u-1}] &= n(1-r_{u-1}^n)\kappa_{u-1}^n(1-\kappa_{u-1}^n) + (n(1-r_{u-1}^n)\kappa_{u-1}^n)^2 \\ &\leq n(1-r_{u-1}^n)\kappa_{u-1}^n + (n(1-r_{u-1}^n)\kappa_{u-1}^n)^2 \\ &\leq \vartheta j_{u-1} + (\vartheta j_{u-1})^2. \end{aligned}$$

Applying this inequality iteratively, we obtain

$$\begin{aligned}E[(m_u^n)^2] &\le \vartheta(E[m_{u-1}^n] + \vartheta E[(m_{u-1}^n)^2]) \\ &\le \vartheta(E[m_{u-1}^n] + \vartheta^2(E[m_{u-2}^n] + \vartheta E[(m_{u-2}^n)^2])) \\ &= \sum_{i=1}^{u} \vartheta^{2i-1} E[m_{u-i}^n] + \vartheta^{2u} E[(m_0^n)^2].\end{aligned}$$

The Law of Iterated Expectations and (85) imply that $E[m_{u-i}^n] \le \vartheta^{u-i} E[m_0^n]$. Substituting this into the previous inequality yields

$$E[(m_u^n)^2] \le \vartheta^u E[m_0^n] \left(\sum_{i=1}^{u} \vartheta^{i-1} \right) + \vartheta^{2u} E[(m_0^n)^2] \le \vartheta^u \frac{E[m_0^n]}{1-\vartheta} + \vartheta^{2u} E[(m_0^n)^2],$$

and thus,

$$\sum_{u=0}^{n} E[(m_u^n)^2] \le \frac{E[m_0^n]}{1-\vartheta} \left(\sum_{u=0}^{n} \vartheta^u \right) + E[(m_0^n)^2] \left(\sum_{u=0}^{n} \vartheta^{2u} \right) \le \frac{E[m_0^n]}{(1-\vartheta)^2} + \frac{E[(m_0^n)^2]}{1-\vartheta^2}.$$

Using $E[m_0^n] \le \theta_0(s) \le \theta \le \vartheta$ and $E[(m_0^n)^2] \le \theta_0(s) + \theta_0^2(s)$, we obtain

$$E[(L^n)^2] \le \frac{2}{1-\vartheta} \left(\frac{\theta}{(1-\vartheta)^2} + \frac{\theta + \theta^2}{1-\vartheta^2} \right) \le \frac{2\vartheta(2-\vartheta)}{(1-\vartheta)^3}. \qquad \square$$

Remark 3. *We know from Proposition 8 that* $\lim_{\pi\to\infty}(q(\pi)^{\mu/\pi} - 1) = \delta^+\eta\mu$. *Hence, if* $\delta^+ < 1/(\eta\mu)$, *we have* $\Theta = (q^{\mu/\pi} - 1)\theta < 1$ *for sufficiently large* π, *and hence the condition for Lemma 9 holds. The condition of Lemma 9 is also satisfied under our calibrated parameter values for various values of* π. *Figure 5 shows* Θ *and* $\theta F(r^*)$ *under US-calibrated parameters except for* π. *We observe that* $\theta F(r^*) < 1$ *holds for* $\Theta > 1$.

A.13 Proof of Proposition 7(d)

First we establish the following lemma.

Lemma 10. *The mean and variance of* L *conditional on* $m_0 = 1$ *are* $1/(1-\theta)$ *and* $\theta/(1-\theta)^3$, *respectively.*

Proof. The probability generating function $\Psi(z)$ of the sum L of a branching process with initial value 1 is a fixed point of a functional equation $\Psi(z) = z\Phi(\Psi(z))$, where Φ is a probability generating function of the number of children born from a parent. In our case, Φ is the probability generating function of a Poisson distribution with mean θ. By the property

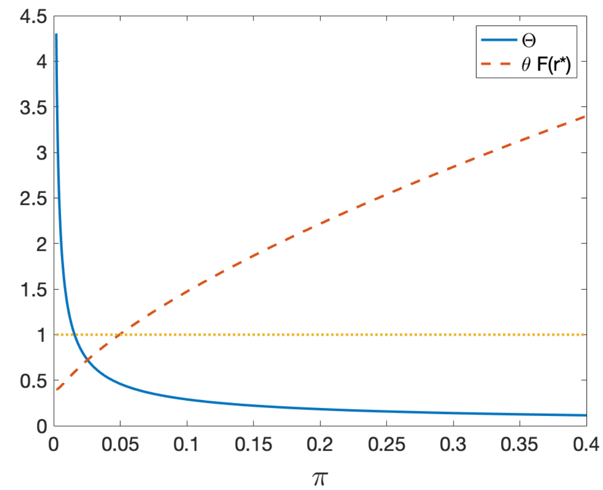

Figure 5: Conditions for assumption of Lemma 9 for parameter values calibrated in Section 5

of a probability generating function, $\Phi(1) = \Psi(1) = 1$, and the mean and variance of the Poisson corresponds to $\Phi'(1) = \theta$ and $\Phi''(1) + \Phi'(1) - (\Phi'(1))^2 = \theta$, respectively. Hence, $\Phi''(1) = \theta^2$.

Using the functional equation, we obtain $\Psi'(z) = \Phi(\Psi(z)) + z\Phi'(\Psi(z))\Psi'(z)$ and $\Psi''(z) = 2\Phi'(\Psi(z))\Psi'(z) + z\Phi''(\Psi(z))(\Psi'(z))^2 + z\Phi'(\Psi(z))\Psi''(z)$. Evaluating at $z = 1$, we obtain the mean of L conditional on $m_0 = 1$ as $\Psi'(1) = 1/(1-\theta)$. For $\Psi''(1)$, we have

$$\Psi''(1) = \frac{2\theta}{(1-\theta)^2} + \frac{\Phi''(1)}{(1-\theta)^3} = \frac{2\theta}{(1-\theta)^2} + \frac{\theta^2}{(1-\theta)^3}.$$

Hence, the variance of L conditional on $m_0 = 1$ is

$$\Psi''(1) + \Psi'(1) - (\Psi'(1))^2 = \frac{2\theta - \theta^2}{(1-\theta)^3} + \frac{1}{1-\theta} - \frac{1}{(1-\theta)^2} = \frac{\theta}{(1-\theta)^3}. \qquad \square$$

Proof of Proposition 7(d). Using (40) and Lemma 10, we obtain $E[L] = E[E[L|m_0]] = E[m_0 E[L|m_0 = 1]] = \theta_0/(1-\theta)$ and

$$\begin{aligned} V(L) &= E\left[V\left(L|m_0\right)\right] + V\left(E\left[L|m_0\right]\right) = E[m_0 V(L|m_0 = 1)] + V(m_0 E[L|m_0 = 1]) \\ &= E[m_0]V(L|m_0 = 1) + V(m_0)E[L|m_0 = 1]^2 = \frac{\theta_0\theta}{(1-\theta)^3} + \frac{V(m_0)}{(1-\theta)^2}. \end{aligned}$$

With notation $\sigma_{\theta_0}^2 := V^{G_s}(\theta_0(s))$, we have

$$V(m_0) = E^{G_s}[V(m_0 \mid s)] + V^{G_s}(E[m_0 \mid s]) = E^{G_s}[\theta_0(s)] + V^{G_s}(\theta_0(s)) = \theta_0 + \sigma_{\theta_0}^2.$$

Hence, we obtain

$$V(L) = \frac{\theta_0 \theta}{(1-\theta)^3} + \frac{\theta_0 + \sigma_{\theta_0}^2}{(1-\theta)^2} = \frac{\theta_0}{(1-\theta)^3} + \frac{\sigma_{\theta_0}^2}{(1-\theta)^2}.$$

Furthermore, applying (36) gives

$$\sigma_{\theta_0}^2 = V^{G_s}(\theta_0(s)) = V^{G_s}\left(\frac{q^{1-\eta} - q^{(1-\eta)s}}{(1-\eta)\varphi(q, 1-\eta+\mu/\pi)}\right) = \frac{E^{G_s}[q^{2s(1-\eta)}] - E^{G_s}[q^{(1-\eta)s}]^2}{(1-\eta)^2 \varphi^2(q, 1-\eta+\mu/\pi)}.$$

We note that $E^G[q^{sx}]$ for any x is derived as follows:

$$\int_0^1 q^{sx} \frac{q^{s\mu/\pi} \log q}{\varphi(q, \mu/\pi)} ds = \frac{\log q}{\varphi(q, \mu/\pi)} \frac{q^{s(x+\mu/\pi)}|_0^1}{(x+\mu/\pi)\log q} = \frac{\varphi(q, x+\mu/\pi)}{\varphi(q, \mu/\pi)}. \tag{86}$$

Substituting in $\sigma_{\theta_0}^2$ above, we obtain

$$\sigma_{\theta_0}^2 = \frac{1}{(1-\eta)^2 \varphi^2(q, \mu/\pi)} \left(\frac{\varphi(q, 2(1-\eta)+\mu/\pi)\varphi(q, \mu/\pi)}{\varphi^2(q, 1-\eta+\mu/\pi)} - 1\right).$$

A.14 Volatility of $\log P_t - \log P_{t\text{-}}$

The main determinants of the variance of a price jump $\log P_t - \log P_{t\text{-}}$ are L_t^n and $h(\pi) = \frac{\varphi(q,\mu/\pi)\varphi(q,1-\eta)}{\varphi(q,1-\eta+\mu/\pi)}$, the extensive and intensive margins of aggregate price adjustment due to menu-cost paying firms. For the intensive margin, we note that $h(\pi) = \varphi(q(\pi), \mu/\pi)\theta(\pi)$. In A.10, we showed that θ is increasing in π for large π. Also, $\varphi(q(\pi), \mu/\pi) = (e^{\mu \log(q(\pi))/\pi} - 1)\pi/\mu$ asymptotically increases linearly in π, since $\log(q(\pi))/\pi$ converges to a constant as $\pi \to \infty$. Hence, $h(\pi)$ is increasing for large π.

For the extensive margin, Proposition 7(d) showed that the variance of L conditional on s is

$$\sigma_L^2 = \frac{\theta_0}{(1-\theta)^3} + \frac{\sigma_{\theta_0}^2}{(1-\theta)^2},$$

where

$$\sigma_{\theta_0}^2 = \frac{1}{(1-\eta)^2 \varphi^2(q, \mu/\pi)} \left(\frac{\varphi(q, 2(1-\eta)+\mu/\pi)\varphi(q, \mu/\pi)}{\varphi^2(q, 1-\eta+\mu/\pi)} - 1\right).$$

By (37), $\theta_0 = \varphi(p^*, 1-\eta)/\varphi(q, \mu/\pi)$. In A.10, we showed that $1-\theta = \varphi(p^*, 1-\eta)\mu/\pi$. Thus,

$$\frac{\theta_0}{(1-\theta)^3} = \frac{1}{(1-\theta)^2} \frac{1}{e^{\mu \log(q)/\pi} - 1}.$$

Since $(\log q(\pi))/\pi$ converges to a constant, and since $1/(1-\theta(\pi))^2 \to \infty$ as $\pi \to \infty$, $\theta_0/(1-\theta)^3$ is asymptotically increasing in π.

Next, we show that $\sigma^2_{\theta_0}/(1-\theta)^2$ is also asymptotically increasing in π. Note that $\varphi(q, 2(1-\eta) + \mu/\pi)$ and $\varphi(q, 1-\eta+\mu/\pi)$ are bounded uniformly on q for any $\pi > \mu/(\eta-1)$, whereas $\varphi(q(\pi), \mu/\pi)$ increases asymptotically linearly in π. Hence, $\sigma^2_{\theta_0} \sim 1/((1-\eta)^2\varphi(q(\pi), \mu/\pi))$ as $\pi \to \infty$. Note that $\varphi(q(\pi), \mu/\pi)(1-\theta(\pi)) = (e^{\mu \log(q(\pi))/\pi} - 1)\varphi(p^*(\pi), 1-\eta)$ is uniformly bounded and converges to $\delta^+\eta\mu/(\eta-1)$ as $\pi \to \infty$. Thus,

$$\frac{\sigma^2_{\theta_0}(\pi)}{(1-\theta(\pi))^2} \sim \frac{1}{(\eta-1)\eta\mu\delta^+}\frac{1}{1-\theta(\pi)} \quad \text{as } \pi \to \infty.$$

Hence, σ^2_L is increasing in π for sufficiently large π.

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