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A Remark on Approximation of the Solutions to Partial Differential Equations in Finance *

Akihiko Takahashi † and Toshihiro Yamada ‡

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Abstract

This paper proposes a general approximation method for the solution to a second-order parabolic partial differential equation(PDE) widely used in finance through an extension of *Léandre's approach*(Léandre (2006,2008)) and the *Bismut identiy*(e.g. chapter IX-7 of Malliavin (1997)) in Malliavin calculus. We present two types of its applications, approximations of derivatives prices and short-time asymptotic expansions of the heat kernel. In particular, we provide approximate formulas for option prices under local and stochastic volatility models. We also derive short-time asymptotic expansions of the heat kernel under general time-homogenous local volatility and local-stochastic volatility models in finance, which include Heston (Heston (1993)) and (λ-)SABR models (Hagan et.al. (2002), Labordere (2008)) as special cases. Some numerical examples are shown.

Keywords: Malliavin calculus, Bismut indentity, Integration-by-parts, Semigroup, Asymptotic expansion, Short time asymptotics, Heat kernel expansions, Derivatives pricing, Stochastic volatility, Local volatility, SABR model, λ-SABR models, Heston model

1 Introduction

This paper proposes a new method for the approximation to the solutions of second-order parabolic partial differential equations (PDEs), which has been widely used for pricing and hedging derivatives in finance since Black and Scholes (1973) and Merton (1973). In particular, we derive an approximation formula as Theorem 2.1 based on an asymptotic expansion of the solutions to the second-order parabolic PDEs by Léandre's Approach (Léandre (2006, 2008)) and an application of Malliavin calculus effectively: the approximation formula is derived through an extension of Léandre's "elementary integration by parts formula" (Theorem 2.2 in Léandre (2006)) presented in Proposition 2.1, and an application of the Bismut identity (e.g. chapter IX-7 of Malliavin (1997)). Also, this derivation can be regarded as an extension of the PDE weight method in Malliaivn-Thalmaier (2006) to an asymptotic expansion of the solutions of the PDEs. As for explanation of Léandre's approach and its connection with our method, please see Takahashi and Yamada (2010).

Moreover, our method has an advantage in a sense that our computational scheme can be applied in a unified way to obtaining derivatives' prices and *Greeks* under various (multi-dimensional) diffusion models.

In addition, we apply this method to deriving a short-time asymptotic expansion of the heat kernel under the general diffusion setting which includes general time-homogenous local volatility, Heston and (λ-)SABR models as special cases; for the local volatility model, we also show how to compute the coefficients in the expansion by using the Lie bracket. Furthermore, we note that the similar method can be applied to a certain class of non-linear parabolic partial differential equations though this paper explicitly deals with the linear PDEs. (Please see Remark 2.1.)

There are many approaches for approximations of heat kernels through certain asymptotic expansions: for instance, there are recent works such as Baudoin (2009), Gatheral, Hsu, Laurence, Ouyang and Wang

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(2009), Ben Arous and Laurence (2009), Ilhan, Jonsson and Sircar (2004), Takahashi, Takehara and Toda (2012) and Takahashi and Yamada (2012).

Related to our work on approximation of the solutions to second-order parabolic equations and its applications to option pricing, Cheng et al. (2010, 2011) have been developing a new method called Dyson-Taylor Commutator method. Moreover, Kato, Takahashi and Yamada (2012) has developed an asymptotic expansion for solutions of Cauchy-Dirichlet problem for second-order parabolic PDEs; as an application, they have derived a new approximation formula for pricing barrier options under stochastic volatility setting. (Please see Remark 2.2 and Remark 3.1 below.) Furthermore, Fujii and Takahashi (2011) has developed a new approximation method for the solutions to the nonlinear PDEs associated with the four step scheme for solving forward backward stochastic differential equations (FBSDEs).

The organization of the paper is as follows: the next section derives an integration by parts formula as an extension of a Léandre's theorem and then provides an approximation to the solution of second-order linear parabolic PDEs. Section 3 applies the method developed in the previous section to finance; we derive approximate formulas for the option price and vega under local/stochastic volatility environment. Section 4 derives a *short-time asymptotic expansion* using integration by parts formula. Section 5 shows examples of the short-time asymptotic expansion under general time-homogeneous local volatility, stochastic volatility model with log-normal local volatility and general local-stochastic volatility models. We also provide numerical examples of the short-time asymptotic expansion under Heston model.

Finally, Appendix summarizes the calculation of the second order approximation in Section 5.1.

2 Integration by Parts Formula and Asymptotic Expansion of the Solution to Parabolic PDEs

Léandre (2006, 2008) reveals the connections between the semigroup theory and Malliavin calculus. In particular, his "elementary integration by parts formula" (Theorem 2.2 in Léandre (2006)) provides a nice idea for an approximation of the solutions to second-order parabolic PDEs.

In this section, we will extend Léandre's "elementary integration by parts formula" to Proposition 2.1 below, and present an approximation formula ((2.20) in Theorem 2.1) of the solution to a second-order linear parabolic partial differential equation:

Let $X^{(\epsilon)}$ be the unique solution to the following n-dimensional perturbed SDE: for $\epsilon \in [0,1]$,

$$\begin{aligned} dX_t^{(\epsilon)} &= \sum_{k=1}^{d} V_k(\epsilon, X_t^{(\epsilon)}) \circ dW_t^k + V_0(\epsilon, X_t^{(\epsilon)})dt, \\ X_0 &= x \in \mathbf{R}^n, \end{aligned} \tag{2.1}$$

or

$$\begin{aligned} dX_t^{(\epsilon)} &= \sum_{k=1}^{d} V_k(\epsilon, X_t^{(\epsilon)}) dW_t^k + \tilde{V}_0(\epsilon, X_t^{(\epsilon)})dt, \\ X_0 &= x \in \mathbf{R}^n, \end{aligned} \tag{2.2}$$

where $V_k = (V_k^1, \cdots, V_k^n)$ $(k = 0, 1, \cdots, d)$ have bounded derivatives of any orders in the variables (ϵ, x) and

$$\tilde{V}_0^i(\epsilon, x) = V_0^i(\epsilon, x) + \frac{1}{2}\sum_{l=1}^{n}\sum_{k=1}^{d} \partial_l V_k^i(\epsilon, x) V_k^l(\epsilon, x).$$

Here, "$\circ$" indicates the stochastic differential in the Stratonovich sense.

Also, consider the following $n \times n$ matrix-valued process, $\{U_t^{(\epsilon)} : (U_j^{(\epsilon),i}(t)), 1 \le i, j \le n, 0 \le t\}$,

$$\begin{aligned} dU_j^{(\epsilon),i}(t) &= \sum_{l=1}^{d}\sum_{k=1}^{n} A_{k,l}^{(\epsilon),i}(s) U_j^{(\epsilon),k}(s) \circ dW_t^l + \sum_{k=1}^{n} B_k^{(\epsilon),i}(s) U_j^{(\epsilon),k}(s) ds, \\ U_j^{(\epsilon),i}(0) &= \delta_j^i, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} A_{k,l}^{(\epsilon),i}(s) &= \partial_k V_l^i(\epsilon, X_s^{(\epsilon)}), \\ B_k^{(\epsilon),i}(s) &= \partial_k V_0^i(\epsilon, X_s^{(\epsilon)}) \end{aligned}$$

and δ^i_j is the Kronecker's delta, that is $U^{(\epsilon)}_0 = I$(the identity matrix). Specifically, for $U^{(0)}_t$,

$$\begin{aligned} A^{(0),i}_{k,l}(s) &= \left[\partial_k V^i_l(\epsilon, X^{(\epsilon)}_s)\right]\Big|_{\epsilon=0}, \\ B^{(0),i}_{k}(s) &= \left[\partial_k V^i_0(\epsilon, X^{(\epsilon)}_s)\right]\Big|_{\epsilon=0}. \end{aligned}$$

Let

$$X^{(1,\epsilon)}_t := \frac{\partial}{\partial \epsilon} X^{(\epsilon)}_t.$$

Then, we have

$$X^{(1,\epsilon)}_t = U^{(\epsilon)}_t \int_0^t \left(U^{(\epsilon)}_s\right)^{-1} \left(\sum_{k=1}^d \partial_\epsilon V_k(\epsilon, X^{(\epsilon)}_s) \circ dW^k_s + \partial_\epsilon V_0(\epsilon, X^{(\epsilon)}_s) ds\right), \tag{2.4}$$

where ∂_ϵ means $\frac{\partial}{\partial\epsilon}$. In particular,

$$\begin{aligned} X^{(1)}_t &:= X^{(1,0)}_t := \frac{\partial}{\partial\epsilon} X^{(\epsilon)}_t\Big|_{\epsilon=0} \\ &= U^{(0)}_t \int_0^t \left(U^{(0)}_s\right)^{-1} \left(\sum_{k=1}^d \left[\partial_\epsilon V_k(\epsilon, X^{(\epsilon)}_s)\right]\Big|_{\epsilon=0} \circ dW^k_s + \left[\partial_\epsilon V_0(\epsilon, X^{(\epsilon)}_s)\right]\Big|_{\epsilon=0} ds\right). \end{aligned}$$

Next, let $a^\epsilon(s)^i_k$, $1 \le i \le n$, $1 \le k \le d$, be the process;

$$a^\epsilon(s)^i_k = (U^\epsilon(s)^{-1} V_k(\epsilon, X^{(\epsilon)}_s))^i.$$

Then, the reduced Malliavin covariance $V^{(\epsilon)}_*(t) = \{(V^\epsilon_*(t))^{ij}\}_{i,j}$ is expressed as

$$(V^\epsilon_*(t))^{ij} = \sum_{k=1}^d \int_0^t a^\epsilon(s)^i_k a^\epsilon(s)^j_k ds. \tag{2.5}$$

Throughout this section, we assume the following non-degeneracy of the reduced Malliavin covariance:

$$[\mathrm{A1}] \ \sup_{\epsilon\in[0,1]} \mathbf{E}[(\det(V^{(\epsilon)}_*(t)))^{-p}] < \infty \text{ for } 1 < p < \infty. \tag{2.6}$$

Then, by Theorem 9.2 in Ikeda and Watanabe (1989), we obtain a smooth density, $y \mapsto p^\epsilon(t,x,y)$ associated with (2.1)((2.2)). Moreover, according to Remark 2.2 and Remark 2.3 in Watanabe (1987) as well as Proposition 2.2 in Ikeda and Watanabe (1989), we can see $p^\epsilon(t,x,y)$ is smooth in x and ϵ as well.

We next define $\hat{V}^\epsilon_k$ as

$$\hat{V}^\epsilon_k = \sum_{i=1}^n V^i_k(\epsilon, x)\frac{\partial}{\partial x_i}, \qquad k = 0,1,\cdots,d.$$

and

$$\mathcal{L}^\epsilon = \frac{1}{2}\sum_{k=1}^d (\hat{V}^\epsilon_k)^2 + \hat{V}^\epsilon_0.$$

Next, for $f \in C_b(\mathbf{R}^n)$, let

$$u^\epsilon(t,x) := \mathbf{P}^\epsilon_t f(x) := \mathbf{E}\left[f(X^{(\epsilon)}_t)\right] = \int_{\mathbf{R}^n} f(y) p^\epsilon(t,x,y) dy. \tag{2.7}$$

Then, $u^\epsilon(t,x)$ is the solution to the following PDE:

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \mathcal{L}^\epsilon\right) u^\epsilon(t,x) &= 0, \\ u^\epsilon(0,x) &= f(x). \end{aligned} \tag{2.8}$$

Also, let

$$u^0(t,x) := \mathbf{P}_t^0 f(x) := \mathbf{E}\left[f(X_t^{(0)})\right] = \int_{\mathbf{R}^n} f(y)p^0(t,x,y)dy, \tag{2.9}$$

where $p^0(t,x,y)$ is the smooth density for (2.1) with $\epsilon = 0$. Then, $u^0(t,x)$ is the solution to the following PDE:

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \mathcal{L}^0\right) u^0(t,x) &= 0, \\ u^0(0,x) &= f(x). \end{aligned} \tag{2.10}$$

2.1 Integration by Parts Formula

In this subsection, we will give the formula for $u^1(t,x) = \frac{\partial}{\partial \epsilon} u^{(\epsilon)}(t,x)\Big|_{\epsilon=0}$, and show that $u^1(t,x)$ satisfies the following PDE:

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \mathcal{L}^0\right) u^1(t,x) &= \mathcal{L}^1 u^0(t,x), \\ u^1(0,x) &= 0, \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} \mathcal{L}^1 := \frac{\partial}{\partial \epsilon}\mathcal{L}^\epsilon\Big|_{\epsilon=0} &= \frac{1}{2}\sum_{i,j=1}^n \sum_{k=1}^d \frac{\partial}{\partial \epsilon}\left[V_k^i(\epsilon,x)V_k^j(\epsilon,x)\right]\Big|_{\epsilon=0} \frac{\partial^2}{\partial x_i \partial x_j} \\ &+ \sum_{i=1}^n \frac{\partial}{\partial \epsilon} V_0^i(\epsilon,x)\Big|_{\epsilon=0} \frac{\partial}{\partial x_i}. \end{aligned} \tag{2.12}$$

For $X_t \in \mathbf{D}_1^2$, we denote $D_{s,k}X_t$ as the Malliavin derivative acting on the Brownian motion W^k, $k = 1, \cdots, d$. (Please see p.97 in Takahashi and Yamada (2012) for the details.) Then, we obtain the following proposition.

Proposition 2.1 *Let $\zeta^{0,(1)}(t)$ be the process given by*

$$\zeta^{0,(1)}(t)^l = \left(V_*^0(t)^{-1}U^0(t)^{-1}X_t^{(1)}\right)^l, \ 1 \le l \le n.$$

Then, the following formula holds:

$$u^1(t,x) = \int_0^t \mathbf{P}_{t-s}^0 \mathcal{L}^1[\mathbf{P}_s^0 f](x)ds = \tag{2.13}$$

$$E\left[f(X_t^{(0)})\sum_{l=1}^n\sum_{k=1}^d\left\{\zeta^{0,(1)}(t)^l\int_0^t a^0(s)_k^l dW_s^k - \int_0^t D_{s,k}\zeta^{0,(1)}(t)^l a^0(s)_k^l ds\right\}\right] = \tag{2.14}$$

$$\int_{\mathbf{R}^n} f(y)w(y)p^0(t,x,y)dy, \tag{2.15}$$

where $y \mapsto w(y)$ is a smooth function given by

$$w(y) = E\left[\sum_{l=1}^n\sum_{k=1}^d\left\{\zeta^{0,(1)}(t)^l\int_0^t a^0(s)_k^l dW_s^k - \int_0^t D_{s,k}\zeta^{0,(1)}(t)^l a^0(s)_k^l ds\right\}|X_t^{(0)} = y\right].$$

(Proof)
Let $\{f_n\}_n \subset C_b^\infty(\mathbf{R}^n)$ be a sequence such that $f_n \to f$ as $n \to \infty$. For $E[f_n(X_t^{(\epsilon)})]$, we can differentiate with respect to ϵ (and set $\epsilon = 0$) as follows;

$$\begin{aligned} &\frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} E[f_n(X_t^{(\epsilon)})] \\ = &\sum_{i=1}^n E\left[\frac{\partial}{\partial x_i} f_n(X_t^{(0)})\ \frac{\partial}{\partial \epsilon}X_{it}^{(\epsilon)}\Big|_{\epsilon=0}\right] \end{aligned}$$

$$
\begin{aligned}
&= E\left[\nabla f_n(X_t^{(0)}) \cdot X_t^{(1)}\right] \\
&= E\left[\nabla f_n(X_t^{(0)}) \cdot U_t V_*^0(t) V_*^0(t)^{-1} U_t^{-1} X_t^{(1)}\right] \\
&= \sum_{i=1}^n \sum_{m=1}^n \sum_{l=1}^n E\left[\frac{\partial}{\partial x_i} f_n(X_t^{(0)}) U(t)_m^i (V_*^0(t))_l^m \left((V_*^0(t))^{-1} U_t^{-1} X_t^{(1)}\right)^l\right] \\
&= \sum_{i=1}^n \sum_{m=1}^n \sum_{l=1}^n E\left[\frac{\partial}{\partial x_i} f_n(X_t^{(0)}) U(t)_m^i \left(\sum_{k=1}^d \int_0^t (U_s^{-1} V_k(X_s^{(0)}))^m (U_s^{-1} V_k(X_s^{(0)}))^l ds\right) \left((V_*^0(t))^{-1} U_t^{-1} X_t^{(1)}\right)^l\right] \\
&= \sum_{i=1}^n \sum_{l=1}^n \sum_{k=1}^d E\left[\frac{\partial}{\partial x_i} f_n(X_t^{(0)}) \int_0^t \left(\sum_{m=1}^n U(t)_m^i (U_s^{-1} V_k(X_s^{(0)}))^m\right) (U_s^{-1} V_k(X_s^{(0)}))^l ds \left((V_*^0(t))^{-1} U_t^{-1} X_t^{(1)}\right)^l\right] \\
&= \sum_{i=1}^n \sum_{l=1}^n \sum_{k=1}^d E\left[\frac{\partial}{\partial x_i} f_n(X_t^{(0)}) \int_0^t (U(t) U_s^{-1} V_k(X_s^{(0)}))^i (U_s^{-1} V_k(X_s^{(0)}))^l ds \left((V_*^0(t))^{-1} U_t^{-1} X_t^{(1)}\right)^l\right] \\
&= \sum_{l=1}^n \sum_{k=1}^d E\left[\int_0^t \sum_{i=1}^n \frac{\partial}{\partial x_i} f_n(X_t^{(0)}) (U(t) U_s^{-1} V_k(X_s^{(0)}))^i \left((V_*^0(t))^{-1} U_t^{-1} X_t^{(1)}\right)^l (U_s^{-1} V_k(X_s^{(0)}))^l ds\right] \\
&= \sum_{l=1}^n \sum_{k=1}^d E\left[\int_0^t [D_{s,k} f_n(X_t^{(0)})] \zeta^{0,(1)}(t)^l a^0(s)_k^l ds\right].
\end{aligned}
$$

In the above equality, $U \equiv U^0$, and we used the following relation.

$$
D_{s,k} f_n(X_t^{(0)}) = \sum_{i=1}^n \frac{\partial}{\partial x_i} f_n(X_t^{(0)}) (U(t) U_s^{-1} V_k(X_s^{(0)}))^i.
$$

For $g = (g^1, \cdots, g^n)$, $g^l = f_n(X_t^{(0)}) \zeta^{0,(1)}(t)^l$, we have

$$
\sum_{l=1}^n \sum_{k=1}^d E\left[g^l \int_0^t a^0(s)_k^l dW_s^k\right] = \sum_{l=1}^n \sum_{k=1}^d E\left[\int_0^t D_{s,k} g^l a^0(s)_k^l ds\right],
$$

and

$$
D_{s,k} g^l = D_{s,k}[f_n(X_t^{(0)}) \zeta^{0,(1)}(t)^l] = [D_{s,k} f_n(X_t^{(0)})] \zeta^{0,(1)}(t)^l + f_n(X_t^{(0)}) [D_{s,k} \zeta^{0,(1)}(t)^l].
$$

Then,

$$
\begin{aligned}
&\sum_{l=1}^n \sum_{k=1}^d E\left[\int_0^t [D_{s,k} f_n(X_t^{(0)})] \zeta^{0,(1)}(t)^l a^0(s)_k^l ds\right] \\
&= \sum_{l=1}^n \sum_{k=1}^d E\left[f_n(X_t^{(0)}) \left\{\zeta^{0,(1)}(t)^l \int_0^t a^0(s)_k^l dW_s^k - \int_0^t D_{s,k} \zeta^{0,(1)}(t)^l a^0(s)_k^l ds\right\}\right].
\end{aligned}
$$

Therefore, we obtain the following formula.

$$
\begin{aligned}
\frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} E[f_n(X_t^{(\epsilon)})] &= \sum_{l=1}^n \sum_{k=1}^d E\left[f_n(X_t^{(0)}) \left\{\zeta^{0,(1)}(t)^l \int_0^t a^0(s)_k^l dW_s^k - \int_0^t D_{s,k} \zeta^{0,(1)}(t)^l a^0(s)_k^l ds\right\}\right] \\
&= \int_{\mathbf{R}^n} f_n(y) w(y) p^0(t, x, y) dy,
\end{aligned}
$$

where

$$
w(y) = E\left[\sum_{l=1}^n \sum_{k=1}^d \left\{\zeta^{0,(1)}(t)^l \int_0^t a^0(s)_k^l dW_s^k - \int_0^t D_{s,k} \zeta^{0,(1)}(t)^l a^0(s)_k^l ds\right\} |X_t^{(0)} = y\right].
$$

The following estimates hold:

$$
\begin{aligned}
&|E[f(X_t^{(0)})] - E[f_n(X_t^{(0)})]| \leq \|f - f_n\|_\infty, \\
&\left|\frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} E[f_n(X_t^{(\epsilon)})] - E[f(X_t^{(0)}) \pi]\right| \leq \|f - f_n\|_\infty \|\pi\|_{L^1},
\end{aligned}
$$

where

$$\pi = \sum_{l=1}^{n}\sum_{k=1}^{d}\left\{\zeta^{0,(1)}(t)^l \int_0^t a^0(s)_k^l dW_s^k - \int_0^t D_{s,k}\zeta^{0,(1)}(t)^l a^0(s)_k^l ds\right\}.$$

Therefore, we obtain as $n \to \infty$,

$$\begin{aligned}
u^1(t,x) &= \left.\frac{\partial}{\partial \epsilon}\right|_{\epsilon=0} E[f(X_t^{(\epsilon)})] \\
&= \sum_{l=1}^{n}\sum_{k=1}^{d} E\left[f(X_t^{(0)})\left\{\zeta^{0,(1)}(t)^l \int_0^t a^0(s)_k^l dW_s^k - \int_0^t D_{s,k}\zeta^{0,(1)}(t)^l a^0(s)_k^l ds\right\}\right] \\
&= \int_{\mathbf{R}^n} f(y)w(y)p^0(t,x,y)dy.
\end{aligned}$$

Alternatively, let $\Xi_s = \mathbf{P}_{t-s}^0\mathbf{P}_s^\epsilon f(x)$. Then, we have

$$\mathbf{P}_t^\epsilon f(x) - \mathbf{P}_t^0 f(x) = \Xi_t - \Xi_0 = \int_0^t \frac{d}{ds}(\Xi_s)ds = \int_0^t \mathbf{P}_{t-s}^0[\mathcal{L}^\epsilon - \mathcal{L}^0]\mathbf{P}_s^\epsilon f(x)ds.$$

Hence, using (2.12), we obain

$$\begin{aligned}
u^1(t,x) = \left.\frac{\partial}{\partial \epsilon}u^\epsilon(t,x)\right|_{\epsilon=0} &= \lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\left[\mathbf{P}_t^\epsilon f(x) - \mathbf{P}_t^0 f(x)\right] \\
&= \lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\int_0^t \mathbf{P}_{t-s}^0\left[\mathcal{L}^\epsilon - \mathcal{L}^0\right]\mathbf{P}_s^\epsilon f(x)ds \\
&= \int_0^t \mathbf{P}_{t-s}^0\mathcal{L}^1\mathbf{P}_s^0 f(x)ds.
\end{aligned} \tag{2.16}$$

Also, we easily see that

$$\frac{\partial}{\partial t}\left(\int_0^t \mathbf{P}_{t-s}^0\mathcal{L}^1\mathbf{P}_s^0 f(x)ds\right) = \mathcal{L}^0\int_0^t \mathbf{P}_{t-s}^0\mathcal{L}^1\mathbf{P}_s^0 f(x)ds + \mathcal{L}^1\mathbf{P}_t^0 f(x),$$

and hence, $F(t,x) := \int_0^t \mathbf{P}_{t-s}^0\mathcal{L}^1\mathbf{P}_s^0 f(x)ds$ satisfies (2.11) with starting condition 0.

2.2 Asymptotic Expansion

Let $H_{(1)}(X_t^{(\epsilon),x},\cdot) : \mathbf{D}_\infty \to \mathbf{D}_\infty$ be the divergence operator (Malliavin weight) defined by *the Bismut identity* (pp.247-248 in Malliavin (1997)):

$$H_{(1)}(X_t^{(\epsilon),x},\Psi_t) = \sum_{i=1}^{n}\sum_{k=1}^{d}\left[\zeta^\epsilon(t)^i \int_0^t a^\epsilon(s)_k^i dW_s^k - \int_0^t D_{s,k}\zeta^\epsilon(t)^i a^\epsilon(s)_k^i ds\right], \tag{2.17}$$

where Ψ_t is a smooth functional in the Malliavin sense, $\Psi_t \in \mathbf{D}_\infty$, and

$$\zeta^\epsilon(t)^i = \left(V_*^\epsilon(t)^{-1}U^\epsilon(t)^{-1}\Psi_t\right)^i.$$

The iterated Malliavin weight H_k is recursively defined as follows:

$$H_k(X_t^{(\epsilon),x},\Psi_t) = H_{(1)}(X_t^{(\epsilon),x}, H_{k-1}(X_t^{(\epsilon),x},\Psi_t)),$$

with

$$H_0(X_t^{(\epsilon),x},\Psi_t) \equiv \Psi_t.$$

The next theorem is our main result in this section.

Theorem 2.1 *Consider the following PDE with its initial condition $f \in C_b(\mathbf{R}^n)$:*

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \mathcal{L}^\epsilon\right) u^\epsilon(t,x) &= 0, \\ u^\epsilon(0,x) &= f(x). \end{aligned} \tag{2.18}$$

Then, its solution

$$u^\epsilon(t,x) = \mathbf{P}_t^\epsilon f(x) = \mathbf{E}\left[f(X_t^{(\epsilon)})\right] = \int_{\mathbf{R}^n} f(y)p^\epsilon(t,x,y)dy, \tag{2.19}$$

has an asymptotic expansion in $\mathbf{R}$*:*

$$\mathbf{P}_t^\epsilon f(x) = \left\{\mathbf{P}_t^0 f(x) + \sum_{j=1}^N \epsilon^j a_j(x)\right\} + O(\epsilon^{N+1}), \tag{2.20}$$

where

$$\begin{aligned} & a_j(x) \\ = & \sum_{k=1}^{j} \sum_{\beta_1+\cdots+\beta_k=j,\beta_i\geq 1} \qquad\qquad (2.21) \\ & \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} \mathbf{P}_{t-t_1}^0 \mathcal{L}^{\beta_1} \mathbf{P}_{t_1-t_2}^0 \mathcal{L}^{\beta_2} \cdots \mathbf{P}_{t_{k-1}-t_k}^0 \mathcal{L}^{\beta_k} \mathbf{P}_{t_k}^0 f(x) dt_k \cdots dt_2 dt_1 \\ = & E\left[f(X_t^{(0),x}) \sum_k^{(j)} H_k(X_t^{(0),x}, \prod_{l=1}^k X_{\alpha_l t}^{0,\beta_l})\right] \qquad\qquad (2.22) \\ = & \int_{\mathbf{R}^n} f(y) w_j(y) p^0(t,x,y) dy, \end{aligned}$$

with $\mathcal{L}^k := \frac{1}{k!}\frac{d^k}{d\epsilon^k}\mathcal{L}^\epsilon|_{\epsilon=0}$, $k \in \mathbf{N}$, *and*

$$\sum_k^{(j)} \equiv \sum_{k=1}^j \sum_{\beta_1+\cdots+\beta_k=j,\beta_i\geq 1} \frac{1}{k!}.$$

Here, the so called Malliavin weight $H_k(X_t^{(0),x}, \prod_{l=1}^k X_{\alpha_l t}^{0,\beta_l})$ *is defined by (2.17) and the push-down of the Malliavin weight* $w_j \in \mathcal{S}$ *is given by*

$$w_j(y) = E\left[\sum_k^{(j)} H_k(X_t^{(0),x}, \prod_{l=1}^k X_{\alpha_l t}^{0,\beta_l}) | X_t^{(0),x} = y\right], \tag{2.23}$$

where $X_{it}^{(0),k} := \frac{1}{k!}\frac{d^k}{d\epsilon^k}X_{it}^{(\epsilon)}|_{\epsilon=0}$, $k \in \mathbf{N}$, $i = 1, \cdots, n$. *Moreover, we obtain a heat kernel expansion in* $\mathbf{R}$*:*

$$p^\epsilon(t,x,y) = p^0(t,x,y) + \sum_{j=1}^N \epsilon^j w_j(y) p^0(t,x,y) + O(\epsilon^{N+1}). \tag{2.24}$$

(Proof)
We can recursively apply the integration by parts in **Proposition 2.1**

$$u^j(t,x) := \frac{1}{j!}\frac{\partial^j}{\partial \epsilon^j}\mathbf{P}_t^\epsilon f(x)|_{\epsilon=0} = \int_{\mathbf{R}^n} f(y) w_j(y) p^0(t,x,y) dy,$$

where

$$w_j(y) = E\left[\sum_k^{(j)} H_k(X_t^{(0),x}, \prod_{l=1}^k X_{\alpha_l t}^{0,\beta_l}) | X_t^{(0),x} = y\right].$$

Then, we have

$$\mathbf{P}^{\epsilon}_{t}f(x) = \mathbf{P}^{0}_{t}f(x) + \sum_{j=1}^{N}\epsilon^{j}u^{j}(t,x) + \epsilon^{N+1}\mathbf{R}_{N}(\epsilon).$$

where the remainder terms $\mathbf{R}_N(\epsilon)$,

$$\mathbf{R}_N(\epsilon) \quad = \quad \int_0^1 \frac{(1-v)^N}{N!}E\left[f(X_t^{(\epsilon v)})\sum_{k}^{(N+1)}H_k(X_t^{(\epsilon v)},\prod_{l=1}^{k}X_{\alpha_l t}^{(\epsilon v),\beta_l}))\right]dv,$$

which satsfies

$$E[|\mathbf{R}_N(\epsilon)|] \leq C(T)\|f\|_\infty E[(\det(V_*^{(\epsilon)}(t)))^{-\gamma}]^\beta < \infty,$$

for some $C(T),\gamma,\beta$. (See P.102 in Nualart (2006) for instance.)

Alternatively, we can recursively obtain the following expression of $u^j(t,x)$ in the similar way for obtaining (2.16) in the proof of **Proposition 2.1**:

$$u^j(t,x) = \sum_{k=1}^{j}\sum_{\beta_1+\cdots+\beta_k=j,\beta_i\geq 1}\int_0^t\int_0^{t_1}\cdots\int_0^{t_{k-1}}\mathbf{P}^0_{t-t_1}\mathcal{L}^{\beta_1}\mathbf{P}^0_{t_1-t_2}\mathcal{L}^{\beta_2}\cdots\mathbf{P}^0_{t_{k-1}-t_k}\mathcal{L}^{\beta_k}\mathbf{P}^0_{t_k}f(x)dt_k\cdots dt_2dt_1.$$

Also, it is easily seen that $u^j(t,x)$ satisfies the following equation:

$$\left(\frac{\partial}{\partial t}-\mathcal{L}^0\right)u^j(t,x) = \mathcal{L}^1u^{j-1}(t,x)+\cdots+\mathcal{L}^ju^0(t,x).$$

Moreover, if we take a sequence $\{f_n\}_{n\in\mathbf{N}}$ such that $f_n\in\mathcal{S}$, $f_n\to\delta_y$ as $n\to\infty$, we have

$$\mathbf{P}^{\epsilon}_{t}f_n(x) = {}_{\mathcal{S}'}\langle f_n, p^\epsilon(t,x,\cdot)\rangle_{\mathcal{S}} \to {}_{\mathcal{S}'}\langle \delta_y, p^\epsilon(t,x,\cdot)\rangle_{\mathcal{S}} = p^\epsilon(t,x,y),\ n\to\infty.$$

Then, the following heat kernel expansion holds :

$$p^\epsilon(t,x,y) = p^0(t,x,y) + \sum_{j=1}^{N}\epsilon^j w_j(y)p^0(t,x,y) + O(\epsilon^{N+1}),$$

Therefore, we obtain the results.

Remark 2.1 *Let us consider the solution of the PDE:*

$$\left\{\begin{array}{l}\left(\frac{\partial}{\partial t}+\mathcal{L}^\epsilon\right)u^\epsilon(t,x)=0,\\ u^\epsilon(T,x)=f(x).\end{array}\right. \tag{2.25}$$

Suppose $u^\epsilon(t,x)$ is expanded by a perturbation method as

$$u^\epsilon(t,x) = u^0(t,x)+\epsilon u^1(t,x)+\epsilon^2u^2(t,x)+\cdots.$$

In order to obtain $u^i(t,x)$, $i=0,1,2$ for instance, we formally expand the PDE:

$$\left(\frac{\partial}{\partial t}+\mathcal{L}^0+\epsilon\mathcal{L}^1+\epsilon^2\mathcal{L}^2+\cdots\right)\left(u^0(t,x)+\epsilon u^1(t,x)+\epsilon^2u^2(t,x)+\cdots\right)=0,$$

where $\mathcal{L}^i = \frac{1}{i!}\frac{\partial^i\mathcal{L}^\epsilon}{\partial\epsilon^i}|_{\epsilon=0}$.

Then, $u^i(t,x)$, $i=0,1,2$ satisfy the following PDEs:

$$\left\{\begin{array}{l}(\frac{\partial}{\partial t}+\mathcal{L}^0)u^0(t,x)=0,\\ u^0(T,x)=f(x),\end{array}\right.$$

$$\left\{\begin{array}{l}(\frac{\partial}{\partial t}+\mathcal{L}^0)u^1(t,x)=-\mathcal{L}^1u^0(t,x),\\ u^1(T,x)=0,\end{array}\right.$$

$$\left\{\begin{array}{l}(\frac{\partial}{\partial t}+\mathcal{L}^0)u^2(t,x)=-(\mathcal{L}^1u^1(t,x)+\mathcal{L}^2u^0(t,x)),\\ u^2(T,x)=0.\end{array}\right.$$

Theorem 2.1 provides a solution to this problem. We note that the same method can be applied, at least formally to a certain class of non-linear parabolic partial differential equations although Theorem 2.1 explicitly deals with the linear ones. A simple example is as follows:

$$(\partial_t + \mathcal{L}^\epsilon)u^\epsilon(t,x) = 0,\ (t < T);\ u^\epsilon(T,x) = f(x) \tag{2.26}$$

$$\mathcal{L}^\epsilon = \frac{1}{2}\sigma(u^\epsilon, \partial_x u^\epsilon)^2 \partial_{xx}, \tag{2.27}$$

$$\sigma(u^\epsilon, \partial_x u^\epsilon) = 1 + \epsilon(u^\epsilon + \partial_x u^\epsilon), \tag{2.28}$$

In this case, we have

$$\begin{aligned}
\mathcal{L}^0 &= \frac{1}{2}\partial_{xx}, \\
\mathcal{L}^1 &= (u^0(t,x) + \partial_x u^0(t,x))\partial_{xx}, \\
\mathcal{L}^2 &= \frac{1}{2}\left\{(u^0(t,x) + \partial_x u^0(t,x))^2 + 2(u^1 + \partial_x u^1)\right\}\partial_{xx}.
\end{aligned}$$

Hence,

$$(\partial_t + \frac{1}{2}\partial_{xx})u^0(t,x) = 0;\ u^0(T,x) = f(x), \tag{2.29}$$

$$(\partial_t + \frac{1}{2}\partial_{xx})u^1(t,x) = -(u^0(t,x) + \partial_x u^0(t,x))\partial_{xx}u^0(t,x);\ u^1(T,x) = 0, \tag{2.30}$$

$$(\partial_t + \frac{1}{2}\partial_{xx})u^2(t,x) = -(\mathcal{L}^1 u^1(t,x) + \mathcal{L}^2 u^0(t,x));\quad u^2(T,x) = 0. \tag{2.31}$$

$u^0(t,x)$ is easily solved by (2.29):

$$u^0(t,x) \quad = \quad \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(T-t)}} e^{\frac{-(z-x)^2}{2(T-t)}} f(z)dz.$$

Then, given $u^0(t,x)$, the right hand side of (2.30) is easily computed and so $u^1(t,x)$ is solved, too:

$$\begin{aligned}
u^1(t,x) \quad &= \quad \mathbf{E}^{(t,x)}\left[\int_t^T g(s, W_s)ds\right] \\
&= \quad \int_t^T \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(s-t)}} e^{\frac{-(z-x)^2}{2(s-t)}} g(s,z)dz\right) ds,
\end{aligned} \tag{2.32}$$

where

$$g(s,z) \quad = \quad (u^0(s,z) + \partial_x u^0(s,z))\partial_{xx}u^0(s,z).$$

Recursively, given $u^0(t,x)$ and $u^1(t,x)$, $u^2(t,x)$ is obtained by (2.31).

Moreover, please see Fujii and Takahashi (2011) for the details, which has developed a new general approximation method for the solutions to the nonlinear PDEs associated with the four step scheme for solving forward backward stochastic differential equations (FBSDEs).

Remark 2.2 *If we try to derive the closed form approximation of the solution to Cauchy-Dirichlet problem for second order parabolic PDEs, which is an expectation including the exit time τ of a domain D such as $E[f(X_t)\mathbf{1}_{\{\tau > t\}}]$, the Malliavin calculus approach fails because Malliavin derivative $D_t\tau$ does not exist (see Fournié et al. (2001)). Therefore, we cannot approximate analytically the solution to Cauchy-Dirichlet problem by applying the Malliavin's integration by parts (2.22). However, Kato, Takahashi and Yamada (2012) has developed an asymptotic expansion for solutions of Cauchy-Dirichlet problem for second order parabolic PDEs and showed a similar formula as (2.20) with (2.21) still holds.*

3 Perturbations around Closed Form Solutions : Application to Options

In this section, we derive approximation formulas for an option's vega and price in local/stochastic volatility models using the expansion methods of semi-group developed in Section 2. Hereafter, we use the notation $\int T(x)p(x)dx$ for $T \in \mathcal{S}'(\mathbf{R}^n)$ and $p \in \mathcal{S}(\mathbf{R}^n)$ meaning that ${}_{\mathcal{S}'}\langle T, p\rangle_{\mathcal{S}}$.

3.1 Vega Weight

Fournié et al. (1999) derive the greeks weights using Malliavin calculus. In this subsection, we obtain the Malliavin weight for the plain-vanilla option's Vega(Vega weight) by the Bismut identity and show how to derive the analytic approximation of option price using the Vega weight. Let us consider the following asset price dynamics:

$$dS_t = \sigma(S_t)dW_t, \tag{3.1}$$

where S_0 is a constant and $\sigma(x) > 0$. We also consider the perturbed diffusion with $\sigma^{(\epsilon)}(x) = \sigma(x) + \epsilon\tilde{\sigma}(x)$, where $\tilde{\sigma}(x) = c \cdot \sigma(x)$ for some positive constant c :

$$\begin{aligned} dS_t^{(\epsilon)} &= \sigma^{(\epsilon)}(S_t^{(\epsilon)})dW_t, \\ S_0^{(\epsilon)} &= S_0. \end{aligned} \tag{3.2}$$

Then, the vega of the plain-vanilla (call) option is defined as

$$vega^{LV} := \frac{\partial}{\partial\epsilon} E[(S_T^{(\epsilon)} - K)^+]|_{\epsilon=0}. \tag{3.3}$$

Under appropriate conditions, $vega^{LV}$ is given by

$$\begin{aligned} vega^{LV} &= E\left[\partial(S_T^{(0)} - K)^+ \frac{\partial}{\partial\epsilon} S_T^{(\epsilon)}|_{\epsilon=0}\right] \\ &= E\left[(S_T^{(0)} - K)^+ H_{(1)}\left(S_T^{(0)}, \frac{\partial}{\partial\epsilon} S_T^{(\epsilon)}|_{\epsilon=0}\right)\right] \\ &= \int_{\mathbf{R}} (z-K)^+ \vartheta(z) p^{S^{(0)}}(T-t,s,z)dz, \end{aligned} \tag{3.4}$$

where $H_{(1)}\left(S_T^{(0)}, \frac{\partial}{\partial\epsilon} S_T^{(\epsilon)}|_{\epsilon=0}\right)$ is the Malliavin weight for $vega^{LV}$, $\vartheta(z)$ is its push-down, and $p^{S^{(0)}}(T-t,s,z)$ is the density function of $S_T^{(0)}$ given $S_t^{(0)} = s$.

Hence, a European call option price for its underlying asset price $S^{(\epsilon)}$ with maturity T and strike K is approximated as follows:

$$\begin{aligned} & C^{\epsilon}(T-t,s,K) = E_{(t,s)}[(S_T^{(\epsilon)} - K)^+] \\ \sim & \int_{\mathbf{R}} (z-K)^+ p^{S^{(0)}}(T-t,s,z)dz + \epsilon \cdot vega^{LV}, \end{aligned} \tag{3.5}$$

where we assume zero interest and dividend rates.

We illustrate this by using a simple case, $\sigma^{(\epsilon)}(x) = (\sigma+\epsilon)x$:

$$\begin{aligned} dS_t^{(\epsilon)} &= (\sigma+\epsilon)S_t^{(\epsilon)}dW_t, \\ S_0^{(\epsilon)} &= S_0. \end{aligned} \tag{3.6}$$

The logarithmic process of $S_t^{(\epsilon)}$ is given by,

$$\begin{aligned} dX_t^{(\epsilon)} &= (\sigma+\epsilon)dW_t - \frac{1}{2}(\sigma+\epsilon)^2 dt. \\ X_0^{(\epsilon)} &= \log S_0. \end{aligned}$$

The associated partial differential equation is given by

$$\begin{aligned} (\partial_t + \mathcal{L}^{\epsilon})u^{\epsilon}(t,x) &= 0, \\ u^{(\epsilon)}(T,x) &= f(e^x), \end{aligned}$$

where $\mathcal{L}^{\epsilon}$ is the generator of $X_t^{(\epsilon)}$, *i.e.*

$$\mathcal{L}^{\epsilon} = \frac{1}{2}(\sigma+\epsilon)^2\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right),$$

and $f \in C_b^{\infty}$.

The Vega is calculated in the following way. Let us consider the process,

$$\begin{aligned} dU_u &= \sigma U_u dW_u, \\ U_t &= 1. \end{aligned}$$

and introduce the process $a(u)$,

$$a(u) = U_u^{-1} \sigma S_u^{(0)},$$

with

$$S_u^{(0)} = s e^{\sigma(W_u - W_t) - \frac{1}{2}\sigma^2 (u-t)}.$$

Let $C(T)$ be the reduced Malliavin covariance,

$$C(T) \quad := \quad \int_t^T a(u)^2 ds = \int_t^T \left(\frac{s}{S_u^{(0)}} \sigma S_u^{(0)} \right)^2 du = (s\sigma)^2 (T-t).$$

Next, we differentiate the underlying asset price at time T with respect to ϵ at $\epsilon = 0$:

$$\frac{\partial}{\partial \epsilon} S_T^{(\epsilon)}|_{\epsilon=0} \quad = \quad S_T^{(0)} (W_T - W_t - \sigma(T-t)).$$

We define the process $\xi^{0,(1)}(t)$ and $\zeta^{0,(1)}(t)$ as

$$\begin{aligned} \xi^{0,(1)}(T) &:= U_T^{-1} \frac{\partial}{\partial \epsilon} S_T^{(\epsilon)}|_{\epsilon=0} = \frac{s}{S_T^{(0)}} S_T^{(0)} (W_T - W_t - \sigma(T-t)) = s(W_T - W_t - \sigma T), \\ \zeta^{0,(1)}(T) &:= C(T)^{-1} \xi^{0,(1)}(T) = \frac{1}{s\sigma^2 (T-t)} (W_T - W_t - \sigma(T-t)). \end{aligned}$$

Then, the Malliavin derivative of $\zeta^{0,(1)}(t)$ is given by

$$D_{u,1} \zeta^{0,(1)}(T) = \frac{1}{s\sigma^2 (T-t)} \mathbf{1}_{t<u\le T}.$$

By the integration by parts derived in section **3.2**, Vega is calculated as follows.

$$\begin{aligned} & \frac{\partial}{\partial \epsilon} E_{(t,s)} [(S_T^{(\epsilon)} - K)^+]|_{\epsilon=0} \\ = \;& E_{(t,s)} \left[(S_T^{(0)} - K)^+ \left\{ \zeta^{0,(1)}(T) \int_t^T a(u) dW_u - \int_t^T D_{u,1} \zeta^{0,(1)}(T) a(u) du \right\} \right] \\ = \;& E_{(t,s)} \left[(S_T^{(0)} - K)^+ \left\{ \frac{1}{s\sigma^2(T-t)} (W_T - W_t - \sigma(T-t)) \int_t^T s\sigma dW_u - \int_t^T \frac{1}{s\sigma^2 (T-t)} s\sigma du \right\} \right] \\ = \;& E_{(t,s)} \left[(S_T^{(0)} - K)^+ \left\{ \frac{1}{\sigma(T-t)} (W_T - W_t)^2 - (W_T - W_t) - \frac{1}{\sigma} \right\} \right] \\ = \;& \int_{\mathbf{R}} (e^z - K)^+ E\left[\frac{1}{\sigma(T-t)} W_{T-t}^2 - W_{T-t} - \frac{1}{\sigma} | X_T^{t,x} = z \right] \frac{1}{\sqrt{2\pi\sigma^2 (T-t)}} e^{-\frac{(z-x-\frac{1}{2}\sigma^2(T-t))^2}{2\sigma^2(T-t)}} dz \\ = \;& \int_{\mathbf{R}} (e^z - K)^+ \vartheta(z) \frac{1}{\sqrt{2\pi\sigma^2 (T-t)}} e^{-\frac{(z-x-\frac{1}{2}\sigma^2(T-t))^2}{2\sigma^2(T-t)}} dz, \end{aligned}$$

where

$$\begin{aligned} \vartheta(z) &:= E\left[\frac{1}{\sigma(T-t)} W_{T-t}^2 - W_{T-t} - \frac{1}{\sigma} | X_T^{t,x} = z \right] \\ &= \frac{1}{\sigma^3 (T-t)} \left(z - x + \frac{1}{2}\sigma^2 (T-t) \right)^2 - \frac{1}{\sigma} \left(z - x + \frac{1}{2}\sigma^2(T-t) \right) - \frac{1}{\sigma}. \end{aligned} \tag{3.7}$$

Equivalently, we can calculate Vega by differentiating the semi-group. Recall that $\mathcal{L}^{(\epsilon)}$ is the generator of $X_t^{(\epsilon)}$,

$$\mathcal{L}^{\epsilon} = \frac{1}{2} (\sigma + \epsilon)^2 \left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} \right),$$

We define the differential operators $\mathcal{L}^0$, $\mathcal{L}^1$ as follows;

$$\begin{aligned}\mathcal{L}^0 &= \frac{1}{2}\sigma^2\left(\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial x}\right),\\ \mathcal{L}^1 &= \frac{\partial}{\partial\epsilon}\mathcal{L}^\epsilon|_{\epsilon=0}=\sigma\left(\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial x}\right).\end{aligned}$$

Using semi-group theory, the Vega is given by

$$\begin{aligned}&\frac{\partial}{\partial\epsilon}u^{(\epsilon)}(t,x)|_{\epsilon=0}\\ =&\int_t^T \mathbf{P}^0_{u-t}\mathcal{L}^1\mathbf{P}^0_{T-u}f(e^x)du\\ =&\int_t^T\int_{\mathbf{R}}\int_{\mathbf{R}} p^{X^{(0)}}(u-t,x,y)\sigma\left(\frac{\partial^2}{\partial y^2}-\frac{\partial}{\partial y}\right)p^{X^{(0)}}(T-u,y,z)dyf(e^z)dzdu\\ =&\int_t^T\left(\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial x}\right)\int_{\mathbf{R}}\int_{\mathbf{R}} p^{X^{(0)}}(u-t,x,y)p^{X^{(0)}}(T-u,y,z)dyf(e^z)dzdu\\ =&\left(\int_t^T du\right)\sigma\left(\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial x}\right)\int_{\mathbf{R}} f(e^z)p^{X^{(0)}}(T-t,x,z)dz\\ =&(T-t)\mathcal{L}^1\int_{\mathbf{R}} f(e^z)p^{X^{(0)}}(T-t,x,z)dz.\end{aligned}$$

Note that

$$\begin{aligned}&\frac{\partial}{\partial x}p^{X^{(0)}}(T-t,x,z)\\ =&\frac{\partial}{\partial x}\frac{1}{\sqrt{2\pi(T-t)}\sigma}\exp\left\{-\frac{1}{2\sigma^2 t}\left(z-x+\frac{1}{2}\sigma^2(T-t)\right)^2\right\}\\ =&\frac{1}{\sqrt{2\pi(T-t)}\sigma}\exp\left\{-\frac{1}{2\sigma^2(T-t)}(z-x+\frac{1}{2}\sigma^2(T-t))^2\right\}\frac{1}{\sigma^2(T-t)}(z-x+\frac{1}{2}\sigma^2(T-t)),\end{aligned}$$

and

$$\begin{aligned}&\frac{\partial^2}{\partial x^2}p^{X^{(0)}}(T-t,x,z)\\ =&\frac{\partial^2}{\partial x^2}\frac{1}{\sqrt{2\pi(T-t)}\sigma}\exp\left\{-\frac{1}{2\sigma^2(T-t)}\left(z-x+\frac{1}{2}\sigma^2(T-t)\right)^2\right\}\\ =&\frac{1}{\sqrt{2\pi(T-t)}\sigma}\exp\left\{-\frac{1}{2\sigma^2(T-t)}\left(z-x+\frac{1}{2}\sigma^2(T-t)\right)^2\right\}\\ &\times\left\{\left(\frac{1}{\sigma^2(T-t)}(z-x+\frac{1}{2}\sigma^2(T-t))\right)^2-\frac{1}{\sigma^2(T-t)}\right\}.\end{aligned}$$

Then, we obtain

$$\begin{aligned}&\frac{\partial}{\partial\epsilon}u^{(\epsilon)}(t,x)|_{\epsilon=0}\\ =&\sigma(T-t)\int_{\mathbf{R}} f(e^z)p^{X^{(0)}}(T-t,x,z)\left\{\left(\frac{z-x+\frac{1}{2}\sigma^2(T-t))}{\sigma^2(T-t)}\right)^2-\frac{1}{\sigma^2(T-t)}-\frac{\left(z-x+\frac{1}{2}\sigma^2(T-t)\right)}{\sigma^2(T-t)}\right\}dz\\ =&\int_{\mathbf{R}} f(e^z)p^{X^{(0)}}(T-t,x,z)\left\{\frac{1}{\sigma^3(T-t)}\left(z-x+\frac{1}{2}\sigma^2(T-t)\right)^2-\frac{1}{\sigma}-\frac{1}{\sigma}\left(z-x+\frac{1}{2}\sigma^2(T-t)\right)\right\}dz.\end{aligned}$$

We obtain the Malliavin weight for the Vega,

$$\vartheta(z)=\frac{1}{\sigma^3(T-t)}\left(z-x+\frac{1}{2}\sigma^2(T-t)\right)^2-\frac{1}{\sigma}-\frac{1}{\sigma}\left(z-x+\frac{1}{2}\sigma^2(T-t)\right). \tag{3.8}$$

Finally, we remark that the Vega we have just evaluated is equivalent to the well-known Black-Scholes Vega.

3.2 Pricing Options under Stochastic Volatility Model

This subsection derives an approximate solution of the partial differential equation (PDE) in stochastic volatility model by a perturbation method. We consider the following stochastic volatility model (S_t, σ_t):

$$\begin{aligned} dS_t^{(\epsilon)} &= \sigma_t^{(\epsilon)} S_t^{(\epsilon)} dW_{1,t}, \\ d\sigma_t^{(\epsilon)} &= \epsilon\sigma_t^{(\epsilon)}(\rho dW_{1,t} + \sqrt{1-\rho^2}dW_{2,t}), \\ S_0 &= S_0^{(0)} > 0, \\ \sigma_0 &= \sigma_0^{(0)} > 0, \end{aligned} \tag{3.9}$$

where $\epsilon \in [0,1]$. The purpose of this subsection is to evaluate a European option price:

$$C^{SV}(T-t, s, K) = E_{(t,s)}[(S_T^{(\epsilon)} - K)^+], \text{ given } S_t^{(\epsilon)} = s.$$

Let $(X_t^{(\epsilon)})$ denotes the logarithmic process of the underlying asset $(S_t^{(\epsilon)})$. We also define

$$\mathbf{P}_t^\epsilon f(x) = E[f(X_t^{(\epsilon)})], \qquad f \in C_b^\infty,$$

and a generator

$$\mathcal{L}^{(\epsilon)} = \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2} - \frac{1}{2}\sigma^2\frac{\partial}{\partial x} + \epsilon\rho\sigma^2\frac{\partial^2}{\partial x\partial\sigma} + \epsilon^2\frac{1}{2}\sigma^2\frac{\partial^2}{\partial\sigma^2}.$$

We decompose the generator in three parts, *i.e.*

$$\mathcal{L}^\epsilon = \mathcal{L}^0 + \epsilon\mathcal{L}_1 + \epsilon^2\mathcal{L}^2,$$

where

$$\begin{aligned} \mathcal{L}^0 &= \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2} - \frac{1}{2}\sigma^2\frac{\partial}{\partial x}, \\ \mathcal{L}^1 &= \rho\sigma^2\frac{\partial^2}{\partial x\partial\sigma}, \\ \mathcal{L}^2 &= \frac{1}{2}\sigma^2\frac{\partial^2}{\partial\sigma^2}. \end{aligned}$$

Note that $\mathcal{L}_0$ is the (logarithmic) Black-Scholes operator. For $f \in C_b^\infty$, $u^\epsilon(t,x) = \mathbf{P}_{T-t}^\epsilon f(e^x)$ satisfies the following PDE:

$$\begin{cases} \left(\frac{\partial}{\partial t} + \mathcal{L}^\epsilon\right) u^\epsilon(t,x) = 0, \\ u^\epsilon(T,x) = f(e^x). \end{cases} \tag{3.10}$$

Let (U_u) be the first variation process defined by $U_u := \frac{\partial}{\partial s}S_u^{(0)}$, *i.e.*

$$\begin{aligned} dU_u &= U_u\sigma_u^{(0)}dW_{1,u}, \\ U_t &= 1, \end{aligned}$$

and $\mathcal{C}(T)$ be the reduced Malliavin (co)variance of $S_t^{(\epsilon)}$ at $\epsilon = 0$, *i.e.*,

$$\mathcal{C}(T) = \int_t^T a(u)^2 du,$$

where

$$a(u) = (U_u)^{-1}\sigma_u^{(0)}S_u^{(0)}.$$

We introduce the following expressions:

$$\begin{aligned} S_{kT}^0 &= \frac{\partial^k}{\partial\epsilon^k}S_T^{(\epsilon)}|_{\epsilon=0}, \\ \xi^{0,(\beta_1,\cdots,\beta_k)}(T) &= (U_T)^{-1}\prod_{i=1}^k S_{\beta_i T}^0, \\ \zeta^{0,(\beta_1,\cdots,\beta_k)}(T) &= \mathcal{C}(T)^{-1}\xi^{0,(\beta_1,\cdots,\beta_k)}(T), \end{aligned}$$

where $\beta_l \geq 1$ satisfy

$$\sum_{l=1}^{k} \beta_l = j, \quad j \in \mathbf{N}, \quad 1 \leq k \leq j.$$

Theorem 3.1 *For $f \in C_b^\infty$, we have an asymptotic expansion of the solution to the PDE (3.10) :*

$$\begin{aligned}
&\mathbf{P}^\epsilon_{T-t} f(e^x) = \\
&\mathbf{P}^0_{T-t} f(e^x) + \sum_{j=1}^{N} \epsilon^j \sum^{j} \int_t^T \int_{t_1}^T \cdots \int_{t_k}^T \mathbf{P}^0_{t_{k_j}-t} \mathcal{L}^{\beta_{k_j}} \cdots \mathbf{P}^0_{t_1-t_2} \mathcal{L}^{\beta_1} \mathbf{P}^0_{T-t_1} f(e^x) dt_{k_j} \cdots dt_2 dt_1 + O(\epsilon^{N+1}) = \\
&\mathbf{P}^0_{T-t} f(e^x) + \sum_{j=1}^{N} \epsilon^j \int_{\mathbf{R}} f(e^y) w_j(t,T,x,y) p^{X^{(0)}}(T-t,x,y) dy + O(\epsilon^{N+1}),
\end{aligned}$$

where

$$\begin{aligned}
\sum^{j} &\equiv \sum_{\beta_1+\cdots+\beta_{k_j}=j, \beta_i = 1 or 2, k_j \geq 1}, \\
w_j(t,T,x,y) &= \frac{1}{j!} E[\vartheta^{(0)}_{(j)} | X_T^{(0),t,x} = y], \\
\vartheta^{(0)}_{(j)} &= \sum_{k=1}^{j} \sum_{\beta_1+\cdots+\beta_k = j, \beta_i \geq 1} \vartheta_k(\zeta^{0,(\beta_1,\cdots,\beta_k)}(T)), \\
\vartheta_1(\zeta^{0,(\beta_1,\cdots,\beta_k)}(T)) &= \zeta^{0,(\beta_1,\cdots,\beta_k)}(T) \int_t^T a(u) dW_{1,u} - \int_t^T [D_{u,1} \zeta^{0,(\beta_1,\cdots,\beta_k)}(T)] a(u) du, \\
\vartheta_k(\zeta^{0,(\beta_1,\cdots,\beta_k)}(T)) &= \vartheta_1 \circ \vartheta_{k-1}(\zeta^{0,(\beta_1,\cdots,\beta_k)}(T)),
\end{aligned}$$

and $p^{X^{(0)}}(t,x,y)$ is the transition density of $X^{(0)}$ and $\mathbf{P}^0$ is the Black-Scholes semigroup with the generator $\mathcal{L}^0$.

(Proof)

Under the condition of $\sigma_0 = \sigma_0^{(0)} > 0$, $\mathbf{P}^\epsilon_{T-t} f(e^x)$ has an asymptotic expansion around $\epsilon = 0$. The result follows from Takahashi and Yamada (2012). The expansion coefficients are obtained by the following way. The limiting (ϵ^0-order) term, $\mathbf{P}^0_{T-t}$ is the (logarithmic) Black-Scholes semi-group with the generator $\mathcal{L}^0$. The coefficients of the asymptotic expansion of the solution to PDE are calculated as following way. First,

$$\begin{aligned}
\frac{\partial}{\partial \epsilon} \mathbf{P}^\epsilon_{T-t} f(e^x)|_{\epsilon=0} &= E_{(t,s)}[f'(S_T^{(0)}) S_{1T}] \\
&= E_{(t,s)}[f'(S_T^{(0)}) U_T \mathcal{C}(T) \zeta^{0,(1)}(T)].
\end{aligned}$$

By the chain rule of Malliavin calculus, for $u \in [t,T]$, we have

$$\begin{aligned}
D_{u,1} f(S_T^{(0)}) &= f'(S_T^{(0)}) U_T U_u^{-1} \sigma_u^{(0)} S_u^{(0)}, \\
D_{u,2} f(S_T^{(0)}) &= 0.
\end{aligned}$$

Then, following the same way in the proof of Proposition (2.1), we obtain

$$\frac{\partial}{\partial \epsilon} \mathbf{P}^\epsilon_{T-t} f(e^x)|_{\epsilon=0} = E_{(t,s)} \left[\int_t^T \{ [D_{u,1} f(S_T^{(0)})] \zeta^{0,(1)}(T) \} a_1(u) du \right].$$

Note that, for $u \leq T$,

$$D_{u,1} \left\{ f(S_T^{(0)}) \zeta^{0,(1)}(T) \right\} = [D_{u,1} f(S_T^{(0)})] \zeta^{0,(1)}(T) + f(S_T^{(0)}) [D_{u,1} \zeta^{0,(1)}(T)].$$

By the integration by parts formula,

$$\begin{aligned}
&E \left[\int_t^T \{ [D_{u,1} f(S_T^{(0)})] \zeta^{0,(1)}(T) \} a(u) du \right] \\
= \; &E \left[f(e^{X_T^{(0)}}) \left\{ \zeta^{0,(1)}(T) \int_t^T a(u) dW_{1,u} - \int_t^T [D_{u,1} \zeta^{0,(1)}(T)] a(u) du \right\} \right],
\end{aligned}$$

and we obtain

$$
\begin{aligned}
&\frac{\partial}{\partial\epsilon}\mathbf{P}^{\epsilon}_{T-t}f(e^x)|_{\epsilon=0}\\
=\ &\int_{\mathbf{R}} f(e^y)E\left[\zeta^{0,(1)}(T)\int_t^T a(u)dW_{1,u}-\int_t^T [D_{u,1}\zeta^{0,(1)}(T)]a(u)du|X_T^{(0),t,x}=y\right]p^{X^{(0)}}(t,x,y)dy.
\end{aligned}
$$

The higher order approximation terms of the expansion is given as follows;

$$
u^j(t,x):=\frac{1}{i!}\frac{\partial^i}{\partial\epsilon^i}\mathbf{P}^{\epsilon}_{T-t}f(e^x)|_{\epsilon=0} \quad = \quad \frac{1}{i!}\int_{\mathbf{R}} f(e^y)E[\vartheta^{(0)}_{(i)}|X_T^{(0),t,x}=y]p^{X^{(0)}}(T-t,x,y)dy,
$$

where $\vartheta^{(0)}_{(i)}\in\mathbf{D}_\infty$. Then, we obtain an asymptotic expansion formula of the solution to PDE of the stochastic volatility model around the Black-Scholes solution,

$$
\mathbf{P}^{\epsilon}_{T-t}f(e^x)=\mathbf{P}^{0}_{T-t}f(e^x)+\sum_{j=1}^{N}\epsilon^j u^j(t,x)+O(\epsilon^{N+1}).
$$

$u^j(t,x)$ satisfies

$$
\begin{aligned}
\left(\frac{\partial}{\partial t}+\mathcal{L}^0\right)u^j(t,x) &= -\mathcal{L}^1u^{j-1}(t,x)-\mathcal{L}^2u^{j-2}(t,x),\\
u^j(T,x) &= 0.
\end{aligned}
$$

Therefore, we have

$$
u^j(t,x)=\sum^{j}\int_t^T\int_{t_1}^T\cdots\int_{t_{k-1}}^T \mathbf{P}^0_{t_{k_j}-t}\mathcal{L}^{\beta_{k_j}}\cdots\mathcal{L}^{\beta_2}\mathbf{P}^0_{t_1-t_2}\mathcal{L}^{\beta_1}\mathbf{P}^0_{T-t_1}f(e^x)dt_{k_j}\cdots dt_2dt_1.
$$

Specifically, **Corollary 3.1** below derives the first order approximation formula of European option under the stochastic volatility model.

Corollary 3.1 *The following approximation formula holds.*

$$
C^{SV}(T-t,e^x,K)=C^{BS}(T-t,e^x,K)+\epsilon C_1(T-t,e^x,K)+O(\epsilon^2), \tag{3.11}
$$

where $C^{BS}(T-t,z,K)$ denotes the Black-Scholes European option price (with time-to-maturity $T-t$, spot price z and strike price K) and

$$
\begin{aligned}
C_1(T-t,e^x) &= \int_{\mathbf{R}}(e^z-K)^+w_1(t,T,x,z)p^{X^{(0)}}(T-t,x,z)dz \qquad (3.12)\\
&= \frac{(T-t)}{2}\rho\sigma_0e^xn(d_1)(-d_2),
\end{aligned}
$$

with

$$
\begin{aligned}
&w_1(t,T,x,z)\\
=\ &\rho\sigma^3\frac{(T-t)^2}{2}\left(\frac{(z-x+\frac{1}{2}\sigma^2(T-t))^3}{(\sigma^2(T-t))^3}-\frac{3(z-x+\frac{1}{2}\sigma^2(T-t))}{(\sigma^2(T-t))^2}-\frac{(z-x+\frac{1}{2}\sigma^2(T-t))^2}{(\sigma^2(T-t))^2}-\frac{1}{\sigma^2(T-t)}\right),\\
&d_1=\frac{\log(e^x/K)+\sigma_0^2(T-t)/2}{\sigma_0\sqrt{T-t}},\\
&d_2=d_1-\sigma\sqrt{T-t},\\
&n(d_1)=\frac{1}{\sqrt{2\pi}}\exp\left(\frac{-d_1^2}{2}\right).
\end{aligned}
$$

(Proof)
By **Theorem 3.1** ,

$$
\begin{aligned}
&\frac{\partial}{\partial\epsilon}\mathbf{P}^{\epsilon}_{T-t}f(e^x)|_{\epsilon=0}\\
=\ &\int_t^T \mathbf{P}^0_{u-t}\mathcal{L}^1\mathbf{P}^0_{T-u}f(e^x)du\\
=\ &\int_{\mathbf{R}} f(e^z)E\left[\zeta^{0,(1)}(T)\int_t^T a(u)dW_{1,u}-\int_t^T [D_{u,1}\zeta^{0,(1)}(T)]a(u)du|X_T^{(0),t,x}=z\right]p^{X^{(0)}}(T-t,x,z)dz.
\end{aligned}
$$

The conditional expectation above is evaluated as follows;

$$\begin{aligned}
& E\left[\zeta^{0,(1)}(T)\int_t^T a(u)dW_{1,u} - \int_t^T [D_{u,1}\zeta^{0,(1)}(T)]a(u)du | X_T^{(0),t,x} = z\right] \\
= & \rho\sigma^3\frac{(T-t)^2}{2}\left(\frac{(z-x+\frac{1}{2}\sigma^2(T-t))^3}{(\sigma^2(T-t))^3} - \frac{3(z-x+\frac{1}{2}\sigma^2(T-t))}{(\sigma^2(T-t))^2} - \frac{(z-x+\frac{1}{2}\sigma^2(T-t))^2}{(\sigma^2(T-t))^2} - \frac{1}{\sigma^2(T-t)}\right).
\end{aligned}$$

Equivalently, we can proceed as follows: Note first that

$$\mathcal{L}^1\mathbf{P}^0_{T-t}f(e^x) = \rho\sigma_0^3(T-t)\left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2}\right)\mathbf{P}^0_{T-t}f(e^x).$$

We also remark that $\mathcal{L}^1\mathbf{P}^0_{T-t}f(e^x)$ is closely related to one of the Greeks in Black-Scholes model, *Vanna* which is a second order derivative of the option value, once to the underlying spot price and once to volatility. Therefore,

$$\begin{aligned}
& \int_t^T \mathbf{P}^0_{u-t}\mathcal{L}^1\mathbf{P}^0_{T-u}f(e^x)du \\
= & \rho\sigma_0^2\int_t^T\int_{\mathbf{R}}\left(\frac{\partial^2}{\partial y\partial\sigma_0}\mathbf{P}^0_{T-u}f(e^y)\right)p^{X^{(0)}}(s-t,x,y)dydu \\
= & \rho\sigma_0^3\int_t^T\int_{-\infty}^{\infty}(T-u)\left(\left(\frac{\partial^3}{\partial y^3} - \frac{\partial^2}{\partial y^2}\right)\mathbf{P}^0_{T-u}f(e^y)\right)p^{X^{(0)}}(u-t,x,y)dydu, \\
= & \rho\sigma_0^3\int_t^T(T-u)\int_{\mathbf{R}}\int_{\mathbf{R}}p^{X^{(0)}}(u-t,x,y)\left(\frac{\partial^3}{\partial y^3} - \frac{\partial^2}{\partial y^2}\right)p^{X^{(0)}}(T-u,y,z)dyf(e^z)dzdu \\
= & \rho\sigma_0^3\int_t^T(T-u)\left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2}\right)\int_{\mathbf{R}}\int_{\mathbf{R}}p^{X^{(0)}}(u-t,x,y)p^{X^{(0)}}(T-u,y,z)dyf(e^z)dzdu \\
= & \rho\sigma_0^3\left(\int_t^T(T-u)du\right)\left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2}\right)\int_{\mathbf{R}}p^{X^{(0)}}(T-t,x,z)f(e^z)dz \\
= & \rho\sigma_0^3\frac{(T-t)^2}{2}\left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2}\right)\mathbf{P}^0_{T-t}f(e^x).
\end{aligned}$$

Take a sequence $\{f_n\}_n$ such that $f_n \in \mathcal{S}$, $f_n \to (\cdot - K)^+$ *in* $\mathcal{S}'$ $(n\to\infty)$, we have

$$\begin{aligned}
\int_{\mathbf{R}} f_n(e^z)p^{X^{(\epsilon)}}(T-t,x,z)dz &\to C^{SV}(T-t,x,K), \\
\int_{\mathbf{R}} f_n(e^z)p^{X^{(0)}}(T-t,x,z)dz &\to C^{BS}(T-t,e^x,K), \\
\int_{\mathbf{R}} f_n(e^z)w_1(t,T,x,z)p^{X^{(0)}}(T-t,x,z)dz &\to {}_{\mathcal{S}'}\langle(e^{\cdot}-K)^+, w_1(t,T,x,\cdot)p^{X^{(0)}}(T-t,x,\cdot)\rangle_{\mathcal{S}},
\end{aligned}$$

as $n\to\infty$.

Then, in sum, we obtain

$$\begin{aligned}
& C^{SV}(T-t,e^x,K) \\
= & C^{BS}(T-t,e^x,K) + \epsilon\rho\sigma_0^3\frac{(T-t)^2}{2}\left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2}\right)C^{BS}(T-t,x,K) + O(\epsilon^2) \\
= & C^{BS}(T-t,e^x,K) + \epsilon\frac{(T-t)}{2}\rho\sigma_0 e^x n(d_{1t})(-d_{2t}) + O(\epsilon^2) \\
= & C^{BS}(T-t,e^x,K) + \epsilon\int_{\mathbf{R}}(e^z-K)^+ w_1(t,T,x,z)p^{X^{(0)}}(T-t,x,z)dz + O(\epsilon^2).
\end{aligned}$$

Remark 3.1 *Kato, Takahashi and Yamada (2012) has derived a new approximation formula for pricing barrier options under stochastic volatility setting as an application of an asymptotic expansion for solutions of Cauchy-Dirichlet problem for second order parabolic PDEs. To summarize Kato, Takahashi and Yamada (2012), consider the above stochastic volatility model and define* $\tau^\epsilon := \inf\{t; X_t^{(\epsilon)} = \log B\}$, *where* B *is a* *constant barrier, and* $\mathbf{P}^\epsilon f(e^x) := E[f(X_t^{(\epsilon)})\mathbf{1}_{\{\tau^\epsilon > T\}}] = C^{SV.Barrier}(t,x,B)$*. Kato, Takahashi and Yamada (2012) has derived the following formula:*

$$
\begin{aligned}
& C^{SV.Barrier}(t,x) \simeq \mathbf{P}^0 f(e^x) + \epsilon \int_t^T \mathbf{P}^0_{T-s}\mathcal{L}^1\mathbf{P}^0_s f(e^x)dx \\
= \; & C^{BS.Barrier}(t,x) + \epsilon\left\{\frac{(T-t)}{2}\mathcal{L}^1 C^{BS.Barrier}(t,x) + \int_t^T \frac{(T-s)}{2}\mathcal{L}^1 C^{BS.Barrier}(s,x)h(s)ds\right\},
\end{aligned}
$$

where $h(s)$ *is the density of the first hitting-time of* $X_t^{(0)}$ *to* $\log B$.

4 Short-Time Heat Kernel Asymptotic Expansion

This section derives a short-time asymptotic expansion under multi-dimensional diffusion setting: in particular, the asymptotic expansion formula developed in Theorem 2.1 is effectively applied.

Consider the following SDE on $\mathbf{R}^n$ over the d-dimensional Wiener space $(\mathcal{W}, P)$:

$$
\begin{aligned}
dX_t^i &= \sum_{k=1}^d V_k^i(X_t)\circ dW_t^k + V_0^i(X_t)dt, && (4.1)\\
X_0^i &= x_0^i \in \mathbf{R},\ i = 1,\cdots,n,
\end{aligned}
$$

or

$$
\begin{aligned}
dX_t^i &= \sum_{k=1}^d V_k^i(X_t)dW_t^k + \tilde{V}_0^i(X_t)dt, && (4.2)\\
X_0^i &= x_0^i \in \mathbf{R},\ i = 1,\cdots,n.
\end{aligned}
$$

where $V_k = (V_k^1,\cdots,V_k^n)$ with $V_k^i \in C_b^\infty$ and

$$
\tilde{V}_0^i(x) = V_0^i(x) + \frac{1}{2}\sum_l\sum_k \partial_l V_k^i(x)V_k^l(x). \tag{4.3}
$$

We assume that $\sigma(x) = [\sigma^{ij}(x)]$ where $\sigma^{ij}(x) = \sum_{k=1}^d V_k^i(x)V_k^j(x)$ is positive definite at $x = x_0$. We also define $\hat{V}_k$ as

$$
\hat{V}_k = \sum_{i=1}^n V_k^i(x)\frac{\partial}{\partial x_i}, \qquad k = 0,1,\cdots,d. \tag{4.4}
$$

and

$$
\mathcal{L} = \frac{1}{2}\sum_{k=1}^d \hat{V}_k^2 + \hat{V}_0.
$$

Let $\mathbf{i} = (i_1,\cdots,i_m) \in \{0,1,\cdots,d\}^m$, we set $\alpha(\mathbf{i}) = \#\{i : i_l = 0\}$ and $\|\mathbf{i}\| = \alpha(\mathbf{i}) + m$. The following stochastic Taylor expansion holds (e.g. p.4 in Baudoin (2009)):

$$
\begin{aligned}
X_t \;=\; & x_0 + \sum_{k=1}^N \sum_{\mathbf{i},\|\mathbf{i}\|=k}\left(\hat{V}_{i_k}\circ\cdots\circ\hat{V}_{i_2}\right)(V_{i_1})(x_0)\int_0^t \circ dW_{t_1}^{i_1}\circ\int_0^{t_1}\circ dW_{t_2}^{i_2}\cdots\int_0^{t_{m-1}}\circ dW_{t_m}^{i_m}\\
& +\mathbf{R}_N(t,x),
\end{aligned}
$$

for some remainder term $\mathbf{R}_N(t,x)$ which satisfies

$$
\sup_{x\in\mathbf{R}^n} E[\mathbf{R}_N(t,x)^2]^{1/2} \le C_N t^{(N+1)/2}\sup_{\mathbf{i},k+\alpha(\mathbf{i})=N+1 or N+2}\|\left(\hat{V}_{i_k}\circ\cdots\circ\hat{V}_{i_2}\right)(V_{i_1})\|_\infty.
$$

We first consider the scaling SDE in order to obtain a short-time heat kernel expansion:

$$
\begin{aligned}
dX_t^\epsilon &= \epsilon\sum_{l=1}^d V_k(X_t^\epsilon)\circ dW_t^k + \epsilon^2 V_0(X_t^\epsilon)dt, && (4.5)\\
X_0 &= x_0 \in \mathbf{R}^n,
\end{aligned}
$$

where $\epsilon \in (0,1]$. Note that X_t^ϵ is equivalent in law to $X^1_{\epsilon^2 t}$, *i.e.*

$$X_t^\epsilon \sim^{\mathcal{L}} X^1_{\epsilon^2 t},$$

and that X_1^ϵ has an asymptotic expansion:

$$X_1^\epsilon \sim x_0 + \sum_{k=1}^{\infty} \epsilon^k X_1^{(k)} \quad in \quad \mathbf{D}_\infty(\mathbf{R}^n),$$

where $X_1^{(k)} = (X_{11}^{(k)}, \cdots, X_{n1}^{(k)})$, $k \in \mathbf{N}$ is expressed as the coefficient in the stochastic Taylor expansion at $t = 1$, *i.e.*

$$X_{i1}^{(k)} = \sum_{\mathbf{i}, \|\mathbf{i}\|=k} \left(\hat{V}_{i_m} \circ \cdots \circ \hat{V}_{i_2}\right)(V_{i_1}^i)(x_0) \int_0^1 \circ dW_{t_1}^{i_1} \circ \int_0^{t_1} \circ dW_{t_2}^{i_2} \cdots \int_0^{t_{m-1}} \circ dW_{t_m}^{i_m}.$$

Next, set

$$Y_1^{\sqrt{t}} = f_{\sqrt{t}}(X_1^{\sqrt{t}}) := \frac{1}{\sqrt{t}}(X_1^{\sqrt{t}} - x_0). \tag{4.6}$$

Then, we have

$$p^X(t, x_0, x) = p^{X^{\sqrt{t}}}(1, x_0, x) = p^{Y^{\sqrt{t}}}\left(1, 0, \frac{x - x_0}{t^{1/2}}\right) t^{-n/2}. \tag{4.7}$$

Note also that the (i,j)-element of the Malliavin covariance matrix of $Y_1^0 = \sum_{k=1}^d \int_0^1 V_k(x_0) \circ dW_t^k$ is given as:

$$\begin{aligned} \sigma^{ij}_{Y_1^0} &= \sum_{k=1}^d \int_0^1 D_{t,k} Y_{i1}^0 D_{t,k} Y_{j1}^0 dt \qquad (4.8) \\ &= \sum_{k=1}^d V_k^i(x_0) V_k^j(x_0) = \sigma^{ij}(x_0). \end{aligned}$$

Since $Y_1^{\sqrt{t}}$ is uniformly non-degenerate by the assumption that $\sigma(x_0)$ is positive definite, the smooth density, $p^{Y^{\sqrt{t}}}(1, y_0, y)$ for the law of $Y_1^{\sqrt{t}}$ exists.

Thus, $p^{Y^{\sqrt{t}}}(1, y_0, y)$ has an asymptotic expansion by setting $\epsilon = \sqrt{t}$ for Y^ϵ, where

$$Y_1^\epsilon \quad := \quad \frac{(X_1^\epsilon - x_0)}{\epsilon} \sim \sum_{i=1}^\infty \epsilon^{i-1} X_1^{(i)} \ in \ \mathbf{D}_\infty(\mathbf{R}^n). \tag{4.9}$$

In particular,

$$Y_1^0 = X_1^{(1)} = \sum_{k=1}^d \int_0^1 V_k(x_0) \circ dW_t^k. \tag{4.10}$$

Let Y_{i1}^ϵ denotes the i-th element of Y_1^ϵ, that is $Y_1^\epsilon = (Y_{11}^\epsilon, Y_{21}^\epsilon, \cdots, Y_{n1}^\epsilon)$, and define $Y_{i1}^{0,k}$, $k \in \mathbf{N}, i = 1, \cdots, n$ as

$$Y_{i1}^{0,k} \quad = \quad \frac{1}{k!} \frac{d^k}{d\epsilon^k} Y_{i1}^\epsilon |_{\epsilon=0} = X_{i1}^{(k+1)}. \tag{4.11}$$

Then, applying Theorem 2.1 especially, (2.24), we obtain an asymptotic expansion of $p^{Y^\epsilon}(1, 0, y)$:

$$\begin{aligned} p^{Y^\epsilon}(1, 0, y) &= p^{Y^0}(1, 0, y) \left(\sum_{j=0}^N \epsilon^j \sum_k^{(j)} E[H_k(Y_1^0, \prod_{l=1}^k Y_{\alpha_l 1}^{0,\beta_l}) | Y_1^0 = y] \right) \qquad (4.12) \\ &\quad + O(\epsilon^{N+1}), \end{aligned}$$

where

$$\sum_k^{(j)} \equiv \sum_{k=1}^j \sum_{\beta_1 + \cdots + \beta_k = j, \beta_i \geq 1} \frac{1}{k!}.$$

Here, it is easily seen that the density of Y_1^0 is given by

$$p^{Y^0}(1,0,y) = (2\pi)^{-N/2}\det(\sigma(x_0))^{-1/2}e^{-\frac{y^T\sigma(x_0)^{-1}y}{2}}, \tag{4.13}$$

where $\sigma(x_0) = (\sum_{k=1}^d V_k^i(x_0)V_k^j(x_0))_{(1\leq i,j\leq n)}$.

Consequently, by (4.7), we obtain the following theorem that presents a short-time off-diagonal heat kernel expansion.

Theorem 4.1 *As $t \downarrow 0$, we have a short-time asymptotic expansion of the density $p^X(t,x_0,x)$:*

$$p^X(t,x_0,x) \sim \frac{1}{(2\pi t)^{n/2}}\det\sigma(x_0)^{-1/2}e^{-\frac{(x-x_0)^T\sigma(x_0)^{-1}(x-x_0))}{2t}}\left(\sum_{j=0}^N t^{j/2}\zeta_j\left(t^{-1/2}(x-x_0)\right)\right), \tag{4.14}$$

where $\sigma(x_0) = (\sum_{k=1}^d V_k^i(x_0)V_k^j(x_0))_{(1\leq i,j\leq n)}$,
and $\zeta_j\left(t^{-1/2}(x-x_0)\right)$ is the j-th push-down of the Malliavin weights defined by

$$\begin{aligned}
\zeta_j\left(t^{-1/2}(x-x_0)\right) &= \sum_k^{[j]} E\left[H_k(X_1^{(1)},\prod_{l=1}^k X_{\alpha_l 1}^{(\beta_l)})|X_1^{(1)} = t^{-1/2}(x-x_0)\right] \\
&= \sum_k^{(j)} E\left[H_k(Y_1^0,\prod_{l=1}^k Y_{\alpha_l 1}^{0,\beta_l})|Y_1^0 = t^{-1/2}(x-x_0)\right].
\end{aligned} \tag{4.15}$$

Here, Y_1^0 and $Y_{i1}^{0,k}$ are given by (4.10) and (4.11), respectively, and $X_1^{(1)}$ and $X_{i1}^{(k)}$ are given by

$$\begin{aligned}
X_1^{(1)} &= \sum_{k=0}^d \int_0^1 V_k(x_0)\circ dW_t^k, \\
X_{i1}^{(k)} &= \sum_{\mathbf{i},\|\mathbf{i}\|=k}\left(\hat{V}_{i_m}\circ\cdots\circ\hat{V}_{i_2}\right)(V_{i_1}^i)(x_0)\int_0^1 \circ dW_{t_1}^{i_1}\circ\int_0^{t_1}\circ dW_{t_2}^{i_2}\cdots\int_0^{t_{m-1}}\circ dW_{t_m}^{i_m}.
\end{aligned}$$

Also,

$$\sum_k^{[j]} \equiv \sum_{k=1}^j \sum_{\beta_1+\cdots+\beta_k=j,\beta_i\geq 2}\frac{1}{k!},$$

and

$$\sum_k^{(j)} \equiv \sum_{k=1}^j \sum_{\beta_1+\cdots+\beta_k=j,\beta_i\geq 1}\frac{1}{k!}.$$

Remark 4.1 *In the diagonal case, the diagonal heat kernel $p^X(t,x_0,x_0)$ is approximated by*

$$\frac{1}{(2\pi t)^{n/2}}\det\sigma(x_0)^{-1/2}\left(\sum_{j=0}^N t^j\zeta_j(0)\right),$$

where

$$\begin{aligned}
\zeta_j(0) &= \sum_k^{[j]} E\left[H_k(X_1^{(1)},\prod_{l=1}^k X_{\alpha_l 1}^{(\beta_l)})|X_1^{(1)} = 0\right] \\
&= \sum_k^{(j)} E\left[H_k(Y_1^0,\prod_{l=1}^k Y_{\alpha_l 1}^{0,\beta_l})|Y_1^0 = 0\right].
\end{aligned}$$

Next, we provide alternative methods to obtain the coefficients of the expansion.

Let $\mathcal{A}$ be the perturbed generator associated with (4.5):

$$\mathcal{A} = \frac{1}{2}\sum_{k=1}^{d}(\epsilon\hat{V}_k)^2 + \epsilon^2\hat{V}_0.$$

Then, the generator $\mathcal{L}^\epsilon$ associated with the process after the transformation, $Y_t^\epsilon = \frac{(X_t^\epsilon - x_0)}{\epsilon}$ is given by

$$\mathcal{L}^\epsilon = \frac{1}{2}\sum_{k=1}^{d}(L_k^\epsilon)^2 + \epsilon L_0^\epsilon. \tag{4.16}$$

where

$$L_k^\epsilon = \sum_{i=1}^{n} V_k^i(x_0 + \epsilon y)\frac{\partial}{\partial y_i}, \qquad k = 0, 1, \cdots, d. \tag{4.17}$$

Hence, by applying (2.20) in Theorem 2.1, we have for $f \in C_b(\mathbf{R}^n)$,

$$\mathbf{P}_1^\epsilon f(0) = \mathbf{P}_1^0 f(0) + \sum_{j=1}^{N}\epsilon^j\xi_j(y) + \epsilon^{N+1}R_N(y), \tag{4.18}$$

where

$$\begin{aligned}\xi_j(y) \quad = \quad & \sum_{k=1}^{j}\sum_{\beta_1+\cdots+\beta_k=j,\beta_i\geq 1}\int_0^1\int_0^{t_1}\cdots\int_0^{t_k} \\ & \mathbf{P}^0_{(1-t_1)}\mathcal{L}^{\beta_1}\mathbf{P}^0_{(t_1-t_2)}\mathcal{L}^{\beta_2}\cdots\mathbf{P}^0_{(t_k-t_{k-1})}\mathcal{L}^{\beta_k}\mathbf{P}^0_{t_k}f(y_0)dt_k\cdots dt_2dt_1|_{y_0=0},\end{aligned} \tag{4.19}$$

with $\mathcal{L}^k := \frac{1}{k!}\frac{d^k}{d\epsilon^k}\mathcal{L}^\epsilon|_{\epsilon=0}$, $k \in \mathbf{N}$, $i = 1, \cdots, n$.

5 Applications of Short-Time Asymptotic Expansion

This section shows three examples of Theorem 4.1 in the previous section. In particular, we explicitly derive short-time asymptotic expansions under stochastic volatility model with log-normal local volatility and general local-stochastic volatility models. Moreover, we applies (4.15) and (4.19) in Section 4 to computing the coefficients in the expansions. In addition, for local volatility model in Section 5.1 and Appendix, we compute the expansion coefficients $\xi_j(y)$ ($j \in \mathbf{N}$), $j = 1, 2$ in (4.19) by using *Lie brackets.*(Lie bracket $[A, Z]$ stands for $[A, Z] = AZ - ZA$ where A and Z are vector fields.)

5.1 Short-Time Asymptotic Expansion for Local Volatility Model

Consider the following time-homogenous local volatility model.

$$\begin{aligned} dX_t &= \mu(X_t)dt + \sigma(X_t)dW_t, \\ X_0 &= x_0. \end{aligned} \tag{5.1}$$

Proposition 5.1 *We $t \downarrow 0$, we have*

$$p(t, x_0, x) \sim \frac{1}{\sqrt{2\pi\sigma(x_0)^2 t}}\exp\left\{-\frac{(x-x_0)^2}{\sigma(x_0)^2 t}\right\}\left(1 + \sqrt{t}\vartheta_1(t, x_0, x) + t\vartheta_2(t, x_0, x)\right)$$

where

$$\begin{aligned} & \vartheta_1(t, x_0, x) \\ = \quad & \mu(x_0)\frac{h_1((x-x_0)/\sqrt{t}, \sigma^2(x_0)}{\sigma^2(x_0)} + \frac{1}{2}\sigma(x_0)^3\partial\sigma(x_0)\frac{h_3((x-x_0)/\sqrt{t}, \sigma^2(x_0))}{(\sigma^2(x_0))^3}, \end{aligned} \tag{5.2}$$

and

$$
\begin{aligned}
&\vartheta_2(t,x_0,x)\\
=\;& \frac{1}{8}\partial\sigma(x_0)^2\sigma^6(x_0)\frac{h_6((x-x_0)/\sqrt{t},\sigma^2(x_0))}{(\sigma^2(x_0))^6}\\
&+\frac{1}{6}\left(\partial^2\sigma(x_0)\sigma(x_0)^5+4\partial\sigma(x_0)^2\sigma(x_0)+3\mu(x_0)\partial\sigma(x_0)\right)\frac{h_4((x-x_0)/\sqrt{t},(\sigma^2(x_0)))}{(\sigma^2(x_0))^4}\\
&+\frac{1}{4}\left(\partial^2\sigma(x_0)\sigma(x_0)^3+2\mu'(x_0)\sigma(x_0)^2+2\mu(x_0)\partial\sigma(x_0)\sigma(x_0)+\partial\sigma(x_0)^2\sigma(x_0)^2+2\mu(x_0)^2\right)\\
&\times\frac{h_2((x-x_0)/\sqrt{t},\sigma^2(x_0))}{(\sigma^2(x_0))^2}.
\end{aligned}
\tag{5.3}
$$

Here, $h_n(x,\Sigma)$ stands for the Hermite polynomial of degree n with Σ, that is

$$h_n(x;\Sigma)=(-\Sigma)^n e^{x^2/(2\Sigma)}\frac{d^n}{dx^n}e^{-x^2/(2\Sigma)}.$$

(Proof)
We apply (4.15) and (4.19) in computation of the coefficients of the expansion.

First, we have the following stochastic Taylor expansion

$$X_t \;=\; x_0+X_{1t}+X_{2t}+X_{3t}+\mathbf{R}_3(t), \tag{5.4}$$

where

$$
\begin{aligned}
X_{1t} &= \int_0^t \sigma(x_0)dW_s,\\
X_{2t} &= \int_0^t \mu(x_0)ds+\int_0^t \partial\sigma(x_0)\int_0^s \sigma(x_0)dW_u dW_s.\\
X_{3t} &= \int_0^t \partial\mu(x_0)\int_0^s \sigma(x_0)dW_u ds\\
&+\frac{1}{2}\int_0^t \partial^2\sigma(x_0)(\int_0^s \sigma(x_0)dW_u)^2 dW_s\\
&+\int_0^t \partial\sigma(x_0)\int_0^s \mu(x_0)du dW_s+\int_0^t \partial\sigma(x_0)\int_0^s \partial\sigma(x_0)\int_0^u \sigma(x_0)dW_v dW_u dW_s,
\end{aligned}
$$

and $\mathbf{R}_3(t)$ is a remainder term.

Let X_t^ϵ be the solution of the following scaling SDE.

$$
\begin{aligned}
dX_t^\epsilon &= \epsilon^2\mu(X_t^\epsilon)dt+\epsilon\sigma(X_t^\epsilon)dW_t,\\
X_0^\epsilon &= x_0.
\end{aligned}
$$

Let $\mathcal{A}^\epsilon$ be the generator of X_t^ϵ defined by

$$\mathcal{A}^\epsilon=\epsilon^2\frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2}+\epsilon^2\mu(x)\frac{\partial}{\partial x}.$$

Consider a transform

$$Y_t^\epsilon=f_\epsilon(X_t^\epsilon)=\frac{1}{\epsilon}(X_t^\epsilon-x_0), \tag{5.5}$$

then the generator $\mathcal{L}^\epsilon$ of Y_t^ϵ has the following form

$$\mathcal{L}^\epsilon=\frac{1}{2}\sigma^2(x_0+\epsilon y)\frac{\partial^2}{\partial y^2}+\epsilon\mu(x_0+\epsilon y)\frac{\partial}{\partial y}.$$

First, we apply the *push-down of the Malliavin weights* to computing the coefficients of the expansion. Note that X_t^ϵ and Y_t^ϵ are expanded in $\mathbf{D}_\infty$ as follows.

$$
\begin{aligned}
X_t^\epsilon &= x_0+\epsilon X_{1t}+\epsilon^2X_{2t}+\epsilon^3X_{3t}+O(\epsilon^4),\\
Y_t^\epsilon &= Y_{0t}+\epsilon Y_{1t}+\epsilon^2Y_{2t}+O(\epsilon^3),
\end{aligned}
$$

where

$$Y_{0t} \quad = \quad X_{1t} = \frac{\partial}{\partial \epsilon} X_t^\epsilon|_{\epsilon=0} = \int_0^t \sigma(x_0) dW_s,$$

$$Y_{1t} \quad = \quad X_{2t} = \frac{1}{2!} \frac{\partial^2}{\partial \epsilon^2} X_t^\epsilon|_{\epsilon=0} = \int_0^t \mu(x_0) ds + \int_0^t \partial\sigma(x_0) \int_0^s \sigma(x_0) dW_u dW_s.$$

$$\begin{aligned} Y_{2t} &= X_{3t} = \frac{1}{3!} \frac{\partial^3}{\partial \epsilon^3} X_t^\epsilon|_{\epsilon=0} \\ &= \int_0^t \partial\mu(x_0) \int_0^s \sigma(x_0) dW_u ds + \frac{1}{2} \int_0^t \partial^2\sigma(x_0) (\int_0^s \sigma(x_0) dW_u)^2 dW_s \\ &\quad + \int_0^t \partial\sigma(x_0) \int_0^s \mu(x_0) du dW_s + \int_0^t \partial\sigma(x_0) \int_0^s \partial\sigma(x_0) \int_0^u \sigma(x_0) dW_v dW_u dW_s. \end{aligned}$$

Note that Y_t^ϵ is uniformly non-degenrate.

The following relation holds,

$$p^X(t, x_0, x) = p^Y(1, f_{\sqrt{t}}(x_0), f_{\sqrt{t}}(x)) \frac{1}{\sqrt{t}}. \tag{5.6}$$

1. Using Bismut identity,

$$\begin{aligned} & \frac{\partial}{\partial \epsilon} p^{Y^\epsilon}(1, y_0, y) \\ = \quad & \frac{\partial}{\partial \epsilon} E\left[\delta_y\left(Y_1^{(\epsilon)}\right)\right]|_{\epsilon=0} \ (\delta_y(\cdot) \text{ is a delta function at } y.) \\ = \quad & E\left[\delta_y\left(Y_1^{(0)}\right) H_1\left(Y_1^{(0)}, \frac{\partial}{\partial \epsilon} Y_1^{(\epsilon)}|_{\epsilon=0}\right)\right] \\ = \quad & E\left[\delta_y\left(Y_1^{(0)}\right) \frac{1}{\sigma^2(x_0)} \left\{\frac{\partial}{\partial \epsilon} Y_1^{(\epsilon)}|_{\epsilon=0} \int_0^1 \sigma(x_0) dW_u - \int_0^1 D_{u,1} \frac{\partial}{\partial \epsilon} Y_1^{(\epsilon)}|_{\epsilon=0} \sigma(x_0) du\right\}\right] \\ = \quad & E\left[\delta_y\left(Y_1^{(0)}\right) \frac{1}{\sigma^2(x_0)} \left\{\left(\mu(x_0) + \sigma(x_0)\partial\sigma(x_0) \int_0^1 W_s dW_s\right) \sigma(x_0) W_1 - \sigma^2(x_0)\partial\sigma(x_0) W_1\right\}\right] \\ = \quad & \int_{\mathbf{R}} \delta_y(v) \\ & \times E\left[\frac{1}{\sigma^2(x_0)} \left\{\left(\mu(x_0) + \sigma(x_0)\partial\sigma(x_0) \int_0^1 W_s dW_s\right) \sigma(x_0) W_1 - \sigma^2(x_0)\partial\sigma(x_0) W_1\right\} |\sigma(x_0) W_1 = v\right] \\ & \times \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}v^2} dv \\ = \quad & E\left[\frac{1}{\sigma^2(x_0)} \left\{\left(\mu(x_0) + \sigma(x_0)\partial\sigma(x_0) \int_0^1 W_s dW_s\right) \sigma(x_0) W_1 - \sigma^2(x_0)\partial\sigma(x_0) W_1\right\} |\sigma(x_0) W_1 = y\right] \\ & \times \frac{1}{\sqrt{2\pi\sigma(x_0)^2}} e^{-\frac{1}{2}\left(\frac{y}{\sigma(x_0)}\right)^2} \end{aligned}$$

Note that

$$D_{u,1} \int_0^1 \sigma(x_0) \partial_{x_0}\sigma(x_0) W_s dW_s = \sigma(x_0) \partial_{x_0}\sigma(z) \{W_u + \int_u^1 dW_s\} = \sigma(x_0)\partial\sigma(x_0) W_1$$

Note that

$$E\left[\int_0^1 W_s dW_s | \sigma(z) W_1 = y\right] \quad = \quad \left(\int_0^1 s ds\right) \left(\frac{y^2}{\sigma(x_0)^4} - \frac{1}{\sigma(x_0)^2}\right) = \frac{1}{2} \left(\frac{y^2}{\sigma(x_0)^4} - \frac{1}{\sigma(x_0)^2}\right).$$

Then, we obtain

$$
\begin{aligned}
& \frac{\partial}{\partial \epsilon} E\left[\delta_y\left(Y_1^{(\epsilon)}\right)\right]|_{\epsilon=0} \qquad (5.7)\\
= \; & \left\{\mu(x_0)\frac{y}{\sigma(x_0)^2} + \frac{1}{2}\sigma(x_0)^3\partial\sigma(x_0)\left(\frac{y^3}{\sigma(x_0)^6} - \frac{3y}{\sigma(x_0)^4}\right)\right\} \\
& \times \frac{1}{\sqrt{2\pi\sigma(x_0)^2}} e^{-\frac{1}{2}\left(\frac{y}{\sigma(x_0)}\right)^2}.
\end{aligned}
$$

2. Alternatively, we can evaluate the coefficients of the expansion in the following way. Let

$$
\begin{aligned}
\mathcal{L}^0 &= \frac{1}{2}\sigma^2(x_0)\frac{\partial^2}{\partial y^2}, \\
\mathcal{L}^1 &= \sigma(x_0)\partial\sigma(x_0)y\frac{\partial^2}{\partial y^2} + \mu(x_0)\frac{\partial}{\partial y}, \\
\mathcal{L}^2 &= \frac{1}{2}((\partial\sigma(x_0))^2 + \sigma(x_0)\partial^2\sigma(x_0))y^2\frac{\partial^2}{\partial y^2} + \partial\mu(x_0)y\frac{\partial}{\partial y},
\end{aligned}
$$

then

$$
\frac{\partial}{\partial \epsilon}|_{\epsilon=0}\mathbf{P}_1^{\epsilon}f(y_0)|_{y_0=0} = \int_0^1 \mathbf{P}^0_{(1-s)}\mathcal{L}^1\mathbf{P}^0_s f(y_0)ds|_{y_0=0}.
$$

Let h be a map $y \mapsto h(y)$ such that

$$
\begin{aligned}
h(y) &= \mathcal{L}^1\mathbf{P}^0_s f(y) = \mathcal{L}^1\mathbf{P}^0_{1-(1-s)}f(y) \\
&= \mathcal{L}^1 E[f(Y_1^0)|Y^0_{1-s} = y] = \mathcal{L}^1\int_{\mathbf{R}} p^{Y^0}(s,y,z)f(z)dz \\
&= \left(\sigma(x_0)\partial\sigma(x_0)y\frac{\partial^2}{\partial y^2} + \mu(x_0)\frac{\partial}{\partial y}\right)\int_{\mathbf{R}} p^{Y^0}(s,y,z)f(z)dz.
\end{aligned}
$$

Then, we explicitly evaluate (4.19) for $j = 1$.

$$
\begin{aligned}
& \mathbf{P}^0_{(1-s)}\mathcal{L}^1\mathbf{P}^0_s f(y_0)|_{y_0=0} \\
= \; & \int_{\mathbf{R}} p^{Y^0}(1-s,y_0,y)h(y)dy|_{y_0=0} \\
= \; & \int_{\mathbf{R}} p^{Y^0}(1-s,y_0,y)\left(\mathcal{L}^1\int_{\mathbf{R}} p^{Y^0}(s,y,z)f(z)dz\right)dy|_{y_0=0} \\
= \; & \int_{\mathbf{R}} p^{Y^0}(1-s,y_0,y)\left(\left(\sigma(x_0)\partial\sigma(x_0)(y-y_0)\frac{\partial^2}{\partial y^2} + \mu(x_0)\frac{\partial}{\partial y}\right)\int_{\mathbf{R}} p^{Y^0}(s,y,z)f(z)dz\right)dy|_{y_0=0}.
\end{aligned}
$$

Note that

$$
p^{Y^0}(1-s,y_0,y)(y-y_0) = (1-s)\sigma(x_0)^2 p^{Y^0}(1-s,y_0,y)\frac{\partial}{\partial y_0}. \qquad (5.8)
$$

Therefore, we have

$$
\begin{aligned}
& \mathbf{P}^0_{(1-s)}\mathcal{L}^1\mathbf{P}^0_s f(y_0)|_{y_0=0} \\
= \; & \int_{\mathbf{R}} \frac{\partial}{\partial y_0}p^{Y^0}(1-s,y_0,y)\left(\left((1-s)\sigma(x_0)^3\partial\sigma(x_0)\frac{\partial^2}{\partial y^2} + \mu(x_0)\frac{\partial}{\partial y}\right)\int_{\mathbf{R}} p^{Y^0}(s,y,z)f(z)dz\right)dy|_{y_0=0} \\
= \; & \int_{\mathbf{R}} \frac{\partial^3}{\partial y^2\partial y_0}p^{Y^0}(1-s,y_0,y)\left(\left((1-s)\sigma(x_0)^3\partial\sigma(x_0)\right)\int_{\mathbf{R}} p^{Y^0}(s,y,z)f(z)dz\right)dy|_{y_0=0} \\
& - \int_{\mathbf{R}} \frac{\partial}{\partial y}p^{Y^0}(1-s,y_0,y)\left(\mu(x_0)\int_{\mathbf{R}} p^{Y^0}(s,y,z)f(z)dz\right)dy|_{y_0=0} \\
= \; & (1-s)\sigma(x_0)^3\partial\sigma(x_0)\int_{\mathbf{R}} \frac{\partial^3}{\partial y_0^3}p^{Y^0}(1-s,y_0,y)\left(\int_{\mathbf{R}} p^{Y^0}(s,y,z)f(z)dz\right)dy|_{y_0=0}
\end{aligned}
$$

$$
\begin{aligned}
& +\mu(x_0)\int_{\mathbf{R}}\frac{\partial}{\partial y_0}p^{Y^0}(1-s,y_0,y)\left(\int_{\mathbf{R}}p^{Y^0}(s,y,z)f(z)dz\right)dy|_{y_0=0}\\
= \;& (1-s)\sigma(x_0)^3\partial\sigma(x_0)\int_{\mathbf{R}}\frac{\partial^3}{\partial y_0^3}\left(\int_{\mathbf{R}}p^{Y^0}(1-s,y_0,y)p^{Y^0}(s,y,z)dy\right)f(z)dz|_{y_0=0}\\
& +\mu(x_0)\int_{\mathbf{R}}\frac{\partial}{\partial y_0}\left(\int_{\mathbf{R}}p^{Y^0}(1-s,y_0,y)p^{Y^0}(s,y,z)dy\right)f(z)dz|_{y_0=0}\\
= \;& (1-s)\sigma(x_0)^3\partial\sigma(x_0)\frac{\partial^3}{\partial y_0^3}\int_{\mathbf{R}}p^{Y^0}(1,y_0,z)f(z)dz|_{y_0=0}+\mu(x_0)\frac{\partial}{\partial y_0}\int_{\mathbf{R}}p^{Y^0}(1,y_0,z)f(z)dz|_{y_0=0}\\
= \;& (1-s)\sigma(x_0)^3\partial\sigma(x_0)\frac{\partial^3}{\partial y_0^3}\mathbf{P}_1^0f(y_0)|_{y_0=0}+\mu(x_0)\frac{\partial}{\partial y_0}\mathbf{P}_1^0f(y_0)|_{y_0=0}.
\end{aligned}
$$

Then, the first order approximation term is given by

$$
\begin{aligned}
& \int_0^1\mathbf{P}_{(1-s)}^0\mathcal{L}^1\mathbf{P}_s^0f(y_0)ds|_{y_0=0}\\
= \;& \left(\int_0^1(1-s)ds\right)\sigma(x_0)^3\partial\sigma(x_0)\frac{\partial^3}{\partial y_0^3}\mathbf{P}_1^0f(y_0)|_{y_0=0}+\mu(x_0)\frac{\partial}{\partial y_0}\mathbf{P}_1^0f(y_0)|_{y_0=0}\\
= \;& \int_{\mathbf{R}}\left\{\frac{1}{2}\sigma(x_0)^3\partial\sigma(x_0)\left(\frac{z^3}{\sigma(x_0)^6}-\frac{3z}{\sigma(x_0)^4}\right)+\mu(x_0)\frac{z}{\sigma(x_0)^2}\right\}p^{Y^0}(1,0,z)f(z)dz.
\end{aligned}
$$

3. Moreover, the coefficient is computed by using the Lie bracket.

$$
\begin{aligned}
& \int_0^1\mathbf{P}_{(1-t_1)}^0\mathcal{L}^1\mathbf{P}_{t_1}^0f(y_0)dt_1|_{y_0=0}\\
= \;& \int_0^1\left(\sum_{i=0}^{\infty}\frac{(1-t_1)^i}{i!}[\mathcal{L}^0,[\mathcal{L}^0,\cdots,[\mathcal{L}^0,\mathcal{L}^1]]]\right)\mathbf{P}_1^0f(y_0)dt_1|_{y_0=0}\\
= \;& \left(\mathcal{L}^1+\frac{1}{2}[\mathcal{L}^0,\mathcal{L}^1]\right)\mathbf{P}_1^0f(y_0)|_{y_0=0},
\end{aligned}
$$

because $[\mathcal{L}^0,[\mathcal{L}^0,\mathcal{L}^1]]=0$ and hence all the terms in (5.9) for $i\geq 2$ are equal to 0.
The Lie bracket $[\mathcal{L}^0,\mathcal{L}^1]$ is explicitly computed as follows.

$$
\begin{aligned}
\mathcal{L}^0\mathcal{L}^1 &= \frac{1}{2}\sigma(x_0)^2\partial^2\left(\sigma(x_0)\partial\sigma(x_0)y\partial^2\right)\\
&= \frac{1}{2}\sigma^3(x_0)\partial\sigma(x_0)\left(\partial(\partial^2+y\partial^3)\right)\\
&= \frac{1}{2}\sigma^3(x_0)\partial\sigma(x_0)\left(2\partial^3+y\partial_x^4\right).\\
\mathcal{L}^1\mathcal{L}^0 &= \sigma(x_0)\partial\sigma(x_0)y\partial^2\left(\frac{1}{2}\sigma(x_0)^2\partial^2\right)\\
&= \frac{1}{2}\sigma^3(x_0)\partial\sigma(x_0)y\partial^4.
\end{aligned}
$$

Then

$$
\begin{aligned}
[\mathcal{L}^0,\mathcal{L}^1] &= \mathcal{L}^0\mathcal{L}^1-\mathcal{L}^1\mathcal{L}^0\\
&= \sigma(x_0)^3\partial\sigma(x_0)\partial^3,
\end{aligned}
$$

Then we have

$$
\begin{aligned}
& \left(\mathcal{L}^1+\frac{1}{2}[\mathcal{L}^0,\mathcal{L}^1]\right)\frac{1}{\sqrt{2\pi\sigma(x_0)^2}}e^{-\frac{1}{2}\left(\frac{y-y_0}{\sigma(x_0)}\right)^2}|_{y_0=0} \qquad (5.9)\\
= \;& \left\{\mu(x_0)\frac{y}{\sigma(x_0)^2}+\frac{1}{2}\sigma(x_0)^3\partial_{x_0}\sigma(x_0)\left(\frac{y^3}{\sigma(x_0)^6}-\frac{3y}{\sigma(x_0)^4}\right)\right\}\\
& \times\frac{1}{\sqrt{2\pi\sigma(x_0)^2}}e^{-\frac{1}{2}\left(\frac{y}{\sigma(x_0)}\right)^2}.
\end{aligned}
$$

The calculation of the second term approximation is given in Appendix.

5.2 Short Time Asymptotics for Stochastic Volatility Model with Log-normal Local Volatility

Consider the following stochastic volatility model with log-normal local volatility which includes the Heston type model:

$$
\begin{aligned}
dS_t &= rS_t dt + \sqrt{v_t} S_t dW_t, \\
S_0 &= s_0 > 0, \\
dv_t &= a(v_t)dt + b(v_t)dZ_t, \\
v_0 &= v > 0,
\end{aligned}
$$

where W_t and Z_t are two standard Brownian motions with correlation ρ.
We have a short-time expansion of density for the logarithmic process.

Proposition 5.2 *When $t \downarrow 0$, we have*

$$
p^X(t, x_0, x) \sim \frac{1}{\sqrt{2\pi v_0 t}} \exp\left(-\frac{(x-x_0)^2}{2v_0 t} \right) \left\{ 1 + \sqrt{t} w_1(t, x_0, x) \right\}, \tag{5.10}
$$

where $x = \log s$, $x_0 = \log s_0$,

$$
w_1(t, x_0, x) = \frac{1}{4}\rho\sqrt{v_0} b(v_0) \frac{h_3((x-x_0)/\sqrt{t}, v_0)}{v_0^3} + \left(r - \frac{1}{2}v_0\right) \frac{h_1((x-x_0)/\sqrt{t}, v_0)}{v_0}.
$$

Also, the following approximation formula of the option price holds:

$$
\begin{aligned}
C(t, K) \sim & \int_{\log(K)}^{\infty} (e^x - K) \frac{1}{\sqrt{2\pi v_0 t}} \exp\left(-\frac{(x-x_0)^2}{2v_0 t} \right) dx \\
& + \sqrt{t} \int_{\log(K)}^{\infty} (e^x - K) w_1(t, x_0, x) \frac{1}{\sqrt{2\pi v_0 t}} \exp\left(-\frac{(x-x_0)^2}{2v_0 t} \right) dx.
\end{aligned}
$$

(Proof)
We will apply (4.15) and (4.19) in computation. First, we have the following stochastic Taylor expansion

$$
X_t = x_0 + X_{1t} + X_{2t} + \mathbf{R}_2(t), \tag{5.11}
$$

where

$$
\begin{aligned}
X_{1t} &= \int_0^t \sqrt{v_0} dW_s, \\
X_{2t} &= \int_0^t (r - \frac{1}{2}v_0) ds + \frac{1}{2} \int_0^t \frac{1}{\sqrt{v_0}} \int_0^s b(v_0) dZ_u dW_s.
\end{aligned} \tag{5.12}
$$

Next, we introduce a time scaling parameter $\epsilon = \sqrt{t}$,

$$
\begin{aligned}
dX_t^\epsilon &= \epsilon^2 (r - \frac{1}{2}v_t^\epsilon) dt + \epsilon \sqrt{v_t^\epsilon} dW_t, \\
dv_t^\epsilon &= \epsilon^2 a(v_t^\epsilon) dt + \epsilon b(v_t^\epsilon) dZ_t.
\end{aligned}
$$

The generator of the above diffusion is

$$
\mathcal{A}^\epsilon = \epsilon^2 \frac{1}{2} v \frac{\partial^2}{\partial x^2} + \epsilon^2 (r - \frac{1}{2}v) \frac{\partial}{\partial x} + \epsilon^2 \rho \sqrt{v} b(v) \frac{\partial^2}{\partial x \partial v} + \epsilon^2 \frac{1}{2} b(v) \frac{\partial^2}{\partial v^2} + \epsilon^2 a(v) \frac{\partial}{\partial v}.
$$

Consider a transform $Y = f^\epsilon(X) = \frac{1}{\epsilon}(X - x_0)$, then the generator of (Y, v) is given by,

$$
\mathcal{L}^\epsilon = \frac{1}{2} v \frac{\partial^2}{\partial y^2} + \epsilon (r - \frac{1}{2}v) \frac{\partial}{\partial y} + \epsilon \rho \sqrt{v} b(v) \frac{\partial^2}{\partial y \partial v} + \epsilon^2 \frac{1}{2} b(v) \frac{\partial^2}{\partial v^2} + \epsilon^2 a(v) \frac{\partial}{\partial v}.
$$

X and Y are expanded in $\mathbf{D}_\infty$,

$$
\begin{aligned}
X_t^\epsilon &= x_0 + \epsilon X_{1t} + \epsilon^2 X_{2t} + O(\epsilon^3), \\
Y_t^\epsilon &= Y_{0t} + \epsilon Y_{1t} + O(\epsilon^3),
\end{aligned}
$$

where

$$\begin{aligned} Y_{0t} &= X_{1t} = \int_0^t \sqrt{v_0} dW_s, \\ Y_{1t} &= X_{2t} = \int_0^t (r - \frac{1}{2}v_0)ds + \frac{1}{2}\int_0^t \frac{1}{\sqrt{v_0}} \int_0^s b(v_0)dZ_u dW_s. \end{aligned}$$

Note that

$$p^X(t, x_0, x) = p^{Y^{\sqrt{t}}}(1, f^{\sqrt{t}}(x_0), f^{\sqrt{t}}(x))\frac{1}{\sqrt{t}}. \tag{5.13}$$

1. Using the Bismut identity, the first order approximation term is given as

$$\begin{aligned} &\frac{\partial}{\partial \epsilon} p^{Y^\epsilon}(1, 0, y)|_{\epsilon=0} \\ =\ & E[H_1(Y_{01}, Y_{11})|Y_{01} = y]p^{Y^0}(1, 0, y) \\ =\ & \frac{1}{v_0} E[Y_{11} \int_0^1 \sqrt{v_0} dW_{1,t} - \int_0^1 D_{t,1}Y_{11}\sqrt{v_0}dt|Y_{01} = y]p^{Y^0}(1, 0, y) \\ =\ & \left\{\frac{1}{4}\rho\sqrt{v_0}b(v_0)\left(\frac{y^3}{v_0^3} - 3\left(\frac{y}{v_0}\right)\right) + \left(r - \frac{1}{2}v_0\right)\left(\frac{y}{v_0}\right)\right\} p^{Y^0}(1, 0, y). \end{aligned}$$

Then we have a approximation formula of the density

$$p^{Y^{\sqrt{t}}}(1, 0, y) \sim \frac{1}{\sqrt{2\pi v_0}} \exp\left\{-\frac{y^2}{2v_0}\right\}\left(1 + \sqrt{t}\zeta_1(1, 0, y)\right), \tag{5.14}$$

where

$$\zeta_1(1, 0, y) = \frac{1}{4}\rho\sqrt{v_0}b(v_0)\frac{h_3(y, v_0)}{v_0^3} + \left(r - \frac{1}{2}v_0\right)\frac{h_1(y, v_0)}{v_0}.$$

By (5.13) and (5.14), we have the formula (5.10).

2. Alternatively, we have

$$\frac{\partial}{\partial \epsilon}|_{\epsilon=0}\mathbf{P}_1^0 f(y_0)|_{y_0=0} = \int_0^1 \mathbf{P}_{(1-s)}^0 \mathcal{L}^1 \mathbf{P}_s^0 f(y_0)ds|_{y_0=0}$$

with

$$\mathcal{L}^1 = \frac{\partial}{\partial \epsilon}\mathcal{L}^\epsilon|_{\epsilon=0} = (r - \frac{1}{2}v)\frac{\partial}{\partial y} + \rho\sqrt{v}b(v)\frac{\partial^2}{\partial y \partial v}. \tag{5.15}$$

Note that

$$\frac{\partial}{\partial v}p^{Y^0}(s, y, z) = \frac{1}{2}s\frac{\partial^2}{\partial y^2}p^{Y^0}(s, y, z).$$

Let g be a map $y \mapsto g(y)$ such that

$$\begin{aligned} g(y) &= \mathcal{L}^1\mathbf{P}_s^0 f(y) = \mathcal{L}^1\mathbf{P}_{1-(1-s)}^0 f(y) \\ &= \mathcal{L}^1 E[f(Y_1^0)|Y_{1-s}^0 = y] = \mathcal{L}^1\int_{\mathbf{R}} p^{Y^0}(s, y, z)f(z)dz. \end{aligned}$$

Then, we explicitly evaluate (4.19) for $j = 1$.

$$\begin{aligned} &\mathbf{P}_{(1-s)}^0\mathcal{L}^1\mathbf{P}_s^0 f(y_0)|_{y_0=0} \\ =\ & \int_{\mathbf{R}} p^{Y^0}(1-s, y_0, y)g(y)dy|_{y_0=0} \\ =\ & \int_{\mathbf{R}} p^{Y^0}(1-s, y_0, y)\left(\mathcal{L}^1\int_{\mathbf{R}} p^{Y^0}(s, y, z)f(z)dz\right)dy|_{y_0=0} \\ =\ & \int_{\mathbf{R}} p^{Y^0}(1-s, y_0, y)\left(\left(\frac{1}{2}s\rho\sqrt{v}b(v)\frac{\partial^3}{\partial y^3} + (r - \frac{1}{2}v)\frac{\partial}{\partial y}\right)\int_{\mathbf{R}} p^{Y^0}(s, y, z)f(z)dz\right)dy|_{y_0=0} \\ =\ & \frac{1}{2}s\rho\sqrt{v}b(v)\frac{\partial^3}{\partial y_0^3}\mathbf{P}_1^0 f(y_0)|_{y_0=0} + (r - \frac{1}{2}v)\frac{\partial}{\partial y_0}\mathbf{P}_1^0 f(y_0)|_{y_0=0}. \end{aligned}$$

Therefore, we have

$$
\begin{aligned}
& \int_0^1 \mathbf{P}^0_{(1-s)} \mathcal{L}^1 \mathbf{P}^0_s f(y_0) ds|_{y_0=0} \\
= & \int_{\mathbf{R}} \left\{ \frac{1}{4}\rho\sqrt{v_0} b(v_0) \left(\frac{y^3}{v_0^3} - 3\left(\frac{y}{v_0}\right)\right) + \left(r - \frac{1}{2}v_0\right)\left(\frac{y}{v_0}\right) \right\} p^{Y^0}(1,0,y)dy.
\end{aligned}
$$

5.3 Short Time Asymptotics for Local-Stochastic Volatility Model

Consider the following diffusion:

$$
\begin{aligned}
dX_t &= \sigma_t c(X_t) dW_t, & (5.16)\\
X_0 &= x_0 > 0, \\
d\sigma_t &= a(\sigma_t)dt + b(\sigma_t)dZ_t, & (5.17)\\
\sigma_0 &= \sigma > 0,
\end{aligned}
$$

where W_t and Z_t are two standard Brownian motions with correlation ρ.

Proposition 5.3 *When $t \downarrow 0$, we have*

$$
p(t,x_0,x) \sim \frac{1}{\sqrt{2\pi\sigma_0^2 c(x_0)^2 t}} \exp\left(-\frac{(x_0-x)^2}{2\sigma_0^2 c(x_0)^2 t}\right)\left(1+\sqrt{t}\eta(t,x_0,x)\right),
$$

where

$$
\eta(t,x_0,x) = \frac{1}{2}\left(\rho b(\sigma) + \sigma^2 \partial c(x_0)\right)\sigma^2 c(x_0)^3 \left\{ \frac{h_3((x-x_0)/\sqrt{t}, \sigma^2 c(x_0)^2)}{(\sigma^2 c(x_0)^2)^3} \right\}.
$$

(Proof)
We compute the coefficient of the first order in the expansion by applying (4.15) and (4.19).

First, we introduce the time scaling parameter $\epsilon = \sqrt{t}$,

$$
\begin{aligned}
dX_t &= \epsilon\sigma_t c(X_t) dW_t, & (5.18)\\
d\sigma_t &= \epsilon^2 a(\sigma_t)dt + \epsilon b(\sigma_t)dZ_t. & (5.19)
\end{aligned}
$$

The generator $\mathcal{A}^\epsilon$ associated with X is given by

$$
\mathcal{A}^\epsilon = \epsilon^2 \frac{1}{2}\sigma^2 c(x)^2 \frac{\partial^2}{\partial x^2} + \epsilon^2 \rho\sigma b(\sigma) c(x) \frac{\partial^2}{\partial x \partial \sigma} + \epsilon^2 a(\sigma)\frac{\partial}{\partial \sigma} + \epsilon^2 \frac{1}{2} b(\sigma)^2 \frac{\partial^2}{\partial \sigma^2}.
$$

When $\epsilon \downarrow 0$, $\mathcal{A}^\epsilon$ is degenerate. We consider the following transform,

$$
Y = \frac{X - x_0}{\epsilon}. \tag{5.20}
$$

Then the generator $\mathcal{L}^\epsilon$ associated with Y is elliptic under $\epsilon \downarrow 0$ and is given by

$$
\mathcal{L}^\epsilon = \frac{1}{2}\sigma^2 c(x_0+\epsilon y)^2 \frac{\partial^2}{\partial y^2} + \epsilon\rho\sigma b(\sigma) c(x_0+\epsilon y)\frac{\partial^2}{\partial y \partial \sigma} + \epsilon^2 a(\sigma)\frac{\partial}{\partial \sigma} + \epsilon^2 \frac{1}{2} b(\sigma)^2 \frac{\partial^2}{\partial \sigma^2},
$$

and

$$
u^\epsilon(t,y_0) = \mathbf{P}^\epsilon_t f(y_0) = \int_{\mathbf{R}} f(y) p^{Y^\epsilon}(t,y_0,dy), \tag{5.21}
$$

is the fundamental solution to the following equation,

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} - \mathcal{L}^\epsilon\right) u^\epsilon(t,y_0) &= 0, & (5.22)\\
u^\epsilon(t_0,y_0) &= f(y_0).
\end{aligned}
$$

By differentiating (5.21) at $t = 1$, we have

$$
\frac{\partial}{\partial \epsilon}|_{\epsilon=0} \mathbf{P}^\epsilon_1 f(y_0) = \int_{\mathbf{R}} f(y) w_1(y) p^{Y^\epsilon}(1,y_0,y)dy, \tag{5.23}
$$

where the map $y \mapsto w_1(y)$ is the first order PDE weight.

1. Using the integration by pats formula, we derive the first order PDE weight $w_1(y)$,

$$
\begin{aligned}
& \frac{\partial}{\partial\epsilon}|_{\epsilon=0}\mathbf{P}_1^{\epsilon}f(y_0) = \int_0^1 \mathbf{P}_{(1-s)}^0\mathcal{L}^1\mathbf{P}_s^0 f(y_0)ds \\
= & \int_{\mathbf{R}} f(y)\frac{1}{\sigma^2 c(x_0)^2}E\left[Y_{11}\int_0^1 \sigma c(x_0)dW_{1,t} - \int_0^1 D_{t,1}Y_{11}\sigma c(x_0)dt|Y_{01}=y\right]p(1,y_0,y)dy.
\end{aligned}
$$

$\mathcal{L}^0$ and $\mathcal{L}^1$ are given as follows;

$$
\begin{aligned}
\mathcal{L}^0 &= \frac{1}{2}\sigma^2 c(x_0)^2\frac{\partial^2}{\partial y^2}, \\
\mathcal{L}^1 &= \sigma^2 c(x_0)\partial c(x_0)y\frac{\partial^2}{\partial y^2} + \rho\sigma b(\sigma)c(x_0)\frac{\partial^2}{\partial y\partial\sigma}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\frac{\partial}{\partial\sigma}p^{Y^0}(s,y,z) &= \sigma s c(x_0)^2\frac{\partial^2}{\partial y^2}p^{Y^0}(s,y,z), \\
p^{Y^0}(1-s,y_0,y)y &= (1-s)\sigma^2 c(x_0)^2\frac{\partial}{\partial y_0}p^{Y^0}(1-s,y_0,y)|_{y_0=0}.
\end{aligned}
$$

Let g be a map $y\mapsto g(y)$ such that

$$
g(y) = \mathcal{L}^1\mathbf{P}_s^0 f(y) = \mathcal{L}^1\mathbf{P}_{1-(1-s)}^0 f(y) = \mathcal{L}^1 E[f(Y_1^0)|Y_{1-s}^0 = y] = \mathcal{L}^1\int_{\mathbf{R}} p^{Y^0}(s,y,z)f(z)dz.
$$

Then, we explicitly evaluate (4.19) for $j=1$.

$$
\begin{aligned}
& \mathbf{P}_{(1-s)}^0\mathcal{L}^1\mathbf{P}_s^0 f(y_0)|_{y_0=0} \\
= & \int_{\mathbf{R}} p^{Y^0}(1-s,y_0,y)g(y)dy|_{y_0=0} \\
= & \int_{\mathbf{R}} p^{Y^0}(1-s,y_0,y)\left(\mathcal{L}^1\int_{\mathbf{R}} p^{Y^0}(s,y,z)f(z)dz\right)dy|_{y_0=0} \\
= & \int_{\mathbf{R}}\left(\int_{\mathbf{R}}(1-s)\sigma^4 c(x_0)^3\partial c(x_0)\left(\frac{\partial}{\partial y_0}p^{Y^0}(1-s,y_0,y)\right)\frac{\partial^2}{\partial y^2}p^{Y^0}(s,y,z)dy\right)f(z)dz|_{y_0=0} \\
& + \int_{\mathbf{R}}\left(\int_{\mathbf{R}} p^{Y^0}(1-s,y_0,y)s\rho\sigma^2 b(\sigma)c(x_0)^3\frac{\partial^3}{\partial y^3}p^{Y^0}(s,y,z)dy\right)f(z)dz|_{y_0=0} \\
= & \int_{\mathbf{R}}\left(\int_{\mathbf{R}}(1-s)\sigma^4 c(x_0)^3\partial c(x_0)\left(\frac{\partial^3}{\partial y^2\partial y_0}p^{Y^0}(1-s,y_0,y)\right)p^{Y^0}(s,y,z)dy\right)f(z)dz|_{y_0=0} \\
& + \int_{\mathbf{R}}\left(\int_{\mathbf{R}}\left(-\frac{\partial^3}{\partial y^3}p^{Y^0}(1-s,y_0,y)\right)s\rho\sigma^2 b(\sigma)c(x_0)^3 p^{Y^0}(s,y,z)dy\right)f(z)dz|_{y_0=0} \\
= & (1-s)\sigma^4 c(x_0)^3\partial c(x_0)\frac{\partial^3}{\partial y_0^3}\int_{\mathbf{R}}\left(\int_{\mathbf{R}} p^{Y^0}(1-s,y_0,y)p^{Y^0}(s,y,z)dy\right)f(z)dz|_{y_0=0} \\
& + s\rho\sigma^2 b(\sigma)c(x_0)^3\frac{\partial^3}{\partial y_0^3}\int_{\mathbf{R}}\left(\int_{\mathbf{R}} p^{Y^0}(1-s,y_0,y)p^{Y^0}(s,y,z)dy\right)f(z)dz|_{y_0=0} \\
= & (1-s)\sigma^4 c(x_0)^3\partial c(x_0)\frac{\partial^3}{\partial y_0^3}\int_{\mathbf{R}} p^{Y^0}(1,y_0,z)f(z)dz \\
& + s\rho\sigma^2 b(\sigma)c(x_0)^3\frac{\partial^3}{\partial y_0^3}\int_{\mathbf{R}} p^{Y^0}(1,y_0,z)f(z)dz|_{y_0=0} \\
= & (1-s)\sigma^4 c(x_0)^3\partial c(x_0)\frac{\partial^3}{\partial y_0^3}\mathbf{P}_1 f(y_0)|_{y_0=0} + s\rho\sigma^2 b(\sigma)c(x_0)^3\frac{\partial^3}{\partial y_0^3}\mathbf{P}_1 f(y_0)|_{y_0=0}.
\end{aligned}
$$

Therefore, we have

$$\begin{aligned}
&\int_0^1 \mathbf{P}^0_{(1-s)}\mathcal{L}^1\mathbf{P}^0_s f(y_0)ds|_{y_0=0}\\
=\;& \frac{1}{2}\left(\sigma^4 c(x_0)^3\partial c(x_0)\frac{\partial^3}{\partial y_0^3}+\rho\sigma^2 b(\sigma)c(x_0)^3\frac{\partial^3}{\partial y_0^3}\right)\mathbf{P}^0_1 f(y_0)|_{y_0=0}.
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial\epsilon}|_{\epsilon=0}p^{Y^\epsilon}(1,y_0,y) &= \frac{1}{2}\left(\sigma^4 c(x_0)^3\partial c(x_0)\frac{\partial^3}{\partial y_0^3}+\rho\sigma^2 b(\sigma)c(x_0)^3\frac{\partial^3}{\partial y_0^3}\right)p^{Y^0}(1,y_0,y)\\
&= \frac{1}{2}\left(\rho b(\sigma)+\sigma^2\partial c(x_0)\right)\sigma^2 c(x_0)^3\frac{h_3(y,\sigma^2 c(x_0)^2)}{(\sigma^2 c(x_0)^2)^3}p^{Y^0}(1,y_0,y).
\end{aligned}$$

Setting $y_0=0$, we obtain the result.

2. Next, we compute the first order PDE weight by applying (4.15) for $j=1$ in the following way. First, X is approximated by stochastic Taylor expansion,

$$X_t = x_0+X_{1t}+X_{2t}+R_3(t),$$

where

$$\begin{aligned}
X_{1t} &= \int_0^t \sigma c(x_0)dW_s,\\
X_{2t} &= \int_0^t c(x_0)\int_0^s b(\sigma)dZ_u dW_s+\int_0^t \sigma\partial c(x_0)\int_0^s \sigma c(x_0)dW_u dW_s.
\end{aligned}$$

$X^{(\epsilon)}$ and $Y^{(\epsilon)}$ are expanded in $\mathbf{D}_\infty$,

$$\begin{aligned}
X_t^{(\epsilon)} &= x_0+\epsilon X_{1t}+\epsilon^2 X_{2t}+O(\epsilon^3),\\
Y_t^{(\epsilon)} &= Y_{0t}+\epsilon Y_{1t}+O(\epsilon^2),
\end{aligned}$$

where

$$Y_{0t} = X_{1t}=\frac{\partial}{\partial\epsilon}X_t^\epsilon|_{\epsilon=0}=\int_0^t \sigma c(x_0)dW_s,$$

$$Y_{1t} = X_{2t}=\frac{1}{2}\frac{\partial^2}{\partial\epsilon^2}X_t^\epsilon|_{\epsilon=0}=\int_0^t c(x_0)\int_0^s b(\sigma)dZ_u dW_s+\int_0^t \sigma\partial c(x_0)\int_0^s \sigma c(x_0)dW_u dW_s.$$

Then, the PDE weight is caluculated as follows,

$$\begin{aligned}
w_1(y) &= \frac{1}{\sigma^2 c(x_0)^2}E\left[Y_{11}\int_0^1 \sigma c(x_0)dW_{1,t}-\int_0^1 D_{t,1}Y_{11}\sigma c(x_0)dt|Y_{01}=y\right]\\
&= \left(\frac{y}{\sigma^2 c(x_0)^2}-\frac{\partial}{\partial y}\right)(\rho\sigma^2 b(\sigma)c(x_0)^3\int_0^t\int_0^s duds\frac{h_2(y,\sigma^2 c(x_0)^2)}{(\sigma^2 c(x_0)^2)^2}\\
&\quad +\sigma^4 c(x_0)^3\partial c(x_0)\int_0^t\int_0^s duds\frac{h_2(y,\sigma^2 c(x_0)^2)}{(\sigma^2 c(x_0)^2)^2})\\
&= \frac{1}{2}\rho\sigma^2 b(\sigma)c(x_0)^3\frac{h_3(y,\sigma^2 c(x_0)^2)}{(\sigma^2 c(x_0)^2)^3}+\frac{1}{2}\sigma^4 c(x_0)^3\partial c(x_0)\frac{h_3(y,\sigma^2 c(x_0)^2)}{(\sigma^2 c(x_0)^2)^3}.\\
&= \frac{1}{2}\left(\rho b(\sigma)+\sigma^2\partial c(x_0)\right)\sigma^2 c(x_0)^3\frac{h_3(y,\sigma^2 c(x_0)^2)}{(\sigma^2 c(x_0)^2)^3}.
\end{aligned}$$

The following formula holds,

$$p^\epsilon(1,x_0,x)=p(1,0,f(x))\frac{1}{\epsilon},$$

and we have

$$p(t,x_0,x) = p^{\sqrt{t}}(1,x_0,x).$$

Then, we obtain a short time off-diagonal asymptotic expansion of heat kernel,

$$p(t,x_0,x) \quad \sim \quad \frac{1}{\sqrt{2\pi\sigma_0^2 c(x_0)^2 t}} \exp\left(-\frac{(x_0-x)^2}{2\sigma_0^2 c(x_0)^2 t}\right)\left(1+\sqrt{t}\eta(t,x_0,x)\right),$$

where

$$\eta(t,x_0,x) = \frac{1}{2}\left(\rho b(\sigma) + \sigma^2 \partial c(x_0)\right)\sigma^2 c(x_0)^3 \frac{h_3((x-x_0)/\sqrt{t}, \sigma^2 c(x_0)^2)}{(\sigma^2 c(x_0)^2)^3}.$$

5.4 Numerical Example

This subsection provides an numerical example for option pricing under the short-time asymptotic expansion. In particular, we use the following Heston model:

$$\begin{aligned} dS_t &= \sqrt{v_t} S_t dW_{1,t}, &(5.24)\\ dv_t &= \kappa(\theta - v_t)dt + \nu\sqrt{v_t}(\rho dW_{1,t} + \sqrt{1-\rho^2} dW_{2,t}), \end{aligned}$$

with parameters $S_0 = 100$, $v_0 = 0.16$, $\kappa = 1.0$, $\theta = 0.16$, $\nu = 0.1$, $\rho = -0.5$.

A call option price with strike K and maturity t is approximated as follows;

$$C(t,K) = E[(S_t - K)^+] \sim C_0(t,K) + \sqrt{t}C_1(t,K) + tC_2(t,K), \tag{5.25}$$

$$\tag{5.26}$$

where

$$\begin{aligned} C_0(t,K) &= \int_{\mathbf{R}} (e^x - K)^+ p(t,x_0,x)dx,\\ C_1(t,K) &= \int_{\mathbf{R}} (e^x - K)^+ w_1(t,x_0,x) p(t,x_0,x)dx,\\ C_2(t,K) &= \int_{\mathbf{R}} (e^x - K)^+ w_2(t,x_0,x) p(t,x_0,x)dx, \end{aligned} \tag{5.27}$$

and $p(t,x_0,x)$, $w_1(t,x_0,x)$, $w_2(t,x_0,x)$ are obtained in the similar manner as in Subsection 5.2. Also, put option prices are computed by the Put-Call parity.

Table 1: Short time asymptotics $T = 0.1$

Strike	Benchmark	HKE order 2	HKE order 1	HKE order 0
70 Put	0.01	0.01(-17.17%)	0.00 (-55.56%)	0.81 (8079.80%)
80 Put	0.19	0.19 (-1.35%)	0.18 (-4.47%)	0.95 (396.41%)
90 Put	1.38	1.38 (-0.35%)	1.37 (-0.91%)	2.02 (46.06%)
100 Call	5.04	5.03 (-0.14%)	5.02 (-0.30%)	5.47 (8.66%)
110 Call	1.70	1.69 (-0.38%)	1.68 (-0.84%)	1.93 (13.95%)
120 Call	0.44	0.43 (-0.83%)	0.43 (-2.33%)	0.53 (22.82%)
130 Call	0.09	0.09 (-1.48%)	0.08 (-6.04%)	0.12 (36.91%)

A Second Order Approximation in Section 5.1

1. Applying Bismut identity, the weights of second order approximations are calculated as follows.

$$\frac{1}{2}\frac{\partial^2}{\partial\epsilon^2} p^{Y^\epsilon}(1,0,y)|_{\epsilon=0} = \left\{\frac{1}{2}E[H_2(Y_{01}, Y_{11}^2)] + E[H_1(Y_{01}, Y_{21})]\right\} p^{Y^0}(1,0,y).$$

Table 2: Short time asymptotics $T = 0.2$

Strike	Benchmark	HKE order 2	HKE order 1	HKE order 0
70 Put	0.15	0.14 (-5.86%)	0.12 (-18.18%)	1.72 (1041.94%)
80 Put	0.86	0.84 (-1.39%)	0.82 (-4.09%)	2.29 (167.57%)
90 Put	2.94	2.93 (-0.59%)	2.90 (-1.43%)	4.14 (40.75%)
100 Call	7.12	7.09 (-0.31%)	7.07 (-0.66%)	8.02 (12.71%)
110 Call	3.46	3.44 (-0.64%)	3.42 (-1.35%)	4.07 (17.67%)
120 Call	1.50	1.48 (-1.17%)	1.46 (-2.75%)	1.87 (24.73%)
130 Call	0.59	0.58 (-1.91%)	0.56 (-5.41%)	0.79 (34.48%)

Table 3: Short time asymptotics $T = 0.3$

Strike	Benchmark	HKE order 2	HKE order 1	HKE order 0
70 Put	0.44	0.42 (-4.64%)	0.38 (-14.33%)	2.74 (516.85%)
80 Put	1.63	1.61 (-1.63%)	1.56 (-4.60%)	3.71 (127.14%)
90 Put	4.26	4.22 (-0.84%)	4.18 (-2.00%)	6.02 (41.33%)
100 Call	8.70	8.66 (-0.49%)	8.61 (-1.06%)	10.10 (16.01%)
110 Call	4.93	4.89 (-0.90%)	4.84 (-1.91%)	5.96 (20.89%)
120 Call	2.61	2.58 (-1.50%)	2.53 (-3.38%)	3.33 (27.30%)
130 Call	1.31	1.27 (-2.31%)	1.23 (-5.83%)	1.77 (35.59%)

Iterating the Bismut identity, the terms of $\frac{1}{2}H_2(Y_{01}, Y_{11}^2)$ are calculated as follows:

$$\begin{aligned} & E\left[H_2\left(Y_t^{(0)}, \frac{1}{2}(\mu(x_0)t)^2\right)\right) |\sigma(x_0)W_t = y\Big] \\ = & \frac{1}{2}(\mu(x_0)t)^2 \frac{h_2(y, \sigma^2(x_0))}{(\sigma^2(x_0))^2}, \end{aligned}$$

$$\begin{aligned} & E\left[H_2\left(Y_t^{(0)}, \mu(x_0)t\sigma(x_0)\partial\sigma(x_0)\int_0^t W_s dW_s\right) |\sigma(x_0)W_t = y\right] \\ = & \frac{1}{2}t^2\mu(x_0)t\sigma^3(x_0)\partial\sigma(x_0)\frac{h_4(y, \sigma^2(x_0))}{(\sigma^2(x_0))^4}, \end{aligned}$$

$$\begin{aligned} & E\left[H_2\left(Y_t^{(0)}, \frac{1}{2}(\sigma(x_0)\partial\sigma(x_0))^2\left(\int_0^t W_s dW_s\right)^2\right) |\sigma(x_0)W_t = y\right] \\ = & \frac{1}{8}t^2(\partial\sigma(x_0))^2\sigma^6(x_0)\frac{h_6(y, \sigma^2(x_0))}{(\sigma^2(x_0))^6} \\ & +\frac{1}{2}\left\{t^3(\partial\sigma(x_0))^2\sigma^4(x_0)\right\}\frac{h_4(y, \sigma^2(x_0))}{(\sigma^2(x_0))^4} \\ & +\frac{1}{4}t^2(\sigma(x_0)\partial\sigma(x_0))^2\frac{h_2(y, \sigma^2(x_0))}{(\sigma^2(x_0))^2}. \end{aligned}$$

The terms of $H_1(Y_{01}, Y_{21})$ are calculated as follows,

$$\begin{aligned} & E\left[H_1\left(Y_t^{(0)}, \partial\mu(x_0)\sigma(x_0)\int_0^t W_s ds\right) |\sigma(x_0)W_t = y\right] \\ = & \frac{1}{2}\partial\mu(x_0)\sigma^2(x_0)t^2\frac{h_2(y, \sigma^2(x_0))}{(\sigma^2(x_0))^2}, \end{aligned}$$

$$E\left[H_1\left(Y_t^{(0)}, \frac{1}{2}\sigma^2(x_0)\partial^2\sigma(x_0)\int_0^t W_s^2 dW_s\right) |\sigma(x_0)W_t = y\right]$$

Table 4: Short time asymptotics $T = 0.4$

Strike	Benchmark	HKE order 2	HKE order 1	HKE order 0
70 Put	0.82	0.79 (-4.46%)	0.71 (-13.68%)	3.83 (364.08%)
80 Put	2.41	2.37 (-1.92%)	2.29 (-5.29%)	5.13 (112.42%)
90 Put	5.41	5.35 (-1.09%)	5.27 (-2.61%)	7.74 (43.03%)
100 Call	10.04	9.97 (-0.69%)	9.88 (-1.51%)	11.94 (18.94%)
110 Call	6.21	6.14 (-1.17%)	6.06 (-2.49%)	7.69 (23.79%)
120 Call	3.68	3.61 (-1.84%)	3.53 (-4.07%)	4.77 (29.87%)
130 Call	2.09	2.03 (-2.71%)	1.95 (-6.49%)	2.87 (37.41%)

$$
\begin{aligned}
= \quad & \frac{1}{6}\sigma^2(x_0)\partial^2\sigma(x_0)\sigma^3(x_0)t^3\frac{h_4(y,\sigma^2(x_0))}{(\sigma^2(x_0))^4} \\
& +\frac{1}{4}\sigma^2(x_0)\partial^2\sigma(x_0)\sigma(x_0)t^2\frac{h_2(y,\sigma^2(x_0))}{(\sigma^2(x_0))^2},
\end{aligned}
$$

$$
\begin{aligned}
& E\left[H_1\left(Y_t^{(0)}, \mu(x_0)\partial\sigma(x_0)\int_0^t s dW_s\right)|\sigma(x_0)W_t = y\right] \\
= \quad & \frac{1}{2}\mu(x_0)\partial\sigma(x_0)\sigma(x_0)t^2\frac{h_1(y,\sigma^2(x_0))}{(\sigma^2(x_0))^2},
\end{aligned}
$$

$$
\begin{aligned}
& E\left[H_1\left(Y_t^{(0)}, (\partial\sigma(x_0))^2\sigma(x_0)\int_0^t\int_0^s W_u dW_u dW_s\right)|\sigma(x_0)W_t = y\right] \\
= \quad & \frac{1}{6}\partial\sigma(x_0)^2\sigma(x_0)^4t^3\frac{h_4(y,\sigma^2(x_0))}{(\sigma^2(x_0))^4}
\end{aligned}
$$

Therefore, we obtain the second approximation term

$$
\begin{aligned}
& \frac{\partial^2}{\partial\epsilon^2}E[\delta_y(Y_1^{(\epsilon)})]|_{\epsilon=0} \\
= \quad & \{\frac{1}{8}\partial\sigma(x_0)^2\sigma^6(x_0)\frac{h_6(y,\sigma^2(x_0)))}{(\sigma^2(x_0))^6} \\
& +\frac{1}{6}\left(\partial^2\sigma(x_0)\sigma(x_0)^5 + 4\partial\sigma(x_0)^2\sigma(x_0) + 3\mu(x_0)\partial\sigma(x_0)\right)\frac{h_4(y,\sigma^2(x_0))}{(\sigma^2(x_0)))^4} \\
& +\frac{1}{4}(\partial^2\sigma(x_0)\sigma(x_0)^3 \\
& +2\mu'(x_0)\sigma(x_0)^2 + 2\mu(x_0)\partial\sigma(x_0)\sigma(x_0) + \partial\sigma(x_0)^2\sigma(x_0)^2 + 2\mu(x_0)^2)\frac{h_2(x,\sigma^2(x_0))}{(\sigma^2(x_0))^2}\} \\
& \times p(1, x_0, y).
\end{aligned}
$$

2. Using the Lie bracket, the second order term is calculated as follows.

$$
\begin{aligned}
\frac{1}{2}\frac{\partial^2}{\partial\epsilon^2}|_{\epsilon=0}\mathbf{P}_t^\epsilon f(y_0)|_{y_0=0} \quad = \quad & \int_0^t\int_0^{t_1}\mathbf{P}^0_{(t-t_1)}\mathcal{L}^1\mathbf{P}^0_{(t_1-t_2)}\mathcal{L}^1\mathbf{P}^0_{t_2}f(y_0)dt_2dt_1|_{y_0=0} \\
& +\int_0^t\mathbf{P}^0_{(t-t_1)}\mathcal{L}^2\mathbf{P}^0_{t_1}f(y_0)dt_1|_{y_0=0}.
\end{aligned}
$$

The first term is given by

$$
\int_0^t\int_0^{t_1}(\mathcal{L}^1 + (t-t_1)[\mathcal{L}^0,\mathcal{L}^1])(\mathcal{L}^1 + (t-t_2)[\mathcal{L}^0,\mathcal{L}^1])\mathbf{P}^0_t f(y_0)dt_2dt_1|_{y_0=0},
$$

since $[\mathcal{L}^0,[\mathcal{L}^0,\mathcal{L}^1]] = 0$.

The second term is given by

$$\int_0^t (\mathcal{L}^2 + (t-t_1)[\mathcal{L}^0, \mathcal{L}^2] + \frac{1}{2}(t-t_1)^2[\mathcal{L}^0, [\mathcal{L}^0, \mathcal{L}^2]])\mathbf{P}_t^0 f(y_0) dt_1|_{y_0=0},$$

because $[\mathcal{L}^0, [\mathcal{L}^0, [\mathcal{L}^0, \mathcal{L}^2]]] = 0$.
Then we have

$$\begin{aligned} & \frac{1}{2}\frac{\partial^2}{\partial \epsilon^2}|_{\epsilon=0} p^{Y^\epsilon}(1,0,y) = \\ = \quad & (\mathcal{L}^2 + \frac{1}{2}[\mathcal{L}^0, \mathcal{L}^2] + \frac{1}{6}[\mathcal{L}^0, [\mathcal{L}^0, \mathcal{L}^2]] + (\mathcal{L}^1)^2 \\ & + \frac{1}{2}\mathcal{L}^1[\mathcal{L}^0, \mathcal{L}^1] + \frac{1}{6}[\mathcal{L}^0, \mathcal{L}^1]\mathcal{L}^1 + \frac{1}{8}[\mathcal{L}^0, \mathcal{L}^1]^2) p^{Y^0}(1, y_0, y)|_{y_0=0}, \end{aligned} \tag{A.1}$$

where

$$p^{Y^0}(1, y_0, y) = \frac{1}{\sqrt{2\pi\sigma(x_0)^2}} e^{-\frac{1}{2}\left(\frac{y-y_0}{\sigma(x_0)}\right)^2}.$$

Each terms are calculated as follows.

$$\mathcal{L}^2 p^{Y^0}(1, y_0, y)|_{y_0=0} = 0,$$

$$\frac{1}{2}[\mathcal{L}^0, \mathcal{L}^2] p^{Y^0}(1, y_0, y)|_{y_0=0} = \frac{1}{2}\sigma(x_0)^2 \left(\partial\mu(x_0) + \frac{1}{2}\partial\sigma(x_0)^2 + \frac{1}{2}\partial^2\sigma(x_0)\sigma(x_0)\right) \partial^2 p^{Y^0}(1, y_0, y)|_{y_0=0},$$

$$\frac{1}{6}[\mathcal{L}^0, [\mathcal{L}^0, \mathcal{L}^2]] p^{Y^0}(1, y_0, y)|_{y_0=0} = \frac{1}{6}\sigma(x_0)^4 \left(\partial\sigma(x_0)^2 + \partial^2\sigma(x_0)\sigma(x_0)\right) \partial^4 p^{Y^0}(1, y_0, y)|_{y_0=0},$$

$$(\mathcal{L}^1)^2 p(1, y_0, y)|_{y_0=0} = \frac{1}{2}(\partial\sigma(x_0)\sigma(x_0)\mu(x_0) + \mu(x_0)^2)\partial^2 p^{Y^0}(1, y_0, y)|_{y_0=0}$$

$$\frac{1}{2}\mathcal{L}^1[\mathcal{L}^0, \mathcal{L}^1] p^{Y^0}(1, y_0, y)|_{y_0=0} = \frac{1}{3}\mu(x_0)\sigma(x_0)^3\partial\sigma(x_0)\partial^4 p^{Y^0}(1, y_0, y)|_{y_0=0},$$

$$\frac{1}{6}[\mathcal{L}^0, \mathcal{L}^1]\mathcal{L}^1 p^{Y^0}(1, y_0, y)|_{y_0=0} = \frac{1}{6}(\mu(x_0) + 3\sigma(x_0)\partial\sigma(x_0))\sigma(x_0)^3\partial\sigma(x_0)\partial^4 p^{Y^0}(1, y_0, y)|_{y_0=0},$$

$$\frac{1}{8}[\mathcal{L}^0, \mathcal{L}^1]^2 p^{Y^0}(1, y_0, y)|_{y_0=0} = \frac{1}{8}\sigma(x_0)^6\partial\sigma(x_0)^2\partial^6 p^{Y^0}(1, y_0, y)|_{y_0=0}.$$

Hence, we have

$$\begin{aligned} & \frac{1}{2}\frac{\partial^2}{\partial \epsilon^2}|_{\epsilon=0} p^{Y^\epsilon}(1,0,y) \\ = \quad & \{\frac{1}{8}\partial\sigma(x_0)^2\sigma^6(x_0)\frac{h_6(y, \sigma^2(x_0))}{(\sigma^2(x_0))^6} \\ & + \frac{1}{6}\left(\partial^2\sigma(x_0)\sigma(x_0)^5 + 4\partial\sigma(x_0)^2\sigma(x_0) + 3\mu(x_0)\partial\sigma(x_0)\right)\frac{h_4(y, \sigma^2(x_0))}{(\sigma^2(x_0))^4} \\ & + \frac{1}{4}\left(\partial^2\sigma(x_0)\sigma(x_0)^3 + 2\mu'(x_0)\sigma(x_0)^2 + 2\mu(x_0)\partial\sigma(x_0)\sigma(x_0) + \partial\sigma(x_0)^2\sigma(x_0)^2 + 2\mu(x_0)^2\right)\frac{h_2(y, \sigma^2(x_0))}{(\sigma^2(x_0))^2}\} \\ & \times p^{Y^0}(1, 0, y). \end{aligned} \tag{A.2}$$

Therefore, we obtain the result.

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