# CARF Working Paper

CARF-F-232

# Hysteresis in Dynamic General Equilibrium Models with Cash-in-Advance Constraints

Kazuya Kamiya
University of Tokyo
Takashi Shimizu
Kansai University

October 2010

CARF is presently supported by Bank of Tokyo-Mitsubishi UFJ, Ltd., Citigroup, Dai-ichi Mutual Life Insurance Company, Meiji Yasuda Life Insurance Company, Nippon Life Insurance Company, Nomura Holdings, Inc. and Sumitomo Mitsui Banking Corporation (in alphabetical order). This financial support enables us to issue CARF Working Papers.

# Hysteresis in Dynamic General Equilibrium Models with Cash-in-Advance Constraints*

Kazuya Kamiya† and Takashi Shimizu‡

October 2010

**Abstract**

In this paper, we investigate equilibrium cycles in dynamic general equilibrium models with cash-in-advance constraints. Our findings are two-fold. First, in such models, if an equilibrium cycle exists, then there also exists a continuum of equilibrium cycles in its neighborhood. Second, the limit cycle, to which a dynamic path converges, varies continuously according to the initial distribution of the money holdings. Thus, temporary shocks that affect the initial distribution have permanent effects in such models; that is, such models exhibit hysteresis. Furthermore, we also explore the logic behind the results.

Keywords: Dynamic General Equilibrium Models, Cash-in-Advance, Cycles, Hysteresis.

JEL Classification Number: D51, E40, E50, E60.

## 1 Introduction

In this paper, we investigate equilibrium cycles in dynamic general equilibrium models with cash-in-advance constraints, wherein each agent's money holding varies over time. We first show that a continuum of equilibrium cycles exists in a specific model with cash-in-advance constraints, and that the limit cycle, to which a dynamic path converges, varies continuously according to the initial distribution of money holdings. Thus, temporary shocks that affect the initial distribution have permanent effects on such models; that is, these models exhibit hysteresis. Then, using a general framework, we also explore the logic behind the results.

*The authors would like to acknowlegde Pedro Gomis Porqueras for his helpful comments. This research was financially supported by the Grant-in-Aid for Scientific Research from JSPS and MEXT and Kansai University's Overseas Research Program for the year of 2009. Of course, any remaining errors are our own.

†Faculty of Economics, University of Tokyo, Bunkyo-ku, Tokyo 113-0033 JAPAN (E-mail: kkamiya@e.u-tokyo.ac.jp)

‡Faculty of Economics, Kansai University, 3-3-35 Yamate-cho, Suita-shi, Osaka 564-8680 JAPAN (E-mail: tshimizu@ipcku.kansai-u.ac.jp)

Our finding on a continuum of equilibrium cycles is new to the literature on this subject. In optimal growth models, a continuum of equilibrium cycles has never been found as a nondegenerate case, although a finite number of cycles have been observed. (See, for example, Mitra and Nishimura [12].) In random matching models with fiat money, a continuum of stationary (non-cycle) equilibria has been found in both specific and general models. (See, for example, Green and Zhou [4], [5], Kamiya and Shimizu [6], Matsui and Shimizu [11], and Zhou [13].)[1] However, even in such models, a continuum of equilibrium cycles has never been found.

Our finding on hysteresis is also new to the literature on monetary economics.[2] In monetary models with Walrasian markets, there typically exist a finite number of stationary equilibria. Hysteresis cannot be found in the case of unique stationary equilibrium; that is, if the equilibrium is stable, then the dynamic paths converge to the stationary equilibrium from any initial point. In the case of multiple equilibria, only large shocks at the initial point can change the limit point, and thus, hysteresis cannot be found for a small shock. In a random matching model with money, Green and Zhou [5] find a continuum of stationary equilibria, and show that any stationary equilibrium can be reached from any initial point; that is, there is indeterminacy in dynamic paths. Therefore, in their model, any temporary shock does not have an effect.

Blanchard and Summers [2] demonstrate that unemployment hysteresis arises from insider-dominated wage determination. In their model, they assume that wage determination is dominated by inside workers. Hysteresis arises since wages depend on the number of inside workers, which in turn depends on past employment. On the other hand, Baldwin [1] shows that hysteresis arises from large exchange rate swings under the assumption that market entry costs are sunk. A large temporary rise in the exchange rate induces foreign firms to enter the market. When the exchange rate falls to the original level, some new entrants remain in the market because of sunk costs. These logics are clearly very different from ours; we demonstrate that if fiat money has value and an equilibrium cycle exists, then hysteresis arises.

In this paper, we first show that in a specific model, there exists a continuum of equilibrium cycles that exhibits hysteresis. Then, using a general model, we explore the logic behind the results. In this model, there is a continuum of agents and the number of goods is $L \geq 1$, and in each time period, a Walrasian market with cash-in-advance constraints is open for each good. Each consumer is characterized by a net demand function, $z(\eta, p_1, p_2, \dots)$, where $\eta$ is the

[1] Kamiya and Shimizu [8] also construct models in which centralized auction markets have a continuum of stationary equilibria, but Walrasian markets with cash-in-advance constraints have a unique stationary equilibrium.

[2] For a survey on hysteresis in economics, see Franz [3] among others.

consumer's money holding at the beginning of the period and $p_t \in R^L_{++}, t = 1, 2, \dots,$ is a price vector in period $t$. In other words, $z(\eta, p_1, p_2, \dots) \in R^L$ is the first period net consumption vector when she maximizes a utility stream under some conditions, including budget constraints and cash-in-advance constraints. We demonstrate that if an equilibrium cycle exists, then there is a continuum of equilibrium cycles under some conditions. We also show that if a dynamic path converges to an equilibrium cycle from an initial money holdings distribution, then under some conditions, the limit cycle continuously depends on the initial money holdings. Thus, a temporal policy shock that affects the initial money holdings distribution also has a permanent effect; that is, hysteresis occurs.

In Section 2, we first investigate a specific model, and show that a continuum of equilibrium cycles exists and that the limit cycle continuously depends on the initial money holdings distribution. Then, in Section 3, even in a rather general framework, we obtain the same results. In Section 4, we discuss some specific assumptions in the model in Section 2. Finally, we conclude the paper in Section 5.

## 2 A Model with Cycles

We use a simple framework that is similar to Kiyotaki and Wright [9]. Time is discrete, denoted by $t = 1, 2, \dots$. There is a continuum of agents, whose measure is one. There are $\mathcal{T} \geq 3$ types of agents with equal fractions and the same number of types of goods. We assume that the goods are perishable and divisible. A type $\tau$ agent can produce good $\tau + 1$ for $\tau = 1, \dots, \mathcal{T} - 1$, and a type $\mathcal{T}$ agent can produce good 1. Throughout this section, we assume that each agent can produce just one unit of her production good with production cost $c \geq 0$ in each time period.[3] A type $\tau$ agent obtains utility $U(q)$ only when she consumes $q$ amount of good $\tau$. In this section, we consider a linear utility function $U(q) = aq$, where $a > c$. Let $\delta \in (0, 1)$ be the discount factor. Our framework includes divisible and durable fiat money, whose nominal stock is $M > 0$.

At each time period, a competitive spot market is open. Purchases of goods are subject to a cash-in-advance constraint. We also assume a *participation constraint*: in each time period an agent can visit only one market; that is, she must choose to be either a buyer or a seller in each time period. $\chi = 1$ means that she only consumes her consumption good, and $\chi = 0$ means that

[3] In this paper, we distinguish between the terms "period" and "time period"; "Period" means a period in a cycle, while "time period" means a period in an entire sequence. For example, when a sequence of prices is $(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, \dots) = (p_a, p_b, p_c, p_d, p_c, p_d, p_c, p_d, \dots)$, then the price in the second time period is $p_b$ and the price in the second period in the cycle is $p_d$.

she only produces her production good. Each agent solves the following optimization problem with respect to $(\chi_1, q_1), (\chi_2, q_2), \dots$:

$$\begin{aligned}
&\max \sum_{t=1}^{\infty} \delta^{t-1}(\chi_t U(q_t) - (1-\chi_t)c) \\
\text{s.t.} \quad & \chi_t \tilde{p}_t q_t + \eta_{t+1} = \eta_t + (1-\chi_t)\tilde{p}_t, \quad t = 1, 2, \dots, \\
& \chi_t \tilde{p}_t q_t \leq \eta_t, \eta_t \geq 0, \quad t = 1, 2, \dots, \\
& \eta_1 \geq 0 \text{ given,}
\end{aligned}$$

where $\eta_t$ is the agent's money holding at the beginning of time period $t$, $\tilde{p}_t$ is the given price of her consumption good at time period $t$, and $q_t$ is the amount of consumption at time period $t$. Note that the agent can choose to "do nothing" by choosing $(\chi, q) = (1, 0)$. A sequence of price $(\tilde{p}_1, \tilde{p}_2, \dots)$ is said to be an *equilibrium price vector* if each consumer solves the above problem and all spot markets clear. Below, we focus on equilibria such that the consumers' policies and prices of goods are symmetric with respect to types; that is, $\tilde{p}_t$ and the optimum policies are the same across types.

For simplicity, we make two assumptions: there is a participation constraint, and each agents can produce only one unit of her production good. In Section 4 and Appendix, we show that these assumptions are not necessary for obtaining the same results.

## 2.1 Equilibrium with a 2-Period Cycle

Here, we demonstrate there is a continuum of 2-*period equilibrium cycles*; that is, the equilibrium price vector satisfies $\tilde{p}_t = \tilde{p}_{t+2}$ for $t = 1, 2, \dots$ and money holdings of each agent alternate between $\eta_0$ and $\eta_1$, or $\eta_0'$ and $\eta_1'$, where $\eta_0$ and $\eta_0'$ are money holdings in even periods, and $\eta_1$ and $\eta_1'$ are money holdings in odd periods. That is, in even periods, some agents have $\eta_0$ and the others have $\eta_0'$, and in odd periods, the former have $\eta_1$, and the latter have $\eta_1'$. Moreover, the prices are the same in even (odd) periods.

**Theorem 1** Suppose $-1 + \delta + \delta^2 < \frac{c}{a} < \delta$. Then, a continuum of 2-period equilibrium cycles exists.

**Proof:**
We first construct a stationary equilibrium; that is, the case that $\tilde{p}_t$ is the same for all $t$. We

then show that there is a continuum of 2-period equilibrium cycles in a neighborhood of the stationary equilibrium.

We consider the following candidate for stationary equilibria:

- There exists a real number $p > 0$, such that $(p, p, \dots)$ is an equilibrium price vector.
- The policy of each agent is as follows: there exists $\bar{\eta} \in (0, p)$, such that
  - an agent with $\eta \in [0, \overline{\eta}]$ sells her production good, and
  - an agent with $\eta \in (\overline{\eta}, \infty)$ spends all her money.
- The stationary money holdings distribution is as follows:
  - the measure of agents without money is 1/2, and
  - the measure of agents with $p$ is 1/2.
- The value function is continuous.

Since half of the agents have $p$ amount of money,

$$M = \frac{1}{2}p$$

holds, then

$$p = \frac{2}{M}.$$

Since agents with $p$ amount of money spend all of it, and agents without money want to sell, the market clearing condition for goods is

$$\frac{1}{2} = \frac{\frac{1}{2}p}{p},$$

where the LHS is the supply of goods and the RHS is the demand for goods. Clearly, it is an identity.

By the above policy, the value function is expressed as

$$V(\eta) = \begin{cases} -c + \delta V(\eta + p), & \text{for } \eta \in [0, \bar{\eta}]\,, \\ \frac{a}{p}\eta + \delta V(0), & \text{for } \eta \in (\bar{\eta}, \infty)\,. \end{cases}$$

Then, we obtain

$$V(\eta) = \begin{cases} \frac{1}{1-\delta^2}(a\delta - c) + \frac{a\delta}{p}\eta, & \text{for } \eta \in [0, \overline{\eta}], \\ \frac{\delta}{1-\delta^2}(a\delta - c) + \frac{a}{p}\eta, & \text{for } \eta \in (\overline{\eta}, \infty). \end{cases}$$

The continuity of $V$ at $\overline{\eta}$ implies

$$\overline{\eta} = \frac{p}{a(1-\delta^2)}(a\delta - c).$$

Clearly, $\overline{\eta} \in (0, p)$ follows from

$$-1 + \delta + \delta^2 < \frac{c}{a} < \delta. \tag{1}$$

Next, we show that the above policy is indeed optimal by stating the following inequality:

- $V(\eta) \geq 0$ for $\eta$.
- $V(\eta) \geq \frac{a}{p}\eta' + \delta V(\eta - \eta')$ for $\eta \in [0, \overline{\eta}]$ and $\eta' \in [0, \eta]$.
- $V(\eta) \geq -c + \delta V(\eta + p)$ for $\eta \in (\overline{\eta}, \infty)$.
- $V(\eta) \geq \frac{a}{p}\eta' + \delta V(\eta - \eta')$ for $\eta \in (\overline{\eta}, \infty)$ and $\eta' \in [0, \eta)$ .

By (1), we can easily verify that the above conditions are satisfied with strict inequalities. Thus, we have shown that the above candidate is indeed a stationary equilibrium under (1).

Next, we demonstrate that there exists is a continuum of 2-period equilibrium cycles in a neighborhood of the stationary equilibrium. We denote the price in even periods by $p_0$ and that in odd periods by $p_1$. Let $h_0$ be the measure of agents with $p_0$ amount of money at the beginning of odd periods and with no money at the beginning of even periods, and let $h_1$ be the measure of agents with no money at the beginning of odd periods and with $p_1$ amount of money at the beginning of even periods. Clearly, $h_0 > 0$, $h_1 > 0$ and

$$h_0 + h_1 = 1 \tag{2}$$

must be satisfied.

Since the total amount of money is $M$,

$$M = h_0 p_0 \text{ and } M = h_1 p_1 \tag{3}$$

hold. Thus,

$$p_0 = \frac{M}{h_0} \text{ and } p_1 = \frac{M}{h_1}$$

hold. The condition for market clearing is

$$h_0 p_0 = h_1 p_1.$$

That is, in even periods, the LHS is the value of supply and the RHS is the total expenditure, whereas in odd periods, the LHS is the total expenditure and the RHS is the value of supply. Thus, (3) clearly implies the market clearing condition. In other words, this condition is redundant. Note that this argument applies to rather general cases. (See Section 3.1.)

We denote the value function in even periods by $V_0$ and that in odd periods by $V_1$. Then, $V_0$ and $V_1$ satisfy

$$V_0(\eta) = \begin{cases} -c + \delta V_1(\eta + p_0), & \text{for } \eta \in [0, \bar{\eta}_0]\,, \\ \frac{a}{p_0}\eta + \delta V_1(0), & \text{for } \eta \in (\bar{\eta}_0, \infty)\,, \end{cases}$$
$$V_1(\eta) = \begin{cases} -c + \delta V_0(\eta + p_1), & \text{for } \eta \in [0, \bar{\eta}_1]\,, \\ \frac{a}{p_1}\eta + \delta V_0(0), & \text{for } \eta \in (\bar{\eta}_1, \infty)\,. \end{cases}$$

Thus, the value functions are expressed as

$$V_0(\eta) = \begin{cases} \frac{a\delta h_1 - ch_0}{(1-\delta^2)h_0} + \frac{a\delta}{p_1}\eta, & \text{for } \eta \in [0, \bar{\eta}_0], \\ \delta\frac{a\delta h_0 - ch_1}{(1-\delta^2)h_1} + \frac{a}{p_0}\eta, & \text{for } \eta \in (\bar{\eta}_0, \infty), \end{cases}$$
$$V_1(\eta) = \begin{cases} \frac{a\delta h_0 - ch_1}{(1-\delta^2)h_1} + \frac{a\delta}{p_0}\eta, & \text{for } \eta \in [0, \bar{\eta}_1], \\ \delta\frac{a\delta h_1 - ch_0}{(1-\delta^2)h_0} + \frac{a}{p_1}\eta, & \text{for } \eta \in (\bar{\eta}_1, \infty). \end{cases}$$

By the continuity of $V_0$ and $V_1$,

$$\bar{\eta}_0 = \frac{M(a\delta h_1^2 + c\delta h_0 h_1 - a\delta^2 h_0^2 - ch_0 h_1)}{ah_0 h_1 (h_0 - \delta h_1)(1-\delta^2)} \quad \text{and} \tag{4}$$
$$\bar{\eta}_1 = \frac{M(a\delta h_0^2 + c\delta h_0 h_1 - a\delta^2 h_1^2 - ch_0 h_1)}{ah_0 h_1 (h_1 - \delta h_0)(1-\delta^2)}. \tag{5}$$

Note that if $h_0 = h_1 = \frac{1}{2}$, then the above is equal to the value function of the stationary equilibrium. Recall that the optimality conditions are satisfied with strict inequalities under (1). Therefore, for sufficiently small $\epsilon > 0$, the above value functions with $(h_0, h_1) = (\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon)$ constitute an equilibrium under (1). ∎

## 2.2 $T$-Period Equilibrium Cycles where $T \geq 3$

We show that the model also has a continuum of $T$-period equilibrium cycles, where $T \geq 3$. As in Section 2.1, we first construct a stationary equilibrium with $T$ states, and then transform it into a continuum of $T$-period equilibrium cycles.

**Theorem 2** Suppose $\delta - (T-1)(1-\delta^2) < \frac{c}{a} < \delta - (T-2)(1-\delta^2)$. Then, a continuum of $T$-period equilibrium cycles exists.

**Proof:**
First, we consider the following candidate for stationary equilibria:

- There exists a real number $p > 0$, such that $(p, p, \dots)$ is an equilibrium price vector.
- The policy of each agent is as follows: there exists $\overline{\eta} \in ((T-2)p, (T-1)p)$ such that
  - an agent with $\eta \in [0, \overline{\eta}]$ sells her production good, and
  - an agent with $\eta \in (\overline{\eta}, \infty)$ spends all her money.
- The support of a stationary money holdings distribution is $\{0, p, \dots, (T-1)p\}$, and the measure of agents with money holdings $ip$ is $1/T$ for $i = 0, 1, \dots, T-1$.
- The value function is continuous.

The total amount of money must be equal to $M$, and the goods market must clear; that is,

$$M = \sum_{i=1}^{T-1} \frac{ip}{T},$$

and

$$p = \frac{\frac{(T-1)p}{T}}{(T-1)\frac{1}{T}}.$$

The former condition implies

$$p = \frac{2}{T-1} M,$$

and the latter condition is automatically satisfied.

The value function satisfies

$$V(\eta) = \begin{cases} -c + \delta V(\eta + p), & \text{if } \eta \in [0, \bar{\eta}]\,, \\ \frac{a}{p}\eta + \delta V(0), & \text{if } \eta \in (\bar{\eta}, \infty)\,. \end{cases}$$

Then, $V$ is obtained as follows:

$$V(\eta) = \begin{cases} \frac{1}{1-\delta^2}\left(a\delta - c\right) + \frac{a\delta}{p}\eta, & \text{if } \eta \in [0, \bar{\eta}]\,, \\ \frac{\delta}{1-\delta^2}\left(a\delta - c\right) + \frac{a}{p}\eta, & \text{if } \eta \in (\bar{\eta}, \infty)\,. \end{cases} \tag{6}$$

The continuity of $V$ at $\overline{\eta}$ implies

$$\overline{\eta} = \frac{p}{a(1-\delta^2)}(a\delta - c).$$

$\overline{\eta} \in ((T-2)p, (T-1)p)$ follows from the following condition:

$$\delta - (T-1)(1-\delta^2) < \frac{c}{a} < \delta - (T-2)(1-\delta^2). \tag{7}$$

The condition for the optimality of the specified policy is as follows:

- $V(\eta) \geq 0$ for any $\eta$.
- $V(\eta) \geq \frac{a}{p}\eta' + \delta V(\eta - \eta')$ for any $\eta \in [0, \overline{\eta})$ and any $\eta' \in [0, \eta]$.
- $V\left(\overline{\eta}\right) \geq \frac{a}{p}\eta' + \delta V\left(\overline{\eta} - \eta'\right)$ for any $\eta' \in [0, \overline{\eta})$.
- $V(\eta) \geq -c + \delta V(\eta + p)$ for any $\eta \in (\overline{\eta}, \infty)$.
- $V(\eta) \geq \frac{a}{p}\eta' + \delta V(\eta - \eta')$ for any $\eta \in (\overline{\eta}, \infty)$ and any $\eta' \in [0, \eta)$ .

By (6), it is easily verified that the above optimality conditions are satisfied with strict inequalities under (7). Thus, we have shown that the specified money holdings distribution and policy constitutes an equilibrium under (7).

Next, we transform the stationary equilibrium with $T$ states into a continuum of equilibria with a $T$-period cycle by perturbing the money holdings distribution. We denote the price at time period $Tn + i$ ($i = 0, \ldots, T-1$) by $p_i$. For ease of exposition, let $(i) = i \mod T$. Let $h_{ij}$ be the measure of agents with $\sum_{k=1}^{j} p_{(i+T-k)}$ amount of money at time period $Tn + i$. (Let $h_{i0}$ be the measure of agents with no money at time $Tn + i$.[4])

[4]Throughout this paper, let $n$ be the generic symbol of natural numbers including 0.

The condition for a stationary cycle is that for any $i$ and $j$, $h_{ij} = h_{(i+1).(j+1)} = \cdots = h_{(i+T-1).(j+T-1)}$.[5] Therefore, if there is a vector $h = (h_0, \dots, h_{T-1})$ such that

$$
\begin{aligned}
& h_i > 0, \quad \forall i, \\
& \sum_{i=0}^{T-1} h_i = 1, \\
& h_i = h_{0i} = h_{1.(i+1)} = \cdots = h_{T-1.(i+T-1)}, \quad i = 0, \dots, T-1,
\end{aligned}
$$

then the condition holds.

The total money holding must be equal to $M$:

$$
M = \sum_{j=1}^{T-1} h_{ij} \sum_{k=1}^{j} p_{(i+T-k)}, \quad i = 0, \dots, T-1. \tag{8}
$$

It is verified that $(p_0, p_1, \dots, p_{T-1})$ is uniquely determined if each $h_{ij}$ is sufficiently close to $1/T$.

The condition for market clearing is

$$
p_i = \frac{h_{i.T-1} \sum_{j=1}^{T-1} p_{(i+T-j)}}{\sum_{j=0}^{T-2} h_{ij}}, \quad i = 0, \dots, T-1.
$$

Then, it is easily verified that the conditions for the stationary cycle and (8) imply the condition for market clearing. In other words, the latter is redundant.

Recall that the optimality condition is satisfied with strict inequalities under (7). Let

$$
h_i = \begin{cases} \frac{1}{T} - \frac{\epsilon - \epsilon^T}{1-\epsilon}, & \text{if } i = 0, \\ \frac{1}{T} + \epsilon^i, & \text{if } i \neq 0. \end{cases}
$$

Then, by redefining $\overline{\eta}$ such that the value function is continuous at $\overline{\eta}$, it is easily verified that the above policy and $h_i$ constitute an equilibrium cycle under (7). ∎

**Remark 1** For $s = 1, 2, \dots$, it is verified that there exists a unique $\delta \in (0,1)$ such that $\delta - (s-1)(1-\delta^2) = \frac{c}{a}$. We denote such a $\delta$ by $\hat{\delta}_s$. Then, Theorems 1 and 2 imply that there exists a continuum of $T(\geq 2)$-period equilibrium cycles if $\delta \in (\hat{\delta}_{T-1}, \hat{\delta}_T)$. Clearly, $\hat{\delta}_1 < \hat{\delta}_2 < \cdots < 1$ and $\lim_{s\to\infty} \hat{\delta}_s = 1$ hold. Therefore, for almost every $\delta > \frac{c}{a}$, there exists a $T \geq 2$ such that a continuum of $T$-period equilibrium cycles also exists, and such a $T$ is unique as long as the equilibria in Theorem 2 are considered.

[5] We have used $h_{i.j}$ instead of $h_{ij}$ when the latter expression may be confusing.

**Remark 2** The indeterminacy in the above theorem is real since the distributions of utilities are different across $\epsilon$. However, the welfare that is defined as the weighted average of agents' values is the same for all $\epsilon$, since the utility function is linear and the cost function is simple. In Appendix, we show that the welfare can be different in the case of a strictly convex cost function.

### 2.3 Dynamic Equilibria Leading to 2-Period Cycles

In this section, we analyze a dynamic path converging to a 2-period equilibrium cycle. Suppose, at the beginning of period 1, the money holdings distribution is expressed by the following density function:

$$f_1(\eta) = \begin{cases} 0, & \eta < 0, \\ \frac{1}{2M}, & \eta \in [0, 2M], \\ 0, & \eta > 2M. \end{cases}$$

Below, we investigate a path from the above distribution converging to a 2-period cycle. More precisely, in this subsection and in the next, we show that the limit cycle depends on the initial distribution; that is, if we slightly perturb the initial distribution, then the limit cycle changes slightly.

First, we briefly explain the process of obtaining the equilibrium path. As in the previous section, we focus on the equilibria with the following policy: in each time period, there exists a threshold $\tilde{\eta} > 0$ such that

- an agent with $\eta \in [0, \tilde{\eta}]$ sells her production good, and
- an agent with $\eta \in (\tilde{\eta}, \infty)$ spends all her money.

In the first time period, $\tilde{\eta}_1 \in (0, 2M)$ is a threshold. Note that $\tilde{\eta}_1 \in (0, 2M)$ will be shown later. Thus, the money holdings distribution at the beginning of the second time period is such that agents with measure $\int_{\tilde{\eta}_1}^{2M} \frac{1}{2M} d\eta$ do not have any money, and the distribution of money holdings of the other agents is expressed by the following density function:

$$f_2(\eta) = \begin{cases} \frac{1}{2M}, & \eta \in [\tilde{p}_1, \tilde{\eta}_1 + \tilde{p}_1], \\ 0, & \text{otherwise}, \end{cases}$$

where $\tilde{p}_1$ is the price in the first time period. In the second time period, we suppose that a threshold $\tilde{\eta}_2$ is in $[0, \tilde{p}_1)$. Note that $\tilde{\eta}_2 \in [0, \tilde{p}_1)$ will be shown later. Thus, the money holdings distribution at the beginning of the third time period is such that agents with measure $\int_{\tilde{\eta}_1}^{2M} \frac{1}{2M} d\eta$

have $\tilde{p}_2$ amount of money and the other agents do not have any money, where $\tilde{p}_2$ is the price in the second time period. We will demonstrate that an equilibrium cycle starts from the third time period. Thus, from $t \geq 3$, in odd periods, agents with measure $h_0 = \int_{\tilde{\eta}_1}^{2M} \frac{1}{2M} d\eta$ have a positive amount of money, and in even periods, agents with measure $h_1 = 1 - h_0$ have a positive amount of money.

We now obtain the equilibrium path by backward induction. As shown in the previous subsection, in the cycle, the value function satisfies

$$V_0(0) = \frac{a\delta h_1 - ch_0}{(1-\delta^2)h_0}, \quad V_0(p_1) = \frac{ah_0 - c\delta h_1}{(1-\delta^2)h_1},$$
$$V_1(0) = \frac{a\delta h_0 - ch_1}{(1-\delta^2)h_1}, \quad V_1(p_0) = \frac{ah_1 - c\delta h_0}{(1-\delta^2)h_0},$$

where $p_0$ and $p_1$ are equilibrium prices in the cycle. $\tilde{p}_2$ must be equal to the price in even periods in the cycle, $p_0$, since in the second time period and in even periods in the cycle, agents with a positive amount of money use all of it, and the measure of the agent who sells the good is the same. Thus, the value function in time period 2 satisfies

$$\tilde{V}_2(0) = V_0(0) = \frac{a\delta h_1 - ch_0}{(1-\delta^2)h_0}, \tag{9}$$
$$\tilde{V}_2(\eta) = V_0(\eta) = a\frac{\eta}{p_0} + \delta V_1(0) = \frac{ah_0\eta}{M} + \delta\frac{a\delta h_0 - ch_1}{(1-\delta^2)h_1}, \quad \text{if } \eta > \tilde{\eta}_2. \tag{10}$$

The value function in time period 1 is expressed as follows:

$$\tilde{V}_1(\eta) = \begin{cases} -c + \delta\tilde{V}_2(\eta + \tilde{p}_1), & \text{if } \eta \leq \tilde{\eta}_1, \\ a\frac{\eta}{\tilde{p}_1} + \delta\tilde{V}_2(0), & \text{if } \eta > \tilde{\eta}_1. \end{cases}$$

The market clearing condition at time period 1 is

$$\tilde{p}_1 = \frac{4M^2 - \tilde{\eta}_1^2}{2\tilde{\eta}_1} \tag{11}$$

since the measure of sellers is $\int_0^{\tilde{\eta}_1} \frac{1}{2M} d\eta = \frac{\tilde{\eta}_1}{2M}$ and the total amount of money of the buyers is $\int_{\tilde{\eta}_1}^{2M} \frac{\eta}{2M} d\eta = \frac{4M^2 - \tilde{\eta}_1^2}{4M}$. Moreover, as shown in the above, the measure of the sellers must be $h_1$; that is,

$$\frac{\tilde{\eta}_1}{2M} = h_1. \tag{12}$$

Note that $\tilde{\eta}_1 \in (0, 2M)$ is automatically satisfied for $h_1 \in (0,1)$. Moreover, by the continuity of $\tilde{V}_1$ at $\eta = \tilde{\eta}_1$,

$$-c + \delta\tilde{V}_2(\tilde{\eta}_1 + \tilde{p}_1) = a\frac{\tilde{\eta}_1}{\tilde{p}_1} + \delta\tilde{V}_2(0) \tag{13}$$

must hold. Substituting (9)–(12) into the above equation, we obtain

$$\frac{2ah_1^2}{1-h_1^2} - \delta a h_0 h_1 - \frac{\delta a h_0 - ch_1}{(1-\delta^2)h_1} + \delta \frac{\delta a h_1 - ch_0}{(1-\delta^2)h_0} = 0.$$

This is equivalent to

$$\xi(h_0, \delta) = \frac{c}{a}, \tag{14}$$

where

$$\begin{aligned}\xi(h_0,\delta) &= \frac{\delta(h_0^2 - \delta h_1^2)}{(1-\delta)h_0 h_1} + \delta(1+\delta)h_0 h_1 - \frac{2(1+\delta)h_1^2}{1-h_1^2} \\ &= \frac{-2(1-h_0)^3 + h_0^2(2-h_0)(2-2h_0+h_0^2)\delta - h_0(1-h_0)^2\delta^2 - h_0^2(1-h_0)^2(2-h_0)\delta^3}{(1-\delta)h_0(1-h_0)(2-h_0)}.\end{aligned}$$

Since it is verified that $\xi$ is strictly increasing in $h_0$, and

$$\lim_{h_0 \to 0} \xi(h_0, \delta) = -\infty \quad \text{and} \quad \lim_{h_0 \to 1} \xi(h_0, \delta) = \infty,$$

a unique $h_0$ satisfying (14) exists.

Below, we check the conditions for $\tilde{\eta}_2$, $\bar{\eta}_0$, and $\bar{\eta}_1$. First, $\tilde{\eta}_2 = \bar{\eta}_0$ holds, and $\bar{\eta}_0$ is determined by (4). Thus, $0 \le \tilde{\eta}_2 < \tilde{p}_1$ must be satisfied in equilibria since $\tilde{p}_1 < p_1$. Under (14), this is equivalent to

$$\frac{\delta(h_1^2 - \delta h_0^2)}{h_0 h_1 (1-\delta)} \ge \frac{c}{a} > \frac{\delta(h_1^2 - \delta h_0^2) - (1-\delta^2)h_0(1-h_1^2)(h_0 - \delta h_1)}{h_0 h_1 (1-\delta)} \tag{15}$$

since

$$\frac{1}{1+\delta} > h_1.$$

Similarly, $\bar{\eta}_1$ is determined by (5), and thus, $0 \le \bar{\eta}_1 < p_0$ must be satisfied in equilibria. Under (14), this is equivalent to

$$\frac{\delta(h_0^2 - \delta h_1^2)}{h_0 h_1 (1-\delta)} \ge \frac{c}{a} > \frac{\delta(h_0^2 - \delta h_1^2) - (1-\delta^2)h_1(h_1 - \delta h_0)}{h_0 h_1 (1-\delta)} \tag{16}$$

since

$$\frac{1}{1+\delta} > h_0.$$

Thus, an equilibrium exists if and only if the $h_0$, which is uniquely determined by (14), satisfies (16) and (15). We numerically verified that a non-empty set of parameters satisfying the conditions exists. For example, $(h_0, \delta, a, c) = (.44059, .4, 10, 1)$ satisfies them.

### 2.4 Dynamic Equilibria Leading to 3-Period Cycles

The initial distribution is the same as that in Section 2.3. We investigate dynamic equilibria leading to a 3-period cycle. More precisely, we focus on the following policy: for some $\tilde{\eta}_t > 0, t = 1, 2, \dots,$

- an agent with $\eta \in [0, \tilde{\eta}_t]$ sells her production good at time period $t$,
- an agent with $\eta \in (\tilde{\eta}_t, \infty)$ spends all her money at time period $t$, and
- each agent spends all her money only once in the first three time periods, say $k$. Moreover, she spends all her money at $k + 3$.

We suppose that the above policy is optimal. By the above, each agent's money holding becomes zero only once in the first three periods, and thus, the agents are classified into three groups: the agents whose money holdings become zero at the end of the first, the second, and the third periods, respectively. Let the measures of each group of agents be denoted by $h_1, h_0$, and $h_2 = 1 - h_0 - h_1$, respectively. We now show that the market clearing conditions for the first six periods determine $(h_1, h_0)$ and the equilibrium prices of the first six periods. Then, we show that the equilibrium prices of $t = 7, 8, \dots$ are determined by a difference equation, and it converges to a limit cycle.

The maximization problem can be written as follows: for a given sequence $(\tilde{p}_1, \tilde{p}_2, \dots)$,

$$
\begin{aligned}
&\max \sum_{t=1}^{k+3} \delta^{t-1}(\chi_t a q_t - (1-\chi_t)c) + \delta^{k+3} V_{k+3+1}(\eta_{k+3+1}) \\
&\text{s.t.} \quad \chi_t \tilde{p}_t q_t + \eta_{t+1} = \eta_t + (1-\chi_t)\tilde{p}_t, \quad t = 1, 2, \dots, k+3, \\
&\qquad\; \chi_t \tilde{p}_t q_t \leq \eta_t, \eta_t \geq 0, \quad t = 1, 2, \dots, k+3, \\
&\qquad\; \eta_1 \geq 0 \text{ given,}
\end{aligned}
$$

where $V_{k+3+1}(\eta_{k+3+1})$ is the value at $k + 3 + 1$. By the above assumption, $\eta_{k+3+1} = 0$. We suppose that the constraint $\eta_{k+3+1} \geq 0$ is binding; that is, the corresponding Lagrange multiplier is positive. Thus, a small change of $(\tilde{p}_{k+3+1}, \tilde{p}_{k+3+2}, \dots)$ does not affect the optimal choice in the above problem.

For some $\eta_a$ and $\eta_b$ such that $0 < \eta_a < \eta_b < 2M$, the threshold $\tilde{\eta}_t$ is given by

$$\begin{aligned}\tilde{\eta}_1 &= \eta_b,\\ \tilde{\eta}_2 &= \eta_a + \tilde{p}_1,\\ \tilde{\eta}_t &\in (\tilde{p}_{t-1}, \tilde{p}_{t-2} + \tilde{p}_{t-1}) \quad \forall t \geq 3.\end{aligned}$$

That is, in the first time period, agents with $\eta \in [\eta_b, \infty)$ spend all their money, and the measure of such agents is $h_1$. In the second time period, agents with $\eta \in [\eta_a + \tilde{p}_1, \infty)$ spend all their money, and the measure of such agents is $h_0$. Note that by the above argument $\eta_a$ and $\eta_b$ do not locally depend on $(\tilde{p}_7, \tilde{p}_8, \dots)$ but depend only on $(\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_6)$; that is, a small change in $(\tilde{p}_7, \tilde{p}_8, \dots)$ does not affect $\eta_a$ and $\eta_b$. Clearly,

$$\eta_a = 2Mh_2 \text{ and } \eta_b = 2M(1 - h_1)$$

hold, where $h_2 = 1 - h_0 - h_1$, and $h_0$ and $h_1$ only depend on $(\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_6)$. Then, by the market clearing conditions, the sequence of prices is determined by

$$\begin{aligned}\tilde{p}_1 &= \frac{h_1}{1-h_1} M(2 - h_1),\\ \tilde{p}_2 &= \frac{h_0}{1-h_0} \left(M(1-h_1) + Mh_2 + \tilde{p}_1\right),\\ \tilde{p}_3 &= \frac{h_2}{1-h_2} \left(Mh_2 + \tilde{p}_1 + \tilde{p}_2\right), \quad \text{and}\\ \tilde{p}_t &= \frac{h_{2-i}}{1-h_{2-i}} (\tilde{p}_{t-2} + \tilde{p}_{t-1}) \quad \forall t = 3n + i \geq 4.\end{aligned}$$

By the above arguments, $(\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_6)$ is determined by the first 6 equations, and thus, $h_0$ and $h_1$ are determined. Now, let $P_n = [\tilde{p}_{3n+1}, \tilde{p}_{3n+2}, \tilde{p}_{3n+3}]'$ for $n \geq 1$. Then,

$$P_{n+1} = AP_n$$

holds, where

$$A = \begin{bmatrix} 0 & H_1 & H_1 \\ 0 & H_0 H_1 & H_0(1 + H_1) \\ 0 & H_1 H_2 (1 + H_0) & H_2(H_0 + H_1 + H_0 H_1) \end{bmatrix}$$

and $H_i = h_i/(1 - h_i)$. Let $\lambda_j$ be the eigenvalues of $A$ and $x_j$ be the corresponding eigenvectors. Then, we obtain

$$A = [x_1\ x_2\ x_3] \begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix} [x_1\ x_2\ x_3]^{-1}.$$

It is verified that the characteristic equation of $A$ is

$$-t^3 + (H_0H_1 + H_1H_2 + H_2H_0 + H_0H_1H_2)t^2 + H_0H_1H_2t = 0.$$

By solving the above, we obtain

$$\begin{aligned}
&\lambda_1 = 1, \quad \lambda_2 = 0, \lambda_3 = -H_0H_1H_2 > -1,\\
&x_1 = [(1+H_0)H_1, H_0(1+H_1), 1-H_0H_1]', \quad x_2 = [1,0,0]', \quad \text{and}\\
&x_3 = [1-H_1H_2, -H_0(1+H_1)H_2, H_0H_1H_2(1+H_2)].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{n\to\infty} A^n &= \begin{bmatrix}(1+H_0)H_1 & 1 & 1-H_1H_2\\ H_0(1+H_1) & 0 & -H_0(1+H_1)H_2\\ 1-H_0H_1 & 0 & H_0H_1H_2(1+H_2)\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}(1+H_0)H_1 & 1 & 1-H_1H_2\\ H_0(1+H_1) & 0 & -H_0(1+H_1)H_2\\ 1-H_0H_1 & 0 & H_0H_1H_2(1+H_2)\end{bmatrix}^{-1}\\
&= \begin{bmatrix}0 & \frac{(1+H_0)H_1^2(1+H_2)}{(1+H_1)(1+H_0H_1H_2)} & \frac{(1+H_0)H_1}{1+H_0H_1H_2}\\ 0 & \frac{H_0H_1(1+H_1)(1+H_2)}{(1+H_1)(1+H_0H_1H_2)} & \frac{H_0(1+H_1)}{1+H_0H_1H_2}\\ 0 & \frac{(1-H_0H_1)H_1(1+H_2)}{(1+H_1)(1+H_0H_1H_2)} & \frac{1-H_0H_1}{1+H_0H_1H_2}\end{bmatrix}
\end{aligned}$$

holds. Let the limit be $[p_1, p_2, p_0]' = \lim_{n\to\infty} P_n$. Then, it is verified that $(p_1, p_2, p_0)$ satisfies the condition that the total money holding is equal to $M$ when the money holdings distribution is $(h_0, h_1, h_2)$. In other words, the dynamic path converges to a 3-period cycle. Note that it does not converge in finite time.

### 2.5 Policy on Initial Distribution

In this subsection, we investigate a permanent effect of a redistribution policy. More precisely, by slightly changing the initial money holdings distribution, the limit cycle also slightly changes; that is, hysteresis occurs.

We focus on a dynamic path leading to a 2-period cycle. For a small $\epsilon > 0$, we consider the following initial distribution:

$$f_1(\eta) = \begin{cases} 0, & \eta \le 2M\epsilon,\\ \frac{1}{2M}, & \eta \in (2M\epsilon, 2M(1-\epsilon)],\\ \frac{1}{M}, & \eta \in [2M(1-\epsilon), 2M],\\ 0, & \eta > 2M.\end{cases}$$

Table 1: The case of $\delta = 0.4$, $a = 10$, and $c = 1$

| $\epsilon$ | 0 | 0.001 | 0.01 | 0.05 |
|---|---|---|---|---|
| $h_0$ | 0.55941 | 0.55930 | 0.55841 | 0.55514 |

Note that in the case of $\epsilon = 0$, $f_1$ coincides with that in Section 2.3. As in Section 2.3,

$$\tilde{\eta}_1 = 2M(h_1 + \epsilon) \tag{17}$$

is obtained from

$$h_1 = \int_{2M\epsilon}^{\tilde{\eta}_1} \frac{1}{2M} d\eta;$$

$h_0$ is determined by (2), (3), (9), (10), (13), (17); and the market clearing condition at time period 1 is expressed as follows:

$$\tilde{p}_1 = \frac{\int_{\tilde{\eta}_1}^{2M(1-\epsilon)} \frac{\eta}{2M} d\eta + \int_{2M(1-\epsilon)}^{2M} \frac{\eta}{M} d\eta}{\int_{2M\epsilon}^{\tilde{\eta}_1} \frac{1}{2M} d\eta}.$$

Table 1 illustrates how a change of $\epsilon$ induces a change of $h_0$. In other words, the policy that affects the initial money holdings distribution has a permanent effect.

## 2.6 Policy on Stationary Equilibrium

In this subsection, we consider an effect of a tax-subsidy scheme, which is analyzed in a random search environment in Kamiya and Shimizu [7], on the equilibrium with a cycle. More precisely, in the model, we consider that the government levies $s$ amount of money as a tax from $g$ measure of agents with money holdings more than $s$ and gives $s$ amount of money as a subsidy to $g$ measure of agents with money holdings less than $s$, where $g$ is a small positive number.

We show that the size of $s$ affects the existence of the equilibria with a 2-period cycle. We assume that (1) holds throughout this section. First, it is clear that a very small $s$ does not affect the trading pattern. Next, we set $s = M$. Then, using the notations in Section 2.1, the condition for the stationary cycle is

$$h_1 = (1-g)\{(1-g)h_1 + gh_0\} + g\{(1-g)h_0 + gh_1\}.$$

Then, we obtain a unique distribution $h_0 = h_1 = 1/2$. Clearly, this is not a cycle.

The logic is simple: If a redistribution policy is sufficiently large, then the transition of money holdings distributions becomes ergodic. It is well known that an ergodic stochastic process has a unique limit distribution.

# 3 A General Model

In this section, we consider the logic behind the existence of a continuum of equilibrium cycles and hysteresis in the previous section. More precisely, we show that if an equilibrium cycle exists, then under a regularity condition, there also exists a continuum of equilibrium cycles in a rather general framework, and that the limit cycle depends on the initial money holdings distribution. Thus, a policy that affects the initial money holdings distribution has a permanent effect.

We begin with the excess demand functions. Note that the cash-in-advance constraint does not appear explicitly, but it implicitly guarantees that money has a positive value. In other words, in any framework in which money has a positive value, the following argument applies.

## 3.1 Equilibria with Cycles

There is a continuum of agents whose measure is one. The number of goods is $L \geq 1$. There exists completely divisible and durable fiat money of which nominal stock is $M > 0$.

In each time period, a Walrasian market with a cash-in-advance constraint is open for each good. Agents have the same net demand function, denoted by $z(\eta, p_1, p_2, \dots) \in R^L$, where $\eta$ is the agent's money holding at the beginning of the time period and $p_t \in R^L_{++}, t = 1, 2, \dots,$ is a price vector in time period $t$. In other words, $z(\eta, p_1, p_2, \dots)$ is the first time period net consumption vector when the agent maximizes a utility stream under some conditions such as budget constraints and cash-in-advance constraints. Similarly, $z(\eta, p_t, p_{t+1}, \dots)$ is the $t$-th time period net consumption vector when the agent has $\eta$ amount of money at the beginning of time period $t$. We do not specify the domain of an infinite sequence $(p_1, p_2, \dots)$ since we focus on the case of cycles, and it can be considered as a finite dimensional case. We assume that for any given $(p_1, p_2, \dots)$, $z$ is a Borel measurable function of $\eta$.

The money holding distribution at the beginning of time period 1 is a Borel probability measure denoted by $g_1(\cdot)$ on $R_+$; that is, $g_1(A)$ is the measure of agents whose money holdings are in a Borel set $A \subset R_+$. The transition from time period $t$ money holding to time period $t+1$

money holding is defined as

$$Q_t(\eta_t, p_t, p_{t+1}, \dots) = \eta_t - p_t \cdot z(\eta_t, p_t, p_{t+1}, \dots), \quad t = 1, 2, \dots. \tag{18}$$

**Definition 1** For given $z$ and $g_1$, a pair of $(p_1, p_2, \dots)$ and money holdings distributions $(g_2, g_3, \dots)$ is said to be an equilibrium if the following conditions hold:

- $Q_t(\eta, p_t, p_{t+1}, \dots) \geq 0$ for all $\eta \in R_+$ and all $t = 1, 2, \dots$.
- for a Borel set $A \subset R_+$,

  $$g_{t+1}(A, p) = g_t(\{\eta | Q_t(\eta, p_t, p_{t+1}, \dots) \in A\}, p), \quad t = 1, 2, \dots$$

  is satisfied, where $g_1(\cdot, p) = g_1(\cdot)$.
- the market clearing conditions for goods and money hold: for $t = 1, 2, \dots$,

  $$\int z(\eta, p_t, p_{t+1}, \dots) g_t(d\eta, p) = 0 \quad \text{and}$$
  $$\int \eta g_t(d\eta, p) = M.$$

We now investigate equilibria in which a sequence of money holdings of each agent is in some finite set

$$\Omega = \{(\eta_1^1, \eta_2^1, \dots), \dots, (\eta_1^K, \eta_2^K, \dots)\}.$$

That is, each agent's sequence of money holdings is one of the elements in the above set; for each agent, there exists a $k \in \{1, \dots, K\}$ such that her sequence of money holdings is $(\eta_1^k, \eta_2^k, \dots)$. We investigate a $T$-cycle equilibrium, where $T \geq 2$. More precisely, $T$-cycle equilibria are defined as follows.

**Definition 2** An equilibrium is said to be a $T$-cycle equilibrium if there exists a set $\Omega = \{(\eta_1^1, \eta_2^1, \dots), \dots, (\eta_1^K, \eta_2^K, \dots)\}$ such that

- for any $\eta_1$ in the support of $g_1$, the sequence of money holdings derived from (18), denoted by $(\eta_1, \eta_2, \dots)$, is in $\Omega$,
- $\eta_t^k = \eta_{t+T}^k$ for $t = 1, 2, \dots$,
- $p_t = p_{t+T}$ for $t = 1, 2, \dots$, and

- $K \geq 2$.

Let $p = (p_1, \ldots, p_T)$. In parallel with this, we denote $z(\eta, p)$ and $(\eta_1^k, \ldots, \eta_T^k)$ instead of $(\eta_1^k, \eta_2^k, \ldots)$. We indicate the distribution of agents on the set of sequences of money holdings by $(h_1, \ldots, h_K)$, where $h_k$ is a measure of agents with $(\eta_1^k, \ldots, \eta_T^k)$. The following conditions must hold:

$$h_k \geq 0, \quad k = 1, \ldots, K,$$
$$\sum_{k=1}^{K} h_k = 1.$$

Let

$$(t) = t \bmod T.$$

Then, the conditions for a $T$-cycle equilibrium are as follows:

$$\eta_t^k - \eta_{(t+1)}^k = p_t \cdot z(\eta_t^k, p_t, \ldots, p_{(t+T-1)}), \quad t = 1, \ldots, T, \ k = 1, \ldots, K, \tag{19}$$
$$\sum_{k=1}^{K} h_k z(\eta_t^k, p_t, \ldots, p_{(t+T-1)}) = 0, \quad t = 1, \ldots, T, \tag{20}$$
$$\sum_{k=1}^{K} h_k \eta_t^k = M, \quad t = 1, \ldots, T. \tag{21}$$

Note that (20) is the condition for market clearing and (21) is the condition that the total money holding is equal to $M$.

From (21),

$$\sum_{k=1}^{K} h_k(\eta_t^k - \eta_{(t+1)}^k) = 0, \quad t = 1, \ldots, T$$

is obtained. Then, by (19),

$$\sum_{k=1}^{K} h_k p_t \cdot z(\eta_t^k, p_t, \ldots, p_{(t+T-1)}) = 0, \quad t = 1, \ldots, T \tag{22}$$

holds. Thus, if (19) and (21) hold, the market clearing conditions for goods $1, \ldots, L-1$ are sufficient for (20). Thus, the number of linearly independent equations in (19)–(21) is $TK + T(L-1) + T = TL + TK$. On the other hand, considering that $p$ and $\eta_t^k$ are endogenous variables, the system has $TL+TK$ variables. If $z$ is of class $C^1$ and the Jacobian matrix of the $TL+TK$ equations with respect to $p$ and $\eta_t^k$ is nonsingular, then by the implicit function theorem, $p$ and $\eta_t^k$ are locally expressed by $C^1$ functions of $(h_1, \dots, h_{K-1})$. Thus, there is a continuum of equilibrium cycles. In other words, (22) plays the role of Walras' law in each period; that is, there are $T$ Walras' laws in total. As is well known, in intertemporal models without money, only one intertemporal Walras' law is observed.

**Theorem 3** Let $(p^*, (\eta_t^{k*})_{t=1,\dots,T,k=1,\dots,K}, h_1^*, \dots, h_{K-1}^*, 1 - \sum_{k=1}^{K-1} h_k^*) \in R_{++}^{TL+TK+K}$ be a solution to (19)–(21). Suppose the system is of class $C^1$ and that the Jacobian matrix of the system at the solution with respect to $p$ and $\eta_t^k$ is of rank $TL+TK$. Then, there is an open set $A \subset R_{++}^{K-1}$ and a $C^1$ function $\xi : A \to R_{++}^{TL+TK}$ such that $(h_1^*, \dots, h_{K-1}^*) \in A$, $\xi(h_1^*, \dots, h_{K-1}^*) = (p^*, (\eta_t^{k*})_{t=1,\dots,T,k=1,\dots,K})$, and $(\xi(h_1, \dots, h_{K-1}), h_1, \dots, h_{K-1}, 1 - \sum_{k=1}^{K-1} h_k)$ form a solution to the system.

**Remark 3** Note that if money has a positive value, that is , $p^* \in R_{++}^{TL}$, then the above theorem holds. That is, this theorem applies to any framework, aside from economies with cash-in-advance constraints, wherein money has a positive value.

### 3.2 Policy and Dynamics

In this subsection, we generalize the arguments in Subsection 2.4. More precisely, in the general model, we show that the limit cycle varies according to the initial money holdings distribution. Thus, a policy that affects the initial money holdings distribution has a permanent effect.

Below, we assume that each agent's cash-in-advance constraint becomes binding only once in the first $T$ time periods. We introduce a function $f_s(\eta, p_s, p_{s+1}, \dots) \in \{0, 1\}$ such that $f_s(\eta, p_s, p_{s+1}, \dots) = 1$ implies that the agents with $(\eta, p_s, p_{s+1}, \dots)$ spend all their money $\eta$ in time period $s$ (the cash-in-advance constraint is binding), and $f_s(\eta, p_s, p_{s+1}, \dots) = 0$ implies that the agents with $(\eta, p_s, p_{s+1}, \dots)$ do not spend all their money in time period $s$.

We focus on the following policy: for all $t = 1, 2, \dots$, there exists an $\eta_t > 0$ such that

- an agent with $\eta \in [0, \bar{\eta}_t]$ sells her production good at time period $t$, and
- an agent with $\eta \in (\bar{\eta}_t, \infty)$ spends all her money at time period $t$.

**Assumption 1** • The above policy is optimal.

- For a given $\eta_1 \geq 0$, let the sequence of money holdings derived from (18) be $(\eta_1, \eta_2, \dots)$. Then, for each $\eta_1$ in the support of $g_1$, there exist only two time periods, $t \in \{1, \dots, T\}$ and $t + T$, in the first $2T$ time periods, such that $f_t(\eta_t, p_t, p_{t+1}, \dots) = f_{t+T}(\eta_{t+T}, p_{t+T}, p_{t+T+1}, \dots) = 1$.
- For all $s = 1, \dots, T$, the measure of agents with $\eta$ such that $f_s(\eta, p_s, p_{s+1}, \dots) = 1$ is positive.

All agents whose cash-in-advance constraints become binding in time period $t$ have the same amount of money from $t+1$ onwards since such agents spend all their money at $t$ and they take the same consumption behavior from $t+1$ onwards. Thus, in the cycle, a sequence of money holdings of an agent is in a finite set $\{(\eta^1_{T+1}, \eta^1_{T+2}, \dots), \dots, (\eta^K_{T+1}, \eta^K_{T+2}, \dots)\}$. Let the measure of agents on the set be denoted by $h = (h_1, \dots, h_K)$; that is, $h_i$ is the measure of agents whose sequence of money holdings is $\{(\eta^i_{T+1}, \eta^i_{T+2}, \dots)\}$. By Assumption 1, $K = T$ clearly holds.

Let $p = (p_1, p_2, \dots)$ be an equilibrium sequence of prices in the cycle. By Assumption 1, the measure of agents who spend all their money at time period 1 is equal to $h_T$:

$$h_T = \int_{\{\eta | f_1(\eta, p) = 1\}} g_1(d\eta, p). \tag{23}$$

Similarly, for $t = 2, \dots, T-1$,

$$h_t = \int_{\{\eta | f_{T+1-t}(\eta, p) = 1\}} g_{T+1-t}(d\eta, p). \tag{24}$$

We make the following assumption.

**Assumption 2** Suppose in a sequence of optimal money holdings $(\eta_1, \eta_2, \dots)$, $f_k(\eta_k, p_k, p_{k+1}, \dots) = f_{k+T}(\eta_{k+T}, p_{k+T}, p_{k+T+1}, \dots) = 1$ holds. Then, $f_k$ and $f_{k+T}$ do not depend on $(p_{k+T+1}, p_{k+T+2}, \dots)$.

The above assumption is typically satisfied in cash-in-advance models. Indeed, the maximization problem in Section 2 can be written as follows: for a given sequence $(p_1, p_2, \dots)$,

$$\begin{aligned} \max \sum_{t=1}^{k+T} & \delta^{t-1}(\chi_t U(q_t) - (1-\chi_t)c) + \delta^{k+T} V_{k+T+1}(\eta_{k+T+1}) \\ \text{s.t.} \quad & \chi_t p_t q_t + \eta_{t+1} = \eta_t + (1-\chi_t)p_t, \quad t = 1, 2, \dots, k+T, \\ & \chi_t p_t q_t \leq \eta_t, \eta_t \geq 0, \quad t = 1, 2, \dots, k+T, \\ & \eta_1 \geq 0 \text{ given,} \end{aligned}$$

where $V_{k+T+1}(\eta_{k+T+1})$ is the value at $k+T+1$. Suppose $\eta_{k+T+1}=0$ and that the constraint $\eta_{k+T+1}\geq 0$ is binding; that is, the corresponding Lagrange multiplier is positive. Then, a small change of $(p_{k+T+1}, p_{k+T+2}, \dots)$ does not affect the optimal choice in the above problem.

By Assumption 2, $(h_1, \dots, h_T)$ does not depend on $(p_{2T+1}, p_{2T+2}, \dots)$, but it depends on the initial money holdings distribution $g_1$. Then, from period $T+1$, there exist only $T$-types of money holdings: in time period $t \geq T$, agents with mass $h_i$ have the same money holding for $i = 1, \dots, T$. Suppose the economy converges to the $T$-cycle. Since $(h_1, \dots, h_T)$ depends on $g_1$, temporary shocks on $g_1$ have permanent effects; that is, the model exhibits hysteresis.

# 4 Discussion

In the previous sections, we have demonstrated that there is a continuum of equilibrium cycles and hysteresis occur in both specific and general models. However, the results in the general model depend on the existence of an equilibrium cycle and the regularity conditions. One might think that some special structures of the specific model guarantee the conditions. In this section, we discuss the special structures in the model in Section 2 and show that our results do not depend on such structures.

In the model, we have adopted a specific utility function and a specific production function: the utility function is linear, and an agent can produce just one unit of good. Even if we change these functions, the outline of the proof does not change greatly. Indeed, in Appendixes A.1 and A.2, we demonstrate that there is a continuum of equilibrium cycles in models with a nonlinear utility function and a convex production function.

It is well known that if the stochastic process is ergodic, then it has a unique stationary distribution. One might think that a similar argument applies; if the economy has a small stochastic shock, then the set of equilibrium cycles is not a continuum. In Appendix A.3, we refute this argument; that is, even if there is a small stochastic shock in utility, a continuum of equilibrium cycles exists.

It is possible that the participation constraint may be considered as the most restrictive assumption in the model. Therefore, in Appendix A.4, we present a model with durable goods, in which agents can be a seller and a buyer simultaneously, and we show that a continuum of equilibrium cycles exists even in this case.

## 5 Conclusion

In this paper, we have investigated dynamic general equilibrium models with cash-in-advance constraints, wherein each consumer's money holding varies over time. We first demonstrated that a continuum of equilibrium cycles exists in a specific model and that the model exhibits hysteresis; that is, the limit cycle, to which a dynamic path converges, depends on the initial distribution of money holdings. Then, we have shown that even in a rather general framework, the same result can be obtained; that is, if an equilibrium cycle exists, then there is a continuum of these cycles, and the limit cycle depends on the initial distribution of money holdings.

Furthermore, we have explored the logic behind the results. The market clearing condition for money implies Walras' law in each period, and thus, there are $T-1$ degrees of freedom in the equilibrium condition for a $T$-period cycle. Moreover, in the dynamic path to an equilibrium $T$-cycle, there are $T-1$ equations that determine $(h_1, \dots, h_{T-1})$.

## A Appendix

### A.1 Nonlinear Utility Functions

In the model in Section 2, we change the utility function as follows:

$$U(q) = \begin{cases} aq, & \text{if } q \leq \frac{2}{3}, \\ \frac{2}{3}a, & \text{if } q > \frac{2}{3}. \end{cases}$$

Then, we demonstrate that a continuum of equilibria with 3-period cycles exists. We first construct a stationary equilibrium with 3 states and then transform it into a continuum of equilibria with 3-period cycles.

First, we show that there exists a stationary equilibrium with the following features:

- A policy is characterized by $\overline{\eta}$ as follows:
  - an agent with $\eta \in [0, \overline{\eta}]$ sells her production good,
  - an agent with $\eta \in \left(\overline{\eta}, \frac{2}{3}p\right]$ spends all her money, and
  - an agent with $\eta \in \left(\frac{2}{3}p, \infty\right)$ spends only $\frac{2}{3}p$.
- A stationary money holdings distribution is discrete with 3 states:
  - the measure of agents without money is 1/3,

- the measure of agents with $\frac{1}{3}p$ is 1/3, and
- the measure of agents with $p$ is 1/3.

- $\overline{\eta} \in \left(0, \frac{1}{3}p\right)$.
- $V$ is continuous.

The condition that the total money holding is equal to $M$ and the market clearing condition are

$$M = \frac{1}{3}p + \frac{1}{3} \cdot \frac{1}{3}p \quad \text{and}$$
$$\frac{1}{3}p = \frac{1}{3} \cdot \frac{2}{3}p + \frac{1}{3} \cdot \frac{1}{3}p \quad \text{respectively.}$$

Then, the former condition implies

$$p = \frac{9}{4}M.$$

The latter condition is automatically satisfied.

The value function is defined as

$$V(\eta) = \begin{cases} -c + \delta V(\eta + p), & \text{if } \eta \in [0, \overline{\eta}], \\ \frac{a}{p}\eta + \delta V(0), & \text{if } \eta \in \left(\overline{\eta}, \frac{2}{3}p\right], \\ \frac{2a}{3} + \delta V\left(\eta - \frac{2}{3}p\right), & \text{if } \eta \in \left(\frac{2}{3}p, \infty\right). \end{cases}$$

We decompose $\eta \geq 0$ into an multiple of $\frac{2}{3}p$ and a residual; that is, $\eta = \frac{2np}{3} + \iota$ such that $\iota \in \left[0, \frac{2}{3}p\right)$. Then we obtain

$$V\left(\frac{2np}{3} + \iota\right) = \begin{cases} \frac{1}{1-\delta}\left\{\frac{2a}{3} - \delta^n\left[\frac{2a}{3} - \frac{1}{1+\delta+\delta^2}\left(-c + \frac{a\delta(2+\delta)}{3}\right) - \frac{a\delta^2(1-\delta)}{p}\iota\right]\right\}, & \text{if } \iota \leq \overline{\eta}, \\ \frac{1}{1-\delta}\left\{\frac{2a}{3} - \delta^n\left[\frac{2a}{3} - \frac{\delta}{1+\delta+\delta^2}\left(-c + \frac{a\delta(2+\delta)}{3}\right) - \frac{a(1-\delta)}{p}\iota\right]\right\}, & \text{if } \iota > \overline{\eta}. \end{cases}$$

The continuity of $V$ at $\overline{\eta}$ implies

$$\overline{\eta} = \frac{p}{a(1+\delta)(1-\delta^3)}\left(-c + \frac{a\delta(2+\delta)}{3}\right).$$

$\overline{\eta} \in \left(0, \frac{1}{3}p\right)$ follows from the following condition:

$$-3 - \delta + \delta^2 + 3\delta^3 + 3\delta^4 < \frac{3c}{a} < \delta(2+\delta). \tag{25}$$

Note that this interval is non-empty.

The optimality condition is stated as follows:

- $V(\eta) \geq 0$ for any $\eta$.
- $V(\eta) \geq \frac{a}{p}\eta' + \delta V(\eta - \eta')$ for any $\eta \in [0, \overline{\eta})$ and any $\eta' \in [0, \eta]$.
- $V(\overline{\eta}) \geq \frac{a}{p}\eta' + \delta V(\overline{\eta} - \eta')$ for any $\eta' \in [0, \overline{\eta})$.
- $V(\eta) \geq -c + \delta V(\eta + p)$ for any $\eta \in (\overline{\eta}, \infty)$.
- $V(\eta) \geq \frac{a}{p}\eta' + \delta V(\eta - \eta')$ for any $\eta \in \left(\overline{\eta}, \frac{2}{3}p\right]$ and any $\eta' \in [0, \eta)$.
- $V(\eta) \geq \frac{a}{p}\eta' + \delta V(\eta - \eta')$ for any $\eta \in \left(\frac{2}{3}p, \infty\right)$ and any $\eta' \in \left[0, \frac{2}{3}p\right)$.

Under (25), it is verified that this entire condition is satisfied with strict inequalities. Then, we have shown that the specified money holdings distribution and policy constitute an equilibrium under (25).

Next, we transform the stationary equilibrium into a continuum of 3-period equilibrium cycles. We denote the price at time period $3n + i$ ( $i = 0, 1, 2$) by $p_i$. For ease of exposition, let $p_3 = p_0$ and $p_4 = p_1$. We express the money holdings distribution by using notations $h_{ij}$ $(i, j = 0, 1, 2)$, defined as follows:

- $h_{i0}$: the measure of agents with no money at time period $3n + i$,
- $h_{i1}$: the measure of agents with $p_{i+1} - \frac{2}{3}p_{i+2}$ money at time period $3n + i$, and
- $h_{i2}$: the measure of agents with $p_{i+2}$ money at time period $3n + i$.

The condition for a stationary cycle is satisfied if there exists an $h = (h_0, h_1, h_2)$ such that

- $h_0 = h_{00} = h_{12} = h_{21}$,
- $h_1 = h_{01} = h_{10} = h_{22}$,
- $h_2 = h_{02} = h_{11} = h_{20}$, and
- $h_0 + h_1 + h_2 = 1$.

Note that if $h_0 = h_1 = h_2 = 1/3$, then they form the stationary equilibrium constructed above.

The condition that the total money holding should be equal to $M$ is

$$M = h_{01}\left(p_1 - \frac{2}{3}p_2\right) + h_{02}p_2,$$
$$M = h_{11}\left(p_2 - \frac{2}{3}p_0\right) + h_{12}p_0,$$
$$M = h_{21}\left(p_0 - \frac{2}{3}p_1\right) + h_{22}p_1.$$

If the condition for stationary cycle holds, this is equivalent to

$$p_0 = \frac{3h_1(9h_2 + 4h_0 - 6h_1)}{D}M,$$
$$p_1 = \frac{3h_2(9h_0 + 4h_1 - 6h_2)}{D}M,$$
$$p_2 = \frac{3h_0(9h_1 + 4h_2 - 6h_0)}{D}M,$$

where

$$D = 27h_0h_1h_2 + (3h_0 - 2h_2)(3h_2 - 2h_1)(3h_1 - 2h_0).$$

$D$ is not zero if all $h_{ij}$ are sufficiently close to $1/3$.

The condition for market clearing is expressed as follows:

$$M = h_{01}\left(p_1 - \frac{2}{3}p_2\right) + h_{02}p_2,$$
$$M = h_{11}\left(p_2 - \frac{2}{3}p_0\right) + h_{12}p_0,$$
$$M = h_{21}\left(p_0 - \frac{2}{3}p_1\right) + h_{22}p_1,$$
$$h_{00}p_0 = h_{01}\left(p_1 - \frac{2}{3}p_2\right) + h_{02}\frac{2}{3}p_0,$$
$$h_{10}p_1 = h_{11}\left(p_2 - \frac{2}{3}p_0\right) + h_{12}\frac{2}{3}p_1,$$
$$h_{20}p_2 = h_{21}\left(p_0 - \frac{2}{3}p_1\right) + h_{22}\frac{2}{3}p_2.$$

Then, it is easily verified that the conditions for a stationary cycle and a constant stock of fiat money imply the condition for market clearing; in other words, the latter is redundant.

Recall that the optimality conditions are satisfied with strict inequalities. Let

- $h_0 = \frac{1}{3} - \epsilon(1 + \epsilon)$,

- $h_1 = \frac{1}{3} + \epsilon$, and
- $h_2 = \frac{1}{3} + \epsilon^2$

for sufficiently small $\epsilon$. Then, by redefining $\overline{\eta}$ such that the value function is continuous at $\overline{\eta}$, it is verified that $(h_0, h_1, h_2)$ also constitutes an equilibrium under condition (25).

## A.2 Convex Cost Functions

We modify the model in Section 2 by assuming that an agent can produce any amount of her production good with a convex cost function. Let the cost function be $C(q) = bq^2$. We show that there is a continuum of equilibria with a 2-period cycle even in this environment.

First, we consider the following stationary equilibrium with 2 states:

- A policy is characterized by $\overline{\eta}$ and $Q(\eta)$ as follows:
  - an agent with $\eta \in [0, \overline{\eta}]$ sells $Q(\eta)$ amount of her production good, and
  - an agent with $\eta \in (\overline{\eta}, \infty)$ spends all her money.
- The support of a stationary money holdings distribution comprises 2 states; that is,
  - the measure of agents without money is 1/2, and
  - the measure of agents with $pQ(0)$ is 1/2.
- $\eta + pQ(\eta) > \overline{\eta}$ for any $\eta \in [0, \overline{\eta}]$.
- $\overline{\eta} > 0$.
- $V$ is continuous.

The value function is defined as

$$V(\eta) = \begin{cases} -b(Q(\eta))^2 + \delta V(\eta + pQ(\eta)), & \text{if } \eta \in [0, \overline{\eta}], \\ a\frac{\eta}{p} + \delta V(0), & \text{if } \eta \in (\overline{\eta}, \infty). \end{cases}$$

If $\eta + pQ(\eta) > \overline{\eta}$ holds, then

$$Q(\eta) = \arg\max_q \left\{ -bq^2 + \delta \left[ aq + a\frac{\eta}{p} + \delta V(0) \right] \right\}$$

must hold. It follows that

$$Q(\eta) = \frac{a\delta}{2b}.$$

Then we obtain the value function

$$V(\eta) = \begin{cases} a\delta\frac{\eta}{p} + \frac{a^2\delta^2}{4b(1-\delta^2)}, & \text{if } \eta \in [0, \overline{\eta}], \\ a\frac{\eta}{p} + \frac{a^2\delta^3}{4b(1-\delta^2)}, & \text{if } \eta \in (\overline{\eta}, \infty). \end{cases}$$

The continuity of $V$ at $\bar{\eta}$ implies

$$\overline{\eta} = \frac{a\delta^2}{4b(1-\delta^2)}p.$$

Then, the condition $\eta + pQ(\eta) > \bar{\eta}$ is satisfied for all $\eta \in [0, \bar{\eta}]$ if

$$2 - \delta - 2\delta^2 > 0. \tag{26}$$

Given $V$ as specified above, we check the optimality condition. First, we consider the conditions that producing $Q(\eta)$ is optimal. We denote the value of one shot deviation by choosing to be a seller and selling $q$ amount of goods by $\tilde{V}(\eta, q)$; that is,

$$\tilde{V}(\eta, q) = \begin{cases} -bq^2 + a\delta^2 q + a\delta^2\frac{\eta}{p} + \frac{a^2\delta^3}{4b(1-\delta^2)}, & \text{if } q \leq \frac{\bar{\eta}-\eta}{p}, \\ -bq^2 + a\delta q + a\delta\frac{\eta}{p} + \frac{a^2\delta^4}{4b(1-\delta^2)}, & \text{if } q > \frac{\bar{\eta}-\eta}{p}. \end{cases}$$

If $q$ is in $(\frac{\bar{\eta}-\eta}{p}, \infty)$, then $\tilde{V}(\eta, q)$ attains the maximum value $V(\eta)$ at $q = \frac{a\delta}{2b} = Q(\eta)$ since

$$\frac{\partial \tilde{V}}{\partial q} = -2bq + a\delta,$$

and $\eta + \frac{a\delta}{2b}p > \bar{\eta}$ hold. Then, it suffices to find a condition for

$$\sup_{q \leq \frac{\bar{\eta}-\eta}{p}} \tilde{V}(\eta, q) \leq V(\eta)$$

for all $\eta \in [0, \bar{\eta}]$. For $q \in [0, \frac{\bar{\eta}-\eta}{p}]$,

$$\frac{\partial \tilde{V}}{\partial q} = -2bq + a\delta^2$$

holds. Thus, the condition is that for all $\eta \in [0, \bar{\eta}]$,

$$\left[-2bq + a\delta^2\right]_{q=\frac{\bar{\eta}-\eta}{p}} \geq 0$$

holds or

$$\left[-2bq + a\delta^2\right]_{q=\frac{\bar{\eta}-\eta}{p}} < 0 \text{ and}$$
$$V(\eta) \geq \tilde{V}\left(\eta, \frac{a\delta^2}{2b}\right)$$

hold. We can show that the above is equivalent to

$$1 - 2\delta^2 \geq 0.$$

Moreover, this implies (26). Hereafter, we assume that this condition holds with strict inequality; that is,

$$\delta < \frac{1}{\sqrt{2}}. \tag{27}$$

In addition, it is easily verified that (27) implies that there is no incentive for

- an agent with $\eta \in [0, \bar{\eta}]$ to be a buyer,
- an agent with $\eta \in (\bar{\eta}, \infty)$ to be a seller, and
- an agent with $\eta \in (\bar{\eta}, \infty)$ to be a buyer but not to spend all her money.

The condition that the total money holding is equal to $M$ and the market clearing condition are

$$M = \frac{1}{2} \cdot \frac{a\delta}{2b} p \quad \text{and}$$
$$\frac{1}{2} \cdot \frac{a\delta}{2b} p = \frac{1}{2} \cdot \frac{a\delta}{2b} p \quad \text{respectively.}$$

Then, the former condition implies

$$p = \frac{4b}{a\delta} M.$$

The latter condition is automatically satisfied.

We perturb the stationary equilibrium as follows:

$$h_0 = \frac{1}{2} - \epsilon \quad \text{and}$$
$$h_1 = \frac{1}{2} + \epsilon.$$

Then, the discounted sum of the utility stream in the $(2n+i)$-th time period is

$$W_i = \frac{a^2\delta}{4b(1-\delta^2)} \left[ \delta h_i \left( \frac{h_{i+1}}{h_i} \right)^{2/3} + (2-\delta^2) h_{i+1} \left( \frac{h_i}{h_{i+1}} \right)^{2/3} \right].$$

Then, their weighted average of them is

$$\frac{1}{2}W_0 + \frac{1}{2}W_1 = \frac{a^2\delta(2-\delta-\delta^2)}{8b(1-\delta^2)} \left[ \delta h_1 \left( \frac{h_0}{h_1} \right)^{2/3} + h_0 \left( \frac{h_0}{h_1} \right)^{2/3} \right].$$

This is unimodal with the peak at $\epsilon = 0$. This implies that the indeterminacy is real.

### A.3 Small Stochastic Shocks

Now, we introduce a preference shock into the model in Section 2; that is,

$$U(q) = \theta q,$$

where $\theta$ is uniformly distributed on $[a-\Theta, a+\Theta]$ for some $\Theta \in (0, a)$.

We first construct a stationary equilibrium with 2 states. The value function satisfies

$$V(\eta, \theta) = \begin{cases} -c + \delta E_{\tilde{\theta}} V(\eta + p, \tilde{\theta}), & \text{if } \eta \in [0, \bar{\eta}(\theta)], \\ \frac{\theta}{p}\eta + \delta E_{\tilde{\theta}} V(0, \tilde{\theta}), & \text{if } \eta \in (\bar{\eta}(\theta), \infty). \end{cases}$$

Then, we obtain

$$V(\eta, \theta) = \begin{cases} \frac{a\delta}{p}\eta + \frac{a\delta - c}{1-\delta^2}, & \text{if } \eta \in [0, \bar{\eta}(\theta)], \\ \frac{\theta}{p}\eta + \frac{\delta(a\delta - c)}{1-\delta^2}, & \text{if } \eta \in (\bar{\eta}(\theta), \infty). \end{cases}$$

If

$$\Theta < a(1-\delta) \tag{28}$$

holds, then $\bar{\eta}(\theta)$ is determined as

$$\bar{\eta}(\theta) = \frac{a\delta - c}{(1+\delta)(\theta - a\delta)} p.$$

The condition $\forall\theta$, $\bar{\eta}(\theta) \in (0, p)$ is equivalent to

$$\frac{c}{a} < \delta \quad \text{and} \tag{29}$$

$$\Theta < \frac{a(1-\delta-\delta^2) + c}{1+\delta}. \tag{30}$$

By (28)–(30), the equilibrium condition can be expressed by (30) and

$$-1 + \delta + \delta^2 < \frac{c}{a} < \delta. \tag{31}$$

In other words, if the size of the noise is sufficiently small, there exists a stationary equilibrium with 2 states. Thus, it can be transformed into a continuum of 2-period equilibrium cycles, as in the previous sections.

**Remark 4** It is worthwhile to compare the above result with that of Lucas [10], who shows that a stochastic version of a cash-in-advance model has a unique limit distribution. This result is obtained from Assumption (2.2) in Lucas [10]. In our environment, this assumption is equivalent to

$$\lim_{\theta \to \bar{\theta}} \frac{\partial U(q, \theta)}{\partial q} = \infty,$$

where $U$ is the instantaneous utility and $\bar{\theta}$ is the upper bound of realization of a stochastic element $\theta$. It implies that, for any agent, there always exists a small probability that she consumes some amount of goods *irrespective of the amount of fiat money she has.* This renders an economy-wide money holdings transition ergodic.

## A.4 Durable Goods

There exist a perishable good $x$ and a durable good $y$; we assume that both goods are divisible. The durable good can be consumed for two time periods; that is, the depreciation rates in the second and third time periods are zero and one respectively. We assume that if an agent has a positive amount of the durable good, she cannot buy the good.[6] An agent obtains utility $ay$ when she consumes $y$ amount of the durable good, where $a > 0$. Each agent is endowed with one unit of the perishable good, and the durable good is produced by production function $y = \alpha x$, where $\alpha > 0$. Moreover, we do not impose the participation constraint; an agent can buy and sell simultaneously.

We show that a continuum of equilibria with 2-period cycles exists. We first construct a stationary equilibrium and then transform it into a continuum of equilibria with 2-period cycles.

First, we consider the following candidate for stationary equilibrium. Let the prices of $x$ and $y$ be $p$ and $q$ respectively. Then, by the zero profit condition, $q = \frac{p}{\alpha}$.

[6] Even if we assume that she can buy the durable good, we obtain similar results.

- The policy is characterized by $\overline{\eta}$ as follows:
  - an agent with $\eta \in [0, \overline{\eta}]$ only sells one unit of $x$ and does not buy $y$;
  - an agent with $\eta \in (\overline{\eta}, \infty)$ and without durable good sells one unit of $x$ and spends all her money for $y$ good; and
  - an agent with positive $y$ only sells one unit of $x$.
- A stationary money holdings distribution is discrete with 2 states:
  - the measure of agents with $p$ money and with $2\alpha$ amount of the durable good is 1/2, and
  - the measure of agents with $2p$ money and without durable good is 1/2.
- $\overline{\eta}$ is in $\overline{\eta} \in (0, p)$.
- $V$ is continuous.

The condition that the total money holding is equal to $M$ and the market clearing condition are

$$M = \frac{1}{2}(p + 2p) \quad \text{and}$$
$$\alpha = \frac{2p}{q}\frac{1}{2} \quad \text{respectively.}$$

Then, the former condition implies

$$p = \frac{2}{3}M.$$

The latter condition is automatically satisfied.

The value function satisfies

$$V(p, 2\alpha) = 2a\alpha + \delta V(2p, 0) \quad \text{and}$$
$$V(2p, 0) = 2a\alpha + \delta V(p, 2\alpha).$$

Then, we obtain

$$V(p, 2\alpha) = V(2p, 0) = \frac{2a\alpha}{1 - \delta}.$$

$\bar{\eta}$ satisfies

$$a\frac{\bar{\eta}}{q} + a\delta\frac{\bar{\eta}}{q} + \delta^2 V(2p, 0) = a\delta\frac{\bar{\eta}+p}{q} + a\delta^2\frac{\bar{\eta}+p}{q} + \delta^3 V(2p, 0).$$

Thus, we obtain

$$\bar{\eta} = \frac{\delta p}{1+\delta}.$$

Then, $\bar{\eta} < p$ is automatically satisfied.

By the policy, for $\eta > \bar{\eta}$,

$$V(\eta, 0) = \frac{a\eta}{q} + \delta\frac{a\eta}{q} + \delta^2 V(2p, 0)$$

holds. Thus, for $\eta > \bar{\eta}$,

$$V(\eta, 0) = \frac{a\alpha\eta}{p} + \delta\frac{a\alpha\eta}{p} + \delta^2\frac{2a\alpha}{1-\delta}.$$

Thus, for $\eta \in [0, \bar{\eta}]$,

$$V(\eta, 0) = \delta V(\eta + p, 0) = \frac{a\alpha(\eta+p)}{p} + \delta\frac{a\alpha(\eta+p)}{p} + \delta^2\frac{2a\alpha}{1-\delta}.$$

Similarly, for $y > 0$,

$$V(\eta, y) = ay + \delta V(\eta + p, 0) = ay + \delta\left(\frac{a\alpha(\eta+p)}{p} + \delta\frac{a\alpha(\eta+p)}{p} + \delta^2\frac{2a\alpha}{1-\delta}\right).$$

Next, we transform the stationary equilibrium with 2 states into a continuum of equilibria with 2-period cycles by perturbing the money holdings distribution. Let the prices at time $2n$ and $2n+1$ be $p_0$ and $p_1$ respectively. Let $h_0$ be the measure of agents with $p_1$ in even periods and $p_0 + p_1$ in odd periods, and let $h_1$ be the measure of agents with $p_0$ in odd periods and $p_0 + p_1$ in even periods. Obviously, $h_0 + h_1 = 1$ holds.

The condition that the total money holding is equal to $M$ is

$$M = p_1 h_0 + (p_0 + p_1)h_1, \tag{32}$$

$$M = p_0 h_1 + (p_0 + p_1)h_0. \tag{33}$$

Subtracting (33) from (32), we obtain

$$p_0 h_0 = p_1 h_1. \tag{34}$$

The conditions for market clearing are

$$p_0 = (p_0 + p_1)h_1 \quad \text{and}$$
$$p_1 = (p_0 + p_1)h_0.$$

The above conditions are clearly obtained from (34). Thus, the conditions for a stationary cycle and a constant stock of fiat money imply the condition for market clearing; in other words, the latter is redundant.

Let

$$h_0 = \frac{1}{2} - \epsilon,$$
$$h_1 = \frac{1}{2} + \epsilon$$

for sufficiently small $\epsilon$. Then, by redefining $\overline{\eta}$ such that the value function is continuous at $\overline{\eta}$, it is verified that they also constitute an equilibrium.

# References


[1] Richard Baldwin. Hysteresis in imported prices: The beachhead effect. *American Economic Review*, 78(4):773–785, 1988.

[2] Olivier J. Blanchard and Lawrence H. Summers. Hysteresis in unemployment. *European Economic Review*, 31(1/2):288–295, 1987.

[3] W. Franz. Hysteresis in economic relationships: An overview. *Empirical Economics*, 15(2):109–125, 1990.

[4] Edward J. Green and Ruilin Zhou. A rudimentary random-matching model with divisible money and prices. *Journal of Economic Theory*, 81(2):252–271, 1998.

[5] Edward J. Green and Ruilin Zhou. Dynamic monetary equilibrium in a random matching economy. *Econometrica*, 70(3):929–969, 2002.

[6] Kazuya Kamiya and Takashi Shimizu. Real indeterminacy of stationary equilibria in matching models with divisible money. *Journal of Mathematical Economics*, 42(4):594–617, 2006.

[7] Kazuya Kamiya and Takashi Shimizu. On the role of the tax-subsidy scheme in money search models. *International Economic Review*, 48(2):575–606, 2007.

[8] Kazuya Kamiya and Takashi Shimizu. Dynamic auction markets with fiat money. mimeo., 2008.

[9] Nobuhiro Kiyotaki and Randall Wright. On money as a medium of exchange. *Journal of Political Economy*, 97(4):927–954, 1989.

[10] Robert E. Lucas, Jr. Equilibrium in a pure currency economy. *Economic Inquiry*, 18(2):203–220, 1980.

[11] Akihiko Matsui and Takashi Shimizu. A theory of money and marketplaces. *International Economic Review*, 46(1):35–59, 2005.

[12] Tapan Mitra and Kazuo Nishimura. Discounting and long-run behavior: Global bifurcation analysis of a family of dynamical systems. *Journal of Economic Theory*, 96(1/2):256–293, 2001.

[13] Ruilin Zhou. Individual and aggregate real balances in a random-matching model. *International Economic Review*, 40(4):1009–1038, 1999.