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A New Computational Scheme for Computing Greeks by the Asymptotic Expansion Approach

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Abstract

We developed a new scheme for computing "Greeks" of derivatives by *an asymptotic expansion approach*. In particular, we derived analytical approximation formulae for Deltas and Vegas of plain vanilla and average European call options under general Markovian processes of underlying asset prices. Moreover, we introduced a new variance reduction method of Monte Carlo simulations based on the asymptotic expansion scheme. Finally, several numerical examples under CEV processes confirmed the validity of our method.

Keywords: Asymptotic expansion approach, Option pricing, Greeks,CEV process, Monte Carlo Simulation, Variance reduction method

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1 Introduction

We propose new approximation formulae for computing Greeks, such as the Delta and the Vega that indicate risk indices, the first derivatives of the value of an asset with respect to its parameters. In particular, we derive analytic approximation formulae of the Delta and the Vega of plain vanilla and average options where the underlying assets' prices follow general diffusion processes. Our method is based on the asymptotic expansion approach developed by Takahashi[1995,1999], Kunitomo and Takahashi[1992,2001,2003a] and Takahashi and Yoshida[2004]. We also introduce a new variance reduction method, which is an extension of Takahashi and Yoshida[2005] to increase efficiency of Monte Carlo simulation in computation of Greeks. Moreover, we present series of numerical examples where the underlying prices follow the constant elasticity of variance (CEV) processes and showed effectiveness of our method.

Monitoring and controlling the risks of derivative securities are as important as pricing derivative securities in the practical world. In Black-Scholes(BS) model, option prices and their Greeks are obtained analytically. However, in the more realistic models, it is usually very hard to evaluate option prices and their Greeks analytically. Then, numerical methods are applied.

We suppose that the underlying asset's price S_t follows a stochastic differential equation(SDE);

$$\begin{cases} dS_t = \mu\, S_t\, dt + \sigma^*\left(S_t\right)\, dw_t \\ S_0 = s_0 \; (>0) \end{cases}, \tag{1}$$

where w_t is a one-dimensional Brownian motion, $\mu = r - q$. r and q denote a risk-free rate and a dividend rate respectively which are assumed to be constants. Next, we consider a derivative security whose payoff function at maturity time T is given by ϕ such as

$$\begin{aligned} \phi\left(S_T\right) &= \left(S_T - K\right)_+ \quad \text{plain vanilla European call option,} \\ \text{or,}\quad \phi\left(\tilde{S}_T\right) &= \left(\tilde{S}_T - K\right)_+ \quad \text{average European call option,} \end{aligned} \tag{2}$$

where $\tilde{S}_T = \frac{1}{T}\int_0^T S_t\, dt$. The price of the derivative security can be represented by

$$\begin{aligned} u(s_0) &= E\left[e^{-rT}\,\phi\left(S_T\right)\right] \\ &= e^{-rT} E\left[\phi\left(S_T\right)\right], \end{aligned}$$

where $E\left[\cdot\right]$ denotes the expectation operator under the risk-neutral measure. To obtain the first order derivative of the price with respect to the underlying asset's price at the initial price s_0, a natural method is computing

$$u'\left(s_0\right) \sim \frac{u\left(s_0+\delta\right) - u\left(s_0-\delta\right)}{2\delta},$$

where δ is a sufficiently small positive number, and $u\left(s_0-\delta\right)$ and $u\left(s_0+\delta\right)$ are generated by Monte Carlo simulations. However, its convergence is sometimes very slow and convergence to the true value may not be achieved under some conditions.

To overcome the problem, we may utilize a representation such as

$$u'\left(s_0\right) = e^{-rT} E\left[\phi'\left(S_T\right) Y_T\right],$$

where Y_t satisfies the following SDE:

$$\begin{cases} dY_t = \mu\, Y_t\, dt + \sigma^{*\prime}\left(S_t\right)\, Y_t\, dw_t \\ Y_0 = 1 \end{cases}.$$

Although this representation makes Monte Carlo simulations more efficient, using Monte Carlo simulation itself may be time-consuming, which cause many difficulties in the practical world such as trading business. Therefore, in the case that analytic formulae can not be obtained, ideally we need analytic approximation schemes which can generate values precise enough for practical purpose.

The asymptotic expansion approach have been applied successfully to a broad class of Itô processes appearing in finance. Takahashi[1999] presented a third-order pricing formulae for plain options and second-order formulae for more complicated derivatives such as average options, basket options, and options with stochastic volatility in a general Markovian setting. Kunitomo and Takahashi[2001] provided pricing formulae for bond options(swap options) and average options based on an interest rate model in the class of Heath-Jarrow-Morton[1992] which is not necessarily Markovian. Takahashi[1995] also presented a second order scheme for average options on foreign exchange rates with stochastic interest rates in Heath-Jarrow-Morton framework.

Moreover, Takahashi and Yoshida[2004] extended the method to dynamic portfolio problems. Recently, Takahashi and Saito[2003] successfully applied the method to American options, and Takahashi and Yoshida[2005] proposed a new variance reduction scheme of Monte Carlo simulation with the asymptotic expansion. For details of mathematical validity based on the Malliavin calculus and of further applications, see Kunitomo and Takahashi[2003a, 2003b, 2004], Takahashi and Uchida[2004] and Takahashi and Yoshida[2004,2005].

The organization of this paper is as follows. In section 2 and 3, we review the asymptotic expansion approach to pricing options and the variance reduction technique by the asymptotic expansion scheme respectively. In section 4, we derive approximation formulae for the Delta of plain-vanilla and average European call options and introduce a variance reduction technique in computation of the Delta. Section 5 treats approximations for computing the Vega of plain vanilla and average European call options. We present the results of numerical experiments in section 6. In appendix, we show another derivation of approximation formulae for the Delta and the Vega.

2 The outline of the asymptotic expansion approach to finance

The asymptotic expansion approach is an unified method of efficient computation justified by Malliavin-Watanabe theory. We show a brief summary of the asymptotic expansion approach mainly based on Takahashi[1999].

2.1 Underlying Assets

Let underlying stock price S_t follow

$$\begin{cases} dS_t &= \mu\, S_t\, dt + \sigma^*\left(S_t\right)\, dw_t \\ S_0 &= s_0\ (>0) \end{cases}, \tag{3}$$

where w_t is a one-dimensional Brownian motion and $\mu = r - q$. r and q denote a risk-free rate and a dividend rate respectively which are assumed to be constants. We introduce a small constant $\varepsilon(0 < \varepsilon \leq 1)$ and let $S_t^{(\varepsilon)}$ follow

$$\begin{cases} dS_t^{(\varepsilon)} &= \mu\, S_t^{(\varepsilon)} dt + \varepsilon\sigma\left(S_t^{(\varepsilon)}\right) dw_t \\ S_0^{(\varepsilon)} &= s_0\,(>0) \end{cases}. \tag{4}$$

where σ is a smooth function with bounded derivatives.

Next, let us consider a familiy of $\mathbf{R}$-valued Wiener functionals $\{F_\varepsilon(w)\}$, $\varepsilon \in (0,1]$. For $k = 1, 2, \cdots$, we say $F_\varepsilon(w) = O(\varepsilon^k; w)$ in D_p^s as $\varepsilon \downarrow 0$ if

$$\limsup_{\varepsilon \downarrow 0} \frac{\|F_\varepsilon\|_{p,s}}{\varepsilon^k} < \infty.$$

Here, $\|f\|_{p,s}$ for a Wiener functional f denotes a norm that is the sum of $L^p(\mathbf{P})$-norms up to the s-th order derivatives in the sense of Malliavin. A Banach space D_p^s is the completion of $\mathcal{P}$ with respect to $\|\cdot\|_{p,s}$-norm where $\mathcal{P}$ is the totality of $\mathbf{R}$-valued polynomials on the Wiener space. Roughly speaking, a Banach space D_p^s can be regarded as the totality of random variables which are finite with respect to $\|\cdot\|_{p,s}$-norm. Moreover, we say that $F_\varepsilon(w)$ has the asymptotic expansion

$$F_\varepsilon(w) \sim f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots$$

in D_∞ as $\varepsilon \downarrow 0$ if for every $p > 1$, $s > 0$ and $k = 1, 2, \cdots$

$$F_\varepsilon(w) - (f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots + \varepsilon^{k-1} f_{k-1}) = O(\varepsilon^k; w)$$

in D_p^s as $\varepsilon \downarrow 0$. See Chapter V of Ikeda and Watanabe[1989] for the detail.

Then, we obtain the next proposition. See Takahashi[1999] for the derivation.

Proposition 1. $S_t^{(\varepsilon)}$ *has an asymptotic expansion at* t*:*

$$S_t^{(\varepsilon)} = A_{0t} + \varepsilon A_{1t} + \frac{\varepsilon^2}{2} A_{2t} + \cdots \tag{5}$$

in D_∞ *as* $\varepsilon \downarrow 0$ *where*

$$\begin{aligned}
A_{0t} &= s_0\, e^{\mu\, t}, && (6)\\
A_{1t} &= \int_0^t e^{\mu(t-s)}\, \sigma\,(A_{0s})\; dw_s, && (7)\\
A_{2t} &= 2\int_0^t e^{\mu(t-s)}\, \sigma'\,(A_{0s})\; A_{1s}\, dw_s, && (8)\\
\sigma'\,(A_{0t}) &\equiv \frac{\mathrm{d}\sigma}{\mathrm{dx}}\,(A_{0t})\;. && (9)
\end{aligned}$$

In order to price the European option, we only need the value of the underlying asset at the maturity. Therefore, it is reasonable to suppose that t is a constant number for calculating the price of the European option. Since the right hand side of (6) is a deterministic function, A_{0t} is a constant number with a given constant t. Concerning with A_{1t}, since $e^{\mu(t-s)}\,\sigma\,(A_{0s})$ is a deterministic function also, A_{01} is a normal distribution with a constant variance.

2.2 Plain vanilla European call option

Let $X_t^{(\varepsilon)} = \left(S_t^{(\varepsilon)} - A_{0t}\right)/\varepsilon$, $g_1 = A_{1T}$ and $g_2 = A_{2T}\,/\,2$. Then, we obtain

$$X_T^{(\varepsilon)} = g_1 + \varepsilon\, g_2 + \cdots$$

in D_∞ as $\varepsilon \downarrow 0$. The asymptotic expansion of the price of the plain vanilla European call option of which payoff at the maturity T is $(S_T - K)^+$ is given by

$$\begin{aligned} e^{-rT} E\left[\left(S_T^{(\varepsilon)} - K\right)_+\right] &= \varepsilon\, e^{-rT} E\left[\left(y + X_T^{(\varepsilon)}\right)_+\right] \\ &= \varepsilon\, e^{-rT}\, E\left[(y + g_1 + \varepsilon\, g_2 + \cdots)\ 1_{\{g_1 + \varepsilon\, g_2 + \cdots \geq -y\}}\right], \end{aligned}$$

where $y = (A_{0T} - K)/\varepsilon$ and $E[\cdot]$ denotes the the expectation operator under the risk-neutral measure. Under an appropriate assumption such as *Assumption 6.2* in Kunitomo and Takahashi[2003b], we obtain its approximated value as

$$\begin{aligned} e^{-rT} E\left[\left(S_T^{(\varepsilon)} - K\right)_+\right] &= \varepsilon\, e^{-rT}\, E\left[(y + g_1)\ 1_{\{g_1 \geq -y\}}\right] + \varepsilon^2\, e^{-rT}\, E\left[g_2\, 1_{\{g_1 \geq -y\}}\right] \\ &\quad + \varepsilon^2\, e^{-rT}\, E\left[(y + g_1)\ g_2\, \delta_{-y}(g_1)\right] + O\left(\varepsilon^3\right), \end{aligned}$$

where $\delta_x(\cdot)$ denotes Dirac's Delta function at x, which means that the derivatives of the step function has a proper mathematical meaning as Schwartz's distribution. And also, the calculation above has a proper mathematical meaning as a generalized Wiener functional (see chapter V of Ikeda and Watanabe[1989] or Watanabe[1987] for the detail discussion). Next, by the property of Brownian Motion, we know the distribution of g_1 follows a normal distribution $N(0, \Sigma)$ of which variance is

$$\begin{aligned} \Sigma &= E\left[\left(\int_0^T e^{\mu\,(T-t)}\, \sigma\left(A_{0t}\right)\, dw_t\right)^2\right] \\ &= \int_0^T e^{2\mu\,(T-t)}\, \sigma\left(A_{0t}\right)^2\, dt. \end{aligned} \tag{10}$$

In addition, the conditional expectation $E\left[g_2 | g_1 = x\right]$ is given by

$$E\left[g_2 | g_1 = x\right] = cx^2 + f,$$

where

$$c = \frac{1}{\Sigma^2} \int_0^T \int_0^s e^{\mu\,(T-s)}\, \sigma\left(A_{0s}\right)\, \sigma'\left(A_{0s}\right)\, e^{2\mu\,(T-v)}\, \sigma\left(A_{0v}\right)^2\, dv\, ds,$$

and $f = -c\,\Sigma$. Therefore,

$$\begin{aligned} & e^{-rT} E\left[\left(S_T^{(\varepsilon)} - K\right)^+\right] \\ &= \varepsilon\, e^{-rT} \int_{-y}^{\infty} (y + x)\ n\left[x; 0, \Sigma\right]\, dx + \varepsilon^2\, e^{-rT} \int_{-y}^{\infty} \left(cx^2 + f\right)\, n\left[x; 0, \Sigma\right]\, dx + O\left(\varepsilon^3\right), \end{aligned} \tag{11}$$

where $n\left[x; 0, \Sigma\right]$ is the density function of $N(0, \Sigma)$, i.e.

$$n\left[x; 0, \Sigma\right] = \frac{1}{\sqrt{2\pi\Sigma}} e^{-\frac{x^2}{2\Sigma}}.$$

Then, we reach the next theorem. For the proof, see Takahashi[1999].

Theorem 1. *The asymptotic expansion of the price of the plain vanilla European call option at time zero with maturity T, $C_E^{(\varepsilon)}(0,T)$, is represented by*

$$\begin{aligned} C_E^{(\varepsilon)}(0,T) &= \varepsilon\, e^{-rT} \int_{-y}^{\infty} (y+x)\; n\left[x;0,\Sigma\right]\; dx + \varepsilon^2\, e^{-rT} \int_{-y}^{\infty} \left(cx^2+f\right)\; n\left[x;0,\Sigma\right]\; dx + O(\varepsilon^3) \\ &= \varepsilon\, e^{-rT} \left(y\, N\left(\frac{y}{\sqrt{\Sigma}}\right) + \Sigma\, n\left[y;0,\Sigma\right] \right) + \varepsilon^2\, e^{-rT}\, f\, y\, n\left[y;0,\Sigma\right] + O(\varepsilon^3), \end{aligned} \tag{12}$$

where $N(x)$ is cumulative distribution function of $N(0,1)$.

2.3 Average European call option

For the average European call option, the similar discussion can be applied. Takahashi [1999] derived

$$\widetilde{S}_t^{(\varepsilon)} = \widetilde{A}_{0t} + \varepsilon \widetilde{A}_{1t} + \frac{\varepsilon^2}{2}\, \widetilde{A}_{2t} + \cdots \tag{13}$$

in D_∞ as $\varepsilon \downarrow 0$ where $\widetilde{S}_t^{(\varepsilon)} = \frac{1}{t}\int_0^t S_u^{(\varepsilon)}\, du$ and $\widetilde{A}_{it} = \frac{1}{t}\int_0^t A_{iu}\, du$ for $i = 0,1,2,\cdots$. Let $\widetilde{X}_t^{(\varepsilon)} = \left(\widetilde{S}_t^{(\varepsilon)} - \widetilde{A}_{0t}\right)/\varepsilon$, $g_1 = \widetilde{A}_{1T}$ and $g_2 = \widetilde{A}_{2T}\,/\,2$, then we obtain

$$\widetilde{X}_T^{(\varepsilon)} = g_1 + \varepsilon\, g_2 + \cdots \tag{14}$$

in D_∞ as $\varepsilon \downarrow 0$. The asymptotic expansion of the price of the average European call option of which payoff at the maturity T is $\left(\widetilde{S}_T - K\right)^+$,where where $\widetilde{S}_t = \frac{1}{t}\int_0^t S_u\, du$, is given by

$$\begin{aligned} e^{-rT} E\left[\left(\widetilde{S}_T^{(\varepsilon)} - K\right)^+\right] &= \varepsilon\, e^{-rT} E\left[\left(y + \widetilde{X}_T^{(\varepsilon)}\right)^+\right] \\ &= \varepsilon\, e^{-rT}\, E\left[\left(y + g_1 + \varepsilon\, g_2 + \cdots\right)\; 1_{\{g_1 + \varepsilon\, g_2 + \cdots \geq -y\}}\right], \end{aligned} \tag{15}$$

where $y = \left(\widetilde{A}_{0T} - K\right)/\varepsilon$. Then, we obtain

$$\begin{aligned} e^{-rT} E\left[\left(\widetilde{S}_T^{(\varepsilon)} - K\right)^+\right] &= \varepsilon\, e^{-rT}\, E\left[(y+g_1)\; 1_{\{g_1 \geq -y\}}\right] + \varepsilon^2\, e^{-rT}\, E\left[g_2\, 1_{\{g_1 \geq -y\}}\right] \\ &\quad + \varepsilon^2\, e^{-rT}\, E\left[(y+g_1)\; g_2\, \delta_{-y}(g_1)\right] + O\left(\varepsilon^3\right), \end{aligned} \tag{16}$$

where Σ, which denotes the variance of the distribution of g_1, is calculated by

$$\begin{aligned} \Sigma &= E\left[\left(\frac{1}{T}\int_0^T \int_0^t e^{\mu\,(t-s)}\, \sigma\left(A_{0s}\right)\, dw_s\, dt\right)^2\right] \\ &= \frac{1}{T^2}\, E\left[\left(\int_0^T \int_s^T e^{\mu\,(t-s)}\, \sigma\left(A_{0s}\right)\, dt\, dw_s\right)^2\right] \\ &= \frac{1}{T^2}\, E\left[\left(\int_0^T \frac{1}{\mu}\, \left(e^{\mu(T-s)} - 1\right)\, \sigma\left(A_{0s}\right)\, dw_s\right)^2\right] \\ &= \frac{1}{\mu^2 T^2} \int_0^T \left(e^{\mu(T-s)} - 1\right)^2\, \sigma\left(A_{0s}\right)^2\, ds. \end{aligned} \tag{17}$$

In addition, the coefficients of the conditional expectation $E\left[g_2|g_1=x\right]=cx^2+f$ are given by

$$
\begin{aligned}
c &= \frac{1}{\Sigma^2 T^3}\int_0^T\int_0^t e^{\mu\,(t-s)}\left[\frac{e^{\mu(T-s)-1}}{\mu}\right]\sigma\left(A_{0s}\right)\,\sigma'\left(A_{0s}\right) \\
&\quad \times\int_0^s e^{\mu\,(s-v)}\left[\frac{e^{\mu(T-v)}-1}{\mu}\right]\sigma\left(A_{0v}\right)^2\,dv\,ds\,dt,
\end{aligned}
$$

and $f=-c\,\Sigma$. Therefore,

$$
\begin{aligned}
& e^{-rT}E\left[\left(\widetilde{S}_T^{(\varepsilon)}-K\right)_+\right] \\
&= \varepsilon\,e^{-rT}\int_{-y}^{\infty}(y+x)\;n\left[x;0,\Sigma\right]\,dx+\varepsilon^2\,e^{-rT}\int_{-y}^{\infty}\left(cx^2+f\right)\,n\left[x;0,\Sigma\right]\,dx+O\left(\varepsilon^3\right). \quad (18)
\end{aligned}
$$

Then, we reach the next proposition. For the proof, see Takahashi[1999].

Proposition 2. *The asymptotic expansion of the price of the average European call option at time zero with maturity* T*,* $C_A^{(\varepsilon)}(0,T)$*, is represented by*

$$
\begin{aligned}
C_A^{(\varepsilon)}\left(0,T\right) &= \varepsilon\,e^{-rT}\int_{-y}^{\infty}(y+x)\;n\left[x;0,\Sigma\right]\,dx+\varepsilon^2\,e^{-rT}\int_{-y}^{\infty}\left(cx^2+f\right)\,n\left[x;0,\Sigma\right]\,dx+O(\varepsilon^3) \\
&= \varepsilon\,e^{-rT}\left(y\,N\left(\frac{y}{\sqrt{\Sigma}}\right)+\Sigma\,n\left[y;0,\Sigma\right]\right)+\varepsilon^2\,e^{-rT}\,f\,y\,n\left[y;0,\Sigma\right]+O(\varepsilon^3). \quad (19)
\end{aligned}
$$

3 Variance reduction technique with asymptotic expansion

The discussion below is the outline of the variance reduction technique with asymptotic expansion, which is mainly based on Takahashi and Yoshida[2005].

3.1 Hybrid method

Suppose that $X_u(s,y)$ $(s\le u\le T)$ follows the stochastic integral equation:

$$X_u^{(\varepsilon)}\left(t,y\right)=y+\int_t^u V_0\left(X_{s-}^{(\varepsilon)}\left(t,y\right),\varepsilon\right)ds+\int_t^u V\left(X_{s-}^{(\varepsilon)}\left(t,y\right),\varepsilon\right)dw_s. \quad (20)$$

For a stochastic approximation of $u\left(0,y\right)\equiv E\left[f\left(X_T^{(\varepsilon)}\left(0,y\right)\right)\right]$, an estimator by crude Mote Carlo simulations with N samples is expressed as

$$G\left(n,N\right)=G\left(n,N;\omega_1,\cdots,\omega_N\right)\equiv\frac{1}{N}\sum_{j=1}^{N}f\left(\bar{X}_T^{(\varepsilon)}(\omega_j)\right), \quad (21)$$

since $G(n,N)$ depends on each sample path $\omega_1,\cdots,\omega_N$. The discretized approximation of $\bar{X}^{(\varepsilon)}$ based on Euler-Maruyama scheme is given by

$$\bar{X}_u^{(\varepsilon)}=y+\int_0^u V_0\left(\bar{X}_{\eta(s)}^{(\varepsilon)},\varepsilon\right)ds+\int_0^u V\left(\bar{X}_{\eta(s)}^{(\varepsilon)},\varepsilon\right)dw_s, \quad (22)$$

where $\eta\left(s\right)=\left[\frac{ns}{T}\right]\frac{T}{n}$.

We introduce a modified new estimator of $u(0,y)$ as

$$
\begin{aligned}
G^*\left(\varepsilon,n,N\right) &= G^*(\varepsilon,n,N;\omega_1,\cdots,\omega_N)\\
&\equiv E\left[\hat{f}\left(X_T^{(0)}\left(0,y\right)\right)\right]+\frac{1}{N}\sum_{j=1}^{N}\left(f\left(\bar{X}_T^{(\varepsilon)}(\omega_j)\right)-\hat{f}\left(\bar{X}_T^{(0)}(\omega_j)\right)\right). \qquad (23)
\end{aligned}
$$

We assume that $E\left[\hat{f}\left(X_T^{(0)}\left(0,y\right)\right)\right]$ is calculated analytically. This estimate can be explained intuitively. If the difference between $f\left(\bar{X}_T^{(\varepsilon)}(\omega_j)\right)-u(0,y)$ and $\hat{f}\left(\bar{X}_T^{(0)}(\omega_j)\right)-E\left[\hat{f}\left(X_T^{(0)}\left(0,y\right)\right)\right]$ is small for each independent copy j, then we can expect that the error of $G^*\left(\varepsilon,n,N\right)$ minus true value $u\left(0,y\right)$ can be small because errors of $f\left(\bar{X}_T^{(\varepsilon)}(\omega_j)\right)$ and $\hat{f}\left(\bar{X}_T^{(0)}(\omega_j)\right)$ can be canceled out.

By definition, we have

$$
\begin{aligned}
G^*\left(\varepsilon,n,N\right)-u(0,y) &= \frac{1}{N}\sum_{j=1}^{N}\left[\left\{f\left(\bar{X}_T^{(\varepsilon)}(\omega_j)\right)-E\left[f\left(X_T^{(\varepsilon)}\left(0,T\right)\right)\right]\right\}\right.\\
&\quad\left.-\left\{\hat{f}\left(\bar{X}_T^{(0)}(\omega_j)\right)-E\left[\hat{f}\left(X_T^{(0)}\left(0,T\right)\right)\right]\right\}\right]. \qquad (24)
\end{aligned}
$$

The correlation between $f\left(\bar{X}_T^{(\varepsilon)}(\omega_j)\right)$ and $\hat{f}\left(\bar{X}_T^{(0)}(\omega_j)\right)$ become positively high, since we generate the same Brownian samples for both of $\bar{X}_T^{(0)}$ and $\bar{X}_T^{(\varepsilon)}$ in the simulation.

This type of estimate is similar to the control variate technique, which has been known in the Monte Carlo simulation. It is difficult to find control variables whose expectation can not be derived analytically. Furthermore, usual variance reduction techniques may use control variables that could apply to very narrow class of processes. The advantage of this technique is due to the asymptotic expansion approach, because it is an unified method in a sense that it is applicable to the broad class of processes. We call this variance reduction method with asymptotic expansion *hybrid method*.

3.2 Pricing average European call option by hybrid method

Suppose that the reference asset price process follows:

$$
\left\{
\begin{aligned}
dS_t^{(\varepsilon)} &= \mu\, S_t^{(\varepsilon)}dt+\varepsilon\sigma\left(S_t^{(\varepsilon)}\right)dw_t\\
S_0^{(\varepsilon)} &= s_0\,(>0)
\end{aligned}
\right. .
$$

The price of an option with the strike price K and the maturity T is expressed by

$$
C_A^{(\varepsilon)}=e^{-rT}E\left[\left(\tilde{S}_T^{(\varepsilon)}-K\right)^+\right].
$$

It is rewritten as

$$
C_A^{(\varepsilon)}=e^{-rT}\varepsilon E\left[\left(\frac{1}{T}X_{2,T}^{(\varepsilon)}+y\right)^+\right],
$$

where

$$
\begin{aligned}
X_{1,t}^{(\varepsilon)} &\equiv \frac{S_t^{(\varepsilon)}-S_t^{(0)}}{\varepsilon},\\
X_{2,t}^{(\varepsilon)} &\equiv \frac{t\left(\tilde{S}_t^{(\varepsilon)}-\tilde{S}_t^{(0)}\right)}{\varepsilon},\\
y &\equiv \frac{\tilde{S}_T^{(0)}-K}{\varepsilon}.
\end{aligned}
$$

We utilize equation (23). Consulting equation (18), we replace $\hat{f}(x)$ by

$$\hat{f}(x) = e^{-rT}\varepsilon\left[x + y + \varepsilon\left(cx^2 + f\right)\right]1_{\{-y<x\}}. \tag{25}$$

And we have

$$S_t^{(0)} = e^{\mu t}s_0, \quad \tilde{S}_t^{(0)} = \frac{1}{t}\frac{s_0}{\mu}\left(e^{\mu t} - 1\right).$$

$X_{1,t}^{(\varepsilon)}$ and $X_{2,t}^{(\varepsilon)}$ follow

$$dX_{1,t}^{(\varepsilon)} = \mu X_{1,t}^{(\varepsilon)}dt + \sigma\left(\varepsilon X_{1,t}^{(\varepsilon)} + S_t^{(0)}, t\right)dw_t, \quad X_{1,0}^{(\varepsilon)} = 0, \tag{26}$$

$$dX_{2,t}^{(\varepsilon)} = X_{1,t}^{(\varepsilon)}dt, \quad X_{2,0}^{(\varepsilon)} = 0. \tag{27}$$

Then, we obtain the modified estimator of $C_A^{(\varepsilon)}$ as

$$E\left[\hat{f}\left(X_{2,T}^{(0)}\right)\right] + \frac{1}{N}\sum_{j=1}^{N}\left(e^{-rT}\varepsilon\left(\frac{1}{T}\bar{X}_{2,T}^{(\varepsilon)}(\omega_j) + y\right)^+ - \hat{f}\left(\bar{X}_{2,T}^{(0)}(\omega_j)\right)\right). \tag{28}$$

$\bar{X}_{2,t}^{(0)}$ denotes the Euler-Maruyama scheme of $X_{2,t}^{(0)}$

$$dX_{1,t}^{(0)} = \mu X_{1,t}^{(0)}dt + \sigma\left(S_t^{(0)}\right)dw_t, \quad X_{1,0}^{(0)} = 0, \tag{29}$$

$$dX_{2,t}^{(0)} = X_{2,t}^{(0)}dt, \quad X_{2,0}^{(0)} = 0. \tag{30}$$

Notice that calculating equation (29) does not require the evaluation of $\sigma\left(S_t^{(0)}\right)$ path by path while crude Monte Carlo scheme requires. Therefore, the amount of calculation of the present technique is as large as the crude Monte Carlo method. The algorithm shown above is based on the second order asymptotic value. For other concrete applications using the first order asymptotic value, see Takahashi and Yoshida[2005]. For other concrete applications using the second order asymptotic value, see Takahashi and Uchida[2004].

4 Delta

We consider derivative securities whose payoff function at maturity time T is given by ϕ such as

$$\begin{aligned} \phi(S_T) &= (S_T - K)^+ && \text{plain vanilla European call option,} \\ \phi\left(\tilde{S}_T\right) &= \left(\tilde{S}_T - K\right)^+ && \text{average European call option.} \end{aligned}$$

The price of them can be represented by

$$\begin{aligned} u(s_0) &= E\left[e^{-rT}\phi(S_T)\right] \\ &= e^{-rT}E\left[\phi(S_T)\right], \end{aligned}$$

where $E[\cdot]$ denotes the expectation operator under the risk-neutral measure.

In order to obtain the first order derivative of the price with respect to the underlying asset's price at the initial time, s_0, computing finite difference value defined by

$$u'(s_0) \sim \frac{u(s_0 + \delta) - u(s_0 - \delta)}{2\delta}, \tag{31}$$

is one of the most practical way. In fact, equation (31) does not require the analytic formulae of $u(x)$. If δ is a sufficiently small positive number, generating $u(s_0 - \delta)$ and $u(s_0 + \delta)$ by Monte Carlo simulations gives a reasonable estimate of the first order derivative. However, its convergence is sometimes very slow due to the irregularity of function ϕ. Moreover, if the size of the δ or the size of time slice of Euler-Maruyama scheme is not sufficiently small, convergence to the true value may not be guaranteed even with large number of simulations.

To avoid the problem, we utilize a representation such as

$$u'(s_0) = e^{-rT} E\left[\phi'(S_T) Y_T\right], \tag{32}$$

where $Y_t \equiv \frac{\partial S_t}{\partial s_0}$ (*a stochastic flow*) satisfies the following SDE:

$$\begin{cases} dY_t = r(t)\, Y_t\, dt + \sigma^{*\prime}(S_t)\; Y_t\; dw_t \\ Y_0 = 1 \end{cases} . \tag{33}$$

For the derivation of equation (32), see Fourni$\acute{e}$ et al.[1999] or Imamura et al.[2004] for instance.

4.1 Asymptotic Expansion of Stochastic flow

As briefly discussed above, we have defined Y_t as $\frac{\partial S_t}{\partial s_0}$. Here, we also define $Y_t^{(\varepsilon)}$ as $\frac{\partial S_t^{(\varepsilon)}}{\partial s_0}$. Then, we utilize the asymptotic expansion of the stochastic flow in order to compute the first order derivative of the option price.

Proposition 3. *The asymptotic expansion of the stochastic flow is represented by:*

$$Y_t^{(\varepsilon)} = \frac{\partial A_{0t}}{\partial s_0} + \varepsilon \frac{\partial A_{1t}}{\partial s_0} + \varepsilon^2 \frac{\partial A_{2t}}{\partial s_0} + \cdots, \tag{34}$$

in D_∞ *as* $\varepsilon \downarrow 0$ *where*

$$\frac{\partial A_{0t}}{\partial s_0} = e^{\mu t}, \tag{35}$$

$$\begin{aligned} \frac{\partial A_{1t}}{\partial s_0} &= \int_0^t e^{\mu(t-s)} \sigma'(A_{0s}) \frac{\partial A_{0s}}{\partial s_0} dw_s \\ &= e^{\mu t} \int_0^t \sigma'(A_{0s})\, dw_s, \end{aligned} \tag{36}$$

$$\begin{aligned} \frac{\partial A_{2t}}{\partial s_0} &= 2\left(\int_0^t e^{\mu(t-s)} \sigma''(A_{0s}) \frac{\partial A_{0s}}{\partial s_0} A_{1s} dw_s + \int_0^t e^{\mu(t-s)} \sigma'(A_{0s}) \frac{\partial A_{1s}}{\partial s_0} dw_s\right) \\ &= 2e^{\mu t}\left(\int_0^t \sigma''(A_{0s})\, A_{1s} dw_s + \int_0^t \sigma'(A_{0s}) \int_0^s \sigma'(A_{0u})\, dw_u dw_s\right). \end{aligned} \tag{37}$$

Proof. Differentiate equations (6), (7) and (8) with respect to s_0 and we obtain equations (35), (36) and (37), respectively. ■

4.2 Asymptotic expansion of $\frac{\partial X_T^{(\varepsilon)}}{\partial s_0}$

The payoff of the plain vanilla European call option at the maturity is expressed by

$$(S_T^{(\varepsilon)} - K)^+ = \varepsilon (X_T^{(\varepsilon)} + y)^+ ,$$

where

$$X_T^{(\varepsilon)} \equiv \frac{S_T^{(\varepsilon)} - S_T^{(0)}}{\varepsilon}, \tag{38}$$

$$y = \frac{S_T^{(0)} - k}{\varepsilon}. \tag{39}$$

Differentiating equation (38), we have

$$\begin{aligned} \frac{\partial X_T^{(\varepsilon)}}{\partial s_0} &= \frac{1}{\varepsilon}\left(\frac{\partial S_T^{(\varepsilon)}}{\partial s_0} - \frac{\partial S_T^{(0)}}{\partial s_0}\right) \\ &= \frac{1}{\varepsilon}Y_T^{(\varepsilon)} - \frac{1}{\varepsilon}\frac{\partial S_T^{(0)}}{\partial s_0} \\ &= \frac{1}{\varepsilon}Y_T^{(\varepsilon)} - d, \end{aligned} \tag{40}$$

where

$$d \equiv \frac{1}{\varepsilon}\frac{\partial S_T^{(0)}}{\partial s_0} = \frac{1}{\varepsilon}e^{\mu T}.$$

Therefore, we obtain

$$\begin{aligned} Y_T^{(\varepsilon)} &= \varepsilon\left(\frac{\partial X_T^{(\varepsilon)}}{\partial s_0} + \frac{1}{\varepsilon}\frac{\partial S_T^{(0)}}{\partial s_0}\right) \\ &= \varepsilon\left(\frac{\partial X_T^{(\varepsilon)}}{\partial s_0} + d\right). \end{aligned} \tag{41}$$

Utilizing proposition 3 and equation (40), we obtain

$$\frac{\partial X_T^{(\varepsilon)}}{\partial s_0} = \frac{1}{\varepsilon}\left(\frac{\partial S_T^{(\varepsilon)}}{\partial s_0} - \frac{\partial S_T^{(0)}}{\partial s_0}\right) = \frac{\partial g_1}{\partial s_0} + \varepsilon\frac{\partial g_2}{\partial s_0} + \cdots, \tag{42}$$

in D_∞ as $\varepsilon \downarrow 0$ where

$$\frac{\partial g_1}{\partial s_0} = e^{\mu T}\int_0^T \sigma'\left(A_{0s}\right) dw_s, \tag{43}$$

$$\frac{\partial g_2}{\partial s_0} = e^{\mu T}\int_0^T \sigma''\left(A_{0s}\right) A_{1s} dw_s + e^{\mu T}\int_0^T \sigma'\left(A_{0s}\right)\int_0^s \sigma'\left(A_{0u}\right) dw_u dw_s. \tag{44}$$

4.3 Asymptotic expansion of Delta

In order to obtain the concrete formulae of Delta, we need calculate the following expectation:

$$
\begin{aligned}
&E\left[Y_T^{(\varepsilon)}\cdot 1_{\left\{S_T^{(\varepsilon)}\geq K\right\}}\right]\\
&=\varepsilon E\left[\left\{\left(d+\tfrac{\partial g_1}{\partial s_0}\right)+\varepsilon\tfrac{\partial g_2}{\partial s_0}+\cdots\right\}\cdot 1_{\{g_1+\varepsilon g_2+\cdots\geq -y\}}\right]\\
&=\varepsilon\Big\{E\left[\left(d+\tfrac{\partial g_1}{\partial s_0}\right)\cdot 1_{\{g_1\geq -y\}}\right]+\varepsilon E\left[\tfrac{\partial g_2}{\partial s_0}\cdot 1_{\{g_1\geq -y\}}\right]\\
&\quad+\varepsilon E\left[\left(d+\tfrac{\partial g_1}{\partial s_0}\right)\cdot\delta_{-y}(g_1)\ \tfrac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0}(g_1+\varepsilon g_2+\cdots)\right]\Big\}\\
&=\varepsilon\Big\{E\left[\left(d+\tfrac{\partial g_1}{\partial s_0}\right)\cdot 1_{\{g_1\geq -y\}}\right]+\varepsilon E\left[\tfrac{\partial g_2}{\partial s_0}\cdot 1_{\{g_1\geq -y\}}\right]\\
&\quad+\varepsilon E\left[\left(d+\tfrac{\partial g_1}{\partial s_0}\right)\cdot g_2\cdot\delta_{-y}(g_1)\right]\Big\}+O\left(\varepsilon^3\right).\\
&=\varepsilon E\left[d\cdot 1_{\{g_1\geq -y\}}\right]+\varepsilon E\left[\tfrac{\partial g_1}{\partial s_0}\cdot 1_{\{g_1\geq -y\}}\right]+\varepsilon^2 E\left[\tfrac{\partial g_2}{\partial s_0}\cdot 1_{\{g_1\geq -y\}}\right]\\
&\quad+\varepsilon^2 E\left[d\cdot g_2\cdot\delta_{-y}(g_1)\right]+\varepsilon^2 E\left[\tfrac{\partial g_1}{\partial s_0}\cdot g_2\cdot\delta_{-y}(g_1)\right]+O\left(\varepsilon^3\right),
\end{aligned}
$$

where $y=\frac{S_T^{(0)}-k}{\varepsilon}$. We notice that there are five parts.

For the first part,

$$
\begin{aligned}
E\left[d\cdot 1_{\{g_1\geq -y\}}\right] &= \varepsilon e^{\mu T}\int_{-\infty}^{\infty}E\left[d\,|g_1=x\right]1_{\{g_1\geq -y\}}dx &(45)\\
&= \varepsilon\cdot d\cdot e^{\mu T}\int_{-y}^{\infty}n\left[x;0,\Sigma\right]dx &(46)\\
&= \varepsilon\cdot d\cdot N\left(\frac{y}{\sqrt{\Sigma}}\right). &(47)
\end{aligned}
$$

For the second part,

$$
\begin{aligned}
&E\left[\frac{\partial g_1}{\partial s_0}\cdot 1_{\{g_1\geq -y\}}\right]\\
&=\varepsilon e^{\mu T}E\left[\int_0^T\sigma'\left(A_{0s}\right)dw_s\cdot 1_{\{g_1\geq -y\}}\right]\\
&=\varepsilon e^{\mu T}\int_{-\infty}^{\infty}E\left[\int_0^T\sigma'\left(A_{0s}\right)dw_s\,|g_1=x\right]\ 1_{\{g_1\geq -y\}}dx &(48)\\
&=\varepsilon e^{\mu T}\int_{-y}^{\infty}\left\{\int_0^T e^{\mu(T-s)}\sigma'\left(A_{0s}\right)\sigma\left(A_{0s}\right)ds\right\}\frac{x}{\Sigma}n\left[x;0,\Sigma\right]dx &(49)\\
&=\varepsilon e^{2\mu T}\left\{\int_0^T e^{\mu(T-s)}\sigma'\left(A_{0s}\right)\sigma\left(A_{0s}\right)ds\right\}n\left[y;0,\Sigma\right]. &(50)
\end{aligned}
$$

For the forth part

$$
\begin{aligned}
&\varepsilon^2 E\left[d\cdot g_2\cdot\delta_{-y}(g_1)\right]\\
&=\varepsilon^2 d\int_{-\infty}^{\infty}E\left[g_2\,|g_1=x\right]\delta_{-y}(x)dx &(51)\\
&=\varepsilon^2\left(d\,c\,y^2+f\right)n\left[y;0,\Sigma\right]. &(52)
\end{aligned}
$$

For the third part, considering $\frac{\partial g_2}{\partial s_0} = e^{\mu T} \int_0^T \sigma'' (A_{0s}) A_{1s} dw_s + e^{\mu T} \int_0^T \sigma' (A_{0s}) \int_0^s \sigma' (A_{0u}) dw_u dw_s$, we modify as following:

$$\begin{aligned}
&\varepsilon^2 E \left[e^{\mu T} \int_0^T \sigma'' (A_{0s}) A_{1s} dw_s \cdot 1_{\{g_1 \geq -y\}} \right] \\
&\quad = \varepsilon^2 e^{\mu T} \int_{-\infty}^{\infty} E \left[\int_0^T \sigma'' (A_{0s}) A_{1s} dw_s \,| g_1 = x \right] 1_{\{g_1 \geq -y\}} dx \qquad (53) \\
&\quad = \varepsilon^2 e^{\mu T} \int_{-y}^{\infty} \left\{ \int_0^T e^{\mu(T-s)} \sigma (A_{0s}) \sigma'' (A_{0s}) \int_0^s e^{\mu(s-u)} \sigma (A_{0u}) e^{\mu(T-u)} \sigma (A_{0u}) duds \right\} \\
&\qquad\qquad \left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma} \right) n [x; 0, \Sigma] \, dx \qquad (54) \\
&\quad = -\varepsilon^2 e^{3\mu T} \left\{ \int_0^T \sigma (A_{0s}) \sigma'' (A_{0s}) \int_0^s e^{-2\mu u} (\sigma (A_{0u}))^2 duds \right\} \frac{y}{\Sigma} n [y; 0, \Sigma] . \qquad (55)
\end{aligned}$$

And also,

$$\begin{aligned}
&\varepsilon^2 E \left[e^{\mu T} \int_o^T \sigma' (A_{0s}) \int_o^s \sigma' (A_{0u}) dw_u dw_s \cdot 1_{\{g_1 \geq -y\}} \right] \\
&\quad = \varepsilon^2 \int_{-\infty}^{\infty} E \left[e^{\mu T} \int_o^T \sigma' (A_{0s}) \int_o^s \sigma' (A_{0u}) dw_u dw_s \,| g_1 = x \right] 1_{\{g_1 \geq -y\}} dx \qquad (56) \\
&\quad = \varepsilon^2 \int_{-y}^{\infty} e^{\mu T} \left\{ \int_0^T \sigma' (A_{0s}) e^{\mu(T-s)} \sigma (A_{0s}) \int_0^s \sigma' (A_{0u}) e^{\mu(T-u)} \sigma (A_{0u}) duds \right\} \qquad (57) \\
&\qquad\qquad \left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma} \right) n [x; 0, \Sigma] \, dx \\
&\quad = -\varepsilon^2 e^{3\mu T} \left\{ \int_0^T e^{-\mu s} \sigma' (A_{0s}) \sigma (A_{0s}) \int_0^s e^{-\mu u} \sigma' (A_{0u}) \sigma (A_{0u}) duds \right\} \frac{y}{\Sigma} n [y; 0, \Sigma] . \qquad (58)
\end{aligned}$$

For the fifth part,

$$\begin{aligned}
&\varepsilon^2 E \left[\frac{\partial g_1}{\partial S_0} \cdot g_2 \cdot \delta_{-y}(g_1) \right] \\
&\quad = \varepsilon^2 e^{2\mu T} \int_{-y}^{\infty} \xi (x) \, n [x; 0, \Sigma] \, \delta_{-y}(x) dx \qquad (59) \\
&\quad = \varepsilon^2 e^{2\mu T} \{ \xi (-y) \} \, n [y; 0, \Sigma] , \qquad (60)
\end{aligned}$$

where

$$
\begin{aligned}
&\xi\left(x\right)\\
&=\left\{\int_{o}^{T}\sigma'\left(A_{ot}\right)e^{\mu T}e^{-\mu t}\sigma\left(A_{0t}\right)\int_{o}^{t}e^{\mu T}e^{-2\mu s}\left(\sigma\left(A_{0s}\right)\right)^{2}dsdt\right\}\\
&\times\left\{\int_{0}^{T}e^{\mu T}e^{-\mu u}\sigma'\left(A_{0u}\right)\sigma\left(A_{0u}\right)du\right\}\\
&\times\left(\frac{x^{3}}{\Sigma^{3}}-\frac{3x}{\Sigma^{2}}\right)\\
&+\left\{\int_{0}^{T}e^{\mu T}e^{-\mu t}\sigma\left(A_{0t}\right)\sigma'\left(A_{0t}\right)\int_{0}^{t}e^{-\mu s}\sigma\left(A_{0s}\right)\sigma'\left(A_{0s}\right)dsdt\right\}\frac{x}{\Sigma}\\
&+\left\{\int_{o}^{T}\left(\sigma'\left(A_{0t}\right)\right)^{2}\int_{o}^{t}e^{\mu T}e^{-2\mu s}\sigma\left(A_{0s}\right)^{2}dsdt\right\}\frac{x}{\Sigma}.
\end{aligned}
\tag{61}
$$

Therefore, as for coefficient of $y\cdot n\left[y;0,\Sigma\right]$, we have:

$$
\begin{aligned}
&\varepsilon^{2}\Big\{-\frac{1}{\Sigma}e^{3\mu T}\left[\int_{0}^{T}e^{-\mu s}\sigma'\left(A_{0s}\right)\sigma\left(A_{0s}\right)\int_{0}^{s}e^{-\mu u}\sigma'\left(A_{0u}\right)\sigma\left(A_{0u}\right)duds\right]\\
&-\frac{1}{\Sigma}e^{3\mu T}\left[\int_{0}^{T}\left(\sigma'\left(A_{0s}\right)\right)^{2}\int_{0}^{s}e^{-2\mu u}\left(\sigma\left(A_{0u}\right)\right)^{2}duds\right]\\
&\frac{3}{\Sigma^{2}}e^{5\mu T}\left\{\int_{0}^{T}e^{-\mu s}\sigma'\left(A_{0s}\right)\sigma\left(A_{0s}\right)ds\right\}\\
&\times\left\{\int_{0}^{T}e^{-\mu s}\sigma'\left(A_{0s}\right)\sigma\left(A_{0s}\right)\int_{0}^{s}e^{-2\mu u}\left(\sigma\left(A_{0u}\right)\right)^{2}duds\right\}\Big\}.
\end{aligned}
\tag{62}
$$

And coefficient of $y^{3}\cdot n\left[y;0,\Sigma\right]$ is:

$$
\begin{aligned}
&\varepsilon^{2}\Big\{-\frac{1}{\Sigma^{3}}e^{5\mu T}\left\{\int_{0}^{T}e^{-\mu s}\sigma'\left(A_{0s}\right)\sigma\left(A_{0s}\right)ds\right\}\\
&\quad\times\left\{\int_{0}^{T}e^{-\mu s}\sigma'\left(A_{0s}\right)\sigma\left(A_{0s}\right)\int_{0}^{s}e^{-2\mu u}\left(\sigma\left(A_{0u}\right)\right)^{2}duds\right\}\Big\}.
\end{aligned}
\tag{63}
$$

Finally, we reach the next theorem.

Theorem 2. *Delta of European Call Option $D(0,T)$ is represented by*

$$
\begin{aligned}
D(0,T) &= e^{-rT}\left\{\varepsilon^{2}(A+By+Cy^{2}+Dy^{3})n[y;0,\Sigma]\right.\\
&+ \left.\varepsilon(d\cdot N(\frac{y}{\sqrt{\Sigma}})+E\cdot n[y,0,\Sigma])\right\}+O\left(\varepsilon^{3}\right),
\end{aligned}
\tag{64}
$$

where

$$
\begin{aligned}
&A,B,C,D\,\text{and}\,E\,\text{are constant},\\
y &= \tfrac{S_{T}^{(0)}-K}{\varepsilon},\\
\Sigma &= E\left[\left(\textstyle\int_{0}^{T}e^{\mu\,(T-t)}\,\sigma\left(A_{0t}\right)\,dw_{t}\right)^{2}\right]\\
&= \textstyle\int_{0}^{T}e^{2\mu\,(T-t)}\,\sigma\left(A_{0t}\right)^{2}\,dt,\\
c &= \tfrac{1}{\Sigma^{2}}\textstyle\int_{0}^{T}\int_{0}^{s}e^{\mu\,(T-s)}\,\sigma\left(A_{0s}\right)\,\sigma'\left(A_{0s}\right)\,e^{2\mu\,(T-v)}\,\sigma\left(A_{0v}\right)^{2}\,dv\,ds,\\
f &= -c\Sigma.
\end{aligned}
$$

Proof. See the calculation which is already shown above. ■

Remark 1. It is easy to check that the following equation is valid with some mathematical condition,

$$\frac{\partial}{\partial s_0}\varepsilon\mathbf{E}[(X_T^{(\varepsilon)}+y)_+] = \varepsilon\mathbf{E}\left[(\frac{\partial X_T^{(\varepsilon)}}{\partial s_0}+\frac{\partial y}{\partial s_0})1_{\{X_T^{(\varepsilon)}\geq -y\}}\right]. \tag{65}$$

The discussion above is to calculate the right hand side of the equation (65). Of course, in order to obtain the same result, we can compute the derivative with respect to the initial value, s_0, directly using the equation (12) (to calculate the left hand side of the equation (65)). For the readers' convenience, we show the result of this calculation in Appendix.

However, the former way is more suitable for applying the variance reduction method which we propose in this paper than the latter, because we could obtain the equation (68) more intuitive way.

4.4 Delta of average European call option

Similar argument can be applied for European average option. Let $\widetilde{S}_t^{(\varepsilon)}$ and $\widetilde{A}_{\cdot t}$ be

$$\begin{aligned}\widetilde{S}_t^{(\varepsilon)} &= \frac{1}{t}\int_0^t S_u^{(\varepsilon)}\,du, &(66)\\ \widetilde{A}_{it} &= \frac{1}{t}\int_0^t A_{iu}\,du \qquad i=0,1,2,\cdots \quad . &(67)\end{aligned}$$

We have

$$\begin{aligned}\tilde{A}_{0T} &= \frac{1}{T}\int_0^T s_0 e^{\mu t}dt = s_0\frac{1}{T}\left(\frac{e^{\mu T}-1}{\mu}\right),\\ \tilde{A}_{1T} &= \frac{1}{T}\int_0^T\int_0^t e^{\mu(t-s)}\sigma\left(A_{0s}\right)dw_s dt\\ &= \frac{1}{T}\int_0^T e^{-\mu s}\sigma\left(A_{0s}\right)\int_s^T e^{\mu t}dt dw_s\\ &= \frac{1}{T}\int_0^T \sigma\left(A_{0s}\right)\left(\frac{e^{\mu(T-s)}-1}{\mu}\right)dw_s,\\ \tilde{A}_{2T} &= \frac{2}{T}\int_0^T\int_0^t e^{\mu(t-s)}\sigma'\left(A_{0s}\right)A_{1s}dw_s dt,\end{aligned}$$

Then, we obtain

$$\begin{aligned}\tilde{A}_{0T} &= \frac{1}{T}\int_0^T s_0 e^{\mu t}dt = s_0\frac{1}{T}\left(\frac{e^{\mu T}-1}{\mu}\right),\\ \tilde{A}_{1T} &= \frac{1}{T}\int_0^T\int_0^t e^{\mu(t-s)}\sigma\left(A_{0s}\right)dw_s dt\\ &= \frac{1}{T}\int_0^T e^{-\mu s}\sigma\left(A_{0s}\right)\int_s^T e^{\mu t}dt dw_s\\ &= \frac{1}{T}\int_0^T \sigma\left(A_{0s}\right)\left(\frac{e^{\mu(T-s)}-1}{\mu}\right)dw_s,\\ \tilde{A}_{2T} &= \frac{2}{T}\int_0^T\int_0^t e^{\mu(t-s)}\sigma'\left(A_{0s}\right)A_{1s}dw_s dt.\end{aligned}$$

We only need to apply the same discussion plugging above terms into the corresponding locations.

4.5 Variance reduction technique for Delta

We utilize equation (23). In order to construct the function $\hat{f}(x)$, we only need to consult equation (45), (48), (51), (53), (56) and (59). Also, referring the equation (46), (49), (54) and (57), we obtain the following formulae:

$$\hat{f}(x) = \varepsilon\{(d + d_1 x) + \varepsilon(d_2 + d_3 x^2)\}1_{\{x \geq -y\}} + \varepsilon^2 d_4, \tag{68}$$

where d,d_1, d_2, d_3 and d_4 are constants. And, we can apply the analytic formulae of the asymptotic expansion which is given by theorem 2 for computing $E\left[\hat{f}\left(X_T^{(0)}(0,y)\right)\right]$.

5 Vega

Although this method is quite similar to the case of Delta, we need some adjustments. We calculate the "differentiation with respect to the volatility coefficient ε" of $S_t^{(\varepsilon)}$ satisfying the following SDE:

$$\begin{cases} dS_t^{(\varepsilon)} = \mu\, S_t^{(\varepsilon)}\, dt + \varepsilon\sigma\left(S_t^{(\varepsilon)}\right)\, dw_t \\ S_0^{(\varepsilon)} = s_0\ (> 0) \end{cases} . \tag{69}$$

In this paper, we define the Vega for European call option as:

$$Vega \equiv \frac{\partial}{\partial\varepsilon} E[e^{-rT}(S_T^{(\varepsilon)} - K)^+] \,. \tag{70}$$

We also define $V_t^{(\varepsilon)}$ as a stochastic process which satisfies the SDE:

$$\begin{cases} dV_t^{(\varepsilon)} = \mu V_t^{(\varepsilon)}\, dt + \left[\sigma\left(S_t^{(\varepsilon)}\right) + \varepsilon\sigma'\left(S_t^{(\varepsilon)}\right) V_t^{(\varepsilon)}\right] dw_t \\ V_0^{(\varepsilon)} = 0 \end{cases} \tag{71}$$

Then the value of Vega for European call option is obtained by $E\left[e^{-rT}\, V_T^{(\varepsilon)}\, 1_{\{S_T \geq K\}}\right]$, but it is sometimes impossible to calculate it analytically. To apply asymptotic expansion of Vega gives us a sufficiently useful calculation method. We can apply hybrid method as well.

5.1 Vega of plain vanilla European call option

Introducing $y = (A_{0T} - K)/\varepsilon$, we obtain Vega as

$$\begin{aligned} & E\left[e^{-\mu T}\, V_T\, 1_{\{s_T \geq K\}}\right] \\ = \ & e^{-\mu T}\, E\left[g_1\, 1_{\{g_1 \geq -y\}}\right] + \varepsilon\, e^{-\mu T}\, E\left[2g_2\, 1_{\{g_1 \geq -y\}}\right] \\ & + \varepsilon\, e^{-\mu T}\, E\left[g_1\, g_2\, \delta_{-y}(g_1)\right] + O\left(\varepsilon^2\right). \end{aligned}$$

The approximated value of Vega $V_{ega,E}^{(\varepsilon)}(0,T)$ is expressed as

$$\begin{aligned} & V_{ega,E}^{(\varepsilon)}(0,T) \\ = \ & e^{-rT} E\left[(g_1 + 2\varepsilon\, g_2)\ 1_{\{g_1 \geq -y\}}\right] + \varepsilon e^{-rT} E\left[g_1\, g_2\, \delta_{-y}(g_1)\right] + O(\varepsilon^2) \\ = \ & e^{-rT} E\left[(g_1 + 2\varepsilon\, E\left[g_2|g_1\right])\ 1_{\{g_1 \geq -y\}}\right] + \varepsilon e^{-rT} E\left[g_1\, E\left[g_2|g_1\right]\ \delta_{-y}(g_1)\right] + O(\varepsilon^2) \\ = \ & e^{-rT} E\left[\left(g_1 + 2\varepsilon\ \left(c\, g_1^2 + f\right)\right)\ 1_{\{g_1 \geq -y\}}\right] + \varepsilon e^{-rT} E\left[g_1\ \left(c\, g_1^2 + f\right)\ \delta_{-y}(g_1)\right] + O(\varepsilon^2) \qquad (72) \\ = \ & e^{-rT}\left(\Sigma + \varepsilon\left(fy + \frac{fy^3}{\Sigma}\right)\right) n[y;0,\Sigma] + O(\varepsilon^2). \qquad (73) \end{aligned}$$

We obtain the next theorem.

Theorem 3. *The asymptotic expansion of the Vega of plain vanilla European call option with maturity T is represented by*

$$V_{ega,E}^{(\varepsilon)} = e^{-rT}\left(\Sigma + \varepsilon\left(f\,y + \frac{f\,y^3}{\Sigma}\right)\right)\,n[y;0,\Sigma] + O(\varepsilon^2). \tag{74}$$

where

$$y = \frac{s_0 e^{\mu T} - K}{\varepsilon},$$

$$\begin{aligned}\Sigma &= E\left[\left(\int_0^T e^{\mu\,(T-t)}\,\sigma\left(A_{0t}\right)\,dw_t\right)^2\right]\\ &= \int_0^T e^{2\mu\,(T-t)}\,\sigma\left(A_{0t}\right)^2\,dt,\end{aligned}$$

$$c = \frac{1}{\Sigma^2}\int_0^T\int_0^s e^{\mu\,(T-s)}\,\sigma\left(A_{0s}\right)\,\sigma'\left(A_{0s}\right)\,e^{2\mu\,(T-v)}\,\sigma\left(A_{0v}\right)^2\,dv\,ds,$$

and

$$f = -c\Sigma.$$

Proof. See the discussion above. ■

Remark 2. Of course, in order to obtain the same result, we can compute the derivative with respect to the parameter, ε, directly using the equation (12). For the readers' convenience, we show the result of this calculation in Appendix. However, concerning with the variance reduction method, we can obtain the equation (75) more intuitively.

5.2 Variance reduction technique for Vega

We utilize equation (23). In order to construct the function $\hat{f}(x)$, we only need to consult equation (72). Then, we obtain the following formulae:

$$\hat{f}(x) = e^{-\mu T}\,\left(x + 2\varepsilon\,\left(c\,x^2 + f\right)\right)\,1_{\{x\geq -y\}} - e^{-\mu\,T}\,y\,\left(c\,y^2 + f\right)\,n\left[y;0,\Sigma\right], \tag{75}$$

And, we can apply the analytic formulae of the asymptotic expansion which is given by theorem 3 for computing $E\left[\hat{f}\left(X_T^{(0)}\left(0,y\right)\right)\right]$.

Concerning with average European call option's case, we can follow almost the same calculation as plain vanilla European call option's case.

6 Numerical experiments

In this section, we assume that the price processes of the underlying assets follow BS or CEV processes. BS process is represented by

$$\begin{cases} dS_t^{(\varepsilon)} = \mu\,S_t^{(\varepsilon)}\,dt + \varepsilon\sigma S_t^{(\varepsilon)}\,dw_t \\ S_0^{(\varepsilon)} = s_0\ (>0)\end{cases}, \tag{76}$$

CEV process is represented by

$$\begin{cases} dS_t^{(\varepsilon)} = \mu\,S_t^{(\varepsilon)}\,dt + \varepsilon\sigma\left(S_t^{(\varepsilon)}\right)^{\gamma}\,dw_t \quad ,0<\gamma<1 \\ S_0^{(\varepsilon)} = s_0\ (>0)\end{cases},$$

Hence, the volatility function of these models are expressed by $\sigma(S_t) = S_t^{\gamma}\ (0 < \gamma \le 1)$. Then, $Y_t = \frac{\partial S_t}{\partial s_0}$ follows the SDE:

$$\begin{cases} dY_t = \mu\, Y_t\, dt + \varepsilon\, \gamma\, S_t^{\gamma-1}\, Y_t\ dw_t & \text{on } \{0 \le \mathrm{t} < \tau\}, \\ Y_0 = 1 & \\ Y_t \equiv 0 & \text{on } \{\tau \le \mathrm{t}\}, \end{cases} \tag{77}$$

where $\tau \equiv \inf\{t > 0;\ S_t = 0\}$.

6.1 Model of plain vanilla European call option

The asymptotic expansion of the Delta and Vega of plain vanilla European call option, $D_E^{(\varepsilon)}(0,T)$ and $V_{ega,E}^{(\varepsilon)}(0,T)$ respectively, whose payoff function at maturity T are expressed by $(S_T - K)^+$ are given by

$$\begin{aligned} D_E^{(\varepsilon)}(0,T) &= \varepsilon\, e^{-rT}\left(d\, N\left(\frac{y}{\sqrt{\Sigma}}\right) + \frac{\gamma\,\Sigma}{s_0} n\left[y;0,\Sigma\right]\right) \\ &\quad + \varepsilon^2\, e^{-rT}\ \left(\frac{(2\gamma-1)}{s_0} f\, y + \left(c\, y^2 + f\right)\left(d - \frac{\gamma}{s_0} y\right)\right) n\left[y;0,\Sigma\right] + O(\varepsilon^3), \\ V_{ega,E}^{(\varepsilon)}(0,T) &= e^{-rT}\left(\Sigma + \varepsilon\left(f\, y + \frac{f\, y^3}{\Sigma}\right)\right) n\left[y;0,\Sigma\right] + O(\varepsilon^2), \end{aligned}$$

where

$$y = \frac{s_0 e^{\mu T} - K}{\varepsilon},$$

$$d = \frac{1}{\varepsilon} e^{\mu T},$$

$$\begin{aligned} \Sigma &= E\left[\left(\int_0^T e^{\mu\,(T-t)}\,\sigma\left(A_{0t}\right)\, dw_t\right)^2\right] \\ &= \int_0^T e^{2\mu\,(T-t)}\,\sigma\left(A_{0t}\right)^2\ dt \\ &= \begin{cases} \frac{s_0^{2\gamma}}{2\mu\,(1-\gamma)}\left(e^{2\mu T} - e^{2\mu\gamma T}\right) & \text{if } 0 < \gamma < 1 \\ s_0^2\, T\, e^{2\mu T} & \text{if } \gamma = 1, \end{cases} \end{aligned}$$

$$\begin{aligned} c &= \frac{1}{\Sigma^2}\int_0^T\int_0^s e^{\mu\,(T-s)}\,\sigma\left(A_{0s}\right)\ \sigma'\left(A_{0s}\right)\ e^{2\mu\,(T-v)}\,\sigma\left(A_{0v}\right)^2\ dv\, ds \\ &= \begin{cases} \frac{\gamma\, s_0^{4\gamma-1}}{\Sigma^2}\, e^{3\mu T}\, \frac{\left(1-e^{2\mu(1-\gamma)T}\right)^2}{8\,\mu^2\,(1-\gamma)^2} & \text{if } 0 < \gamma < 1 \\ \frac{s_0^3}{\Sigma^2}\, \frac{T^2}{2}\, e^{3\mu T} & \text{if } \gamma = 1, \end{cases} \end{aligned}$$

and

$$f = -c\Sigma.$$

The coefficients Σ, c, and f are the function of s_0, T, γ and μ. Paying attention to γ, we represent $\Sigma = \Sigma(\gamma)\,,\ c = c(\gamma)$. We can easily check the continuity at $\gamma = 1$, i.e. $\Sigma(\gamma) \to \Sigma(1)$ and $c(\gamma) \to c(1)$ as $\gamma \to 1$.

For the variance reduction technique of Delta, we replace equation (23) by

$$\hat{f}(x) = \varepsilon\, e^{-rT} \left(d + \frac{\gamma}{s_0} x + \frac{2\,\gamma - 1}{s_0} \varepsilon \, \left(c\, x^2 + f\right)\right) 1_{\{x \geq -y\}} + \varepsilon^2\, e^{-rT} \left(d - \frac{\gamma}{s_0} y\right) \left(c\, y^2 + f\right)\, n\left[y; 0, \Sigma\right].$$

For the variance reduction technique of Vega, we replace equation (23) by

$$\hat{f}(x) = e^{-\mu T} \left(x + 2\varepsilon \, \left(c\, x^2 + f\right)\right) 1_{\{x \geq -y\}} - e^{-\mu T}\, y\, \left(c\, y^2 + f\right)\, n\left[y; 0, \Sigma\right].$$

6.2 Model of average European call option

The asymptotic expansion of the Delta and Vega of average European call option, $D_A^{(\varepsilon)}(0,T)$ and $V_{ega,A}^{(\varepsilon)}(0,T)$ respectively, whose payoff function at maturity T are expressed by $\left(\tilde{S}_T - K\right)^+$ are given by

$$\begin{aligned} D_A^{(\varepsilon)}(0,T) &= \varepsilon\, e^{-rT} \left(d\, N\left(\frac{y}{\sqrt{\Sigma}}\right) + \frac{\gamma\,\Sigma}{s_0} n\left[y;0,\Sigma\right]\right) \\ &\quad + \varepsilon^2\, e^{-rT} \left(\frac{(2\gamma-1)}{s_0} f\, y + \left(c\, y^2 + f\right)\left(d - \frac{\gamma}{s_0} y\right)\right) n\left[y;0,\Sigma\right] + O(\varepsilon^3), \\ V_{ega,E}^{(\varepsilon)}(0,T) &= e^{-rT} \left(\Sigma + \varepsilon \left(f\, y + \frac{f\, y^3}{\Sigma}\right)\right) n\left[y;0,\Sigma\right] + O(\varepsilon^2), \end{aligned} \tag{78}$$

where

$$y = \frac{s_0 \left(e^{\mu T} - 1\right)/\mu T - K}{\varepsilon},$$

$$d = \frac{e^{\mu T} - 1}{\varepsilon \mu T},$$

$$\begin{aligned} \Sigma &= \int_0^T \left(\frac{1}{T}\, \frac{e^{\mu\,(T-t)} - 1}{\mu}\, \sigma\left(A_{0t}\right)\right)^2 dt \\ &= \begin{cases} \frac{s_0^{2\gamma}}{\mu^2\, T^2} \left\{\frac{e^{2\mu T}}{2\mu(1-\gamma)} - \frac{2e^{\mu T}}{\mu(1-2\gamma)} - \frac{1}{2\mu T} + \frac{e^{\mu\gamma T}}{2\mu\gamma(1-\gamma)(1-2\gamma)}\right\} & \text{if } 0 < \gamma < 1,\ \gamma \neq \frac{1}{2} \\ \frac{s_0}{\mu^2 T^2} \left(\frac{e^{2\mu T}}{\mu} - 2T\, e^{\mu T} - \frac{1}{\mu}\right) & \text{if } \gamma = \frac{1}{2} \\ \left(\frac{s_0}{\mu\, T}\right)^2 \left\{e^{2\mu T}\left(T - \frac{3}{2\mu}\right) + \frac{2e^{\mu T}}{\mu} - \frac{1}{2\mu}\right\} & \text{if } \gamma = 1, \end{cases} \end{aligned}$$

$$
\begin{aligned}
c &= \frac{1}{\Sigma^2 T^3} \int_0^T \int_0^t e^{\mu(t-s)} \left[\frac{e^{\mu(T-s)-1}}{\mu}\right] \sigma(A_{0s})\, \sigma'(A_{0s}) \\
&\quad \times \int_0^s e^{\mu(s-v)} \left[\frac{e^{\mu(T-v)}-1}{\mu}\right] \sigma(A_{0v})^2 \, dv\, ds\, dt \\
&= \begin{cases}
\frac{\gamma s_0^{4\gamma-1}}{\Sigma^2 \mu^5 T^3} \begin{pmatrix} \left(\left(-1+e^{\mu T}\right)\left(2-2\gamma+e^{\mu T}(-1+2\gamma)\right)\right)^2 \\ +(e^{\mu(4\gamma-1)T}\gamma(-7+10\gamma) \\ +4(-1+\gamma)^2\left(3-13+12\gamma^2\right) \\ +e^{\mu(1+2\gamma)T}\left(6-44\gamma+96\gamma^2-64\gamma^3\right) \\ +4e^{2\mu\gamma T}\left(-3+19\gamma-32\gamma^2+16\gamma^3\right) \\ +e^{2\mu T}\gamma\left(-5+36-76\gamma^2+48\gamma^3\right) \\ -2e^{\mu T}\left(3-28\gamma+89\gamma^2-112\gamma^3+48\gamma^4\right)) \\ /\gamma\left(3-16\gamma+16\gamma^2\right))/8\left(1-3\gamma+2\gamma^2\right)^2 \end{pmatrix} & \text{if } 0<\gamma<1,\ \gamma\neq\frac{1}{2} \\
\frac{s_0}{2\Sigma^2\mu^2 T^3}\left\{\frac{e^{3\mu T}}{2\mu^3}+e^{2\mu T}\left(\frac{1}{\mu^3}-\frac{2T}{\mu^2}\right)+e^{\mu T}\left(-\frac{1}{2\mu^3}-\frac{T}{\mu^2}+\frac{T^2}{\mu}\right)-\frac{1}{\mu^3}\right\} & \text{if } \gamma=\frac{1}{2} \\
\frac{s_0^3}{2\mu^2\mu^5 T^3}\left\{e^{3\mu T}\left(\mu^2T^2-3\mu T+\frac{17}{6}\right)+e^{2\mu T}(2\mu T-5)+\frac{5}{2}e^{\mu T}-\frac{1}{3}\right\} & \text{if } \gamma=1,
\end{cases}
\end{aligned}
$$

and

$$f = -c\Sigma.$$

As in the plain vanilla case, the coefficients Σ, c, and f are the function of s_0, T, γ and μ. We can easily check the continuity at $\gamma = 1$, i.e. $\Sigma(\gamma) \to \Sigma(1)$ and $c(\gamma) \to c(1)$ as $\gamma \to 1$.

For the variance reduction technique of Delta, we replace equation (23) by

$$\hat{f}(x) = \varepsilon\, e^{-rT} \left(d + \frac{\gamma}{s_0}x + \frac{2\gamma-1}{s_0}\varepsilon\, \left(c\, x^2 + f\right)\right) 1_{\{x\geq -y\}} + \varepsilon^2\, e^{-rT} \left(d - \frac{\gamma}{s_0}y\right) \left(c\, y^2 + f\right)\, n\left[y; 0, \Sigma\right].$$

For the variance reduction technique of Vega, we replace equation (23) by

$$\hat{f}(x) = e^{-\mu T}\, \left(x + 2\varepsilon\, \left(c\, x^2 + f\right)\right)\, 1_{\{x\geq -y\}} - e^{-\mu T}\, y\, \left(c\, y^2 + f\right)\, n\left[y; 0, \Sigma\right].$$

6.3 Parameter settings

In the numerical simulation, we suppose two kind of options. The first are interest rate options (six month caplets) under Libor Market Model (LMM) settings. This is an example of plain vanilla European call options. We check the accuracy of the approximate formulas in $\gamma = 1$ (BS process) cases comparing with the result of LMM and the efficiency in $\gamma = 0.1, 0.4, 0.6$ and 0.9 cases. When $\gamma \neq 1$, we usually do not know the exact value of Deltas nor Vegas. Therefore, we use the value of the numerical simulation with 1,000,000 paths as a benchmark. In this case, we can apply zero as 'drift term,' μ. At the same time, we need to check the validity of the computation when the maturities are rather long in the practical sense. Therefore, for interest rate options, we put 0.05 as s_0, zero as μ, and 1,5,7,10 year(s) as T.

The second are average (stock) options. While we need the 'non-zero' drift, the maturities are at most one year practically. So, for average European call options, we put 100 as s_0, 0.05 as μ, and 1 year as T. We check the accuracy and efficiency with $\gamma = 0.1, 0.4, 0.6, 0.9$ and 1.

For both case, we put 20% as $\varepsilon\sigma s_0^\gamma$. And strike price is set from 20% in the money(ITM) to 40% out of the money(OTM).

6.4 Computing results

Table 1 compares the Deltas of plain vanilla European call options between by LMM and our approximation with $\gamma = 1$. Colum (A), (B) and (C) indicate Delta given by LMM, by asymptotic expansion formulae and the error defined by $((B) - (A))/(A)$ respectively. Around the at the money(ATM), the accuracy of the approximation is very good. Even in $T = 10$ case, the error is below 1%. We can think approximations from 20% ITM to 20% OTM are acceptable for almost all purposes. However, we can find that we may not satisfy the accuracy of the approximations especially in very far ITM or OTM cases.

Table 2 compares the Vegas of plain vanilla European call options between by LMM and our approximation with $\gamma = 1$. Colum (A), (B) and (C) indicate Vega given by LMM, by asymptotic expansion formulae and the error defined by $((B) - (A))/(A)$ respectively. Even around the ATM, the accuracy of the approximation may not be satisfactory in the longer maturity cases. However, We also notice that the level of both Vega and the difference between colum (A) and (B) are very small. Considering this fact, we can think that the approximation works well.

On table 3 and 4, we check the accuracy and efficiency of the hybrid method. Table 3 indicates the result of the Deltas of plain vanilla European call options. Table 4 indicates the result of the Vegas of plain vanilla European call options. Colum (A), (B) and (C) indicate Delta (Vega) given by LMM, by asymptotic expansion formulae and the error defined by $((B) - (A))/(A)$ respectively. Colum (D) and (G) shows the result simulation by crude Monte Carlo method and hybrid method respectively. These figures are given by 1,000,000 paths simulation. Colum (E) and (H) are the error of (D) and (G) defined by $((D) - (A)/(A))$ and $((G) - (A)/(A))$ respectively. Colum (F) and (I) indicates the variance of the simulations. For each method, we compute Delta (Vega) with 10,000 paths and repeat them 100 times. Then, we have distributions of Delta (Vega) which consist of 100 values. From the distributions, we compute variance and then calculate the value of $\text{Var}/(\text{A})^2$.

As for table 3, colum (C) and (E) indicate the crude Monte Carlo method with 1,000,000 paths gives us better accuracy than asymptotic expansion. Colum (E) and (H) indicate hybrid method with 1,000,000 paths gives better accuracy than crude Monte Carlo method. The value of (F) and (I) indicates that even in 10,000 paths simulation, the accuracy is satisfactory in all cases, especially in hybrid method. The value (F)/(I) indicates the ratio of the speed of convergence. The larger value implies that hybrid method achieves the faster convergence. We find that hybrid method is at least five or six times faster than crude monte Carlo method and at most 74 times faster.

As for table 4, the tendency is very similar to table 3. We find that hybrid method is at least five times faster than crude monte Carlo method and at most 335 times faster.

On table 5 and 6, we check the efficiency of the hybrid method in $\gamma = 0.1, 0.4, 0.6$ and 0.9 cases. Colum (A) and (C) on table 5 and 6 corresponds to colum (D) and (F) on table 3 and 4 respectively. Only the difference is that colum (C) uses value of colum (A) as the mean value on table 5 and 6, while Colum (F) uses value of colum (A) on table 3 and 4. Similarly, Colum (D) and (F) on table 5 and 6 corresponds to colum (G) and (I) on table 3 and 4 respectively. Colum (B) and (E) on table 5 and 6 is the raw data of variance made by 100 values.

As for the table 5, we can find the asymptotic expansion works well as in $\gamma = 1$ case. We also notice that the accuracy of the asymptotic expansion is the better in the shorter maturity cases. Moreover, the accuracy is the better in the asymptotic expansion case, the ratio of the speed of convergence is the larger. In other words, the better approximation formulae gives the better reduction of the computing resources in the hybrid method. Hybrid method is at least three times faster than crude monte Carlo method and at most 976 times faster.

The tendency on table 6 is very similar to table 5. We find that acceleration ratio by hybrid method is larger than in the case of table 5.

On table 7 and 8, we check the accuracy of our approximation and both the accuracy and efficiency of hybrid method in average option's case. Colum (A) indicates the result of asymptotic expansion formulae. colum (B) to (G) corresponds to colum (A) to (F) on table 5 and 6.

We find the accuracy of asymptotic expansion is well and the efficiency of the hybrid method is very well. We notice that the ratio of the acceleration is higher than plain vanilla European options cases.

A Appendix

A.1 Direct derivation of plain vanilla European call option's Delta

Consider the derivative of equation (12) with respect to s_0:

$$
\begin{aligned}
&\frac{\partial}{\partial s_0} C_E^{(\varepsilon)}(0,T) \\
=\;& \varepsilon\, e^{-rT} \left(d\, N\left(\frac{y}{\sqrt{\Sigma}}\right) + \left(y \left(d - \frac{y}{2\,\Sigma}\frac{\partial \Sigma}{\partial s_0}\right) + \frac{\partial \Sigma}{\partial s_0} + \Sigma \left(-\frac{1}{2\Sigma}\frac{\partial \Sigma}{\partial s_0} - \frac{yd}{\Sigma} + \frac{y^2}{2\Sigma^2}\frac{\partial \Sigma}{\partial s_0}\right)\right) n\,[y;0,\Sigma] \right) \\
&+\varepsilon^2\, e^{-rT} \left(\frac{\partial f}{\partial s_0}\, y + f\, d + f\, y \left(-\frac{1}{2\Sigma}\frac{\partial \Sigma}{\partial s_0} - \frac{yd}{\Sigma} + \frac{y^2}{2\Sigma^2}\frac{\partial \Sigma}{\partial s_0}\right)\right) n\,[y;0,\Sigma] + O(\varepsilon^3) \\
=\;& \varepsilon\, e^{-rT} \left(d\, N\left(\frac{y}{\sqrt{\Sigma}}\right) + \frac{1}{2}\frac{\partial \Sigma}{\partial s_0} n\,[y;0,\Sigma] \right) \\
&+\varepsilon^2\, e^{-rT} \left(\frac{\partial f}{\partial s_0}\, y + f\, d + c\, y \left(\frac{1}{2}\frac{\partial \Sigma}{\partial s_0} + yd - \frac{y^2}{2\Sigma}\frac{\partial \Sigma}{\partial s_0}\right)\right) n\,[y;0,\Sigma] + O(\varepsilon^3),
\end{aligned}
$$

where $d = \frac{\partial y}{\partial s_0} = e^{\mu T}/\varepsilon$. The derivatives of coefficients are calculated as follows:

$$
\begin{aligned}
\frac{\partial \Sigma}{\partial s_0} &= \frac{\partial}{\partial s_0}\int_0^T e^{2\mu\,(T-t)}\,\sigma\,(A_{0t})^2\; dt \\
&= 2\int_0^T e^{2\mu\,(T-t)}\, e^{\mu\, t}\,\sigma\,(A_{0t})\;\sigma'\,(A_{0t})\; dt \\
&= 2\int_0^T e^{2\mu\, T-\mu\, t}\,\sigma\,(A_{0t})\;\sigma'\,(A_{0t})\; dt
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial c}{\partial s_0} &= \frac{\partial}{\partial s_0}\left\{ \frac{1}{\Sigma^2}\int_0^T\int_0^s e^{\mu\,(T-s)}\,\sigma\,(A_{0s})\;\sigma'\,(A_{0s})\; e^{2\mu\,(T-v)}\,\sigma\,(A_{0v})^2\; dv\, ds \right\} \\
&= -\frac{2}{\Sigma^3}\frac{\partial \Sigma}{\partial s_0}\int_0^T\int_0^s e^{\mu\,(T-s)}\,\sigma\,(A_{0s})\;\sigma'\,(A_{0s})\; e^{2\mu\,(T-v)}\,\sigma\,(A_{0v})^2\; dv\, ds \\
&\quad +\frac{1}{\Sigma^2}\int_0^T\int_0^s e^{\mu\, T}\,\sigma'\,(A_{0s})^2\; e^{2\mu\,(T-v)}\,\sigma\,(A_{0v})^2\; dv\, ds \\
&\quad +\frac{1}{\Sigma^2}\int_0^T\int_0^s e^{\mu\, T}\,\sigma\,(A_{0s})\;\sigma''\,(A_{0s})\; e^{2\mu\,(T-v)}\,\sigma\,(A_{0v})^2\; dv\, ds \\
&\quad +\frac{2}{\Sigma^2}\int_0^T\int_0^s e^{\mu\,(T-s)}\,\sigma\,(A_{0s})\;\sigma'\,(A_{0s})\; e^{2\mu\, T-\mu\, v}\,\sigma\,(A_{0v})\;\sigma'\,(A_{0v})\; dv\, ds
\end{aligned}
$$

$$\frac{\partial f}{\partial s_0} = -\frac{\partial}{\partial s_0}(c\Sigma) = -\frac{\partial c}{\partial s_0}\Sigma - c\frac{\partial \Sigma}{\partial s_0}$$

A.2 Direct derivation of average option's Delta

Consider the derivative of equation (19) with respect to s_0:

$$\begin{aligned}
&\frac{\partial}{\partial s_0} C_A^{(\varepsilon)}(0,T) \\
= \ & \varepsilon\, e^{-rT} \left(d\, N\left(\frac{y}{\sqrt{\Sigma}}\right) + \left(y \left(d - \frac{y}{2\,\Sigma}\frac{\partial \Sigma}{\partial s_0}\right) + \frac{\partial \Sigma}{\partial s_0} + \Sigma \left(-\frac{1}{2\Sigma}\frac{\partial \Sigma}{\partial s_0} - \frac{yd}{\Sigma} + \frac{y^2}{2\Sigma^2}\frac{\partial \Sigma}{\partial s_0}\right)\right) n\,[y;0,\Sigma] \right) \\
& + \varepsilon^2\, e^{-rT} \left(\frac{\partial f}{\partial s_0}\, y + f\, d + f\, y \left(-\frac{1}{2\Sigma}\frac{\partial \Sigma}{\partial s_0} - \frac{yd}{\Sigma} + \frac{y^2}{2\Sigma^2}\frac{\partial \Sigma}{\partial s_0}\right)\right) n\,[y;0,\Sigma] + O(\varepsilon^3) \\
= \ & \varepsilon\, e^{-rT} \left(d\, N\left(\frac{y}{\sqrt{\Sigma}}\right) + \frac{1}{2}\frac{\partial \Sigma}{\partial s_0} n\,[y;0,\Sigma] \right) \\
& + \varepsilon^2\, e^{-rT} \left(\frac{\partial f}{\partial s_0}\, y + f\, d + c\, y \left(\frac{1}{2}\frac{\partial \Sigma}{\partial s_0} + yd - \frac{y^2}{2\Sigma}\frac{\partial \Sigma}{\partial s_0}\right)\right) n\,[y;0,\Sigma] + O(\varepsilon^3),
\end{aligned}$$

where $d = \frac{\partial y}{\partial s_0} = \left(e^{\mu T} - 1\right)/(\mu T \varepsilon)$. The derivatives of coefficients are calculated as

$$\begin{aligned}
\frac{\partial \Sigma}{\partial s_0} &= \frac{1}{\mu^2 T^2}\frac{\partial}{\partial s_0}\int_0^T \left(e^{\mu(T-s)} - 1\right)^2 \sigma\left(A_{0s}\right)^2\, ds \\
&= \frac{2}{\mu^2 T^2}\int_0^T \left(e^{\mu(T-s)} - 1\right)^2 e^{\mu\, s}\, \sigma\left(A_{0s}\right)\, \sigma'\left(A_{0s}\right)\, ds \\
&= \frac{2}{\mu^2 T^2}\int_0^T \left(e^{2\mu T - \mu s} - 2e^{\mu T} + e^{\mu\, s}\right) \sigma\left(A_{0s}\right)\, \sigma'\left(A_{0s}\right)\, ds,
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial c}{\partial s_0} &= \frac{\partial}{\partial s_0}\{\frac{1}{\Sigma^2 T^3}\int_0^T \int_0^t e^{\mu\,(t-s)}\left[\frac{e^{\mu(T-s)-1}}{\mu}\right]\sigma\left(A_{0s}\right)\,\sigma'\left(A_{0s}\right) \\
&\quad\times\int_0^s e^{\mu\,(s-v)}\left[\frac{e^{\mu(T-v)}-1}{\mu}\right]\sigma\left(A_{0v}\right)^2\,dv\,ds\,dt\} \\
&= -\frac{2}{\Sigma^3}\frac{\partial\Sigma}{\partial s_0}\frac{1}{T^3}\int_0^T \int_0^t e^{\mu\,(t-s)}\left[\frac{e^{\mu(T-s)-1}}{\mu}\right]\sigma\left(A_{0s}\right)\,\sigma'\left(A_{0s}\right) \\
&\quad\times\int_0^s e^{\mu\,(s-v)}\left[\frac{e^{\mu(T-v)}-1}{\mu}\right]\sigma\left(A_{0v}\right)^2\,dv\,ds\,dt \\
&\quad+\frac{1}{\Sigma^2 T^3}\int_0^T \int_0^t e^{\mu\,t}\left[\frac{e^{\mu(T-s)-1}}{\mu}\right]\sigma'\left(A_{0s}\right)^2 \\
&\quad\times\int_0^s e^{\mu\,(s-v)}\left[\frac{e^{\mu(T-v)}-1}{\mu}\right]\sigma\left(A_{0v}\right)^2\,dv\,ds\,dt \\
&\quad+\frac{1}{\Sigma^2 T^3}\int_0^T \int_0^t e^{\mu\,t}\left[\frac{e^{\mu(T-s)-1}}{\mu}\right]\sigma\left(A_{0s}\right)\,\sigma''\left(A_{0s}\right) \\
&\quad\times\int_0^s e^{\mu\,(s-v)}\left[\frac{e^{\mu(T-v)}-1}{\mu}\right]\sigma\left(A_{0v}\right)^2\,dv\,ds\,dt \\
&\quad+\frac{2}{\Sigma^2 T^3}\int_0^T \int_0^t e^{\mu\,(t-s)}\left[\frac{e^{\mu(T-s)-1}}{\mu}\right]\sigma\left(A_{0s}\right)\,\sigma'\left(A_{0s}\right) \\
&\quad\times\int_0^s e^{\mu\,(s-v)}\left[\frac{e^{\mu(T-v)}-1}{\mu}\right]\sigma\left(A_{0v}\right)\,\sigma'\left(A_{0v}\right)\,dv\,ds\,dt,
\end{aligned}
$$

and

$$\frac{\partial f}{\partial s_0} = -\frac{\partial}{\partial s_0}\left(c\Sigma\right) = -\frac{\partial c}{\partial s_0}\Sigma - c\frac{\partial\Sigma}{\partial s_0}.$$

A.3 Direct derivation of plain vanilla European call option's Vega

Consider the derivative of equation (19) with respect to ε:

$$
\begin{aligned}
&\frac{\partial}{\partial\varepsilon}C_E^{(\varepsilon)}\left(0,T\right) \\
&= \frac{\partial}{\partial\varepsilon}\left(\varepsilon\,e^{-rT}\left(y\,N\left(\frac{y}{\sqrt{\Sigma}}\right)+\Sigma\,n\left[y;0,\Sigma\right]\right)+\varepsilon^2\,e^{-rT}\,f\,y\,n\left[y;0,\Sigma\right]\right)+O(\varepsilon^2) \\
&= e^{-rT}\left(y\,N\left(\frac{y}{\sqrt{\Sigma}}\right)+\Sigma\,n\left[y;0,\Sigma\right]-y\,N\left(\frac{y}{\sqrt{\Sigma}}\right)-y^2\,n\left[y;0,\Sigma\right]+y^2\,n\left[y;0,\Sigma\right]\right)+O(\varepsilon^2) \\
&\quad+2\varepsilon\,e^{-rT}\,f\,y\,n\left[y;0,\Sigma\right]-\varepsilon\,e^{-rT}\,f\,y\,n\left[y;0,\Sigma\right]+\varepsilon\,\frac{f\,y^3}{\Sigma}\,n\left[y;0,\Sigma\right] \\
&= e^{-rT}\left(\Sigma+\varepsilon\,\left(f\,y+\frac{f\,y^3}{\Sigma}\right)\right)n\left[y;0,\Sigma\right]+O(\varepsilon^2).
\end{aligned}
$$

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K	T	(A) BS	(B) AE	(C) error	K	T	(A) BS	(B) AE	(C) error
0.020	10	0.961219	0.938009	-2.414656%	0.020	5	0.988472	0.972526	-1.613206%
0.025	10	0.921053	0.906544	-1.575285%	0.025	5	0.961930	0.945815	-1.675213%
0.030	10	0.869475	0.864533	-0.568426%	0.030	5	0.914007	0.902952	-1.209412%
0.035	10	0.810619	0.812861	0.276495%	0.035	5	0.846410	0.842503	-0.461523%
0.040	10	0.748268	0.753689	0.724543%	0.040	5	0.765028	0.766271	0.162493%
0.045	10	0.685387	0.690209	0.703533%	0.045	5	0.676955	0.679387	0.359328%
0.050	10	0.624085	0.626157	0.331927%	0.050	5	0.588468	0.589206	0.125390%
0.055	10	0.565736	0.565199	-0.094925%	0.055	5	0.504184	0.503321	-0.171172%
0.060	10	0.511150	0.510319	-0.162513%	0.060	5	0.426977	0.427450	0.110767%
0.065	10	0.460725	0.463336	0.566622%	0.065	5	0.358281	0.364071	1.616089%
0.070	10	0.414579	0.424674	2.435083%	0.070	5	0.298483	0.312316	4.634273%
0.075	10	0.372640	0.393425	5.577795%	0.075	5	0.247290	0.269054	8.800625%
0.080	10	0.334721	0.367667	9.842707%	0.080	5	0.204018	0.230578	13.018336%
0.085	10	0.300567	0.344960	14.769694%	0.085	5	0.167795	0.194111	15.683386%
0.090	10	0.269890	0.322893	19.638454%	0.090	5	0.137697	0.158570	15.158197%
0.020	7	0.977044	0.955619	-2.192875%	0.020	1	0.999999	1.000689	0.069018%
0.025	7	0.942314	0.925343	-1.800956%	0.025	1	0.999819	1.001021	0.120243%
0.030	7	0.890641	0.881668	-1.007524%	0.030	1	0.996024	0.995607	-0.041915%
0.035	7	0.826039	0.824727	-0.158771%	0.035	1	0.970175	0.966544	-0.374333%
0.040	7	0.753731	0.756777	0.404108%	0.040	1	0.887954	0.884899	-0.343963%
0.045	7	0.678564	0.681961	0.500633%	0.045	1	0.734606	0.734590	-0.002079%
0.050	7	0.604332	0.605550	0.201653%	0.050	1	0.539828	0.539894	0.012288%
0.055	7	0.533653	0.532809	-0.158220%	0.055	1	0.353254	0.353426	0.048725%
0.060	7	0.468126	0.467849	-0.059196%	0.060	1	0.208508	0.211889	1.621285%
0.065	7	0.408562	0.412812	1.040284%	0.065	1	0.112790	0.117643	4.302139%
0.070	7	0.355208	0.367619	3.494111%	0.070	1	0.056784	0.058384	2.818767%
0.075	7	0.307946	0.330353	7.276561%	0.075	1	0.026969	0.024395	-9.544832%
0.080	7	0.266429	0.298082	11.880355%	0.080	1	0.012224	0.008175	-33.123037%
0.085	7	0.230191	0.267838	16.354659%	0.085	1	0.005338	0.002137	-59.960186%
0.090	7	0.198710	0.237435	19.488212%	0.090	1	0.002263	0.000430	-80.979700%

Table 1: Deltas of Plain Vanilla European Call Options ($\gamma = 1$)

K	T	(A) BS	(B) AE	(C) error	K	T	(A) BS	(B) AE	(C) error
0.035	10	0.042821	0.046009	7.446638%	0.035	5	0.026481	0.027870	5.244776%
0.040	10	0.050429	0.053402	5.895150%	0.040	5	0.034355	0.035516	3.377900%
0.045	10	0.056138	0.059102	5.279271%	0.045	5	0.040140	0.041218	2.685837%
0.050	10	0.060002	0.063078	5.127110%	0.050	5	0.043502	0.044603	2.531512%
0.055	10	0.062220	0.065487	5.251220%	0.055	5	0.044601	0.045786	2.657013%
0.060	10	0.063054	0.066602	5.627723%	0.060	5	0.043854	0.045202	3.073360%
0.065	10	0.062772	0.066724	6.295099%	0.065	5	0.041758	0.043363	3.842481%
0.070	10	0.061627	0.066105	7.265933%	0.070	5	0.038784	0.040662	4.841814%
0.075	10	0.059836	0.064897	8.458397%	0.075	5	0.035323	0.037304	5.607524%
0.080	10	0.057584	0.063146	9.658748%	0.080	5	0.031676	0.033367	5.339922%
0.085	10	0.055022	0.060812	10.523134%	0.085	5	0.028056	0.028924	3.093111%
0.090	10	0.052269	0.057820	10.619138%	0.090	5	0.024605	0.024134	-1.915774%
0.035	7	0.033972	0.036032	6.065064%	0.035	1	0.003386	0.003319	-1.970762%
0.040	7	0.041702	0.043521	4.362319%	0.040	1	0.009527	0.009679	1.596742%
0.045	7	0.047396	0.049156	3.714451%	0.045	1	0.016390	0.016503	0.692609%
0.050	7	0.050960	0.052775	3.561971%	0.050	1	0.019848	0.019947	0.501252%
0.055	7	0.052587	0.054526	3.686180%	0.055	1	0.018582	0.018703	0.654121%
0.060	7	0.052607	0.054753	4.079353%	0.060	1	0.014350	0.014518	1.174357%
0.065	7	0.051383	0.053848	4.797066%	0.065	1	0.009572	0.009633	0.637254%
0.070	7	0.049260	0.052124	5.813721%	0.070	1	0.005704	0.005399	-5.343853%
0.075	7	0.046535	0.049752	6.914621%	0.075	1	0.003114	0.002465	-20.834878%
0.080	7	0.043448	0.046776	7.659233%	0.080	1	0.001587	0.000886	-44.145205%
0.085	7	0.040188	0.043175	7.433523%	0.085	1	0.000766	0.000246	-67.899358%
0.090	7	0.036891	0.038950	5.580414%	0.090	1	0.000355	0.000052	-85.282090%

Table 2: Vegas of Plain Vanilla European Call Options ($\gamma = 1$)

K	T	(A) BS	(B) AE	(C) error	(D) Crude	(E) error	(F) Var/(Mean^2)	(G) Hybrid	(H) error	(I) Var/(Mean^2)	(F)/(I)
0.03	10	0.869475	0.864533	-0.568426%	0.86877719	-0.080279%	0.00010990	0.86972539	0.028775%	0.00000213	51.635
0.04	10	0.748268	0.753689	0.724543%	0.74689552	-0.183403%	0.00017490	0.74807987	-0.025124%	0.00001052	16.622
0.05	10	0.624085	0.626157	0.331927%	0.62235008	-0.278013%	0.00017737	0.62419739	0.017990%	0.00002138	8.296
0.06	10	0.511150	0.510319	-0.162513%	0.51089100	-0.050643%	0.00036690	0.51056803	-0.113828%	0.00005721	6.413
0.07	10	0.414579	0.424674	2.435083%	0.41446397	-0.027758%	0.00041712	0.41425951	-0.077075%	0.00006836	6.102
0.08	10	0.334721	0.367667	9.842707%	0.33272196	-0.597315%	0.00066018	0.33450235	-0.065412%	0.00006310	10.463
0.09	10	0.269890	0.322893	19.638454%	0.26947796	-0.152789%	0.00098681	0.26985832	-0.011858%	0.00002661	37.086
0.04	7	0.753731	0.756777	0.404108%	0.75380784	0.010223%	0.00011434	0.75364738	-0.011066%	0.00000882	12.963
0.05	7	0.604332	0.605550	0.201653%	0.60422292	-0.017976%	0.00018079	0.60411111	-0.036477%	0.00002149	8.411
0.06	7	0.468126	0.467849	-0.059196%	0.46953214	0.300286%	0.00032621	0.46751679	-0.130228%	0.00004837	6.744
0.07	7	0.355208	0.367619	3.494111%	0.35559455	0.108827%	0.00050760	0.35496267	-0.069063%	0.00005779	8.783
0.04	5	0.765028	0.766271	0.162493%	0.76572928	0.091626%	0.00007590	0.76514766	0.015600%	0.00000541	14.031
0.05	5	0.588468	0.589206	0.125390%	0.58905289	0.099337%	0.00017059	0.58855133	0.014105%	0.00001911	8.928
0.06	5	0.426977	0.427450	0.110767%	0.42718140	0.047944%	0.00022749	0.42671627	-0.060992%	0.00004273	5.323
0.07	5	0.298483	0.312316	4.634273%	0.29798904	-0.165550%	0.00058010	0.29870112	0.073016%	0.00005523	10.503
0.04	1	0.887954	0.884899	-0.343963%	0.88749403	-0.051768%	0.00001843	0.88794510	-0.000969%	0.00000027	67.142
0.05	1	0.539828	0.539894	0.012288%	0.54021864	0.072383%	0.00009329	0.54004679	0.040549%	0.00001566	5.958
0.06	1	0.208508	0.211889	1.621285%	0.20813464	-0.179206%	0.00061038	0.20844899	-0.028445%	0.00000818	74.597
0.07	1	0.056784	0.058384	2.818767%	0.05650152	-0.496713%	0.00273819	0.05659569	-0.330873%	0.00089689	3.053

Table 3: Deltas of Plain Vanilla European Call Options by Hybrid method ($\gamma = 1$)

K	T	(A) BS	(B) AE	(C) error	(D) Crude	(E) error	(F) Var/(Mean^2)	(G) Hybrid	(H) error	(I) Var/(Mean^2)	(F)/(I)
0.03	10	0.033542	0.037184	10.858815%	0.03417980	1.902788%	0.00358509	0.03356008	0.055176%	0.00048757	7.353
0.04	10	0.050429	0.053402	5.895150%	0.05056428	0.268498%	0.00209694	0.05049383	0.128796%	0.00032698	6.413
0.05	10	0.060002	0.063078	5.127110%	0.06003373	0.052961%	0.00110998	0.05998620	-0.026252%	0.00018898	5.874
0.06	10	0.063054	0.066602	5.627723%	0.06288745	-0.263632%	0.00110244	0.06307760	0.037946%	0.00011779	9.359
0.07	10	0.061627	0.066105	7.265933%	0.06174325	0.189025%	0.00132503	0.06156455	-0.100947%	0.00012446	10.646
0.08	10	0.057584	0.063146	9.658748%	0.05777334	0.328328%	0.00139169	0.05768212	0.169910%	0.00019660	7.079
0.09	10	0.052269	0.057820	10.619138%	0.05219150	-0.148309%	0.00134616	0.05224904	-0.038240%	0.00016437	8.190
0.04	7	0.041702	0.043521	4.362319%	0.04162488	-0.185186%	0.00140858	0.04174095	0.093157%	0.00014873	9.471
0.05	7	0.050960	0.052775	3.561971%	0.05084451	-0.226490%	0.00079223	0.05101622	0.110474%	0.00007417	10.681
0.06	7	0.052607	0.054753	4.079353%	0.05268027	0.140080%	0.00094428	0.05259670	-0.018781%	0.00005709	16.540
0.07	7	0.049260	0.052124	5.813721%	0.04921219	-0.096668%	0.00101367	0.04928743	0.056059%	0.00007059	14.361
0.04	5	0.034355	0.035516	3.377900%	0.03422500	-0.378543%	0.00109475	0.03434293	-0.035278%	0.00007967	13.741
0.05	5	0.043502	0.044603	2.531512%	0.04366824	0.382493%	0.00070214	0.04351783	0.036745%	0.00002799	25.089
0.06	5	0.043854	0.045202	3.073360%	0.04382771	-0.059493%	0.00066501	0.04386263	0.020142%	0.00002259	29.444
0.07	5	0.038784	0.040662	4.841814%	0.03888714	0.266153%	0.00095613	0.03881078	0.069270%	0.00005127	18.650
0.04	1	0.009527	0.009679	1.596742%	0.00959848	0.753318%	0.00248930	0.00953222	0.057823%	0.00002006	124.075
0.05	1	0.019848	0.019947	0.501252%	0.01981001	-0.189541%	0.00040210	0.01984706	-0.002879%	0.00000120	334.501
0.06	1	0.014350	0.014518	1.174357%	0.01429368	-0.390582%	0.00075063	0.01434524	-0.031272%	0.00000531	141.402
0.07	1	0.005704	0.005399	-5.343853%	0.00563689	-1.174929%	0.00267239	0.00569887	-0.088233%	0.00052857	5.056

Table 4: Vegas of Plain Vanilla European Call Options by Hybrid method ($\gamma = 1$)

K	T	gamma	AE	(A) Crude	(B) Var	(C) Var/(Mean^2)	(D) Hybrid	(E) Var	(F) Var/(Mean^2)	(C)/(F)
0.04	10	0.1	0.639422	0.640105	0.002123%	0.000052	0.639183	0.000145%	0.000004	14.56
0.05	10	0.1	0.512616	0.512301	0.002945%	0.000112	0.513403	0.000160%	0.000006	18.47
0.06	10	0.1	0.386979	0.387733	0.002833%	0.000188	0.388599	0.000114%	0.000008	24.91
0.07	10	0.1	0.274452	0.276818	0.002862%	0.000374	0.276580	0.000075%	0.000010	38.01
0.04	10	0.4	0.682263	0.680254	0.004499%	0.000097	0.680634	0.000469%	0.000010	9.61
0.05	10	0.4	0.550463	0.551200	0.003500%	0.000115	0.551433	0.000590%	0.000019	5.94
0.06	10	0.4	0.423340	0.427675	0.003643%	0.000199	0.427742	0.000357%	0.000019	10.22
0.07	10	0.4	0.314115	0.319046	0.002898%	0.000285	0.318106	0.000393%	0.000039	7.33
0.04	10	0.6	0.708184	0.704140	0.004679%	0.000094	0.704880	0.000457%	0.000009	10.27
0.05	10	0.6	0.575694	0.574634	0.005289%	0.000160	0.575937	0.000731%	0.000022	7.27
0.06	10	0.6	0.450221	0.453449	0.003731%	0.000181	0.453954	0.000753%	0.000037	4.97
0.07	10	0.6	0.346341	0.347590	0.004681%	0.000387	0.347881	0.000571%	0.000047	8.21
0.04	10	0.9	0.743105	0.738024	0.007891%	0.000145	0.738090	0.000540%	0.000010	14.61
0.05	10	0.9	0.613541	0.612073	0.006969%	0.000186	0.612398	0.000932%	0.000025	7.48
0.06	10	0.9	0.494503	0.496566	0.007986%	0.000324	0.496948	0.000853%	0.000035	9.37
0.07	10	0.9	0.403356	0.396081	0.007441%	0.000474	0.396335	0.000962%	0.000061	7.74
0.04	7	0.1	0.660244	0.660811	0.002676%	0.000061	0.660017	0.000102%	0.000002	26.07
0.05	7	0.1	0.510555	0.511298	0.002867%	0.000110	0.511020	0.000148%	0.000006	19.38
0.06	7	0.1	0.362219	0.363209	0.002497%	0.000189	0.363395	0.000102%	0.000008	24.51
0.07	7	0.1	0.234636	0.235917	0.001696%	0.000305	0.235886	0.000029%	0.000005	57.75
0.04	7	0.4	0.696465	0.695186	0.002372%	0.000049	0.695054	0.000362%	0.000007	6.55
0.05	7	0.4	0.542220	0.543470	0.003170%	0.000107	0.542805	0.000388%	0.000013	8.16
0.06	7	0.4	0.393386	0.396231	0.002898%	0.000185	0.396642	0.000302%	0.000019	9.61
0.07	7	0.4	0.269989	0.272047	0.002695%	0.000364	0.272048	0.000175%	0.000024	15.37
0.04	7	0.6	0.718366	0.714988	0.003072%	0.000060	0.715657	0.000335%	0.000007	9.20
0.05	7	0.6	0.563330	0.564012	0.004612%	0.000145	0.563470	0.000560%	0.000018	8.22
0.06	7	0.6	0.416410	0.420451	0.003361%	0.000190	0.419302	0.000518%	0.000029	6.45
0.07	7	0.6	0.298544	0.297994	0.003634%	0.000409	0.297663	0.000284%	0.000032	12.75
0.04	7	0.9	0.747848	0.744886	0.006046%	0.000109	0.744472	0.000501%	0.000009	12.05
0.05	7	0.9	0.594995	0.594271	0.006361%	0.000180	0.594502	0.000727%	0.000021	8.76
0.06	7	0.9	0.454316	0.454936	0.005252%	0.000254	0.455291	0.001056%	0.000051	4.98
0.07	7	0.9	0.348855	0.341004	0.006107%	0.000525	0.340066	0.000535%	0.000046	11.35
0.04	5	0.1	0.683746	0.683594	0.002289%	0.000049	0.683526	0.000088%	0.000002	25.99
0.05	5	0.1	0.508921	0.509843	0.002132%	0.000082	0.509232	0.000108%	0.000004	19.76
0.06	5	0.1	0.335626	0.336210	0.001841%	0.000163	0.336490	0.000053%	0.000005	34.48
0.07	5	0.1	0.194349	0.194875	0.001833%	0.000483	0.195174	0.000010%	0.000003	181.76
0.04	5	0.4	0.714742	0.713803	0.002890%	0.000057	0.713654	0.000246%	0.000005	11.73
0.05	5	0.4	0.535682	0.535940	0.003103%	0.000108	0.536485	0.000343%	0.000012	9.06
0.06	5	0.4	0.362747	0.365572	0.003237%	0.000242	0.365042	0.000284%	0.000021	11.38
0.07	5	0.4	0.225921	0.226498	0.002532%	0.000494	0.226157	0.000068%	0.000013	37.18
0.04	5	0.6	0.733468	0.731799	0.003040%	0.000057	0.731636	0.000276%	0.000005	11.01
0.05	5	0.6	0.553524	0.553649	0.003618%	0.000118	0.553675	0.000609%	0.000020	5.94
0.06	5	0.6	0.382765	0.385749	0.002848%	0.000191	0.384824	0.000450%	0.000030	6.30
0.07	5	0.6	0.251275	0.248343	0.002531%	0.000410	0.248500	0.000132%	0.000021	19.21
0.04	5	0.9	0.758652	0.757366	0.004961%	0.000086	0.757047	0.000365%	0.000006	13.57
0.05	5	0.9	0.580286	0.579701	0.004741%	0.000141	0.579584	0.000545%	0.000016	8.70
0.06	5	0.9	0.415697	0.416918	0.006380%	0.000367	0.416067	0.000703%	0.000041	9.04
0.07	5	0.9	0.295764	0.287106	0.004736%	0.000575	0.285101	0.000429%	0.000053	10.87
0.04	1	0.1	0.846571	0.846721	0.001304%	0.000018	0.846571	0.000002%	0.000000	699.81
0.05	1	0.1	0.503989	0.504452	0.003009%	0.000118	0.504007	0.000036%	0.000001	83.18
0.06	1	0.1	0.163108	0.163105	0.001371%	0.000515	0.163190	0.000001%	0.000001	976.05
0.07	1	0.1	0.025342	0.025361	0.000236%	0.003663	0.025186	0.000016%	0.000247	14.85
0.04	1	0.4	0.861090	0.861357	0.001360%	0.000018	0.861532	0.000007%	0.000000	188.00
0.05	1	0.4	0.515958	0.516048	0.002301%	0.000086	0.516017	0.000191%	0.000007	12.04
0.06	1	0.4	0.177626	0.177599	0.001517%	0.000481	0.177366	0.000009%	0.000003	171.40
0.07	1	0.4	0.034412	0.033753	0.000321%	0.002814	0.033606	0.000100%	0.000881	3.19
0.04	1	0.6	0.869801	0.870741	0.001175%	0.000015	0.870797	0.000012%	0.000000	101.59
0.05	1	0.6	0.523937	0.523230	0.002680%	0.000098	0.524055	0.000203%	0.000007	13.26
0.06	1	0.6	0.188272	0.187722	0.001923%	0.000546	0.187240	0.000010%	0.000003	184.59
0.07	1	0.6	0.041539	0.040353	0.000458%	0.002814	0.040453	0.000128%	0.000783	3.59
0.04	1	0.9	0.881415	0.883591	0.001727%	0.000022	0.883775	0.000016%	0.000000	106.11
0.05	1	0.9	0.535905	0.536393	0.002851%	0.000099	0.536088	0.000302%	0.000011	9.42
0.06	1	0.9	0.205694	0.203798	0.002474%	0.000596	0.203094	0.000035%	0.000008	70.32
0.07	1	0.9	0.053849	0.052412	0.000746%	0.002714	0.052503	0.000244%	0.000884	3.07

Table 5: Deltas of Plain Vanilla European Call Options by Hybrid method

K	T	gamma	AE	(A) Crude	(B) Var	(C) Var/(Mean^2)	(D) Hybrid	(E) Var	(F) Var/(Mean^2)	(C)/(F)
0.04	10	0.1	0.879607	0.861035	0.021539%	0.000291	0.864740	0.000542%	0.000007	40.11
0.05	10	0.1	0.934990	0.924037	0.022415%	0.000263	0.923509	0.000197%	0.000002	113.69
0.06	10	0.1	0.899173	0.887686	0.021924%	0.000278	0.889660	0.000083%	0.000001	263.87
0.07	10	0.1	0.786939	0.776204	0.020838%	0.000346	0.777378	0.000141%	0.000002	147.76
0.04	10	0.4	0.346132	0.332214	0.003875%	0.000351	0.331046	0.000104%	0.000010	36.91
0.05	10	0.4	0.380626	0.368111	0.005939%	0.000438	0.367596	0.000052%	0.000004	113.19
0.06	10	0.4	0.377993	0.364255	0.004388%	0.000331	0.365533	0.000027%	0.000002	163.08
0.07	10	0.4	0.346532	0.332137	0.004576%	0.000415	0.332827	0.000075%	0.000007	61.26
0.04	10	0.6	0.185748	0.175984	0.001575%	0.000508	0.176293	0.000078%	0.000025	20.21
0.05	10	0.6	0.209070	0.200064	0.001863%	0.000465	0.200219	0.000023%	0.000006	81.44
0.06	10	0.6	0.211999	0.202938	0.002332%	0.000566	0.202840	0.000010%	0.000002	239.69
0.07	10	0.6	0.199929	0.188973	0.001750%	0.000490	0.189374	0.000033%	0.000009	52.67
0.04	10	0.9	0.072945	0.068770	0.000524%	0.001108	0.068885	0.000064%	0.000134	8.24
0.05	10	0.9	0.085110	0.080945	0.000603%	0.000921	0.081007	0.000047%	0.000071	12.99
0.06	10	0.9	0.088975	0.084689	0.000609%	0.000849	0.084359	0.000049%	0.000069	12.30
0.07	10	0.9	0.087243	0.081478	0.000637%	0.000960	0.081601	0.000030%	0.000044	21.66
0.04	7	0.1	0.720018	0.713115	0.019853%	0.000390	0.711890	0.000184%	0.000004	107.69
0.05	7	0.1	0.782269	0.775435	0.019357%	0.000322	0.776295	0.000011%	0.000000	1,720.95
0.06	7	0.1	0.736665	0.729612	0.014244%	0.000268	0.731420	0.000050%	0.000001	288.13
0.07	7	0.1	0.606333	0.602280	0.016252%	0.000448	0.600522	0.000073%	0.000002	221.05
0.04	7	0.4	0.282947	0.274848	0.003370%	0.000446	0.274031	0.000069%	0.000009	48.42
0.05	7	0.4	0.318454	0.310110	0.004045%	0.000421	0.311077	0.000016%	0.000002	249.98
0.06	7	0.4	0.310055	0.303208	0.003253%	0.000354	0.302619	0.000017%	0.000002	193.23
0.07	7	0.4	0.269396	0.261565	0.003282%	0.000480	0.260690	0.000047%	0.000007	68.79
0.04	7	0.6	0.151694	0.145432	0.001431%	0.000677	0.145883	0.000046%	0.000022	31.21
0.05	7	0.6	0.174921	0.169719	0.001349%	0.000468	0.169731	0.000012%	0.000004	110.99
0.06	7	0.6	0.174030	0.168778	0.001634%	0.000574	0.168580	0.000007%	0.000002	250.32
0.07	7	0.6	0.156236	0.149747	0.001201%	0.000536	0.149536	0.000020%	0.000009	59.10
0.04	7	0.9	0.059480	0.056817	0.000317%	0.000982	0.056932	0.000023%	0.000071	13.75
0.05	7	0.9	0.071209	0.068668	0.000462%	0.000980	0.068792	0.000018%	0.000038	26.02
0.06	7	0.9	0.073119	0.070200	0.000315%	0.000640	0.070306	0.000010%	0.000021	30.21
0.07	7	0.9	0.068648	0.064895	0.000379%	0.000901	0.064982	0.000016%	0.000038	23.51
0.04	5	0.1	0.591044	0.586723	0.011927%	0.000346	0.586618	0.000043%	0.000001	274.90
0.05	5	0.1	0.661138	0.656247	0.012246%	0.000284	0.657688	0.000004%	0.000000	3,297.30
0.06	5	0.1	0.605401	0.602459	0.009753%	0.000269	0.602176	0.000031%	0.000001	315.18
0.07	5	0.1	0.459128	0.454799	0.008122%	0.000393	0.455315	0.000025%	0.000001	323.45
0.04	5	0.4	0.231841	0.226130	0.002346%	0.000459	0.226382	0.000057%	0.000011	41.44
0.05	5	0.4	0.269143	0.264623	0.002288%	0.000327	0.264731	0.000007%	0.000001	346.14
0.06	5	0.4	0.255220	0.250845	0.002410%	0.000383	0.250705	0.000016%	0.000003	152.73
0.07	5	0.4	0.206391	0.200419	0.002278%	0.000567	0.200576	0.000019%	0.000005	117.72
0.04	5	0.6	0.124135	0.120797	0.001015%	0.000696	0.120521	0.000028%	0.000019	35.93
0.05	5	0.6	0.147835	0.145035	0.000677%	0.000322	0.144689	0.000004%	0.000002	190.62
0.06	5	0.6	0.143398	0.140526	0.000979%	0.000496	0.139965	0.000006%	0.000003	168.01
0.07	5	0.6	0.120502	0.116014	0.000750%	0.000557	0.115999	0.000009%	0.000007	80.38
0.04	5	0.9	0.048574	0.046921	0.000213%	0.000966	0.047019	0.000012%	0.000054	17.99
0.05	5	0.9	0.060182	0.058627	0.000228%	0.000662	0.058698	0.000006%	0.000018	36.72
0.06	5	0.9	0.060336	0.058527	0.000214%	0.000625	0.058596	0.000005%	0.000014	45.18
0.07	5	0.9	0.053412	0.051223	0.000243%	0.000927	0.051027	0.000006%	0.000025	37.50
0.04	1	0.1	0.175746	0.174284	0.002957%	0.000973	0.175239	0.000001%	0.000000	2,066.88
0.05	1	0.1	0.295670	0.295556	0.001649%	0.000189	0.295383	0.000000%	0.000000	30,554.28
0.06	1	0.1	0.182919	0.182899	0.001895%	0.000567	0.182503	0.000001%	0.000000	1,794.19
0.07	1	0.1	0.044016	0.043862	0.000788%	0.004095	0.043586	0.000032%	0.000168	24.33
0.04	1	0.4	0.067164	0.066130	0.000433%	0.000990	0.066500	0.000001%	0.000001	773.62
0.05	1	0.4	0.120364	0.119951	0.000338%	0.000235	0.119981	0.000000%	0.000000	2,696.85
0.06	1	0.4	0.078845	0.078115	0.000281%	0.000461	0.078264	0.000001%	0.000001	535.99
0.07	1	0.4	0.022805	0.022592	0.000199%	0.003901	0.022545	0.000030%	0.000594	6.57
0.04	1	0.6	0.035288	0.034903	0.000146%	0.001201	0.034832	0.000000%	0.000004	342.98
0.05	1	0.6	0.066114	0.066098	0.000106%	0.000243	0.065840	0.000000%	0.000000	1,506.38
0.06	1	0.6	0.044912	0.044633	0.000122%	0.000613	0.044473	0.000000%	0.000002	328.30
0.07	1	0.6	0.014316	0.014280	0.000070%	0.003441	0.014391	0.000013%	0.000622	5.53
0.04	1	0.9	0.013386	0.013177	0.000035%	0.002015	0.013189	0.000000%	0.000009	215.24
0.05	1	0.9	0.026914	0.026835	0.000026%	0.000366	0.026784	0.000000%	0.000001	393.93
0.06	1	0.9	0.019263	0.019016	0.000019%	0.000526	0.019041	0.000000%	0.000003	158.26
0.07	1	0.9	0.006921	0.007270	0.000011%	0.002127	0.007228	0.000003%	0.000658	3.23

Table 6: Vegas of Plain Vanilla European Call Options by Hybrid method

	K	gamma	(A) AE	(B) Crude	(C) Var	(D) Var/(Mean^2)	(E) Hybrid	(F) Var	(G) Var/(Mean^2)	(D)/(G)
5%ITM	97.41508	0.1	0.654064	0.654330	0.0000194	0.004542%	0.653742	0.0000003	0.000068%	66.500
ATM	102.54219	0.1	0.489509	0.489068	0.0000219	0.009173%	0.489299	0.0000002	0.000099%	92.549
5%OTM	107.66930	0.1	0.325548	0.324583	0.0000210	0.019935%	0.325466	0.0000002	0.000149%	133.888
10%OTM	112.79641	0.1	0.189790	0.189376	0.0000169	0.047143%	0.189655	0.0000001	0.000210%	224.649
20%OTM	123.05063	0.1	0.041959	0.041995	0.0000045	0.255775%	0.041864	0.0000001	0.007417%	34.487
5%ITM	97.41508	0.4	0.659954	0.659781	0.0000285	0.006557%	0.659627	0.0000008	0.000174%	37.754
ATM	102.54219	0.4	0.494935	0.495165	0.0000254	0.010346%	0.494545	0.0000014	0.000555%	18.642
5%OTM	107.66930	0.4	0.332288	0.331281	0.0000256	0.023290%	0.332243	0.0000008	0.000768%	30.316
10%OTM	112.79641	0.4	0.198466	0.197969	0.0000145	0.037005%	0.198286	0.0000003	0.000639%	57.941
20%OTM	123.05063	0.4	0.049476	0.048816	0.0000056	0.233946%	0.048903	0.0000005	0.019119%	12.236
5%ITM	97.41508	0.6	0.663747	0.663530	0.0000218	0.004948%	0.663514	0.0000009	0.000202%	24.468
ATM	102.54219	0.6	0.498563	0.497843	0.0000185	0.007449%	0.498433	0.0000015	0.000596%	12.502
5%OTM	107.66930	0.6	0.336938	0.336738	0.0000228	0.020094%	0.336880	0.0000011	0.000985%	20.393
10%OTM	112.79641	0.6	0.204595	0.203896	0.0000170	0.040778%	0.204242	0.0000004	0.000925%	44.078
20%OTM	123.05063	0.6	0.055044	0.053580	0.0000047	0.164805%	0.054227	0.0000007	0.022433%	7.346
5%ITM	97.41508	0.9	0.669235	0.668938	0.0000227	0.005077%	0.669115	0.0000019	0.000432%	11.738
ATM	102.54219	0.9	0.504023	0.503286	0.0000296	0.011696%	0.503920	0.0000026	0.001010%	11.575
5%OTM	107.66930	0.9	0.344149	0.343168	0.0000258	0.021909%	0.343782	0.0000016	0.001384%	15.828
10%OTM	112.79641	0.9	0.214308	0.212834	0.0000191	0.042070%	0.213365	0.0000006	0.001317%	31.940
20%OTM	123.05063	0.9	0.064253	0.062386	0.0000073	0.186694%	0.062693	0.0000014	0.034685%	5.383
5%ITM	97.41508	1	0.671010	0.670881	0.0000255	0.005661%	0.670938	0.0000015	0.000344%	16.469
ATM	102.54219	1	0.505848	0.505434	0.0000227	0.008880%	0.505868	0.0000026	0.001029%	8.630
5%OTM	107.66930	1	0.346617	0.346289	0.0000241	0.020068%	0.346284	0.0000016	0.001329%	15.102
10%OTM	112.79641	1	0.217686	0.215948	0.0000174	0.037318%	0.216507	0.0000007	0.001483%	25.164
20%OTM	123.05100	1	0.067555	0.065428	0.0000066	0.154573%	0.065870	0.0000015	0.035633%	4.338

Table 7: Deltas of Average European Call Options ($T = 1$)

	K	gamma	(A) AE	(B) Crude	(C) Var	(D) Var/(Mean^2)	(E) Hybrid	(F) Var	(G) Var/(Mean^2)	(D)/(G)
5%ITM	97.41508	0.1	0.321133	0.320411	0.0000373	0.036306%	0.321095	0.0000000	0.000033%	1,089.109
ATM	102.54219	0.1	0.354286	0.353064	0.0000257	0.020634%	0.354159	0.0000000	0.000000%	53,319.866
5%OTM	107.66930	0.1	0.323419	0.323477	0.0000267	0.025502%	0.323195	0.0000000	0.000039%	650.252
10%OTM	112.79641	0.1	0.245120	0.244207	0.0000331	0.055580%	0.244750	0.0000000	0.000075%	741.937
20%OTM	123.05063	0.1	0.081607	0.080836	0.0000170	0.260092%	0.081219	0.0000003	0.004460%	58.319
5%ITM	97.41508	0.4	1.270425	1.263062	0.0005453	0.034184%	1.268738	0.0000024	0.000149%	230.160
ATM	102.54219	0.4	1.415718	1.411778	0.0005507	0.027628%	1.414053	0.0000002	0.000010%	2,723.779
5%OTM	107.66930	0.4	1.307002	1.303195	0.0005270	0.031029%	1.304241	0.0000015	0.000090%	344.386
10%OTM	112.79641	0.4	1.012752	1.012073	0.0004192	0.040927%	1.009453	0.0000030	0.000292%	140.000
20%OTM	123.05063	0.4	0.374925	0.367922	0.0002836	0.209505%	0.372758	0.0000160	0.011500%	18.218
5%ITM	97.41508	0.6	3.177614	3.155467	0.0031814	0.031952%	3.172798	0.0000198	0.000197%	162.393
ATM	102.54219	0.6	3.565024	3.554975	0.0033793	0.026739%	3.559398	0.0000027	0.000022%	1,233.098
5%OTM	107.66930	0.6	3.315839	3.290550	0.0036781	0.033970%	3.307924	0.0000149	0.000137%	248.741
10%OTM	112.79641	0.6	2.606188	2.575140	0.0033317	0.050241%	2.595140	0.0000237	0.000352%	142.616
20%OTM	123.05063	0.6	1.027001	1.024212	0.0018046	0.172033%	1.021342	0.0001545	0.014812%	11.615
5%ITM	97.41508	0.9	12.568705	12.489924	0.0722371	0.046306%	12.551482	0.0007059	0.000448%	103.346
ATM	102.54219	0.9	14.246137	14.184014	0.0700261	0.034807%	14.218662	0.0001599	0.000079%	440.040
5%OTM	107.66930	0.9	13.397888	13.282798	0.0728660	0.041300%	13.360984	0.0005436	0.000305%	135.628
10%OTM	112.79641	0.9	10.750092	10.642989	0.0594140	0.052452%	10.693793	0.0004542	0.000397%	132.056
20%OTM	123.05063	0.9	4.606249	4.585882	0.0384366	0.182768%	4.620611	0.0043734	0.020484%	8.922
5%ITM	97.41508	1	19.876665	19.829848	0.1715604	0.043629%	19.839990	0.0025972	0.000660%	66.123
ATM	102.54219	1	22.607037	22.509795	0.2058188	0.040620%	22.567062	0.0007547	0.000148%	274.116
5%OTM	107.66930	1	21.339049	21.226979	0.1723755	0.038256%	21.275983	0.0013481	0.000298%	128.455
10%OTM	112.79641	1	17.236851	16.935203	0.1312096	0.045749%	17.147439	0.0021928	0.000746%	61.346
20%OTM	123.05100	1	7.577571	7.533425	0.1219983	0.214966%	7.624605	0.0136844	0.023539%	9.132

Table 8: Vegas of Average European Call Options ($T = 1$)