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A General Computation Scheme for a High-Order Asymptotic Expansion Method *

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Abstract

This paper presents a new computational scheme for an asymptotic expansion method of an arbitrary order.

The asymptotic expansion method in finance initiated by Kunitomo and Takahashi [9], Yoshida [34] and Takahashi [20], [21] is a widely applicable methodology for an analytic approximation of expectation of a certain functional of diffusion processes. Hence, not only academic researchers but also many practitioners have used the methodology for a variety of financial issues such as pricing or hedging complex derivatives under high-dimensional underlying stochastic environments. In practical applications of the expansion, a crucial step is calculation of conditional expectations for a certain kind of Wiener functionals. [20], [21] and Takahashi and Takehara [23] provided explicit formulas for those conditional expectations necessary for the asymptotic expansion up to the third order.

This paper presents the new method for computing an arbitrary-order expansion in a general diffusion-type stochastic environment, which is powerful especially for high-order expansions: We develops a new calculation algorithm for computing coefficients of the expansion through solving a system of ordinary differential equations that is equivalent to computing the conditional expectations directly. To demonstrate its effectiveness, the paper gives numerical examples of the approximation for a λ-SABR model up to the fifth order and a cross-currency Libor market model with a general stochastic volatility model of the spot foreign exchange rate up to the fourth order.

Keywords: Asymptotic Expansion, Malliavin Calculus, Approximation Formula, Stochastic Volatility, λ-SABR Model, Libor Market Model, Currency

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1 Introduction

This paper presents a new scheme for computation in the method so-called "an asymptotic expansion approach" by developing a new calculation algorithm.

To our best knowledge, the 'asymptotic expansion method' was firstly introduced to a financial literature by [9] and [20] with an application to the evaluation of an average option that is a popular derivative in commodity markets. They derive the approximation formulas for the average option by the asymptotic expansion method based on log-normal approximations of a distribution of an average price when an underlying asset price follows a geometric Brownian motion. [34] applies a formula derived through the asymptotic expansion of certain statistical estimators for small diffusion processes to approximating average option prices. Thereafter, the asymptotic expansion have been applied to a broad class of problems in finance: See [21], [22], Kunitomo and Takahashi [10], [11], Matsuoka, Takahashi and Uchida [14], Takahashi and Yoshida [30], [31], Muroi [15], and Takahashi and Takehara [23], [24], [25]. It is notable that the method has flexible applicability to a broad class of diffusion-type stochastic settings in a unified way, and mathematical justification by Watanabe theory(Watanabe [32], Yoshida [33]) in Malliavin calculus.

There are also other various approaches for approximation of solutions to pricing PDEs, Greeks and heat kernels through certain asymptotic expansions: for instance, there are recent works such as Fouque, Papanicolaou and Sircar [4], [5], Hagan, Kumar, Lesniewski and Woodward [7], Henry-Labordere [12], [13], Siopacha and Teichmann [19], Ben Arous and Laurence [3] and Gatheral, Hsu, Laurence, Ouyang and Wang [6].

Recently, not only academic researchers but also many practitioners such as Antonov and Misirpashaev [1] or Andersen and Hutchings [2] have used the asymptotic expansion method based on Watanabe theory in or combined with their techniques for a variety of financial issues. e.g. pricing or hedging complex derivatives under high-dimensional underlying stochastic environments. These methods fully or partially rely on the framework developed by [9], [20], [21] in a financial literature.

In theory, this method provides us the expansion, which has a proper meaning in the limit of some ideal situations such as cases where these processes would be deterministic, of underlying stochastic processes (for the detail see [32], [33] or [11]).

In practice, however, we are often interested in cases far from those situations, where the underlying processes are highly volatile as seen in recent financial markets especially after the crisis on 2008. Then from view points of accuracy or stability of the techniques in practical uses, it is desirable to investigate behaviors of its estimators especially with expansion up to high orders in such environments.

In the existing application of the asymptotic expansion based on Watanabe theory, they calculated certain conditional expectations which appear in their expansions and which play key roles in computation, by formulas up to the third order given explicitly in [20], [21] and [23]. In many applications, these formulas give sufficiently accurate approximation, but in some cases, for example in cases with long maturities or/and with highly volatile underlying variables, the approximations up to the third order may not provide satisfactory accuracies. Thus, formulas for higher-order computations are desirable. But to our knowledge, the asymptotic expansion formulas higher than the third order in a general setting have not been given yet.

This paper provides a new scheme for computing unconditional expectations which is completely equivalent to direct calculation of the conditional expectations. This enables us to derive the high-order approximation formulas in an automatic manner. Consequently, our approximation generally shows sufficient accuracy in computation of high-order expansions, which is confirmed by numerical experiments. In those experiments, comparing our method to another existing one, we also see its advantageous applicability in financial practice.

Recently, Takehara, Takahashi and Toda [27] introduced a computational scheme for the conditional expectations as well as an equivalent scheme for the unconditional ones in the same manner to this work, and gave some formulas for the fourth order expansion. That paper and Takehara, Toda and Takahashi [28] applied those to option pricing in the λ-SABR model and in a cross-currency setting with long maturities. However, in those papers key ideas of the methods were heuristically shown only in a very simple setting while their applications were out of that framework. Thus, in this sense, this paper can be considered to give justification and generalization of those works to a much broader class of models.

Organization of this paper is as follows: After a brief explanation of the asymptotic expansion in Section 2, Section 3 introduces our new computation algorithm and derives the formulas for the asymptotic expansion. Section 4 applies our algorithm described in the previous section to the concrete financial models, and confirms effectiveness of the high-order expansions by numerical examples in the (λ-)SABR model and a cross-currency Libor market model with a general stochastic volatility model of the spot foreign exchange rate. Finally, Appendix shows proofs of Lemma 1 and Theorem 1 in Section 3, which are omitted in the main text. Due to the limitation of space, some related results and some of concrete data of the experiments in Section 4 are omitted and left in our online working paper CARF-F-149 [26] (http://www.carf.e.u-tokyo.ac.jp/pdf/workingpaper/fseries/154.pdf).

2 An Asymptotic Expansion in a General Diffusion Setting

This section briefly describes an asymptotic expansion method in a general diffusion setting.

Let (W, P) be a r-dimensional Wiener space. We consider a d-dimensional diffusion process $X_t^{(\epsilon)} = (X_t^{(\epsilon),1}, \cdots, X_t^{(\epsilon),d})'$ which is the solution to the following stochastic differential equation:

$$\begin{aligned} dX_t^{(\epsilon),j} &= V_0^j(X_t^{(\epsilon)}, \epsilon)dt + \epsilon V^j(X_t^{(\epsilon)})dW_t \quad (j = 1, \cdots, d) \\ X_0^{(\epsilon)} &= x_0 \in \mathbf{R}^d \end{aligned} \tag{1}$$

where $W = (W^1, \cdots, W^r)'$ is a r-dimensional standard Wiener process, and $\epsilon \in (0, 1]$ is a known parameter.

Suppose that $V_0 = (V_0^1, \cdots, V_0^d)' : \mathbf{R}^d \times (0, 1] \mapsto \mathbf{R}^d$ and $V = (V^1, \cdots, V^d)$: $\mathbf{R}^d \mapsto \mathbf{R}^d \otimes \mathbf{R}^r$ satisfy some regularity conditions(for example, V_0 and V are smooth functions with bounded derivatives of all orders).

Next, let a function $g : \mathbf{R}^d \mapsto \mathbf{R}$ be smooth and all of its derivatives have polynomial growth. Then, a smooth Wiener functional $g(X_T^{(\epsilon)})$ has its asymptotic expansion:

$$g(X_T^{(\epsilon)}) \sim g_{0T} + \epsilon g_{1T} + \epsilon^2 g_{2T} + \cdots$$

in $\mathbf{D}^\infty$ as $\epsilon \downarrow 0$ where $g_{0T}, g_{1T}, g_{2T}, \cdots \in \mathbf{D}^\infty$. For any $k \in \mathbf{N}$, $q \in (1,\infty)$ and $s > 0$, this expansion means that

$$\frac{1}{\epsilon^k}\|g(X_T^{(\epsilon)}) - (g_{0T} + \epsilon g_{1T} + \cdots + \epsilon^{k-1} g_{k-1,T})\|_{q,s} = O(1) \ (as\ \epsilon \downarrow 0),$$

where $\|G\|_{q,s}$ represents the sum of L^q -norms of Malliavin derivatives of a Wiener functional G up to the s-th order. Further, a Banach space $\mathbf{D}_{q,s} = \mathbf{D}_{q,s}(\mathbf{R})$ can be regarded as the totality of random variables bounded with respect to (q,s)-norm $\|\cdot\|_{q,s}$, and $\mathbf{D}^\infty = \cap_{s>0}\cap_{1<q<\infty}\mathbf{D}_{q,s}$.

Coefficients $g_{nT} \in \mathbf{D}^\infty (n = 0, 1, \cdots)$ in the expansion can be obtained by Taylor's formula and represented based on multiple Wiener-Itô integrals. See chapter V of Ikeda and Watanabe [8] for the detail.

In financial applications, for example $X^{(\epsilon)}$ consists of two stocks, $X^{(\epsilon)} = (S_1^{(\epsilon)}, S_2^{(\epsilon)})'$ and $g(\cdot)$ is their average $g(x) = \frac{x_1+x_2}{2}$ for $x = (x_1, x_2)'$: As another example, we can set $X^{(\epsilon)}$ is a vector of N discrete Libor forward rates, $X^{(\epsilon)} = (L_1^{(\epsilon)}, \cdots, L_N^{(\epsilon)})'$, and

$$g(X_T^{(\epsilon)}) = SR_T^{(\epsilon)} = \frac{1 - \prod_{j=1}^N \frac{1}{1+\tau L_{jT}^{(\epsilon)}}}{\tau \sum_{i=1}^N \prod_{j=1}^i \frac{1}{1+\tau L_{jT}^{(\epsilon)}}},$$

that is a swap rate with maturity $T_N = T + N\tau$.

Let $A_{kt} = \frac{1}{k!}\frac{\partial^k X_t^{(\epsilon)}}{\partial \epsilon^k}|_{\epsilon=0}$ and A_{kt}^j, $j = 1, \cdots, d$ denote the j-th elements of A_{kt}. In particular, A_{1t} is represented by

$$A_{1t} = \int_0^t Y_t Y_u^{-1}\left(\partial_\epsilon V_0(X_u^{(0)}, 0)du + V(X_u^{(0)})dW_u\right) \tag{2}$$

where Y denotes the solution to the ordinary differential equation:

$$dY_t = \partial V_0(X_t^{(0)}, 0)Y_t dt;\ Y_0 = I_d.$$

Here, ∂V_0 denotes the $d \times d$ matrix whose (j,k)-element is $\partial_k V_0^j = \frac{\partial V_0^j(x,\epsilon)}{\partial x_k}$, V_0^j is the j-th element of V_0, and I_d denotes the $d \times d$ identity matrix.

For $k \geq 2$, A_{kt}^j, $j = 1, \cdots, d$ is recursively determined by the following equation:

$$\begin{aligned} A_{kt}^j &= \frac{1}{k!}\int_0^t \partial_\epsilon^k V_0^j(X^{(0)}, 0)du \\ &\quad + \sum_{l=1}^k \sum_{\vec{l}_\beta, \vec{d}_\beta}^{(l)} \frac{1}{(k-l)!}\frac{1}{\beta!}\int_0^t \left(\prod_{j=1}^\beta A_{l_j u}^{d_j}\right)\partial_{\vec{d}_\beta}^\beta \partial_\epsilon^{k-l} V_0^j(X_u^{(0)}, 0)du \\ &\quad + \sum_{\vec{l}_\beta, \vec{d}_\beta}^{(k-1)} \frac{1}{\beta!}\int_0^t \left(\prod_{j=1}^\beta A_{l_j u}^{d_j}\right)\partial_{\vec{d}_\beta}^\beta V^j(X_u^{(0)})dW_u \end{aligned} \tag{3}$$

where $\partial_\epsilon^l = \frac{\partial^l}{\partial \epsilon^l}$, $\partial_{\vec{d}_\beta}^\beta = \frac{\partial^\beta}{\partial x_{d_1}\cdots\partial x_{d_\beta}}$,

$$L_{n,\beta} = \left\{\vec{l}_\beta = (l_1, \cdots, l_\beta); \sum_{j=1}^\beta l_j = n, l_j \geq 1, j = 1, \cdots, \beta\right\} \tag{4}$$

and

$$\sum_{\vec{l}_\beta,\vec{d}_\beta}^{(n)} = \sum_{\beta=1}^{n} \sum_{\vec{l}_\beta \in L_{n,\beta}} \sum_{\vec{d}_\beta \in \{1,\cdots,d\}^\beta}$$

for $n \geq 1$, and

$$\sum_{\vec{l}_\beta,\vec{d}_\beta}^{(0)} = \sum_{\beta=0} \sum_{\vec{l}_0=(\emptyset)} \sum_{\vec{d}_0=(\emptyset)} .$$

Then, g_{0T} and g_{1T} can be written as

$$\begin{aligned} g_{0T} &= g(X_T^{(0)}), \\ g_{1T} &= \sum_{j=1}^{d} \partial_j g(X_T^{(0)}) A_{1T}^j. \end{aligned}$$

For $n \geq 2$, g_{nT} is expressed as follows:

$$g_{nT} = \sum_{\vec{l}_\beta,\vec{d}_\beta}^{(n)} \frac{1}{\beta!} \partial_{\vec{d}_\beta}^{\beta} g(X_T^{(0)}) A_{l_1 T}^{d_1} \cdots A_{l_\beta T}^{d_\beta}. \tag{5}$$

Next, let normalize $g(X_T^{(\epsilon)})$ to

$$G^{(\epsilon)} = \frac{g(X_T^{(\epsilon)}) - g_{0T}}{\epsilon}$$

for $\epsilon \in (0,1]$. Then, we have

$$G^{(\epsilon)} \sim g_{1T} + \epsilon g_{2T} + \cdots$$

in $\mathbf{D}^\infty$. Moreover, let

$$\hat{V}(x,t) = (\partial g(x))' [Y_T Y_t^{-1} V(x)]$$

and make the following assumption:

$$\text{(Assumption 1)} \quad \Sigma_T = \int_0^T \hat{V}(X_t^{(0)},t) \hat{V}(X_t^{(0)},t)' dt > 0.$$

Note that g_{1T} follows a normal distribution with variance Σ_T; the density function of g_{1T} denoted by $f_{g_{1T}}(x)$ is given by

$$f_{g_{1T}}(x) = \frac{1}{\sqrt{2\pi\Sigma_T}} \exp\left(-\frac{(x-C)^2}{2\Sigma_T} \right)$$

where

$$C := \left(\partial g(X_T^{(0)}) \right)' \int_0^T Y_T Y_t^{-1} \partial_\epsilon V_0(X_t^{(0)}, 0) dt. \tag{6}$$

Hence, Assumption 1 means that the distribution of g_{1T} does not degenerate. In application, it is easy to check this condition in most cases. Hereafter, Let $\mathcal{S}$ be the real Schwartz space of rapidly decreasing $\mathbf{C}^\infty$-functions on $\mathbf{R}$ and $\mathcal{S}'$ be its dual space that is the space of the Schwartz tempered distributions. Next, take $\Phi \in \mathcal{S}'$. Then, by Watanabe theory(Watanabe [32], Yoshida [33]) a generalized Wiener functional $\Phi(G^{(\epsilon)})$ has an asymptotic expansion in $\mathbf{D}^{-\infty}$ as $\epsilon \downarrow 0$ where

$\mathbf{D}^{-\infty}$ denotes the set of generalized Wiener functionals. See chapter V of Ikeda and Watanabe [8] for the detail. Hence, the expectation of $\Phi(G^{(\epsilon)})$ is expanded around $\epsilon = 0$ as follows: For $N = 0, 1, 2, \cdots$,

$$\begin{aligned}
\mathbf{E}[\Phi(G^{(\epsilon)})] &= \sum_{n=0}^{N} \epsilon^n \sum_{\vec{k}_\delta}^{(n)} \frac{1}{\delta!} \mathbf{E}\left[\Phi^{(\delta)}(g_{1T}) \left(\prod_{j=1}^{\delta} g_{(k_j+1)T}\right)\right] + o(\epsilon^N) \\
&= \sum_{n=0}^{N} \epsilon^n \sum_{\vec{k}_\delta}^{(n)} \frac{1}{\delta!} \mathbf{E}\left[\Phi^{(\delta)}(g_{1T}) X^{\vec{k}_\delta}\right] + o(\epsilon^N) \\
&= \sum_{n=0}^{N} \epsilon^n \sum_{\vec{k}_\delta}^{(n)} \frac{1}{\delta!} \int_{\mathbf{R}} \Phi^{(\delta)}(x) \mathbf{E}[X^{\vec{k}_\delta} | g_{1T} = x] f_{g_{1T}}(x) dx + o(\epsilon^N) \\
&= \sum_{n=0}^{N} \epsilon^n \sum_{\vec{k}_\delta}^{(n)} \frac{1}{\delta!} \int_{\mathbf{R}} \Phi(x)(-1)^\delta \frac{d^\delta}{dx^\delta} \left\{\mathbf{E}[X^{\vec{k}_\delta} | g_{1T} = x] f_{g_{1T}}(x)\right\} dx + o(\epsilon^N)
\end{aligned} \tag{7}$$

where $\Phi^{(\delta)}(g_{1T}) = \left.\frac{\partial^\delta \Phi(x)}{\partial x^\delta}\right|_{x=g_{1T}}$,

$$X^{\vec{k}_\delta} = \prod_{j=1}^{\delta} g_{(k_j+1)T}$$

for $\vec{k}_\delta \in L_{n,\delta}$, and

$$\sum_{\vec{k}_\delta}^{(n)} = \sum_{\delta=1}^{n} \sum_{\vec{k}_\delta \in L_{n,\delta}}.$$

In the preceding works on application of the asymptotic expansion, conditional expectations in (7) were directly computed with some formulas given in [21] or [23] (for example, see Appendix B of [23]). Recently, while the formulas had been given up to the third order by those papers, [26] developed a high-order computation scheme for the conditional expectations using the fact that each of these $\{A^j_{k,t}\}_{j,k}$, $\{g_{nT}\}_n$ and also $\{X^{\vec{k}_\delta}\}_{\vec{k}_\delta}$ can be decomposed into a finite sum of iterated multiple Wiener-Itô integrals by Itô's formula, and a certain property of iterated multiple Wiener-Itô integrals (see Nualart, Üstünel and Zakai [17] and Section 4 of [26]). On the other hand, as shown in the next section, this paper develops a new method computing unconditional expectations instead of the conditional ones.

3 A New Computational Scheme

In this section we propose the new computational scheme in the asymptotic expansion, which is an alternative to the direct calculation method for the conditional expectations given by [26].

To compute the conditional expectations in the right hand side of (7), we use the following lemma which can be derived from a property of Hermite polynomials and leads us to compute the unconditional expectations instead of the conditional ones.

Lemma 1 *Let (Ω, F, P) be a probability space. Suppose that $X \in L^2(\Omega, P)$ and Z is a random variable with Gaussian distribution with mean 0 and variance Σ. Then, the conditional expectation $E[X|Z = x]$ has following expansion in $L^2(\mathbf{R}, \mu)$ where μ is the Gaussian measure on $\mathbf{R}$ with mean 0 and variance Σ:*

$$E[X|Z = x] = \sum_{n=0}^{\infty} \frac{a_n}{\Sigma^n} H_n(x; \Sigma) \tag{8}$$

where $H_n(x; \Sigma)$ is the Hermite polynomial of degree n which is defined as

$$H_n(x; \Sigma) = (-\Sigma)^n e^{x^2/2\Sigma} \frac{d^n}{dx^n} e^{-x^2/2\Sigma}$$

and coefficients a_n are given by

$$a_n = \frac{1}{n!} \frac{1}{i^n} \frac{\partial^n}{\partial \xi^n}\bigg|_{\xi=0} \left\{ e^{\frac{\xi^2}{2}\Sigma} \mathbf{E}[e^{i\xi Z} X] \right\}. \tag{9}$$

(proof) See Section 5.1. □

Here, let we define $\hat{g}_1 = \{\hat{g}_{1t}; t \in \mathbf{R}^+\}$ and $Z^{\langle\xi\rangle} = \{Z_t^{\langle\xi\rangle}; t \in \mathbf{R}^+\}$ as the stochastic processes

$$\hat{g}_{1t} = \int_0^t \hat{V}(X_u^{(0)}, u) dW_u$$

and

$$Z_t^{\langle\xi\rangle} = \exp\left(i\xi \hat{g}_{1t} + \frac{\xi^2}{2} \Sigma_t \right),$$

respectively.

Then, from Lemma 1, the conditional expectations appearing in the right hand side of the equation (7) is expressed as

$$\begin{aligned} \mathbf{E}[X^{\vec{k}_\delta} | g_{1T} = x] &= \mathbf{E}[X^{\vec{k}_\delta} | \hat{g}_{1T} = x - C] \\ &= \sum_{l=0}^{\infty} \frac{a_l^{\vec{k}_\delta}}{\Sigma_T^l} H_l(x - C, \Sigma_T) \end{aligned} \tag{10}$$

where

$$a_l^{\vec{k}_\delta} = \frac{1}{l!} \frac{1}{i^l} \frac{\partial^l}{\partial \xi^l} \mathbf{E}[X^{\vec{k}_\delta} Z_T^{\langle\xi\rangle}]\bigg|_{\xi=0}. \tag{11}$$

Here it is noted that with this expression we now need to compute unconditional expectations $\mathbf{E}[X^{\vec{k}_\delta} Z_T^{\langle\xi\rangle}]$ instead of the conditional expectations.

3.1 The Asymptotic Expansion of Density Function

In this subsection, we explain the new computational method through deriving a general formula for the expansion (7) with an arbitrary specification of its order N. In particular, we show that the coefficients in the expansion are obtained through a system of ordinary differential equations that is solved easily.

First, let we define $\eta_{\vec{l}_\beta}^{\vec{d}_\beta}(t; \xi)$ for $\vec{l}_\beta \in L_{n,\beta}$ and $\vec{d}_\beta \in \{1, \cdots, d\}^\beta$ $(n \geq \beta \geq 1)$ as

$$\eta_{\vec{l}_\beta}^{\vec{d}_\beta}(t; \xi) = \mathbf{E}\left[\left(\prod_{j=1}^{\beta} A_{l_j t}^{d_j} \right) Z_t^{\langle\xi\rangle} \right], \tag{12}$$

and for $n=0$ as

$$\eta_{(\emptyset)}^{(\emptyset)}(t;\xi)=\mathbf{E}\left[Z_t^{\langle\xi\rangle}\right]. \tag{13}$$

Then, unconditional expectations $\mathbf{E}[X^{\vec{k}_\delta}Z_T^{\langle\xi\rangle}]$ appearing in the definition of $a_l^{\vec{k}_\delta}$ (11) can be written in terms of η as follows:

$$\begin{aligned}
\mathbf{E}[X^{\vec{k}_\delta}Z_T^{\langle\xi\rangle}] &= \mathbf{E}\left[\left(\prod_{j=1}^{\delta} g_{(k_j+1)T}\right)Z_T^{\langle\xi\rangle}\right]\\
&= \mathbf{E}\left[\left(\prod_{j=1}^{\delta}\sum_{\vec{l}^j_{\beta_j},\vec{d}^j_{\beta_j}}^{(k_j+1)}\frac{1}{\beta_j!}\partial^{\beta_j}_{\vec{d}^j_{\beta_j}}g(X_T^{(0)})A^{d_1^j}_{l_1^jT}\cdots A^{d^j_{\beta_j}}_{l^j_{\beta_j}}\right)Z_T^{\langle\xi\rangle}\right]\\
&= \sum_{\vec{l}^1_{\beta_1},\vec{d}^1_{\beta_1}}^{(k_1+1)}\cdots\sum_{\vec{l}^\delta_{\beta_\delta},\vec{d}^\delta_{\beta_\delta}}^{(k_\delta+1)}\left(\prod_{j=1}^{\delta}\frac{1}{\beta_j!}\partial^{\beta_j}_{\vec{d}^j_{\beta_j}}g(X_T^{(0)})\right)\eta^{\vec{d}^1_{\beta_1}\otimes\cdots\otimes\vec{d}^\delta_{\beta_\delta}}_{\vec{l}^1_{\beta_1}\otimes\cdots\otimes\vec{l}^\delta_{\beta_\delta}}(T;\xi)
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
\vec{d}^i_{\beta_i}\otimes\vec{d}^j_{\beta_j} &:= (d_1^i,\cdots,d^i_{\beta_i},d_1^j,\cdots,d^j_{\beta_j}),\\
\vec{l}^i_{\beta_i}\otimes\vec{l}^j_{\beta_j} &:= (l_1^i,\cdots,l^i_{\beta_i},l_1^j,\cdots,l^j_{\beta_j}).
\end{aligned}$$

So, we have to calculate $\eta^{\vec{d}_\beta}_{\vec{l}_\beta}(T;\xi)$ to evaluate the asymptotic expansion (7). In the following, we derive a system of ODEs satisfied by these $\{\eta^{\vec{d}_\beta}_{\vec{l}_\beta}\}$.

Before showing a general result, we first derive the ODEs for few leading-low-order terms explicitly to give a better intuition of a key idea of our method. Consider the evaluation of $\eta^j_{(2)}(T;\xi)=E[A^j_{2T}Z_T^{\langle\xi\rangle}]$ which appears in the ϵ-order. Here, for simplicity, we assume that V_0 does not depend on ϵ, and write $V_0(x,\epsilon)$ as $V_0(x)$. First, applying Itô's formula to $A^j_{2t}Z_t^{\langle\xi\rangle}$, we have

$$\begin{aligned}
d(A^j_{2t}Z_t^{\langle\xi\rangle}) &= A^j_{2t}dZ_t^{\langle\xi\rangle}+Z_t^{\langle\xi\rangle}dA^j_{2t}+d\langle A^j_2,Z^{\langle\xi\rangle}\rangle_t\\
&= \left\{(i\xi)\sum_{j'=1}^{d}A^{j'}_{1t}Z_t^{\langle\xi\rangle}\hat{V}(X_t^{(0)},t)\partial_{j'}V^j(X_t^{(0)})'+\sum_{j'=1}^{d}A^{j'}_{2t}Z_t^{\langle\xi\rangle}\partial_{j'}V_0^j(X_t^{(0)})\right.\\
&\quad\left.+\frac{1}{2}\sum_{j',k'=1}^{d}A^{j'}_{1t}A^{k'}_{1t}Z_t^{\langle\xi\rangle}\partial_{j'}\partial_{k'}V_0^j(X_t^{(0)})\right\}dt\\
&\quad+\left\{(i\xi)A^j_{2t}Z_t^{\langle\xi\rangle}\hat{V}(X_t^{(0)},t)+\sum_{j'=1}^{d}A^{j'}_{1t}Z_t^{\langle\xi\rangle}\partial_{j'}V^j(X_t^{(0)})\right\}dW_t.
\end{aligned}$$

Since the last term is a martingale, taking expectation on both sides, we have the following ordinary differential equation for $\eta^j_{(2)}$:

$$\begin{aligned}
\frac{d}{dt}\eta^j_{(2)}(t;\xi) &= (i\xi)\sum_{j'=1}^{d}\eta^{j'}_{(1)}(t;\xi)\hat{V}(X_t^{(0)},t)\partial_{j'}V^j(X_t^{(0)})'\\
&\quad+\sum_{j'=1}^{d}\eta^{j'}_{(2)}(t;\xi)\partial_{j'}V_0^j(X_t^{(0)})+\frac{1}{2}\sum_{j',k'=1}^{d}\eta^{j',k'}_{(1,1)}(t;\xi)\partial_{j'}\partial_{k'}V_0^j(X_t^{(0)}).
\end{aligned}$$

Here, $\eta^{j}_{(1)}(j=1,\cdots,d)$ appearing in the right hand side of the above ODE are evaluated in the similar manner:

$$\begin{aligned} d(A^j_{1t}Z^{\langle\xi\rangle}_t) &= A^j_{1t}dZ^{\langle\xi\rangle}_t + Z^{\langle\xi\rangle}_t dA^j_{1t} + d\langle A^j_1, Z^{\langle\xi\rangle}\rangle_t \\ &= \left\{(i\xi)Z^{\langle\xi\rangle}_t \hat{V}(X^{(0)}_t,t)V^j(X^{(0)}_t)' + \sum_{j'=1}^{d} A^{j'}_{1t}Z^{\langle\xi\rangle}_t \partial_{j'}V^j_0(X^{(0)}_t)\right\}dt \\ &\quad + \left\{(i\xi)A^j_{1t}Z^{\langle\xi\rangle}_t\hat{V}(X^{(0)}_t,t) + Z^{\langle\xi\rangle}_t V^j(X^{(0)}_t)\right\}dW_t, \end{aligned}$$

hence, we have

$$\frac{d}{dt}\eta^j_{(1)}(t;\xi) = (i\xi)\hat{V}(X^{(0)}_t,t)V^j(X^{(0)}_t)' + \sum_{j'=1}^{d}\eta^{j'}_{(1)}(t;\xi)\partial_{j'}V^j_0(X^{(0)}_t).$$

$\eta^{j,k}_{(1,1)}$ and other higher-order terms can be evaluated in the same way.

The key observation is that each ODE does not involve any higher-order terms, and only lower- or the same order- terms appear in the right hand side of the ODE. So, one can easily solve (analytically or numerically) the system of ODEs and evaluate the expectations.

The following theorem provides a way to calculate general $\eta^{\vec{d}_\beta}_{\vec{l}_\beta}(T;\xi)$ as a solution to the system of the ordinary differential equations:

Theorem 1 *For $\eta^{\vec{d}_\beta}_{\vec{l}_\beta}(t;\xi)$ defined in (12), the following system of ordinary differential equations is satisfied:*

$$\begin{aligned} \frac{d}{dt}\left\{\eta^{\vec{d}_\beta}_{\vec{l}_\beta}(t;\xi)\right\} &= \sum_{k=1}^{\beta}\frac{1}{l_k!}\left\{\eta^{\vec{d}_{\beta/k}}_{\vec{l}_{\beta/k}}(t;\xi)\right\}\left\{\partial^{l_k}_\epsilon V^{d_k}_0(X^{(0)}_t,0)\right\} \\ &+ \sum_{k=1}^{\beta}\sum_{l=1}^{l_k}\sum_{\vec{m}_\gamma,\vec{\tilde{d}}_\gamma}^{(l)}\frac{1}{(l_k-l)!}\frac{1}{\gamma!}\left\{\eta^{(\vec{d}_{\beta/k})\otimes\vec{\tilde{d}}_\gamma}_{(\vec{l}_{\beta/k})\otimes\vec{m}_\gamma}(t;\xi)\right\}\left\{\partial^{\gamma}_{\vec{\tilde{d}}_\gamma}\partial^{l_k-l}_\epsilon V^{d_k}_0(X^{(0)}_t,0)\right\} \\ &+ \sum_{\substack{k,m=1\\k<m}}^{\beta}\sum_{\vec{m}_\gamma,\vec{\tilde{d}}_\gamma}^{(l_k-1)}\sum_{\vec{m}_\delta,\vec{\tilde{d}}_\delta}^{(l_m-1)}\frac{1}{\gamma!\delta!}\left\{\eta^{(\vec{d}_{\beta/k,m})\otimes\vec{\tilde{d}}_\gamma\otimes\vec{\tilde{d}}_\delta}_{(\vec{l}_{\beta/k,m})\otimes\vec{m}_\gamma\otimes\vec{m}_\delta}(t;\xi)\right\}\left\{\partial^{\gamma}_{\vec{\tilde{d}}_\gamma}V^{d_k}(X^{(0)}_t)\right\}\left\{\partial^{\delta}_{\vec{\tilde{d}}_\delta}V^{d_m}(X^{(0)}_t)\right\} \\ &+ (i\xi)\sum_{k=1}^{\beta}\sum_{\vec{m}_\gamma,\vec{\tilde{d}}_\gamma}^{(l_k-1)}\frac{1}{\gamma!}\left\{\eta^{(\vec{d}_{\beta/k})\otimes\vec{\tilde{d}}_\gamma}_{(\vec{l}_{\beta/k})\otimes\vec{m}_\gamma}(t;\xi)\right\}\left\{\partial^{\gamma}_{\vec{\tilde{d}}_\gamma}V^{d_k}(X^{(0)}_t)\right\}\hat{V}(X^{(0)}_t,t) \end{aligned} \tag{15}$$

where

$$\begin{aligned} \vec{l}_{\beta/k} &:= (l_1,\cdots,l_{k-1},l_{k+1},\cdots,l_\beta) \\ \vec{l}_{\beta/k,n} &:= (l_1,\cdots,l_{k-1},l_{k+1},\cdots,l_{n-1},l_{n+1},\cdots,l_\beta),\ 1\le k<n\le\beta \\ \vec{l}_\beta\otimes\vec{m}_\gamma &:= (l_1,\cdots,l_\beta,m_1,\cdots,m_\gamma) \end{aligned}$$

for $\vec{l}_\beta=(l_1,\cdots,l_\beta)$ and $\vec{m}_\gamma=(m_1,\cdots,m_\gamma)$.

(Proof) See Section 5.2. □

Remark 1 *Due to the hierarchical structure of the ODEs with respect to $n=\sum_{j=1}^{\beta}l_j$ and $\eta^{(\emptyset)}_{(\emptyset)}(t;\xi)=\mathbf{E}[Z^{\langle\xi\rangle}_t]=1$, one can easily solve these ODEs successively from lower-order terms to higher-order terms with initial conditions $\eta^{\vec{d}_\beta}_{\vec{l}_\beta}(0;\xi)=0$ for $(\vec{l}_\beta,\vec{d}_\beta)\neq(\emptyset,\emptyset)$.*

Remark 2 *Further, due to the structure of the system of the differential equations, it is easily shown by induction that each* $\eta_{\vec{l}_\beta}^{\vec{d}_\beta}(t;\xi)$ *is expressed as a polynomial of degree* $n = \sum_{j=1}^{\beta} l_j$ *with respect to* $(i\xi)$*. Then, we can also show that* $\mathbf{E}[X^{\vec{k}_\delta} Z_T^{\langle\xi\rangle}]$ *is a polynomial of degree* $(n+\delta)$ *with respect to* $(i\xi)$*, and thus* $a_l^{\vec{k}_\delta} = 0 (l > n+\delta)$ *for* $\vec{k}_\delta \in L_{n,\delta}$*. This ensures a convergence of the infinite sum in (10).*

Then, from Lemma 1 and (7), we have the following expression of $\mathbf{E}[\Phi(G^{(\epsilon)})]$:

$$
\begin{aligned}
\mathbf{E}[\Phi(G^{(\epsilon)})] &= \sum_{n=0}^{N} \epsilon^n \sum_{\vec{k}_\delta}^{(n)} \frac{1}{\delta!} \int_{\mathbf{R}} \Phi(x)(-1)^\delta \frac{d^\delta}{dx^\delta} \left\{ \sum_{l=0}^{n+\delta} \frac{a_l^{\vec{k}_\delta}}{\Sigma_T^l} H_l(x-C,\Sigma_T) f_{g_{1T}}(x) \right\} dx + o(\epsilon^N) \\
&= \sum_{n=0}^{N} \epsilon^n \sum_{\vec{k}_\delta}^{(n)} \frac{1}{\delta!} \int_{\mathbf{R}} \Phi(x) \left\{ \sum_{l=0}^{n+\delta} \frac{a_l^{\vec{k}_\delta}}{\Sigma_T^{l+\delta}} H_{l+\delta}(x-C,\Sigma_T) f_{g_{1T}}(x) \right\} dx + o(\epsilon^N)
\end{aligned}
$$

Here we used the relation

$$
\frac{d^\delta}{dx^\delta} \left\{ H_l(x-C,\Sigma_T) f_{g_{1T}}(x) \right\} = \frac{1}{\Sigma_T^\delta} H_{l+\delta}(x-C,\Sigma_T) f_{g_{1T}}(x).
$$

In particular, let Φ be the delta function at $x \in \mathbf{R}$, δ_x, we obtain the asymptotic expansion of the density of $G^{(\epsilon)}$:

$$
\begin{aligned}
f_{G^{(\epsilon)}}(x) &= \mathbf{E}[\delta_x(G^{(\epsilon)})] \\
&= \sum_{n=0}^{N} \epsilon^n \sum_{\vec{k}_\delta}^{(n)} \frac{1}{\delta!} \sum_{l=0}^{n+\delta} \frac{a_l^{\vec{k}_\delta}}{\Sigma_T^{l+\delta}} H_{l+\delta}(x-C,\Sigma_T) f_{g_{1T}}(x) + o(\epsilon^N). \quad (16)
\end{aligned}
$$

We summarize the discussion above as the following theorem:

Theorem 2 *The asymptotic expansion of the density function of* $G^{(\epsilon)}$ *up to* ϵ^N*-order is given by*

$$
f_{G^{(\epsilon)}}(x) = f_{g_{1T}}(x) + \sum_{n=1}^{N} \epsilon^n \left(\sum_{m=0}^{3n} C_{nm} H_m(x-C,\Sigma_T) \right) f_{g_{1T}}(x) + o(\epsilon^N) \tag{17}
$$

where

$$
\begin{aligned}
C_{nm} = & \frac{1}{\Sigma_T^m} \sum_{\delta=1}^{m} \sum_{\vec{k}_\delta \in L_{n,\delta}} \sum_{\vec{l}_{\beta_1}^1, \vec{d}_{\beta_1}^1}^{(k_1+1)} \cdots \sum_{\vec{l}_{\beta_\delta}^\delta, \vec{d}_{\beta_\delta}^\delta}^{(k_\delta+1)} \frac{1}{\delta!(m-\delta)!} \\
& \left(\prod_{j=1}^{\delta} \frac{1}{\beta_j!} \partial_{\vec{d}_{\beta_j}^j}^{\beta_j} g(X_T^{(0)}) \right) \frac{1}{i^{m-\delta}} \frac{\partial^{m-\delta}}{\partial \xi^{m-\delta}} \left\{ \eta_{\vec{l}_{\beta_1}^1 \otimes \cdots \otimes \vec{l}_{\beta_\delta}^\delta}^{\vec{d}_{\beta_1}^1 \otimes \cdots \otimes \vec{d}_{\beta_\delta}^\delta}(T;\xi) \right\} \Bigg|_{\xi=0}
\end{aligned} \tag{18}
$$

and $\eta_{\vec{l}_\beta}^{\vec{d}_\beta}(T;\xi)$ *are obtained as a solution to the system of ODEs given in Theorem 1.*

3.2 Asymptotic Expansion of Option Prices

We apply the asymptotic expansion to option pricing. We consider a plain vanilla option on the underlying asset $g(X_T^{(\epsilon)})$ whose dynamics is given by (1).

For example, an asymptotic expansion up to $\epsilon^{(N+1)}$ of a call option price at time 0 with maturity T and strike price K where $K = X_T^{(0)} - \epsilon y$ for arbitrary $y \in \mathbf{R}$ is given by

$$\begin{aligned} C(K,T) &= P(0,T)\mathbf{E}[\max(g(X_T^{(\epsilon)}) - K, 0)] \\ &= \epsilon P(0,T)\int_{-y}^{\infty}(x+y)f_{G^{(\epsilon)},N}(x)dx + o(\epsilon^{(N+1)}). \end{aligned} \tag{19}$$

Here, $P(0,T)$ denotes the price at time 0 of a zero coupon bond with maturity T and $f_{G^{(\epsilon)},N}$ is the asymptotic expansion of the density of $G^{(\epsilon)}$ up to ϵ^N-order given by (17):

$$f_{G^{(\epsilon)},N}(x) = f_{g_{1T}}(x) + \sum_{n=1}^{N}\epsilon^n\left(\sum_{m=0}^{3n}C_{nm}H_m(x-C,\Sigma_T)\right)f_{g_{1T}}(x)$$

Integrals appearing in the right hand side of (19) can be calculated by following formulas related to the Hermite polynomials

$$\begin{aligned} \int_{-y}^{\infty}H_k(x;\Sigma)f_{g_{1T}}(x)dx &= \Sigma H_{k-1}(-y;\Sigma)f_{g_{1T}}(y)\ (k \geq 1), \\ \int_{-y}^{\infty}xH_k(x;\Sigma)f_{g_{1T}}(x)dx &= -\Sigma y H_{k-1}(-y;\Sigma)f_{g_{1T}}(y) \\ &\quad + \Sigma^2 H_{k-2}(-y;\Sigma)f_{g_{1T}}(y)\ (k \geq 2). \end{aligned}$$

3.3 A Log-Normal Asymptotic Expansion

In this subsection, we develop a slightly different expansion from that introduced in the previous.

Suppose that an underlying one-dimensional asset process $S^{(\epsilon)}$ and d-dimensional stochastic process $X^{(\epsilon)}$ follow

$$\begin{aligned} dS_t^{(\epsilon)} &= g(X_t^{(\epsilon)})S_t^{(\epsilon)}\bar{\sigma}dW_t; \quad S_0^{(\epsilon)} = s_0, \\ dX_t^{(\epsilon)} &= V_0(X_t^{(\epsilon)},\epsilon)dt + \epsilon V(X_t^{(\epsilon)})dW_t; \quad X_0^{(\epsilon)} = x_0 \in \mathbf{R}^d \end{aligned}$$

respectively, where g:$\mathbf{R}^d \to \mathbf{R}$ and $\bar{\sigma}$ is a constant vector in $\mathbf{R}^r$.

First, let we define $\hat{X}^{(\epsilon)}$ as

$$\hat{X}_t^{(\epsilon)} = \log\left(\frac{S_t^{(\epsilon)}}{s_0}\right).$$

Then, we have

$$\hat{X}_t^{(\epsilon)} = -\frac{|\bar{\sigma}|^2}{2}\int_0^t g(X_u^{(\epsilon)})^2du + \int_0^t g(X_u^{(\epsilon)})\bar{\sigma}dW_u,$$

and note that

$$\hat{X}_T^{(0)} \sim N(\hat{\mu}_T, \hat{\Sigma}_T),$$

where

$$\begin{aligned}\hat{\mu}_T &= -\frac{|\bar{\sigma}|^2}{2}\int_0^T g(X_u^{(0)})^2 du = -\frac{1}{2}\hat{\Sigma}_T,\\ \hat{\Sigma}_T &= |\bar{\sigma}|^2\int_0^T g(X_u^{(0)})^2 du.\end{aligned}$$

Moreover, an asymptotic expansion of $\hat{X}_T^{(\epsilon)}$ up to ϵ^N-order is expressed as

$$\hat{X}_T^{(\epsilon)} = \hat{X}_T^{(0)} + \sum_{n=1}^{N}\epsilon^n \hat{A}_{nT} + o(\epsilon^N),$$

where $\hat{A}_{nt} = \frac{1}{n!}\frac{\partial^n \hat{X}_t^{(\epsilon)}}{\partial \epsilon^n}|_{\epsilon=0}$. Note that $S_T^{(\epsilon)}$ is now expanded around a log-normal distribution since $\hat{X}_T^{(0)}$ has the Gaussian distribution (hereafter we call this expansion 'the log-normal asymptotic expansion' of $S_T^{(\epsilon)}$ in contrast to calling the expansion in the previous subsection 'the normal asymptotic expansion').

Next, define $Z^{\langle\xi\rangle} = \{Z_t^{\langle\xi\rangle}; t \in \mathbf{R}^+\}$ as

$$Z_t^{\langle\xi\rangle} = \exp\left(i\xi\int_0^t g(X_u^{(0)})\bar{\sigma}dW_u\right).$$

Then, the result in the previous subsection is applied to deriving the density function of $\hat{X}_T^{(\epsilon)}$ with replacement of $G^{(\epsilon)}$ by $\hat{X}_T^{(\epsilon)}$.

Similar to the normal case, the log-normal asymptotic expansion of the price of the call option on $\hat{X}_T^{(\epsilon)}$ is given by

$$C(K,T) \;=\; P(0,T)\int_{\log\frac{K}{s_0}}^{\infty}(s_0 e^x - K) f_{\hat{X}_T^{(\epsilon)}}(x)dx.$$

We also remark that as shown in Takahashi and Yamada [29], the asymptotic expansion approach can be applied to a shifted log-normal model, sometimes called a displaced diffusion model (Rubinstein [18]), under stochastic volatility environments. Hence, the method for high-order expansions introduced in this paper can be applied to the model. More specifically, a stochastic volatility version of this model in the asymptotic expansion framework is expressed as follows:

$$\begin{aligned}dS_t^{(\epsilon)} &= g(X_t^{(\epsilon)})(S_t^{(\epsilon)}+\alpha)\bar{\sigma}dW_t; \quad S_0^{(\epsilon)} = s_0,\\ dX_t^{(\epsilon)} &= V_0(X_t^{(\epsilon)},\epsilon)dt + \epsilon V(X_t^{(\epsilon)})dW_t; \quad X_0^{(\epsilon)} = x_0 \in \mathbf{R}^d\end{aligned}$$

where α is a constant. Once we define $\hat{S}_t^{(\epsilon)} := S_t^{(\epsilon)} + \alpha$, the same method as above is easily applied since the model is now described as

$$\begin{aligned}d\hat{S}_t^{(\epsilon)} &= g(X_t^{(\epsilon)})\hat{S}_t^{(\epsilon)}\bar{\sigma}dW_t; \quad \hat{S}_0^{(\epsilon)} = s_0 + \alpha,\\ dX_t^{(\epsilon)} &= V_0(X_t^{(\epsilon)},\epsilon)dt + \epsilon V(X_t^{(\epsilon)})dW_t; \quad X_0^{(\epsilon)} = x_0 \in \mathbf{R}^d\end{aligned}$$

and the payoffs of call and put options with strike K and maturity T are expressed as $(\hat{S}_T^{(\epsilon)} - (K+\alpha))^+$ and $((K+\alpha) - \hat{S}_T^{(\epsilon)})^+$ respectively.

4 Numerical Examples

4.1 λ-SABR model

In this section, we test effectiveness of the asymptotic expansion method described in the previous section through numerical examples. Also, we compare approximation accuracy of our method with that of another existing approximation method.

4.1.1 Asymptotic Expansion of the λ-SABR Model

To test efficiency of the expansion, we first consider a European plain-vanilla call and put prices under the following λ-SABR model [12] (interest rate=0%) :

$$
\begin{aligned}
dS^{(\epsilon)}(t) &= \epsilon\sigma^{(\epsilon)}(t)(S^{(\epsilon)}(t))^{\beta}dW_t^1, \\
d\sigma^{(\epsilon)}(t) &= \lambda(\theta - \sigma^{(\epsilon)}(t))dt + \epsilon\nu_1\sigma^{(\epsilon)}(t)dW_t^1 + \epsilon\nu_2\sigma^{(\epsilon)}(t)dW_t^2, \\
S^{(\epsilon)}(0) &= S_0, \quad \sigma^{(\epsilon)}(0) = \sigma_0,
\end{aligned}
$$

where $\nu_1 = \rho\nu,\ \nu_2 = (\sqrt{1-\rho^2})\nu$ (an instantaneous correlation between $S^{(\epsilon)}$ and $\sigma^{(\epsilon)}$ is $\rho \in [-1,1]$). Note that when $\lambda = 0$ the model becomes the original SABR model [7]. Rigorously speaking, this model dose not satisfy the regularity conditions stated in pp.3-4 of Section 2 since the coefficient function $V^1(\sigma, s) = \sigma s^{\beta}$ is unbounded and has non-smooth derivatives at $s = 0$. However, as seen in the following, our method is (formally) applicable to this model and gives better accuracies for approximate prices in higher-order expansions for various range of strikes and parameters.

To compute an option price on $S^{(\epsilon)}$, we need the density function of $S_T^{(\epsilon)}$ whose asymptotic expansion is given by (17) with setting $g(S,\sigma) = S$. The asymptotic expansion of the density function is obtained by solving the system of the ordinary differential equations given in Theorem 1. For example, the corresponding differential equations up to the second order are given by

$$
\begin{aligned}
\frac{d}{dt}\eta_{(1)}^S(t;\xi) &= (i\xi)(S_t^{(0)})^{2\beta}(\sigma_t^{(0)})^2, \\
\frac{d}{dt}\eta_{(1)}^\sigma(t;\xi) &= (i\xi)\nu_1(S_t^{(0)})^{\beta}(\sigma_t^{(0)})^2 - \lambda\eta_{(1)}^\sigma(t;\xi), \\
\frac{d}{dt}\eta_{(2)}^S(t;\xi) &= (i\xi)\beta(S_t^{(0)})^{2\beta-1}(\sigma_t^{(0)})^2\eta_{(1)}^S(t;\xi) + (i\xi)(S_t^{(0)})^{2\beta}\sigma_t^{(0)}\eta_{(1)}^\sigma(t;\xi),
\end{aligned}
$$

where $S_t^{(0)} = S_0$ and $\sigma_t^{(0)} = e^{-\lambda t}(\sigma_0 - \theta) + \theta$. Since these equations are linear and have the hierarchical structure, one can easily integrate them as

$$
\begin{aligned}
\eta_{(1)}^S(t;\xi) &= (i\xi)\int_0^t (S_{t_1}^{(0)})^{2\beta}(\sigma_{t_1}^{(0)})^2 dt_1, \\
\eta_{(1)}^\sigma(t;\xi) &= (i\xi)\int_0^t e^{-\lambda(t-t_1)}\nu_1(S_{t_1}^{(0)})^{\beta}(\sigma_{t_1}^{(0)})^2 dt_1, \\
\eta_{(2)}^S(t;\xi) &= (i\xi)^2\int_0^t\int_0^{t_1} \beta(S_{t_1}^{(0)})^{2\beta-1}(\sigma_{t_1}^{(0)})^2(S_{t_2}^{(0)})^{2\beta}(\sigma_{t_2}^{(0)})^2 dt_2 dt_1 \\
&\quad + (i\xi)^2\int_0^t\int_0^{t_1} e^{-\lambda(t_1-t_2)}(S_{t_1}^{(0)})^{2\beta}\sigma_{t_1}^{(0)}\nu_1(S_{t_2}^{(0)})^{\beta}(\sigma_{t_2}^{(0)})^2 dt_2 dt_1.
\end{aligned}
$$

Integrals appearing in the right hand side are analytically evaluated, which is omitted due to the limitation of the space (they are available upon request).

Then, from Theorem 2 the asymptotic expansion of the density function of $G^{(\epsilon)} = \frac{S_T^{(\epsilon)} - S_T^{(0)}}{\epsilon}$ can be expressed as

$$
f_{G^{(\epsilon)}}(x) \sim f_{g_{1T}}(x) + \epsilon C_{13}H_3(x;\Sigma_T)f_{g_{1T}}(x) + \cdots \tag{20}
$$

where

$$
f_{g_{1T}}(x) = \frac{1}{\sqrt{2\pi\Sigma_T}}\exp\left(-\frac{x^2}{2\Sigma_T}\right)
$$

with

$$
\Sigma_T = \int_0^T (S_t^{(0)})^{2\beta}(\sigma_t^{(0)})^2 dt
$$

and

$$C_{13} = \frac{1}{\Sigma_T^3}\int_0^T\int_0^{t_1}\beta(S_{t_1}^{(0)})^{2\beta-1}(\sigma_{t_1}^{(0)})^2(S_{t_2}^{(0)})^{2\beta}(\sigma_{t_2}^{(0)})^2dt_2dt_1$$
$$+\frac{1}{\Sigma_T^3}\int_0^T\int_0^{t_1}e^{-\lambda(t_1-t_2)}(S_{t_1}^{(0)})^{2\beta}\sigma_{t_1}^{(0)}\nu_1(S_{t_2}^{(0)})^{\beta}(\sigma_{t_2}^{(0)})^2dt_2dt_1.$$

Note also that C_{13} is calculated in closed form; the expression is omitted, which is available upon request. Moreover, by a similar calculation to that in Section 3.2, an approximate price of a call option on $S^{(\epsilon)}$ at time 0 with maturity T and strike $K = S_T^{(0)} - \epsilon y$ up to ϵ^2-order is given by

$$C(K,T) = \epsilon P(0,T)\left(\Sigma_T f_{g_{1T}}(y) + yN\left(\frac{y}{\sqrt{\Sigma_T}}\right)\right)$$
$$-\epsilon^2 P(0,T)C_{13}\Sigma_T^2 y f_{g_{1T}}(y) + o(\epsilon^2) \quad (21)$$

where $N(\cdot)$ is a cumulative distribution function of the standard normal distribution.

Higher-order asymptotic expansions can be calculated in a similar manner.

4.1.2 Numerical Example: $\lambda = 0$ (SABR case)

First, we consider European plain-vanilla call and put prices under the original SABR case ($\lambda = 0$ in the λ-SABR model). We calculate approximated prices by the asymptotic expansion method up to the fifth order. Note that all the solutions to the differential equations are obtained in closed form. Thus, the computation is very fast(e.g. the computation time is within 10^{-3} second for the fifth-order expansion). We also calculate approximated prices by Hagan et al.[7] to compare accuracy of its approximation with ours. Benchmark values are computed by Monte Carlo simulations. In the simulations for the benchmark values, we use Euler-Maruyama scheme as a discretization scheme with 1024 time steps, and generate 10^8 paths in each simulation. ϵ is set to be one and other parameters used in the test are given in Table 1.

Table 1: Parameters used in the SABR ($\lambda = 0$) case

Parameter	$S(0)$	β	$\sigma(0)$	ν	ρ	T
i	100	0.5	3.0	0.3	-0.7	10

Results are in Table 3 and Figure 1. From the results, we can see that the higher-order asymptotic expansion almost always improves accuracy of the approximation by the lower ones. While sometimes the third-order approximation does not perform well, particularly in OTM options, the fifth-order one approximates the prices almost perfectly in these settings. This strongly supports importance of computing high-order terms, and hence of our method. We also see the fifth-order expansion has equal or smaller approximation errors than Hagan et al.[7]'s formula. Moreover, as seen in the next example, the asymptotic expansion method can be easily extend to the λ-SABR ($\lambda \neq 0$) case.

4.1.3 Numerical Example: $\lambda \neq 0$

Next, we consider the European option prices under the λ-SABR model with $\lambda \neq 0$. Parameters used in the test are given in Table 2 (and $\epsilon = 1$ as well as in the previous examples).

Table 2: Parameters used in the λ-SABR ($\lambda \neq 0$) case

Parameter	$S(0)$	β	$\sigma(0)$	λ	θ	ν	ρ	T
ii	100	0.5	3.0	0.1	3.0	0.3	-0.7	10
iii	100	1.0	0.3	0.1	0.3	0.3	-0.7	10

We calculate approximated prices by the asymptotic expansion method up to the fifth order. Note that all the solutions to the differential equations are obtained analytically. Further, for the case of $\beta = 1$ in the λ-SABR model (case iii), we can also apply the log-normal asymptotic expansion method given in Section 3.3. This gives the slightly different approximation formula from that with the normal asymptotic expansion method. Note also that the system of ODEs appearing in the log-normal expansion formula are solved analytically as in the normal asymptotic expansion case. We calculate approximated prices by the log-normal asymptotic expansion up to the forth order. We also calculate option prices by Hagan et al.[7]'s formula by setting $\lambda = 0$ in the model which can be thought as the SABR approximation to the λ-SABR model. Benchmark prices are computed by Monte Carlo simulations with Euler-Maruyama discretization scheme with 1024 time steps, and we generate 10^8 paths in each simulation.

Results for the normal asymptotic expansion are in Table 3 and Figure 2 and 3 and results for the log-normal expansion for case iii are in Table 4 and Figure 4. Note that the 0th-order log-normal expansion (indicated by 'LogNormal' in these table and figures) gives a simple log-normal approximation of the model.

From the results, in each case, as well as the examples in the original SABR model the higher-order expansion or log-normal expansion almost always improve accuracy of the approximation by the lower-order expansions. On the other hand, a naive application of Hagan et al.[7]'s formula to λ-SABR model($\lambda \neq 0$) seems to fail to capture the underlying distribution and the resulting option prices. This might be caused by the fact that it cannot be directly applied to the λ-SABR setting while our method is applicable to a general setting in the unified manner. Further, unlike Hagan et al.[7]'s one whose high-order expansions are difficult to calculate, our method easily provides us the approximation with an arbitrary-high order as we have already seen. These results support flexibility of ours in financial practice.

Table 3: Monte Carlo prices and standard errors (S.E.) and approximated prices, relative errors, and average absolute relative errors by asymptotic expansions (A.E.) and Hagan et al.[7]'s formula (Hagan) in the SABR model (case i) and the λ-SABR model with $\beta = 0.5$ (case ii) and $\beta = 1$ (case iii).

Case	Strike	M.C.	Approximation Prices (Relative Errors)					
	(C/P)	Price (S.E.)	A.E. 1st	A.E. 2nd	A.E. 3rd	A.E. 4th	A.E. 5th	Hagan(SABR)
	50 Put	12.859 (0.002)	17.987 (39.88%)	19.634 (52.68%)	16.628 (29.31%)	14.679 (14.15%)	13.112 (1.97%)	15.031 (16.89%)
	80 Put	23.824 (0.003)	28.686 (20.41%)	29.426 (23.51%)	26.179 (9.88%)	25.443 (6.80%)	23.978 (0.64%)	25.891 (8.67%)
i	100 Call	32.971 (0.004)	37.847 (14.79%)	37.847 (14.79%)	34.554 (4.80%)	34.554 (4.80%)	33.108 (0.42%)	34.791 (5.52%)
	120 Call	23.887 (0.004)	28.686 (20.09%)	27.945 (16.99%)	24.698 (3.40%)	25.434 (6.48%)	23.968 (0.34%)	25.334 (6.06%)
	150 Call	13.619 (0.003)	17.987 (32.07%)	16.340 (19.98%)	13.334 (-2.09%)	15.284 (12.23%)	13.718 (0.73%)	14.597 (7.19%)
Average Error		-	25.45%	25.59%	9.90%	8.89%	0.82%	8.87%
	50 Put	13.058 (0.002)	17.987 (37.75%)	18.110 (38.69%)	15.423 (18.11%)	14.177 (8.57%)	13.370 (2.39%)	15.031 (15.11%)
	80 Put	24.670 (0.003)	28.686 (16.28%)	28.741 (16.51%)	25.990 (5.35%)	25.499 (3.36%)	24.838 (0.68%)	25.891 (4.95%)
ii	100 Call	34.325 (0.004)	37.847 (10.26%)	37.847 (10.26%)	35.087 (2.22%)	35.087 (2.22%)	34.452 (0.37%)	34.791 (1.36%)
	120 Call	25.654 (0.005)	28.686 (11.82%)	28.630 (11.60%)	25.879 (0.88%)	26.370 (2.79%)	25.709 (0.21%)	25.334 (-1.25%)
	150 Call	15.611 (0.004)	17.987 (15.22%)	17.863 (14.43%)	15.175 (-2.79%)	16.421 (5.19%)	15.614 (0.02%)	14.597 (-6.49%)
Average Error		-	18.27%	18.30%	5.87%	4.43%	0.73%	5.83%
	50 Put	9.429 (0.002)	17.985 (90.74%)	13.991 (48.39%)	8.982 (-4.74%)	8.721 (-7.51%)	8.989 (-4.66%)	8.867 (5.95%)
	80 Put	22.043 (0.003)	28.685 (30.13%)	26.890 (21.99%)	21.618 (-1.93%)	21.505 (-2.44%)	21.771 (-1.23%)	21.350 (-1.91%)
iii	100 Call	32.974 (0.004)	37.847 (14.78%)	37.847 (14.78%)	32.539 (-1.32%)	32.539 (-1.32%)	32.805 (-0.51%)	32.598 (-3.71%)
	120 Call	25.664 (0.006)	28.685 (11.77%)	30.480 (18.77%)	25.209 (-1.77%)	25.321 (-1.33%)	25.587 (-0.30%)	25.892 (-7.47%)
	150 Call	17.427 (0.005)	17.985 (3.20%)	21.979 (26.12%)	16.969 (-2.63%)	17.231 (-1.13%)	17.499 (0.41%)	18.723 (-14.20%)
Average Error		-	30.12%	26.01%	2.48%	2.75%	1.42%	6.65%

Table 4: Monte Carlo prices and standard errors (S.E.) and approximated prices, relative errors, and average absolute relative errors by log-normal (LogNormal) and log-normal asymptotic expansions (LN-A.E.) in the λ-SABR model with $\beta = 1$ (case iii).

Case	Strike	M.C.	Approximation Prices (Relative Errors)					
	(C/P)	Price (S.E.)	LogNormal	LN-A.E. 1st	LN-A.E. 2nd	LN-A.E. 3rd	LN-A.E. 4th	Hagan(SABR)
	50 Put	9.429 (0.002)	8.532 (-9.51%)	9.679 (2.65%)	9.899 (4.99%)	9.206 (-2.37%)	9.450 (0.22%)	8.867 (5.95%)
	80 Put	22.043 (0.003)	23.661 (7.34%)	21.942 (-0.46%)	22.455 (1.87%)	21.851 (-0.87%)	22.080 (0.17%)	21.350 (-1.91%)
iii	100 Call	32.974 (0.004)	36.474 (10.62%)	32.555 (-1.27%)	33.350 (1.14%)	32.808 (-0.50%)	33.022 (0.15%)	32.598 (-3.71%)
	120 Call	25.664 (0.006)	30.804 (20.03%)	24.882 (-3.04%)	25.993 (1.29%)	25.520 (-0.56%)	25.718 (0.21%)	25.892 (-7.47%)
	150 Call	17.427 (0.005)	24.332 (39.62%)	16.004 (-8.17%)	17.681 (1.46%)	17.333 (-0.54%)	17.503 (0.43%)	18.723 (-14.20%)
Average Error		-	17.42%	3.12%	2.15%	0.97%	0.24%	6.65%

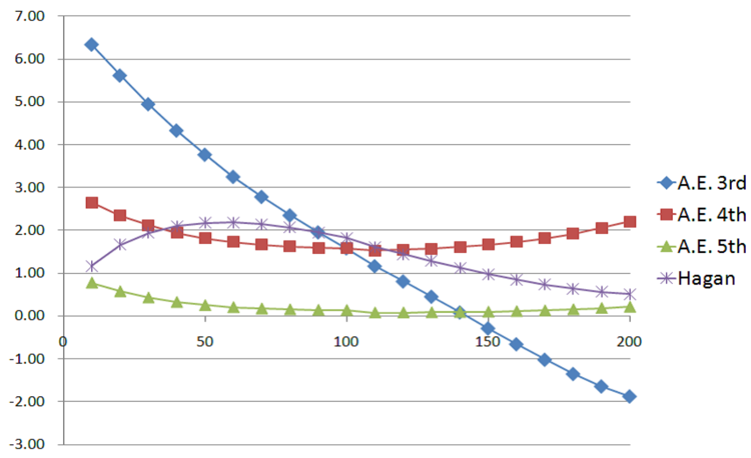

Figure 1: Approximation errors in price in case i

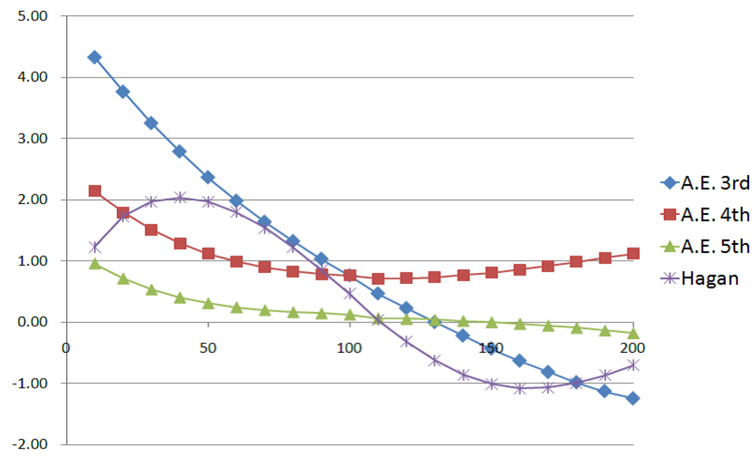

Figure 2: Approximation errors in price in case ii

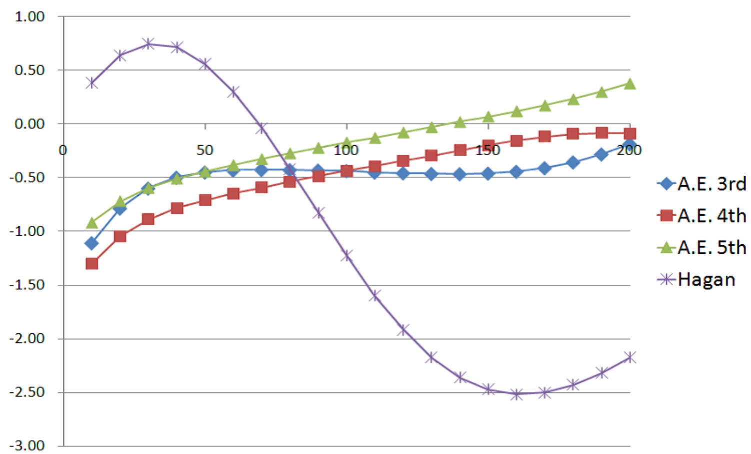

Figure 3: Approximation errors in price in case iii

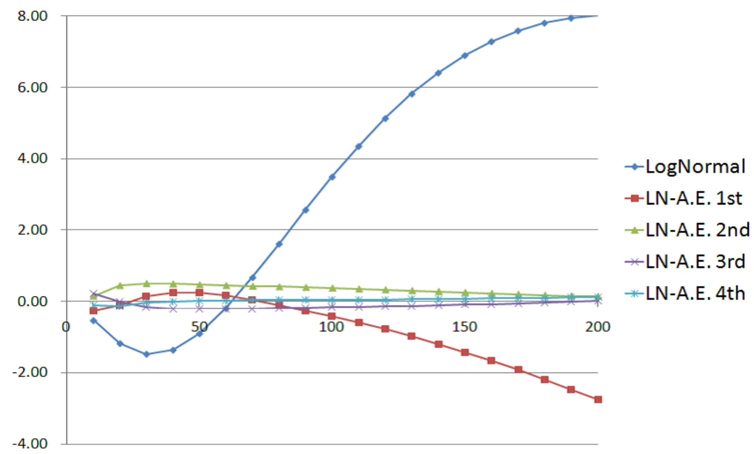

Figure 4: Approximation errors in price in case iii (log-normal asymptotic expansions)

4.2 Currency Options under a Libor Market Model of Interest Rates and a Stochastic Volatility of a Spot Exchange Rate

In this subsection, we apply our methods to pricing options on currencies under Libor Market Models(LMMs) of interest rates and a stochastic volatility of the spot foreign exchange rate(Forex). Due to limitation of space, only the structure of the stochastic differential equations of our model is described here. For details of the underlying model, see Takahashi and Takehara [23].

4.2.1 Cross-Currency Libor Market Models

Let $(\Omega, \mathcal{F}, \tilde{P}, \{\mathcal{F}_t\}_{0\le t\le T^*<\infty})$ be a complete probability space with a filtration satisfying the usual conditions. We consider the following pricing problem for the call option with maturity $T \in (0, T^*]$ and strike rate $K > 0$;

$$V^C(0;T,K) = P_d(0,T) \times \mathbf{E}^P\left[(S(T)-K)^+\right] = P_d(0,T) \times \mathbf{E}^P\left[(F_T(T)-K)^+\right] \tag{22}$$

where $V^C(0;T,K)$ denotes the value of an European call option at time 0 with maturity T and strike rate K, $S(t)$ denotes the spot exchange rate at time $t \ge 0$ and $F_T(t)$ denotes the time t value of the forex forward rate with maturity T. Similarly, for the put option we consider

$$V^P(0;T,K) = P_d(0,T) \times \mathbf{E}^P\left[(K-S(T))^+\right] = P_d(0,T) \times \mathbf{E}^P\left[(K-F_T(T))^+\right]. \tag{23}$$

It is well known that the arbitrage-free relation between the forex spot rate and the forex forward rate are given by $F_T(t) = S(t)\frac{P_f(t,T)}{P_d(t,T)}$ where $P_d(t,T)$ and $P_f(t,T)$ denote the time t values of domestic and foreign zero coupon bonds with maturity T respectively. $\mathbf{E}^P[\cdot]$ denotes an expectation operator under EMM(Equivalent Martingale Measure to $\tilde{P}$) P whose associated numeraire is the domestic zero coupon bond maturing at T. We sometimes call this probability measure 'the domestic terminal measure'.

For these pricing problems, a market model and a stochastic volatility model are applied to modeling interest rates' and the spot exchange rate's dynamics respectively.

We first define domestic and foreign forward interest rates as $f_{dj}(t) = \left(\frac{P_d(t,T_j)}{P_d(t,T_{j+1})} - 1\right)\frac{1}{\tau_j}$ and $f_{fj}(t) = \left(\frac{P_f(t,T_j)}{P_f(t,T_{j+1})} - 1\right)\frac{1}{\tau_j}$ respectively, where $j = n(t), n(t)+1, \cdots, N$, $\tau_j = T_{j+1} - T_j$, and $P_d(t,T_j)$ and $P_f(t,T_j)$ denote the prices of domestic/foreign zero coupon bonds with maturity T_j at time $t(\le T_j)$ respectively; $n(t) = \min\{i : t \le T_i\}$. We also define spot interest rates to the nearest fixing date denoted by $f_{d,n(t)-1}(t)$ and $f_{f,n(t)-1}(t)$ as $f_{d,n(t)-1}(t) = \left(\frac{1}{P_d(t,T_{n(t)})} - 1\right)\frac{1}{(T_{n(t)}-t)}$ and $f_{f,n(t)-1}(t) = \left(\frac{1}{P_f(t,T_{n(t)})} - 1\right)\frac{1}{(T_{n(t)}-t)}$. Finally, we set $T = T_{N+1}$ and will abbreviate $F_{T_{N+1}}(t)$ to $F_{N+1}(t)$ in what follows.

Under the framework of the asymptotic expansion in the standard cross-currency libor market model, we have to consider the following system of stochastic differential equations(henceforth called SDEs) under the domestic terminal measure P to price options. For detailed arguments on the framework of these SDEs see [23].

As for the domestic and foreign interest rates we assume forward market models; for $j = n(t)-1, n(t), n(t)+1, \cdots, N$,

$$f_{dj}^{(\epsilon)}(t) = f_{dj}(0) + \epsilon^2 \sum_{i=j+1}^{N} \int_0^t g_{di}^{0,(\epsilon)}(u)' \gamma_{dj}(u) f_{dj}^{(\epsilon)}(u) du + \epsilon \int_0^t f_{dj}^{(\epsilon)}(u) \gamma_{dj}'(u) dW_u, \quad (24)$$

$$\begin{aligned} f_{fj}^{(\epsilon)}(t) &= f_{fj}(0) - \epsilon^2 \sum_{i=0}^{j} \int_0^t g_{fi}^{0,(\epsilon)}(u)' \gamma_{fj}(u) f_{fj}^{(\epsilon)}(u) du + \epsilon^2 \sum_{i=0}^{N} \int_0^t g_{di}^{0,(\epsilon)}(u)' \gamma_{fj}(u) f_{fj}^{(\epsilon)}(u) du \\ &\quad -\epsilon^2 \int_0^t \sigma^{(\epsilon)}(u) \bar{\sigma}' \gamma_{fj}(u) f_{fj}^{(\epsilon)}(u) du + \epsilon \int_0^t f_{fj}^{(\epsilon)}(u) \gamma_{fj}'(u) dW_u, \quad (25) \end{aligned}$$

where

$$g_{dj}^{0,(\epsilon)}(t) := \frac{-\tau_j f_{dj}^{(\epsilon)}(t)}{1+\tau_j f_{dj}^{(\epsilon)}(t)} \gamma_{dj}(t), \quad g_{fj}^{0,(\epsilon)}(t) := \frac{-\tau_j f_{fj}^{(\epsilon)}(t)}{1+\tau_j f_{fj}^{(\epsilon)}(t)} \gamma_{fj}(t);$$

x' denotes the transpose of x and W is a r-dimensional standard Wiener process under the domestic terminal measure P; $\gamma_{dj}(s)$, $\gamma_{fj}(s)$ are r-dimensional vector-valued functions of time-parameter s; $\bar{\sigma}$ denotes a r-dimensional constant vector satisfying $||\bar{\sigma}|| = 1$ and $\sigma^{(\epsilon)}(t)$, the volatility of the spot exchange rate, is specified to follow a $\mathbf{R}_{++}$-valued general time-inhomogeneous Markovian process as follows:

$$\sigma^{(\epsilon)}(t) = \sigma(0) + \int_0^t \mu(u, \sigma^{(\epsilon)}(u)) du + \epsilon^2 \sum_{j=1}^{N} \int_0^t g_{dj}^{0,(\epsilon)}(u)' \omega(u, \sigma^{(\epsilon)}(u)) du + \epsilon \int_0^t \omega'(u, \sigma^{(\epsilon)}(u)) dW_u, \quad (26)$$

where $\mu(s,x)$ and $\omega(s,x)$ are functions of s and x.

Finally, we consider the process of the forex forward $F_{N+1}(t)$. Since $F_{N+1}(t) \equiv F_{T_{N+1}}(t)$ can be expressed as $F_{N+1}(t) = S(t)\frac{P_f(t,T_{N+1})}{P_d(t,T_{N+1})}$, we easily notice that it is a martingale under the domestic terminal measure. Thus, we conclude that $F_{N+1}^{(\epsilon)}$, explicitly dependent on ϵ as well, follows

$$F_{N+1}^{(\epsilon)}(t) = F_{N+1}(0) + \epsilon \int_0^t \sigma_F^{(\epsilon)}(u)' F_{N+1}^{(\epsilon)}(u) dW_u \quad (27)$$

where

$$\sigma_F^{(\epsilon)}(t) := \sum_{j=0}^{N} \left(g_{fj}^{0,(\epsilon)}(t) - g_{dj}^{0,(\epsilon)}(t) \right) + \sigma^{(\epsilon)}(t) \bar{\sigma}.$$

4.2.2 Numerical Examples

We here specify our model and parameters, and confirm effectiveness of our method in this cross-currency framework.

First of all, the processes of domestic and foreign forward interest rates and of the volatility of the spot exchange rate are specified. We suppose $r = 4$, that is the dimension of a Brownian motion is set to be four; it represents the uncertainty of domestic and foreign interest rates, the spot exchange rate, and its volatility. Note that in this framework correlations among all factors are allowed. We also suppose $S(0) = 100$.

Next, we specify the volatility process of the spot exchange rate in (26) with

$$\begin{cases} \mu(s,x) = \kappa(\theta - x), \\ \omega(s,x) = \omega x, \end{cases} \quad (28)$$

Table 5: Initial domestic/foreign forward interest rates and their volatilities

	f_d	γ_d^*	f_f	γ_f^*
case (i)	0.05	0.12	0.05	0.12
case (ii)	0.02	0.3	0.05	0.12
case (iii)	0.05	0.12	0.02	0.3
case (iv)	0.02	0.3	0.02	0.3

where θ and κ represent the level and speed of its mean-reversion respectively, and ω denotes a volatility vector on the volatility. In this subsection the parameters are set as follows; $\epsilon = 1$, $\sigma(0) = \theta = 0.1$, and $\kappa = 0.1$; $\omega = \omega^* \bar{v}$ where $\omega^* = 0.3$ and $\bar{v}$ denotes a four dimensional constant vector given below.

We further suppose that initial term structures of domestic and foreign forward interest rates are flat, and their volatilities also have flat structures and are constant over time: that is, for all $j, f_{dj}(0) = f_d$, $f_{fj}(0) = f_f$, $\gamma_{dj}(t) = \gamma_d^* \bar{\gamma}_d 1_{\{t<T_j\}}(t)$ and $\gamma_{fj}(t) = \gamma_f^* \bar{\gamma}_f 1_{\{t<T_j\}}(t)$. Here, f_d, f_f, γ_d^* and γ_f^* are constant scalars, and $\bar{\gamma}_d$ and $\bar{\gamma}_f$ denote four dimensional constant vectors. Moreover, given a correlation matrix $\underline{C}$ among all four factors, the constant vectors $\bar{\gamma}_d$, $\bar{\gamma}_f$, $\bar{\sigma}$ and $\bar{v}$ can be determined to satisfy $||\bar{\gamma}_d|| = ||\bar{\gamma}_f|| = ||\bar{\sigma}|| = ||\bar{v}|| = 1$ and $V'V = \underline{C}$ where $V := (\bar{\gamma}_d, \bar{\gamma}_f, \bar{\sigma}, \bar{v})$.

In this subsection, we consider four different cases for f_d, γ_d^*, f_f and γ_f^* as in Table 5. For correlations, four sets of parameters are considered: In the case "Corr.1", all the factors are independent: In "Corr.2", there exists only the correlation of -0.5 between the spot exchange rate and its volatility (i.e. $\bar{\sigma}'\bar{v} = -0.5$) while there are no correlations among the others: In "Corr.3", the correlation between interest rates and the spot exchange rate are allowed while there are no correlations among the others; the correlation between domestic ones and the spot forex is 0.5($\bar{\gamma}_d'\bar{\sigma} = 0.5$) and the correlation between foreign ones and the spot forex is -0.5($\bar{\gamma}_f'\bar{\sigma} = -0.5$): Finally in "Corr.4", more intricately correlated structure is considered; $\bar{\gamma}_d'\bar{\sigma} = 0.5$, $\bar{\gamma}_f'\bar{\sigma} = -0.5$ between interest rates and the spot forex; and $\bar{\sigma}'\bar{v} = -0.5$ between the spot forex and its volatility. It is well known that (both of exact and approximate)evaluation of the long-term options is a hard task in cases with complex structures of correlations such as in "Corr.3" or "Corr.4". Thus, this again shows flexibility and generality of our method compared to other approximation methods.

Lastly, we make an assumption that $\gamma_{dn(t)-1}(s)$ and $\gamma_{fn(t)-1}(s)$, volatilities of the domestic and foreign interest rates applied to the period from t to the next fixing date $T_{n(t)}$, are equal to be zero for arbitrary $s \in [t, T_{n(t)}]$ and $t \in [0, T]$.

In Figure 4.2.2, we compare our estimations of the values of call and put options by the asymptotic expansion up to the fourth order to the benchmarks estimated by 10^6 trials of Monte Carlo simulation with discretization by Euler-Maruyama scheme with time step 0.05 and application of the Antithetic Variable Method. For the moneynesses(defined by $K/F_{N+1}(0)$) less than one, the prices of put options are shown; otherwise, the prices of call options are displayed. Detailed data are omitted due to limitation of space and available in [26].

As seen in these graphs, in general the estimators show more accuracy as the order of the expansion increases. Especially, for the deep OTM options the fourth order approximation performs much better and is stabler than the approximation with the lower orders.

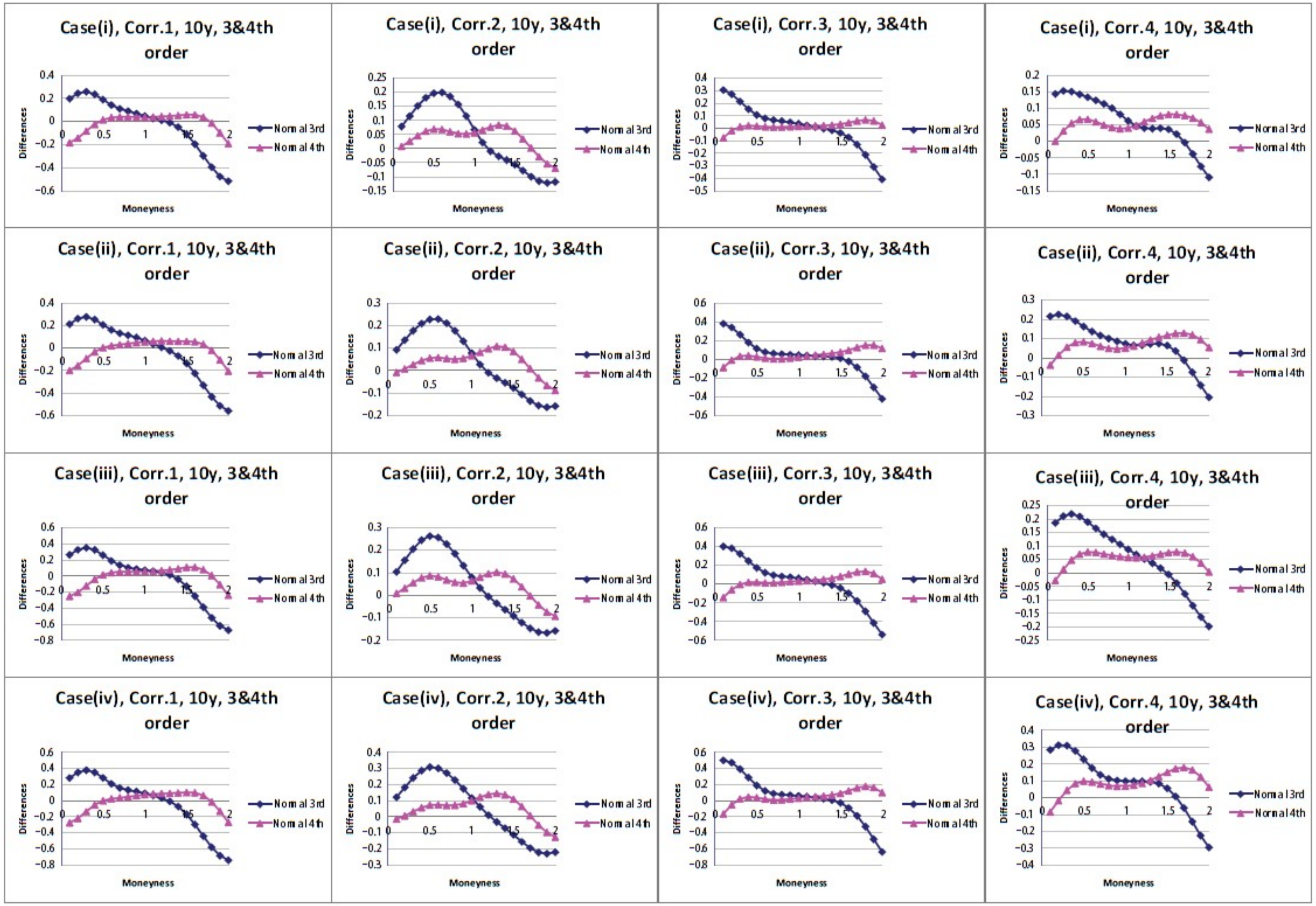

Figure 5: Approximation errors in the cross-currency Libor market model

References

[1] Antonov, A. and Misirpashaev, T.[2009], "Projection on a Quadratic Model by Asymptotic Expansion with an Application to LMM Swaption," Working Paper.

[2] Andersen, L.B.G. and Hutchings, N.A.[2009], "Parameter Averaging of Quadratic SDEs With Stochastic Volatility," Working Paper.

[3] Ben Arous, G., Laurence, P.[2009], "Second order expansion for implied volatility in two factor local stochastic volatility models and applications to the dynamic λ-SABR model," Working Paper.

[4] Fouque, J.-P., Papanicolaou, G. and Sircar, K. R.[1999], "Financial Modeling in a Fast Mean-reverting Stochastic Volatility Environment, "*Asia-Pacific Financial Markets*, Vol. 6(1), pp.37-48.

[5] Fouque, J.-P., Papanicolaou, G. and Sircar, K. R.[2000], *Derivatives in financial Markets with Stochastic Volatility*, Cambridge University Press.

[6] Gatheral, J., Hsu, E.P., Laurence, P., Ouyang, C. and Wang, T-H.[2009], "Asymptotics of implied volatility in local volatility models," Working Paper.

[7] Hagan, P.S., Kumar, D., Lesniewski, A.S., and Woodward, D.E. [2002], "Managing smile risk," *Willmott Magazine*, 15, 84-108.

[8] Ikeda, N. and Watanabe, S. [1989], *Stochastic Differential Equations and Diffusion Processes*, Second Edition, North-Holland/Kodansha, Tokyo.

[9] Kunitomo, N. and Takahashi, A. [1992], "Pricing Average Options," *Japan Financial Review*, Vol. 14, 1-20. (in Japanese).

[10] Kunitomo, N. and Takahashi, A. [2001], "The Asymptotic Expansion Approach to the Valuation of Interest Rate Contingent Claims," *Mathematical Finance*, Vol. 11, 117-151.

[11] Kunitomo, N. and Takahashi, A. [2003a], "On Validity of the Asymptotic Expansion Approach in Contingent Claim Analysis," *Annals of Applied Probability* Vol. 13-3, 914-952.

[12] Labordere, P.H. [2005a], "A General Asymptotic Implied Volatility for Stochastic Volatility Models", cond-mat/0504317.

[13] Labordere, P.H. [2005b], "Solvable Local and Stochastic Volatility Models: Supersymmetric Methods in Option Pricing," Working Paper.

[14] Matsuoka, R. Takahashi, A. and Uchida, Y. [2004], "A New Computational Scheme for Computing Greeks by the Asymptotic Expansion Approach," *Asia-Pacific Financial Markets*, Vol.11, 393-430.

[15] Muroi, Y. [2005], "Pricing Contingent Claims with Credit Risk: Asymptotic Expansion Approach," *Finance and Stochastics*, Vol. 9(3), 415-427.

[16] Nualart, D. [1995], "The Malliavin Calculus and Related Topics," Springer.

[17] Nualart, D., Üstünel A. S. and Zakai M. [1988], "On the moments of a multiple Wiener-Itô integral and the space induced by the polynomials of the integral," *Stochastics*, Vol. 25, 233-340.

[18] Rubinstein, M.[1983], “Displaced Diffusion Option Pricing,” *Journal of Finance*, Vol.38-1.

[19] Siopacha, M. and Teichmann, J.[2007], “Weak and Strong Taylor Methods for Numerical Solutions of Stochastic Differential Equations,” Working paper.

[20] Takahashi, A. [1995], “Essays on the Valuation Problems of Contingent Claims,” Unpublished Ph.D. Dissertation, Haas School of Business, University of California, Berkeley.

[21] Takahashi, A. [1999], “An Asymptotic Expansion Approach to Pricing Contingent Claims,” *Asia-Pacific Financial Markets*, Vol. 6, 115-151.

[22] Takahashi, A. [2009], ”On an Asymptotic Expansion Approach to Numerical Problems in Finance,” *Selected Papers on Probability and Statistics*, pp. 199-217, 2009, American Mathematical Society.

[23] Takahashi, A. and Takehara, K.[2007], “An Asymptotic Expansion Approach to Currency Options with a Market Model of Interest Rates under Stochastic Volatility Processes of Spot Exchange Rates,” *Asia-Pacific Financial Markets*, Vol.14 , pp. 69-121.

[24] Takahashi, A. and Takehara, K.[2008a], “Fourier Transform Method with an Asymptotic Expansion Approach: an Applications to Currency Options,” *International Journal of Theoretical and Applied Finance*, Vol. 11(4), pp. 381-401.

[25] Takahashi, A. and Takehara, K. [2008b], “A Hybrid Asymptotic Expansion Scheme: an Application to Currency Options,” Working paper, CARF-F-116, the University of Tokyo, http://www.carf.e.u-tokyo.ac.jp/workingpaper/

[26] Takahashi, A., Takehara, K. and Toda, M. [2009], “Computation in an Asymptotic Expansion Method,” Working paper, CARF-F-149, the University of Tokyo, http://www.carf.e.u-tokyo.ac.jp/workingpaper/

[27] Takehara, K., Takahashi, A. and Masashi Toda [2010], ”New Unified Computation Algorithm in a High-Order Asymptotic Expansion Scheme,” Recent Advances in Financial Engineering (the Proceedings of KIER-TMU International Workshop on Financial Engineering 2009), pp. 231-251.

[28] Takehara, K., Toda, M. and Takahashi, A. [2011], ”Application of a High-Order Asymptotic Expansion Scheme to Long-Term Currency Options,” *International Journal of Business and Finance Research*, vol. 5-3, pp. 87-100.

[29] Takahashi, A. and Yamada, T. [2009], “An Asymptotic Expansion with Push Down Malliavin Weights,” Working paper, CARF-F-194, the University of Tokyo, http://www.carf.e.u-tokyo.ac.jp/workingpaper/

[30] Takahashi, A. and Yoshida, N. [2004], “An Asymptotic Expansion Scheme for Optimal Investment Problems,” *Statistical Inference for Stochastic Processes*, Vol. 7-2, 153-188.

[31] Takahashi, A. and Yoshida, N. [2005], “Monte Carlo Simulation with Asymptotic Method,” *The Journal of Japan Statistical Society*, Vol. 35-2, 171-203.

[32] Watanabe, S. [1987], "Analysis of Wiener Functionals (Malliavin Calculus) and its Applications to Heat Kernels," *The Annals of Probability*, Vol. 15, 1-39.

[33] Yoshida, N. [1992a], "Asymptotic Expansion for Small Diffusions via the Theory of Malliavin-Watanabe," *Probability Theory and Related Fields*, Vol. 92, 275-311.

[34] Yoshida, N. [1992b], "Asymptotic Expansions for Statistics Related to Small Diffusions," *The Journal of Japan Statistical Society*, Vol. 22, 139-159.

5 Appendix

5.1 Proof of Lemma 1 in Section 3

Since the system of Hermite polynomials $\{H_n(x;\Sigma)\}$ is an orthogonal basis of $L^2(\mathbf{R},\mu)$, and $E[X|Z=x] \in L^2(\mathbf{R},\mu)$, we have the following unique expansion of $E[X|Z=x]$ in $L^2(\mathbf{R},\mu)$:

$$E[X|Z=x] = \sum_{n=0}^{\infty} \frac{a_n}{\Sigma^n} H_n(x;\Sigma).$$

Since we have another Taylor expansion

$$e^{i\xi x} = e^{-\frac{\xi^2}{2}\Sigma} \sum_{n=0}^{\infty} \frac{H_n(x;\Sigma)}{n!}(i\xi)^n,$$

then,

$$\begin{aligned}
e^{\frac{\xi^2}{2}\Sigma}\mathbf{E}[e^{i\xi Z}X] &= e^{\frac{\xi^2}{2}\Sigma}\int_{\mathbf{R}} e^{i\xi x}\mathbf{E}[X|Z=x]\mu(dx) \\
&= \int_{\mathbf{R}} \sum_{m=0}^{\infty} \frac{H_m(x;\Sigma)}{m!}(i\xi)^m \sum_{n=0}^{\infty} a_n H_n(x;\Sigma)\mu(dx) \\
&= \sum_{n=0}^{\infty} a_n (i\Sigma)^n \xi^n.
\end{aligned}$$

Comparing to the coefficients of the Taylor series of $e^{\frac{\xi^2}{2}\Sigma}\mathbf{E}[e^{i\xi Z}X]$ around 0 with respect to ξ, we see that a_n can be written as (9).□

5.2 Proof of Theorem 1 in Section 3.1

First, Applying Itô's formula to $\left(\prod_{j=1}^{\beta} A_{l_j t}^{d_j}\right)$, we have

$$
\begin{aligned}
d\left(\prod_{j=1}^{\beta} A_{l_j t}^{d_j}\right) &= \sum_{k=1}^{\beta}\left(\prod_{\substack{j=1\\ j\neq k}}^{\beta} A_{l_j t}^{d_j}\right) dA_{l_k t}^{d_k} + \sum_{\substack{k,m=1\\ k<m}}^{\beta}\left(\prod_{\substack{j=1\\ j\neq k,m}}^{\beta} A_{l_j t}^{d_j}\right) d\langle A_{l_k}^{d_k}, A_{l_m}^{d_m}\rangle_t \\
&= \sum_{k=1}^{\beta}\left(\prod_{\substack{j=1\\ j\neq k}}^{\beta} A_{l_j t}^{d_j}\right)\frac{1}{l_k!}\partial_\epsilon^{l_k} V_0^{d_k}(X_t^{(0)},0)dt \\
&+ \sum_{k=1}^{\beta}\sum_{l=1}^{l_k}\left(\prod_{\substack{j=1\\ j\neq k}}^{\beta} A_{l_j t}^{d_j}\right)\sum_{\vec{m}_\gamma,\vec{\tilde{d}}_\gamma}^{(l)}\frac{1}{(l_k-l)!}\frac{1}{\gamma!}\left(\prod_{j'=1}^{\gamma} A_{m_{j'} t}^{\tilde{d}_{j'}}\right)\partial_{\tilde{d}_\gamma}^{\gamma}\partial_\epsilon^{l_k-l} V_0^{d_k}(X_t^{(0)},0)dt \\
&+ \sum_{k=1}^{\beta}\left(\prod_{\substack{j=1\\ j\neq k}}^{\beta} A_{l_j t}^{d_j}\right)\sum_{\vec{m}_\gamma,\vec{\tilde{d}}_\gamma}^{(l_k-1)}\frac{1}{\gamma!}\left(\prod_{j'=1}^{\gamma} A_{m_{j'} t}^{\tilde{d}_{j'}}\right)\partial_{\tilde{d}_\gamma}^{\gamma} V^{d_k}(X_t^{(0)})dW_t \\
&+ \sum_{\substack{k,m=1\\ k<m}}^{\beta}\left(\prod_{\substack{j=1\\ j\neq k,m}}^{\beta} A_{l_j t}^{d_j}\right)\sum_{\vec{m}_\gamma,\vec{\tilde{d}}_\gamma}^{(l_k-1)}\sum_{\vec{m}_\delta,\vec{\hat{d}}_\delta}^{(l_m-1)}\frac{1}{\gamma!\delta!} \\
&\quad\times\left(\prod_{j'=1}^{\gamma} A_{m_{j'} t}^{\tilde{d}_{j'}}\right)\partial_{\tilde{d}_\gamma}^{\gamma} V^{d_k}(X_t^{(0)})\left(\prod_{j'=1}^{\delta} A_{m_{j'} t}^{\hat{d}_{j'}}\right)\partial_{\hat{d}_\delta}^{\delta} V^{d_m}(X_t^{(0)})dt.
\end{aligned} \tag{29}
$$

Note also that

$$
dZ_t^{\langle\xi\rangle} = (i\xi)\hat{V}(X_t^{(0)},t)Z_t^{\langle\xi\rangle}dW_t. \tag{30}
$$

Then, applying Itô's formula again to $\left(\prod_{j=1}^{\beta} A_{l_j t}^{d_j} Z_t^{\langle\xi\rangle}\right)$ and taking expectations on both sides, we obtain the result. □