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An Asymptotic Expansion for Solutions of Cauchy-Dirichlet Problem for Second Order Parabolic PDEs and its Application to Pricing Barrier Options

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Abstract

This paper develops a rigorous asymptotic expansion method with its numerical scheme for the Cauchy-Dirichlet problem in second order parabolic partial differential equations (PDEs). As an application, we propose a new approximation formula for pricing a barrier option under a certain type of stochastic volatility model including the log-normal SABR model.

Keywords: Asymptotic expansion, The Cauchy-Dirichlet problem, Second order parabolic PDEs, Barrier options, Stochastic volatility model.

1 Introduction

Numerical methods for the Cauchy-Dirichlet problem have been a topic of great interest in stochastic analysis and its applications. For example, in mathematical finance the Cauchy-Dirichlet problem naturally arises in valuation of continuously monitoring barrier options:

\[ C_{\text{Barrier}}(T, x) = \mathbb{E}[f(X_T^x)1_{\{\tau > T\}}] = \mathbb{E}[f(X_T^x)1_{\{\min_{t \in [0,T]} X_t > L\}}]. \tag{1.1} \]

Here, \( T > 0 \) is a maturity of the option, and \((X_t^x)\) denotes a price process of the underlying asset starting from \( x \) (usually given as the solution of a certain stochastic differential equation

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\( L \) stands for a constant lower barrier, that is \( L < x \), and \( \tau \) is the hitting time to \( L \):

\[
\tau = \inf\{ t \in [0, T] : X_t^\varepsilon \leq L \}. \tag{1.2}
\]

It is well-known that a possible approach in computation of \( C_{\text{Barrier}}(T, x) \) is the Euler scheme, which stores the sample paths of the process \((X_t^\varepsilon)\) through an \( n \)-time discretization with the step size \( T/n \). In applying this scheme to pricing a continuously monitoring barrier option, one kills the simulated process (say, \((\bar{X}_t^\varepsilon)\)) if \( \bar{X}_t^\varepsilon \) exits from the domain \((L, \infty)\) until the maturity \( T \). The usual Euler scheme is suboptimal since it does not control the diffusion paths between two successive dates \( t_i \) and \( t_{i+1} \): the diffusion paths could have crossed the barriers and come back to the domain without being detected. It is also known that the error between \( C_{\text{Barrier}}(T, x) \) and \( \bar{C}_{\text{Barrier}}(T, x) \) (the barrier option price obtained by the Euler scheme) is of order \( \sqrt{T/n} \), as opposed to the order \( T/n \) for standard plain-vanilla options. (See [7]) Therefore, to improve the order of the error, many schemes for the Monte-Carlo method have been proposed. (See [16] for instance.)

One of the other tractable approaches for calculating \( C_{\text{Barrier}}(T, x) \) is to derive an analytical approximation. If we obtain a closed form approximation formula, then it is a powerful tool for evaluation of continuously monitoring barrier options because we do not have to rely on Monte-Carlo simulations anymore. However, from a mathematical viewpoint, deriving an approximation formula by applying stochastic analysis is not an easy task since the Malliavin calculus cannot be directly applied. It is due to the non-existence of the Malliavin derivative \( D_t \tau \) (see [4]) and to the fact that the minimum (maximum) process of the Brownian motion has only first-order differentiability in the Malliavin sense. Thus, neither approach in [11] nor in [19] can be applied directly to valuation of continuously monitoring barrier options while they are applicable to pricing discrete barrier options. (See [18] for the detail.)

This paper proposes a new general method for approximating the solution to the Cauchy-Dirichlet problem. Roughly speaking, our objective is to pricing barrier options when the dynamics of the underlying asset price is described by the following perturbed SDE:

\[
\begin{align*}
\frac{dX_t^\varepsilon}{dt} &= b(X_t^\varepsilon, \varepsilon)dt + \sigma(X_t^\varepsilon, \varepsilon)dB_t, \\
X_0^\varepsilon &= x,
\end{align*}
\tag{1.3}
\]

where \( \varepsilon \) is a small parameter, which will be defined precisely later in the paper. In this case, the barrier option price (1.1) is characterized as a solution of the Cauchy-Dirichlet problem:

\[
\begin{align*}
\partial_t u^\varepsilon(t, x) + \mathcal{L}^\varepsilon u^\varepsilon(t, x) &= 0, \quad (t, x) \in [0, T) \times (L, \infty), \\
u^\varepsilon(T, x) &= f(x), \quad x > L, \\
u^\varepsilon(t, L) &= 0, \quad t \in [0, T],
\end{align*}
\tag{1.4}
\]

where the differential operator \( \mathcal{L}^\varepsilon \) is determined by the diffusion coefficients \( b \) and \( \sigma \). Next, we introduce an asymptotic expansion formula:

\[
u^\varepsilon(t, x) = u^0(t, x) + \varepsilon v_1^0(t, x) + \cdots + \varepsilon^{n-1} v_{n-1}^0(t, x) + O(\varepsilon^n),
\tag{1.5}
\]

where \( O \) denotes the Landau symbol. The function \( u^0(t, x) \) is the solution of (1.4) with \( \varepsilon = 0 \): if \( b(x, 0) \) and \( \sigma(x, 0) \) have some simple forms such as constants (as in the Black-Scholes model),
we already know the closed form of \( u^0(t, x) \) and hence obtain the price. Then, we are able to get the approximate value for \( u^\varepsilon(t, x) \) through evaluation of \( v^0_1(t, x), \ldots, v^0_{n-1}(t, x) \). In fact, they are also characterized as the solution of a certain PDE with the Dirichlet condition. By formal asymptotic expansions, (1.5) and 
\[
\mathcal{L}^\varepsilon = \mathcal{L}^0 + \varepsilon \mathcal{L}^0_1 + \cdots + \varepsilon^{n-1} \mathcal{L}^0_{n-1} + \cdots,
\]
we can derive the PDEs corresponding to \( v^0_k(t, x) \) of the form:
\[
\begin{cases}
\frac{\partial}{\partial t} v^0_k(t, x) + \mathcal{L}^0 v^0_k(t, x) + g^0_k(t, x) = 0, & (t, x) \in [0, T) \times (L, \infty), \\
v^0_k(T, x) = 0, & x > L, \\
v^0_k(t, L) = 0, & t \in [0, T],
\end{cases}
\tag{1.6}
\]
where \( g^0_k(t, x) \) will be given explicitly later in this paper. Moreover, by applying the Feynman-Kac approach, we obtain their stochastic representations. We will justify the above argument in a mathematically rigorous manner in Section 2-4.

The theory of the Cauchy-Dirichlet problem for this kind of second order parabolic PDE is well understood for the case of bounded domains (see [5], [6] and [14] for instance). As for an unbounded domain case such as (1.4), [17] provides the existence and uniqueness results for a solution of the PDE and the Feynman-Kac type formula (cited as Theorem 1 below). However, some mathematical difficulty exists for applying the results of [17] to the PDE (1.6). More precisely, the function \( g^0_k(t, x) \) may be divergent at \( t = T \). (If \( g^0_k(t, x) \) is continuous on \([0, T] \times [L, \infty)\), the existence and uniqueness of (1.6) are guaranteed: see [5].) To overcome this difficulty, we generalize the Levi’s parametrix method (which is used to construct a classical solution of the PDE) in Theorem 2. Furthermore, we show another representation of \( v^0_k(t, x) \) by using the corresponding semi-group. We notice that such a form is convenient for evaluation of \( v^0_k(t, x) \) in concrete examples.

We also apply our method to pricing a barrier option in a stochastic volatility model:
\[
\begin{align*}
dS^\varepsilon_t &= (c - q)S_t^\varepsilon dt + \sigma^\varepsilon_t S_t dB^1_t, \quad S^\varepsilon_0 = S > 0, \\
d\sigma^\varepsilon_t &= \varepsilon \lambda(\theta - \sigma^\varepsilon_t)dt + \varepsilon \nu \sigma^\varepsilon_t (\rho dB^1_t + \sqrt{1 - \rho^2} dB^2_t), \quad \sigma^\varepsilon_0 = \sigma > 0,
\end{align*}
\]
where \( c, q > 0, \varepsilon \in [0, 1), \lambda, \theta, \nu > 0, \rho \in [-1, 1] \) and \( B = (B^1, B^2) \) is a two dimensional Brownian motion. Then, we obtain a new approximation formula: for \( x = \log S \),
\[
C^{SV, \varepsilon}_{\text{Barrier}}(T, e^x) = \mathbb{E} \left[ f(S^e_T)1_{\{\min_{t \leq T} S^e_t > L\}} \right] \\
\approx P_T^D f(x) + \varepsilon \int_0^T P_{T-t}^D \mathcal{L}^0_1 P_t^D f(x)dr,
\]
where \( S^e_t = \exp(X^\varepsilon_t), (P^D_t)_t \) is a semi-group defined in Section 2, \( f \) is a payoff function and \( f(x) = f(e^x) \). Here, \( P^D_t f(x) \) is regarded as the down-and-out barrier option price, \( C^{SV, \varepsilon}_{\text{Barrier}}(T, e^x) \) in the Black-Scholes model. Moreover, we confirm practical validity of our method through a numerical example given in Section 5. Note also that rigorously speaking, our example does not satisfy the assumptions introduced in Section 2. Thus, in Section 4 we generalize our main result with weaker version of the assumptions.
Finally, we remark that there exist the previous works on barrier option pricing such as [2], [3], [8], [9], which start with some specific models (the Black-Scholes model or some type of fast mean-reversion model) and derive approximation formulas for (discretely or continuously monitoring) barrier option prices. Our approach is to firstly develop a general asymptotic expansion scheme for the Cauchy-Dirichlet problem under multi-dimensional diffusion setting; then, as an application we provide a new approximation formula under a certain class of stochastic volatility model that can be widely applied in practice (e.g. in currency option markets).

The organization of this paper is as follows: After the next section prepares the assumptions and gives basic results, Section 3 shows an asymptotic expansion. Section 4 generalizes the result in Section 3 for an expansion of barrier option prices in a concrete example of Section 5. Section 5 shows numerical examples under log-normal SABR stochastic volatility model. Finally, Appendix provides the proofs of the results in the main text.

2 Preparation

Suppose first that the underlying asset price is described by the following perturbed SDE:

\[
\begin{align*}
\frac{dX_t^{\varepsilon,x}}{dt} & = b(X_t^{\varepsilon,x}, \varepsilon) dt + \sigma(X_t^{\varepsilon,x}, \varepsilon) dB_t, \\
X_0^{\varepsilon,x} & = x,
\end{align*}
\]

where \(\varepsilon\) is a small parameter, which will be defined precisely later in the paper. Let \(b : \mathbb{R}^d \times I \to \mathbb{R}^d\) and \(\sigma : \mathbb{R}^d \times I \to \mathbb{R}^d \otimes \mathbb{R}^m\) be Borel measurable functions \((d, m \in \mathbb{N})\) where \(I\) is an interval on \(\mathbb{R}\) including the origin 0 (for instance \(I = (-1, 1)\)). We consider the SDE (2.1) for any \(x \in \mathbb{R}^d\) and \(\varepsilon \in I\); in subsection 2.1 we will introduce the assumptions for existence and uniqueness of a weak solution of (2.1).

We are interested in evaluation of the following: for a small \(\varepsilon\),

\[
\begin{align*}
{u}^{\varepsilon}(t, x) & = E \left[ \exp \left( - \int_0^{T-t} c(X_r^{\varepsilon,x}, \varepsilon) dr \right) f(X_{T-t}^{\varepsilon,x}) 1_{\{\tau_D(X^{\varepsilon,x}) \geq T-t\}} \right], \quad (t, x) \in [0, T] \times \bar{D}
\end{align*}
\]

for Borel measurable functions \(f : \mathbb{R}^d \to \mathbb{R}\) and \(c : \mathbb{R}^d \times I \to \mathbb{R}\), a positive real number \(T > 0\) and a domain \(D \subset \mathbb{R}^d\); \(\bar{D} \subset \mathbb{R}^d\) is the closure of \(D\) and \(\tau_D(w)\), \(w \in C([0, T]; \mathbb{R}^d)\), stands for the first exit time from \(D\), that is

\[
\tau_D(w) = \inf\{t \in [0, T]; w(t) \notin D\}.
\]

Let us define a second order differential operator \(\mathcal{L}^{\varepsilon}\) by

\[
\mathcal{L}^{\varepsilon} = \frac{1}{2} \sum_{i,j=1}^{d} a^{ij}(x, \varepsilon) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{d} b^i(x, \varepsilon) \frac{\partial}{\partial x^i} - c(x, \varepsilon),
\]

where \(a^{ij} = \sum_{k=1}^{d} \sigma^{ik} \sigma^{jk}\). We consider the following Cauchy-Dirichlet problem for a PDE of
parabolic type:

\[
\begin{align*}
\frac{\partial}{\partial t} u^\varepsilon(t, x) + \mathcal{L}^\varepsilon u^\varepsilon(t, x) &= 0, \quad (t, x) \in [0, T) \times D, \\
u^\varepsilon(T, x) &= f(x), \quad x \in D, \\
u^\varepsilon(t, x) &= 0, \quad (t, x) \in [0, T] \times \partial D.
\end{align*}
\]

Our purpose is to present an asymptotic expansion such that

\[
u^\varepsilon(t, x) = u^0(t, x) + \varepsilon v^0_1(t, x) + \cdots + \varepsilon^{n-1} v^0_{n-1}(t, x) + O(\varepsilon^n), \quad \varepsilon \to 0.
\]

(2.4)

To study an asymptotic expansion of \(u^\varepsilon(t, x)\), we put the following assumptions.

### 2.1 Assumptions

[A] There is a positive constant \(A_1\) such that

\[
|\sigma^{ij}(x, \varepsilon)|^2 + |b^i(x, \varepsilon)|^2 \leq A_1(1 + |x|^2), \quad x \in \mathbb{R}^d, \quad \varepsilon \in I, \quad i, j = 1, \ldots, d.
\]

Moreover, for each \(\varepsilon \in I\) it holds that \(\sigma^{ij}(\cdot, \varepsilon), b^i(\cdot, \varepsilon) \in \mathcal{L}\) for \(i, j = 1, \ldots, d\), where \(\mathcal{L}\) is the set of locally Lipschitz continuous functions defined on \(\mathbb{R}^d\):

\[
\mathcal{L} = \{ f \in C([-1, 1] \times \mathbb{R}^d, \mathbb{R}) : \text{for any compact set } K \subset \mathbb{R}^d, \exists C_K > 0 \text{ such that } |f(t, x) - f(s, y)|^2 \leq C_K(|t - s| + |x - y|^2), (t, x, s, y) \in [0, T] \times K \}.
\]

**Remark 1.** Note that under [A], the existence and uniqueness of a solution of (2.1) are guaranteed on any filtered probability space equipped with a standard \(d\)-dimensional Brownian motion, and Corollary 2.5.12 in [10] and Lemma 3.2.6 in [15] imply

\[
E[\sup_{0 \leq r \leq t} |X^x_{\tau^r} - x|^2] \leq C_l t^{l-1}(1 + |x|^2), \quad (t, x) \in [0, T] \times \mathbb{R}^d, \quad l \in \mathbb{N}
\]

for some \(C_l > 0\) which depends only on \(l\) and \(A_1\). Moreover, \((X^x_r)_{r}\) has the strong Markov property.

[B] The function \(f(x)\) is continuous on \(\bar{D}\) and there are \(C_f > 0\) and \(m \in \mathbb{N}\) such that

\[
|f(x)| \leq C_f(1 + |x|^{2m}), \quad x \in \mathbb{R}^d.
\]

Moreover, \(f(x) = 0\) on \(\mathbb{R}^d \setminus D\).

**Remark 2.** The assumption [B] guarantees the continuity of a solution of (2.3) on the so called parabolic boundary \(\Sigma = \partial D \times [0, T) \cup \bar{D} \times \{T\}\), in addition to the continuity and polynomial growth of \(f\).

[C] There is a positive constant \(A_2\) such that \(c(x, \varepsilon) \geq 0\) for \(x \in \bar{D}, \varepsilon \in I\). Moreover, for each \(\varepsilon \in I\), it holds that \(c(\cdot, \varepsilon) \in \mathcal{L}\).

[D] The boundary \(\partial D\) has the outside strong sphere property, that is, for each \(x \in \partial D\) there is a closed ball \(E\) such that \(E \cap D = \emptyset\) and \(E \cap \bar{D} = \{x\}\).
Remark 3. The assumption [D] provides the regularity of each point in $\partial D$. (c.f.[5]) Also, [17] points out that [D] with the ellipticity of the matrix $(a^i_j(x, \varepsilon))_{ij}$ in [E] below gives
\[ P(\tau_D(X_t^x) = \tau_D(X_{t^x})) = 1. \]

[E] (i) The matrix $(a^i_j(x, \varepsilon))_{ij}$ is elliptic in the sense that for each $\varepsilon \in I$ and compact set $K \subset \mathbb{R}^d$ there is a positive number $\mu_{\varepsilon, K}$ such that $\sum_{i,j=1}^d a^i_j(x, \varepsilon)\xi^i\xi^j \geq \mu_{\varepsilon, K}|\xi|^2$ for any $x \in K$ and $\xi \in \mathbb{R}^d$.

(ii) For the case of $\varepsilon = 0$, we further assume
\[ \mu_0|\xi|^2 \leq \sum_{i,j=1}^d a^i_j(x, 0)\xi^i\xi^j \leq \mu_0^{-1}|\xi|^2, \quad x \in \bar{D}, \xi \in \mathbb{R}^d \]
for some $\mu_0 > 0$.

The following conditions [F]–[H] are necessary assumptions for the asymptotic expansion in Section 3. For details, see Remark 6.

[F] For each $i, j = 1, \ldots, d$ the functions $\sigma^i_j(x, 0), b^i(x, 0)$ and $c(x, 0)$ are bounded on $[0, T] \times \bar{D}$, and there exist constants $A_3 > 0$ and $\alpha \in (0, 1]$ such that
\[ |\sigma^i_j(x, 0) - \sigma^i_j(y, 0)| + |b^i(x, 0) - b^i(y, 0)| + |c(x, 0) - c(y, 0)| \leq A_3|x - y|^{\alpha}, \quad x, y \in \bar{D}. \]

[G] Let $n \in \mathbb{N}$. The functions $a^i_j(x, \varepsilon), b^i(x, \varepsilon)$ and $c(x, \varepsilon)$ are $n$-times continuously differentiable in $\varepsilon$. Furthermore, each of derivatives $\partial^k a^i_j / \partial \varepsilon^k, \partial^k b^i / \partial \varepsilon^k, \partial^k c / \partial \varepsilon^k$, $k = 1, \ldots, n - 1$, has a polynomial growth rate in $x \in \mathbb{R}^d$ uniformly in $\varepsilon \in I$.

Remark 4. By [G], we can define \( \mathcal{L}^0_k \), $k \in \mathbb{N}$ as
\[ \mathcal{L}^0_k = \frac{1}{k!} \left\{ \sum_{i,j=1}^d \frac{\partial^2}{\partial x^i \partial x^j} (x, 0) + \sum_{i=1}^d \frac{\partial^k b^i}{\partial \varepsilon^k} (x, 0) \right\}, \]
which will appear as the equation (3.4) for the asymptotic expansion in the next section.

To state the existence of the functions $u^0_k(t, x)$ in the asymptotic expansion (2.4), equivalently (3.1) below, we prepare the following set.

Definition 1. The set $\mathcal{G}^{m, \alpha, p}$ of $g \in C([0, T] \times \bar{D})$ is defined to satisfy the following conditions:

- There is some $M^g \in C([0, T]) \cap L^p([0, T], dt)$ such that
  \[ |g(t, x)| \leq M^g(t)(1 + |x|^{2m}), \quad t \in [0, T], \quad x, y \in \bar{D}. \]

- For any compact set $K \subset D$ there is some $\tilde{M}^{g, K} \in C([0, T]) \cap L^p([0, T], dt)$ such that
  \[ |g(t, x) - g(t, y)| \leq \tilde{M}^{g, K}(t)|x - y|^\alpha, \quad t \in [0, T], \quad x, y \in K. \]

Then, we put the next condition on $u^0$.

[H] $u^0 \in \mathcal{G}^{m, \alpha, p}$ for some $p > 1/\alpha$, where
\[ \mathcal{G}^{m, \alpha, p} = \left\{ g \in C^{1,2}([0, T] \times \bar{D}) \cap C([0, T] \times \bar{D}) ; \quad \frac{\partial g}{\partial x^i} \in \mathcal{H}^{m, 0, 2}, \quad \frac{\partial^2 g}{\partial x^i \partial x^j} \in \mathcal{H}^{m, \alpha, p}, \quad i, j = 1, \ldots, d \right\}. \]
2.2 Basic Results

In the first place, we have the following existence and uniqueness result due to Theorem 3.1 in [17].

**Theorem 1.** Assume [A]–[E]. For each \(\varepsilon \in I\), \(u^\varepsilon(t, x)\) is a (classical) solution of (2.3) and

\[
\sup_{(t, y) \in [0, T] \times D} |u^\varepsilon(t, x)|/(1 + |x|^{2m}) < \infty. \tag{2.7}
\]

Moreover, if \(w^\varepsilon(t, x)\) is also a solution of (2.3) satisfying the growth condition

\[
\sup_{(t, y) \in [0, T] \times D} |w^\varepsilon(t, x)|/(1 + |x|^{2m'}) < \infty
\]

for some \(m' \in \mathbb{N}\), then \(u^\varepsilon = w^\varepsilon\).

Next, we construct a semi-group corresponding to \((X^0_{t,x})_t\) and \(D\). Let \(C^0_b(\bar{D})\) be the set of bounded continuous functions \(f : \bar{D} \to \mathbb{R}\) such that \(f(x) = 0\) on \(\partial D\). Obviously, \(C^0_b(\bar{D})\) equipped with the sup-norm becomes a Banach space.

For \(t \in [0, T]\) and \(f \in C^0_b(\bar{D})\), we define \(P^D_t f : \bar{D} \to \mathbb{R}\) by

\[
P^D_t f(x) = E \left[ \exp \left( - \int_0^t c(X^0_{v,x}, 0) dv \right) f(X^0_{t,x}) 1_{\{v \in \tau_D(X^0,x) \geq t\}} \right], \tag{2.8}
\]

where \(c(x, 0)\) is non-negative. We notice that \(P^D_t f(x)\) is equal to \(u^0(T - t, x)\) with the payoff function \(f\). Then, we have the following:

**Proposition 1.** Under the assumptions [A]–[F], the mapping \(P^D_t : C^0_b(\bar{D}) \to C^0_b(\bar{D})\) is well-defined and \((P^D_t)_0 \leq t \leq T\) is a contraction semi-group.

**Proof.** Let \(f \in C^0_b(\bar{D})\). The relations \(P^D_0 f = f\), \(P^D_0 f|_{\partial D} = 0\) and \(\sup_{\bar{D}} |P^D_t f| \leq \sup_{\bar{D}} |f|\) are obvious. The continuity of \(P^D_t f\) is by Lemma 4.3 in [17]. The semi-group property is verified by a straightforward calculation. \(\Box\)

**Remark 5.** Note that \((P^D_t)_t\) also becomes a semi-group on the set \(C^0_b(\bar{D})\) of continuous functions \(f\), each of which has a polynomial growth rate and satisfies \(f(x) = 0\) on \(\partial D\).

3 Asymptotic Expansion

As noted in the previous section, our purpose is to present an asymptotic expansion:

\[
u^\varepsilon(t, x) = u^0(t, x) + \varepsilon v^0_1(t, x) + \cdots + \varepsilon^{n-1} v^0_{n-1}(t, x) + O(\varepsilon^n), \ \varepsilon \to 0. \tag{3.1}
\]

Here, \(v^0_k(t, x), k = 1, \ldots, n - 1\) are given as the solution of

\[
\begin{cases}
\frac{\partial}{\partial t} v^0_k(t, x) + D^0 v^0_k(t, x) + g^0_k(t, x) = 0, \quad (t, x) \in [0, T) \times D, \\
v^0_k(T, x) = 0, \quad x \in D, \\
v^0_k(t, x) = 0, \quad (t, x) \in [0, T] \times \partial D,
\end{cases} \tag{3.2}
\]
where \( g_k^0(t, x) \) is given inductively by

\[
g_k^0(t, x) = \mathcal{L}_k^0 u^0(t, x) + \sum_{l=1}^{k-1} \mathcal{L}_{k-l}^0 v_l^0(t, x),
\]

(3.3)

and

\[
\mathcal{L}_k^0 = \frac{1}{k!} \left\{ \frac{1}{2} \sum_{i,j=1}^{d} \partial^k u^{ij}(x,0) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{l=1}^{d} \partial^k v_l(x,0) \frac{\partial}{\partial x^l} - \partial^k v_l(x,0) \right\}.
\]

(3.4)

The next theorem shows the existence and the uniqueness for the PDE of the type (3.2). The proof is given in Section 6.1 of Appendix.

**Theorem 2.** Assume \([A]–[G]\). Let \( g \in \mathcal{H}^{n,\alpha,p} \) for some \( p > 1/\alpha \). Then, the following PDE

\[
\begin{cases}
\frac{\partial}{\partial t} v(t, x) + \mathcal{L}^0 v(t, x) + g(t, x) = 0, & (t, x) \in [0,T] \times D, \\
v(T, x) = 0, & x \in D, \\
v(t, x) = 0, & (t, x) \in [0,T] \times \partial D
\end{cases}
\]

(3.5)

has a classical solution \( v \) such that

\[
|v(t, x)| \leq C(1 + |x|^{2m})
\]

(3.6)

for some \( C > 0 \) which depends only on \( a(\cdot,0), b(\cdot,0), c(\cdot,0), D \) and \( M^p \). Moreover, if \( w \) is another classical solution of (3.5) which satisfies \( |w(t, x)| \leq C' \exp(\beta|x|^2), (t, x) \in [0,T] \times \bar{D} \), for some \( C', \beta > 0 \), then \( v = w \).

**Remark 6.** It is easy to see that the assumptions \([F]–[H]\) imply \( g_1^0 \in \mathcal{H}^{m_1,\alpha,p} \) for some \( m_1 \in \mathbb{N} \). Therefore (3.2) with \( k = 1 \) has a unique classical solution \( v_1^0 \) under \([A]–[H]\). Similarly, if \( v_1^0, \ldots, v_0^0 \) exist and are subject to \( \mathcal{G}^{m_k,\alpha,p} \) for some \( m_k \in \mathbb{N} \), then the unique classical solution \( v_{k+1}^0 \) of (3.2) exists.

This remark leads to the following as our final assumption.

**[I]** It holds that \( v_k^0 \in \mathcal{G}^{m_n,\alpha,p}, k = 1, \ldots, n-1 \) for some \( m_n \in \mathbb{N} \).

Now, we are able to state our main result whose proof is given in Section 6.2 of Appendix.

**Theorem 3.** Assume \([A]–[I]\). There are positive constants \( C_n \) and \( \tilde{m}_n \) which are independent of \( \varepsilon \) such that

\[
|u^\varepsilon(t, x) - (u^0(t, x) + \sum_{k=1}^{n-1} \varepsilon^k v_k^0(t, x))| \leq C_n(1 + |x|^{2\tilde{m}_n})\varepsilon^n, \quad (t, x) \in [0,T] \times \bar{D}.
\]

**Remark 7.** We remark that \( v_k^0(t, x) \) has the stochastic representation:

\[
v_k^0(t, x) = E \left[ \int_0^{(T-t)\wedge \tau_D(X^0,x)} \exp \left( -\int_0^r c(X^0_{e}, x, 0)dv \right) g_k^0(t + r, X_r^{0,x})dr \right]
\]

(3.7)

for \( k = 1, \ldots, n-1 \) under \([I]\). The proof is almost the same as Theorem 5.1.9 in [13].
Next, we provide another representation of (3.7) based on the semi-group introduced in (2.8). See Section 6.3 in Appendix for the proof.

**Proposition 2.** Under Assumptions \([A]–[I]\), for each \(k = 1, \ldots, n - 1\)
\[
v_k^0(T - t, x) = \sum_{l=1}^{k} \sum_{(\beta^l)_{l-1} \in \mathbb{N}^l} \sum_{\gamma^l = k} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{l-1}} P_{t - t_1} \tilde{Z}_{\beta^l} P_{t_1 - t_2} \tilde{Z}_{\beta^2} \cdots P_{t_{l-1} - t_l} \tilde{Z}_{\beta^l} P_{t_l} f(x) dt_1 \cdots dt_l.
\]

(3.8)

4 Generalization

There are several cases in practice that our assumptions in the previous sections are not satisfied: for instance, see asymptotic expansions for barrier option prices in Section 5. Hence, we need to weaken the assumptions so as to deal with those cases.

Particularly, let \(d' \in \{1, \ldots, d\}\), and we regard \(X^\varepsilon,x,i_t\) as logarithm of the underlying asset prices for \(i \leq d'\), and as parameter processes (e.g. a stochastic volatility and a stochastic interest rate) for \(i > d'\). Also, we assume \(I \subset [0, \infty)\) in this section for a technical reason introduced later.

Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_t, P)\) be a filtered space equipped with a standard Brownian motion \((B_t)_t\).

Set
\[
\hat{b}^i(y, \varepsilon) = \begin{cases} 
  y^i \left\{ b^i(\pi(y), \varepsilon) + \frac{1}{2} \sum_{j=1}^d (\sigma^{ij}(\pi(y), \varepsilon))^2 \right\}, & i \leq d', \\
  b^i(\pi(y), \varepsilon), & i > d',
\end{cases}
\]
\[
\hat{\sigma}^{ij}(y, \varepsilon) = \begin{cases} 
  y^i \sigma^{ij}(\pi(y), \varepsilon), & i \leq d', \\
  \sigma^{ij}(\pi(y), \varepsilon), & i > d',
\end{cases}
\]
where \(\pi(y) = (\log y^1, \ldots, \log y^{d'}, y^{d'+1}, \ldots, y^d) \in \mathbb{R}^d\).

In the following subsections we present two cases of generalization.

4.1 Expansion for Logarithmic Processes

In the first place, \([A'] - [C']\) below with \([D] - [E]\) in Section 2.1 are necessary for the existence and uniqueness of the viscosity solution of the relevant PDEs (2.3) and (4.5) : in particular, \([A']-\ [B']\) are for the existence in Theorem 4, and \([A']\ - [C']\) with \([D] - [E]\) are for the uniqueness in Theorem 5 below.

Next, \([A'] - [C']\) and \([H'] - [J']\) below with \([D] - [G]\) in Section 2.1 are necessary for the first generalization (Theorem 6) of the asymptotic expansion.

\([A']\) For each \(\varepsilon \in I\) it holds that \(\sigma^{ij}(\cdot, \varepsilon), \hat{b}^i(\cdot, \varepsilon) \in \mathcal{L}\), and that \(\hat{\sigma}^{ij}(\cdot, \varepsilon), \hat{b}^i(\cdot, \varepsilon)\) and \(c(\pi(\cdot), \varepsilon)\) are also in \(\mathcal{L}\). Here, \(\mathcal{L}\) is defined in the assumption \([A]\) of Section 2.1, that is the set of locally Lipschitz continuous functions defined on \(\mathbb{R}^d\).
Moreover, there exists a solution \((X^ε,t)\) of SDE (2.1) and for any \(m > 0\) there are \(m', C > 0\) such that
\[
\sup_{0 \leq r \leq t} E[|Y^ε,υ|^2m] \leq C t^{m-1} (1 + |υ|^{2m'}), \quad (t,υ) \in [0,T] \times [0,\infty)^d \times \mathbb{R}^{d-d'}, \quad \epsilon \in I, \quad (4.3)
\]
where
\[
Y^ε,υ_t = \iota(X^ε,\pi(υ)), \quad (4.4)
\]
\(\iota(x) = (e^{x^1}, \ldots, e^{x^{d'}}, x^{d'+1}, \ldots, x^d) \in \mathbb{R}^d\).

**Remark 8.** Note that Ito’s formula implies that \((Y^ε,υ)_t\) is a solution of
\[
\begin{cases}
\frac{dY^ε,υ_t}{dt} = \hat{b}(Y^ε,υ_t, \epsilon) dt + \hat{σ}(Y^ε,υ_t, \epsilon) dB_t, \\
Y^ε,υ_0 = υ.
\end{cases}
\]

\([B']\) The function \(f(x)\) is represented by the continuous function \(\hat{f} : \mathbb{R}^d \rightarrow \mathbb{R}\) as \(f(x) = \hat{f}(\iota(x))\). There exists \(C_\hat{f} > 0\) such that \(\|\hat{f}(υ)\| \leq C_\hat{f}(1 + |υ|^{2m}), \quad υ \in \mathbb{R}^d\). Moreover, \(f(x) = 0\) on \(\mathbb{R}^d \setminus D\).

\([C']\) In addition to the condition \([C]\) (There is a positive constant \(A_2\) such that \(c(x, \epsilon) \geq 0\) for \(x \in \bar{D}, \epsilon \in I\). Moreover, for each \(\epsilon \in I\), it holds that \(c(\cdot, \epsilon) \in \mathcal{L}\)), there is a constant \(A^\epsilon_2 > 0\) such that \(c(t, x, \epsilon) \leq A^\epsilon_2(1 + |x|^{2m})\).

We note that Theorem 3.1 in [17] no longer works under \([A']–[B']\). However, in the viscosity sense we are still able to characterize \(u^\epsilon(t, x)\) as the solution of (2.3):
\[
\begin{cases}
\frac{∂}{∂t} u^\epsilon(t, x) + \mathcal{L}^\epsilon u^\epsilon(t, x) = 0, \quad (t, x) \in [0,T) \times D, \\
u^\epsilon(T, x) = f(x), \quad x \in D, \\
u^\epsilon(t, x) = 0, \quad (t, x) \in [0,T] \times \partial D.
\end{cases}
\]
To see this, set
\[
\hat{L}^\epsilon = \frac{1}{2} \sum_{i,j=1}^{d} \hat{a}_{ij}(υ, \epsilon) \frac{∂^2}{∂y^i∂y^j} + \sum_{i=1}^{d} \hat{b}^i(υ, \epsilon) \frac{∂}{∂y^i} - c(\pi(υ), \epsilon),
\]
where \(\hat{a}_{ij} = \sum_{k=1}^{d} \hat{a}^{ik}\hat{a}^{jk}\). Moreover, define
\[
\hat{D} = \{ y \in \mathbb{R}^d ; \ y^i > 0, \ i = 1, \ldots, d' \ \text{and} \ \pi(y) \in D \}
\]
and \(\hat{u}^\epsilon(t, y) = u^\epsilon(t, \pi(y)) \ ((t, y) \in [0,T] \times \hat{D}), \ 0 \ ((t, y) \in [0,T] \times \partial \hat{D})\). Then, we obtain the following existence result.
Theorem 4. Assume \([A']- [B']\). Then, \(u^e(t, x)\) is a viscosity solution of (2.3). Moreover, \(\hat{u}^e(t, y)\) is a viscosity solution of

\[
\begin{aligned}
\frac{\partial}{\partial t} \hat{u}^e(t, y) - \mathcal{L}^e \hat{u}^e(t, y) &= 0, \quad (t, y) \in [0, T) \times \hat{D} , \\
\hat{u}^e(T, y) &= \hat{f}(y), \quad x \in \hat{D}, \\
\hat{u}^e(t, y) &= 0, \quad (t, y) \in [0, T) \times \partial \hat{D}.
\end{aligned}
\] (4.5)

satisfying

\[
\sup_{(t, y) \in [0, T) \times \hat{D}} \frac{|\hat{u}^e(t, y)|}{(1 + |y|^{2m'})} < \infty.
\] (4.6)

Proof. The latter assertion is by the similar argument to the proof of Proposition 8. Then, the simple calculation gives the former assertion. 

Also, applying Theorem 8.2 in [1] and Theorem 7.7.2 in [15] to (4.5), we have the following uniqueness theorem.

Theorem 5. Assume \([A']-[C']\) and \([D]-[E]\). If \(\hat{u}^e(t, y)\) is a viscosity solution of (4.5) satisfying the growth condition (4.6), then \(\hat{u}^e = \hat{w}^e\).

For the first generalization of the asymptotic expansion stated as Theorem 6 below, we need to modify the assumptions \([H]\) and \([I]\) in the previous sections.

In the first place, in order to state the existence of a function \(v_k^0(t, x)\), we prepare the following set which slightly modifies \(G^{m,\alpha,p}\) in Definition 1. Moreover, we define \(\mathcal{G}^{m,\alpha,p}\) similarly to \(G^{m,\alpha,p}\), replacing \(H^{m,0,2}\) and \(H^{m,\alpha,p}\) in the definition with \(H^{m,0,2}\) and \(\hat{H}^{m,\alpha,p}\), respectively.

\([H']\) The condition \([H]\) holds replacing \(G^{m,\alpha,p}\) with \(\mathcal{G}^{m,\alpha,p}\). That is, \(u^0 \in \mathcal{G}^{m,\alpha,p}\) for some \(p > 1/\alpha\), where

\[
\mathcal{G}^{m,\alpha,p} = \left\{ g \in C^{1,2}((0, T) \times D) \cap C([0, T] \times \hat{D}); \right. \\
\left. \frac{\partial g}{\partial x^i} \in H^{m,0,2}, \quad \frac{\partial^2 g}{\partial x^i \partial x^j} \in H^{m,\alpha,p}, \quad i, j = 1, \ldots, d \right\},
\]

and the set \(\hat{H}^{m,\alpha,p}\) of \(g \in C([0, T] \times \hat{D})\) is given by the following:

Definition 2. The set \(\hat{H}^{m,\alpha,p}\) of \(g \in C([0, T] \times \hat{D})\) is defined to satisfy the following conditions:

- There is some \(\bar{M}^g \in C([0, T]) \cap L^p([0, T), dt)\) such that

\[
|g(t, x)| \leq \bar{M}^g(t)(1 + |x|^2m), \quad t \in [0, T), \quad x, y \in \hat{D}.
\] (4.7)

- For any compact set \(K \subset D\) there is some \(\bar{M}^{g,K} \in C([0, T]) \cap L^p([0, T), dt)\) such that

\[
|g(t, x) - g(t, y)| \leq \bar{M}^{g,K}(t)|x - y|^\alpha, \quad t \in [0, T), \quad x, y \in K.
\]

Accordingly, the condition \([I]\) is replaced by the following:
Theorem 6. Assume \([A'] - [C']\), \([D'] - [G]\) and \([H'] - [I']\). Then, there are positive constants \(C_n\) and \(\hat{m}_n\) which are independent of \(\varepsilon\) such that

\[
\left| u^\varepsilon(t, x) - (u^0(t, x) + \sum_{k=1}^{n-1} \varepsilon^k v_k^0(t, x)) \right| \leq C_n(1 + |u(x)|^{2\hat{m}_n})\varepsilon^n, \quad (t, x) \in [0, T] \times \bar{D}.
\]

4.2 Expansion for Logarithmic Processes under Partially Elliptic when \(\varepsilon = 0\)

Let us move to another generalization, which is directly relevant with barrier option pricing in Section 5. \([A'] - [C']\) in the previous subsection, \([D'] - [F']\) and \([H''] - [I'']\) below with \([G]\) in Section 2.1 are necessary for the second generalization of the asymptotic expansion: particularly, \([A'] - [F']\) with \([G]\) are for Theorem 7; \([A'] - [F']\) and \([H''] - [I'']\) with \([G]\) are for Theorem 8.

[D'] The domain \(D\) is given as \(D = U \times \mathbb{R}^{d-d'}\), where \(U\) is a domain in \(\mathbb{R}^{d'}\) whose boundary \(\partial U\) satisfies the outside strong sphere property.

[E'] The condition \([E]\) holds for \(\varepsilon \neq 0\), that is, the matrix \((a^{ij}(x, \varepsilon))_{ij}\) is elliptic in the sense that for each \(\varepsilon \in I\) and compact set \(K \subset \mathbb{R}^d\) there is a positive number \(\mu_{\varepsilon, K}\) such that

\[
\sum_{i,j=1}^{d} a^{ij}(x, \varepsilon)\xi^i \xi^j \geq \mu_{\varepsilon, K} |\xi|^2 \text{ for any } x \in K \text{ and } \xi \in \mathbb{R}^d.
\]

Moreover, \(\sigma^{ij}(x, 0) = b^i(x, 0) = 0\) for \(i = d' + 1, \ldots, d, j = 1, \ldots, d\) and for each compact set \(K \subset D\) there is a positive constant \(\mu_{0,K}\) such that \(\mu_{0,K} |\xi|^2 \leq \sum_{i,j=1}^{d} a^{ij}(x, 0) \leq \mu_{0,K}^{-1} |\xi|^2\) for \(x \in K\) and \(\xi \in \mathbb{R}^d\).

[F'] For each \(y \in \mathbb{R}^{d-d'}\), the inequality

\[
\max_{i,j} \sup_{x \in U} \{ |\sigma^{ij}((x, y), 0)| + |b^i((x, y), 0)| + |c((x, y), 0)| \} \leq \infty
\]

holds, where \((x, y) = (x^1, \ldots, x^{d'}, y^1, \ldots, y^{d-d'}) \in \mathbb{R}^d\) and there exist \(A_3(y) > 0\) and \(\alpha \in (0, 1]\) such that

\[
|\sigma^{ij}((x, y), 0) - \sigma^{ij}((x', y), 0)| + |b^i((x, y), 0) - b^i((x', y), 0)| + |c((x, y), 0) - c((x', y), 0)| \\
\leq A_3(y)|x - x'|^\alpha, \quad (t, x), (s, x') \in [0, T] \times \bar{U}, y \in \mathbb{R}^{d-d'}.
\]
\[H^*\] The condition \([H]\) holds replacing \(G^{m,\alpha,p}\) with \(\overline{G}^{m,\alpha,p}: u^0 \in \overline{G}^{m,\alpha,p}\) for some \(p > 1/\alpha\), where

\[
\overline{G}^{m,\alpha,p} = \left\{ g \in C^{1,2}([0,T) \times D) \cap C([0,T] \times \bar{D}) : \right. \\
\left. \frac{\partial g}{\partial x}(\cdot, \cdot, y) \in \tilde{H}^{m,0,2}_{U}, \quad \frac{\partial^2 g}{\partial x^i \partial x^j}(\cdot, \cdot, y) \in \tilde{H}^{m,\alpha,p}_{U}, \right. \\
\left. i, j = 1, \ldots, d, \quad y \in \mathbb{R}^{d-d'} \right\}
\]

Here, \(h(\cdot, \cdot, y)\) denotes the function \([0,T) \times \bar{U} \ni (t, x) \mapsto h(t, (x, y)) \in \mathbb{R}\) for \(h = \partial g/\partial x, \partial^2 g/\partial x^i \partial x^j\). Also, \(\tilde{H}^{m,\alpha,p}_{U}\) is the same as \(\tilde{H}^{m,\alpha,p}\) replacing \(D \subset \mathbb{R}^d\) in the definition with \(U \subset \mathbb{R}^{d'}\) as follows:

**Definition 3.** The set \(\tilde{H}^{m,\alpha,p}_{U}\) of \(g \in C([0,T) \times \bar{U})\) is defined to satisfy the following conditions:

- **There is some** \(M^g \in C([0,T)) \cap L^p([0,T), dt)\) such that
  \[
  |g(t, x)| \leq M^g(t)(1 + |x|^{2m}), \quad t \in [0,T), \quad x, y \in \bar{U}. \tag{4.8}
  \]
- **For any compact set** \(K \subset U\) there is some \(\tilde{M}^g_{K} \in C([0,T)) \cap L^p([0,T), dt)\) such that
  \[
  |g(t, x) - g(t, y)| \leq \tilde{M}^g_{K}(t)|x - y|^\alpha, \quad t \in [0,T), \quad x, y \in K.
  \]

**[I*] The condition \([I]\) holds replacing \(G^{m,\alpha,p}\) with \(\overline{G}^{m,\alpha,p}: \right\}

It holds that \(v^0_k \in \overline{G}^{m,\alpha,p}, k = 1, \ldots, n - 1\) for some \(m_n \in \mathbb{N}\).

Finally, using assumptions \([A']-[F']\) and \([H^*]-[I^*]\) with \([G]\), we obtain the next two theorems, which will be applied to the asymptotic expansion of barrier option prices in the following section. First, Theorem 2 in Section 3 implies the next theorem:

**Theorem 7.** Assume \([A']-[F']\) and \([G]\). Let \(g \in H^{m,\alpha,p}_{U}\) for some \(p > 1/\alpha\). Then for each fixed \(y = (x')_{1=d'+1}\), the following PDE

\[
\begin{cases}
\frac{\partial}{\partial t} v(t, x) + \mathcal{L}^0_y v(t, x) + g(t, x; y) = 0, \quad (t, x) \in [0,T) \times U, \\
v(T, x) = 0, \quad x \in U, \\
v(t, x) = 0, \quad (t, x) \in [0,T) \times \partial U
\end{cases}
\tag{4.9}
\]

has a classical solution \(v\) satisfying

\[
|v(t, x)| \leq C(1 + |x|^{2m})
\tag{4.10}
\]

for some \(C > 0\), where

\[
\mathcal{L}^0_y = \frac{1}{2} \sum_{i,j=1}^{d'} a^ij((x, y), 0) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{d'} b^i((x, y), 0) \frac{\partial}{\partial x^i} - c((x, y), 0), \quad x \in \mathbb{R}^{d'}.
\]

Moreover, if \(w\) is another classical solution of (4.9) which satisfies \(|w(t, x)| \leq C' \exp(\beta |x|^2)\), \((t, x) \in [0,T) \times \bar{U}\), for some \(C', \beta > 0\), then \(v = w\).

Applying Theorem 7 above, we obtain the following expansion theorem similarly to Theorem 6.

**Theorem 8.** Assume \([A']-[F'], [G]\) and \([H^*]-[I^*]\). Then, the same assertion of Theorem 6 holds: There are positive constants \(C_n\) and \(\bar{m}_n\) which are independent of \(\varepsilon\) such that

\[
\left| u^\varepsilon(t, x) - (u^0(t, x) + \sum_{k=1}^{n-1} \varepsilon^k v^0_k(t, x)) \right| \leq C_n(1 + |x|^{2\bar{m}_n})\varepsilon^n, \quad (t, x) \in [0,T] \times \bar{D}.
\]
5 Application to Barrier Option Pricing under Stochastic Volatility

Consider the following stochastic volatility model:

\[
\begin{align*}
    dS_t^e &= (c - q)S_t^e dt + \sigma_t^e S_t^e dB_t^1, \quad S_0^e = S, \\
    d\sigma_t^e &= \varepsilon \lambda (\theta - \sigma_t^e) dt + \varepsilon \nu \sigma_t^e (\rho dB_t^1 + \sqrt{1 - \rho^2} dB_t^2), \quad \sigma_0^e = \sigma,
\end{align*}
\]  
(5.1)

where \( c, q > 0, \varepsilon \in [0, 1], \lambda, \theta, \nu > 0, \rho \in [-1, 1] \) and \( B = (B^1, B^2) \) is a two dimensional Brownian motion. Here \( c \) and \( q \) represent a domestic interest rate and a foreign interest rate, respectively when we consider the currency options. Clearly, applying Itô’s formula, we have its logarithmic process:

\[
\begin{align*}
    dX_t^e &= (c - q - \frac{1}{2}(\sigma_t^e)^2) dt + \sigma_t^e dB_t^1, \quad X_0^e = \log S, \\
    d\sigma_t^e &= \varepsilon \lambda (\theta - \sigma_t^e) dt + \varepsilon \nu \sigma_t^e (\rho dB_t^1 + \sqrt{1 - \rho^2} dB_t^2), \quad \sigma_0^e = \sigma.
\end{align*}
\]  
(5.2)

Also, its generator is expressed as

\[
\mathcal{L}^e = \left( c - q - \frac{1}{2} \sigma_t^2 \right) \frac{\partial}{\partial t} + \frac{1}{2} \sigma_t^2 \frac{\partial^2}{\partial x^2} + \varepsilon \rho \sigma_t \frac{\partial}{\partial x} + \varepsilon \lambda (\theta - \sigma_t) \frac{\partial}{\partial \sigma} + \varepsilon^2 \frac{1}{2} \nu^2 \sigma_t^2 \frac{\partial^2}{\partial \sigma^2}.
\]  
(5.3)

In this case, \( \mathcal{L}_1^0 \) defined by (3.4) is given as

\[
\mathcal{L}_1^0 = \rho \sigma^2 \frac{\partial^2}{\partial x \partial \sigma} + \lambda (\theta - \sigma) \frac{\partial}{\partial \sigma}.
\]  
(5.4)

We will apply Theorem 8 to (5.1) with \( d = 2 \) and \( d' = 1 \) and give an approximation formula for a barrier option of which value is given under a risk-neutral probability measure as

\[
C_{\text{Barrier}}^{SV, \varepsilon} (T - t, e^x) = \mathbb{E} \left[ e^{-\varepsilon(T-t)} f(S_{T-t}^e) \right| \tau_{L, \infty} > T-t \] 
(5.5)

where \( f \) stands for a payoff function and \( L(< S) \) is a barrier price.

\[
u^x(t, x) = C_{\text{Barrier}}^{SV, \varepsilon} (T - t, e^x) \]

satisfies the following PDE:

\[
\begin{align*}
    &\left\{ \left( \frac{\partial}{\partial t} + \mathcal{L}^e - c \right) u^x(t, x) = 0, \quad (t, x) \in (0, T] \times D, \\
    &u^x(T, x) = \bar{f}(x), \quad x \in \bar{D}, \\
    &u^x(t, l) = 0, \quad t \in [0, T],
\end{align*}
\]  
(5.5)

where \( \bar{f}(x) = \max\{e^x - K, 0\}, D = (l, \infty) \) and \( l = \log L \). We obtain the 0-th order \( u^0 \) as

\[
u^0(t, x) = \rho^D_{T-t} \bar{f}(x) = \mathbb{E}[e^{-c(T-t)} \bar{f}(X^D_{T-t}) 1_{\tau_D(X^D, e^x) > T-t}].
\]  
(5.6)

Set \( \alpha = c - q \). Then \( \rho^D_{T-t} \bar{f}(x) = C_{\text{Barrier}}^{BS} (T - t, e^x, \alpha, \sigma, L) \) is the price of the down-and-out barrier call option under the Black-Scholes model:

\[
C_{\text{Barrier}}^{BS} (T - t, e^x, \alpha, \sigma, L) = C^{BS} (T - t, e^x, \alpha, \sigma) - \left( \frac{e^x}{L} \right)^{1 - \frac{2\alpha}{L}} C^{BS} \left( T - t, \frac{L^2}{e^x}, \alpha, \sigma \right).
\]  
(5.7)
Here, we recall that the price of the plain vanilla option under the Black-Scholes model is given as

\[ C^{BS}(T - t, x, \sigma) = e^{-q(T-t)}e^x N(d_1(T - t, x, \alpha)) - e^{-c(T-t)} KN(d_2(T - t, x, \alpha)), \quad (5.8) \]

where

\[
\begin{align*}
    d_1(t, x, \alpha) &= \frac{x - \log K + \alpha t}{\sigma \sqrt{t}} + \frac{1}{2}\sigma \sqrt{t}, \\
    d_2(t, x, \alpha) &= d_1(t, x, \alpha) - \sigma \sqrt{t} \\
    N(x) &= \int_{-\infty}^{x} n(y)dy, \\
    n(y) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}.
\end{align*}
\]

Note also that

\[ P(\tau_D(X^{0,x}) \geq t|X^{0,x}_t) = 1 - \exp \left( -\frac{2(x-l)(X^{0,x}_t-l)}{\sigma^2 t} \right) \text{ on } \{X^{0,x}_t > l\}. \]

Therefore, for \( g \in C^0(\bar{D}) \) we have

\[ P^D_t g(x) = E[P(\tau_D(X^{0,x}) \geq t|X^{0,x}_t)g(X^{0,x}_t)1_{\{X^{0,x}_t > l\}}] = \int_{t}^{\infty} g(y)p(t, x, y)dy, \quad (5.9) \]

where

\[ p(t, x, y) = \frac{1}{\sqrt{2\pi\sigma^2 t}} (1 - e^{-\frac{2(x-l)(y-l)}{\sigma^2 t}}) e^{-\frac{(y-x-\mu t)^2}{2\sigma^2 t}}. \quad (5.10) \]

Then, we show the following main result in this section.

**Theorem 9.** We obtain an approximation formula for the down-and-out barrier call option under the stochastic volatility model (5.1):

\[ C^{SV,\epsilon}_{\text{Barrier}}(T, e^x) = C^{BS}_{\text{Barrier}}(T, e^x, \alpha, \sigma, L) + \epsilon v^0_1(0, x) + O(\epsilon^2), \quad (5.11) \]

where

\[ v^0_1(0, x) = e^{-cT} \int_0^T \int_{l}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 s}} (1 - e^{-\frac{2(x-l)(y-l)}{\sigma^2 s}}) e^{-\frac{(y-x-\mu s)^2}{2\sigma^2 s}} \vartheta(s, y)dyds, \quad (5.12) \]

\[
\begin{align*}
\vartheta(t, x) &= e^{\alpha(T-t)} \rho \nu \sigma e^x n(d_1(T - t, x, \alpha))(-d_2(T - t, x, \alpha)) \\
&+ 2e^{\alpha(T-t)} \rho \nu \alpha \left( \frac{e^x}{L} \right)^{-\frac{2\alpha}{\sigma^2}} \ln(c_1(T - t, x, \alpha)) \sqrt{T-t} \\
&- e^{\alpha(T-t)} \rho \nu \sigma \left( \frac{e^x}{L} \right)^{-\frac{2\alpha}{\sigma^2}} \ln(c_1(T - t, x, \alpha)) c_1(T - t, x, \alpha)
\end{align*}
\]
\[-e^{c(T-t)\rho v}4\frac{4\alpha}{\sigma}(\frac{e^x}{L})^{1-\frac{2\alpha}{\sigma^2}}\times \left\{C_{BS}\left(T-t, \frac{L^2}{e^x}, \alpha, \sigma\right)\left\{1 + (x - \log L)\left(1 - \frac{2\alpha}{\sigma^2}\right)\right\} - (x - \log L)e^{-q(T-t)}\frac{L^2}{e^x}N(c_1(T-t, x, \alpha))\right\} + \\
\lambda(\theta - \sigma)e^{\alpha(T-t)}e^x n(d_1(T-t, x, \alpha))\sqrt{T - t}\]
\[-\lambda(\theta - \sigma)\left(\frac{e^x}{L}\right)^{-\frac{2\alpha}{\sigma^2}}e^{\alpha(T-t)}Ln(c_1(T-t, x, \alpha))\sqrt{T - t}\]
\[-e^{c(T-t)\lambda(\theta - \sigma)}\frac{4\alpha}{\sigma^3}\left(\log \frac{e^x}{L}\right)\left(\frac{e^x}{L}\right)^{1-\frac{2\alpha}{\sigma^2}}C_{BS}\left(T-t, \frac{L^2}{e^x}, \alpha, \sigma\right),\]

and
\[c_1(t, x, \alpha) = \frac{2l - x - \log K + \alpha t}{\sigma \sqrt{l}} + \frac{1}{2} \sigma \sqrt{l}.
\]

**Proof.** Firstly, note that when \(k = 1\) in Proposition 2, we have
\[v^0_1(T-t, x) = \int_0^T P^D_{t-r} \bar{Z}^0_1 P^D_r f(x) dr.
\]
Thus, by Theorem 8, we see the expansion
\[C_{SV, \alpha}^{\text{Barrier}}(T-t, e^x) = C_{\text{Barrier}}^{\text{BS}}(T-t, e^x, \alpha, \sigma, L) + \varepsilon \int_0^{T-t} P^D_s \bar{Z}^0_1 P^D_{T-t-s} f(x) ds + O(\varepsilon^2).\]

The first-order approximation term \(v^0_1(t, x)\) is given by
\[v^0_1(t, x) = \int_0^{T-t} E[e^{-cs} \bar{Z}^0_1 P^D_{T-t-s} f(X_{T-t-s})] ds
\]
\[= \int_0^{T-t} E[e^{-cs} \bar{Z}^0_1 e^{-c(T-t-s)} P^D_{T-t-s} f(X_{T-t-s})] ds
\]
\[= e^{-c(T-t)} \int_0^{T-t} P^D_s \bar{Z}^0_1 P^D_{T-t-s} f(x) ds,
\]
where \(P^D_t\) is defined by (5.9) with the density (5.10), that is,
\[P^D_t f(x) = \int_1^\infty \frac{1}{\sqrt{2\pi} \sigma^2 s} (1 - e^{-\frac{2(x-t)(y-t)}{\sigma^2 s^2}}) e^{-\frac{(y-x-x_1^2)}{2\sigma^2 s^2}} f(y) dy.
\]
Define \(\vartheta(t, x)\) as
\[\vartheta(t, x) = \bar{Z}^0_1 P^D_{T-t} f(e^x
\]
\[= e^{c(T-t)\rho v}4\frac{4\alpha}{\sigma}\frac{\partial^2}{\partial x \partial \sigma}C_{\text{Barrier}}^{\text{BS}}(T-t, e^x, \alpha, \sigma, L) + e^{c(T-t)\lambda(\theta - \sigma)}\frac{\partial}{\partial \sigma}C_{\text{Barrier}}^{\text{BS}}(T-t, e^x, \alpha, \sigma, L).
\]
A straightforward calculation shows that the above function agrees with the right-hand side of (5.13). Then we get the assertion. \(\blacksquare\)
Remark that through numerical integrations with respect to time $s$ and space $y$ in (5.12), we easily obtain the first order approximation of the down-and-out option prices.

Next, as a special case of (5.1) we consider the following stochastic volatility model with no drifts:

$$
\begin{align*}
    dS_t^\varepsilon &= \sigma_t^\varepsilon S_t^\varepsilon dB^1_t, \quad S_0^\varepsilon = S > 0, \\
    d\sigma_t^\varepsilon &= \varepsilon \nu \sigma_t^\varepsilon (\rho dB_t^1 + \sqrt{1-\rho^2} dB_t^2), \quad \sigma_0^\varepsilon = \sigma > 0.
\end{align*}
$$

where $\varepsilon \in [0, 1), \rho \in [-1, 1]$ and $B = (B^1, B^2)$ is a two dimensional Brownian motion. In this case, we can give a slightly simple approximation formula compared with Theorem 9.

By Itô’s formula, the following logarithmic model is obtained.

$$
\begin{align*}
    dX_t^\varepsilon &= -\frac{1}{2} (\sigma_t^\varepsilon)^2 dt + \sigma_t^\varepsilon dB_t^1, \quad X_0^\varepsilon = x = \log S, \\
    d\sigma_t^\varepsilon &= \varepsilon \nu \sigma_t^\varepsilon (\rho dB_t^1 + \sqrt{1-\rho^2} dB_t^2), \quad \sigma_0^\varepsilon = \sigma.
\end{align*}
$$

This model is regarded as a SABR model with $\beta = 1$ and known as the log-normal SABR (see [12]). Again, the barrier option price is given by

$$
C_{\text{Barrier}}^{\text{SV,}\varepsilon}(T, e^x) = \mathbb{E} \left[ f(S_T^\varepsilon) 1_{\{\min_{0 \leq \sigma \leq T} S_0^\varepsilon > L\}} \right],
$$

where $f$ stands for a payoff function and $L(< S)$ is a barrier price.

The differentiation operators $\mathcal{L}^{\varepsilon}$, $\mathcal{L}_1^{\varepsilon}$ and the PDE are same as (5.3)–(5.5) with $c = q = 0$ and $\lambda = 0$. Also, the barrier option price in the Black-Scholes model coincides with (5.7) with no drift, that is,

$$
C_{\text{Barrier}}^{\text{BS}}(T, S) = C^{\text{BS}}(T, S) - \left( \frac{S}{L} \right) C^{\text{BS}} \left( T, \frac{L^2}{S} \right),
$$

where $C^{\text{BS}}(T, S)$ is the driftless Black-Scholes formula of the European call option given by

$$
C^{\text{BS}}(T, S) = SN(d_1(T, \log S)) - KN(d_2(T, \log S))
$$

with

$$
\begin{align*}
    d_1(t, x) &= d_1(t, x, 0) = \frac{x - \log K + \sigma^2 t/2}{\sigma \sqrt{t}}, \\
    d_2(t, x) &= d_2(t, x, 0) = d_1(t, x) - \sigma \sqrt{t}.
\end{align*}
$$

Then, we reach the following expansion formula which only needs 1-dimensional numerical integration.

**Theorem 10.** $C_{\text{Barrier}}^{\text{SV,}\varepsilon}(T, e^x) = C_{\text{Barrier}}^{\text{BS}}(T, e^x) + \varepsilon v_1^0(0, x) + O(\varepsilon^2)$, where

$$
\begin{align*}
    v_1^0(0, x) &= -\frac{1}{2} T \nu \rho \sigma \left\{ e^x n(d_1(T, x)) d_2(T, x) + \text{Ln}(c_1(T, x)) c_1(T, x) \right\} \\
    &\quad + \frac{\nu \rho L(x - l) \log(L/K)}{2 \pi \sigma} \int_0^T \left( \frac{T - s}{s^{3/2}} \right)^{1/2} \exp \left( -\frac{c_2(T - s, L/K) + c_2(s, L/e^x)}{2} \right) ds,
\end{align*}
$$

$$
\begin{align*}
    c_1(t, x) &= \frac{\log(L^2/e^x K) + \sigma^2 t/2}{\sigma \sqrt{t}}, \\
    c_2(t, y) &= \left( \frac{\log y + \sigma^2 t/2}{\sigma \sqrt{t}} \right)^2.
\end{align*}
$$
Proof. By Theorem 8 and the equality in Proposition 2 with \( k = 1 \), we see that the expansion

\[
C_{\text{Barrier}}^{SV,\varepsilon}(T, e^x) = C_{\text{Barrier}}^{BS,\varepsilon}(T, e^x) + \varepsilon v_1^0(0, x) + O(\varepsilon^2)
\]

holds with

\[
v_1^0(t, x) = \int_0^{T-t} P_{T-t-r}^{D} \mathcal{L}_1 P_{T-r}^{D} \bar{f}(x) dr.
\] (5.17)

Then, we have the following proposition for an expression of \( v_1^0(0, x) \). The proof is given in Section 6.5.

**Proposition 3.**

\[
v_1^0(0, x) = \frac{T}{2} \nu \rho \sigma^2 \frac{\partial^2}{\partial x \partial \sigma} P_T^{D} \bar{f}(x) - \frac{1}{2} \mathbb{E}[(T - \tau_D(X^{0,x})) \nu \rho \sigma^2 \frac{\partial^2}{\partial x \partial \sigma} P_{T-\tau_D(X^{0,x})}^{D}(l) 1_{\{\tau_D(X^{0,x}) < T}\}].
\]

We remark that the expectation in the above equality can be represented as

\[
\frac{1}{2} \mathbb{E}[(T - \tau_D(X^{0,x})) \nu \rho \sigma^2 \frac{\partial^2}{\partial x \partial \sigma} P_{T-\tau_D(X^{0,x})}^{D}(l) 1_{\{\tau_D(X^{0,x}) < T}\}]
= \int_0^T \frac{(T-s)}{2} \nu \rho \sigma^2 \frac{\partial^2}{\partial x \partial \sigma} P_{T-s}^{D}(l) h(s, x-l) ds,
\] (5.18)

where \( h(s, x-l) \) is the density function of the first hitting time to \( l \) defined by

\[
h(s, x-l) = \frac{-(l-x)}{\sqrt{2\pi\sigma^2s^3}} \exp \left( -\frac{(l-x + \sigma^2 s/2)^2}{2\sigma^2 s} \right).
\] (5.19)

Now we evaluate

\[
\nu \rho \sigma^2 \frac{\partial^2}{\partial x \partial \sigma} P_T^{D} \bar{f}(x) = \nu \rho \sigma^2 \frac{\partial^2}{\partial x \partial \sigma} C^{BS}(t, e^x) - \nu \rho \sigma^2 \frac{\partial^2}{\partial x \partial \sigma} \left\{ \left( \frac{e^x}{L} \right) C^{BS} \left( t, \frac{L^2}{e^x} \right) \right\}.
\]

Note that

\[
\frac{\partial}{\partial \sigma} C^{BS} (t, e^x) = e^x n(d_1(t, x)) \sqrt{t},
\] (5.20)

and

\[
\frac{\partial}{\partial \sigma} \left\{ \left( \frac{e^x}{L} \right) C^{BS} \left( t, \frac{L^2}{e^x} \right) \right\} = L n(c_1(t, x)) \sqrt{t}.
\] (5.21)

Then we have

\[
\nu \rho \sigma^2 \frac{\partial^2}{\partial x \partial \sigma} C^{BS} (t, e^x) = \nu \rho \sigma^2 e^x n(d_1(t, x)) \sqrt{t} \left\{ 1 - \frac{d_1(t, x)}{\sigma \sqrt{t}} \right\}
= -\nu \rho \sigma e^x n(d_1(t, x)) d_2(t, x)
\] (5.22)

and

\[
\nu \rho \sigma^2 \frac{\partial^2}{\partial x \partial \sigma} \left\{ \left( \frac{e^x}{L} \right) C^{BS} \left( t, \frac{L^2}{e^x} \right) \right\} = \nu \rho \sigma L n(c_1(t, x)) c_1(t, x).
\] (5.23)
Combining (5.20), (5.22) and (5.23), we get

\[ \nu \rho \sigma^2 \frac{\partial^2}{\partial x \partial \sigma} P_t^D \bar{f}(x) = \nu \rho \sigma \{ \varepsilon n(d_1(t, x))(-d_2(t, x)) - L n(c_1(t, x)) c_1(t, x) \}. \] (5.24)

Substituting (5.24) into (5.18), we have

\[ \nu \rho \sigma^2 \frac{\partial^2}{\partial x \partial \sigma} P_t^D \bar{f}(l) = \nu \rho \sigma L n(d_1(t, l))(-d_2(t, l)) - \rho \sigma L n(c_1(t, l)) c_1(t, l) \]
\[ = \nu \rho \sigma L n(d_1(t, l))(-d_1(t, l) + d_2(t, l)) \]
\[ = \nu \rho \sigma \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{(l - \log K + \frac{1}{2} \sigma^2 t)^2}{2 \sigma^2 t} \right) \left( -2(l - \log K) \right). \]

Thus we obtain

\[ -\frac{1}{2} \mathbb{E}[(T - \tau_D(X^{0,x}))\nu \rho \sigma^2 \frac{\partial^2}{\partial x \partial \sigma} P_{T-\tau_D(X^{0,x})}^D \bar{f}(l) 1\{\tau_D(X^{0,x}) < T\}] \]
\[ = -\int_0^T \frac{1}{2} \nu \rho \sigma L e^{-\frac{(1 - \log K + \frac{1}{2} \sigma^2 (T - s))^2}{2 \sigma^2 (T - s)}} \left( -2(l - \log K) \right) \]
\[ \times \frac{-(l - x)}{\sqrt{2\pi \sigma^2 s^3}} e^{-\frac{(l - x + \sigma^2 / 2)}{2 \sigma^2}} ds \]
\[ = \frac{\nu \rho L(x - l) \log(L/K)}{2 \pi \sigma} \int_0^T \frac{(T - s)^{1/2}}{s^{3/2}} \exp \left( -\frac{c_2(T - s, L/K) + c_2(s, L/e^x)}{2} \right) ds. \] (5.25)

By Proposition 3, (5.18), (5.24) and (5.25), we reach the assertion. \( \blacksquare \)

Finally, we show a simple numerical example of European down-and-out barrier call prices as an illustrative purpose. Denote \( u^0 = C_{\text{Bar}}^{BS}(T, S) \) and \( v_1^0 = v_1^0(0, \log S) \). Then we see

\[ C_{\text{Bar}}^{SV,\varepsilon}(T, S) \approx u^0 + \varepsilon v_1^0. \]

We list the numerical examples below, where the numbers in the parentheses show the error rates (%) relative to the benchmark prices of \( C_{\text{Bar}}^{SV,\varepsilon}(T, S) \); they are computed by Monte Carlo simulations with 100,000 time steps and 1,000,000 trials. We check the accuracy of our approximations by changing the model parameters. Case 1–6 show the results for the stochastic volatility model with drifts (5.1), and case 7 shows the result for the lognormal SABR model (5.15).

Apparently, our approximation formula \( u^0 + \varepsilon v_1^0 \) improves the accuracy for \( C_{\text{Bar}}^{SV,\varepsilon}(T, S) \), and it is observed that \( \varepsilon v_1^0 \) accurately compensates for the difference between \( C_{\text{Bar}}^{SV,\varepsilon}(T, S) \) and \( C_{\text{Bar}}^{BS}(T, S) \), which confirms the validity of our method.

1.

\[ S = 100, \sigma = 0.15, c = 0.01, q = 0.0, \varepsilon \nu = 0.2, \rho = -0.5, \]
\[ \varepsilon \lambda = 0.00, \theta = 0.00, L = 95, T = 0.5, K = 100, 102, 105. \]
Table 1: Down-and-Out Barrier Option

<table>
<thead>
<tr>
<th>Strike</th>
<th>Benchmark</th>
<th>Our Approximation ($u^0 + \varepsilon v^n$)</th>
<th>Barrier Black-Scholes ($u^b$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>3.468</td>
<td>3.466 (-0.05%)</td>
<td>3.495 (0.80%)</td>
</tr>
<tr>
<td>102</td>
<td>2.822</td>
<td>2.822 (0.00%)</td>
<td>2.866 (1.57%)</td>
</tr>
<tr>
<td>105</td>
<td>1.986</td>
<td>1.986 (0.01%)</td>
<td>2.052 (3.36%)</td>
</tr>
</tbody>
</table>

2.

$S = 100, \sigma = 0.15, c = 0.01, q = 0.0, \varepsilon\nu = 0.35, \rho = -0.7,$
$\varepsilon\lambda = 0.00, \theta = 0.00, L = 95, T = 0.5, K = 100, 102, 105.$

Table 2: Down-and-Out Barrier Option

<table>
<thead>
<tr>
<th>Strike</th>
<th>Benchmark</th>
<th>Our Approximation ($u^0 + \varepsilon v^n$)</th>
<th>Barrier Black-Scholes ($u^b$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>3.421</td>
<td>3.423 (0.07%)</td>
<td>3.495 (2.18%)</td>
</tr>
<tr>
<td>102</td>
<td>2.753</td>
<td>2.757 (0.18%)</td>
<td>2.866 (4.13%)</td>
</tr>
<tr>
<td>105</td>
<td>1.885</td>
<td>1.890 (0.23%)</td>
<td>2.052 (8.88%)</td>
</tr>
</tbody>
</table>

3.

$S = 100, \sigma = 0.15, c = 0.05, q = 0.0, \varepsilon\nu = 0.35, \rho = -0.7,$
$\varepsilon\lambda = 0.00, \theta = 0.00, L = 95, T = 0.5, K = 100, 102, 105.$

Table 3: Down-and-Out Barrier Option

<table>
<thead>
<tr>
<th>Strike</th>
<th>Benchmark</th>
<th>Our Approximation ($u^0 + \varepsilon v^n$)</th>
<th>Barrier Black-Scholes ($u^b$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>4.352</td>
<td>4.349 (-0.07%)</td>
<td>4.399 (1.06%)</td>
</tr>
<tr>
<td>102</td>
<td>3.585</td>
<td>3.586 (0.02%)</td>
<td>3.665 (2.24%)</td>
</tr>
<tr>
<td>105</td>
<td>2.560</td>
<td>2.563 (0.11%)</td>
<td>2.696 (5.31%)</td>
</tr>
</tbody>
</table>

4.

$S = 100, \sigma = 0.15, c = 0.05, q = 0.1, \varepsilon\nu = 0.2, \rho = -0.5,$
$\varepsilon\lambda = 0.00, \theta = 0.00, L = 95, T = 0.5, K = 100, 102, 105.$
Table 4: Down-and-Out Barrier Option

<table>
<thead>
<tr>
<th>Strike</th>
<th>Benchmark</th>
<th>Our Approximation $(u^0 + \varepsilon v^0_1)$</th>
<th>Barrier Black-Scholes $(u^0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>2.231</td>
<td>2.224 (-0.31%)</td>
<td>2.268 (1.64%)</td>
</tr>
<tr>
<td>102</td>
<td>1.758</td>
<td>1.754 (-0.27%)</td>
<td>1.812 (3.02%)</td>
</tr>
<tr>
<td>105</td>
<td>1.172</td>
<td>1.168 (-0.31%)</td>
<td>1.243 (6.05%)</td>
</tr>
</tbody>
</table>

5.

$S = 100, \sigma = 0.15, c = 0.01, q = 0.0, \varepsilon \nu = 0.2, \rho = -0.5, \varepsilon \lambda = 0.2, \theta = 0.25, L = 95, T = 0.5, K = 100, 102, 105.$

Table 5: Down-and-Out Barrier Option

<table>
<thead>
<tr>
<th>Strike</th>
<th>Benchmark</th>
<th>Our Approximation $(u^0 + \varepsilon v^0_1)$</th>
<th>Barrier Black-Scholes $(u^0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>3.523</td>
<td>3.517 (-0.16%)</td>
<td>3.495 (-0.77%)</td>
</tr>
<tr>
<td>102</td>
<td>2.891</td>
<td>2.888 (-0.09%)</td>
<td>2.866 (-0.85%)</td>
</tr>
<tr>
<td>105</td>
<td>2.066</td>
<td>2.065 (-0.06%)</td>
<td>2.052 (-0.64%)</td>
</tr>
</tbody>
</table>

6.

$S = 100, \sigma = 0.15, c = 0.01, q = 0.0, \varepsilon \nu = 0.2, \rho = -0.5, \varepsilon \lambda = 0.5, \theta = 0.25, L = 95, T = 0.5, K = 100, 102, 105.$

Table 6: Down-and-Out Barrier Option

<table>
<thead>
<tr>
<th>Strike</th>
<th>Benchmark</th>
<th>Our Approximation $(u^0 + \varepsilon v^0_1)$</th>
<th>Barrier Black-Scholes $(u^0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>3.587</td>
<td>3.594 (0.20%)</td>
<td>3.495 (-2.55%)</td>
</tr>
<tr>
<td>102</td>
<td>2.976</td>
<td>2.987 (0.39%)</td>
<td>2.866 (-3.68%)</td>
</tr>
<tr>
<td>105</td>
<td>2.170</td>
<td>2.183 (0.59%)</td>
<td>2.052 (-5.41%)</td>
</tr>
</tbody>
</table>

7.

$S = 100, \sigma = 0.15, c = 0.0, q = 0.0, \varepsilon \nu = 0.2, \rho = -0.5, \varepsilon \lambda = 0.0, \theta = 0.0, L = 95, T = 0.5, K = 100, 102, 105.$
Table 7: Down-and-Out Barrier Option

<table>
<thead>
<tr>
<th>Strike</th>
<th>Benchmark</th>
<th>Our Approximation ($u_0^0 + \varepsilon v_0^0$)</th>
<th>Barrier Black-Scholes ($u^0$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>3.261</td>
<td>3.258 (-0.09%)</td>
<td>3.290 (0.90%)</td>
</tr>
<tr>
<td>102</td>
<td>2.640</td>
<td>2.639 (-0.02%)</td>
<td>2.686 (1.78%)</td>
</tr>
<tr>
<td>105</td>
<td>1.841</td>
<td>1.841 (0.01%)</td>
<td>1.911 (3.77%)</td>
</tr>
</tbody>
</table>

References


Appendix

6.1 Proof of Theorem 2

We consider the following PDE which is equivalent to (3.5) with changing variable $t$ to $T-t$

$$
\begin{cases}
    \frac{\partial}{\partial t} v(t, x) + \mathcal{L}^0 v(t, x) + g(t, x) = 0, & (t, x) \in (0, T] \times D, \\
    v(0, x) = 0, & x \in D, \\
    v(t, x) = 0, & (t, x) \in [0, T] \times \partial D.
\end{cases}
$$

We define $\tilde{H}^{m,\alpha,p}$ as the same as $H^{m,\alpha,p}$ replacing $[0, T)$ in the definition with $(0, T]$.

We divide the proof of Theorem 2 into the following two propositions.

**Proposition 4.** For any $g$, a classical solution of (6.1) is unique in the following sense: if $v$ and $w$ are classical solutions of (6.1) and $|v(t, x)| + |w(t, x)| \leq C \exp(\beta|x|^2)$ for some $C, \beta > 0$, then $v = w$.

Proposition 4 is obtained by the same argument as the proof of Theorem 2.4.9 in [5].

**Proposition 5.** There exists a classical solution $v$ of (6.1) for $g \in \tilde{H}^{m,\alpha,p}$ with $p > 1/\alpha$. Moreover, (3.6) holds.

**Proof.** By Levi’s parametrix method, we can construct the fundamental solution $\Gamma(t, x; \tau, \xi)$ for the operator $L = -\partial/\partial t + \mathcal{L}^0$, that is,

$$
W_g(t, x) = \int_0^t \int_D \Gamma(t, x; \tau, \xi) g(\tau, \xi) d\xi d\tau
$$

is continuous in $(t, x)$, continuously differentiable in $x$ for $g \in C([0, T] \times \bar{D})$. When $g$ is Hölder continuous in $x$ uniformly in $t \in [0, T]$, then we see that $W_g$ is a solution of (6.1) (See Theorem
However functions $U_t, x$ may not have the regularity at $t = 0$. So we generalize the argument in Chapter 1 of [5]. We remark that $\Gamma(t, x; \tau, \xi)$ is given by

$$\Gamma(t, x; \tau, \xi) = Z(t, x; \tau, \xi) + \int_0^T \int_D Z(t, x; \sigma, \eta) \Phi(\sigma, \eta; \tau, \xi) d\eta d\sigma,$$

where

$$Z(t, x; \tau, \xi) = \frac{\sqrt{\text{det}(a(x, 0))}}{(4\pi(t - \tau))^{d/2}} \exp \left( - \sum_{i,j=1}^d a^{ij}(\xi, 0)(x^i - \xi^i)(x^j - \xi^j) \right) / 4(t - \tau)$$

and $\Phi(t, x; \tau, \xi)$ is the solution of

$$\Phi(t, x; \tau, \xi) = LZ(t, x; \tau, \xi) + \int_0^T \int_D LZ(t, x; \sigma, \eta) \Phi(\sigma, \eta; \tau, \xi) d\eta d\sigma.$$

Fix any $g \in \mathcal{H}^{m, \alpha, p}$. We can divide $W_g$ as $W_g = V_g + U_g$, where

$$V_g(t, x) = \int_0^t \int_D Z(t, x; \tau, \xi) g(\tau, \xi) d\xi d\tau,$$

$$U_g(t, x) = \int_0^t \int_D Z(t, x; \tau, \xi) \hat{g}(\tau, \xi) d\xi d\tau, \quad \hat{g}(t, x) = \int_0^t \int_D \Phi(t, x; \tau, \xi) g(\tau, \xi) d\xi d\tau.$$

We remark that $V_g, U_g$ and $\hat{g}$ are well-defined by virtue of (4.9) and (4.15) in [5] and the property of $g$. Take $\beta \in (\alpha - 1/p, \alpha)$. By Theorem 1.4.8 in [5], we see that

$$|\Phi(t, x; \tau, \xi) - \Phi(t, y; \tau, \xi)| \leq \frac{C|x - y|^{\beta}}{(t - \tau)^{(d - 2(\alpha - \beta))}/2} \left\{ \exp \left( - \frac{\lambda|x - \xi|^2}{t - \tau} \right) + \exp \left( - \frac{\lambda|y - \xi|^2}{t - \tau} \right) \right\}$$

for some $C, \lambda > 0$. Hence,

$$|\hat{g}(t, x) - \hat{g}(t, y)| \leq C' \int_0^t \frac{M^g(\tau)}{(t - \tau)^{1 - (\alpha - \beta)/2}} d\tau |x - y|^{\beta}$$

$$\leq C' \left( \int_0^T (M^g(\tau))^{p} d\tau \right)^{1/p} \left( \int_0^T (t - \tau)^{-(1 - (\alpha - \beta)/2)q} d\tau \right)^{1/q} |x - y|^{\beta}, \quad t \in (0, T], \ x, y \in D$$

for some $C' > 0$ by virtue of the Hölder inequality, where $q > 1$ is given by $1/p + 1/q = 1$. Since $(1 - (\alpha - \beta)/2)q$ is smaller than 1, we see that $\hat{g}(t, x)$ is $\beta$-Hölder continuous in $x$ uniformly in $t \in (0, T]$. Then, Theorem 1.3.3–1.3.6 and the equality (4.2) in Chapter 1 of [5] imply that $U_g(t, x) \in C^{1, 2}((0, T] \times D)$ and

$$LU_g(t, x) = -\hat{g}(t, x) + \int_0^t \int_D \left\{ \frac{1}{2} \sum_{i,j=1}^d (a^{ij}(x, 0) - a^{ij}(\xi, 0)) \frac{\partial^2}{\partial x_i \partial x_j} + b^i(x, 0) \frac{\partial}{\partial x_i} - c(x, 0) \right\} Z(t, x; \tau, \xi) \hat{g}(\tau, \xi) d\xi d\tau$$

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\[
\dot{g}(t, x) + \int_0^t \int_D LZ(t, x; \tau, \xi) \dot{g}(\tau, \xi) d\xi d\tau.
\]  

(6.2)

For the volume potential \( V_g \), we follow the proof of Theorem 1.3.4 in [5] to find that for any compact set \( K \subset D \)

\[
\sum_{i,j=1}^{d} \left| \frac{\partial^2}{\partial x^i \partial x^j} J_g(t, x, \tau) \right| \leq \frac{C_K M^{g,K}(\tau)}{(t - \tau)^\mu}, \quad t \in (0, T], \ x \in K, \ \tau \in (0, t), \ \mu \in (1 - \alpha/2, 1)
\]

for some \( C_K > 0 \), where

\[
J_g(t, x, \tau) = \int_D Z(t, x; \tau, \xi) g(\tau, \xi) d\xi.
\]

Hence, the dominated convergence theorem implies that \( V_g(t, x) \) is twice continuously differentiable in \( x \). Similarly, we get \( V_g \in C^{1,2}((0, T] \times D) \) and

\[
LV_g(t, x) = -g(t, x) + \int_0^t \int_D LZ(t, x; \tau, \xi) g(\tau, \xi) d\xi d\tau.
\]

(6.3)

Combining (6.2)–(6.3), we obtain

\[
LW_g(t, x) = -g(t, x) - \int_0^t \int_D \Phi(t, x; \tau, \xi) - LZ(t, x; \tau, \xi) g(\tau, \xi) d\tau + \int_0^t \int_D \Phi(t, x; \tau, \xi) - LZ(t, x; \tau, \xi) \Phi(\sigma, \eta; \tau, \xi) d\eta d\sigma d\xi d\tau
\]

\[
= -g(t, x),
\]

which implies that \( W_g \) is a solution of (6.1). Moreover, since \( g \in \mathcal{H}^{m,\alpha,p} \), using the inequality (6.12) in p.24 of [5], we get

\[
|v(t, x)| \leq C'' \int_0^t M^g(\tau) \left( 1 + |\xi|^{2m} \right) \exp \left( -\frac{\lambda' |x - \xi|^2}{t - \tau} \right) d\tau
\]

\[
\leq C''' \left( \int_0^T (M^g(\tau))^p d\tau \right)^{1/p} (1 + |x|^{2m})
\]

for some \( C'', C''' > 0 \). Then, we complete the proof of Proposition 5.

\[\Box\]

### 6.2 Proof of Theorem 3

First, we generalize the definitions of \( \mathcal{L}_k^0, g_k^0 \) and \( v_k^0 \). We define

\[
\mathcal{L}_k^\varepsilon = \frac{1}{(k - 1)!} \left\{ \frac{1}{2} \sum_{i,j=1}^{d} \int_0^1 (1 - \tau)^{k-1} \partial^k a_i^j(x, \tau \varepsilon) d\tau \right. - \frac{\partial^2}{\partial x^i \partial x^j}
\]

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We consider the following Cauchy-Dirichlet problem:

\[ g_n^\varepsilon(t, x) = \mathcal{L}_n^\varepsilon u^0(t, x) + \sum_{k=1}^{n-1} \mathcal{L}_{n-k}^0 v_k^0(t, x) + \sum_{k=1}^{n-2} \varepsilon^k \left\{ \mathcal{L}_n^\varepsilon v_k^0(t, x) + \sum_{l=k+1}^{n-1} \mathcal{L}_{n-k-l}^0 v_l^0(t, x) \right\} + \varepsilon^{n-1} \mathcal{L}_{n-1}^0 v_{n-1}^0(t, x). \]

We consider the following Cauchy-Dirichlet problem:

\[
\begin{cases}
-\frac{\partial}{\partial t} v(t, x) - \mathcal{L}^\varepsilon u(t, x) - g_n^\varepsilon(t, x) = 0, & (t, x) \in [0, T) \times D, \\
v(T, x) = 0, & x \in D, \\
v(t, x) = 0, & (t, x) \in [0, T] \times \partial D.
\end{cases}
\] (6.4)

For \( \varepsilon \neq 0 \), we define \( v_n^\varepsilon = [u^\varepsilon - \{u^0 + \varepsilon^k v_k^0(t, x)\}] / \varepsilon^n \). Obviously, we see

\[ u^\varepsilon(t, x) = u^0(t, x) + \sum_{k=1}^{n-1} \varepsilon^k v_k^0(t, x) + \varepsilon^n v_n^\varepsilon(t, x). \] (6.5)

**Proposition 6.** The function \( v_n^\varepsilon \) is a solution of (6.4).

**Proof.** It is obvious that \( v_n^\varepsilon(T, x) = 0 \) for \( x \in D \) and \( v_n^\varepsilon(t, x) = 0 \) for \( (t, x) \in [0, T) \times \partial D \).

Apply Taylor’s theorem to (2.3) to observe

\[ \mathcal{L}^\varepsilon u^\varepsilon(t, x) = \left\{ \mathcal{L}^0 + \varepsilon^k \mathcal{L}_k^0 + \varepsilon^n \mathcal{L}_n^\varepsilon \right\} u^\varepsilon(t, x). \] (6.6)

Since \( u^0 \) is the solution of (2.3) with \( \varepsilon = 0 \), we get

\[ \frac{\partial}{\partial t} u^0(t, x) + \mathcal{L}^0 u^0(t, x) = 0. \] (6.7)

Similarly, by Theorem 2, we have

\[ \frac{\partial}{\partial t} v_k^0(t, x) + \mathcal{L}^0 v_k^0(t, x) + \mathcal{L}_k^0 u^0(t, x) + \sum_{l=1}^{k-1} \mathcal{L}_{k-l}^0 v_l^0(t, x) = 0. \] (6.8)

Combining (6.5)–(6.8) and Theorem 1, we obtain

\[ \varepsilon^n \left\{ \frac{\partial}{\partial t} v_n^\varepsilon(t, x) + \mathcal{L}^0 v_n^\varepsilon(t, x) + \mathcal{L}_n^\varepsilon u^0(t, x) + \sum_{l=1}^{n-1} \mathcal{L}_{n-l}^0 v_l^0(t, x) \right\} \]
\[ + \sum_{k=n+1}^{2n-2} \varepsilon^k \left\{ \mathcal{L}_{n-k}^0 v_n^\varepsilon(t, x) + \mathcal{L}_n^\varepsilon v_{k-n}^0(t, x) + \sum_{l=k-n+1}^{n-1} \mathcal{L}_{k-l}^0 v_l^0(t, x) \right\} \]
\[ + \varepsilon^{2n-1} \left\{ \mathcal{L}_{n-1}^0 v_n^\varepsilon(t, x) + \mathcal{L}_n^\varepsilon v_{n-1}^0(t, x) \right\} + \varepsilon^{2n} \mathcal{L}_n^\varepsilon v_n^\varepsilon(t, x) = 0, \]
and thus,
\[ \frac{\partial}{\partial t} v_\varepsilon^t(t, x) + \mathcal{L}^\varepsilon v_\varepsilon^t(t, x) + g_\varepsilon^t(t, x) = 0. \]

This implies the assertion.

Set
\[ \tilde{v}_\varepsilon^t(t, x) = E \left[ \int_0^{\tau_D(X^\varepsilon,t) \wedge (T-t)} \exp \left( - \int_0^r c(X^\varepsilon_{\nu}, \varepsilon) dv \right) g_\varepsilon^t(r, X^\varepsilon_{\nu}) dr \right]. \]

By [F–I], we find that there are \( C_n > 0, \tilde{m}_n \in \mathbb{N} \) which are independent of \( \varepsilon \) and the function \( M_n \in C([0, T)) \cap L^p([0, T), dt) \) determined by \( u^0, v^0_1, \ldots, v^0_{n-1} \) such that
\[ \left| g_\varepsilon^t(t, x) \right| \leq C_n M_n(t)(1 + |x|^{2\tilde{m}_n}). \tag{6.9} \]

The inequalities (2.5) and (6.9) imply
\[ |\tilde{v}_\varepsilon^t(t, x)| \leq C_n' \int_t^T M_n(r) dr (1 + |x|^{2\tilde{m}_n}) \tag{6.10} \]
for some \( C_n' > 0 \) which is also independent of \( \varepsilon \).

**Proposition 7.** \( v_\varepsilon^t = \tilde{v}_\varepsilon^t \).

**Proof.** The assertion is easily obtained by the similar argument to Theorem 5.1.9 in [13].

**Proof of Theorem 3.** By (6.5) and Proposition 7, we have \( u^\varepsilon(t, x) - (u^0(t, x) + \sum_{k=1}^{n-1} \varepsilon^k v^0_k(t, x)) = \varepsilon^n \tilde{v}_\varepsilon^t(t, x) \). Our assertion is now immediately obtained by the inequality (6.10).

**6.3 Proof of Proposition 2**

1. Firstly, let us consider the case for \( k = 1 \). Let \( g \in \mathcal{H}^{m, \alpha, p} \). Observe that
\[ \int_0^{(T-t) \wedge \tau_D(X^0, x)} \exp \left( - \int_0^r c(X^0_{\nu}, 0) dv \right) g(t + r, X^0_{\nu}) dr = \int_0^{T-t} \exp \left( - \int_0^r c(X^0_{\nu}, 0) dv \right) g(t + r, X^0_{\nu}) 1_{\{\tau_D(X^0, \cdot) \geq r\}} dr, \]
and we obtain
\[ E \left[ \int_0^{(T-t) \wedge \tau_D(X^0, x)} \exp \left( - \int_0^r c(X^0_{\nu}, 0) dv \right) g(t + r, X^0_{\nu}) dr \right] = \int_0^{T-t} E \left[ \exp \left( - \int_0^r c(X^0_{\nu}, 0) dv \right) g(t + r, X^0_{\nu}) 1_{\{\tau_D(X^0, \cdot) \geq r\}} \right] dr \]
\[ = \int_0^{T-t} P^D_r g(t + r, \cdot)(x) dr. \]
Thus, under the assumption [H], we see

\[
v_1^0(T - t, x) = E \left[ \int_0^t \exp \left( - \int_0^r c(X_0^{0,x}, 0) dv \right) g_1^0(T - t + r, X_0^{0,x}) 1_{\{r_D(X_0^{0,x}) \geq r\}} dr \right]
\]

\[
= \int_0^t P^D \tilde{L}_1^0 u^0(T - t + r, \cdot)(x) dr
\]

\[
= \int_0^t P^D \tilde{L}_1^0 P^D_{t-r} f(x) dr = \int_0^t P^D_{t-r} P^D_{t} f(x) dr.
\]  
(6.11)

Thus, we have the assertion for \( k = 1 \).

2. If the assertion holds for \( 1, \ldots, k - 1 \), then

\[
v_k^0(T - t, x) = \int_0^t P^D_{t} \{ \tilde{L}_k^0 u^0 + \sum_{l=1}^{k-1} \tilde{L}_k^0 \} (T - t + t_0, \cdot)(x) dt_0
\]

\[
= \int_0^t P^D_{t-t_0} \tilde{L}_k^0 P^D_{t_0} f(x) dt_0
\]

\[
+ \sum_{l=1}^{k-1} \sum_{m=1}^{l} \sum_{(\beta^m)_{l=1}^m C N^m, \sum_{l=1}^{m} \beta^l = l} \int_0^t \int_0^{t_0} \int_0^{t_1} \cdots \int_0^{t_{l-1}} P^D_{t-t_0} \tilde{L}_{k-l}^0 P^D_{t_0-t_l} \tilde{L}_{k-l+1}^0 \cdots P^D_{t_{l-1}-t_0} \tilde{L}_1^0 P^D_{t_0} f(x) dt_1 \cdots dt_1 dt_0
\]

\[
= \int_0^t P^D_{t-t_0} \tilde{L}_k^0 P^D_{t_0} f(x) dt_0
\]

\[
+ \sum_{l=2}^{k} \sum_{m=1}^{l} \sum_{(\beta^m)_{l=1}^m C N^m, \sum_{l=1}^{m} \beta^l = l} \int_0^t \int_0^{t_0} \int_0^{t_1} \cdots \int_0^{t_{l-1}} P^D_{t-t_0} \tilde{L}_{k-l}^0 P^D_{t_0-t_l} \tilde{L}_{k-l+1}^0 \cdots P^D_{t_{l-1}-t_0} \tilde{L}_1^0 P^D_{t_0} f(x) dt_1 \cdots dt_1
\]

\[
= \sum_{l=1}^{k} \sum_{(\beta^l)_{l=1}^l C N^l, \sum_{l=1}^{l} \beta^l = l} \int_0^t \int_0^{t_0} \cdots \int_0^{t_{l-1}} P^D_{t-t_0} \tilde{L}_{k-l}^0 P^D_{t_0-t_l} \tilde{L}_{k-l+1}^0 \cdots P^D_{t_{l-1}-t_0} \tilde{L}_1^0 P^D_{t_0} f(x) dt_1 \cdots dt_1.
\]

Thus, our assertion is also true for \( k \). Then we complete the proof of Proposition 2 by mathematical induction.

\section{6.4 Proof of Theorem 6}

Let \( v_n^\varepsilon \) and \( \tilde{v}_n^\varepsilon \) be as in Section 6.2. Thanks for the assumption \( I \subset [0, \infty) \), the same argument as the proof of Proposition 6 tells us that \( v_n^\varepsilon \) is a viscosity solution of (6.4). Moreover, we have the next proposition.

\begin{proposition}
The function \( \tilde{v}_n^\varepsilon \) is a viscosity solution of (6.4).
\end{proposition}

\begin{proof}
Until the end of the proof we suppress \( \varepsilon \) in the notation. First, we check the continuity. By the similar argument to the proof of Lemma 4.2 in [17], we see that \( v_n \) is continuous on

\]
\[0, T) \times \bar{D}. \] Moreover, by (6.10), we get
\[
\sup_{x \in K \cap \bar{D}} |\tilde{v}_n(t, x)| \leq C'_n(1 + \sup_{x \in K} |x|^{2m}) \left\{ \int_0^T M_n(r)dr - \int_0^t M_n(r)dr \right\} \rightarrow 0, \quad t \rightarrow T
\]
for any compact set \( K \subset \mathbb{R}^d \). Thus, \( v_n \) is continuous on \([0, T) \times \bar{D}\).

Next, we show that \( v_n \) is a viscosity subsolution of (6.4). Take any \((t, x) \in [0, T) \times D\) and let \( \varphi \) be \( C^{1,2} \)-function such that \( v_n - \psi \) has a maximum 0 at \((t, x)\). We may assume that \( \varphi \) and its derivatives have polynomial growth rates in \( x \) uniformly in \( t \). By the Markov property, we have
\[
E \left( J(h)\tilde{v}_n(t + h, X^x_h)1_{\{\tau_D(X^x) \geq h\}} \right) = E \left[ \int_0^{(T-t)\wedge \tau_D(X^x)} J(r)g_n(t + r, X^x_r)dr 1_{\{\tau_D(X^x) \geq h\}} \right],
\]
where \( J(r) = \exp \left(-\int_0^r c(X^x_r, \varepsilon)dv\right). \) Since \( \tau_D(X^x_h) = \tau_D(X^x) - h \) on \( \{\tau_D(X^x) \geq h\} \), we obtain
\[
E \left( J(h)\tilde{v}_n(t + h, X^x_h)1_{\{\tau_D(X^x) \geq h\}} \right) = \tilde{v}_n(t, x) - E \left[ \int_0^h J(r)g_n(t + r, X^x_r)dr 1_{\{\tau_D(X^x) \geq h\}} \right] - A_h,
\]
where
\[
A_h = E \left[ \int_0^{(T-t)\wedge \tau_D(X^x)} J(r)g_n(t + r, X^x_r)dr 1_{\{\tau_D(X^x) < h\}} \right].
\]
Therefore,
\[
\varphi(t, x) = \tilde{v}_n(t, x) = E \left( J(h)\tilde{v}_n(t + h, X^x_h)1_{\{\tau_D(X^x) \geq h\}} \right) + E \left[ \int_0^h g_n(t + r, X^x_r)dr 1_{\{\tau_D(X^x) \geq h\}} \right] + A_h
\]
\[
\leq E \left( J(h)\varphi(t + h, X^x_h)1_{\{\tau_D(X^x) \geq h\}} \right) + E \left[ \int_0^h g_n(t + r, X^x_r)dr 1_{\{\tau_D(X^x) \geq h\}} \right] + A_h.
\]
Applying Ito’s formula to \( J(r)\varphi(t + r, X^x_r) \), we get
\[
-\frac{1}{h} \int_0^h E \left[ \left\{ \left( \frac{\partial}{\partial t} + \mathcal{L} \right) \varphi(t + r, X^x_r) + g_n(t + r, X^x_r) \right\} 1_{\{\tau_D(X^x) \geq h\}} \right] dr \leq \frac{A_h - \varphi(t, x)P(\tau_D(X^x) < h)}{h}.
\]
By (6.9) and the Schwarz inequality, we have
\[
|A_h| \leq C''_n(1 + |t(x)|^{2m}) \int_0^T M_r dt P(\tau_D(X^x) < h)^{1/2}
\]
for some \( C''_n > 0 \). Using (4.3) and the Chebyshev inequality, we obtain
\[
P(\tau_D(X^x) < h) \leq E[ \sup_{0 \leq \tau \leq h} |X^x_\tau - x| \geq \text{dist}(x, \partial D)] \leq \frac{C''_n}{\text{dist}(x, \partial D)^8} E[ \sup_{r \in [0, h]} |X^x_r - x|^{8}]\]
\[
\leq \frac{C_{nm}'}{\text{dist}(x, \partial D)^8} (1 + |x|^{2\bar{m}})h^3
\]
for some \(C_{nm}', C_{nm}'' > 0\). Thus, letting \(h \to 0\) in (6.12), we see that

\[
-\frac{\partial}{\partial t} \varphi(t, x) - \mathcal{L} \varphi(t, x) - g_n(t, x) \leq 0.
\]

Hence, \(\bar{v}_n\) is a viscosity subsolution of (6.4). By the similar argument, we also find that \(\bar{v}_n\) is a viscosity supersolution. By the definition of \(\bar{v}_n\), we easily get \(\bar{v}_n(T, x) = 0\) for \(x \in D\) and \(\bar{v}_n(t, x) = 0\) for \((t, x) \in [0, T] \times \partial D\).

To see the equivalence \(v^\varepsilon_n = \bar{v}^\varepsilon_n\), we need to give a new proof of Proposition 7 under the assumptions of Theorem 6.

**Proof of Proposition 7.** Set \(\bar{u}_n^\varepsilon(t, x) = u_0^0(t, x) + \sum_{k=1}^{n-1} \varepsilon^k v_k^0(t, x) + \varepsilon^n \bar{v}_n^\varepsilon(t, x)\). The analogous argument of the proof of Proposition 6 implies that \(\bar{u}_n^\varepsilon\) is a viscosity solutions of (2.3). We easily see that \(\bar{u}_n^\varepsilon\) has a polynomial growth rate in \(x\) uniformly in \(t\). Then, Theorem 5 leads us to \(\bar{u}_n^\varepsilon = u^\varepsilon\). This equality and (6.5) imply the assertion. \(\blacksquare\)

Now, we obtain the assertion of Theorem 6 by the same way as that of Theorem 3.

### 6.5 Proof of Proposition 3

First, we notice the following relation:

\[
\mathcal{L}_1^0 P_t^D \bar{f}(x) = \nu \rho \sigma^2 t \left( \frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2} \right) P_t^D \bar{f}(x). \tag{6.13}
\]

Then, using the relations \(\mathcal{L}^0 \mathcal{L}_1^0 P_t^D \bar{f}(x) = \mathcal{L}_1^0 \mathcal{L}^0 P_t^D \bar{f}(x)\) and

\[
\left( \frac{\partial}{\partial t} + \mathcal{L}^0 \right) P_{T-t}^0 \bar{f}(x) = 0,
\]
we get

\[
\left( \frac{\partial}{\partial t} + \mathcal{L}^0 \right) \frac{T-t}{2} P_{T-t}^0 \bar{f}(x) = -\mathcal{L}_1^0 P_{T-t}^0 \bar{f}(x). \tag{6.14}
\]

Also, we have

\[
\left( \frac{\partial}{\partial t} + \mathcal{L}^0 \right) \int_0^{T-t} P_{T-t-r}^0 \left( \nu \rho \sigma^2 \frac{\partial^2}{\partial x \partial \sigma} P_r^D \bar{f} \right)(x) dr = -\mathcal{L}_1^0 P_{T-t}^0 \bar{f}(x), \quad x \in (l, \infty). \tag{6.15}
\]

Therefore, the function

\[
\eta(t, x) = \int_0^{T-t} P_{T-t-r}^0 \left( \nu \rho \sigma^2 \frac{\partial^2}{\partial x \partial \sigma} P_r^D \bar{f} \right)(x) dr - \frac{T-t}{2} \mathcal{L}_1^0 P_{T-t}^0 \bar{f}(x) \tag{6.16}
\]
satisfies the following PDE

\[
\begin{cases}
\left( \frac{\partial}{\partial t} + \mathcal{L}^0 \right) \eta(t, x) = 0, & (t, x) \in [0, T) \times (l, \infty), \\
\eta(T, x) = 0, & x \in [l, \infty), \\
\eta(t, l) = -\frac{T - t}{2} \tilde{L}_1 P_{T-t} \bar{f}(l), & t \in [0, T).
\end{cases}
\]

Then Theorem 6.5.2 in [6] implies

\[
\eta(0, x) = -\frac{1}{2} E[(T - \tau_D(X^0, x)) \nu \rho \sigma^2 \frac{\partial^2}{\partial x \partial \sigma} P_{T-\tau_D(X^0, x)} \bar{f}(l) 1_{\{\tau_D(X^0, x) < T\}}].
\]

By (6.16) and (6.17), we get the assertion. \(\blacksquare\)