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Kazuya Kamiya
University of Tokyo
Takashi Shimizu
Kansai University

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A New Technique for Proving the Existence of Monetary Equilibria in Matching Models with Divisible Money*

Kazuya Kamiya† and Takashi Shimizu‡

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Abstract

This paper develops a new technique for proving the existence of monetary equilibria in money search models. In money search models with divisible money, the set of equilibria, if it exists, is at least one-dimensional. We develop a method to prove the existence of such a set in a fairly simple way. That is, we first find an endpoint of the set of equilibria and then we prove the existence of a continuum of the set of equilibria from this endpoint. Solving for these equilibria is complicated otherwise than using our method. Thus, our technique is simple but very powerful. Further, we consider a rather complicated bargaining procedure that allows us to prove the existence of equilibria in money search model with perfectly divisible goods and money.

Keywords: Real Indeterminacy, Matching Model, Money, Existence of Monetary Equilibria.

Journal of Economic Literature Classification Number: C78, D51, D83, E40.

1 Introduction

In order to prove the existence of equilibria, fixed point theorems have usually been used in both game theory and general equilibrium theory; for example, the existence of Nash equilibria can be directly proved by Kakutani's fixed point theorem and, converting the equilibrium conditions into a mapping from a compact set to itself, the existence of general equilibrium prices follows from Brouwer's or Kakutani's fixed point theorem. As contrasted with these theories, fixed point theorem has seldom been used in money search models, because we cannot discern whether the equilibrium is monetary or non-monetary.[1] Non-monetary equilibrium is the equilibrium without monetary trades and always exists in such models. What we show in this paper is the existence of

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†Faculty of Economics, University of Tokyo, Bunkyo-ku, Tokyo 113-0033 JAPAN (E-mail: kkamiya@e.u-tokyo.ac.jp)

‡Faculty of Economics, Kansai University, 3-3-35 Yamate-cho, Suita, Osaka 564-8680 JAPAN (E-mail: tshimizu@ipcku.kansai-u.ac.jp)

[1]Notable exceptions are Aiyagari and Wallace [1] and Zhu [13], [14].

the equilibrium with monetary trade, so called, "monetary equilibrium." Of course, some sophisticated technique would prevent finding a non-monetary equilibrium even when fixed point theorems are used. However, even with such a technique, equilibrium strategies cannot explicitly be found.

Therefore, instead of using fixed point theorems, the existence of monetary equilibria has typically been proved by explicitly finding a equilibrium strategy in which monetary trades occur. (See, for example, Kiyotaki and Wright [9], Trejos and Wright [11], and Green and Zhou [4].) That is, we first guess the strategy of a monetary equilibrium and, then verify the incentive condition. More precisely, first we pick a strategy in which monetary trades occur, second we solve the Bellman equation corresponding to the strategy, and finally, given the value function obtained in the second step, we check the incentive to play the strategy.

In early papers, such as Kiyotaki and Wright [9], both goods and money are assumed to be indivisible and an agent is assumed to be able to hold at most one unit of them. Thus it is not very difficult to use the above method of "guess and verify." By the assumptions, the equilibrium price, if it exists, is trivially unity in these models. Subsequently, relaxing these assumptions, Shi [10], Trejos and Wright [11] and Green and Zhou [4] present models in which equilibrium prices are endogenously determined. However, they make much effort at solving the Bellman equations, since these equations have quite complicated structure.[2]

It is known that in money search models with divisible money there exists a continuum of monetary equilibria. More precisely, the equilibrium has the property of one degree of freedom. Moreover, there typically exist equilibria in which most of agents do not have money while a few agents have large amounts of money, and in the limit of such equilibria, no agents has money. Note that the limit point is not an equilibrium, since the total amount of money is zero while the exogenously given amount of money is positive. Note that any point close to the limit point can be an equilibrium. For the details, see Sections 2, 3, and 4.

By fully exploiting the above property, we present a new technique for proving the existence of equilibria in money search models with divisible money. Suppose there is a money search model where the existence of equilibrium has not been proved. Let us pick the point where no agent has any money. This could be the endpoint of the set of equilibria. Since the Bellman equation is typically quite simple at the point, then

[2]Kamiya and Sato [7] find a dual-price equilibrium in Green and Zhou's model. To obtain the equilibrium, they analyze a quite complicated fifth order polynomial equations.

it is easy to obtain a solution with a positive value of money, if it exists. In showing the regularity at the solution and applying the implicit function theorem, it can be extended to solutions in which some agents has money.

Zhu [14] shows the existence of equilibria in a divisible money version of Camera and Corbae [3]'s model. In the paper, since a fixed point theorem is indirectly used, the equilibrium strategy cannot be explicitly found. Moreover, quite complicated nature of the technique seems to limit the applicability to other models. Our technique is simple but very powerful; our technique is sufficiently general to apply to quite complicated models, and equilibrium strategies are always explicitly found. Indeed, in Section 3, we show the existence of stationary equilibrium in a new model, where both money and goods are perfectly divisible, and the bargaining procedure is rather complicated.

The plan of this paper is as follows. In Section 2, we present a simple model and explain the technique. Then in Section 3, we apply it to a new model with divisible money and divisible goods, and in Section 4 we discuss our technique in general. In Section 5, we conclude the paper with some discussion.

2 A Simple Example

We first investigate a simple model, which can be considered as a simplified version of Zhou [12]'s model. We first find an endpoint of the set of equilibria, and then we show the existence of the following set of equilibria from this endpoint. We adopt this model because it has a simple and typical structure of the set of equilibria, although the existence of monetary equilibria can be directly obtained.

There is a continuum of agents with a mass of measure one. There are k types of agents with equal fractions and the same number of types of goods. Let κ be the reciprocal of k. A type $i-1$ agent can produce just one unit of type i good and the production cost is $c > 0$. (We assume that a type k agent produces type 1 good.) A type i agent obtains utility $u > c$ only when she consumes one unit of type i good. Time is continuous and pairwise random matchings take place according to Poisson process with parameter $\mu > 0$. For every matched pair, the seller posts a take-it-or-leave-it price offer without knowing the amount of the buyer's money holdings. Let $M > 0$ be the nominal stock of fiat money, and $\gamma > 0$ is a discount rate.

The conditions for a stationary equilibrium are (i) each agent maximizes the expected value of utility-streams, i.e., the Bellman equation is satisfied, (ii) the money holdings distribution of the economy is stationary, i.e., time-invariant, and (iii) the total amount of money the agents have is equal to M. Since the rigorous definition is rather complicated, then, instead, we present the conditions for stationary equilibria with a specific strategy.

In what follows, we focus on a stationary distribution of money holdings of the agents with the support $\{0, p\}$ for some $p > 0$. For simplicity, we assume that money holdings of the agents are in $[0, 2p)$.[3] Thus the money holdings distribution can be expressed by h_n for $n = 0, 1$, the measure of the set of agents with money holding np. Of course, h must satisfy

$$h_0 + h_1 = 1, \tag{1}$$

$$h_n \geq 0, \quad n = 0, 1. \tag{2}$$

We focus on the equilibrium with the following strategy:

- a seller with money holding $\eta \in [0, p)$ offers p,
- a seller with money holding $\eta \in [p, 2p)$ chooses no trade, and
- a buyer with money holding η accepts offer prices less than or equal to η.

According to the strategy specified above, a type i agent without money makes a sale when she meets a type $i+1$ agent with money. The measure of agents with 0 is h_0 and the probability that they can make a sale is $\mu\kappa h_1$, and thus the set of agents with measure $\mu\kappa h_0 h_1$ moves out from 0, i.e., it is an outflow at 0 as well as an inflow at p. On the other hand, a type i agent with p makes a purchase when she meets a type $i-1$ agent without money. The probability that they can make a purchase is $\mu\kappa h_0$, and thus the set of agents with measure $\mu\kappa h_1 h_0$ moves out from p, i.e., it is an outflow at p as well as an inflow at 0. The stationary condition for $h = (h_0, h_1)$ requires that the time rate of inflow should be equal to the time rate of outflow at $n = 0$ and $n = 1$. Both conditions are the same and expressed as follows:

$$\mu\kappa h_0 h_1 = \mu\kappa h_0 h_1.$$

This is clearly an identity, and therefore any h satisfying (1) and (2) can be a stationary distribution. On the other hand, p is determined by

$$M = p h_1. \tag{3}$$

[3] Without this assumption, we can prove a similar result. See Zhou [12] and Kamiya et al. [6].

Let the value function be denoted by $\mathcal{V} : \mathbb{R}_+ \to \mathbb{R}$. Next, we consider the values at $\{0, p\}$. We denote the value at np by V_n, i.e., $V_n = \mathcal{V}(np)$, then the Bellman equation is as follows:

$$G_0 = V_0 - \frac{1}{\phi + 2}\left[h_1 \left(V_1 - c\right) + h_0 V_0 + V_0\right] = 0, \tag{4}$$

$$G_1 = V_1 - \frac{1}{\phi + 2}\left[V_1 + h_0 \left(u + V_0\right) + h_1 V_1\right] = 0, \tag{5}$$

where $\phi = \frac{\gamma}{\mu\kappa}$.

The incentive conditions to play the strategy in (4) and (5) are as follows:

$$-c + V_1 \geq V_0, \tag{6}$$

$$u + V_0 \geq V_1. \tag{7}$$

The first inequality is the condition that an agent with no money has incentive to sell her production good. The second inequality is the condition that an agent with p has incentive to accept an offer price p. Note that defining $\mathcal{V}(\eta) = V_{\lfloor \eta/p \rfloor}$ the incentive conditions at the other η follow from the above conditions, where $\lfloor n \rfloor$ is the largest integer less than or equal to n. (See Zhou [12].) We should also note that the Bellman equation at $\eta \notin (0, p) \cup (p, 2p)$ is satisfied. Therefore a stationary monetary equilibrium, in which all agents choose the strategy described above, is defined as (h_0, h_1, V_0, V_1, p) satisfying (1)-(7). Note that

Remark 1 We should also check the incentive for agents not to offer non-integer multiple of p. Since we defined $\mathcal{V}(\eta) = V_{\lfloor \eta/p \rfloor}$, it is clearly satisfied. (See also Zhou [12].) Note that similar arguments apply to a general model. See Remark 3 in Section 4.

The first step in our technique is to find a point at $h_0 = 1$ satisfying all conditions except (3). The Bellman equation at the point is as follows:

$$V_0 - \frac{1}{\phi + 2}\left[V_0 + V_0\right] = 0,$$

$$V_1 - \frac{1}{\phi + 2}\left[V_1 + (u + V_0)\right] = 0.$$

Since this system of equation is much simpler than (4) and (5), the solution is easily obtained as $V_0 = 0$, $V_1 = \frac{1}{\phi+1} u$.

Next, we check if this value function satisfies the equilibrium conditions except (3) at $(h_0, h_1) = (1, 0)$. Clearly, (7) is satisfied with strict inequality for any u and ϕ. The necessary and sufficient condition for (6) is

$$\phi + 1 \leq \frac{u}{c}. \tag{8}$$

In what follows, we assume that (8) holds with strict inequality.

Clearly, this solution does not satisfy (3) for any $p > 0$. Then the last step of our technique is to slightly extend the point so that (3) is satisfied. To be more precise, we find $(h_0, h_1) = (1-\epsilon, \epsilon)$ and corresponding (V_0, V_1) satisfying all conditions for stationary equilibrium for a sufficiently small $\epsilon > 0$. Clearly, (3) is satisfied for $p = \frac{M}{\epsilon} > 0$. To find such a point, we can simply apply the implicit function theorem. More precisely, the regularity of the system of equations (4) and (5) at $h_0 = 1$ is satisfied as follows:

$$\begin{aligned}
\det \begin{pmatrix} \frac{\partial G_0}{\partial V_0} & \frac{\partial G_0}{\partial V_1} \\ \frac{\partial G_1}{\partial V_0} & \frac{\partial G_1}{\partial V_1} \end{pmatrix}_{h_0=1} &= \det \begin{pmatrix} 1 - \frac{1}{\phi+2}(1+h_0) & -\frac{1}{\phi+2}(1-h_0) \\ -\frac{1}{\phi+2}h_0 & 1 - \frac{1}{\phi+2}(2-h_0) \end{pmatrix}_{h_0=1} \\
&= \det \begin{pmatrix} \frac{\phi}{\phi+2} & 0 \\ -\frac{1}{\phi+2} & \frac{\phi+1}{\phi+2} \end{pmatrix} \\
&\neq 0.
\end{aligned}$$

Then, by the implicit function theorem, (V_0, V_1) satisfying (4) and (5) can be written as C^1 functions of ϵ, $(V_0(\epsilon), V_1(\epsilon))$. It remains to show that this solution satisfies the incentive condition (6). Because (8) is satisfied with strict inequality, this condition is still satisfied for a sufficiently small $\epsilon > 0$. This concludes that $(h_0, h_1, V_0, V_1, p) = (1-\epsilon, \epsilon, V_0(\epsilon), V_1(\epsilon), \frac{M}{\epsilon})$ is a stationary monetary equilibrium for a sufficiently small $\epsilon > 0$.

Note that the point at $h_0 = 1$ is not a “non-monetary” equilibrium, since fiat money has positive value, i.e., $V_1 > 0$. In other words, agents have incentive to use money if they have.

Simple systems of inequalities as considered above can be solved directly, but in the cases where the models are more complex, the system may not be solved directly. However, our technique is applicable to complicated models as we show in Section 3.

3 A Model with Divisible Goods

In this section, we apply our method to a new model. We adopt the same environment as the previous section besides the following three environments:

(i) We assume that money holdings of the agents are in $[0, (N+1)p)$ for some positive integer N, where N is exogenously given. Thus money holdings distributions have the support $\{0, p, \ldots, Np\}$.

(ii) We assume that commodity goods are perfectly divisible. Let $C(q) = q$ be the cost function and $U(q) = q^{\frac{1}{2}}$ be the utility function.

(iii) When a type i agent (i.e. a seller of good $i+1$) meets a type $i+1$ agent (a buyer of good $i+1$), one of the following events occurs: (1) with probability $\frac{1}{2}$, the former can make a take-it-or-leave-it offer (d_s, q_s), a pair of an amount of fiat money and a quantity of good, (2) with probability $\frac{1}{2}$, the latter can make a take-it-or-leave-it offer (d_b, q_b). Moreover, we assume that an agent can observe the type and the current money holdings of the matched agent at the beginning of the bargaining.

We search for a monetary equilibrium in which there exists $p > 0$ such that

- a seller with money holding $\eta < Np$ always offers (p, q_s) for some $q_s > 0$. The offer is accepted by any buyer with money holdings more than or equal to p,
- a seller with money holding $\eta \geq Np$ offers no trade, and
- a buyer with money holding more than or equal to p always offers (p, q_b) for some $q_b > 0$. The offer is accepted by any seller with money holdings less than Np.

Although the off-equilibrium strategy is not completely specified in the above, it will be determined by the value function. Note that the above is sufficient for finding the equilibrium value function. Then the stationary condition for $h = (h_0, h_1, \ldots, h_N)$ is

$$\sum_{n=0}^{N} h_n - 1 = 0, \tag{9}$$

$$\mu\kappa \left[h_1(1 - h_N) - h_0(1 - h_0) \right] = 0, \tag{10}$$

$$\mu\kappa \left[\{h_{n-1}(1 - h_0) + h_{n+1}(1 - h_N)\} - h_n \{(1 - h_0) + (1 - h_N)\} \right] = 0, \quad 1 \leq n \leq N-1, \tag{11}$$

$$\mu\kappa \left[h_{N-1}(1 - h_0) - h_N(1 - h_N) \right] = 0. \tag{12}$$

Note that (10), (11), and (12) correspond to the stationarity at $n = 0$, $n = 1, \ldots, N-1$, and $n = N$, respectively. As in the previous section, it is easily verified that two equations among them are redundant. Thus in what follows we focus on (9) and (11).

Let

$$F_0 = \sum_{n=0}^{N} h_n - 1 = 0,$$
$$F_n = \{h_{n-1}(1-h_0) + h_{n+1}(1-h_N)\} - h_n \{(1-h_0) + (1-h_N)\} = 0, \quad 1 \le n \le N-1.$$

Then we obtain the following stationary distribution from the stationary condition:

$$h_n = h_0 \left(\frac{1-h_0}{1-h_N} \right)^n, \quad n = 1, \dots, N, \tag{13}$$

where h_N is determined so that

$$h_N (1-h_N)^N = h_0 (1-h_0)^N. \tag{14}$$

Clearly, for any $h_0 \in [0,1]$, there exist $h_1, \dots, h_N \in [0,1]$ satisfying (13) and (14). In other words, for any $h_0 \in [0,1]$, there is the corresponding distribution h satisfying the stationary condition.

Next, by the barging procedure, on the equilibrium path, a seller offers (p, q_s^n) to the matched buyer with np, where q_s^n satisfies

$$U(q_s^n) = V_n - V_{n-1}. \tag{15}$$

Similarly, a buyer bids (p, q_b^n) to the matched seller with np, where q_b^n satisfies

$$C(q_b^n) = V_{n+1} - V_n. \tag{16}$$

Then the Bellman equation can be written as follows:

$$G_0 = V_0 - \frac{1}{\phi+2} \left\{ \frac{1}{2} \sum_{n'=1}^{N} h_{n'} (V_1 - C(q_s^{n'})) + \frac{1}{2} h_0 V_0 + \frac{1}{2} V_0 + V_0 \right\} = 0,$$
$$G_n = V_n - \frac{1}{\phi+2} \left\{ \frac{1}{2} \sum_{n'=1}^{N} h_{n'} (V_{n+1} - C(q_s^{n'})) + \frac{1}{2} h_0 V_n \right.$$
$$\left. + \frac{1}{2} V_n + \frac{1}{2} \sum_{n'=0}^{N-1} h_{n'} (V_{n-1} + U(q_b^{n'})) + \frac{1}{2} h_N V_n + \frac{1}{2} V_n \right\} = 0, \quad n = 1, \dots, N-1,$$
$$G_N = V_N - \frac{1}{\phi+2} \left\{ V_N + \frac{1}{2} \sum_{n'=0}^{N-1} h_{n'} (V_{N-1} + U(q_b^{n'})) + \frac{1}{2} h_N V_N + \frac{1}{2} V_N \right\} = 0.$$

As in the previous section, we investigate a solution at $h_0 = 1$. First, from (13) and (14), $h_0 = 1$ implies $h_1 = \cdots = h_N = 0$. Then, using (16), the Bellman equation is written as follows:

$$G_0 = V_0 - \frac{1}{\phi+2}\left\{\frac{1}{2}V_0 + \frac{1}{2}V_0 + V_0\right\} = 0,$$

$$G_n = V_n - \frac{1}{\phi+2}\left\{\frac{1}{2}V_n + \frac{1}{2}V_n + \frac{1}{2}\left[V_{n-1} + U(V_1 - V_0)\right] + \frac{1}{2}V_n\right\} = 0, \quad n = 1, \ldots, N-1,$$

$$G_N = V_N - \frac{1}{\phi+2}\left\{V_N + \frac{1}{2}\left[V_{N-1} + U(V_1 - V_0)\right] + \frac{1}{2}V_N\right\} = 0.$$

we obtain $q_b^0 = A^2$, $V_0 = 0$, and

$$V_n = A^2 \sum_{k=0}^{n-1} A^k, \quad n = 1, \ldots, N, \tag{17}$$

where $A = \left(\frac{1}{2\phi+1}\right)$.[4] Let $y^* = (h_0, \cdots, h_N, V_0, \cdots, V_N)$.

In this paper, we only investigate the case $N = 2$. Let

$$y^* = \left((1,0,0), \left(0, A^2, A^2(1+A)\right)\right).$$

Let $\Psi : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^2 \times \mathbb{R}^3$ be defined as

$$\Psi\left((h_0, h_1, h_2), (V_0, V_1, V_2)\right) = (F_0, F_1, G_0, G_1, G_2)(h_0, h_1, h_2, V_0, V_1, V_2).$$

Let $y = (h, V)$ and denote by $\det D\Psi(y)$ the Jacobian of Ψ with respect to $(h_1, h_2, V_0, V_1, V_2)$ at y. Then the next step is to verify that $\det D\Psi(y^*) \neq 0$ at y^*. Indeed, $\det D\Psi(y^*)$ is calculated as follows:

$$\det D\Psi(y^*) = \det \begin{pmatrix} \Upsilon_1 & 0 \\ \Upsilon_2 & \Upsilon_3 \end{pmatrix},$$

where

$$\begin{aligned} \det \Upsilon_1 &= \det \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \\ &= 3, \end{aligned}$$

and

$$\begin{aligned} \det \Upsilon_3 &= \det \begin{pmatrix} -\frac{\phi}{\phi+2} & 0 & 0 \\ -\frac{1}{\phi+2} - \frac{1}{4A} & \frac{\phi+1}{\phi+2} + \frac{1}{4A} & 0 \\ -\frac{1}{2(\phi+2)} - \frac{1}{4A} & -\frac{1}{2(\phi+2)} + \frac{1}{4A} & -\frac{\phi+1}{\phi+2} \end{pmatrix} \\ &= \frac{\phi(\phi+1)}{(\phi+2)^2}\left(\frac{\phi+1}{\phi+2} + \frac{1}{4A}\right). \end{aligned}$$

[4] The other solution is $q_b^0 = 0$ and $V_n = 0, n = 0, \ldots, N$.

Then $\det D\Psi(y^*) \neq 0$. By the implicit function theorem, a solution to $\Psi(y) = 0$ can be written as a C^1 function of $\epsilon > 0$, where $h_0 = 1 - \epsilon$. Note that, it follows from (13) that, for a sufficiently small $\epsilon > 0$, the corresponding $y(\epsilon) = ((1-\epsilon, h_1(\epsilon), h_2(\epsilon)), (V_0(\epsilon), V_1(\epsilon), V_2(\epsilon)))$ satisfies $h_1(\epsilon) > 0$ and $h_2(\epsilon) > 0$.

Next, we verify that all the incentive conditions are satisfied at $h_0 = 1$ with strict inequalities. It follows that, for a sufficiently small $\epsilon > 0$, the corresponding $y(\epsilon)$ satisfies the incentive conditions.

In case of $N = 2$, the relevant incentive conditions are as follows:

(a) incentive for a buyer with ip to offer p to a seller with jp, where $i = 1, 2$ and $j = 0, 1$,

(b) incentive for a buyer with $2p$ to offer p to a seller with 0,

(c) incentive for a seller with ip to offer p to a buyer with jp, where $i = 0, 1$ and $j = 1, 2$,

(d) incentive for a seller with 0 to offer p to a buyer with $2p$.

Remark 2 As in the model in the previous section, defining $\mathcal{V}(\eta) = V_{\lfloor \eta/p \rfloor}$ for $\eta \in (0,p) \cup (p, 2p)$, we should show (i) the incentive not to offer a noninteger multiple of p, (ii) the incentive to take the strategy at $\eta \in (0,p) \cup (p,2p)$, which is only partially specified in the above, and (iii) the Bellman equation is satisfied at $\eta \in (0,p) \cup (p,2p)$. (i) clearly follows from the above incentive conditions and the definition of $\mathcal{V}$. (ii) and (iii) are also easily follow from them.

For example, consider an agent with money holdings $1.5p$. Suppose she is a buyer and makes an offer to a seller with η. Her optimal offer is determined by

$$\begin{aligned} &\max_{(d,q)} U(q) + \mathcal{V}(1.5p - d) \\ &\text{s.t.} C(q) = \mathcal{V}(\eta + d) - \mathcal{V}(\eta) \\ &\quad \eta + d < 3p, \ 0 \leq d \leq 1.5p. \end{aligned}$$

On the equilibrium path, the money holdings of the sellers are either $0, p$, or $2p$. Thus we should investigate the cases of $\eta = 0, p, 2p$ in order to check the Bellman equation. We investigate the incentive for her to offer p instead of $\tilde{d} \in (0, p)$. Let $q_{(p_b,k)}$ be the quantity of the commodity good such that the seller with kp is indifferent between accepting

$(p_b, q_{(p_b,k)})$ and rejecting it. Then, since $\mathcal{V}$ is a step function, $q_{(\tilde{d},k)}$ is determined by

$$q_{(\tilde{d},k)} = \mathcal{V}\left(kp + \tilde{d}\right) - \mathcal{V}(kp) = V_k - V_k = 0.$$

Thus offering $\tilde{d}$ is not better than no trade. Similarly, an offer price $\tilde{d} \in (p, 1.5p)$ is not better than p. Therefore, we need to show that she prefers offering p to no trade, i.e.,

$$U\left(q_{(p,k)}\right) + V_0 \geq V_1. \tag{18}$$

(18) is the same as the condition that a buyer with p prefers offering p to no trade. Clearly, (18) is a special case of (a). Similar arguments apply to the case that the agent with $1.5p$ is a buyer, to the case that her partner offers, and to the case that she is a seller. Thus the Bellman equation at $1.5p$ is the same as that at p. Of course, this argument applies to agents with any money holdings.

Note that similar arguments apply to a general model. See Remark 3 in Section 4.

As for (a), the strict incentive condition is

$$u\left(V_{i+j} - V_j\right) + V_{i-1} - V_i > 0.$$

Since

$$\begin{aligned} u\left(V_{i+j} - V_j\right) + V_{i-1} - V_i &= A^{1+.5j}\left(1 - A^{i-.5j}\right) \\ &\geq A^{1+.5j}\left(1 - A^{.5}\right), \end{aligned}$$

the strict incentive is always satisfied.

As for (b), the strict incentive condition is

$$u\left(V_1 - V_0\right) + V_1 - u\left(V_2 - V_0\right) + V_0 > 0.$$

Thus by (a), a sufficient condition is

$$V_2 - u\left(V_2 - V_0\right) + V_0 > 0.$$

Clearly,

$$V_2 - u\left(V_2 - V_0\right) + V_0 = A^2(1 + A)\left(1 - \frac{1}{A(1 + A)^{.5}}\right)$$

is strictly positive when A is sufficiently close 1, i.e., ϕ is sufficiently small. Then the strict incentive condition is satisfied for a sufficiently small ϕ.

Similarly, we can verify that (c) and (d) hold for a sufficiently small ϕ. In other words, for a sufficiently small ϕ, $y(\epsilon)$ is a monetary equilibrium when ϵ is sufficiently small.

For $N > 2$, similar arguments can be applied. However, the incentive at $h_0 = 1$ is not strict in some cases. Thus we need to choose strategies which the agents prefer even for $h_0 = 1 - \epsilon$.

4 The Technique in a General Model

In this section we extend our technique to more general environment. The model we consider in this section is similar to one in Kamiya and Shimizu [8]. (In what follows, we call KS.)

4.1 A General Model

There is a continuum of agents with a mass of measure one. There are $k \geq 3$ types of agents with equal fractions and the same number of types of goods. Let κ be the reciprocal of k. A type i good is produced by a type $i-1$ agent. A type i agent obtains some positive utility only when she consumes type i good. We make no assumption on the divisibility of goods. We assume that fiat money is durable and perfectly divisible. Time is continuous, and pairwise random matchings take place according to Poisson process with parameter $\mu > 0$.

We confine our attention to the case that, for some positive number p, all trades occur with its integer multiple amounts of money. In what follows, we focus on a stationary distribution of economy-wide money holdings on $\{0, \dots, N\}$ expressed by $h = (h_0, \dots, h_N)$, where h_n is the measure of agents with np amount of money, and $N < \infty$ is the upper bound of the distribution. Our model includes the case of exogenously determined N as well as the case of endogenously determined N. Of course, $h_n \geq 0$ and $\sum_{n=0}^{N} h_n = 1$ hold. Let $M > 0$ be a given nominal stock of money. Since p is uniquely determined by $\sum_{n=0}^{N} pnh_n = M$ for a given h for $h_0 \neq 1$, then, deleting p from $\{0, p, \dots, Np\}$, the set $\{0, \dots, N\}$ can be considered as the state space.

Since we adopt a general framework, various types of bargaining procedures are allowed.[5] An agent with n, or an agent with np amount of money, chooses an action in $A_n = \{a_{n1}, \dots, a_{ns_n}\}$. Let $A = \Pi_{n=0}^{N} A_n$. For example, an action consists of an

[5] See Subsection 3 for the details.

offer price and a reservation price. Throughout the paper, we confine our attention to the stationary equilibrium in which all agents choose pure strategies. As for mixed strategy equilibrium, see KS. Let $S = \sum_{n=0}^{N} s_n$. Given an equilibrium action profile $a = (a_0, \dots, a_N)$, where a_n is the action taken at np in the equilibrium, define $\alpha(a) = \{(n, j) \mid a_n = a_{nj}\}$.

The monetary transition resulted from transaction among a matched pair is described by a function f. When an agent with money holdings np and action a_{nj} meets an agent with $n'p$ and $a_{n'j'}$, their states, i.e., money holdings, become $(n+f(n,j;n',j'))p$ and $(n' - f(n,j;n',j'))p$, respectively. That is f maps an ordered pair $(n,j;n',j')$ to a non-negative integer $f(n,j;n',j')$. Here "ordered" means, for example, that the former is a seller and the latter is a buyer. When N is exogenously determined, we assume

$$N \geq n + f(n,j;n',j') \quad \text{and} \quad n' - f(n,j;n',j') \geq 0.$$

When N is endogenously determined, we assume the latter condition while the former one should be satisfied on the equilibrium path.

Let $\theta \in \mathbb{R}^L$ be the parameters of the model.

We adopt Bellman equation approach. Let V_n be the value of state n, $n = 0, \dots, N$. The variables in the model are denoted by $x = (h, V, a)$. Let $W_{nj}(x;\theta)$ be the value of action j at state n. Thus, in equilibria, $W_{nj}(x;\theta) = V_n$ holds for $(n,j) \in \alpha(a)$. Note that $W_{nj}(x;\theta)$ includes the utility and/or the production cost of perishable goods.

4.2 A Property of the Stationary Condition

We define

$$h_{nj} = \begin{cases} h_n & \text{if } a_{nj} = a_n, \\ 0 & \text{if } a_{nj} \neq a_n. \end{cases}$$

Then by the random matching assumption and the definition of f, the inflow I_n into state n and the outflow O_n from state n are defined as follows:

$$I_n(h, a; \theta) = \mu\kappa \left[\sum_{(i,j,i',j') \in X_n} h_{ij} h_{i'j'} + \sum_{(i,j,i',j') \in X'_n} h_{ij} h_{i'j'} \right],$$

$$O_n(h, a; \theta) = \mu\kappa \left[\sum_{(j,i',j') \in Y_n} h_{nj} h_{i'j'} + \sum_{(j,i',j') \in Y'_n} h_{nj} h_{i'j'} \right],$$

where

$$
\begin{aligned}
X_n &= \{(i,j,i',j') \mid f(i,j;i',j') > 0,\ i + f(i,j;i',j') = n\},\\
X'_n &= \{(i,j,i',j') \mid f(i,j,i',j') > 0,\ i' - f(i,j;i',j') = n\},\\
Y_n &= \{(j,i',j') \mid f(n,j;i',j') > 0\},\\
Y'_n &= \{(j,i',j') \mid f(i',j';n,j) > 0\}.
\end{aligned}
$$

We denote $I_n - O_n$ by D_n. Then the condition for stationarity is $D_n = 0$ for $n = 0, \ldots, N$ and $\sum_{n=0}^{N} h_n = 1$. Clearly, $\sum_{n=0}^{N} D_n = 0$ holds as an identity, and thus at least one equation is redundant. The following theorem shows that one more equation is always redundant.

Theorem 1 (Kamiya and Shimizu [8]) For any a,

$$\sum_{n=0}^{N} nD_n(h,a;\theta) = 0, \tag{19}$$

is an identity.

Suppose that two agents, say a buyer and a seller, meet and a monetary trade occurs. Then the amount of money the buyer pays is equal to that of the seller obtains; in other words, the amount of money before trade is equal to that of after trade. Since this holds in each trade, the total amount of money before trades, expressed by $\sum_{n=0}^{N} pnO_n(h,a;\theta)$, is equal to the total amount of money after trades, expressed by $\sum_{n=0}^{N} pnI_n(h,a;\theta)$, and thus $\sum_{n=0}^{N} nD_n(h,a;\theta) = 0$ always holds.

Together with the other identity $\sum_{n=0}^{N} D_n(h,a) = 0$, the above theorem implies that h is a stationary distribution if and only if $D_n(h,a;\theta) = 0, n = 2, \ldots, N$, and $\sum_{n=0}^{N} h_n = 1$ hold. Namely, the condition for stationarity has at least one-degree of freedom. This is the main cause of the indeterminacy.

Now the equilibrium condition is expressed as follows:

Definition 1 Given θ, $x = (h, V, a) \in \mathbb{R}^{N+1} \times \mathbb{R}_{+}^{N+1} \times A$ is a (pure strategy) *stationary* *equilibrium* if it satisfies the following:

$$
\begin{aligned}
h_0 &\neq 1, && && (20)\\
\sum_{n=0}^{N} h_n - 1 &= 0, && && (21)\\
D_n(h, a; \theta) &= 0, && n = 1, \dots, N-1 && (22)\\
V_n - W_{nj}(x; \theta) &= 0, && (n, j) \in \alpha(a) && (23)\\
V_n - W_{nj}(x; \theta) &\geq 0, && (n, j) \notin \alpha(a). && (24)
\end{aligned}
$$

(h, V) is called a stationary equilibrium for a and θ if (h, V, a) is a stationary equilibrium for θ. A stationary equilibrium is called a *monetary equilibrium* if $V_n - V_0 > 0$ for some $n > 0$.

(20) is required for the existence of $p > 0$ satisfying

$$\sum_{n=0}^{N} pnh_n = M. \tag{25}$$

(21)-(22) is the stationary condition. Note that, because of Theorem 1, the stationary conditions at $n = 1$ and N are dropped. (23) is the condition that the equilibrium strategy indeed realizes the value. (24) is the relevant incentive condition.[6] We define

$$
\begin{aligned}
F_0 &= \sum_{n=0}^{N} h_n - 1,\\
F_n &= D_n(h, a; \theta), \quad n = 1, \dots, N-1\\
G_n &= V_n - W_{nj}(x; \theta), \quad (n, j) \in \alpha(a)
\end{aligned}
$$

Remark 3 In addition to the above equilibrium conditions, the following conditions are typically required to be an "equilibrium" in most of matching models with money: (i) the incentive not to choose an action out of our action space,[7] and (ii) the incentive to take the equilibrium strategy at state $\eta \notin \{0, p, \dots, Np\}$. However, they are not very restrictive, for KS presents a sufficient condition to assure that (i) and (ii) hold, and it is satisfied in all of the matching models with divisible money known so far, such as Zhou [12]'s model, a divisible money version of Camera and Corbae [3]'s model, and

[6] For the other incentive conditions, see the discussion in Subsection 3.

[7] For example in Section 3, a seller may offer a price which is not an integer multiple of p.

a divisible money version of Trejos and Wright [11]'s model, as well as the models in Section 2 and 3.

4.3 The Technique

KS shows that there is real indeterminacy of stationary equilibria under some global regularity conditions which restrict the global structure of the set of equilibria. In this paper, we only utilize the local structure of stationary equilibria around $h_0 = 1$.

Given a and θ, let $\Psi : \mathbb{R}^{N+1} \times \mathbb{R}^{N+1} \to \mathbb{R}^N \times \mathbb{R}^{N+1}$ be defined as

$$\Psi\left((h_0, \dots, h_N), (V_0, \dots, V_N)\right) = (F_0, \dots, F_{N-1}, G_0, \dots, G_N)(h_0, \dots, h_N, V_0, \dots, V_N, a; \theta).$$

Denote by $\det D\Psi(h, V)$ the determinant of the following $(2N+1) \times (2N+1)$ matrix:

$$\begin{pmatrix} \frac{\partial F_0}{\partial h_1}(h,V) & \dots & \frac{\partial F_0}{\partial h_N}(h,V) & \frac{\partial F_0}{\partial V_0}(h,V) & \dots & \frac{\partial F_0}{\partial V_N}(h,V) \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial F_{N-1}}{\partial h_1}(h,V) & \dots & \frac{\partial F_{N-1}}{\partial h_N}(h,V) & \frac{\partial F_{N-1}}{\partial V_0}(h,V) & \dots & \frac{\partial F_{N-1}}{\partial V_N}(h,V) \\ \frac{\partial G_0}{\partial h_1}(h,V) & \dots & \frac{\partial G_0}{\partial h_N}(h,V) & \frac{\partial G_0}{\partial V_0}(h,V) & \dots & \frac{\partial G_0}{\partial V_N}(h,V) \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial G_N}{\partial h_1}(h,V) & \dots & \frac{\partial G_N}{\partial h_N}(h,V) & \frac{\partial G_N}{\partial V_0}(h,V) & \dots & \frac{\partial G_N}{\partial V_N}(h,V) \end{pmatrix}.$$

Then, our technique proceeds by the following steps:

(1) Find a candidate strategy a for a stationary monetary equilibrium.

(2) Setting $(h_0, h_1, \dots, h_N) = (1, 0, \dots, 0)$, obtain V satisfying (23). Denote $(h, V) = ((1, 0, \dots, 0), V)$ by y^*.

(3) Verify that $\det D\Psi(y^*)$ is nonzero. Then, by the implicit function theorem, there are C^1 functions $(h_1(\epsilon), \dots, h_N(\epsilon), V_0(\epsilon), \dots, V_N(\epsilon))$ which, together with $h_0 = 1 - \epsilon$, a, and θ, satisfies (20)-(23) for a sufficiently small $\epsilon > 0$. Denote $y^*(\epsilon) = (1 - \epsilon, h_1(\epsilon), \dots, h_N(\epsilon), V_0(\epsilon), \dots, V_N(\epsilon))$.

(4) Verify that $h_n(\epsilon) \geq 0$ for $n = 1, 2, \dots, N$ for sufficiently small $\epsilon > 0$.

(5) Verify the incentive condition (24) for a sufficiently small $\epsilon > 0$.

The advantage of our technique is that it is applicable to various models, since the Bellman equations are typically simple at $h_0 = 1$.

5 Concluding Remarks

5.1 The Case of Indivisible Money

The above argument can be easily applied to models with indivisible money. Suppose Δ is the minimum unit of fiat money, i.e., the reciprocal of Δ stands for the degree of divisibility of money. The set of admissible prices is $\{0, \Delta, 2\Delta, \ldots\}$. From (25), it follows that among a continuum of stationary money holdings distributions only a finite number of them, if any, are in the set. Of course, the smaller Δ is, the larger the number of admissible stationary money holdings distributions is. Therefore, for sufficiently small Δ, we can find stationary equilibrium in the neighborhood of $h_0 = 1$.

5.2 General Structure of Stationary Equilibria

In the above discussion, there exists a one-dimensional manifold, a set of stationary equilibria, with the endpoint corresponding to $h_0 = 1$. In this case, following the manifold, we can find its whole structure; especially, equilibria with h_0 not close to one can be obtained. For the methods to follow one-dimensional manifolds, see, for example, Allgower and Georg [2]. See also Herings, Talman and Yang [5]; they present a method to follow a continuum of price constrained equilibria.

References

[1] A. Rao Aiyagari and Neil Wallace. Existence of steady states with positive consumption in the kiyotaki-wright model. *Review of Economic Studies*, 58(5):901–916, Oct. 1991.

[2] Eugene Allgower and Kurt Georg. Simplicial and continuation methods for approximating fixed points. *SIAM Review*, 22:28–85, 1980.

[3] Gabriele Camera and Dean Corbae. Money and price dispersion. *International Economic Review*, 40(4):985–1008, 1999.

[4] Edward J. Green and Ruilin Zhou. A rudimentary random-matching model with divisible money and prices. *Journal of Economic Theory*, 81(2):252–271, 1998.

[5] Jean-Jacques Herings, Dolf Talman, and Zaifu Yang. The computation of a continuum of constrained equilibria. *Mathematics of Operations Research*, 21(3):675–696, 1996.

[6] Kazuya Kamiya, Noritsugu Morishita, and Takashi Shimizu. The existence of single-price equilibria in a matching model with divisible money and production cost. *International Journal of Economic Theory*, forthcoming.

[7] Kazuya Kamiya and Takashi Sato. Equilibrium price dispersion in a matching model with divisible money. *International Economic Review*, 45:413–430, 2004.

[8] Kazuya Kamiya and Takashi Shimizu. Real indeterminacy of stationary equilibria in matching models with divisible money. 2005.

[9] Nobuhiro Kiyotaki and Randall Wright. On money as a medium of exchange. *Journal of Political Economy*, 97(4):927–954, 1989.

[10] Shouyong Shi. Money and prices: A model of search and bargaining. *Journal of Economic Theory*, 67:467–496, 1995.

[11] Alberto Trejos and Randall Wright. Search, bargaining, money, and prices. *Journal of Political Economy*, 103(1):118–141, 1995.

[12] Ruilin Zhou. Individual and aggregate real balances in a random-matching model. *International Economic Review*, 40(4):1009–1038, 1999.

[13] Tao Zhu. Existence of a monetary steady state in a matching model: Indivisible money. *Journal of Economic Theory*, 112:307–324, 2003.

[14] Tao Zhu. Existence of a monetary steady state in a matching model: Divisible money. *Journal of Economic Theory*, forthcoming.