CARF-F-227

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August 2010

CARF is presently supported by Bank of Tokyo-Mitsubishi UFJ, Ltd., Citigroup, Dai-ichi Mutual Life Insurance Company, Meiji Yasuda Life Insurance Company, Nippon Life Insurance Company, Nomura Holdings, Inc. and Sumitomo Mitsui Banking Corporation (in alphabetical order). This financial support enables us to issue CARF Working Papers.

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August 7, 2010

Abstract

We model the stock market as a timing game, in which arbitrageurs who are not expected to be certainly rational compete over profit by bursting the bubble caused by investors' euphoria. The manager raises money by issuing shares and the arbitrageurs use leverage. If leverage is weakly regulated, it is the unique Nash equilibrium that the bubble persists for a long time. This holds even if the euphoria is negligible and all arbitrageurs are expected to be almost certainly rational. This bubble causes serious harm to the society, because the manager uses the money raised for his personal benefit.

Keywords: Euphoria, Leverage, Rational and Behavioral Arbitrageurs, Harmful Bubble, Unique Nash Equilibrium

JEL Classification Numbers: C720, C730, D820, G140

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*The research of this paper was supported by a grant-in-aid for scientific research (KAKENHI 21330043) from the Japan Society for the Promotion of Science (JSPS) and the Ministry of Education, Culture, Sports, Science and Technology (MEXT) of the Japanese government. I thank all the participants of the seminar at the University of Okayama and the special session on the "Financial Crises" at the 2010 Japanese Economic Association Spring Meeting. I am especially grateful to Kazuhiko Ohashi for his comments on housing bubbles. Of course, all errors are my own.
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1. Introduction

This paper demonstrates a theoretical foundation for the bubble phenomenon in a stock market during which the company's manager pursues his personal benefit using his advantageous condition to raise money. We analyze a modified version of the timing game addressed by Matsushima (2009), and show that a long persistence of the bubble, which causes serious harm to the society, can be described as the unique Nash equilibrium price movement, even if the investors' euphoria as the dynamo of the bubble outbreak is negligible and professional arbitrageurs are expected to be almost certainly rational.

The efficient market hypothesis asserts that for a company the share price in the stock market is immediately adjusted to the company's fundamental value per share.[1] However, there are considerable evidences in support of the phenomenon that the bubble persists for a long time contrary to this hypothesis; that is, the share price increases beyond the fundamental value until it goes into a free fall.[2] Advocates of behavioral finance such as Shiller (2000) and Shleifer (2000) argue that the bubble is sometimes driven by behavioral investors who are slaves of euphoria; they incorrectly perceive that the share price will sell at a high price in the future and, not mindful of the danger that the bubble will crash, continue to reinforce their misperception with increasing momentum.

Although the supporters of the efficient market hypothesis recognize the presence of such investors, they generally object to the behavioral finance argument by claiming that rational arbitrageurs or professional portfolio managers quickly undo this mispricing by selling out their shareholdings; their selling pressures dampen the investors' euphoria, bursting the bubble immediately. It might be the case that rational arbitrageurs, who correctly predict the manner in which the investors reinforce their misperception, examine the possibility of profiting by riding the bubble for a while and selling out later on. Given that multiple arbitrageurs coexist, however, they have to remain in competition with each other over who becomes the winner by selling out

[1] See Friedman (1953) and Fama (1965), for instance.
[2] See Kindleberger (1978), Shiller (2000), Shleifer (2000), Brunnermeier and Nagel (2004), Brunnermeier (2008), and Malkiel (2010), for instance.

before others. Hence, according to the backward induction method, this competition quickly undoes the mispricing caused by the investors' euphoria.

The advocates of behavioral finance counter this claim by pointing out that any arbitrageur is not expected by the other arbitrageurs to be certainly rational; they are expected to be behavioral on some occasions. That is, every arbitrageur remains committed not to burst the bubble of his own accord; thus, the effect that this competition prevents the bubble is quite restrictive.

On the basis of these arguments, Matsushima (2009) formulated the stock market as a timing game among arbitrageurs on the assumption of incomplete information on whether each arbitrageur is rational or behavioral. The author shows the necessary and sufficient condition under which it is the unique Nash equilibrium that the bubble persists until a particular time and each arbitrageur randomizes the time to sell out afterward. The main result of Matsushima (2009) implies that, in order for the bubble to persist for a long time, it is necessary that both the acceleration with which investors reinforce their misperception and the probability that each arbitrageur is behavioral are sufficiently high. Hence, in this timing game, provided that either the investors' euphoria is negligible or each arbitrageur is expected to be almost certainly rational, it is impossible that the bubble will persist for a long time.

The present paper reconsiders the foundation underlying the willingness of rational arbitrageurs to ride the bubble. We demonstrate a new concept that makes the bubble persistent even if the investors' euphoria is negligible and each arbitrageur is expected to be almost certainly rational. We modify the timing game of Matsushima (2009) by taking into account the fact that the company's manager raises money by selling new shares to investors and arbitrageurs if the resulting selling pressure does not dampen the investors' euphoria. Any arbitrageur is permitted to use leverage to purchase these shares; he borrows money from the behavioral investors on the security of the shares that he owns. The main result of this paper provides a justification for the claim of behavioral finance that the persistence of the bubble is much easier than the efficient market hypothesis proposes; whenever arbitrageurs' leverage is only weakly regulated, it is the unique Nash equilibrium that the bubble persists for a long time, even if the investors' euphoria is negligible and all arbitrageurs are expected to be almost certainly rational.

Given an arbitrary strength of the regulation on arbitrageurs' leverage, it is sure that the bubble crashes immediately whenever the investors' euphoria is negligible and all arbitrageurs are expected to be almost certainly rational. The main statement of this paper, however, implies that, by assuming that the regulation on arbitrageurs' leverage is sufficiently weak, we can describe a long persistence of the bubble as the unique Nash equilibrium price movement, even if the investors' euphoria is negligible and all arbitrageurs are expected to be almost certainly rational.

When any rational arbitrageur makes debt contracts with behavioral investors, he can take a viewpoint that is more advantageous to him than these investors; he definitely predicts the manner in which the investors unconsciously reinforce their misperception, and also takes into account the danger that the bubble would crash. On the other hand, the investors are unconscious of the manner of their reinforcement and incorrectly believe that the bubble would never crash. In this case, the arbitrageur can make a non-recourse debt contract that is advantageous to him in that he can force much of the loss from the crash of the bubble on his contract partners. If any arbitrageur fails to sell out before the bubble crashes, the market value of the shares that he owns declines to zero. Because of the non-recourse feature, his debt is substantially reduced in this case; he can avoid his debt just by holding out the valueless security. This is the driving force behind each arbitrageur's willingness to ride the bubble and postpone the time to sell out without being unduly worried that the other arbitrageurs would sell out before he does.

If each arbitrageur continues to increase his personal fund, he could assign this increase for the purchase of new shares instead of using leverage. However, purchasing new shares in this manner does nothing to promote the persistence of the bubble as would leverage; in this case, the arbitrageur does not enjoy any advantageous condition such as a contractual term that reduces his fear of suffering a loss from the crash of the bubble.

Keynes (1936, chapter 12) pointed out that the general public misperceives the share price and that their misperception is apt to change in an unconscious manner. Keynes also argued that professionals who can predict how the general public's misperception changes compete with each other over profit by ascertaining the best time to sell out. In the behavioral finance literature, several works such as Delong et al.

(1990a, 1990b), Shleifer and Vishny (1997), and Abreu and Brunnermeier (2002, 2003) have described the general public as behavioral investors who reinforce their misperception with increasing momentum but are sensitive to a trend of the professionals. The formulations of behavioral investors in Matsushima (2009) and the present paper are the most relevant to Abreu and Brunnermeier (2002, 2003) and Brunnermeier and Morgan (2006). The present paper modifies their formulations by proposing that the selling pressure dampens the euphoria only if the arbitrageurs fail to absorb a particular proportion of this pressure.

It is important from the welfare viewpoint that the bubble induced by a weak regulation on arbitrageurs' leverage causes serious harm to the society; given that the fundamental value equals zero (i.e., it is low even compared with a safe asset), we can implicitly assume that the company's manager uses the money raised for his personal benefit. This implies that the arbitrageurs and investors would waste enormous sums of money by purchasing the shares that the manager issues during the bubble.

Since the share price is higher than the fundamental value, the manager can easily raise money for his personal benefit by issuing new shares. Since behavioral investors incorrectly believe that the bubble never crashes, the arbitrageurs can borrow money from them for the purchase of these shares on conditions that are advantageous to him. These two factors cause the bubble to persist longer, increasing the harm to the society.

The rest of this paper is organized as follows. Section 2 models the timing game among the arbitrageurs as a modified version of Matsushima (2009), in which the manager issues new shares and some arbitrageur uses leverage for the purchase of these shares. Section 3 presents the main theorem of this paper: a sufficient condition under which there exists the unique Nash equilibrium, denoted by $\tilde{F}$, and it causes the bubble to persist for a long while. Section 4 shows a sufficient condition under which there exists the unique Nash equilibrium, denoted by F^*, and it causes the bubble to quickly crash at the initial time. Section 5 investigates a case in which both the sufficient conditions, presented in Sections 3 and 4, do not hold. It is shown that the unique symmetric Nash equilibrium, denoted by $\hat{F}$, is specified as a hybrid of the strategy profiles $\tilde{F}$ and F^*. In Section 6, we apply the analysis of this paper to the case of housing bubbles.

2. The Model

Let us consider the market for a company's stock in a continuous time horizon given by $[0,1]$. For convenience of our argument, we assume that the company's fundamental value is set equal to zero, the market interest rate is set equal to zero, no dividends are paid to the shareholders during the time interval $[0,1]$, the share price is set equal to unity at the initial time 0, and the number of the company's shares is set equal to unity at the initial time 0.

2.1. Arbitrageurs and Behavioral Investors

There exist $n \geq 2$ arbitrageurs each of whom decides the time to sell out his shareholdings. The number of the shares that each arbitrageur owns at the initial time 0 is given by $c \in (0, 1/n)$. The bubble persists as long as each arbitrageur continues to hold a claim to at least $c \times 100\,\%$ of the company's assets. During the bubble, the share price grows exponentially at a constant rate $\rho > 0$ (i.e., the price is considered to be $e^{\rho t}$ at each time $t \in [0,1]$). We assume that once some arbitrageur reduces his claim below $c \times 100\,\%$ of the company's assets, the bubble crashes and the share price declines to zero immediately.[3] We also assume that the bubble automatically crashes just after the terminal time 1 for an exogenous reason, even if no arbitrageur sells out before.

Against the abovementioned background, it can be implicitly assumed that many behavioral investors are slaves to euphoria as the following discussion shows. At each time $t \in [0,1]$, they incorrectly perceive that each share will sell for the price $e^{\rho t}$ in the future, and reinforce their misperception with increasing momentum at the rate ρ. However, once some arbitrageur reduces his claim, the resultant selling pressure forces the investors out of their euphoria; they stop supporting the bubble price immediately.

[3] Abreu and Brunnermeier (2002, 2003) and Brunnermeier and Mogan (2006) investigated a coordinated attack in which two or more arbitrageurs have to sell out in order to burst the bubble. With minor modifications, we can generalize our model to this coordinated attack case. See Matsushima (2009).

They automatically wake up from their euphoria just after the terminal time 1 for an exogenous reason, even if no arbitrageur sells out before.

2.2 Issuing New Shares

During the bubble, the company continues to issue new shares; the number of shares outstanding grows exponentially at a constant rate $\lambda > 0$ (i.e., the number of shares outstanding is considered to be $e^{\lambda t}$ at each time $t \in [0,1]$). As long as the bubble persists, the market value of this company expands with increasing momentum at the rate of $\rho + \lambda$ (i.e., its market value is considered to be $e^{(\rho+\lambda)t}$ at each time $t \in [0,1]$). Once the bubble crashes, the market value declines to zero. In order to keep the bubble alive without dampening the investors' euphoria, each arbitrageur must purchase $c \times 100\%$ of the newly issued shares (i.e., $c\lambda e^{\lambda t}\Delta$ shares) for the price of $e^{\rho t}$ per share in each short interval $[t, t+\Delta]$, where Δ is positive but close to zero. The behavioral investors purchase the rest of the shares that the company issues, i.e., $(1-nc)\lambda e^{\lambda t}\Delta$ shares.

During the bubble, each arbitrageur continues to purchase new shares by debt finance as follows. We assume that at the initial time 0, each arbitrageur already has an obligation of $\frac{L-1}{L}c$ in debts to the behavioral investors on the security of the shares that he owns, where $L \geq 1$ implies the leverage ratio, which is defined as the ratio of the market value of the shares that this arbitrageur owns to the market value of his personal capital. In each short interval $[t, t+\Delta]$ during the bubble, he borrows a sum of $(L-1)\rho c e^{L\rho t}\Delta$ from the behavioral investors in order to purchase $(L-1)\rho c e^{(L-1)\rho t}\Delta$ newly issued shares. Hence, at any time t during the bubble, his debt accumulates to

$$\frac{L-1}{L}c + \int_{\tau=0}^{t}(L-1)\rho c e^{L\rho\tau}d\tau = \frac{L-1}{L}ce^{L\rho t},$$

the number of shares that he owns accumulates to

$$c + \int_{\tau=0}^{t}(L-1)\rho c e^{(L-1)\rho\tau}d\tau = ce^{(L-1)\rho t},$$

and the market value of the shares that he owns is given by

$ce^{L\rho t}$.

More precisely, at each time t during the bubble, every arbitrageur, who owns $ce^{(L-1)\rho t}$ number of the shares, borrows $\frac{L-1}{L}ce^{L\rho t}$ in the very short term Δ on the security of the shares that he owns; at time $t+\Delta$, he repays his debt by transferring either all the shares that he owns or a sum of $\frac{L-1}{L}ce^{L\rho t}$ to the debt holders. During the bubble, he refinances at time $t+\Delta$ by borrowing $\frac{L-1}{L}ce^{L\rho(t+\Delta)}$, and allots the money left over after he repays his debt, i.e., $\frac{L-1}{L}ce^{L\rho(t+\Delta)}-\frac{L-1}{L}ce^{L\rho t}\approx(L-1)\rho ce^{L\rho t}\Delta$, for the purchase of new shares.

In order to keep the bubble alive, the number of shares that each arbitrageur owns (i.e., $ce^{(L-1)\rho t}$) must be more than or equal to $ce^{\lambda t}$, which implies $\lambda\le(L-1)\rho$. Since the company prefers issuing as many shares as possible, it is appropriate to assume that

(1) $\lambda=(L-1)\rho$.

Because the fundamental value is set equal to zero, we can implicitly assume that the manager of the company uses the money raised for his personal benefit. Hence, the total money raised by issuing shares from the initial time 0 up to time $t\in[0,1]$, when the bubble bursts, which is given by

$$C(t)\equiv\int_{\tau=0}^{t}e^{\rho\tau}d(e^{\lambda\tau})=\frac{\lambda}{\rho+\lambda}\{e^{(\rho+\lambda)t}-1\},$$

should be regarded as the *social cost* that is induced by the persistence of the bubble up to time t. Clearly, $C(t)$ is increasing, and $C(0)=0$.

2.3 Behavioral Aspects of Arbitrageurs

Each arbitrageur is expected by the other arbitrageurs to be rational with a probability of $1-\varepsilon>0$; he is expected to be irrational, or behavioral, with the remaining probability, $\varepsilon>0$. If an arbitrageur is behavioral, he continues to purchase new shares and never sells out before the other arbitrageurs do; i.e., he is committed not to burst the bubble of his own accord even at the terminal time 1.

If he is rational, he selects the time to stop purchasing new shares and sells out his shareholdings of his own accord. If he can sell out before the bubble crashes, he receives the capital gain $ce^{L\rho t} = ce^{(\rho+\lambda)t}$ and repays his debt $\frac{L-1}{L}ce^{L\rho t} = \frac{\lambda}{\rho+\lambda}ce^{(\rho+\lambda)t}$. Hence, his payoff in this case is given by

$$ce^{(\rho+\lambda)t} - \frac{\lambda}{\rho+\lambda}ce^{(\rho+\lambda)t} = \frac{\rho}{\rho+\lambda}ce^{(\rho+\lambda)t}.$$ [4]

If he fails to sell out before the bubble crashes, the market value of the shares that he owns declines to zero. Since his debt is substantially reduced in this case, his corresponding payoff is given by zero. Moreover, if he attempts to sell out before the bubble crashes but other $m \in \{1,...,n-1\}$ arbitrageurs also attempt to do so at the same time, he succeeds to sell out before the bubble crashes with just a probability of $\frac{1}{m+1}$.

2.4. Nash Equilibrium

A (mixed) strategy for each arbitrageur $i \in N$ is denoted by a non-decreasing right-continuous function $F_i : [0,1] \to [0,1-\varepsilon]$, where $F(1) = 1-\varepsilon$. Given that arbitrageur i is rational and no other arbitrageur sells out before him, he is going to sell out at or before time t with a probability of $\frac{F(t)}{1-\varepsilon}$. Given that he is behavioral, he never sells out of his own accord even at the terminal time 1. Note from equality $F(1) = 1-\varepsilon$ that any rational arbitrageur certainly sells out at or before the terminal time 1. Let us denote the set of all strategies for arbitrageur i by Φ_i and a strategy profile by $F \equiv (F_i)_{i\in N}$. Further, denote $\Phi \equiv \times_{i\in N} \Phi_i$, $F_{-i} \equiv (F_j)_{j\in N\setminus\{i\}}$, $F_i^-(t) \equiv \lim_{\tau\uparrow t} F(\tau)$ for all $\tau \in (0,1]$, and $F_i^-(0) \equiv 0$. A strategy profile F is said to be *symmetric* if

$$F_i = F_1 \text{ for all } i \in N.$$

[4] This paper assumes that short-sales are prohibited.

Suppose that the other arbitrageurs follow F_{-i}. Then, the expected payoff that arbitrageur i obtains by selling out at time $t \in [0,1]$ is defined as

$$V_i(t;F_{-i}) \equiv \frac{\rho}{\rho+\lambda} c e^{(\rho+\lambda)t} \sum_{H \in 2^{N\setminus\{i\}}} \frac{1}{|H|+1} [\prod_{j \in H} \{F_j(t) - F_j^-(t)\}] \prod_{j \notin H \cup \{i\}} \{1 - F_j(t)\}.$$

If $F_j(t)$ is continuous for all $j \in N \setminus \{i\}$, we can simply write $V_i(t;F_{-i})$ with

$$V_i(t;F_{-i}) = \frac{\rho}{\rho+\lambda} c e^{(\rho+\lambda)t} \prod_{j \in N\setminus\{i\}} \{1 - F_j(t)\}.$$

Let us define the expected payoff for arbitrageur i induced by strategy profile $F \in \Phi$ given that he is rational as

$$V_i(F) \equiv \frac{1}{1-\varepsilon} \{V_i(0;F_{-i})F_i(0) + \int_{t=0}^{1} V_i(t;F_{-i}) dF_i(t)\}.$$

A strategy profile F is said to be a Nash equilibrium if

$$V_i(F) \geq V_i(t;F_{-i}) \text{ for all } i \in N \text{ and all } t \in [0,1].$$

Lemma 1: *If F is a Nash equilibrium, then $F_i(t)$ is continuous for all $i \in N$.*

Proof: Suppose that $F_i(\tau)$ is not continuous, i.e., there exists $\tau' > 0$ such that $F_i^-(\tau') < F_i(\tau')$. Then, any other arbitrageur $j \neq i$ can drastically increase his winning probability by selecting any time that is slightly earlier than τ' instead of τ'; he never selects any time that is either the same as or slightly later than τ'. This implies that arbitrageur i can increase the winner's gain by postponing further sales without decreasing his winning probability. This contradicts the Nash equilibrium property.

Q.E.D.

Lemma 2: *If F is a Nash equilibrium, then, at each time $t \in [\tau^1, 1]$, there exists $i \in N$ such that $F_i(t)$ is increasing, where we specify*

$$\tau^1 \equiv \max\{\tau \in [0,1] : F_i(\tau) = F_i(0) \textit{ for all } i \in N\}.$$

Proof: Suppose that Lemma 2 is not true. Then, there exist $\tau' \in [\tau^1, 1]$ and $\tau'' \in (\tau', 1]$

such that $F_i(\tau') = F_i(\tau'')$ for all $i \in N$, and the time choice τ' is the best response for some arbitrageur. Since no arbitrageur selects any time in the interval (τ', τ''), it follows that by selecting time τ'' instead of τ', any arbitrageur can increase the winner's gain from $\frac{\rho}{\rho+\lambda} ce^{(\rho+\lambda)\tau'}$ to $\frac{\rho}{\rho+\lambda} ce^{(\rho+\lambda)\tau''}$ without decreasing his winning probability. This is a contradiction.

Q.E.D.

From Lemma 2, it follows that if a strategy profile F is a symmetric Nash equilibrium, then for every $i \in N$, F_i increases at any time after time τ^1, where $F_i(\tau) = F_i(0)$ whenever $\tau < \tau^1$.

3. Bubbles and Crashes

Let us suppose that ε, ρ, and λ are large enough to satisfy the following:

(2) $$\varepsilon^{n-1}e^{\rho+\lambda} > 1.$$

Let us define

$$\tilde{\tau} = \tilde{\tau}(n,\varepsilon,\rho,\lambda) \equiv 1 + \frac{(n-1)\ln\varepsilon}{\rho+\lambda}.$$

Note from inequality (2) that $\tilde{\tau}$ is well defined, i.e.,

$$0 < \tilde{\tau} < 1.$$

By assuming inequality (2), we can specify a symmetric and continuous strategy profile $\tilde{F} = (\tilde{F}_i)_{i\in N}$ as follows; for every $i \in N$,

$$\tilde{F}_i(t) = 0 \quad \text{for all} \quad t \in [0,\tilde{\tau}),$$

and

$$\tilde{F}_i(t) = 1 - \varepsilon\exp[\frac{(\rho+\lambda)(1-t)}{n-1}] \quad \text{for all} \quad t \in [\tilde{\tau},1],$$

where, from the specification of $\tilde{\tau}$, $\tilde{F}_i(\tilde{\tau}) = 0$.

According to $\tilde{F}$, any arbitrageur never bursts the bubble of his own accord until time $\tilde{\tau}$. Hence, it is certain that the bubble will persist until time $\tilde{\tau}$. After time $\tilde{\tau}$, any arbitrageur i randomizes the time to sell out according to the hazard rate given by

(3) $$\frac{\frac{d\tilde{F}_i(t)}{dt}}{1-\tilde{F}_i(t)} = \frac{\rho+\lambda}{n-1}.$$

Note that $\tilde{\tau} = \tilde{\tau}(n,\varepsilon,\rho,\lambda)$ is increasing with respect to ε, ρ, and λ, that it is decreasing with respect to n, and that

$$\lim_{\rho\to\infty}\tilde{\tau}(n,\varepsilon,\rho,\lambda) = \lim_{\lambda\to\infty}\tilde{\tau}(n,\varepsilon,\rho,\lambda) = \lim_{\varepsilon\to 1}\tilde{\tau}(n,\varepsilon,\rho,\lambda) = 1.$$

Hence, given that ε, ρ, or λ is sufficiently large, it is almost certain that the bubble will continue almost up to the terminal time 1.

Theorem 3: *With inequality (2), the strategy profile* $\tilde{F}$ *is the unique Nash equilibrium.*

Proof: Since no arbitrageur sells out before time $\tilde{\tau}$, it is clear that for every $i \in N$,

$$V_i(\tilde{\tau}; \tilde{F}_{-i}) > V_i(t; \tilde{F}_{-i}) \quad \text{for all} \quad t \in [0, \tilde{\tau}).$$

From the symmetry of $\tilde{F}$ and equality (3), it follows that for every $i \in N$ and $t \in (\tilde{\tau}, 1)$,

$$\frac{\partial V_i(t; \tilde{F}_{-i})}{\partial t} = \frac{\partial}{\partial t}[\frac{\rho}{\rho+\lambda} c e^{(\rho+\lambda)t} \{1 - \tilde{F}_i(t)\}^{n-1}]$$

$$= \rho c e^{(\rho+\lambda)t} \{1 - \tilde{F}_i(t)\}^{n-1} \{1 - \frac{n-1}{\rho+\lambda} \cdot \frac{\frac{d\tilde{F}_i(t)}{dt}}{1 - \tilde{F}_i(t)}\} = 0.$$

This implies that the first-order condition holds in $[\tilde{\tau}, 1]$. Hence, we have proved that

$$V_i(\tilde{\tau}; \tilde{F}_{-i}) = V_i(t; \tilde{F}_{-i}) \quad \text{for all} \quad i \in N \quad \text{and all} \quad t \in (\tilde{\tau}, 1],$$

which implies that $\tilde{F}$ is a Nash equilibrium.

We will show that $\tilde{F}$ is the unique symmetric Nash equilibrium. We set any symmetric Nash equilibrium F arbitrarily. Inequality (2) implies that for every arbitrageur $i \in N$, the time choice 0 is a dominated strategy, where

$$V_i(0; F_{-i}) \leq \frac{\rho}{\rho+\lambda} c < \varepsilon^{n-1} \frac{\rho}{\rho+\lambda} c e^{\rho+\lambda} \leq V_i(1; F_{-i}).$$

Hence, no arbitrageur ever selects the initial time 0; i.e.,

$$F_i(0) = 0 \quad \text{for all} \quad i \in N,$$

and $\tau^1 < 1$ must hold. From Lemmas 1 and 2 and the symmetry of F, it follows that for every $i \in N$ and every $t \in [\tau^1, 1]$, F_i must be continuous and increasing, and therefore must satisfy the first-order condition given by

$$\frac{\partial V_i(t; F_{-i})}{\partial t} = 0.$$

From the first-order condition and the symmetry of F, it follows that

$$\frac{\frac{dF_i(t)}{dt}}{1 - F_i(t)} = \frac{\rho+\lambda}{n-1},$$

which implies that $F = \tilde{F}$, where $\tau^1 = \tilde{\tau}$.

Finally, we will show that if a strategy profile F is a Nash equilibrium, then it is symmetric. Since the time choice 0 is a dominated strategy for any arbitrageur, it follows that

$$F_i(0)=0 \quad \text{for all} \quad i \in N \text{, and} \quad \tau^1 < 1 .$$

For every $t \in [\tau^1, 1]$ and every $i \in N$, if $\frac{dF_i(t)}{dt} > 0$, then it follows from the first-order condition that

$$\frac{\partial V_i(t;F_{-i})}{\partial t} = \frac{\partial}{\partial t}[\frac{\rho}{\rho+\lambda} ce^{(\rho+\lambda)t} \prod_{h \in N \backslash \{i\}} \{1-F_h(t)\}]$$

$$= \rho ce^{(\rho+\lambda)t}[\prod_{h \in N \backslash \{i\}} \{1-F_h(t)\} - \frac{1}{\rho+\lambda} \sum_{k \in N \backslash \{i\}} \frac{dF_k(t)}{dt} \prod_{h \in N \backslash \{i,k\}} \{1-F_h(t)\}] = 0 ,$$

that is,

$$\sum_{k \in N \backslash \{i\}} \frac{\frac{dF_k(t)}{dt}}{1-F_k(t)} = \rho + \lambda .$$

This implies that for every $t \in [\tau^1, 1]$, every $i \in N$, and every $j \in N \backslash \{i\}$, if $\frac{dF_i(t)}{dt} > 0$ and $\frac{dF_j(t)}{dt} > 0$, then the following must hold:

$$\frac{\frac{dF_i(t)}{dt}}{1-F_i(t)} = \frac{\frac{dF_j(t)}{dt}}{1-F_j(t)} .$$

Hence, if a Nash equilibrium F is not symmetric, then there must exist $i \in N$, $\tau' \in [\tau^1, 1]$, and $\tau'' \in (\tau', 1]$ such that the time choice τ'' is the best response for arbitrageur i and

$$\frac{dF_i(t)}{dt} = 0 \quad \text{for all} \quad t \in (\tau', \tau'') .$$

Lemma 2 implies that at any time $t \in (\tau', \tau'')$ there exists another arbitrageur $j \in N \backslash \{i\}$ such that

$$\frac{dF_j(t)}{dt} > 0 = \frac{dF_i(t)}{dt} .$$

Since arbitrageur j satisfies the first-order condition $\frac{\partial V_j(t;F_{-j})}{\partial t} = 0$, it follows that

$$\sum_{k\in N\backslash\{j\}} \frac{\frac{dF_k(t)}{dt}}{1-F_k(t)} > \sum_{k\in N\backslash\{i\}} \frac{\frac{dF_k(t)}{dt}}{1-F_k(t)} = \rho+\lambda\,,$$

which implies that $\frac{\partial V_i(t;F_{-i})}{\partial t} < 0$. Hence, arbitrageur i prefers selecting time τ' instead of τ''. This is a contradiction.

From the above observations, we have proved that $\tilde{F}$ is the unique Nash equilibrium, even if we take into account any asymmetric strategy profile.

Q.E.D.

Since $\lim_{\lambda\to\infty} \tilde{\tau}(n,\varepsilon,\rho,\lambda) = 1$, it follows from inequality (1) that whenever the regulation on arbitrageurs' leverage is sufficiently weak, i.e., the leverage ratio L is sufficiently large, then the bubble can persist for a long time as the unique Nash equilibrium price movement, even if both ε and ρ are close to zero; i.e., even if all the arbitrageurs are expected to be almost certainly rational and the investors are not euphoric. By slightly postponing the sale from time t to time $t+\Delta$, any arbitrageur i can increase his capital gain per share by $\frac{\partial e^{\rho t}}{\partial t}\cdot\Delta$. On the other hand, the probability that the other arbitrageurs will burst the bubble increases by $\frac{d\prod_{j\in N\backslash\{i\}}\tilde{F}_j(t)}{dt}\cdot\Delta$. In this case, the market value of the shares that he owns, i.e., $ce^{(\rho+\lambda)t}$, declines to zero, while his debt, i.e., $\frac{\lambda}{\rho+\lambda}ce^{(\rho+\lambda)t}$, is reduced substantially. Hence, the amount of loss per share that he really bears is equal to

$$\frac{\rho}{\rho+\lambda}e^{\rho t} = \frac{e^{\rho t}}{L}.$$

The larger the leverage ratio L, the smaller is the amount of loss per share that he really bears. This weak regulation on arbitrageurs' leverage is the driving force behind their willingness to ride the bubble.

We can calculate the expected social cost induced by the strategy profile $\tilde{F}$ as

$$\int_{t=0}^{1} C(t)d[1-\prod_{i\in N}\{1-\tilde{F}_i(t)\}]$$

$$=\frac{\lambda}{\rho+\lambda}[e^{(\rho+\lambda)\tilde{\tau}}+(n-1)\{e^{(\rho+\lambda)\tilde{\tau}}-e^{\frac{(\rho+\lambda)(n\tilde{\tau}-1)}{n-1}}\}-1],$$

which approximates the worst social cost given by $\frac{\lambda}{\rho+\lambda}\{e^{\rho+\lambda}-1\}$, provided that $\tilde{\tau}$ is close to the terminal time 1.

4. Quick Crashes

We denote by F^* the symmetric strategy profile defined by

$$F_i^*(t) = 1-\varepsilon \quad \text{for all } i \in N \text{ and } t \in [0,1].$$

According to this strategy profile, any rational arbitrageur certainly sells out at the initial time 0. Hence, the bubble crashes at the initial time 0 with a probability of $1-\varepsilon^n$. The following theorem states that if $\rho+\lambda$ is close to 0, the specified strategy profile F^* is the unique Nash equilibrium; i.e., it is inevitable that the bubble quickly crashes at the initial time 0 whenever at least one arbitrageur is rational.

Theorem 4: *The strategy profile* F^* *is a Nash equilibrium if and only if*

$$\sum_{m\in\{0,\dots,n-1\}} \frac{1}{m+1}\{\frac{(n-1)!}{m!(n-m-1)!}(1-\varepsilon)^m \varepsilon^{n-m-1}\} \geq \varepsilon^{n-1} e^{\rho+\lambda}. \tag{4}$$

With the strict inequality of (4), F^* *is the unique Nash equilibrium.*

Proof: For every $i \in N$,

$$V_i(F^*) = \frac{\rho}{\rho+\lambda} c \sum_{m\in\{0,\dots,n-1\}} \frac{1}{m+1}\{\frac{(n-1)!}{m!(n-m-1)!}(1-\varepsilon)^m \varepsilon^{n-m-1}\},$$

and for every $t \in (0,1]$,

$$V_i(t;F_{-i}^*) = \varepsilon^{n-1} \frac{\rho}{\rho+\lambda} c e^{(\rho+\lambda)t}.$$

Hence, F^* is a Nash equilibrium if and only if for every $t \in (0,1]$,

$$\frac{\rho}{\rho+\lambda} c \sum_{m\in\{0,\dots,n-1\}} \frac{1}{m+1}\{\frac{(n-1)!}{m!(n-m-1)!}(1-\varepsilon)^m \varepsilon^{n-m-1}\} \geq \varepsilon^{n-1} \frac{\rho}{\rho+\lambda} c e^{(\rho+\lambda)t},$$

which is equivalent to inequality (4).

Suppose that $F \neq F^*$ is a Nash equilibrium. Then, it follows from Lemmas 1 and 2 that $\tau^1 < 1$, and the time choice 1 is the best response for some arbitrageur $i \in N$, where

$$V_i(1;F_{-i}) = \varepsilon^{n-1} \frac{\rho}{\rho+\lambda} e^{\rho+\lambda},$$

and

$$V_i(0;F_{-i}) \geq \frac{\rho}{\rho+\lambda} c \sum_{m\in\{0,\ldots,n-1\}} \frac{1}{m+1} \{\frac{(n-1)!}{m!(n-m-1)!}(1-\varepsilon)^m \varepsilon^{n-m-1}\}.$$

However, the strict inequality of (4) implies that $V_i(1;F_{-i}) < V_i(0;F_{-i})$; i.e., he prefers selecting time 0 instead of time 1. This is a contradiction.

Q.E.D.

5. Hybrid Equilibria

This section assumes that $(\varepsilon,\rho,\lambda)$ does not satisfy either inequality (2) or (4); i.e.,

$$(5)\qquad \sum_{m\in\{0,\dots,n-1\}}\frac{1}{m+1}\{\frac{(n-1)!}{m!(n-m-1)!}(1-\varepsilon)^m\varepsilon^{n-m-1}\}<\varepsilon^{n-1}e^{\rho+\lambda}<1,$$

which implies that neither $\tilde{F}$ or F^* is a Nash equilibrium. This subsection specifies another strategy profile and shows that it is the unique symmetric Nash equilibrium.

Note that with inequality (5), a unique real number $\eta\in(\varepsilon,1)$ exists such that

$$(6)\qquad \sum_{m\in\{0,\dots,n-1\}}\frac{1}{m+1}\{\frac{(n-1)!}{m!(n-m-1)!}(1-\eta)^m\eta^{n-m-1}\}=\varepsilon^{n-1}e^{\rho+\lambda}.$$

Let us define

$$\hat{\tau}\equiv 1+\frac{(n-1)}{\rho+\lambda}\ln\frac{\varepsilon}{\eta}.$$

Note from inequalities (5) and (6) that $\hat{\tau}$ is well defined; i.e.,

$$0<\tilde{\tau}<\hat{\tau}<1.$$

With inequality (5), we can specify a symmetric and continuous strategy profile $\hat{F}=(\hat{F}_i)_{i\in N}$, which is regarded as a hybrid of $\tilde{F}$ and F^*, as follows: for every $i\in N$,

$$\hat{F}_i(t)=1-\eta \quad \text{for all} \quad t\in[0,\hat{\tau}),$$

and

$$\hat{F}_i(t)=\tilde{F}_i(t)=1-\varepsilon\exp[\frac{(\rho+\lambda)(1-t)}{n-1}] \quad \text{for all} \quad t\in[\hat{\tau},1].$$

According to $\hat{F}$, any arbitrageur would sell out at the initial time 0 with a probability of $1-\eta$. Hence, the bubble quickly crashes at the initial time 0 with a probability of $1-\eta^n<1-\varepsilon^n$. After the initial time, no arbitrageur ever bursts the bubble of his own accord until time $\hat{\tau}$. After time $\hat{\tau}$, some arbitrageur i randomizes the time to sell out according to the same hazard rate as $\tilde{F}$; i.e.,

$$(7)\qquad \frac{\frac{\partial\hat{F}_i(t)}{\partial t}}{1-\hat{F}_i(t)}=\frac{\rho+\lambda}{n-1}.$$

Theorem 5: *With inequality (5), the strategy profile* $\hat{F}$ *is the unique symmetric Nash equilibrium.*

Proof: Since, after the initial time 0, any arbitrageur never sells out until time $\hat{\tau}$, it follows that

$$V_i(\tilde{\tau};\hat{F}_{-i}) > V_i(t;\hat{F}_{-i}) \quad \text{for all } i \in N \text{ and all } t \in (0,\hat{\tau}).$$

From the symmetry of $\hat{F}$ and equality (7), it follows in the same manner as in Theorem 3 that for every $i \in N$ and every $t \in (\hat{\tau},1)$, the first-order condition holds; i.e.,

$$\frac{\partial V_i(t;\hat{F}_{-i})}{\partial t} = 0.$$

Hence, we have proved that

$$V_i(\hat{\tau};\hat{F}_{-i}) = V_i(t;\hat{F}_{-i}) \quad \text{for all } i \in N \text{ and all } t \in [\hat{\tau},1].$$

From equality (6),

$$V_i(0;\hat{F}_{-i}) = \frac{\rho}{\rho+\lambda} c \sum_{m\in\{0,\dots,n-1\}} \frac{1}{m+1}\{\frac{(n-1)!}{m!(n-m-1)!}(1-\eta)^m \eta^{n-m-1}\}$$

$$= \varepsilon^{n-1}\frac{\rho}{\rho+\lambda} c e^{\rho+\lambda} = V_i(1;\hat{F}_{-i}),$$

which implies that the time choice 0 is the best response for any arbitrageur. Since no arbitrageur has an incentive to sell out at any time in $(0,\hat{\tau})$, we have proved that $\hat{F}$ is a Nash equilibrium.

We will show that $\hat{F}$ is the unique symmetric Nash equilibrium. We set any symmetric Nash equilibrium F arbitrarily. From inequality (5) and Theorem 4, it follows that $F \neq F^*$; i.e.,

$$F_i(0) < 1-\varepsilon \quad \text{for all } i \in N.$$

From Lemmas 1 and 2, it follows that F is continuous and increasing in $[\tau^1,1]$, where $\tau^1 < 0$ must hold. Hence, at any time $t \in (\tau^1,1)$, F_i must satisfy the first-order condition given by

$$\frac{\frac{\partial F_i(t)}{\partial t}}{1-F_i(t)} = \frac{\rho+\lambda}{n-1}.$$

Since the latter part of inequality (5) implies the non-existence of $\tilde{F}_i$, $F_i \neq \tilde{F}_i$ must hold for all $i \in N$, and therefore,

$$0 < F_i(0) < 1-\varepsilon \quad \text{for all} \quad i \in N.$$

This implies that both the time choices 0 and 1 are the best responses for any arbitrageur; i.e.,

$$V_i(0;F_{-i}) = \frac{\rho}{\rho+\lambda} c \sum_{m\in\{0,\dots,n-1\}} \frac{1}{m+1}\{\frac{(n-1)!}{m!(n-m-1)!} F_i(0)^m \{1-F_i(0)\}^{n-m-1}\}$$

$$= \varepsilon^{n-1} \frac{\rho}{\rho+\lambda} c e^{\rho+\lambda} = V_i(1;F_{-i}),$$

which along with equality (6) implies that

$$F_i(0) = 1-\eta \quad \text{for all} \quad i \in N, \text{ and } \quad \tau^1 = \hat{\tau}.$$

These observations imply that $F = \hat{F}$.

Q.E.D.

6. Concluding Remarks

This paper has shown the possibility that the bubble persists for a long time even if the arbitrageurs are expected to be almost certainly rational. The availability of arbitrageurs' debt finance plays a central role. If the arbitrageurs' leverage is only weakly regulated, the bubble persists for a long time as the unique Nash equilibrium price movement. This holds true even if the investors' euphoria is negligible. This bubble causes serious harm to the society because the manager of the company can raise enormous sums of money and use it for his private benefit.

Although we have investigated the stock market in this paper, we can apply the same arguments to other markets as well. For instance, let us consider the case of housing bubbles, where the company and the arbitrageurs are replaced with a mortgage bank and structured investment vehicles (SIV), respectively. The mortgage bank makes subprime loans; it raises money by issuing ABS-CDOs and selling them to SIVs and behavioral investors. Each SIV invests in newly issued ABS-CDOs, raising money for the purpose by selling short-term securities backed by the ABS-CDOs that it owns to the behavioral investors. In this case, given that the SIVs' leverage is only weakly regulated, the harmful bubble persists for a long time.

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