CARF Working Paper

CARF-F-294

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First version: November 2011
Current version: August 2012

CARF is presently supported by Bank of Tokyo-Mitsubishi UFJ, Ltd., Dai-ichi Mutual Life Insurance Company, Meiji Yasuda Life Insurance Company, Nomura Holdings, Inc. and Sumitomo Mitsui Banking Corporation (in alphabetical order). This financial support enables us to issue CARF Working Papers.

Efficient Combinatorial Exchanges with Opt-Out Types[1]

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First Version: November 11, 2011

This Version: August 7, 2012

Abstract

We investigate combinatorial exchanges as a generalization of auctions and bilateral trades, where multiple heterogeneous commodities are initially possessed not only by a central planner but also by participants. We assume private values, quasi-linearity, risk neutrality, and independent type distribution. Efficiency, Bayesian Incentive Compatibility, and Interim Individual Rationality in a type-dependent manner are required. We introduce a stability notion in the ex-ante term, namely marginal core. By assuming the presence of opt-out types for each player, we show a full characterization in that the central planner inevitably has a deficit if and only if the marginal core is non-empty.

Keywords: Combinatorial Exchanges, Groves Mechanisms, Outside Opportunity, Opt-Out Types, Deficit, Marginal Core

JEL Classification Numbers: D44, D61, D82

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[1] This study was supported by a grant-in-aid for scientific research (KAKENHI 21330043) from the Japan Society for the Promotion of Science (JSPS) and the Ministry of Education, Culture, Sports, Science and Technology (MEXT) of the Japanese government.

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1. Introduction

This paper investigates collective decision problems that have incomplete information, namely, *combinatorial exchanges*. Combinatorial exchanges are regarded to unify and generalize both cases of *bilateral trades* explored by Myerson and Satterthwaite (1983)[3] and *combinatorial auctions* explored by Rassenti, Bulfin, and Smith (1982), Kelso and Crawford (1982), Ausubel and Milgrom (2002), and others[4]. In the same manner as combinatorial auctions, multiple heterogeneous commodities are traded altogether; these commodities are divided into multiple packages to be allocated to participants (players), according to a specified revelation mechanism with side payments, along with these participants' announcements.

Combinatorial auctions generally assume that the central planner (mediator or government) initially possesses all commodities to trade as his (or her) endowment. In realistic situations, however, each participant's valuations of these commodities are dependent on his valuations of commodities that are possessed by other participants or himself as their endowments. Hence, the central planner expects to improve welfare further by exchanging their endowments with each other and allocating the central planner's endowment at the same time.

The framework of combinatorial exchanges allows tradable commodities to be initially possessed not only by the central planner but also by players; each participant sells his endowment and purchases another package of commodities at the same time.[5] However, each player has the *outside opportunity* not to participate in the collective decision and instead to consume his endowment by himself; he could thus have the significant bargaining power over the central planner. Consequently, in order to implement efficient allocations in an incentive-compatible manner, the central planner may have to make considerable subsidies that fulfill their informational rents. As Myerson and Satterthwaite (1983) pointed out, in the opposing case of combinatorial exchanges such as bilateral trades, where the central planner has no initial endowment,

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[3] For related studies such as double auctions, see Chatterjee and Samuelson (1983), Wilson (1985), and Matsushima (2008), for instance.

[4] For a general survey on combinatorial auctions, see Cramton et al (2006).

[5] For the argument about the importance of combinatorial exchanges, see Milgrom (2004, 2007) and Cramton (2011).

it might be inevitable for the central planner who attempts to achieve efficiency to have a deficit[6]. This contrasts with combinatorial auctions, guaranteeing the positivity of the central planner's revenue.

Based on these arguments, the purpose of this paper is to clarify the degree of financial burden on the central planner when implementing efficient allocations in combinatorial exchanges in a manner that is consistent not only with *Bayesian Incentive Compatibility* (BIC) but also with *Interim Individual Rationality* (IIR). IIR requires each player's interim expected payoff to be at least the same as his *type-dependent* outside opportunity value. The main concern of this paper is to clarify what is the necessary and sufficient condition under which it is inevitable for the central planner to have a deficit.

According to the standard auction theory, this paper assumes quasi-linearity, risk-neutrality, private values, and independent type distribution. This paper permits each player's consumption to have an externality effect on other players' welfare. We assume payoff/revenue equivalence, according to which, we can mainly focus on *Groves mechanisms*.

We derive the least upper bound of the central planner's ex-ante expected revenue. We then show a full characterization of the case that the central planner has a deficit from the viewpoint of *stability* in the *ex-ante* term. We introduce a new stability notion, namely the *marginal core*. The marginal core is defined as the collection of all efficient imputations that assign to the central planner zero revenue, but that are unblocked by any coalition that consists of all players but a single player in the ex-ante stage before the players' types being determined.

According to Makowski and Ostroy (1989) and Segal and Whinston (2010), we assume that there is an *opt-out type* in the set of possible types for each player, with which, the consumption of his initial endowment by himself is valuable to the point that the efficient allocation rule will assign it to him irrespective of other players' types. This assumption excludes near-equal share ownerships investigated by Cramton, Gibbons, and Klemperer (1987), which guarantees the non-negativity of revenue even in bilateral trades. The presence of such an opt-out type for each player makes his (or her) bargaining power the strongest. With this assumption, we show a full characterization in

[6] See Cramton, Gibbons, and Klemperer (1987) and Segal and Whinston (2010) for exceptions.

that *the marginal core is non-empty if and only if it is inevitable for the central planner to have a deficit*.

This characterization unifies and generalizes combinatorial auctions and bilateral trades; in combinatorial auctions, where the central planner possesses all commodities, the marginal core is generally empty, implying that the central planner can earn non-negative revenue. In bilateral trades a la Myerson and Satterthwaite, no efficient mechanism satisfies BIC, IIR, or the balanced budgets across participants; in the general case where the central planner possesses no commodity, it is inevitable that the marginal core is non-empty, implying that the central planner has a deficit.

It is generally impossible to make ex-ante stability in terms of the marginal core compatible with BIC and IIR. Whenever a player possesses a sufficient endowment, then the exclusion of him from the collective decision results in a decrease in other players' welfare. By excluding this player, they consequently lose the valuable chance to win the commodities that this excluded player possessed. This makes the marginal core unlikely to be empty, but, at the same time, allows players to have the significant bargaining powers over the central planner, making his revenue negative. Hence, the non-emptiness of the marginal core could be equivalent to the negativity in revenue.

The organization of the remainder of this paper is as follows. Section 2 describes a basic model for general collective decision problems and demonstrates a calculation method for the least upper bound of the ex-ante expected revenue. Section 3 explains combinatorial exchanges. Section 4 explains opt-out types, and describes a tractable characterization of this least upper bound. Section 5 introduces marginal core, and describes a full characterization in that the non-emptiness of the marginal core is necessary and sufficient for the central planner to have a deficit. Section 6 considers various special cases such as bilateral trades, combinatorial auctions, and single seller cases. Section 6 concludes.

2. The Basic Model

Let us consider a collective decision problem that has incomplete information in the following manner. Let $N \equiv \{1,2,...,n\}$ denote the finite set of *players*, where $n \geq 2$. Each player $i \in N$ has a *type* $\omega_i \in \Omega_i$ that is unknown to either other players or a *central planner*, where Ω_i denotes the set of possible types for player i. Let $\Omega \equiv \prod_{i \in N} \Omega_i$ and $\Omega_{-i} \equiv \prod_{j \in N/\{i\}} \Omega_j$. The types ω_i are *independently* distributed across players according to a probability measure that have the full support of Ω. Let A denote the set of all *alternatives* that have typical element a. Each player $i's$ payoff function has a *quasi-linear* and *risk-neutral* form with *private values*, i.e., $v_i(a,\omega_i)+t_i$, where $t_i \in R$ denotes the monetary transfer from the central planner to him and $v_i : A \times \Omega_i \to A$ is his type-dependent valuation function for the alternatives.

For every $i \in N$, let $U_i^* : \Omega_i \to R$ denote player $i's$ *outside opportunity* function, where the outside opportunity for player i with type ω_i is denoted by $U_i^*(\omega_i) \in R$, implying the interim expected payoff that he can receive when he does not participate in this collective decision. Let $U_N^* = (U_i^*)_{i \in N}$.

A *direct mechanism* is defined as (f, x), where $f : \Omega \to A$ denotes an *allocation rule*, $x : \Omega \to R^n$ denotes a *payment rule*, $x = (x_i)_{i \in N}$, and $x_i : \Omega \to R$. When each player $i \in N$ announces $\omega_i \in \Omega_i$, the central planner selects the alternative $f(\omega) \in A$ and makes the transfer payment to each player i, $x_i(\omega) \in R$, where $\omega = (\omega_i)_{i \in N} \in \Omega$ and $x(\omega) = (x_i(\omega))_{i \in N} \in R^n$.

We assume that the allocation rule f is *efficient*; for every $\omega \in \Omega$, the corresponding allocation $f(\omega) \in A$ maximizes the sum of players' valuations in the ex-post term, i.e.,

$$\sum_{i \in N} v_i(f(\omega), \omega_i) = \max_{a \in A} \sum_{i \in N} v_i(a, \omega_i).$$

In order to make the collective decision problem non-trivial, we assume that

(1) $$E[\sum_{i\in N} v_i(f(\omega),\omega_i)] - E[\sum_{i\in N} U_i^*(\omega_i)] > 0 .$$[7]

This assumption implies that the efficient allocation rule f induces a positive net ex-ante expected surplus.

Bayesian Incentive Compatibility (BIC): A direct mechanism (f, x) satisfies *BIC* if for every $i \in N$, every $\omega_i \in \Omega_i$, and every $m_i \in \Omega_i$,

$$E[v_i(f(\omega),\omega_i) + x_i(\omega) \mid \omega_i] \geq E[v_i(f(m_i,\omega_{-i}),\omega_i) + x_i(m_i,\omega_{-i}) \mid \omega_i].$$

Interim Individual Rationality (IIR): A direct mechanism (f, x) and a profile of outside opportunity functions U_N^* satisfy *IIR* if for every $i \in N$ and every $\omega_i \in \Omega_i$,

$$E[v_i(f(\omega)),\omega_i) + x_i(\omega) \mid \omega_i] \geq U_i^*(\omega_i).$$

BIC implies that truth-telling is a Bayesian Nash equilibrium. IIR implies that each player has the incentive to participate in the collective decision irrespective of his type.

Let X denote the set of all payment rules. A payment rule $x \in X$ is said to be a *Groves payment rule* if there exists a function $h_i : \Omega_{-i} \to R$ for each $i \in N$ such that

$$x_i(\omega) = \sum_{j\in N\setminus\{i\}} v_j(f(\omega),\omega_j) + h_i(\omega_{-i}) \quad \text{for all } \omega \in \Omega.$$

Let $X(f) \subset X$ denote the set of all Groves payment rules. Note that a direct mechanism (f, x) is so-called a *Groves mechanism*[8] if and only if $x \in X(f)$. It is evident that any Groves mechanism (f, x) satisfies *strategy-proofness* in the sense that for every $i \in N$ and every $\omega \in \Omega$,

$$v_i(f(\omega),\omega_i) + x_i(\omega) \geq v_i(f(m_i,\omega_{-i}),\omega_i) + x_i(m_i,\omega_{-i}) \quad \text{for all } m_i \in \Omega_i.$$

Strategy-proofness automatically implies BIC.

This paper assumes *payoff equivalence*[9] in that for every payment rule $x \in X$, if

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[7] $E[\cdot]$ denotes the ex-ante expectation operator in terms of $\omega \in \Omega$. $E[\cdot \mid \omega_i]$ denotes the interim expectation operator in terms of $\omega_{-i} \in \Omega_{-i}$ conditional on $\omega_i \in \Omega_i$.

[8] See Groves (1973), Green and Laffont (1977), and Holmstrom (1979).

[9] Krishna and Maenner (2001) showed mild conditions that are sufficient for payoff equivalence in a

(f,x) satisfies BIC, there exists a Groves payment rule $y \in X(f)$ that induces the same interim expected values of transfer payment, i.e., satisfies that for every $i \in N$,

$$E[x_i(\omega) \mid \omega_i] = E[y_i(\omega) \mid \omega_i] \text{ for all } \omega_i \in \Omega_i .$$

The (ex-post) *revenue* for the central planner is defined as the sum of the transfers from all players to the central planner, $-\sum_{i \in N} x_i(\omega)$. Note that payoff equivalence implies *revenue equivalence* in the sense that

$$E[-\sum_{i \in N} x_i(\omega)] = E_{\omega_{-i}}[-\sum_{i \in N} y_i(\omega)].$$

The following proposition shows that for every efficient mechanism (f,x) that satisfies BIC, there exists another efficient mechanism $(f,\tilde{x})$ that satisfies BIC, that induces the same interim expected payoff as that induced by (f,x) for each player irrespective of his type, and that induces the constant ex-post revenue that is equivalent to the ex-ante expected payoff induced by (f,x) irrespective of the type profile.

Proposition 1: *For every payment rule* $x \in X$ *such that* (f,x) *satisfies BIC, there exists another payment rule* $\tilde{x} \in X$ *such that* $(f,\tilde{x})$ *satisfies BIC,*

$$E[\tilde{x}_i(\omega) \mid \omega_i] = E[x_i(\omega) \mid \omega_i] \textit{ for all } i \in N \textit{ and } \omega_i \in \Omega_i ,$$

and

$$\sum_{i \in N} \tilde{x}_i(\omega) = E[\sum_{i \in N} x_i(\omega)] \textit{ for all } \omega \in \Omega .$$

Proof: See the Appendix[10].

Let us denote by $X(f,U_N^*) \subset X(f)$ the set of all Groves payment rules $x \in X(f)$ such that (f,x) and U_N^* satisfy IIR. Let us denote by $r_0 = r_0(U_N^*) \in R$ the *least upper bound* of the ex-ante expected revenue under the constraints of $x \in X(f,U_N^*)$;

broad class of environments with multidimensional types. See also Krishna and Perry (2000), Milgrom and Segal (2002), and Bikhchandani et al. (2006).

[10] The proof of Proposition 1 is closely related to Krishna and Perry (1998) and Krishna (2010, Chapter 5).

$$r_0 \equiv \max_{x \in X(f, U_N^*)} E[-\sum_{i \in N} x_i(\omega)].$$

Proposition 2: *It holds that*

(2) $$r_0 = -(n-1)E[\sum_{i \in N} v_i(f(\omega), \omega_i)] - \sum_{i \in N} \max_{\omega_i \in \Omega_i} \{U_i^*(\omega_i) - E[\sum_{j \in N} v_j(f(\omega), \omega_j) \mid \omega_i]\}.$$

Proof: Let us consider an arbitrary Groves payment rule with IIR, $x \in X(f, U_N^*)$. For every $i \in N$ and every $\omega_i \in \Omega_i$,

$$E[v_i(f(\omega)), \omega_i) + x_i(\omega) \mid \omega_i]$$

$$= E[v_i(f(\omega), \omega_j) + \sum_{j \in N \setminus \{i\}} v_j(f(\omega), \omega_j) + h_i(\omega_{-i}) \mid \omega_i]$$

$$= E[\sum_{j \in N} v_j(f(\omega), \omega_j) \mid \omega_i] + E[h_i(\omega_{-i})].$$

Hence, IIR is equivalent to the inequalities given by

$$E[h_i(\omega_{-i})] \geq \max_{\omega_i \in \Omega_i} \{U_i^*(\omega_i) - E[\sum_{j \in N} v_j(f(\omega), \omega_j) \mid \omega_i]\} \text{ for all } i \in N.$$

Since x is a Groves payment rule, it follows that

$$E[x_i(\omega) \mid \omega_i] \geq E[\sum_{j \in N \setminus \{i\}} v_j(f(\omega), \omega_j) \mid \omega_i]$$

$$+ \max_{\omega_i \in \Omega_i} \{U_i^*(\omega_i) - E[\sum_{j \in N} v_j(f(\omega), \omega_j) \mid \omega_i]\},$$

that is,

$$E[x_i(\omega)] \geq E[\sum_{j \in N \setminus \{i\}} v_j(f(\omega), \omega_j)]$$

$$+ \max_{\omega_i \in \Omega_i} \{U_i^*(\omega_i) - E[\sum_{j \in N} v_j(f(\omega), \omega_j) \mid \omega_i]\}.$$

This implies that

$$E[\sum_{i \in N} x_i(\omega)] \geq (n-1)E[\sum_{i \in N} v_i(f(\omega), \omega_i)]$$

$$+ \sum_{i \in N} \max_{\omega_i \in \Omega_i} \{U_i^*(\omega_i) - E[\sum_{j \in N} v_j(f(\omega), \omega_j) \mid \omega_i]\}.$$

Hence, it follows that

$$r_0 \leq -(n-1)E[\sum_{i \in N} v_i(f(\omega), \omega_i)] - \sum_{i \in N} \max_{\omega_i \in \Omega_i} \{U_i^*(\omega_i) - E[\sum_{j \in N} v_j(f(\omega), \omega_j) \mid \omega_i]\}.$$

For every $i \in N$, let us specify h_i in a manner that

$$E[h_i(\omega_{-i})] = \max_{\omega_i \in \Omega_i}\{U_i^*(\omega_i) - E[\sum_{j \in N} v_j(f(\omega), \omega_j) \mid \omega_i]\},$$

that is,

$$E[x_i(\omega)]$$

$$= E[\sum_{j \in N \setminus \{i\}} v_j(f(\omega), \omega_j) \mid \omega_i] + \max_{\omega_i \in \Omega_i}\{U_i^*(\omega_i) - E[\sum_{j \in N} v_j(f(\omega), \omega_j) \mid \omega_i]\}.$$

Clearly, the specified payment rule x satisfies IIR, and

$$E[\sum_{i \in N} x_i(\omega)]$$

$$= (n-1)E[\sum_{i \in N} v_i(f(\omega), \omega_i)] + \sum_{i \in N} \max_{\omega_i \in \Omega_i}\{U_i^*(\omega_i) - E[\sum_{j \in N} v_j(f(\omega), \omega_j) \mid \omega_i]\},$$

which implies (2).

Q.E.D.

Provided that a Groves payment rule $x \in X(f, U_N^*)$ induces the least upper bound of the ex-ante expected revenue, let us denote by $r_i = r_i(U_N^*)$ the corresponding ex-ante expected payoff for each player $i \in N$. In the proof of Proposition 2, it was shown that for every $i \in N$,

(3) $$r_i = E[\sum_{i \in N} v_i(f(\omega), \omega_i)] - \max_{\omega_i \in \Omega_i}\{U_i^*(\omega_i) - E[\sum_{j \in N} v_j(f(\omega), \omega_j) \mid \omega_i]\}.$$

Note from Proposition 1 that there exists a payment rule $\tilde{x} \in X$ such that $(f, \tilde{x})$ and U_N^* satisfy BIC and IIR,

$$r_i = E[\sum_{i \in N} v_i(f(\omega), \omega_i) + \tilde{x}_i(\omega)] \text{ for all } i \in N,$$

and

$$-\sum_{i \in N} \tilde{x}_i(\omega) = r_0 \text{ for all } \omega \in \Omega.$$

This property has the very important implication; *whenever the central planner can earn a positive ex-ante expected revenue, i.e.,* $r_0 > 0$*, then he can even earn this value* r_0 *as the constant ex-post revenue across type profiles.*

3. Combinatorial Exchanges

From this section on, we focus on *combinatorial exchanges* as a special case of the collective decision problem, in which, both players and the central planner possess multiple commodities as their initial endowments and trade these objects altogether with each other at the same time.

There exist L heterogeneous items. For each $l \in \{1,...,L\}$, the total amount of the $l-th$ item is given by a positive integer $e^l > 0$. Let $e \equiv (e^l)_{l=1}^L \in R_+^L$. We specify the set of all alternatives A as the set of all $nL-$dimensional vectors of nonnegative integers $a = (a_i)_{i\in N}$ satisfying that for every $l \in \{1,...,L\}$,

$$\sum_{i\in N} a_i^l \leq e^l \text{, and } a_i^l \geq 0 \text{ for all } i \in N,$$

where $a_i = (a_i^l)_{l=1}^L$. Let $a = (a_i)_{i\in N} \in A$ and $a_i = (a_i^l)_{l=1}^L$, where $a_i^l \in R$ implies the amount of the $l-th$ item that is allocated to player i. Let us denote $f(\omega) = (f_i(\omega))_{i\in N}$. For every non-empty subset of players, i.e., coalition, $S \subset N$, we denote $a_S \equiv (a_i)_{i\in S} \in R^{|S|L}$.

Let an $L-$dimensional vector of nonnegative integers $e_i = (e_i^l)_{l=1}^L \geq 0$ denote the *initial endowment* for player $i \in N$. Let us denote by $e_N \equiv (e_i)_{i\in N}$ the profile of their initial endowments, where we assume that

$$\sum_{i\in N} e_i^l \leq e^l \text{ for all } l \in \{1,...,L\}.$$

The central planner initially possesses $e^l - \sum_{i\in N} e_i^l \geq 0$ amount of the $l-th$ item for each item $l \in \{1,...,L\}$. According to the standard auction theory, the central planner has zero valuation for any package of commodities.

For every $S \subset N$, let us define a subset $A(S) \subset A$ as the set of all alternatives $a \in A$ such that

$$a_i = e_i \text{ for all } i \in S,$$

implying that any player who belongs to the coalition S, $i \in S$, does not participate in the collective decision. Let us specify a function $f^S : \Omega_{N\setminus S} \to A(S)$, which is regarded as the efficient allocation rule for the difference coalition $N \setminus S$, in a manner that for every $\omega_{N\setminus S} \in \Omega_{N\setminus S}$,

$$\sum_{i\in N\setminus S} v_i(f^S(\omega_{N\setminus S}),\omega_i) = \max_{a\in A(S)} \sum_{i\in N\setminus S} v_i(a,\omega_i),$$

where $\Omega_{N\setminus S} \equiv \times_{i\in N\setminus S} \Omega_i$ and $\omega_{N\setminus S} \equiv (\omega_i)_{i\in N\setminus S} \in \Omega_{N\setminus S}$. According to f^S, the central planner implements efficient allocations for participants, i.e., for players who belong to $N \setminus S$. We assume free disposal in that $v_i(a,\omega_i)$ is non-decreasing with respect to a_i. We permit externality effect in that $v_i(a,\omega_i)$ depends on a_{-i}. We assume that

(4) $$U_i^*(\omega_i) = E[v_i(f^{\{i\}}(\omega_{-i}),\omega_i) \mid \omega_i] \text{ for all } i \in N \text{ and all } \omega_i \in \Omega_i.$$

Each player i has the outside opportunity not to participate in the collective decision and instead to consume his initial endowment e_i by himself. The central planner allocates the remaining commodities $e - e_i$ in order to maximize the sum of other players' (participants') payoffs and his revenue[11].

Theorem 3: *It holds that*

(5) $$r_0 \geq -(n-1)E[\sum_{i\in N} v_i(f(\omega),\omega_i)] + E[\sum_{i\in N}\sum_{j\in N\setminus\{i\}} v_j(f^{\{i\}}(\omega_{-i}),\omega_j)].$$

and for each $i \in N$,

(6) $$r_i \leq E[\sum_{i\in N} v_i(f(\omega),\omega_i)] - E[\sum_{j\in N\setminus\{i\}} v_j(f^{\{i\}}(\omega_{-i}),\omega_j)].$$

Proof: From the definition of $f^{\{i\}}$ and (4), it follows that for every $\omega_i \in \Omega_i$,

$$U_i^*(\omega_i) - E[\sum_{j\in N} v_j(f(\omega),\omega_j) \mid \omega_i]$$

$$= U_i^*(\omega_i) - E[\max_{a\in A} \sum_{j\in N} v_j(a,\omega_j) \mid \omega_i]$$

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[11] Several works such as Jehiel and Moldovanu (1996), Jehiel, Moldovanu, and Stacchetti (1999), and Figueroa and Skreta (2009) have investigated auctions that have externality in different manners. Figueroa and Skreta (2009) assumed that the central planner can make a binding commitment to make inefficient allocations as a device for threatening any player who considers not participating in this auction.

$$\leq U_i^*(\omega_i) - E[\max_{a \in A(\{i\})} \sum_{i \in N \setminus S} v_i(a, \omega_i) \mid \omega_i]$$

$$= U_i^*(\omega_i) - E[\sum_{j \in N} v_j(f^{\{i\}}(\omega_{-i}), \omega_j) \mid \omega_i]$$

$$= -E[\sum_{j \in N \setminus \{i\}} v_j(f^{\{i\}}(\omega_{-i}), \omega_j) \mid \omega_i],$$

which, along with (2) and (3), implies (5) and (6).

Q.E.D.

From Proposition 1 and Theorem 3, it follows that the central planner can earn the ex-post revenue that is greater than or equal to the right hand side of (5) irrespective of the type profile.

4. Opt-Out Types

For every $i \in N$, a type $\tilde{\omega}_i \in \Omega_i$ is said to be an *opt-out type* if

$$f_i(\tilde{\omega}_i, \omega_{-i}) = e_i \text{ for all } \omega_{-i} \in \Omega_{-i}.$$

The notion of out-out type was introduced by Makowski and Ostroy (1989), and further cultivated by Segal and Whinston (2010). When player i has this type, the consumption of his initial endowment e_i by himself is valuable to the point that the efficient allocation rule f assigns it to him irrespective of the other players' types. Whenever there is an opt-out type in the set of possible types Ω_i for each player i, then we can replace Theorem 3 with the following full characterization result.

Theorem 4: *Whenever there exists an opt-out type* $\tilde{\omega}_i \in \Omega_i$ *for each player* $i \in N$, *then*

(7) $$r_0 = -(n-1)E[\sum_{i\in N} v_i(f(\omega), \omega_i)] + E[\sum_{i\in N}\sum_{j\in N\setminus\{i\}} v_j(f^{\{i\}}(\omega_{-i}), \omega_j)],$$

and

(8) $$r_i = E[\sum_{i\in N} v_i(f(\omega), \omega_i)] - E[\sum_{j\in N\setminus\{i\}} v_j(f^{\{i\}}(\omega_{-i}), \omega_j)] \text{ for all } i \in N.$$

Proof: From the definition of opt-out type $\tilde{\omega}_i \in \Omega_i$,

$$E[\sum_{j\in N} v_j(f(\omega), \omega_j) \mid \tilde{\omega}_i] = E[\sum_{j\in N} v_j(f^{\{i\}}(\omega_{-i}), \omega_j)].$$

In the same manner as the proof of Theorem 3,

$$\max_{\omega_i\in\Omega_i}\{U_i^*(\omega_i) - E[\sum_{j\in N} v_j(f(\omega), \omega_j) \mid \omega_i]\} = -E[\sum_{j\in N\setminus\{i\}} v_j(f^{\{i\}}(\omega_{-i}), \omega_j)].$$

This, along with (2) and (3), implies (7) and (8).

Q.E.D.

We can interpret the ex-ante expected payoff for each player i, r_i, which is given by (8), as *his marginal contributions in the ex-ante term*. According to Segal and Whinston (2010), let us define the *coalitional game* $\varpi : 2^N \setminus\{\phi\} \to R$ as assigning to each non-empty coalition $S \subset N$ the maximal ex-ante expected gross surplus in the economy without players who belong to the difference coalition $N \setminus S$, i.e.,

$$\varpi(S) \equiv E[\sum_{j \in S} v_j(f^{N \setminus S}(\omega_S), \omega_j)] \text{ for all } S \in 2^N \setminus \{\phi\}.$$

Note that

$$\varpi(N) \equiv E[\sum_{j \in N} v_j(f(\omega), \omega_j)].$$

It is clear from Theorems 3 and 4 that the following theorem holds.

Theorem 5: *It holds that*

$$r_0 \geq -(n-1)\varpi(N) + \sum_{i \in N} \varpi(N \setminus \{i\}),$$

and for every $i \in N$,

$$r_i \leq \varpi(N) - \varpi(N \setminus \{i\}).$$

Whenever there is an opt-out type for each player, then it holds that

$$r_0 = -(n-1)\varpi(N) + \sum_{i \in N} \varpi(N \setminus \{i\}),$$

and for every $i \in N$,

$$r_i = \varpi(N) - \varpi(N \setminus \{i\}).$$

We can regard $\varpi(N) - \varpi(N \setminus \{i\})$ as the marginal contribution of player i in the ex-ante term. Note that the VCG mechanisms[12], where the out-side opportunity for each player is set equal to zero irrespective of his type, make the ex-post payoff for each player equivalent to his marginal contribution in the ex-post term. Theorem 5 generalizes this equivalence by allowing the outside opportunity for each player to be dependent on his type, and by replacing the ex-post term with the ex-ante term.

From Proposition 1 and Theorem 5, the central planner can earn the constant ex-post revenue that equals $-(n-1)\varpi(N) + \sum_{i \in N} \varpi(N \setminus \{i\})$; there exists $x \in X$ with BIC and IIR such that

$$-\sum_{i \in N} x_i(\omega) = -(n-1)\varpi(N) + \sum_{i \in N} \varpi(N \setminus \{i\}) \text{ for all } \omega \in \Omega.$$

[12] See Vickrey (1961), Clarke (1971), and Groves (1973).

5. Marginal Core

We interpret the full characterization in Theorem 5 from the viewpoint of stability in the *ex-ante* term, namely *marginal core*, in the following manner. Let us consider the situation in which the central planner gives his initial endowment e_0 to the grand coalition N gratis before the players' types being determined; it is implicit to assume that their types become public information, and therefore, the efficient allocations are achievable without requiring any incentive consideration for their truthful revelation.

Let us call any $n-$dimensional vector $\pi = (\pi_i)_{i \in N} \in R^n$ an *imputation*, where π_i implies player $i's$ ex-ante expected payoff. We require an imputation π to satisfy the equality of

(9) $$\sum_{i \in N} \pi_i = \varpi(N).$$

The stability notion namely marginal core, which will be defined below, will permit any size $(n-1)$ coalition to have the option to exclude the single player who does not belong to this coalition; this coalition can conspire to steal the central planner's initial endowment in the ex-ante stage. In terms of possible retaliation measures, the excluded player can cancel his participation by withdrawing his initial endowment from the collective decision, removing the opportunity of its exchange from all members of the coalition. In this case, the imputation could be regarded as being stable if any size $(n-1)$ coalition hesitates to conspire to steal the central planner's initial endowment in the ex-ante stage because they are afraid of the excluded player's subsequent retaliation, i.e., being stable if it is unblocked by any size $(n-1)$ coalition in the ex-ante stage.

Based on the above arguments, we define the *marginal core* as the collection of all imputations π satisfying the equality (9) and

(10) $$\sum_{i \in S} \pi_i > \varpi(N \setminus \{i\}) \text{ for all } i \in N.$$

Proposition 6: *The marginal core is non-empty if and only if*

(11) $$(n-1)\varpi(N) > \sum_{i \in N} \varpi(N \setminus \{i\}).$$

Proof: Suppose that the marginal core is non-empty. Then, there exists $\pi \in R^n$ that satisfies (9) and (10). Then,

$$\varpi(N) = \sum_{i \in N} \pi_i = \frac{1}{n-1} \sum_{i \in N} \sum_{j \in N \setminus \{i\}} \pi_j > \frac{1}{n-1} \sum_{i \in N} \varpi(N \setminus \{i\}),$$

which implies (11).

Suppose that (11) holds. Then, clearly, there exists $\tilde{\pi} = (\tilde{\pi}_i)_{i \in N} \in R^n$ satisfying that

(12) $$\sum_{j \in N \setminus \{i\}} \tilde{\pi}_j = \varpi(N \setminus \{i\}) \text{ for all } i \in N.$$

Let us specify $\pi = (\pi_i)_{i \in N} \in R^n$ by

$$\pi_i = \tilde{\pi}_i + \frac{\varpi(N) - \sum_{j \in N} \tilde{\pi}_j}{n} \text{ for all } i \in N.$$

It is evident that π satisfies (9). From (11) and (12), it follows that

$$\varpi(N) - \sum_{j \in N} \tilde{\pi}_j > 0,$$

and therefore,

$$\pi_i > \tilde{\pi}_i \text{ for all } i \in N,$$

which along with (12) implies (10). Hence, the marginal core is non-empty.

Q.E.D.

The following theorem shows that whenever there is an opt-out type for each player, then *the non-emptiness of the marginal core is necessary and sufficient for the central planner to have a deficit.*

Theorem 7: *If $r_0 < 0$, then the marginal core is non-empty. Whenever there is an opt-out type for each player, then the marginal core is non-empty if and only if $r_0 < 0$.*

Proof: From Theorem 5, it is evident that $r_0 < 0$ implies (11), and that, on the assumption that there is an opt-out type for each player, $r_0 < 0$ is equivalent to (11). This observation, along with Proposition 6, implies this theorem.

Q.E.D.

Segal and Whinston (2010) investigated a case in which the central planner possesses no commodities to trade. They showed that the central planner has a deficit in ex-ante expected revenue if the standard notion of core, which was defined as the collection of all efficient imputations unblocked by any coalition, is non-empty. It would be mostly impossible for the central planner to earn a non-negative ex-ante expected revenue whenever the central planner possesses no initial endowments and there is an opt-out type for each player. Segal and Whinston argued that the exclusion of opt-out types from the set of possible types for each player, such as near-equal share ownerships in Cramton, Gibbons, and Klemperer (1987), is crucial to the settlement of this deficit issue. In contract to Segal and Whinston, this paper emphasizes that the central planner's possession of initial endowment plays the important role in this deficit issue.

From Proposition 1 and Theorem 7, it follows that whenever there is an opt-out type for each player, then the emptiness of the marginal core is necessary and sufficient for the central planner to earn a nonnegative ex-post revenue irrespective of the type profile. That is, the marginal core is empty if and only if there exists a payment rule $x \in X$ such that (f, x) and U_N^* satisfy BIC, IIR, and

$$-\sum_{i \in N} x_i(\omega) = r_0 \geq 0 \quad \text{for all} \quad \omega \in \Omega .$$

6. Special Cases

6.1. Bilateral Trades

Let us consider *bilateral trades* in which $n = 2$, and the central planner possesses no initial endowment, i.e., $e_1 + e_2 = e$. The model of Myerson and Satterthwaite (1983) is a special case, additionally assuming a single object with a single unit. The analysis of Segal and Whinston (2011) on opt-out types is also related to this subsection.

In this case,

$$\varpi(N \setminus \{i\}) = E[U_i^*(\omega_i)] \quad \text{for each} \quad i \in \{1,2\},$$

which, along with Theorem 5, implies that whenever there is an opt-out type for each player, then

$$r_0 = -E[\sum_{i\in\{1,2\}} v_i(f(\omega),\omega_i)] + E[\sum_{i\in\{1,2\}} U_i^*(\omega_i)].$$

Because of (1), it is inevitable that the central planner has a deficit; the central planner loses the amount of money equivalent to the maximal net ex-ante expected surplus in the entire economy.

6.2. Combinatorial Auctions

Let us consider a profile of initial endowments $e_N = \tilde{e}_N = (\tilde{e}_i)_{i\in N}$ such that

$$\tilde{e}_i = 0 \quad \text{for all} \quad i \in N,$$

where the central planner initially possesses all commodities. This subsection makes an assumption that restricts the positivity of the externality effect in a manner that for every $i \in N$,

$$\sum_{j\in N\setminus\{i\}} v_j(f(\omega),\omega_j) < \sum_{j\in N\setminus\{i\}} v_j(f^{\{i\}}(\omega_{-i}),\omega_j), \text{ where } e_i = 0.$$

This assumption implies that players prefer to exclude a single player and consume all commodities by themselves. In this case, the central planner can earn a *positive* ex-ante expected revenue. Since $A(S) = A$ for all $S \subset N$, it follows that

$$\varpi(N\setminus\{i\})=\max_{\substack{a\in A:\\ a_i=0}}\sum_{j\in N\setminus\{i\}}v_i(a,\omega_j)\quad\text{for all}\quad i\in N,$$

which implies that whenever there is an opt-out type for each player, then

(13) $$r_0=r_0(\tilde{e}_N)=-\sum_{i\in N}E[\sum_{j\in N\setminus\{i\}}v_j(f(\omega),\omega_j)-\max_{\substack{a\in A:\\ a_i=0}}\sum_{j\in N\setminus\{i\}}v_i(a,\omega_j)].$$

The assumption of this subsection implies that the right-hand side of (13) is positive.

6.3. Single Seller

Let us consider another profile of initial endowments $e_N=\hat{e}_N=(\hat{e}_i)_{i\in N}$ such that

$$\hat{e}_1=e,\text{ and }\ \hat{e}_i=0\quad\text{for all}\quad i\in N\setminus\{1\},$$

which implies that player 1 initially possesses all commodities. This subsection makes an assumption that restricts the externality effect in a manner that for every $i\in N$, every $\omega_i\in\Omega_i$, and every $a\in A$, player i's valuation of the null package equals zero at all times, i.e.,

$$v_i(a,\omega_i)=0\quad\text{if}\quad a_i=0.$$

Whenever there is an opt-out type for each player, then it is inevitable that the central planner has a deficit. Since

$$A(N\setminus\{1\})=\phi,\text{ and }\ A(N\setminus\{i\})=A\quad\text{for all}\quad i\in N\setminus\{1\},$$

it follows from the assumption of this subsection that

$$v_i(f^{\{1\}}(\omega_{-1}),\omega_i)=0\quad\text{for all}\quad i\in N\setminus\{1\}.$$

Hence,

$$\varpi(N\setminus\{1\})=0,$$

and for every $i\in N\setminus\{1\}$,

$$\varpi(N\setminus\{i\})=\max_{\substack{a\in A:\\ a_i=0}}\sum_{j\in N\setminus\{i\}}v_i(a,\omega_j),$$

implying that

(14) $$r_0=r_0(\hat{e}_N)=-\sum_{i\in N\setminus\{1\}}E[\sum_{j\in N}v_j(f(\omega),\omega_j)-\max_{\substack{a\in A:\\ a_i=0}}\sum_{j\in N\setminus\{i\}}v_i(a,\omega_j)]$$

$$=-(n-1)\varpi(N)+\sum_{i\in N\setminus\{1\}}\varpi(N\setminus\{i\}),$$

which is negative because f is efficient. The interim expected payoff for player 1 is given by

(15) $$E[v_1(f(\omega),\omega_1)+x_1(\omega)\,|\,\omega_1]=E[\sum_{j\in N}v_j(f(\omega),\omega_j)\,|\,\omega_1],$$

and the interim expected payoff for each player $i\in N\setminus\{1\}$ is given by

(16) $$E[v_i(f(\omega),\omega_i)+x_i(\omega)\,|\,\omega_i]=E[\sum_{j\in N}v_j(f(\omega),\omega_j)-\max_{\substack{a\in A:\\ a_i=0}}\sum_{j\in N\setminus\{i\}}v_i(a,\omega_j)\,|\,\omega_i].$$

From (15), the interim expected payoff for player 1 is equivalent to the maximal gross ex-ante expected surplus in the entire economy. Hence, player 1 prefers to invite potential buyers to the collective decision as many times as is possible. From (14) and (16), it follows that the least upper bound of the ex-ante expected revenue is equivalent to the sum of the ex-ante expected payoffs for the players other than player 1. The ex-ante expected revenue does not necessarily increase as the number of players who participate in the collective decision increases. The central planner might not think positively about inviting new traders to the collective decision.

With the assumptions made in this and previous subsections, it follows from (13) and (14) that

$$r(\tilde{e}_N)-r(\hat{e}_N)=E[\max_{a\in A}\sum_{j\in N\setminus\{1\}}v_j(a,\omega_j)],$$

implying that by giving all commodities to player 1 gratis, the central planner must suffer a decrease of $E[\max_{a\in A}\sum_{j\in N\setminus\{1\}}v_j(a,\omega_j)]$ in ex-ante expected revenue; the central planner loses the amount of money equivalent to the *maximal gross surplus in the combinatorial auction that does not have player 1.*

7. Conclusion

This paper investigated combinatorial exchanges in which multiple heterogeneous commodities to trade are initially possessed not only by the central planner but also by players. Each player has the outside opportunity not to participate in the collective decision and to consume his initial endowment by himself. Hence, any player who possesses non-negligible initial endowments has the significant bargaining power over the central planner, making it non-trivial for the central planner to earn a nonnegative revenue.

We assumed that there is an opt-out type for each player, with which, the consumption of his initial endowment by himself is valuable to the point that the efficient allocation rule will assign it to him irrespective of other players' types. With this assumption, we characterized the least upper bound of the ex-ante expected revenue, where the corresponding ex-ante expected payoff for each player was regarded as his marginal contribution in the ex-ante term.

Subsequently, from the viewpoint of stability in the ex-ante term, we showed a full characterization of the case that the central planner had a deficit; the marginal core, which was defined as the collection of all efficient imputations unblocked by any size $(n-1)$ coalition in the ex-ante stage, is empty if and only if the central planner can earn a nonnegative revenue.

It was generally impossible to make stability in terms of marginal core compatible with BIC and IIR. Whenever a player possesses a sufficient initial endowment, the exclusion of this player from the collective decision results in a decrease in other players' welfare; by excluding this player, they consequently lose the valuable chance to win the commodities that this excluded player possessed. This makes the marginal core likely to be non-empty, but, at the same time, allows players to have significant bargaining powers over the central planner, making the revenue negative.

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The Appendix

Proof of Proposition 1: According to Arrow (1979) and d'Aspremont and Gérard-Varet (1979), it is evident from efficiency that there exists a payment rule $x' \in X$ such that (f, x') satisfies BIC and the balanced budgets, i.e.,

$$\sum_{i \in N} x_i'(\omega) = 0 \quad \text{for all} \quad \omega \in \Omega .$$

From payoff equivalence, it is evident that there exists a n-dimensional vector $(b_i)_{i \in N} \in R^n$ such that

$$E[x_i(\omega) \mid \omega_i] = E[x_i'(\omega) \mid \omega_i] + b_i \quad \text{for all} \quad i \in N \quad \text{and all} \quad \omega_i \in \Omega_i .$$

Note that

$$\sum_{i \in N} b_i = E[\sum_{i \in N} x_i(\omega)] .$$

We specify another payment rule $\tilde{x} \in X$ by

$$\tilde{x}_i(\omega) = x_i'(\omega) + b_i \quad \text{for all} \quad i \in N \quad \text{and all} \quad \omega \in \Omega .$$

It is evident from this specification that $(f, \tilde{x})$ satisfies BIC,

$$E[\tilde{x}_i(\omega) \mid \omega_i] = E[x_i'(\omega) \mid \omega_i] + b_i = E[x_i(\omega) \mid \omega_i] \quad \text{for all} \quad i \in N \quad \text{and} \quad \omega_i \in \Omega_i ,$$

and

$$\sum_{i \in N} \tilde{x}_i(\omega) = \sum_{i \in N} x_i'(\omega) + \sum_{i \in N} b_i = E[\sum_{i \in N} x_i(\omega)] \quad \text{for all} \quad \omega \in \Omega .$$

Q.E.D.