CARF Working Paper

CARF-F-087

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December 2006

CARF is presently supported by Bank of Tokyo-Mitsubishi UFJ, Ltd., Dai-ichi Mutual Life Insurance Company, Meiji Yasuda Life Insurance Company, Mizuho Financial Group, Inc., Nippon Life Insurance Company, Nomura Holdings, Inc. and Sumitomo Mitsui Banking Corporation (in alphabetical order). This financial support enables us to issue CARF Working Papers.

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Abstract

We present a Markov perfect equilibrium for a dynamic limit order market. For simplicity, we assume that traders have symmetric information and that limit orders expire in two periods after their submission. In equilibrium, when sellers enter the market consecutively, the best ask decreases tick by tick. Once the best ask reaches a certain level, it jumps more than one tick, creating a hole in the book. A trade-off between price improvement and execution probability in submitting orders causes such quote jumps.

JEL Classification: G19, G29

Keywords: Limit Order Markets, A Price-Time Precedence Rule, Markov Perfect Equilibrium.

*We gratefully acknowledge helpful comments and suggestions by Takao Kobayashi, Makoto Saito, and seminar participants at the Nippon Finance Workshop, University of Tokyo, Nagoya University, and Osaka University. We are responsible for any remaining errors.

1 Introduction

In limit order markets, traders can submit both limit orders that are contingent on price and market orders that are not. Orders are matched and transactions take place according to the trading rule specified by an exchange. The question is what is the optimal order submission strategy under a certain market condition. As a consequence of traders optimally submitting orders, how do quotes change and how do transactions take place? The exchange can affect order submissions and transactions through a trading rule such as the tick size which is the minimum price variation. Is the smaller tick size better for the exchange? These questions are important because limit order markets are prevalent as the execution systems of many financial markets.

To answer these questions, we consider a situation in which sellers and buyers arrive randomly in each period and submit an order to the exchange. We further assume that limit orders automatically expire in two periods after their submission. In general, limit orders incur the cost of uncertain execution, delayed execution, and adverse selection. These assumptions allow us to concentrate on a trade-off between price improvement and execution uncertainty. We set aside the effect of asymmetric information .[1] For example, Chordia *et al.* (2005) report that information is very quickly incorporated in prices for frequently traded stocks in the New York Stock Exchange, which implies considerable amounts of transactions take place with less asymmetric information. In addition, Admati and Pfleiderer (1988) argue that orders of liquidity traders affect order submission strategies of informed traders. Thus, investigating the behavior of liquidity traders seems to be a reasonable step. As we will show, competition among liquidity traders can move quotes in limit order markets even if there is no provision of new information nor any asymmetry of information. This may be in contrast to quote dynamics in dealer markets studied by Easley and O'Hara (1992) where asymmetric information drives quotes to change.

Our model has a pure-strategy Markov perfect equilibrium similar to an Edgeworth cycle. We call it a quote-cutting equilibrium, in which if sellers enter the market consecutively, the best ask initially decreases tick by tick, and then jumps more than one tick. The next quote rebounds to the less aggressive level, and the same cycle starts over again. This cycling continues until a buyer arrives at the market. In a cycle, widening the spread is faster than narrowing the spread.

The reasoning behind these quote dynamics is as follows. In submitting orders, traders face a trade-off between price improvement and execution uncertainty; the more a trader compromises on price, the more certainly he can trade. The first seller arriving at the market submits a limit sell order at a high ask and allows the next seller to undercut it because the cost for deterring quote-cutting is significant enough. The following sellers

[1] Copeland and Galai (1983), Glosten and Milgrom (1985), among others, investigate the problem of adverse selection for market makers.

undercut the best ask by only one tick so as to minimize the cost in price to assure a higher priority because they expect further quote-cutting. When the best ask drops to a certain level, the next seller undercuts the best ask by more than one tick. Such an aggressive order is reasonable because a high execution probability by deterring further quote-cutting compensates for the substantial loss in price. The next seller facing the most aggressive ask submits a limit order behind the market. This order is also reasonable because a low execution probability is compensated by the less aggressive price.

In a quote-cutting equilibrium, quotes jump and "holes" emerge in the book. At a hole there is no limit order even though limit orders currently exist at higher and lower prices on the same side of the book. Holes in the book have been observed by Biais *et al.* (1995) in the Paris Bourse, by Irvine *et al.* (2000) in the Toronto Stock Exchange, and by Sandås (2001) in the Stockholm Stock Exchange. A spread narrows rapidly when the quote jumps, which creates holes. A hole accelerates the widening of a spread when the market order hits the edge of the hole. Spreads and transaction prices can be volatile due to holes. Our model predicts that the size of a hole is greater the more frequently traders arrive because the large cost in quote-cutting is compensated by the large benefit in the execution probability. How holes emerge in the book and what affects their size are issues which remain for future empirical studies.

The tick size is the minimum cost in price for price priority, and affects how traders compete on price. A quote-cutting equilibrium exists if the tick size is small. On the other hand, if it is large, there can be an equilibrium where traders do not compete on price but queue at the same quote. We call it a queuing equilibrium. Such an equilibrium can exist under the large tick size because the high cost for price priority inhibits quote-cutting. Our numerical examples show that if the tick size is large, a queuing equilibrium and an equilibrium with quote-cutting co-exist. Due to the multiplicity of equilibria, the effect of a tick size reduction on spreads can be ambiguous, which is in line with the empirical findings of Bourghelle and Declerck (2004) in the Paris Bourse.

Several studies have investigated limit order markets. Glosten (1994), Chakravarty and Holden (1995), Seppi (1997), Biais *et al.* (2000), Viswanathan and Wang (2002), and Parlour and Seppi (2003) analyze them using static models. Dynamics models are used by Cohen *et al.* (1981), Parlour (1998), Foucault (1999), Goettler *et al.* (2005), Foucault *et al.* (2005), and Rosu (2006). Parlour (1998) presents a model for a limit order market where the spread is the same as the tick size. By contrast, we consider a situation in which the spread is so wide relative to the tick size that traders compete on price. Cohen *et al.* (1981) and Foucault (1999) assume that limit orders expire in one period after their submission. Under such a one-period expiration, the book has at most one limit order, so that limit orders do not directly compete with each other. Foucault *et al.* (2005) assume that limit orders can survive indefinitely, and that traders have to undercut the best quote in submitting limit orders. Foucault *et al.* (2005) and the present study share some results, e.g., the possibility of holes emerging in the book. One difference, however, is that our results suggest that traders place limit orders outside the best quotes, which can make widening the spread faster than narrowing the spread. Submission of such orders is discussed by Cohen *et al.* (1981), and documented by Griffiths *et al.* (2000) in the Toronto Stock Exchange, by Hasbrouck and Saar (2002) in the Island ECN, and by Biais *et al.* (1995) and Bourghelle and Declerck (2004) in the Paris Bourse. Of the other articles, Goettler *et al.* (2005) solve for equilibrium numerically, and Rosu (2006) studies a continuous-time model. By assuming traders can adjust their orders instantaneously, Rosu (2006) investigates the shape of the book where every limit order yields the same expected utility. His traders can move very fast while our traders are so slow that they have to commit to their prices for a while.

Maskin and Tirole (1988) investigate price competition in an oligopolistic market, and Cordella and Foucault (1999) in a dealer market. A quote-cutting equilibrium presented here corresponds to an Edgeworth cycle equilibrium in Maskin and Tirole (1988). They consider how long-lived producers or dealers set prices only on the one side of the market. Our results show that an Edgeworth cycle is observed even if short-lived public traders set prices on both sides of the market. Myopic consideration of a trade-off between price improvement and execution uncertainty can create an Edgeworth cycle.[2]

This article is organized as follows. Section 2 provides the model. Section 3 demonstrates equilibrium when limit orders expire in one period after their submission in order to explain the structure of our model in detail. Sections 4 and 5 present a queuing equilibrium and a quote-cutting equilibrium under the two-period expiration of limit orders, respectively. We discuss the case where limit orders survive longer periods in Section 6. Until then, we assume that traders are homogeneous in patience. We briefly discuss the effect of heterogeneity in patience in Section 7. Section 8 summarizes empirical implications along with the the effect of a tick size reduction. Section 9 contains some concluding remarks. All proofs can be found in the Appendix.

2 The Model

This section provides the model. We explain types of traders, orders traders can choose, the state of the book, the trading rule, and equilibrium concept. At the end of this section, we lay out the assumption about expiration of limit orders. The model is a stochastic game where the state of the book represents the state of the model, the type of trader arriving at the market is stochastic, actions of traders are submitting orders, and the trading rule specifies payoffs for traders and transitions of the state of the book. We will show some numerical examples of equilibria in Section 3.

[2] For example, Eckert (2003) and Noel (2006) observe Edgeworth cycles in Canadian retail gasoline markets.

2.1 Types of traders

There are discrete and infinite periods, which are represented by $\tau \in \{0, +1, \ldots, +\infty\}$. In each period, one potential trader arrives at the exchange and submits an order. A trader is either a seller, s, or a buyer, b. We denote the set of trader types by $\Theta = \{s, b\}$.

The trader arrival is stochastic in the following way. Let $\alpha \in (0, 1]$ be the probability of a trader arriving, and $\beta \in (0, 1)$ be the probability of a seller conditional on a trader arriving. That is, in each period, the trader is a seller with the probability $\pi_s = \alpha\beta \in (0, 1)$, and a buyer with the probability $\pi_b = \alpha(1-\beta) \in (0, 1)$. No trader arrives with the probability $z = 1 - \pi_s - \pi_b \in [0, 1)$. Let $\pi = (\pi_s, \pi_b)$ and $\Pi = \{\pi : 0 < \pi_s < 1, 0 < \pi_b < 1, \pi_s + \pi_b \leq 1\}$. The π represents the trader arrival. We assume that π is exogenous and constant.

Sellers hold one share of the asset and evaluate it as $v_L \geq 0$. Buyers hold no shares and evaluate a share as v_H. We assume $\Delta = v_H - v_L > 0$. A payoff for a seller is $P - v_L$ if he sells a share at a price P. A payoff for a buyer is $v_H - P$ if he buys a share at a price P. If a trader does not trade, he receives zero payoff. Traders choose an order to maximize their expected utilities. The discount factor of every trader is assumed to be one until Section 7. We assume traders are risk neutral.

A trader can submit an order only when he arrives at the market. We assume that a trader himself cannot cancel or modify his order once he submits it. Thus, a trader faces a static problem in choosing an order, which circumvents the complexity of a dynamic problem.[3]

2.2 The orders

An action of a trader is the submission of an order. A set of actions depends on what orders the exchange accepts. In the same way as typical limit order markets, we assume that the exchange accepts a market sell order (MS), a market buy order (MB), a limit sell order (LS), and/or a limit buy order (LB). We restrict the volume of each order to one share.

A trader specifies a price, or a quote, in submitting a limit order. The price of a LS is an ask, say A, and the price of a LB is a bid, say B. The exchange designates the tick size $k > 0$ which is the minimum price variation, and a trader must choose a price from the pricing grid $N_k = \{0, k, 2k, \ldots\}$. We denote $k = 0$ when a real number is allowed for a price. For simplicity, we assume that v_H and v_L are on the pricing grid, $v_L \in N_k$ and $v_H \in N_k$. In summary, the set of available orders is $X = \{\text{no order}, \text{a MS}, \text{a MB}, \text{a LS at } A, \text{a LB at } B : A \in N_k, B \in N_k\}$.

[3]Both Goettler *et al.* (2005) and Foucault *et al.* (2005) exclude, as we do, the possibility of resubmission of orders. An exception at present is Rosu (2006), who assumes that traders can cancel and change orders at will.

2.3 The state of the book

When a limit order is submitted to the exchange, it is stored in the book until its execution or expiration. The book holds information regarding a price of a limit order and when the limit order is submitted. Let ω be a state of the book, or simply a book, and let Ω be the set of all states of the book. See Appendix A.1 for the concrete definition of Ω.

A lower (higher) price for an ask (bid) is called a more aggressive price. In a book, the best ask (bid) is the most aggressive ask (bid). Let $A^*(\omega)$ be the best ask and $B^*(\omega)$ be the best bid of the book ω, respectively.

The spread is the difference between the best ask and the best bid. In order to calculate the spread for the book without a LS or a LB, we assume, similar to Seppi (1997), that a trading crowd implicitly provides LSs at v_H and LBs at v_L. In other words, if the book does not have any LS (LB), the best ask (bid) is assumed to be v_H (v_L). The purpose of this assumption is to calculate the spread for any book, and is irrelevant to an equilibrium (see footnote 5).

2.4 The trading rule

We consider a transparent market with the pure price-time precedence rule and the discriminatory pricing rule. The specific rule used here is as follows: (1) the market is transparent in the sense that traders can observe the book when submitting an order. (2) The exchange treats a MS as the LS at v_L, and a MB as the LB at v_H. (3) A LS at or below the best bid and a LB at or above the best ask are called marketable. (4) If an incoming order is not marketable, it is stored in the book. If it is marketable, it is matched with an unfilled limit order on the opposite side of the book. (5) The priority among limit orders is assigned by price, and by time for limit orders at the same price due to a price-time precedence rule. (6) The transaction price is the quote of the limit order waiting in the book due to the discriminatory pricing rule.

Under this trading rule, a MS, a LS at or below the best bid, a MB, and a LB at or above the best ask are marketable. In what follows, we call marketable orders as market orders. Market orders are executed at the best price in the book immediately after their submission. We call the other orders, a LS above the best bid and a LB below the best ask, as limit orders. Limit orders are stored in the book and wait for future market orders.[4]

A seller never chooses no order, any buy order, a LS at or below v_L, and a LS at or above v_H in an equilibrium.[5] Thus, we restrict the set of orders for a seller facing the

[4]This trading process is similar to that of a bargaining model analyzed by Rubinstein (1982) and Rubinstein and Wolinsky (1985), in the sense that submitting a limit order corresponds to proposing a price and submitting a market order corresponds to accepting it. They study transactions where two persons negotiate a price while we study transactions where many persons propose prices at a given time via the book.

[5]Because a buyer submits a MB to the book with very aggressive LS, there is always a LS yielding the

book ω as $X(s,\omega) = \{\text{a MS}, \text{a LS at } A : A \in (B^*(\omega), v_H) \cap N_k\} \subseteq X$. Symmetrically, the set of orders for a buyer facing the book ω is restricted to $X(b,\omega) = \{\text{a MB}, \text{a LB at } B : B \in (v_L, A^*(\omega)) \cap N_k\} \subseteq X$.

2.5 Equilibrium concept

A strategy of a trader specifies an order he submits. We consider a Markov strategy which depends only on the type of a trader and on the current book, and depends neither on time nor on the history of the book. In addition, we focus only on a pure strategy. We denote a pure Markov strategy of a trader $i \in \Theta$ facing the book $\omega \in \Omega$ as $x(i,\omega) \in X(i,\omega)$. A profile of strategies is denoted as $\mathbf{x} = \{x(i,\omega) : i \in \Theta, \omega \in \Omega\}$.

The expected utility of a seller in submitting an order is as follows. A LS submitted to the book ω changes the book according to the trading rule. After the transition of the book, the next trader arrives at the market according to the trader arrival π, and submits a new order according to the profile of strategies $\mathbf{x}$. Thus, the execution probability of a LS at A depends on ω, π, and $\mathbf{x}$, and we denote it as $\Phi(s, A, \omega, \pi, \mathbf{x})$. The payoff of a MS is $B^*(\omega) - v_L$. Consequently, the expected utility of an order $x \in X(s,\omega)$ for a seller is

$$V(s, x, \omega, \pi, \mathbf{x}) = \begin{cases} \Phi(s, A, \omega, \pi, \mathbf{x})(A - v_L) & \text{if } x \text{ is a LS at } A \\ B^*(\omega) - v_L & \text{if } x \text{ is a MS.} \end{cases}$$

Symmetrically, the expected utility of an order $x \in X(b,\omega)$ for a buyer is

$$V(b, x, \omega, \pi, \mathbf{x}) = \begin{cases} \Phi(b, B, \omega, \pi, \mathbf{x})(v_H - B) & \text{if } x \text{ is a LB at } B \\ v_H - A^*(\omega) & \text{if } x \text{ is a MB.} \end{cases}$$

where $\Phi(b, B, \omega, \pi, \mathbf{x})$ represents the execution probability of a LB at B submitted to the book ω under π and $\mathbf{x}$.

As for equilibrium, we consider a pure-strategy Markov perfect equilibrium. That is, a profile of pure Markov strategies $\mathbf{x}^* = \{x^*(i,\omega) : i \in \Theta, \omega \in \Omega\}$ consists of an equilibrium if

$$x^*(i,\omega) \in \arg\max_{x \in X(i,\omega)} V(i, x, \omega, \pi, \mathbf{x}^*) \qquad \text{for } {}^\forall i \in \Theta,\ {}^\forall \omega \in \Omega.$$

Though a Markov perfect equilibrium allowing mixed strategies exists for which both actions and states are finite, a pure-strategy Markov perfect equilibrium does not necessarily exist. However, Theorem 3 in the Appendix shows that a pure-strategy Markov perfect equilibrium indeed exists under certain conditions. Practically, the problem is not the possibility of non-existence of an equilibrium but rather the multiplicity of equilibria as discussed in Section 3.

positive expected utility for a seller. Thus, sellers never submit orders yielding zero or negative expected utility, which are no order, any buy order, a LS at or below v_L, and a MS to the book with a LB at or below v_L. The orders which buyers never submit are symmetric. Because buyers never submits a MB to the book with a LS at or above v_H, a seller never submits a LS at or above v_H. Symmetrically, a buyer never submits a LB at or below v_L.

2.6 Assumption of order expiration

The analysis of limit order markets is complicated because the number of possible states of the book can be large. To make the set of books simple, we assume that limit orders expire automatically in certain periods after their submission. Foucault (1999) and Section 3 assume that limit orders expire in one period. The main results of this article presented in Sections 4 and 5 assume that limit orders expire in two periods. The two-period expiration is the simplest assumption to investigate direct quote competition of limit orders. In Section 6, we will discuss the case where limit orders expire in longer periods by numerical examples.

Some empirical studies report parts of limit orders are rapidly canceled after their submission if they are not executed. Hasbrouck and Saar (2002) report that about 25% (40%) of limit orders have been canceled within two (ten) seconds after their submission on the Island ECN. Lo, MacKinlay, and Zhang (2002) report that the average time-to-expiration or cancellation of non-executed orders is 34.15 (46.92) minutes for limit sell (buy) orders on the New York Stock Exchange. These studies suggest that limit orders commit prices for short intervals similar to our assumption.

Short-lived limit orders can be interpreted as the consequence of the quick reaction of traders. Even when a trader can adjust his limit order, his slow response may result in competition with many limit orders submitted in the near future. We can consider the above as a situation where limit orders expire in longer periods. On the other hand, if traders react quickly to the transition of the book, limit orders face a smaller number of incoming orders, which can be considered as shorter expiration periods of limit orders.

We can also consider that patience of traders reflects expiration periods of limit orders. Let $\delta(t)$ be the discount factor for t periods ahead. The discount factors are assumed to be $\delta(0) = \delta(1) = \delta(2) = 1$ and $\delta(t) = 0$ for $t \geq 3$ for a trader who can await a transaction only for two periods. This is the case of the two-period expiration of limit orders. In a similar way, we can assume that limit orders expire in longer periods for a trader who can await a transaction for longer periods. [6]

3 Explanation of an equilibrium

Before presenting an equilibrium under the two-period expiration of limit orders, let us revisit an equilibrium under the one-period expiration. Readers not interested in the detail explanation of an equilibrium concept can skip to Section 4. Section 3.1 presents an equilibrium, and Section 3.2 explains levels of quotes in the equilibrium. Most implications in these subsections have been discussed in Foucault (1999). We will apply these implications to the case under the longer-period expiration. Section 3.3 presents numerical examples of pure-strategy Markov perfect equilibria under the positive tick size. We

[6] Though this preference does not exhibit exponential discounting, time inconsistency is not a problem here because a trader can submit an order only once in our model.

discuss multiplicity of equilibria caused by the positive tick size. Section 3.4 presents numerical examples of equilibria under the two-period expiration.

3.1 Equilibrium under one-period expiration

The next theorem provides a unique equilibrium under the one-period expiration of limit orders.

Theorem 1 Suppose that limit orders expire in one period after their submission and that the discount factor of every trader is one. If a limit order can be contingent on a price of a real number, the following profile of strategies is a unique equilibrium. Let A_r and B_r be

$$A_r = \frac{(1-\pi_s)v_H + \pi_s(1-\pi_b)v_L}{1-\pi_s\pi_b}, \qquad B_r = \frac{\pi_b(1-\pi_s)v_H + (1-\pi_b)v_L}{1-\pi_s\pi_b}.$$

The strategy for a seller is to submit a MS if the book has a LB at B such as $B_r \leq B$, but otherwise to submit a LS at A_r. The strategy for a buyer is to submit a MB if the book has a LS at A such as $A \leq A_r$, but otherwise to submit a LB at B_r.

The proof is straightforward, since A_r and B_r are the solutions to the simultaneous equations

$$v_H - A_r = \pi_s(v_H - B_r), \tag{1}$$

$$B_r - v_L = \pi_b(A_r - v_L). \tag{2}$$

The limit prices A_r and B_r are constructed for a buyer to submit a MB to a LS at A_r if a seller submits a MS to a LB at B_r, and vice versa. These equations show that the expected utility by submitting a limit order is equal to the expected utility by submitting a market order. A situation where a limit order and a market order yield the different expected utility does not constitute an equilibrium because traders can choose both types of orders in limit order markets.

Theorem 1 is a special case of Proposition 3 in Foucault (1999) in the sense that the asset value does not change over time. While Foucault (1999) considers a case where the asset value fluctuates over time, we simplify his model by assuming that the asset value does not change, and extend it by stipulating that limit orders survive for longer periods in order to analyze quote competition.

3.2 The level of quotes under the one-period expiration

Under the one-period expiration, the ask is strictly higher than the bid due to the discontinuity of the execution probability in price as the following proposition states.

Proposition 1 Under the equilibrium in Theorem 1, the ask is higher than the bid ($A_r > B_r$).

If the book has a LB at B_r, the execution probability of a LS at $A \in (B_r, A_r]$ is π_b. The execution probability of a sell order is discontinuous in price at B_r because a seller can trade at B_r by a MS whose execution probability is unity. For a LS at an ask slightly above B_r, its gain in price relative to a MS cannot compensate for its loss in execution probability due to discontinuity, which inhibits a seller from submitting a LS at $A \in (B_r, A_r)$. Cohen *et al.* (1981) investigate an effect of this discontinuity on the spread, referring to it as a "gravitational pull." As we will see, the discontinuity of execution probability in price causes holes in the book under the two-period expiration.

Basically, the following relation between the trader arrival and the level of quotes under the one-period expiration is preserved even under the longer-period expiration.

Proposition 2 Under the equilibrium in Theorem 1, (1) the ask A_r decreases in π_s given π_b. The bid side is symmetric. (2) The ask A_r increases in π_b given π_s. The bid side is symmetric. (3) For $\beta > (3 - \sqrt{5})/2 \simeq 0.38$, the ask A_r decreases in α given β. The bid side is symmetric. (4) The expected spread decreases in α given β. (5) The ask A_r decreases in β given α. The bid side is symmetric.

In submitting a limit order, a trader has a monopoly power over future traders. However, he needs to satisfy participation constraints of future traders to extract their market orders. Participation constraints of a future trader require that the expected utility from a market order is equal to or higher than the expected utility from a limit order. Because the higher execution probability π_s of a LB raises the expected utility of a buyer in submitting a LB, a more aggressive ask is required to allure a MB. As a result, the ask A_r decreases in π_s as Proposition 2(1) states.

Symmetrically, the ask A_r increases in π_b as Proposition 2(2) states. The higher execution probability π_b of a LS increases the expected utility of a seller; a buyer submits a more aggressive LB, which reduces his expected utility; a less aggressive LS is sufficient to extract a MB.

Equations (1) and (2) suggest that not α and β but π_s and π_b directly determine the level of quotes, making comparative statics regarding α not simple as Proposition 2(3) implies. The trader arrival rate α raises the execution probability and the expected utility of a limit order. Thus, to extract a MB (MS), a limit order submitters need to submit a more aggressive LS (LB). As a result, the spread becomes narrower as Proposition 2(4) states. At the same time, a more aggressive LB reduces the expected utility of a buyer, and a less aggressive LS is sufficient to extract a MB. This counter effect is strong enough for small β, and the ask A_r does not decrease in α given small β as Proposition 2(3) implies. In what follows, we pay attention to the case of $\beta = 1/2$ partly because comparative statics regarding α can be complicated. Another reason is that $\beta = 1/2$ seems to be reasonable under no asymmetric information among traders.

Proposition 2(5) states the effect of the proportion of sellers and buyers on quotes. When the share of sellers, β, is higher, a buyer gets the higher expected utility by submitting a LB; a seller must submit a more aggressive LS to extract a MB; a buyer can submit a less aggressive LB to extract a MS because a seller suffers from an aggressive LS. As a result, when sellers arrive at the market more frequently, both sellers and buyers post lower quotes.

3.3 Multiplicity of equilibria under the positive tick size

Theorem 1 says that an equilibrium is unique if limit orders expire in one period and if the tick size is zero. If limit orders survive more than one period and if the tick size is zero, the optimal strategy may not exist because the maximum ask undercutting the best ask and the minimum bid overbidding the best bid do not exist due to the openness problem. To ensure the existence of optimal strategies, we need to assume the positive tick size $k > 0$ as in real exchanges. However, the discreteness in price can cause multiplicity of equilibria as we will show in this subsection.

Consider parameter values of $v_H = 3$, $v_L = 0$, $k = 1$, $\alpha = 1$, and $\beta = 1/2$ under one-period expiration of limit orders. Table 1 presents the states of the book and two equilibria, Eq A and Eq B, under these parameter values. The book ω_0 represents an empty book. A seller facing an empty book can choose a LS at 1 or a LS at 2, and a buyer facing an empty book can choose a LB at 1 or a LB at 2. There are the four other states of the book, each of which is represented by an order in the book; ω_1 is the book with a LS at 1, ω_2 is the book with a LS at 2, ω_3 is the book with a LB at 1, and ω_4 is the book with a LB at 2. Eq A and Eq B are profiles of strategies specifying an order each for a seller and a buyer, and for every state of the book. For example, in Eq A, a seller submits a LS at 2 to an empty book ω_0. The strategies denoted by * in the table are those for the states on the equilibrium path.

Eq A in Table 1 is a discretized version of a unique equilibrium in Theorem 1 in which $A_r = 2$ and $B_r = 1$. We can verify that Eq A is indeed an equilibrium by checking if a strategy for $^{\forall} i \in \Theta$ to $^{\forall} \omega \in \Omega$ maximises the expected utility given the profile of strategies of Eq A. Let's check if a LS at 2 is optimal for a seller facing an empty book ω_0. If a seller submits a LS at 1 to ω_0, the book becomes ω_1, the next trader is a buyer with probability 1/2, and the next buyer submits a MB under Eq A. Thus, a LS at 1 to ω_0 yields 1/2 as the expected utility. On the other hand, a LS at 2 to ω_0 yields 1 as the expected utility. Thus, a LS at 2 is optimal for a seller facing ω_0. In the same way, the optimality of strategies in Eq A is checked for $^{\forall} i \in \Theta$ and $^{\forall} \omega \in \Omega$.

The discrete pricing grid can cause multiple equilibria because optimal strategies can be multiple. There are three equilibria for this numerical example. Table 1 presents two of them, and the third equilibrium is symmetric to Eq B. Under Eq A, the optimal order for a seller to ω_3 is either a MS or a LS at 2 because both orders yield 1 as the expected utility. Eq A designates a seller to submit a MS to ω_3. On the other hand, there is another equilibrium, Eq B, which designates a seller to submit a LS at 2 to ω_3. As this example suggests, discreteness in price causes multiplicity of optimal strategies, which can lead to multiple equilibria.

Another source of multiple equilibria is that unconstrained prices may fail on the pricing grid. In this numerical example, if $k \neq 1/n$ for some integer n, A_r and B_r in Theorem 1 are not on the pricing grid. In such a case, there are multiple substitutes for unconstrained prices A_r and B_r, which can lead to multiple equilibria. To avoid this kind of multiplicity, Theorem 3 in the appendix assumes that the critical prices belong to the pricing grid, such as $k = 1$ for the above example.

The above examples suggest that we must be cautious in numerically examining limit order markets because there can be multiple equilibria. We will show that an equilibrium with quote-cutting and a queuing equilibrium coexist if the tick size is large, whereas a queuing equilibrium does not exist if the tick size is small. The small tick size seems to weaken the problem of the positive tick size by circumventing the multiplicity of optimal strategies.[7]

3.4 Equilibrium under the two-period expiration

The model under the one-period expiration differs with the model in the two-period expiration in the set of the states of the book. Table 2 presents three numerical examples of equilibria under the two-period expiration. The parameter values are the same as those for Table 1 except for the expiration period of limit orders. The number of states of the book is 19 under the two-period expiration, whereas the number is 5 under the one-period expiration.

Quotes change under these equilibria in the following way. Under Eq 1, if sellers arrive at the market consecutively, the first seller submits a LS at 2 to an empty book, the next seller submits a LS at 1 which undercuts the best ask by one tick. After that, the next seller submits a LS at 2, and the same cycle starts again. Eq 1 exhibits both quote-cutting and quote-rebounding. In Section 5, we will show how traders undercut the best quote when the tick size is small. In contrast, every seller submits a LS at 2 under Eq 2, and every seller submits a LS at 1 under Eq 3. Eq 2 and Eq 3 are examples of queuing equilibria. The next section will show that a queuing equilibrium can exist if the tick size is large. We will return to the numerical examples in Table 2 in Section 8 to discuss the effect of a tick-size reduction.

[7]A positive tick size generates multiple equilibria in some models. For example, the subgame perfect equilibrium of the bargaining game in Rubinstein (1982) is unique when the set of alternatives is a continuum. In contrast, Van Damme *et al.* (1990) show that the bargaining game has multiple equilibria when the set of alternatives is finite due to a positive tick size. Another example is the model of oligopolistic markets in Maskin and Tirole (1988). They assume a positive tick size, yielding multiple equilibria.

4 Queuing equilibrium

This section demonstrates there can be a queuing equilibrium where traders queue at the same quote if the pricing grid is coarse. To define a queuing equilibrium, let A_q be the ask at which a seller submits a LS to an empty book. Symmetrically, let B_q be the bid at which a buyer submits a LB to an empty book. An equilibrium is called queuing if a seller submits a LS at A_q to the book with the two LSs at A_q, if the symmetric condition for the bid side holds, and if some additional conditions are satisfied. We present the formal definition of our queuing equilibrium in Definition 1 in the Appendix (refer to the remarks followed by Definition 1 for the reasons behind our definition). Eq 2 and Eq 3 in Table 2 are examples of queuing equilibria.

The next theorem states that a queuing equilibrium cannot exist if the tick size, k, is small relative to the difference in valuation between sellers and buyers, $\Delta = v_H - v_L$.

Theorem 2 Suppose that limit orders expire in two periods and that the discount factor of every trader is one. Let k_q^s and k_q^b be defined as

$$k_q^s = (1-\pi_s)(1-\pi_b)/\{1 + (1-\pi_b)(2-\pi_s)(1+2\pi_s - \pi_s^2)\},$$

$$k_q^b = (1-\pi_s)(1-\pi_b)/\{1 + (1-\pi_s)(2-\pi_b)(1+2\pi_b - \pi_b^2)\}.$$

If the tick size k is equal to or smaller than $k_q^s\Delta$ or $k_q^b\Delta$, a queuing equilibrium defined by Definition 1 in the Appendix cannot exist.

A queuing equilibrium can exist under the coarse pricing grid because the large tick size hinders quote-cutting by raising the cost in price to obtain price priority. For example, if $\pi_s = \pi_b = 1/2$, $9k_q^s < 1 < 10k_q^s$, suggesting that a queuing equilibrium can exist when the difference in valuations between sellers and buyers is smaller than ten ticks.

A queuing equilibrium more likely to exist when the trader arrival rate, α, is large, as the following proposition implies.

Proposition 3 Suppose $\beta = 1/2$ and $\pi_s = \pi_b = \alpha/2$. $k_q^s = k_q^b$ and k_q^s decreases in α.

The higher trader arrival rate discourages quote-cutting by lowering the cost of waiting at or behind the market. At the same time, the higher trader arrival rate makes quotes more aggressive by raising the expected utility for traders on the opposite side of the market as Proposition 2 suggests.

The next proposition shows that traders submit very aggressive limit orders in a queuing equilibrium. Recall that if $\alpha = 1$, $\pi_s = \beta$ and $\pi_b = 1 - \beta$.

Proposition 4 Suppose that limit orders expire in two periods and that the discount factor of every trader is one. In addition, suppose $\alpha = 1$. Under a queuing equilibrium, $A_q - B_q \leq (2 + 1/\beta/(1-\beta))k - \Delta$.

Under Eq 3 in Table 2, a seller submits a LS at 1, and symmetrically a buyer submits a LB at 2. That is, the ask is lower than the bid on the equilibrium path. For another example, under the parameter values $v_H = 6$, $v_L = 0$, $k = 1$, $\alpha = 1$, and $\beta = 1/2$, Proposition 4 states $A_q - B_q \leq 0$. In fact, there are numerical examples of queuing equilibria whose quotes are $(A_q, B_q) = (3, 3)$, $(3, 4)$, $(2, 3)$, and $(2, 4)$. For a queuing equilibrium with $(A_q, B_q) = (3, 3)$, sellers and buyers queue at the same quote. Consequently, though the positive spread is observed, every transaction takes place at the same price like a transition in a call auction. This case is similar to that reported in Figure 3b by Biais *et al.* (1995). Propositions 3 and 4 predict that a sequence of transactions at the same price are frequently observed for actively traded stocks.

Aggressive quotes mentioned in Proposition 4 relates to the existence of a queuing equilibrium. If the tick size is small, a seller has a strong incentive to undercut the extant ask because he incurs a small cost in price to obtain price priority. To deprive the future sellers of incentive for quote-cutting, the first seller arriving at an empty book needs to submit a very aggressive LS. If the tick size is sufficiently small, the first seller prefers allowing future quote-cutting to preventing it. In such a case, he posts a high ask, and the next seller undercuts it. As a result, a small tick size eliminates a queuing equilibrium, and a quote-cutting equilibrium emerges. The next section will show how traders undercut quotes in a quote-cutting equilibrium.

5 Quote-cutting equilibrium

This section is devoted to explaining a quote-cutting equilibrium which is an equilibrium when limit orders expire in two periods and when the discount factor of every trader is one. A quote-cutting equilibrium exists if the tick size is small. First, Section 5.1 demonstrates quote dynamics of a quote-cutting equilibrium using a numerical example. Then, a corollary in Section 5.2 formally presents the equilibrium quote dynamics. Next, we study the level of quotes and the size of holes. Section 5.5 discusses the allocational efficiency. Section 5.6 presents other types of equilibria under the small tick size. Refer to Appendix A.4 to A.7 for a complete explanation of a quote-cutting equilibrium.

5.1 A numerical example

In a quote-cutting equilibrium, there are four critical asks, A_l, A_u, A_f, and A_h, defined by π_s, π_b, v_H, and v_L (refer to the Appendix for their definitions). They satisfy $v_L < A_l < A_u < A_f < A_h < v_H$. As we will see, A_f is the first ask submitted to an empty book; A_u is the end of the range of one-tick quote-cutting; A_l is the lowest ask; and A_h is the highest ask submitted on the equilibrium path.

There are three types of equilibria according to the trader arrival $\pi \in \Pi$. Since the equilibrium quote dynamics are essentially the same for all types, this section explains only the case under $\pi_s = \pi_b = 1/2$, that is, $\alpha = 1$ and $\beta = 1/2$. Furthermore, we set

$v_H = 21$, $v_L = 0$, and $k = 1$. Then, there are 1,261 states of the book. The tick size $k = 1$ satisfies the conditions for the existence of a quote-cutting equilibrium. For these parameter values, $A_l = 5$, $A_u = 15/2$, $A_f = 12$, and $A_h = 15$. Since the critical ask A_u is not on the pricing grid, let $A_u^* \in [A_u, A_u + k) \cap N_k$, i.e., A_u^* is the ask on the pricing grid at or just above A_u, and $A_u^* = 8$.

Figure 1 illustrates the equilibrium quote dynamics. The solid lines indicate the asks and the broken lines the bids. The two horizontal lines in one period mean that the book has two limit orders. For example, the book has a LS at 12 in period 0, and LSs at 11 and 12 in period 1. The point indicates a transaction. For example, a transaction takes place at price 10 in period 10.

In Figure 1, the first ten traders are all sellers. After the first ask posted in an empty book at $A_f = 12$, the best ask decreases tick by tick from $A_f = 12$ to $A_u^* = 8$, then it jumps from $A_u^* = 8$ to $A_l = 5$ by three ticks. When the book has a LS at $A_l = 5$ as period 6, the next seller submits a LS at $A_h = 15$, and the best ask remains at 5. Then, the same cycle of submission of LSs starts over again until a buyer arrives, i.e., a LS at $A_f = 12$ is submitted, and a LS at $A_l = 5$ expires, which makes the best ask rebound seven ticks from 5 to 12. After that, the best ask walks down the pricing grid tick-by-tick. In period 10, a buyer appears, a transaction takes place, and the book becomes empty. Quote dynamics of the bid side are symmetric as shown from period 11 to 20.

During quote competition, quotes jump and holes emerge, causing rapid quote changes. In period 28 in Figure 1, the book has a large hole where an old LS at $A_l = 5$ is posted along with a new LS at $A_h = 15$. If a buyer arrives at this book and submits a MB as shown in period 28, the LS at $A_l = 5$ is executed, and the best ask falls back from $A_l = 5$ to $A_h = 15$. If sellers arrive after such a large quote jump, the best ask returns to $A_f = 12$. On the other hand, if two buyers arrive consecutively when the asks in the book are $A_l = 5$ and $A_h = 15$, as in periods 36 and 37, a transaction at 5 is immediately followed by a transaction at 15. This example suggests that widening the spread is faster than narrowing the spread, and that transaction prices can be volatile due to holes in the book.

5.2 Quote dynamics

The next corollary stems from Theorem 3 in the Appendix, and formally presents the quote dynamics of a quote-cutting equilibrium illustrated in Figure 1.

Corollary 1: In a quote-cutting equilibrium, sellers submit the following LSs on the equilibrium path. A seller submits a LS at A_f to the book with no limit order. If sellers arrive consecutively after the submission of a LS at A_f, the ask submitted to the book declines from $A_f - k$ to A_u^* tick by tick, drops down to A_l, jumps back up to A_h, then falls to A_f. After the return to A_f, the same cycle repeats itself until a buyer arrives. A buyer submits a MB if the book has LSs on the equilibrium path. The bid side is symmetric.

On the equilibrium path, the book does not have both LSs and LBs at the same time because a trader submits a limit order to which future traders on the opposite side of the market will submit market orders.[8]

Sellers compete in quotes in the following way. When the tick size is small, a first seller arriving at an empty book allows future quote-cutting and submits a LS at a high ask A_f. After submission of the first ask A_f to an empty book, one-tick quote-cutting occurs up to A_u^*. Because the asks higher than the most aggressive ask, A_l, will be undercut by the future sellers, sellers undercut the best ask by only one tick to minimize the cost in price to acquire price priority. When the best ask reaches A_u^*, the next seller submits the most aggressive ask, A_l, on the equilibrium path, which he makes low enough to deter further quote-cutting. Preventing further quote-cutting provides the LS at A_l with a discontinuously high execution probability, which compensates for the large cost in price. That is, a discontinuity in execution probability causes the best ask to jump more than one tick. In facing A_l as the best ask, the next seller submits the LS at the least aggressive ask, A_h. Such an order awaits a transaction opportunity in case the limit orders with higher priority are cleared from the book. It is reasonable because the gain in price covers the loss in execution probability. The ask A_f is the optimal to an empty book, implying that A_f is also the optimal to the book whose best ask is higher than A_f. Thus, after the submission of a LS at A_h, the next seller submits a LS at A_f and the same cycle repeats itself.

The first ask submitted to an empty book, A_f, is lower than the ask submitted behind the market, A_h, for the following reason. To extract MBs, a seller needs to compensate future buyers for the expected utility from LBs. A buyer facing the book with a LS submitted one period ago does not compete with the next buyer because the next buyer will submit a MB to the existing LS. This makes the expected utility from a LB to the book with a LS submitted one period ago higher than the expected utility from a LB to the book with a LS submitted two periods ago. As a result, A_f which extract a MB from the next buyer is lower than A_h which extract MBs from buyers two periods ahead. In an equilibrium of a limit order market, there is a marginal trader who is indifferent between a market order and a limit order. On the equilibrium path under the two-period expiration, the buyers facing the book whose best ask is A_f or A_h are marginal traders. During quote-cutting, sellers submit more aggressive LSs, and buyers facing such LSs enjoy the high expected utility from MBs. That is, a market order is more advantageous than a limit order in the process of quote-cutting.

Quote-cutting in Corollary 1 is similar to those reported in Cordella and Foucault (1999) and Foucault *et al.* (2005). Cordella and Foucault (1999) investigate quote com-

[8] Rosu (2006) studies a continuous time model of limit order markets with heterogeneous traders in patience. In his model, the book has either LSs or LBs, but not both at once when all traders are patient. This feature is shared by the outcome of our model. The small tick size seems to be one of causes of this feature because the book has both a LS and a LB on the equilibrium path in Eq 2 of Table 2.

petition between two dealers while Foucault *et al.* (2005) investigate quote competition when a seller and a buyer arrive at the market alternatively. Our quote-cutting equilibrium shows that similar quote dynamics can be observed when public traders arrive at the market randomly. However, they exclude the possibility of submitting limit orders at or behind the best quote by their assumptions. Our equilibrium shows that traders reasonably submit such limit orders, which makes widening the spread faster than narrowing the spread. Traders actually submit limit orders outside the spread. For example, Griffiths *et al.* (2000) report that the ratio of the number of limit orders placed outside the best quotes relative to all orders is 13.16% (11.06%) for sell (buy) orders on the Toronto Stock Exchange. Hasbrouck and Saar (2002) report that the ratio is 30.5% in the Island ECN.

5.3 The level of quotes

In an equilibrium under the one-period expiration, the ask submitted to an empty book is higher than the bid submitted to an empty book as Proposition 1 shows. In a quote-cutting equilibrium under the two-period expiration, similar property holds. However, more aggressive quotes are also posted as the next proposition suggests.

Proposition 5 In a quote-cutting equilibrium, (1) the first ask submitted to an empty book, A_f, is higher than the first bid submitted to an empty book. (2) The most aggressive ask submitted on the equilibrium path, A_l, is lower than the most aggressive bid submitted on the equilibrium path.

Proposition 5(2) shows that some bids exceed some asks on the equilibrium path because of quote competition. Thus, an outside dealer can make a profit if he buys an asset when an ask is low and sells it when a bid is high. This profitable opportunity would attract dealers into limit order markets. We leave the investigation into dealing in limit order markets to future research.[9]

Quotes posted in a book depend on the trader arrival. The following proposition states that the relations between the trader arrival and the critical quotes are similar to those under the one-period expiration.

Proposition 6 In a quote-cutting equilibrium, (1) A_l, A_u, A_f, and A_h decrease in π_s given π_b. The bid side is symmetric. (2) A_l, A_u, A_f, and A_h increase in π_b given π_s. The bid side is symmetric. (3) Suppose $\beta = 1/2$ and $\pi_s = \pi_b = \alpha/2$. The asks A_f and A_l decrease in α. Let $a_1 = 2(\sqrt{2} - 1) \simeq 0.83$. The asks A_h and A_u decrease in α for $\alpha < a_1$ but increase in α for $\alpha > a_1$. The bid side is symmetric.

Propositions 6(1) and (2) correspond to Propositions 2(1) and (2), respectively. The asks decrease when the share of sellers is large or the share of buyers is small. Proposition 6(3) considers the effect of the trader arrival on quotes when sellers and buyers arrive

[9] Bloomfield *et al.* (2005) experimentally show an endogenous liquidity provision in limit order markets.

proportionally. The effect under $\pi_s = \pi_b$ is not simple like Proposition 2. When α is higher and more traders arrive at the market, limit orders are more profitable because of a higher execution probability. In order to attract market orders, a trader has to submit a more aggressive limit order. It follows that the asks A_f and A_l decrease in α. At the same time, there is a counter effect; the bids increase in α symmetrically; aggressive bids reduce the expected utility of buyers; a less aggressive ask is required to attract MBs. This counter effect induces A_h and A_u to increase in α if α is high.

It is not easy to calculate the expected spread in general because it depends on the number of steps of one-tick quote-cutting, which depends on the tick size. However, we conjecture that a higher trader arrival rate is related with a narrower spread similar to Proposition 2(4). This is because A_f is the most frequently observed ask and it decreases in the trader arrival rate, as Proposition 6(3) states. For numerical examples, consider the parameter values $v_H = 21$, $v_L = 0$, $k = 1$, and $\beta = 1/2$. For $\alpha = 1 > a_1$, the expected spread is 14.3 for a quote-cutting equilibrium. For $\alpha = 2/3 < a_1$, there is a numerical example of an equilibrium whose expected spread is 15.8. These examples are consistent with our conjecture.

The execution probability of limit orders can be higher when they expire in two periods rather than one. Such a high execution probability induces traders to submit aggressive limit orders to call for market orders. This leads to the following proposition.

Proposition 7 The difference between the first ask submitted to an empty book, A_f, and the first bid submitted to an empty book under the two-period expiration is smaller than the difference between the ask A_r and the bid B_r under the one-period expiration.

Proposition 7 implies that the expected spread is narrower if limit orders expire in longer periods. In Section 6, we will show numerical examples consistent with this conjecture.

5.4 The size of holes

Some empirical studies observe holes in books. A quote-cutting equilibrium offers an explanation for the existence of holes, which emerge in the book by quote jumps. There are three types of holes on the ask side in a quote-cutting equilibrium: (1) a hole with the size of $A_u - A_l$ when the most aggressive ask A_l is submitted to the book as in period 5, Figure 1; (2) a hole with the size of $A_h - A_l$ when the least aggressive ask A_h is submitted behind the best ask as in period 6, Figure 1; and (3) a hole with the size of $A_h - A_f$ when cyclical quote dynamics resume again from A_f as in period 7, Figure 1. The first hole is observed more frequently than the others.

The size of holes depends not on the tick size but on the trader arrival because quote jumps are caused by the discontinuity of the execution probability. The next proposition concerns the relation between the size of holes and the trader arrival.

Proposition 8 Suppose $\beta = 1/2$ and $\pi_s = \pi_b = \alpha/2$. The sizes of the holes $A_u - A_l$ and $A_h - A_f$ increase in α. Let $a_2 = \sqrt{6} - \sqrt{4\sqrt{6} - 6} \simeq 0.50$. The size of the hole $A_h - A_l$ increases in α for $\alpha > a_2$ and decreases in α for $\alpha < a_2$. The price range of one-tick quote-cutting $A_f - A_u$ decreases in α. The bid side is symmetric.

Discontinuity in execution probability creates quote jumps and holes. If traders arrive at the market more frequently, discontinuity is greater, and the size of a hole is larger. When the best ask is A_u, a seller undercuts it to a larger degree to deter further quote-cutting under the higher trader arrival rate because its gain in the execution probability is larger. This order submission widens the hole with the size $A_u - A_l$. Though the hole with the size $A_h - A_l$ under the small trader arrival rate is an exception, such a hole is observed less frequently.

5.5 Allocational Efficiency

A queuing equilibrium can attain the most efficient allocation as the following proposition states.

Proposition 9 Suppose that limit orders expire in two periods and that the discount factor of every trader is one. (1) A queuing equilibrium attains the most efficient allocation if a trader submits a market order to the book with limit orders on the opposite side of the book on the equilibrium path. (2) A quote-cutting equilibrium cannot attain the most efficient allocation. (3) For a quote-cutting equilibrium, the allocation is more efficient if the tick size is larger.

In this model, whenever a seller meets a buyer, there is a gain of trade because their valuations are different. The probability of a transaction perfectly correlates with the allocational efficiency. When traders are matched on a first-come, first-served basis, the allocation is most efficient. In Table 2, a queuing equilibrium of Eq 3 attains the most efficient allocation because a seller (buyer) submits a MS (MB) to the book with LBs (LSs) on the equilibrium path. In contrast, Eq 2 is a queuing equilibrium, but does not attain the most efficient allocation because a seller submits not a MS but a LS at 2 to the book with a LB at 1 submitted one period ago.

Queuing enhances efficiency whereas quote-cutting reduces efficiency because quote-cutting deprives an early trader of a chance for a transaction. If a trader submits a limit order at or behind the market, such inefficiency does not occur. Thus, the large tick size can enhance allocational efficiency for a quote-cutting equilibrium by making one-tick quote-cutting take fewer steps as Proposition 9(3) states.

5.6 A hybrid of queuing and quote-cutting

As discussed in Section 3.3, the large tick size can make optimal strategies multiple, creating multiple equilibria. This subsection shows some numerical examples of hybrid equilibria of queuing and quote-cutting under the large tick size.

Consider the parameter values of $v_H = 7$, $v_L = 0$, $k = 1$, $\alpha = 1$, and $\beta = 1/2$ under the two-period expiration. In an equilibrium, sellers consecutively submit LSs at 4, 3, 2, then queue at 2. This equilibrium displays both quote-cutting and queuing but not rebound of quotes.

Under the above parameter values, there is another equilibrium where sellers subsequently submit LSs at 2, 4, and the same cycle starts again from 2. The first ask submitted to an empty book is the most aggressive ask on the equilibrium path. In this equilibrium, the best ask remains at 2, and a hole emerges and persists behind the best ask. We can observe the hole statically even though cyclic order submissions are creating the hole.

6 Expiration periods of limit orders

We have assumed that limit orders expire in one or two periods after their submission. In this section, we discuss how expiration periods of limit orders affect cyclic quote dynamics and order composition by numerical examples.

6.1 Cyclic quote dynamics

A quote-cutting equilibrium under the two-period expiration has cyclic quote dynamics. The following examples show that a cycle is observed even if limit orders expire in longer periods. Consider the parameter values $v_H = 14$, $v_L = 0$, $k = 1$, $\alpha = 1$, and $\beta = 1/2$. Under the two-period expiration, there is an equilibrium where sellers submit LSs at 8, 7, 4, and 10 consecutively, and the next seller restarts the same cycle by submitting a LS at 8. Under the three-period expiration, there is an equilibrium where sellers submit LSs at 8, 7, 6, 5, and 3 consecutively, and the same cycle starts over from 8. Under the four-period expiration, there is an equilibrium where sellers submit LSs at 7, 6, 5, 4, and 3 consecutively, the next ask rebounds to 8, and the same cycle starts again.

The above examples suggest that, as limit orders survive over longer periods, quotes become more aggressive. This is consistent with Proposition 7 regarding one- and two-period expiration. In addition, the size of the hole at the most aggressive quote is expected to be smaller. Our reasoning is as follows. If limit orders survive over longer periods, their potential execution probabilities are higher, and the expected utilities from limit orders are greater. Thus, to call for market orders, traders need to submit more aggressive limit orders. Because limit orders already yield the lower expected utility, there is less room for quote jumps, which reduces the size of holes.

6.2 Order composition

To investigate order composition numerically, consider parameter values $v_H = 14$, $v_L = 0$, $k = 1$, and $\beta = 1/2$. Table 3 reports the expected spread and order composition for the trader arrival rate α of 1/3, 2/3, and 1 under the two-, three- and four-period expiration of limit orders. We classify market orders as "Market order," limit orders undercutting the best quote by more than one tick as "Undercutting by more than one tick," limit orders undercutting the best quote by one tick as "Undercutting by one tick," limit orders at the best quote as "At the market," and limit orders behind the best quote as "Behind the market." We classify the other orders, limit orders submitted to an empty book, as "Empty" because our traders frequently submit such orders. For all equilibria, strategies for sellers and buyers are symmetric. Thus, the order composition for sell orders, buy orders, and total orders is the same.

Table 3 shows that the share of market orders and limit orders submitted behind the market increases as the trader arrival rate or the expiration periods increase. One reason is that the book tends not to be empty under the larger trader arrival or under the longer expiration periods. On the other hand, the order compositions for $\alpha = 1$ under the three-period expiration indicates that the share of limit orders undercutting the best quote does not necessarily increase in the expiration periods. The longer expiration periods make quotes more aggressive, which leaves no room for further quote-cutting.

In this model, the share of market orders is the same as the transaction probability, and is perfectly correlated with allocational efficiency. Because the share of market orders and the share of limit orders submitted behind the market correlate, both of them can be indicators of allocational efficiency. In addition, the table shows that the expected spread negatively correlates with the share of market orders. Thus, the size of spread can also be a measure of allocational efficiency. Two points to note are that queuing is more efficient as Proposition 9 states and that there can be equilibrium yielding less efficient allocation with narrower spreads because of multiplicity of equilibria.

6.3 Other types of equilibria

In a quote-cutting equilibrium, quotes do not jump in the middle of one-tick quote-cutting, and a trader does not submit a limit order behind the best quote until the best quote reaches the most aggressive level. The following example shows that these features are not necessarily true if limit orders expire in longer periods partly because of the multiplicity of equilibria due to the positive tick size.

Consider the parameter values $v_H = 14$, $v_L = 0$, $k = 1$, $\alpha = 1$, and $\beta = 1/2$. These parameter values are the same as those in Section 6.1, and there is an equilibrium with cyclic quote dynamics. Under the three-period expiration, there is another equilibrium where sellers submit LSs at 9, 7, 6, 5, 4, 8, 3, 8, 7, 6, 5, and 4 consecutively, then return to the first 8, and the same cycle starts over again. During quote competition, the best ask jumps from 9 to 7. In addition, quotes rebound from 4 to 8 even though 4 is not the most aggressive ask.

Foucault *et al.* (2005) demonstrate by their Corollary 4 that quotes jump more likely when the spread is wide under the large share of patient traders, while quotes jump more likely when the spread is narrow under the large share of impatient traders. Our results are consistent with those of Foucault *et al.* (2005) if the length of expiration of limit orders can be considered to represent the degree of patience of traders because a quote-cutting equilibrium displays quote jumps under the narrow spread while the above example displays quote jumps under the wide spread. Their results depend on the assumptions that sellers and buyers arrive at the market alternately with certainty, that traders cannot submit limit orders at or behind the market, that traders are heterogeneous in patience, and that limit orders survive definitely. A quote-cutting equilibrium demonstrates that quotes jump even if their assumptions are relaxed. In particular, our results clarify that quote competition among homogeneous traders can cause quotes to jump. The heterogeneity of patience introduces another effect as we will discuss in the next section.

With respect to equilibria under the longer expiration periods, we do not investigate them formally but merely show some numerical examples. Are cyclic quote dynamics observed even under the longer expiration periods if the tick size is small enough? Under which long or short expiration periods does queueing more likely occur? These questions remain to be pursued.

7 The effect of patience

We have assumed that the discount factor of every trader is one in order to focus on quote competition. This section discusses the effect of patience to compare our results with those of Foucault *et al.* (2005) who assume that traders are heterogeneous in patience. First, Section 7.1 examines how patience affects spreads when traders are homogeneous in patience. In Section 7.2, we investigate how heterogeneous patience affects order submission strategies and spreads. Section 7.3 discusses implications.

To focus on the effect of patience by putting quote competition aside, we assume that limit orders expire in one period and that the expected utility for a seller is

$$V(s,x,\omega,\pi,\mathbf{x}) = \begin{cases} \delta\Phi(s,A,\omega,\pi,\mathbf{x})(A-v_L) & \text{if } x \text{ is a LS at } A \\ B^*(\omega)-v_L & \text{if } x \text{ is a MS} \end{cases}$$

where δ is the discount factor. The expected utility for a buyer is assumed to be symmetric. We assume that traders differ in patience in the following way.

Assumption 1: A trader is either a patient seller *ps*, an impatient seller *is*, a patient buyer *pb*, or an impatient buyer *ib*. The discount factor of patient traders, *ps* and *pb*, is δ_p while the discount factor of impatient traders, *is* and *ib*, is δ_i. We assume $0 < \delta_i < \delta_p \leq 1$. The share of patient traders is $\gamma \in [0,1]$. In addition, limit orders expire in one period after their submission, while limit orders can be contingent on a price of a real number, and $\beta = 1/2$.

Under Assumption 1, a trader is a patient seller with probability $\alpha\gamma/2$, an impatient seller with probability $\alpha(1-\gamma)/2$, a patient buyer with probability $\alpha\gamma/2$, an impatient buyer with probability $\alpha(1-\gamma)/2$, and no trader otherwise. We consider $\gamma = 1$ as the homogeneous case. Theorem 4 in the Appendix presents an equilibrium under Assumption 1.

7.1 Homogeneous patience

If traders are less patient, the expected utility from a limit order is smaller; the less aggressive limit order is sufficient to extract a market order; and the spread is wider. The next proposition states this relation.

Proposition 10 Under Assumption 1 along with $\gamma = 1$, the smaller the discount factor δ_p is, the larger the expected spread is.

Proposition 10 corresponds to Proposition 2(3). In this case, the ask A_p and the bid B_p submitted on the equilibrium path satisfy the following equations:

$$v_H - A_p = \delta_p \alpha (v_H - B_p)/2, \tag{3}$$

$$B_p - v_L = \delta_p \alpha (A_p - v_L)/2. \tag{4}$$

If $\delta_p = 1$, these equations are the same as Equations (1) and (2) under $\pi_s = \pi_b = \alpha/2$. Equations (3) and (4) indicate that the execution risk represented by α and the cost of execution delay represented by δ_p have exactly the same effect on quotes in this model.

7.2 Heterogeneous patience

In an equilibrium under heterogeneous patience, if the degree of heterogeneity in patience is large, only impatient traders submit market orders on the equilibrium path. The following corollary stemming from Theorem 4 in the Appendix states this property. Let $\hat{\alpha} = 2\gamma/(1-\gamma)/(\delta_p - \delta_i)$.

Corollary 2: Under Assumption 1, if $\alpha < \hat{\alpha}$, every trader submits a market order to the book with a limit order on the equilibrium path. If $\alpha > \hat{\alpha}$, only an impatient trader submits a market order to the book with a limit order on the equilibrium path. If $\alpha = \hat{\alpha}$, there are multiple equilibria.

If γ is large or if $\delta_p - \delta_i$ is small, $\alpha < \hat{\alpha}$. In this case, heterogeneity of patience is small, and tradings are similar to tradings under homogeneous patience. On the other hand, if γ is small or if $\delta_p - \delta_i$ is large, $\alpha > \hat{\alpha}$. In this case, heterogeneity of patience is large, and a trader submits a less aggressive limit order to which only impatient traders submit market orders.

The reason is that impatient traders are desperate to trade and are thus inclined to submit market orders. Hence, a seller has two options in submitting a LS, a conservative LS to trade only with impatient buyers and an aggressive LS to trade both with patient buyers and impatient buyers. If the share of impatient buyers $1-\gamma$ is large, the cost in the execution probability to give up trading with patient buyers is small. If the difference of patience $\delta_p - \delta_i$ is large, the benefit in price of submitting a conservative LS is large because more impatient buyers submit market orders to less aggressive LSs. Thus, if $\alpha > \hat{\alpha}$, a seller chooses a conservative LS, and the next patient buyer submits not a MB but a conservative LB. Patient buyers can wait and let transaction opportunities pass in order to trade at a more favorable price with a future impatient seller.

Large heterogeneity in patience makes quotes less aggressive and widens the expected spread as the next proposition states. Let $\hat{\gamma} = \alpha(\delta_p - \delta_i)/(2 + \alpha(\delta_p - \delta_i))$. $\alpha > \hat{\alpha}$ if and only if $\gamma < \hat{\gamma}$.

Proposition 11 Under Assumption 1, (1) suppose that the parameter values except α are constant and that $\hat{\alpha} < 1$. The expected spread under $\alpha = \hat{\alpha} - \epsilon$ is strictly narrower than the expected spread under $\alpha = \hat{\alpha} + \epsilon$ for small $\epsilon > 0$. (2) Suppose the parameter values except γ are constant. The expected spread under $\gamma > \hat{\gamma}$ is narrower than the expected spread under $\gamma < \hat{\gamma}$.

Figure 2 illustrates Proposition 11(1). The figure depicts the relation between the trader arrival rate α and the expected spread under $v_H = 1$, $v_L = 0$, $\beta = 1/2$, $\delta_p = 1$, and $\delta_i = 0.2$. The solid line, broken line, and dash-dotted line depict the expected spread under $\gamma = 0.1$, $\gamma = 0.25$, and $\gamma = 0.6$, respectively. Basically, the higher trader arrival rate relates with the narrower spread as Proposition 2(4) states. Heterogeneity of patience introduces a large change in the expected spread at $\hat{\alpha}$. In addition, because $\hat{\alpha}$ increases in γ, a large change occurs at the higher α if γ is higher as Figure 2 shows. These results are similar to numerical examples of Table 3 in Foucault *et al.* (2005). For empirical implications, we expect that the spreads can change drastically in some situations if heterogeneity in patience is discrete and large.

Figure 3 illustrates Proposition 11(2). The figure depicts the relation between the share of patient traders γ and the expected spread under $v_H = 1$, $v_L = 0$, $\alpha = 1$, $\beta = 1/2$, $\delta_p = 1$, and $\delta_i = 0.2$. The points A, B, and C correspond to the respective points A, B, and C in Figure 2. The figure shows that as the proportion of patient traders increases, the spread narrows drastically at $\hat{\gamma}$.

If the share of patient traders γ is larger than the critical level $\hat{\gamma}$, traders submit aggressive limit orders in order not to miss any transaction opportunity. On the other hand, if γ is smaller than $\hat{\gamma}$, traders submit less aggressive limit orders, which widens the spread and, reduces allocational efficiency as the next proposition states.

Proposition 12 Under Assumption 1, suppose the parameter values except γ are constant. The transaction probability is greater under $\gamma \in (\hat{\gamma}, 1]$ than under $\gamma \in (0, \hat{\gamma})$.

Thus, the narrow spread indicates more efficient allocation under heterogeneity in patience without quote competition.

7.3 Discussion

We have separately analyzed the effect of quote competition and the effect of heterogeneity in patience. Foucault *et al.* (2005) have analyzed both effects simultaneously. They define that the market is more resilient if "the probability that the spread reverts to its competitive level before the next transaction occurs" is larger. It means that the market is resilient if the spread reaches the most aggressive level rapidly. They show that the market is more resilient and the spread is narrower if the share of patient traders is larger.

Figure 3 depicts the relation between the share of patient trader γ and the expected spread without the effect of quote competition. In contrast to the prediction of Foucault *et al.* (2005), the figure does not show the negative relation. The spread is very wide under the middle range of γ like Point B in the figure. This inconsistency suggests that resiliency for the middle share of patient traders depends not only on heterogeneity in patience but also on quote competition. If the shares of patient traders are nearly equal to the share of impatient traders, and if limit orders survive longer periods, patient traders would compete in quote for market orders, which makes the market resilient and the spread narrow.

On the other hand, as Proposition 3 implies, traders do not compete in price but tend to queue at the same price if they expect plentiful market orders. Given α, if γ is small, a certain amount of market orders are submitted by impatient traders. If γ is large, market orders are plentiful because every trader submits a market order. Thus, for small and large γ like Points A and C but not for medium γ like Point B in Figure 3, patient traders do not fiercely compete in quote. This suggests that not quote competition but rather heterogeneous patience can be important for resiliency and spreads under small and large γ.

As a whole, our analysis suggests that heterogeneity in patience is an important factor for weak and strong resiliency and quote competition is an important factor for medium resiliency. These two factors can cause the negative relation between the share of patient traders and resiliency as Foucault *et al.* (2005) predict. This article investigates how homogeneous traders compete in quotes to focus on quote competition. We leave a study on how heterogeneous traders compete in quotes under longer expiration periods of limit orders for future research.[10]

[10] Goettler *et al.* (2005) investigate the effect of heterogeneity in valuation. In our framework, heterogeneity in valuation and heterogeneity in patience yield similar equilibrium at least under the following assumption. Let the valuation of sellers be $v_L - \rho\Delta$, and let the valuation of buyers be $v_H + \rho\Delta$. For each

8 Empirical implications

Now, we summarize empirical implications on quote jumps and holes, and spreads. At the end of this section, we discuss the effect of a tick size reduction.

8.1 Quote jumps and holes

Quote dynamics in a quote-cutting equilibrium show that spreads can change dynamically, and that there can be no single equilibrium value of the spread. During quote-cutting, small quote improvements occur in succession at first, followed by a large quote jump. The sizes of quote jumps are larger when the trader arrival rate is higher. In contrast, widening the spread takes few steps, and is faster than narrowing the spread.

Several empirical studies have observed holes in the book. We predict that some holes are related with quote jumps. The size of a hole is large if the trader arrival rate is high. Holes are observed more frequently just behind the best quote if traders can quickly respond to the transition of the book.

A tick size affects the number of steps of one-tick quote-cutting, and does not affect the size of quote jumps and holes. However, how often holes are observed depends on the tick size. If the tick size is large, quote jumps can be observed less frequently because the extent of a quote jump can match the tick size. On the other hand, if the tick size is too small, quote jumps are observed less frequently because of the long steps involved in one-tick quote-cutting. Thus, we expect that quote jumps and holes are observed frequently for a certain tick size.

As a consequence of holes, distributions of quotes and transaction prices can be fat-tailed. For the example of a quote-cutting equilibrium depicted in Figure 1, transactions at $A_l = 5$ or $A_h = 15$ take place though transactions by MBs at the adjacent asks 6, 7, 13, and 14 do not take place because sellers do not submit LSs at these prices. The kurtosis of transaction prices for this example is 4.9, which is higher than that for the normal distribution, i.e., 3.

Another possible implication of quote jumps relates to quote clustering. Harris (1991) and Chung, Van Ness, and Van Ness (2004), among others, observe that prices in securities markets tend to cluster at prices whose final digits are 0 or 5. Suppose that for some reason there are some traders who prefer prices divisible by 5. Other traders expect that limit orders at quotes ending at 0 will be submitted in the future. When the final digit of the current ask is 5, a seller would submit a LS at an ask ending at 4 to enjoy the high price in spite of expecting future quote-cutting. On the other hand, if the final digit of the current ask is 2, a seller would skip one tick and submit a LS at an ask ending at 0 to obtain the higher execution probability by submitting the clustering price himself. These

seller and buyer, a trader is either a low type with $\rho = 0$ and a high type with $\rho > 0$. The share of the low type traders is γ. Under this setting, a low (high) type trader corresponds to a patient (impatient) trader.

order submission strategies lead the final digits of asks to cluster at 0 followed by 4, 3, 2, and 1, and to cluster at 5 followed by 9, 8, 7, and 6. A symmetric argument predicts that the final digits of bids cluster at 0 followed by 6, 7, 8, and 9, and also cluster at 5 followed by 1, 2, 3, and 4. Chung, Van Ness, and Van Ness (2004) present data partially consistent with this prediction in their Table 4 (but for the NASDAQ and the New York Stock Exchange rather than for pure limit order markets).

8.2 The spreads

If the tick size is small, the spread changes dynamically, and the average spread is narrow for frequently traded stocks. If the tick size is large and if the stocks are traded frequently, the spread can be stable because of queuing. Moreover, transactions can take place at a single price like transactions in a call auction. In such a case, the spread is meaningless for the measure of transaction costs.

If traders differ in patience discretely, the size of spreads can change discontinuously in relative to the change in the trader arrival rate. According to Foucault *et al.* (2005), if traders become more impatient toward the market closing, we may be able to observe the discontinuous increase in the spread near the market closing.

8.3 A reduction in the tick size

Using numerical examples, we discuss the effect of a tick size reduction on spreads. In summary, our model predicts that a large reduction in the tick size would expand spreads by extending the convergence of quotes to the most aggressive level. This widens the average spread and undermine allocational efficiency. These effects are also discussed in Foucault *et al.* (2005). In addition, we show that if the extent of a tick-size reduction is small, the change of spreads can be ambiguous due to multiplicity of equilibria.

Consider the parameter values $v_H = 21$, $v_L = 0$, $\alpha = 1$, and $\beta = 1/2$. Table 4 presents the expected spread and order composition for six equilibria. The classification of order composition is the same as in Table 3. Eq 5 is a quote-cutting equilibrium under $k = 1$ whose quote dynamics Figure 1 depicts. Eq 6 is a quote-cutting equilibrium under $k = 1/2$. We work out equilibrium strategies numerically for $k = 7$ and $k = 3$. Eq 1, Eq 2, and Eq 3 are equilibria under $k = 7$ whose strategies are presented in Table 2.[11] Eq 4 is an equilibrium under $k = 3$. Theorem 2 states that a queuing equilibrium can exist under $k = 7$ and $k = 3$ but cannot exist under $k = 1$ and $k = 1/2$. Indeed, Eq 2 and Eq 3 are queuing equilibria under $k = 7$.[12]

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[11] Table 2 presents equilibria under $v_H = 3$ and $k = 1$ while Table 4 presents equilibria under $v_H = 21$ and $k = 7$.

[12] For example, Biais *et al.* (1995) report that, in the Paris Bourse, the share of limit orders undercutting the best quote to all orders is 17.3%, the share of limit orders at the market is 9.6%, and the share of limit orders behind the market is 14.4%. These shares are similar to those for Eq 1 in Table 4.

As for the expected spread, Table 4 shows that the smaller tick size is related to the wider expected spread if the tick size is smaller than 3. The fine pricing grid delays the convergence of quotes to the most aggressive level, which widens the expected spread. Thus, a tick size reduction would expand the spread if the tick size is already small.

On the other hand, if the tick size is large, the effect of a tick size reduction on the spread can be ambiguous. In Table 4, the expected spread is 13.8, 13.2, or 10.2 under $k = 7$, against 13.6 under $k = 3$. Thus, the effect of the reduction from $k = 7$ to $k = 3$ depends on which equilibrium is attained before the reduction. The results of an empirical investigation by Bourghelle and Declerck (2004) in the Paris Bourse are consistent with this ambiguity.

Though the effect of a tick size reduction is ambiguous, an interesting case is the transition from Eq 1 under $k = 7$ to Eq 4 under $k = 3$. From Eq 1 to Eq 4, the share of limit orders undercutting the best quote increases from 16.2% to 27.5%, the expected spread decreases from 13.8 to 13.6, and the share of limit orders at the market decreases from 8.1% to 0%. Thus, more frequent quote-cutting is related to a narrower spread and smaller depth. This relation is observed by Bacidore (1997) in the Toronto Stock Exchange, by Bourghelle and Declerck (2004) in the Paris Bourse, and by Chung, Charoenwong, and Ding (2004) in the New York Stock Exchange.

Goettler *et al.* (2005) argue that a tick size reduction can be welfare improving, though their arguments are based on numerical examples and the driving force of their results is not clear. Proposition 9(3) predicts that if traders are homogeneous, a large tick size reduction can undermine allocational efficiency by depriving the transaction opportunity from traders who arrive at the market early. On the other hand, if traders differ greatly in patience or valuation, a tick size reduction can improve welfare. Large heterogeneity in patience among traders widens the spread and reduces allocational efficiency as Propositions 11 and 12 suggests. A tick size reduction can promote quote-cutting, and traders submit more aggressive orders, which can improve efficiency. We conjecture that a tick size reduction enhances allocational efficiency if heterogeneity among traders is large and if a tick size reduction is sufficient.

9 Concluding remarks

We investigate how traders submit orders to limit order markets and how quotes change in equilibrium by assuming that limit orders expire in certain periods after their submission. This assumption simplifies the state space of the book, whereas it does not eliminate competition among limit orders.

We present cyclic quote dynamics with quote jumps. A trader undercuts the best extant quote by more than one tick if such an order can prevent future traders from undercutting it further, because its gain in execution probability compensates for its cost in price. When the book has aggressive limit orders, a trader may submit a limit order behind the market because its gain in price pays for its loss in execution probability. These order submission strategies cause holes to emerge in the book. On the other hand, if the tick size is large, traders can queue at the same quote because the cost in price of quote-cutting is high.

Our model yields the following original insight on limit order markets. (1) Quote jumps can create holes at the most aggressive quote, and the sizes of holes are large if the trader arrival rate is high. (2) If holes are frequently observed, distributions of quotes and transaction prices can be fat-tailed. (3) Limit orders submitted behind the market are reasonable, and such orders make widening the spread faster than narrowing it. (4) Queuing of limit orders can be observed if the tick size is large and the trader arrival rate is high. (5) The effect of a tick size reduction on spreads can be ambiguous because of multiplicity of equilibria. (6) If traders are largely heterogeneous in patience, the spread is wide and allocation is not efficient. Investigating the empirical relevance of these features for limit order markets is an interesting topic for future empirical studies.

Finally, we end this article with a discussion of two issues left for future research. We assume that the size of an order is one and that limit orders expire automatically. An important question is how traders compete in price if they submit multiple orders with cancellation and resubmission. However, in such a situation, the decision problem of traders is dynamic, and the model has difficulty in solving it. Another question is how asymmetric information affects order submission strategies and the shape of the book. What information the book has and what information traders try to extract from the book are interesting questions.

Appendix

In this appendix, we first describe some notations to specify the model. Next, we define a queuing equilibrium in Definition 1 together with remarks, and prove Theorem 2. After that, we explain asks in a quote-cutting equilibrium in Theorem 3, followed by the definitions of variables, Theorem 3 itself, along with some relevant remarks, and then its proof. Next, we present Theorem 4 describing an equilibrium under heterogeneity in patience, and prove it. Finally, we prove the propositions.

A.1 Some notations

First, to describe the model specifically, let the decision process of a trader in period τ be divided into two stages. At stage 1, he decides whether or not he will submit a market order. If he submits a market order, period τ terminates and it passes to the next period. If he does not, he proceeds to stage 2 and decides whether or not he will submit a limit order. After his submission of a limit order or no order, period τ terminates and it passes to the next period.

Next we define the set of states of the book under the two-period expiration. When limit orders expire in two periods, the book has at most two limit orders. The possible limit orders in the book in period τ are a LS submitted in $\tau-2$, a LS in $\tau-1$, a LB in $\tau-1$, and a LB in $\tau-2$. Let $A_{-\tau'}$ ($B_{-\tau'}$) be the quote of a LS (LB) submitted in τ' periods before τ. y^s_{-2} (y^s_{-1}) takes the value A_{-2} (A_{-1}) if a LS is submitted in $\tau-2$ ($\tau-1$) and if it is still listed in the book in τ, but takes n otherwise. Similarly, let y^b_{-2} (y^b_{-1}) take B_{-2} (B_{-1}) if a LB submitted in $\tau-2$ ($\tau-1$) is in the book in τ, but n otherwise. By these notations the book is represented by $\omega=(y^b_{-2},y^b_{-1},y^s_{-2},y^s_{-1})$. The set of states of the book Ω is

$$\begin{aligned}\Omega = \{&(n,n,n,n),\ (n,n,n,A_{-1}),\ (n,n,A_{-2},n),\ (n,B_{-1},n,n),\ (B_{-2},n,n,n),\\ &(n,n,A_{-2},A_{-1}),\ (B_{-2},B_{-1},n,n) : A_{-1},A_{-2},B_{-1},B_{-2}\in N_k\cap(v_L,v_H)\}\cup\\ &\{(B_{-2},n,n,A_{-1}),\ (n,B_{-1},A_{-2},n), : A_{-1},A_{-2},B_{-1},B_{-2}\in N_k\cap(v_L,v_H),\\ &\quad B_{-2}<A_{-1},B_{-1}<A_{-2}\}.\end{aligned}$$

We restrict asks and bids in (v_L,v_H) because traders never choose the other quotes in equilibrium as footnote 5 explains. In addition, from Ω we exclude books in which the best bid exceeds the best ask because marketable limit orders are immediately executed under our trading rule.

To indicate traders, let t_0 be a trader in a given period. t_s and t_b denote a seller and a buyer in the following period, respectively. A seller and a buyer after t_s (t_b) are denoted by t_{ss} (t_{bs}) and t_{sb} (t_{bb}), respectively. If no trader arrives following t_0, a seller and a buyer in the two periods ahead of t_0 are denoted by t_{ns} and t_{nb}. Figure 4 illustrates these notations.

A.2 Definition of queuing equilibrium

Definition 1: Let A_q be the ask at which a seller submits a LS to an empty book. Similarly, let B_q be the bid at which a buyer submits a LB to an empty book. We call an equilibrium satisfying the following (q1) to (q8) a queuing equilibrium under the two-period expiration of limit orders.

(q1) A seller submits a LS at A_q to (n, n, A_q, A_q).
(q2) A seller submits a LS at A' such as $A' \geq A_q$ to (n, n, A, A_q) where $A < A_q$.
(q3) A buyer submits a MB to (n, n, A, y^s_{-1}) where $A \leq A_q$ and $^\forall y^s_{-1}$.
(q4) Let A_m be

$$A_m = \max_{A \in N_k} A \ s.t. \quad v_H - A \geq \pi_s(1 + z + \pi_b)(v_H - B_q).$$

When $A_m > A_q$, a seller submits a LS at A' such as $A' < A_m$ to (n, n, A_q, A_m) and a buyer submits a MB to (n, n, A_m, n).
(q5) A buyer submits a LB at B_q to (B_q, B_q, n, n).
(q6) A buyer submits a LB at B' such as $B' \leq B_q$ to (B, B_q, n, n) where $B > B_q$.
(q7) A seller submits a MS to $(B, y^b_{-1} n, n)$ where $B \geq B_q$ and $^\forall y^b_{-1}$.
(q8) Let B_m be

$$B_m = \min_{B \in N_k} B \ s.t. \quad B - v_L \geq \pi_b(1 + z + \pi_s)(A_q - v_L).$$

When $B_m < B_q$, a buyer submits a LB at B' such as $B' > B_m$ to (B_q, B_m, n, n) and a seller submits a MS to (B_m, n, n, n).

Remark 1: The conditions from (q1) to (q4) are those for sellers to queue at A_q. The conditions from (q5) to (q8) are those for buyers to queue at B_q. Though the conditions characterizing queuing itself are (q1) and (q5), we require the other conditions in proving Theorem 2. We explain (q2), (q3), and (q4) in the following Remarks 2, 3, and 4, respectively. We discuss the necessity and sufficiency of these conditions in Remarks 5 and 6, respectively.

Remark 2: The condition (q2) demands that a seller does not follow the quote-cutting that occurred two periods ago. Under (q1), (q2) is not satisfied only if the optimal strategy for a seller facing (n, n, n, A_q) is not unique because the optimal strategy for a seller facing (n, n, y^s_{-2}, A_q) is the same for $^\forall y^s_{-2}$. There are numerical examples of equilibria satisfying (q1) but not (q2) under the large tick size.

Remark 3: The condition (q3) demands that a buyer submits a MB to (n, n, A_q, A_q). In general, a seller calls for MBs by submitting an aggressive order. However, if the tick size is large, the cost in price to submit aggressive orders is so high that a seller reluctantly permits the next buyer not to submit a MB, causing (q3) not to be satisfied. There are numerical examples of equilibria where (q3) are not satisfied under the large tick size.

Remark 4: The condition (q4) demands that a seller submits a LS at A_q if the previous seller submits a LS at A_m such as $A_m > A_q$. Under (q5) and (q7), buyers queue at B_q and sellers submit MSs to the queue. Thus, a buyer facing an empty book obtains the expected utility $\pi_s(1+z+\pi_b)(v_H - B_q)$, which is the highest for buyers on the equilibrium path. We define A_m as the maximum ask for a seller t_0 to trade with the buyers t_{bb} and t_{nb} by giving them this highest expected utility. The condition (q4) also requires that the buyers actually submit MBs to a LS at A_m. There are numerical examples of equilibria where (q4) are not satisfied under the large tick size.

Remark 5: The conditions from (q1) to (q8) are not necessary for transactions to take place only at A_q or B_q. There are numerical examples of equilibria where transactions take place at A_q or B_q even though they do not satisfy every condition of the definition. Our conjecture is that (q2)-(q4) and (q6)-(q8) are satisfied if the tick size is smaller than a certain level and that Theorem 2 or its modified version holds for a queuing equilibrium defined by only (q1) and (q5). It is an open question whether our conditions specifying queuing equilibrium can be relaxed.

Remark 6: The conditions from (q1) to (q8) are not sufficient for transactions to take place only at A_q or B_q. For example, the definition does not require that a seller submit a LS at A_q to (n, n, A_q, n) and (n, n, n, A_q), or that he submits a MS or a LS at A_q to (n, B_q, n, n). The omission of these requirements enables our queuing equilibria to allow transactions at different prices from A_q or B_q. There are numerical examples of such equilibria. Even so, we do not include these requirements in our definition since we do not use them in proving Theorem 2.

A.3 Proof of Theorem 2

The proof proceeds as follows. First, we consider the conditions where a seller has no incentive to submit a LS behind A_q. Second, we consider the conditions where a seller has no incentive to undercut A_q. Combining these conditions yields Theorem 2.

First, we examine the conditions where a seller has no incentive to submit a limit order behind A_q. Suppose $A_m > A_q$. A seller t_0 facing (n, n, n, A_q) gets the expected utility $\pi_b(A_q - v_L)$ by submitting a LS at A_q because his LS is matched with MBs of t_{sb}, t_{nb}, and t_{bb} due to (q1) and (q3). Similarly, t_0 gets the expected utility $\pi_b(z + \pi_b)(A_m - v_L)$ by submitting a LS at A_m due to (q3) and (q4). A seller prefers A_q to A_m if

$$\pi_b(A_q - v_L) \geq \pi_b(z + \pi_b)(A_m - v_L). \tag{5}$$

If $A_m > A_q$, a queuing equilibrium satisfies Condition (5). If $A_m \leq A_q$, Condition (5) is always satisfied. Thus, Condition (5) is satisfied for every queuing equilibrium.

By the definition of A_m, $\pi_s(1 + z + \pi_b)(v_H - B_q) > v_H - A_m - k$. Applying this to Condition (5) yields

$$A_q - v_L > (1 - \pi_s)(\Delta - k) - \pi_s(1 - \pi_s)(2 - \pi_s)(v_H - B_q). \tag{6}$$

The symmetric condition for a buyer is

$$v_H - B_q > (1-\pi_b)(\Delta - k) - \pi_b(1-\pi_b)(2-\pi_b)(A_q - v_L). \tag{7}$$

Every queuing equilibrium satisfies Conditions (6) and (7).

Next, we examine the conditions where a seller has no incentive to undercut A_q. There can be four cases according to whether or not successive quote-cutting occurs after one quote-cutting: (1) successive quote-cutting occurs on neither the ask nor bid side. (2) successive quote-cutting occurs on both the ask and bid side. (3) successive quote-cutting occurs only on the ask side. (4) successive quote-cutting occurs only on the bid side. In any case, a seller facing the book (n, n, y^s_{-2}, A_1) for $^\forall y^s_{-2}$ and $A_1 \in (v_L, A_q]$ gets the expected utility $\pi_b(A_q - v_L)$ by submitting a LS at A_q because of (q1), (q2), and (q3).

Case 1: Suppose there is no successive quote-cutting after one quote-cutting. If a seller t_0 facing (n, n, n, A_q) submits a LS at A such as $A < A_q$ and if t_s does not undercut A_q, t_0 gets the expected utility $\pi_b(1 + z + \pi_s)(A - v_L)$ due to (q3). The optimal ask in this case is $A = A_q - k$. Thus, t_0 does not undercut A_q but queues at A_q if

$$\pi_b(A_q - v_L) \geq \pi_b(1 + z + \pi_s)(A_q - k - v_L).$$

This condition is transformed to

$$A_q \leq v_L + (2-\pi_b)k/(1-\pi_b). \tag{8}$$

The symmetric condition for a buyer not to undercut B_q is

$$B_q \geq v_H - (2-\pi_s)k/(1-\pi_s). \tag{9}$$

The queuing equilibrium of Case 1 satisfies Conditions (6), (7), (8), and (9). Accordingly, if there is no A_q and B_q satisfying these four conditions, a queuing equilibrium does not exist. There is no A_q and B_q satisfying these four conditions if

$$A_x - v_L \leq (1-\pi_s)(\Delta - k) - \pi_s(1-\pi_s)(2-\pi_s)(v_H - B_x) \tag{10}$$

or

$$v_H - B_x \leq (1-\pi_b)(\Delta - k) - \pi_b(1-\pi_b)(2-\pi_b)(A_x - v_L) \tag{11}$$

are satisfied at $A_x = v_L + (2-\pi_b)k/(1-\pi_b)$ and $B_x = v_H - (2-\pi_s)k/(1-\pi_s)$. Condition (10) is transformed to $k \leq k^s_q\Delta$, and Condition (11) is transformed to $k \leq k^b_q\Delta$. Thus, there is no queuing equilibrium in Case 1 if $k \leq k^s_q\Delta$ or $k \leq k^b_q\Delta$.

Case 2: Suppose there is successive quote-cutting both on the ask and bid side. If a seller t_0 facing (n, n, n, A_q) submits a LS at $A_q - n_1k$ for some integer $n_1 \geq 1$ and if the next seller t_s undercuts the ask submitted by t_0, t_0 gets the expected utility $\pi_b(1 + z)(A_q - n_1k - v_L)$ due to (q3). Thus, t_0 queues at A_q if

$$\pi_b(A_q - v_L) \geq \pi_b(1 + z)(A_q - n_1k - v_L). \tag{12}$$

At the same time, there is a LS at $A_q - n_2 k$ for some integer $n_2 > n_1$ which is not undercut further. The execution probability of this LS is $\pi_b(1+z+\pi_s)$. A seller has an incentive to submit this LS if

$$\pi_b(1+z+\pi_s)(A_q - n_2 k - v_L) \geq \pi_b(A_q - v_L). \tag{13}$$

For queuing at A_q, Condition (13) must be satisfied in equality, otherwise a seller prefers $A_q - n_2 k$ to A_q. For successive quote-cutting off the equilibrium path, Condition (12) must be satisfied in equality, otherwise a seller prefers $A_q - n_2 k$ to $A_q - n_1 k$. Moreover, Condition (12) is satisfied in equality for $n_1 = 1$ since

$$\pi_b(1+z)(A_q - k - v_L) > \pi_b(A_q - v_L) = \pi_b(1+z)(A_q - n_1 k - v_L)$$

if $n_1 > 1$, which contradicts a seller's preference for A_q over $A_q - k$.

If $z = 0$, Condition (12) with $n_1 = 1$ is not satisfied in equality, implying there is no queuing equilibrium in Case 2. If $z > 0$,

$$A_q = v_L + (1+z)k/z \tag{14}$$

when Condition (12) with $n_1 = 1$ is satisfied in equality. The symmetric condition for the bid side is $z > 0$ and

$$B_q = v_H - (1+z)k/z. \tag{15}$$

Thus, if $z > 0$, the queuing equilibrium in Case 2 satisfies Conditions (6), (7), (14), and (15).

By the same argument as in Case 1, there is no queuing equilibrium in Case 2 under $z > 0$ if $A_x = v_L + (1+z)k/z$ and $B_x = v_H - (1+z)k/z$ do not satisfy Conditions (10) or (11). These conditions are transformed to

$$k/\Delta \leq z(1-\pi_s)/\{(1+z)(1+\pi_s(1-\pi_s)(2-\pi_s)) + z(1-\pi_s)\} < k_q^s$$

or

$$k/\Delta \leq z(1-\pi_b)/\{(1+z)(1+\pi_b(1-\pi_b)(2-\pi_b)) + z(1-\pi_b)\} < k_q^b.$$

Thus, there is no queuing equilibrium of Case 2 if $k \leq k_q^s \Delta$ or $k \leq k_q^b \Delta$.

Under Case 3, the queuing equilibrium satisfies Conditions (6), (7), (14), and (9). By the same argument as in Case 1, there is no queuing equilibrium in Case 3 if $k \leq k_q^s \Delta$ or $k \leq k_q^b \Delta$. Case 4 is symmetric to Case 3. In brief, there is no queuing equilibrium if $k \leq k_q^s \Delta$ or $k \leq k_q^b \Delta$. Q.E.D.

A.4 Explanation of asks in Theorem 3

This subsection explains how and why the asks in a quote-cutting equilibrium are defined. The next subsection will offer the formal definitions. The following explanation bases on the proof of the optimality of strategies (s1) and (s5) in Theorem 3.

Suppose that a buyer facing an empty book obtains the expected utility $\bar{v}_f^b$ by submitting an optimal LB. Though $\bar{v}_f^b$ is exogenous for a while, we endogenize it in the end. Let $A_h = v_H - \bar{v}_f^b$. A_h is the maximum ask satisfying a participation constraint of a buyer $v_H - A \geq \bar{v}_f^b$. A seller t_0 can trade with the buyers in two periods ahead, t_{sb}, t_{nb}, and t_{bb} by submitting a LS at A such as $A \leq A_h$ if the next traders do not interfere.

Suppose that a buyer t_b facing the book with a LS of a seller t_0 obtains the expected utility $\bar{v}_m^b$ by submitting not a MB but rather an optimal LB. Similar to $\bar{v}_f^b$, $\bar{v}_m^b$ is exogenous for a while, and is endogenized in the end. Let $A_f = v_H - \bar{v}_m^b$. A seller t_0 can trade with the next buyer t_b by submitting a LS at A such as $A \leq A_f$. We suppose $\bar{v}_m^b > \bar{v}_f^m$ and $A_f < A_h$, which is confirmed in the end. In the following, we exclude LSs at A such as $A_h < A$ because no buyer submits a MB to such a high ask.

Let $v_h^s = \pi_b(z+\pi_b)(A_h - v_L)$. A seller obtains at least v_h^s to any book. This is because if a seller t_0 submits a LS at A_h, he cannot trade with t_b because $A_f < A_h$, and he may not be able to trade with t_{sb} because t_s may undercut A_h, but he can trade with t_{nb} and t_{bb}. That is, v_h^s is the reservation utility for a seller.

Using v_h^s, let A_u be the ask satisfying $v_h^s = \pi_b(1+z)(A_u - v_L)$. A LS at A_u yields v_h^s to a seller t_0 by trading with t_b and t_{nb}. Similarly, let A_l be the ask satisfying $v_h^s = \pi_b(1+z+\pi_s)(A_l - v_L)$. A LS at A_l yields v_h^s to a seller t_0 by trading with t_b, t_{nb}, and t_{sb}. In addition, let $v_f^s = \pi_b(1+z)(A_f - v_L)$. v_f^s is the expected utility for a seller facing an empty book as we will show in the following. Figure 5 depicts the relation between these asks and the expected utilities. As the figure shows, $A_l < A_u < A_f < A_h$.

Now, we present the equilibrium strategies for a seller. If the book is empty, the optimal ask for a seller is A_f. If the book has a LS at A_{-1} submitted one period ago, the optimal ask for a seller is A_h if $A_{-1} \leq A_l$, A_l if $A_l < A_{-1} < A_u + k$, $A_{-1} - k$ if $A_u + k \leq A_{-1} \leq A_f$, and A_f if $A_f < A_{-1}$. The optimal ask does not depend on the ask of a LS submitted two periods ago because limit orders expire in two periods. Given these strategies, a seller t_0 can trade with t_{sb} if he submits a LS at A_l because such an aggressive ask deters t_s from further quote-cutting. If a seller submits a LS at A such as $A > A_l$, he can not trade with t_{sb}. In what follows, we show these asks are indeed optimal.

Suppose a seller t_0 faces an empty book. If he submits a LS at A such as $A \leq A_l$, the next seller t_s submits a LS at A_h, t_0 can trade with t_{sb}, t_{nb}, and t_b, and his expected utility is $\pi_b(1+z+\pi_s)(A - v_L)$. Among these asks, A_l yields the maximum expected utility v_h^s. If t_0 submits a LS at A such as $A_l < A \leq A_f$, t_s undercuts his ask, t_0 can trade with t_{nb} and t_b, and his expected utility is $\pi_b(1+z)(A - v_L)$. Among these asks, A_f yields the maximum expected utility v_f^s. If t_0 submits a LS at A such as $A_f < A \leq A_h$, t_s submits a LS at A_f, t_0 can trade with t_{nb} and t_{bb}, and his expected utility is $\pi_b(z+\pi_b)(A-v_L)$. Among these asks, A_h yields the maximum expected utility v_h^s. In Figure 5, the solid lines depict the expected utility for a seller facing an empty book with respect to the ask at which he submits a LS. The figure shows that the optimal order is a LS at A_f yielding v_f^s. The figure also suggests that a seller chooses an ask $A \in \{A_l, A_h\} \cup [A_u, A_f]$ because these asks yield the expected utility equal to or higher than the reservation utility v_h^s.

Suppose a seller t_0 faces the book with a LS at A_{-1} such as $A_{-1} \leq A_l$ submitted in the previous period. We will show that the optimal ask is A_h because the other asks yield the expected utility less than the reservation utility. If he undercuts A_{-1}, the next seller t_s submits a LS at A_h, t_0 can trade with t_{sb}, t_{nb}, and t_b, and his expected utility is $\pi_b(1+z+\pi_s)(A-v_L)$. Among these asks, $A_{-1}-k$ yields the maximum expected utility $\pi_b(1+z+\pi_s)(A_{-1}-k-v_L)$, which is smaller than v_h^s. If he submits a LS at A such as $A_l < A \leq A_h$, t_s undercuts his ask, a MB from t_b is matched with the existing LS, he can trade with t_{nb} and t_{bb}, and his expected utility is $\pi_b(z+\pi_b)(A-v_L)$. Among these asks, A_h yields the maximum expected utility v_h^s. Thus, the optimal ask for t_0 is A_h, and he obtains the reservation utility v_h^s. If the book has the very aggressive LS, the next seller had better to submit a LS behind the market because undercutting the best ask further incurs the cost in price without a sufficient gain in execution probability.

Suppose a seller t_0 faces the book with a LS at A_{-1} such as $A_l < A_{-1} < A_u + k$ submitted in the previous period. If t_0 submits a LS at A such as $A \leq A_l$, the next seller t_s submits a LS at A_h, he can trade with t_{sb}, t_{nb}, and t_b, and his expected utility is $\pi_b(1+z+\pi_s)(A-v_L)$. Among these asks, A_l yields the maximum expected utility v_h^s. If t_0 submits a LS at A such as $A_l < A < A_{-1}$, t_s undercuts his ask, t_0 can trade with t_{nb} and t_b, and his expected utility is $\pi_b(1+z)(A-v_L)$. Among these asks, $A_{-1}-k$ yields the maximum expected utility $\pi_b(1+z)(A_{-1}-k-v_L)$ which is smaller than v_h^s because $\pi_b(1+z)(A_{-1}-k-v_L) < \pi_b(1+z)(A_u-v_L) = v_h^s$. If t_0 submits a LS at A such as $A_{-1} \leq A \leq A_h$, t_s undercuts his ask, a MB from t_b is matched with the existing LS, t_0 can trade with t_{nb} and t_{bb}, and his expected utility is $\pi_b(z+\pi_b)(A-v_L)$. Among these asks, A_h yields the maximum expected utility v_h^s. Thus, the optimal ask for t_0 is A_l and A_h both of which yield the reservation utility v_h^s. A quote-cutting equilibrium designates not A_h but A_l.

Suppose a seller t_0 faces the book with a LS at A_{-1} such as $A_u + k \leq A_{-1} \leq A_f$ submitted in the previous period. In this case, one-tick quote-cutting is optimal for the following reason. Among asks A such as $A \leq A_l$, A_l yields the maximum expected utility v_h^s because A_l is the highest ask deterring further quote-cutting. If t_0 submits a LS at $A_{-1}-k$, t_s undercuts his ask, t_0 can trade with t_{nb} and t_b, and his expected utility is $\pi_b(1+z)(A-v_L)$. Among these asks, $A_{-1}-k$ yields the maximum expected utility $\pi_b(1+z)(A_{-1}-k-v_L)$ which is equal to or greater than v_h^s because $A_{-1} \geq A_u + k$. Among asks A such as $A_{-1} \leq A \leq A_h$, A_h yields the maximum expected utility v_h^s

because the ask is undercut by t_s. Thus, the optimal ask for t_0 is $A_{-1} - k$ yielding $\pi_b(1+z)(A_{-1} - k - v_L) > v_h^s$.

Suppose a seller t_0 faces the book with a LS at A_{-1} such as $A_f < A_{-1}$ submitted in the previous period. The optimal ask for t_0 is A_f by the similar reason for A_f to be optimal to an empty book.

As a whole, the equilibrium strategies described above are indeed an equilibrium. For the final step, we endogenize $\bar{v}_f^b$ and $\bar{v}_m^b$. Let B_f be the symmetric quote to A_f. That is, B_f is the optimal bid to an empty book. $\bar{v}_m^b$ is the expected utility for a buyer t_0 by submitting a LS at B_f to the book with a LS at A_f submitted in the previous period. By a LB at B_f, t_0 can trade with t_{bs} because t_b submits a MB to trade with the existing LS at A_f. t_0 can also trade with t_s and t_{ns}. Thus, $\bar{v}_m^b = \pi_s(1 + z + \pi_b)(v_H - B_f)$. Because we have defined $A_f = v_H - \bar{v}_m^b$,

$$v_H - A_f = \pi_s(1 + z + \pi_b)(v_H - B_f). \tag{16}$$

The symmetric argument leads to

$$B_f - v_L = \pi_b(1 + z + \pi_s)(A_f - v_L). \tag{17}$$

We obtain A_f and B_f by solving Equations (16) and (17). On the whole, A_f and B_f are constructed for a buyer to submit a MB to a LS at A_f as long as a seller submits a MS to a LB at B_f, and vice versa. Because of symmetry, $v_f^s = \pi_b(1+z)(A_f - v_L)$ implies $\bar{v}_f^b = \pi_s(1+z)(v_H - B_f)$. Then, B_f determines $\bar{v}_f^b$, which determines A_h, v_h^s, A_u, and A_l. Finally, $\bar{v}_m^b > \bar{v}_f^b$ because $\bar{v}_m^b = \pi_s(1 + z + \pi_b)(v_H - B_f) > \pi_s(1 + z)(v_H - B_f) = \bar{v}_f^b$. Up to this point, we have explained the optimal order submission strategies of sellers.

A hole at the most aggressive quote is created for the following reason. In equilibrium, the execution probability of a LS at A is $\pi_b(1 + z + \pi_s)$ if $A \leq A_l$, whereas it is equal to or less than $\pi_b(1 + z)$ if $A > A_l$. Quote-cutting creates a discontinuity in the execution probability at A_l. For a LS at an ask slightly above A_l, the gain in price does not compensate for the loss in execution probability because of a discontinuity of execution probability in price. A_u is the minimum price offsetting the loss in execution probability since it is defined as the ask yielding the reservation utility under expectation of future quote-cutting. Thus, a seller never submits a LS at $A \in (A_l, A_u)$, creating a hole.

A.5 Definitions for Theorem 3

In this subsection, we define the variables for the quote-cutting equilibrium in Theorem 3. There are three cases of equilibria according to the trader arrival $\pi \in \Pi$. Let $g_1 = (1-\pi_s)^2 - \pi_s\pi_b(1-\pi_b)$ and $g_2 = (1-\pi_b)^2 - \pi_s\pi_b(1-\pi_s)$. We define $\Pi_1 = \{\pi : g_1 \geq 0,\ g_2 \geq 0\} \cap \Pi$, $\Pi_2 = \{\pi : g_1 < 0\} \cap \Pi$, and $\Pi_3 = \{\pi : g_2 < 0\} \cap \Pi$. The set $\{\Pi_1, \Pi_2, \Pi_3\}$ is the partition of Π and is illustrated in Figure 6. The arrival rate of buyers π_b for $\pi \in \Pi_2$ is relatively small, as is that of sellers π_s for $\pi \in \Pi_3$. These disproportions cause the differences in equilibrium among $\pi \in \Pi_1$, Π_2, and Π_3 in critical price levels and in strategies off the equilibrium path.

The first ask A_f and the first bid B_f submitted to an empty book are defined as follows. For $i \in \{1, 2, 3\}$, let A_{fi} and B_{fi} be

$$A_{f1} = \frac{(1-\pi_s)^2 v_H + \pi_s(2-\pi_s)(1-\pi_b)^2 v_L}{1-\pi_s\pi_b(2-\pi_s)(2-\pi_b)},$$

$$B_{f1} = \frac{\pi_b(1-\pi_s)^2(2-\pi_b)v_H + (1-\pi_b)^2 v_L}{1-\pi_s\pi_b(2-\pi_s)(2-\pi_b)},$$

$$A_{f2} = \frac{((1-\pi_s)^2 + \pi_s\pi_b^2)v_H + \pi_s(2-\pi_s)(1-\pi_b)v_L}{1-\pi_s\pi_b(2-\pi_s-\pi_b)},$$

$$B_{f2} = \frac{\pi_b((1-\pi_s)^2 + \pi_s\pi_b)v_H + (1-\pi_b)v_L}{1-\pi_s\pi_b(2-\pi_s-\pi_b)},$$

$$A_{f3} = \frac{(1-\pi_s)v_H + \pi_s((1-\pi_b)^2 + \pi_s\pi_b)v_L}{1-\pi_s\pi_b(2-\pi_s-\pi_b)},$$

$$B_{f3} = \frac{\pi_b(1-\pi_s)(2-\pi_b)v_H + ((1-\pi_b)^2 + \pi_s^2\pi_b)v_L}{1-\pi_s\pi_b(2-\pi_s-\pi_b)}.$$

Furthermore, let $A_f = A_{fi}$ and $B_f = B_{fi}$ if $\pi \in \Pi_i$ for $i \in \{1, 2, 3\}$. $A_{f1} = A_{f2}$ and $B_{f1} = B_{f2}$ for $\pi \in \Pi$ satisfying $g_1 = 0$. $A_{f1} = A_{f3}$ and $B_{f1} = B_{f3}$ for $\pi \in \Pi$ satisfying $g_2 = 0$. Thus, A_f and B_f are continuous in $\pi \in \Pi$.

We obtain A_f and B_f for $\pi \in \Pi_1$ by solving Equations (16) and (17). For $\pi \in \Pi_2$, the definitions of A_f and B_f are slightly different from those for $\pi \in \Pi_1$ because π_b is relatively small, and the execution probability of a LS is low. Suppose the book has a LB at B_f submitted in $\tau = -1$. At stage 2, a seller t_0 in $\tau = 0$ submits a LS at a higher ask A_h instead of A_f to compensate for the low execution probability due to such a small π_b. By the LS at A_h, the seller t_0 can trade with t_{sb}, t_{nb}, and t_{bb} because t_s submits a MS to the LB at B_f and t_b submits a LB at B_f. That is, the seller t_0 gets the expected utility $\pi_b(A_h - v_L)$ at stage 2. To attract a MS of t_0, the buyer in $\tau = -1$ submits a LB at $B_f = v_L + \pi_b(A_h - v_L)$, leading to

$$B_f - v_L = \pi_b(\Delta - \pi_s(1+z)(v_H - B_f)). \tag{18}$$

Solving Equations (16) and (18) yields A_f and B_f for $\pi \in \Pi_2$. The symmetric equation to Equation (18) is

$$v_H - A_f = \pi_s(\Delta - \pi_b(1+z)(A_f - v_L)). \tag{19}$$

For $\pi \in \Pi_3$, we obtain A_f and B_f by solving Equations (17) and (19).

Once A_f and B_f are defined, the other variables are defined as follows. They will be confirmed in the proof of Theorem 3. $v_f^b = \pi_s(1+z)(v_H - B_f)$ is the expected utility a buyer gets by submitting a LB at B_f to an empty book on the equilibrium path. v_f^b is the maximum payoff for a buyer to submit a LB. So, a buyer would submit a MB to a LS at or below $A_h = v_H - v_f^b$. By submitting a LS at A_h, a seller can get the reservation utility $v_h^s = \pi_b(z + \pi_b)(A_h - v_L)$ in any circumstance. $A_l = v_L + v_h^s/\pi_b/(1 + z + \pi_s)$ is the ask for a seller to get the reservation utility with the highest possible execution probability $\pi_b(1 + z + \pi_s)$. $A_u = v_L + v_h^s/\pi_b/(1 + z)$ is the ask for a seller to get the reservation utility with the execution probability $\pi_b(1 + z)$. A seller gets the reservation utility by submitting a MS to a LB at $B_e = v_L + v_h^s$. Let $A_d(B)$ be

$$A_d(B) = \arg\max A \in N_k \ s.t. \ v_L + (B - v_L)/\pi_b/(1 + z) + k \geq A.$$

Given the bid at B in the book, $A_d(B)$ is the maximum ask that induces the next seller not to undercut $A_d(B)$ by one tick but to submit a MS to a LB at B because

$$B - v_L \geq \pi_b(1 + z)(A_d(B) - k - v_L).$$

Let $B_c = v_L + v_f^s(1 + z)/(1 + z + \pi_s)$. As shown by $v_f^s = \pi_b(1 + z + \pi_s)(A_d(B_c) - v_L)$ under $k = 0$, B_c is the critical bid in the book that equates v_f^s with the expected utility from a LS at $A_d(B)$ with the highest possible execution probability.

The other variables are defined symmetrically. $v_f^s = \pi_b(1+z)(A_f - v_L)$ is the expected utility a seller gets by submitting a LS at A_f to an empty book on the equilibrium path. A seller would submit a MS to a LB at or above $B_l = v_L + v_f^s$, and the reservation utility of a buyer is $v_l^b = \pi_s(z+\pi_s)(v_H - B_l)$. $B_h = v_H - v_l^b/\pi_s/(1+z+\pi_b)$ is the bid for a buyer to get the reservation utility with the highest possible execution probability $\pi_s(1+z+\pi_b)$. $B_u = v_H - v_l^b/\pi_s/(1 + z)$ is the bid for a buyer to get the reservation utility with the execution probability $\pi_s(1 + z)$. A buyer gets the reservation utility by submitting a MB to a LS at $A_e = v_H - v_l^b$. $B_d(A)$ is defined as

$$B_d(A) = \arg\min B \in N_k \ s.t. \ v_H - (v_H - A)/\pi_s/(1 + z) - k \leq B.$$

We define $A_c = v_H - v_f^b(1+z)/(1+z+\pi_b)$, which satisfies $v_f^b = \pi_s(1+z+\pi_b)(v_H - B_d(A_c))$ under $k = 0$.

We indicate variables for $\pi \in \Pi_i$ by a subscript $i \in \{1, 2, 3\}$. For example, $v_{f1}^b = \pi_s(1 + z)(v_H - B_{f1})$ and $A_{h1} = v_H - v_{f1}^b$ denote the variables for $\pi \in \Pi_1$. Let $D_1 = 1 - \pi_s\pi_b(2 - \pi_s)(2 - \pi_b)$ and $D_2 = D_3 = 1 - \pi_s\pi_b(1 + z)$. By definition, $D_1 \in (0, 1)$ and $D_2 = D_3 \in (0, 1)$ for $\pi \in \Pi$.

A.6 Theorem 3

Theorem 3: Suppose that limit orders expire in two period after their submission, that the discount factor of every trader is one, and that $A_j, B_j \in N_k$ for $j \in \{l, f, h, c\}$, $A_u + k \leq A_f$, and $B_f \leq B_u - k$. The following profile of strategies is a pure strategy Markov perfect equilibrium for $\pi \in \Pi$.
(s1): For (n, n, n, n) or (n, n, A_{-2}, n), not depending on A_{-2}, a seller submits a LS at A_f.

(s2): For (B_{-2}, n, n, n), a seller submits a LS at A_f if $B_{-2} < B_l$ and a MS if $B_l \leq B_{-2}$.
(s3): For (n, B_{-1}, A_{-2}, n) or (n, B_{-1}, n, n), not depending on A_{-2}, a seller submits a LS at A_f if $B_{-1} < B_c$, a LS at $A_d(B_{-1})$ if $B_c \leq B_{-1} < B_l$, a LS at A_f (A_h) if $B_l \leq B_{-1} < B_f$ and if $\pi \in \Pi_1 \cup \Pi_3$ ($\pi \in \Pi_2$), and a MS if $B_f \leq B_{-1}$.
(s4): For (B_{-2}, B_{-1}, n, n), a seller submits a LS at A_f if $B_{-1} < B_c$ and if $B_{-2} < B_l$, a LS at $A_d(B_{-1})$ if $B_c \leq B_{-1} < B_l$ and if $B_{-2} < \pi_b(2-\pi_b)(A_d(B_{-1}) - v_L) + v_L$, and a LS at A_f (A_h) if $B_l \leq B_{-1} < B_f$, if $B_{-2} < B_f$, and if $\pi \in \Pi_1 \cup \Pi_3$ ($\pi \in \Pi_2$). In the other cases, a seller submits a MS.
(s5): For (n, n, n, A_{-1}) or (n, n, A_{-2}, A_{-1}), not depending on A_{-2}, a seller submits a LS at A_h if $A_{-1} \leq A_l$, a LS at A_l if $A_l < A_{-1} < A_u + k$, a LS at $A_{-1} - k$ if $A_u + k \leq A_{-1} \leq A_f$, and a LS at A_f if $A_f < A_{-1}$.
(s6): For (B_{-2}, n, n, A_{-1}), a seller submits a LS at A_h if $A_{-1} \leq A_l$ and if $B_{-2} < B_e$, a LS at A_l if $A_l < A_{-1} < A_u + k$ and if $B_{-2} < B_e$, a LS at $A_{-1} - k$ if $A_u + k \leq A_{-1} \leq A_f$ and if $B_{-2} < v_L + \pi_b(1+z)(A_{-1} - k - v_L)$, and a LS at A_f if $A_f < A_{-1}$ and if $B_{-2} < B_l$. In the other cases, a seller submits a MS.
(b1): For (n, n, n, n) or (B_{-2}, n, n, n), not depending on B_{-2}, a buyer submits a LB at B_f.
(b2): For (n, n, A_{-2}, n), a buyer submits a MB if $A_{-2} \leq A_h$ and a LB at B_f if $A_h < A_{-2}$.
(b3): For (B_{-2}, n, n, A_{-1}) or (n, n, n, A_{-1}), not depending on B_{-2}, a buyer submits a MB if $A_{-1} \leq A_f$, a LB at B_f (B_l) if $A_f < A_{-1} \leq A_h$ and if $\pi \in \Pi_1 \cup \Pi_2$ ($\pi \in \Pi_3$), a LB at $B_d(A_{-1})$ if $A_h < A_{-1} \leq A_c$, and a LB at B_f if $A_c < A_{-1}$.
(b4): For (n, n, A_{-2}, A_{-1}), a buyer submits a LB at B_f (B_l) if $A_f < A_{-1} \leq A_h$, if $A_f < A_{-2}$, and if $\pi \in \Pi_1 \cup \Pi_2$ ($\pi \in \Pi_3$). A buyer submits a LB at $B_d(A_{-1})$ if $A_h < A_{-1} \leq A_c$ and if $v_H - \pi_s(2-\pi_s)(v_H - B_d(A_{-1})) < A_{-2}$, and a LB at B_f if $A_c < A_{-1}$ and if $A_h < A_{-2}$. In the other cases, a buyer submits a MB.
(b5): For (n, B_{-1}, n, n) or (B_{-2}, B_{-1}, n, n), not depending on B_{-2}, a buyer submits a LB at B_f if $B_{-1} < B_f$, a LB at $B_{-1} + k$ if $B_f \leq B_{-1} \leq B_u - k$, a LB at B_h if $B_u - k < B_{-1} < B_h$, and a LB at B_l if $B_h \leq B_{-1}$.
(b6): For (n, B_{-1}, A_{-2}, n), a buyer submits a LB at B_f if $B_{-1} < B_f$ and if $A_h < A_{-2}$, a LB at $B_{-1} + k$ if $B_f \leq B_{-1} \leq B_u - k$ and if $v_H - \pi_s(1+z)(v_H - B_{-1} - k) < A_{-2}$, a LB at B_h if $B_u - k < B_{-1} < B_h$ and if $A_e < A_{-2}$, and a LB at B_l if $B_h \leq B_{-1}$ and if $A_e < A_{-2}$. In the other cases, a buyer submits a MB.

Remark 1: The conditions A_j, $B_j \in N_k$ for $j \in \{l, f, h, c\}$ are required to avoid the complexity from discreteness in price as discussed in Section 3.

Remark 2: The condition $A_u + k \leq A_f$ is required for $A_{-1} \in [A_u + k, A_f]$ to exist for strategies in (s5) and (s6). Then, there is at least one one-tick quote-cutting. The condition $B_f \leq B_u - k$ is required for the reasons of symmetry.

Remark 3: The conditions in Theorem 3 mentioned in Remarks 1 and 2 are satisfied if the difference in the valuation $\Delta = v_H - v_L$ is large enough relative to the tick size k. For example, 21 is the minimum integer for v_H to meet the conditions under the parameter values $v_L = 0$, $k = 1$, $\alpha = 1$, and $\beta = 1/2$. We use these parameter values for a numerical example in Section 5.2.

Remark 4: The difference in strategies between $\pi \in \Pi_1$ and $\pi \in \Pi_2$ is the limit price when $B_{-1} \in [B_l, B_f)$ in (s3) and (s4). The difference in strategies between $\pi \in \Pi_1$ and $\pi \in \Pi_3$ is the limit price when $A_{-1} \in (A_f, A_h]$ in (b3) and (b4).

Remark 5: Strategies in (s1) and (s5) relate to the quote dynamics in Corollary 1.

Remark 6: Equilibrium is not necessarily unique. One source of multiplicity is the non-uniqueness of optimal strategies, as the following example shows. Consider the parameter values $v_H = 21$, $v_L = 0$, $k = 1$, and $\pi_s = \pi_b = 1/2$. In this case, $B_l = 6$, and (s6) in Theorem 3 designates a seller to submit a MS to $(6, n, n, 13)$. On the other hand, there is an equilibrium where strategies are the same as those in Theorem 3, but a seller submits not a MS but a LS at $A_f = 12$ to $(6, n, n, 13)$. Because the execution probability of a LS at A_f to $(6, n, n, 13)$ is 1/2, a MS and a LS at A_f yield the same expected utility of 6, which causes multiple equilibria.

Remark 7: The monotonicity of execution probability in quote of a limit order is a favorable feature in comparative statics of limit order markets. For example, Goettler *et al.* (2005) assume it in their Proposition 1. However, the latter equilibrium in Remark 6 is an example of equilibrium where the execution probability is not monotonic in quote. If a seller t_0 submits a LS at 13 to $(n, 6, n, n)$, the book changes to $(6, n, n, 13)$, and t_s submits a LS at 12. If a seller t_0 submits a LS at 14 to $(n, 6, n, n)$, the book changes to $(6, n, n, 14)$, and t_s submits a MS. This difference in the order of t_s makes the execution probability of a LS at 14, 1/2, greater than that of a LS at 13, 1/4. Although, it is not clear whether this example is significant or not because this non-monotonicity occurs off the equilibrium path.

A.7 Proof of Theorem 3

Theorem 3 is proved simply by checking the optimality of a strategy for each type of trader and for every state of the book. The proof is constructed in Steps 1 and 2. In Step 1, we show that for $\pi \in \Pi_1$, optimal strategies for a seller t_0 are strategies from (s1) to (s6) in Theorem 3 if sellers and buyers arriving after t_0 follow strategies in Theorem 3. A symmetric argument is applied to strategies from (b1) to (b6). In Step 2, we prove the case for $\pi \in \Pi_2 \cup \Pi_3$.

Before Step 1, we present Lemmas 1, 2, and 3 to prove the theorem. The proofs of lemmas are straightforward.

Lemma 1: (1) For $\pi \in \Pi$, $0 < v_h^s < v_f^s < \Delta$ and $0 < v_l^b < v_f^b < \Delta$. (2) For $\pi \in \Pi$, $v_L < A_l < A_u < A_f < A_h < A_c \leq A_e < v_H$ and $v_L < B_e \leq B_c < B_l < B_f < B_u < B_h < v_H$. (3) For $\pi \in \Pi$, $B_f < A_f$, $B_e < A_l$, $B_h < A_e$, and $A_u < B_u$.

Lemma 2: For $\pi \in \Pi$, $\pi_b^2(A_c - v_L) < v_f^s$ and $\pi_s^2(v_H - B_c) < v_f^b$.

Lemma 3: (1) For $\pi \in \Pi_1$, $\pi_b(2-\pi_b)(A_{f1} - v_L) \geq \pi_b(A_{h1} - v_L)$ and $\pi_s(2-\pi_s)(v_H - B_{f1}) \geq \pi_s(v_H - B_{l1})$. (2) For $\pi \in \Pi_2$, $\pi_b(2-\pi_b)(A_{f2} - v_L) < \pi_b(A_{h2} - v_L)$ and $\pi_s(2 - \pi_s)(v_H - B_{f2}) > \pi_s(v_H - B_{l2})$. (3) For $\pi \in \Pi_3$, $\pi_b(2-\pi_b)(A_{f3} - v_L) > \pi_b(A_{h3} - v_L)$ and $\pi_s(2-\pi_s)(v_H - B_{f3}) < \pi_s(v_H - B_{l3})$.

Step 1: the case for $\pi \in \Pi_1$.

(s1) Under the book (n, n, n, n) or (n, n, A_{-2}, n), if t_0 submits a LS at A_0 at stage 2, the book becomes (n, n, n, A_0). If $A_0 \leq A_{f1}$, t_b submits a MB according to (b3). The possibility for t_{bb} to submit a MB is as follows: If $A_{f1} < A_0 \leq A_{h1}$, t_b submits a LB at B_{f1} according to (b3). In this case, t_{bb} faces (n, B_{f1}, A_0, n) and submits a MB according to (b6). If $A_{h1} < A_0 \leq A_{c1}$, t_b submits a LB at $B_d(A_0)$, and t_{bb} faces $(n, B_d(A_0), A_0, n)$. For $A_0 \in (A_{h1}, A_{c1}]$, because $A_{c1} \leq A_{e1}$ and because $B_d(A)$ is non-decreasing in A,

$$B_f - k = v_H - \frac{v_H - A_{h1}}{\pi_s(1+z)} - k < v_H - \frac{v_H - A_0}{\pi_s(1+z)} - k \leq B_d(A_0) \leq B_d(A_{e1}) < v_H - \frac{v_H - A_{e1}}{\pi_s(1+z)} = B_{u1},$$

which implies that $A_0 \leq v_H - \pi_s(1+z)(v_H - B_d(A_0) - k)$ and $B_{f1} \leq B_d(A_0) \leq B_{u1} - k$. Thus, t_{bb} submits a MB if $A_{h1} < A_0 \leq A_{c1}$ according to (b6). If $A_{c1} < A_0$, t_b submits a LB at B_{f1} and t_{bb} faces (n, B_{f1}, A_0, n). In that case,

$$v_H - \pi_s(1+z)(v_H - B_{f1} - k) = A_{h1} + \pi_s(1+z)k \leq A_{h1} + k \leq A_{c1} < A_0$$

because $A_{h1}, A_{c1} \in N_k$ and because $A_{h1} < A_{c1}$ implies $A_{h1} + k \leq A_{c1}$. Thus, t_{bb} submits a LB at $B_{f1} + k$ according to (b6). On the whole, t_{bb} submits a MB if $A_{f1} < A_0 \leq A_{c1}$. t_{nb} faces (n, n, A_0, n) and submits a MB if $A_0 \leq A_{h1}$ according to (b2). t_0's LS at A_0 is matched with t_{sb}'s MB if $A_0 \leq A_{l1}$. The reason is as follows: if $A_{l1} < A_0$, t_s submits a LS strictly lower than $min[A_0, A_{f1} + k]$ according to (s5). Then, t_{sb} faces (n, n, A_0, A_1) with $A_1 \leq A_{f1}$ and submits a MB according to (b4). t_{sb}'s MB is matched with t_s's LS because of price precedence. If $A_0 \leq A_{l1}$, t_s submits a LS at A_{h1} according to (s5). Then, t_{sb} faces (n, n, A_0, A_{h1}) and submits a MB, which is matched with t_0's LS.

In brief, if an ask A_0 at which t_0 submits a LS is $A_0 \leq A_{l1}$, its execution probability is $\pi_b(1 + z + \pi_s)$ because he can trade with t_b, t_{nb}, and t_{sb}. If $A_{l1} < A_0 \leq A_{f1}$, its execution probability is $\pi_b(1+z)$ because he can trade with t_b and t_{nb}. If $A_{f1} < A_0 \leq A_{h1}$, its execution probability is $\pi_b(z + \pi_b)$ because he can trade with t_{bb} and t_{nb}. If $A_{h1} < A_0 \leq A_{c1}$, its execution probability is π_b^2 because he can trade with t_{bb}. If $A_{c1} < A_0$, its execution probability is zero.

The optimal asks among those with the same execution probability are A_{l1}, A_{f1}, A_{h1}, and A_{c1}, which yield the expected utilities v_{h1}^s, v_{f1}^s, v_{h1}^s, and $\pi_b^2(A_{c1} - v_L)$, respectively. From Lemmas 1 and 2, it is optimal for t_0 to get v_{f1}^s by submitting a LS at A_{f1}.

(s2) Under the book (B_{-2}, n, n, n) the decision at stage 2 is the same as that in (s1). At stage 1, if $B_{l1} \leq B_{-2}$, $v^s_{f1} \leq B_{-2} - v_L$ and t_0 submits a MS. In the other cases, he submits a LS at A_{f1}.

(s3) Under the book (n, B_{-1}, A_{-2}, n) or (n, B_{-1}, n, n), if t_0 submits a LS at $A_0 > B_{-1}$, the book becomes (B_{-1}, n, n, A_0). For the same reason in (s1), it can be matched with t_b's MB if $A_0 \leq A_{f1}$ and with t_{bb}'s MB if $A_{f1} < A_0 \leq A_{c1}$. t_{nb} submits a MB if $A_0 \leq A_{h1}$ according to (b2). When the book is (B_{-1}, n, n, A_0), t_s follows (s6). In the case where t_s submits a MS, t_{sb} faces (n, n, A_0, n) and submits a MB if $A_0 \leq A_{h1}$ according to (b2). In the case where t_s does not submit a MS, if $A_0 \leq A_{l1}$, t_s submits a LS at A_{h1} and t_{sb} submits a MB, which is matched with t_0's LS. In the other cases, t_0's LS will not be executed.

When $B_{-1} < B_{e1}$, because of $B_{e1} < A_{l1}$ from Lemma 1(3) and for the same reason as in (s1), it is optimal for t_0 to get v^s_{f1} by submitting a LS at A_{f1}. He does not submit a MS because $B_{e1} - v_L < v^s_{f1}$.

When $B_{e1} \leq B_{-1} < B_{l1}$, the optimal asks among those with the same execution probability are $A_d(B_{-1})$, A_{f1}, A_{h1}, and A_{c1}, which yield the expected utilities $\pi_b(2 - \pi_b)(A_d(B_{-1}) - v_L)$, v^s_{f1}, v^s_{h1}, and $\pi_b^2(A_{c1} - v_L)$, respectively. Because of Lemmas 1 and 2, the optimal asks are either A_{f1} or $A_d(B_{-1})$. From the definition of $A_d(B)$, we have

$$v_L + (B_{c1} - v_L)/\pi_b/(1+z) < A_d(B_{c1}), \quad A_d(B_{c1} - k) \leq v_L + (B_{c1} - k - v_L)/\pi_b/(1+z) + k,$$

which imply $v^s_{f1} < \pi_b(1 + z + \pi_s)(A_d(B_{c1}) - v_L)$ and

$$\pi_b(1 + z + \pi_s)(A_d(B_{c1} - k) - v_L) \leq v^s_{f1} + \pi_b(1 + z + \pi_s)(1 - 1/(\pi_b(1+z)))k < v^s_{f1}.$$

Since $A_d(B)$ is non-decreasing in B, it is optimal for t_0 to submit a LS at A_{f1} if $B_{e1} \leq B_{-1} < B_{c1}$ (if $B_e = B_c$ there is no B_{-1} satisfying this condition) and to submit a LS at $A_d(B_{-1})$ if $B_{c1} \leq B_{-1} < B_{l1}$. He does not submit a MS because $B_{c1} - v_L < v^s_{f1}$ and $B_{l1} - v_L = v^s_{f1} < \pi_b(1 + z + \pi_s)(A_d(B_{-1}) - v_L)$ for $B_{c1} \leq B_{-1} < B_{l1}$.

When $B_{l1} \leq B_{-1} < A_{f1}$, the optimal asks among those with the same execution probability are A_{f1}, A_{h1}, and A_{c1}, which yield the expected utilities $\pi_b(1+z+\pi_s)(A_{f1} - v_L)$, $\pi_b(A_{h1} - v_L)$, and $\pi_b^2(A_{c1} - v_L)$, respectively. Because of Lemmas 2 and 3, it is optimal for t_0 to get $\pi_b(1 + z + \pi_s)(A_{f1} - v_L)$ by submitting a LS at A_{f1}. However, if $B_{f1} \leq B_{-1} < A_{f1}$, $\pi_b(1 + z + \pi_s)(A_{f1} - v_L) \leq B_{-1} - v_L$ and t_0 submits a MS. In short, t_0 submits a LS at A_{f1} if $B_{l1} \leq B_{-1} < B_{f1}$, and a MS if $B_{f1} \leq B_{-1} < A_{f1}$.

When $A_{f1} \leq B_{-1}$, t_0 submits a MS because it yields a higher expected utility than a LS at A_{h1} or A_{c1}.

(s4) Under the book (B_{-2}, B_{-1}, n, n), the decision at stage 2 is the same as that in (s3). At stage 1, t_0 submits a MS in the following cases. When $B_{-1} < B_{c1}$, because he gets v^s_{f1} at stage 2, he submits a MS if $B_{-2} \geq v_L + v^s_{f1} = B_{l1}$. When $B_{c1} \leq B_{-1} < B_{l1}$, because he gets $\pi_b(1 + z + \pi_s)(A_d(B_{-1}) - v_L)$ at stage 2, he submits a MS if $B_{-2} \geq v_L + \pi_b(1 + z + \pi_s)(A_d(B_{-1}) - v_L)$. When $B_{l1} \leq B_{-1} < B_{f1}$, because he gets $\pi_b(1+z+\pi_s)(A_{f1} - v_L)$

at stage 2, he submits a MS if $B_{-2} \geq B_{f1}$. When $B_{f1} \leq B_{-1}$, he submits a MS for the same reason as in (s3).

(s5) Under the book (n, n, n, A_{-1}) or (n, n, A_{-2}, A_{-1}), if t_0 submits a LS at A_0, the book becomes (n, n, A_{-1}, A_0). When $A_{h1} < A_{-1}$, it is optimal for t_0 to get v^s_{f1} by submitting a LS at A_{f1} because the LS at A_{f1} is optimal under the book (n, n, n, n) from (s1), and because a higher price priority is assigned to t_0's LS than to the LS at A_{-1}.

When $A_{f1} < A_{-1} \leq A_{h1}$, to get the execution probability π_b^2, t_0 has to submit a LS at an ask lower than A_{c1} because of strategies in (b4). The execution probabilities for other asks are the same as those for (s1). Since the optimal ask is not A_{c1} but A_{f1} in (s1), it is optimal for t_0 to submit a LS at A_{f1}.

Next consider the case where $A_{-1} \leq A_{f1}$. t_b submits a MB according to (b4). If $A_0 < A_{-1}$, t_0's LS can be matched with t_b's MB. If $A_{-1} \leq A_0$, a LS at A_{-1} is matched with t_b's MB. Then, t_{bb} faces the book (n, n, A_0, n) and submits a MB if $A_0 \leq A_{h1}$ according to (b2). t_{nb} submits a MB if $A_0 \leq A_{h1}$ according to (b2). t_s follows (s5). To get a higher price priority over t_s's LS, t_0 must submit a LS such as $A_0 \leq A_{l1}$. In this case, t_{sb} submits a MB according to (b4), which is matched with t_0's LS.

When $A_{-1} \leq A_{l1}$, the optimal asks among those with the same execution probability are $A_{-1}-k$, A_{l1}, and A_{h1}, which yield the expected utilities $\pi_b(2-\pi_b)(A_{-1}-k-v_L) < v^s_{h1}$, $\pi_b(A_{l1} - v_L) < v^s_{h1}$, and v^s_{h1}, respectively. Thus, t_0 submits a LS at A_{h1} to get v^s_{h1}.

When $A_{l1} < A_{-1} < A_{u1} + k$, the optimal asks among those with the same execution probability are A_{l1}, $A_{-1} - k$ if $A_{-1} - k > A_{l1}$, and A_{h1}, which yield the expected utilities v^s_{h1}, $\pi_b(1 + z)(A_{-1} - k - v_L) < \pi_b(1 + z)(A_{u1} - v_L) = v^s_{h1}$, and v^s_{h1}, respectively. He then submits a LS at A_{l1} to get v^s_{h1}.

When $A_{u1} + k \leq A_{-1} \leq A_{f1}$, the optimal asks among those with the same execution probability are A_{l1}, $A_{-1} - k$, and A_{h1}, which yield the expected utilities v^s_{h1}, $\pi_b(1 + z)(A_{-1} - k - v_L) \geq \pi_b(1 + z)(A_{u1} - v_L) = v^s_{h1}$, and v^s_{h1}, respectively. Then, he submits a LS at $A_{-1} - k$ to get $\pi_b(1 + z)(A_{-1} - k - v_L)$.

To summarize, t_0 submits a LS at A_{h1} to get v^s_{h1} if $A_{-1} \leq A_{l1}$, a LS at A_{l1} to get v^s_{h1} if $A_{l1} < A_{-1} < A_{u1}+k$, a LS at $A_{-1}-k$ to get $\pi_b(1+z)(A_{-1}-k-v_L)$ if $A_{u1}+k \leq A_{-1} \leq A_{f1}$, and a LS at A_{f1} to get v^s_{f1} if $A_{f1} < A_{-1}$.

(s6) Under the book (B_{-2}, n, n, A_{-1}), the optimal LS and its expected utility at stage 2 are just the same as those in (s5). At stage 1, when $A_{-1} < A_{u1} + k$, t_0 submits a MS if $v^s_{h1} \leq B_{-2} - v_L$, that is, $B_{e1} \leq B_{-2}$. When $A_{u1} + k \leq A_{-1} \leq A_{f1}$, he submits a MS if $\pi_b(1+z)(A_{-1}-k-v_L) \leq B_{-2}-v_L$. When $A_{f1} < A_{-1}$, he submits a MS if $v^s_{f1} \leq B_{-2}-v_L$, i.e., $B_{l1} \leq B_{-2}$.

The above argument proves that the strategies of sellers in the theorem are optimal for $\pi \in \Pi_1$. We can use a symmetric argument for the strategies of buyers. We have thus proved the theorem for $\pi \in \Pi_1$.

Step 2: the case for $\pi \in \Pi_2 \cup \Pi_3$.

Consider the case $\pi \in \Pi_2$. Between $\pi \in \Pi_1$ and $\pi \in \Pi_2$, strategies differ in (s3) and (s4) due to Lemma 3. First, we will check the optimality of (s3) for $\pi \in \Pi_2$. Under the book (n, B_{-1}, A_{-2}, n) or (n, B_{-1}, n, n), the execution probabilities and optimal strategies for $B_{-1} < B_{l2}$ are just the same as those for $\pi \in \Pi_1$. When $B_{l2} \leq B_{-1}$, the optimal asks among those with the same execution probability are A_{f2}, A_{h2}, and A_{c2}, which yield the expected utilities $\pi_b(1+z+\pi_s)(A_{f2}-v_L)$, $\pi_b(A_{h2}-v_L)$, and $\pi_b^2(A_{c2}-v_L)$, respectively. It is optimal for t_0 to get $\pi_b(A_{h2}-v_L)$ by submitting a LS at A_{h2} if $B_{l2} \leq B_{-1} < B_{f2}$ because of Lemmas 2 and 3(2). He submits a MS if $B_{f2} \leq B_{-2}$ because $\pi_b(A_{h2} - v_L) = B_{f2} - v_L$. Thus, strategies in (s3) are optimal for $\pi \in \Pi_2$. The optimality of (s4) for $\pi \in \Pi_2$ is similar.

Differences in the strategies of sellers in (s3) and (s4) do not affect sellers' other strategies since the expected utilities of sellers are affected by other sellers only through strategies in (s5) and (s6). In addition, it does not affect the optimal strategies of buyers, either. To verify this, consider the symmetric case $\pi \in \Pi_3$. Strategies in (b3) and (b4) for $\pi \in \Pi_3$ are different from those for $\pi \in \Pi_1$. These differences do not affect the strategies of sellers for the following reason. In proving the case $\pi \in \Pi_1$, strategies in (b3) and (b4) are explicitly used to confirm the optimality of strategies in (s1). Concerning (s1), when t_0 submits a LS at A_0 such as $A_{f3} < A_0 \leq A_{h3}$, t_b submits a LB not at B_{f3} but at B_{l3} according to (b3) when $\pi \in \Pi_3$. Then, t_{bb} faces (n, B_{l3}, A_0, n) and submits a MB according to (b6). This does not change t_0's expected utility. Similarly, for the other states of the book, the differences of strategies in (b3) and (b4) between $\pi \in \Pi_1$ and $\pi \in \Pi_3$ do not affect the expected utilities of sellers. By symmetric argument, the differences of strategies in (s3) and (s4) between $\pi \in \Pi_1$ and $\pi \in \Pi_2$ do not affect the optimal strategies of buyers. We have thus proved the theorem for the case $\pi \in \Pi_2$, thereby completing the proof of Theorem 3 since the case for $\pi \in \Pi_3$ is symmetric. Q.E.D.

A.8 Theorem 4

Theorem 4: Let $\hat{\alpha} = 2\gamma/(1-\gamma)/(\delta_p - \delta_i)$, $w_p = \alpha\Delta/(2+\alpha\delta_p)$, and $w_i = \alpha(1-\gamma)\Delta/(2+\alpha(1-\gamma)\delta_i)$. By w_p and w_i, let $A_p = v_H - \delta_p w_p$, $B_p = v_L + \delta_p w_p$, $A_i = v_H - \delta_i w_i$, and $B_i = v_L + \delta_i w_i$. Under Assumption 1, there are the following three types of Markov perfect equilibria depending on the parameter values. If $\alpha < \hat{\alpha}$, a Markov perfect equilibrium is Type 1 and unique. If $\hat{\alpha} < 1$ and $\alpha > \hat{\alpha}$, a Markov perfect equilibrium is Type 2 and unique. If $\alpha = \hat{\alpha} \leq 1$, there are multiple Markov perfect equilibria described in Type 3.

Type 1: A patient seller submits a MS if the book has a LB at $B \geq B_p$, and submits a LS at A_p otherwise. An impatient seller submits a MS if the book has a LB at $B \geq v_L + \delta_i w_p$, and submits a LS at A_p otherwise. A patient buyer submits a MB if the book has a LS at $A \leq A_p$, and submits a LB at B_p otherwise. An impatient buyer submits a MB if the book has a LS at $A \leq v_H - \delta_i w_p$, and submits a LB at B_p otherwise.

Type 2: A patient seller submits a MS if the book has a LB at $B \geq v_L + \delta_p w_i$, and submits a LS at A_i otherwise. An impatient seller submits a MS if the book has a LB at

$B \geq B_i$, and submits a LS at A_i otherwise. A patient buyer submits a MB if the book has a LS at $A \leq v_H - \delta_p w_i$, and submits a LB at B_i otherwise. An impatient buyer submits a MB if the book has a LS at $A \leq A_i$, and submits a LB at B_i otherwise.

Type 3: Let σ_{ps}, σ_{is}, σ_{pb}, and σ_{ib} be some numbers in $[0, 1]$. A patient seller submits a MS if the book has a LB at $B \geq B_p$. Otherwise, he submits a LS at A_p with probability σ_{ps} and submits a LS at A_i with probability $1 - \sigma_{ps}$. An impatient seller submits a MS if the book has a LB at $B \geq B_i$. Otherwise, he submits a LS at A_p with probability σ_{is} and submits a LS at A_i with probability $1 - \sigma_{is}$. A patient buyer submits a MB if the book has a LS at $A \leq A_p$. Otherwise, he submits a LB at B_p with probability σ_{pb} and submits a LB at B_i with probability $1 - \sigma_{pb}$. An impatient buyer submits a MB if the book has a LS at $A \leq A_i$. Otherwise, he submits a LB at B_p with probability σ_{ib} and submits a LB at B_i with probability $1 - \sigma_{ib}$.

A.9 Proof of Theorem 4

To prove the optimality of strategies is straightforward since the quotes in the theorem are the solutions of the following simultaneous equations

$$B_p - v_L = \delta_p \pi_b (A_p - v_L), \quad v_H - A_p = \delta_p \pi_s (v_H - B_p),$$

$$B_i - v_L = \delta_i \pi_b (1 - \gamma)(A_i - v_L), \quad v_H - A_i = \delta_i \pi_s (1 - \gamma)(v_H - B_i).$$

w_p and w_i are defined as $w_p = \pi_b(A_p - v_L)$ and $w_i = \pi_b(1 - \gamma)(A_i - v_L)$. If $\alpha = \hat{\alpha}$, $w_p = w_i$. Thus, if $\alpha = \hat{\alpha}$, a seller is indifferent between A_p and A_i in submitting a LS and a buyer is indifferent between B_p and B_i in submitting a LB. For $\alpha = \hat{\alpha} \leq 1$, $A_p > B_p$ because $A_p - B_p = (2/\hat{\alpha} - \delta_p) w_p > 0$.

To prove the uniqueness of an equilibrium under $\alpha \neq \hat{\alpha}$, we first establish Lemma 4.

Lemma 4. Under Assumption 1, the expected utility obtained by submitting a limit order to an empty book divided by the discount factor is uniquely determined for any trader as w_p if $\alpha < \hat{\alpha}$, $w_p = w_i$ if $\alpha = \hat{\alpha}$, and w_i if $\alpha > \hat{\alpha}$.

Proof of Lemma 4: Let $w^s(A, \mathbf{x}) = \Phi(s, A, \omega_0, \pi, \mathbf{x})(A - v_L)$ and $w^b(B, \mathbf{x}) = \Phi(b, B, \omega_0, \pi, \mathbf{x})(v_H - B)$ where ω_0 represents an empty book. The expected utility by submitting a LS at A to an empty book under a profile of strategies $\mathbf{x}$ is $\delta_p w^s(A, \mathbf{x})$ for a patient seller and $\delta_i w^s(A, \mathbf{x})$ for an impatient seller, respectively. The expected utility by submitting a LB at B to an empty book under a profile of strategies $\mathbf{x}$ is $\delta_p w^b(B, \mathbf{x})$ for a patient buyer and $\delta_i w^b(B, \mathbf{x})$ for an impatient buyer, respectively. Let M^s and m^s be the supremum and the infimum of $w^s(A, \mathbf{x})$ for any Markov perfect equilibrium, respectively. Symmetrically, let M^b and m^b be the supremum and the infimum of $w^b(B, \mathbf{x})$ for any Markov perfect equilibrium, respectively. We shall show that $m^s = M^s = m^b = M^b$.

A patient buyer always submits a MB to the book with a LS at A if $v_H - A > \delta_p M^b$, and an impatient buyer always submits a MB to the book with a LS at A if $v_H - A > \delta_i M^b$. Thus, the infimum of the expected utility of a seller satisfies

$$m^s \geq (\alpha/2)(\Delta - \delta_p M^b), \tag{20}$$

and

$$m^s \geq (\alpha/2)(1-\gamma)(\Delta - \delta_i M^b). \tag{21}$$

A patient buyer never submits a MB to the book with a LS at A if $v_H - A < \delta_p m^b$, and an impatient buyer never submits a MB to the book with a LS at A if $v_H - A < \delta_i m^b$. Thus, the supremum of the expected utility of a seller satisfies

$$M^s \leq (\alpha/2)\max[y_1, y_2] \tag{22}$$

where $y_1 = \Delta - \delta_p m^b$ and $y_2 = (1-\gamma)(\Delta - \delta_i m^b)$. Symmetrically, we have

$$m^b \geq (\alpha/2)(\Delta - \delta_p M^s), \tag{23}$$

$$m^b \geq (\alpha/2)(1-\gamma)(\Delta - \delta_i M^s), \tag{24}$$

$$M^b \leq (\alpha/2)\max[y_3, y_4], \tag{25}$$

where $y_3 = \Delta - \delta_p m^s$ and $y_4 = (1-\gamma)(\Delta - \delta_i m^s)$.

Inequalities (20) through (25) must be satisfied for any Markov perfect equilibrium. We distinguish the following five cases to specify M^s, m^s, M^b, and m^b; Case 1: $y_1 > y_2$ and $y_3 > y_4$; Case 2: $y_1 < y_2$ and $y_3 < y_4$; Case 3: $y_1 = y_2$ and $y_3 = y_4$; Case 4: $y_1 > y_2$ and $y_3 < y_4$; Case 5: $y_1 < y_2$ and $y_3 > y_4$.

Case 1: Inequalities (20) and (25) together imply

$$m^s \geq (\alpha/2)(\Delta - \delta_p M^b) \geq (\alpha/2)(\Delta - \delta_p(\alpha/2)(\Delta - \delta_p m^s)). \tag{26}$$

This can be arranged to give $m^s \geq w_p$. Similarly, Inequalities (22) and (23) give $M^s \leq w_p$. Thus, we have $m^s = M^s = w_p$. Symmetrically, we have $m^b = M^b = w_p$ from Inequalities (20), (22), (23), and (25). In addition, m^s, M^s, m^b, and M^b must satisfy Inequalities (21), (24), $y_1 > y_2$, and $y_3 > y_4$, which are satisfied if $\alpha < \hat{\alpha}$. As a whole, if $\alpha < \hat{\alpha}$, then $w^s(A, \mathbf{x})$ and $w^b(B, \mathbf{x})$ are uniquely determined as w_p for any Markov perfect equilibrium.

Case 2: Inequalities (21), (22), (24), and (25) give $m^s = M^s = m^b = M^b = w_i$. Inequalities (20), (23), $y_1 < y_2$, and $y_3 < y_4$ are satisfied if $\alpha > \hat{\alpha}$. As a whole, if $\alpha > \hat{\alpha}$, then $w^s(A, \mathbf{x})$ and $w^b(B, \mathbf{x})$ are uniquely determined as w_i for any Markov perfect equilibrium.

Case 3: Inequalities (20) through (25) give $m^s = M^s = m^b = M^b = w_p = w_i$. These equations along with $y_1 = y_2$ and $y_3 = y_4$ are satisfied if $\alpha = \hat{\alpha}$. Thus, for any Markov perfect equilibrium, if $\alpha = \hat{\alpha}$, then $w^s(A, \mathbf{x})$ and $w^b(B, \mathbf{x})$ are uniquely determined as w_p, and $w_p = w_i$.

Case 4: Inequalities (20), (22), (24), and (25) give $m^s = M^s = \alpha(2-\alpha(1-\gamma)\delta_p)\Delta/(4-\alpha^2(1-\gamma)\delta_p\delta_i)$ and $m^b = M^b = \alpha(1-\gamma)(2-\alpha\delta_i)\Delta/(4-\alpha^2(1-\gamma)\delta_p\delta_i)$. $y_1 > y_2$ is satisfied under $\alpha < \hat{\alpha}$ and $y_3 < y_4$ is satisfied under $\alpha > \hat{\alpha}$. Thus, there is no equilibrium under Case 4.

Case 5: By reason of the symmetry with Case 4, there is no equilibrium under Case 5. Q.E.D.

Now, we prove the uniqueness of a Markov perfect equilibrium under $\alpha < \hat{\alpha}$. Lemma 4 shows that, if $\alpha < \hat{\alpha}$, a patient seller and a patient buyer obtain the expected utility $\delta_p w_p$ by submitting a limit order, and an impatient seller and an impatient buyer obtain the expected utility $\delta_i w_p$ by submitting a limit order. First, we specify the execution probability, ϕ, of a LS given the expected utility of buyers. Next, we search for the ask, A', which is consistent with the expected utility of a seller and the execution probability ϕ, that is, $w_p = \phi(A' - v_L)$.

The execution probability of a LS is as follows. Because a patient buyer obtains $\delta_p w_p$ by submitting a LB, he does not submit a MB to the book with a LS at A such as $A > v_H - \delta_p w_p = A_p$, and submits a MB to the book with a LS at A such as $A < A_p$. He would submit a MB with some probability $\sigma \in [0,1]$ to the book with a LS at A_p. An impatient buyer obtains $\delta_i w_p$ by submitting a LB, and he submits a MB in the similar way to a patient buyer. Thus, the execution probability of a LS at A is (1) $\alpha/2$ if $A < A_p$, (2) $\alpha(1-\gamma+\gamma\sigma)/2$ if $A = A_p$, where $\sigma \in [0,1]$, (3) $\alpha(1-\gamma)/2$ if $A_p < A < v_H - \delta_i w_p$, (4) $\alpha(1-\gamma)\sigma/2$ if $A = v_H - \delta_i w_p$, where $\sigma \in [0,1]$, and (5) 0 if $A > v_H - \delta_i w_p$.

The ask A' which satisfies $w_p = \phi(A' - v_L)$ is as follows. (1) If $A' < A_p$, $w_p = \alpha(A' - v_L)/2$. This equation leads to $A' = A_p$, which does not satisfy $A' < A_p$. Thus, there is no A' under $A' < A_p$. (2) If $A' = A_p$, $w_p = \alpha(1-\gamma+\gamma\sigma)(A' - v_L)/2$, and $\sigma = 1$ satisfies this equation. Thus, submitting a LS at A_p of a seller and submitting a MB to the book with a LS at A_p of a patient buyer and of an impatient buyer are consistent with Lemma 4. An equilibrium of Type 1 designates these strategies. (3) If $A_p < A' < v_H - \delta_i w_p$, $w_p = \alpha(1-\gamma)(A' - v_L)/2$, and $A' = v_L + 2w_p/\alpha/(1-\gamma)$. $A' > v_H - \delta_i w_p$ because $A' - (v_H - \delta_i w_p) = (\hat{\alpha} - \alpha)(\delta_p - \delta_i)w_p/\alpha > 0$. Thus, there is no A' under $A_p < A' < v_H - \delta_i w_p$. (4) If $A' = v_H - \delta_i w_p$, $w_p = \alpha(1-\gamma)\sigma(A' - v_L)/2$. This equation yields $1 = \sigma(1 - \gamma(1 - \alpha/\hat{\alpha}))$. Because $\alpha < \hat{\alpha}$, $\sigma > 1$, which is not consistent with $\sigma \in [0,1]$. Thus, there is no A' under $A' = v_H - \delta_i w_p$. (5) If $A' > v_H - \delta_i w_p$, the execution probability is zero, which is not consistent with $w_p > 0$. Thus, there is no A' under $A' > v_H - \delta_i w_p$. As a whole, under $\alpha < \hat{\alpha}$, strategies of sellers in Type 1 are unique strategies which are consistent with Lemma 4.

Symmetrically, under $\alpha < \hat{\alpha}$, strategies of buyers in Type 1 are unique strategies which are consistent with Lemma 4. Thus, an equilibrium of Type 1 is a unique Markov perfect equilibrium under $\alpha < \hat{\alpha}$. The similar argument proves that an equilibrium of Type 2 constitutes a unique Markov perfect equilibrium under $\alpha > \hat{\alpha}$. Q.E.D.

A.10 Proofs of Propositions

Proof of Proposition 1: Straightforward. Q.E.D.

Proof of Proposition 2: (1) The direct calculation shows $\partial A_r/\partial \pi_s < 0$. The bid side is symmetric.

(2) The direct calculation shows $\partial A_r/\partial \pi_b > 0$. The bid side is symmetric.

(3) $\partial A_r/\partial \alpha = \psi_1(\alpha,\beta)\beta\Delta/(1-\alpha^2(1-\beta)\beta)^2$, where $\psi_1(\alpha,\beta) = -1+2\alpha(1-\beta)-\alpha^2\beta(1-\beta)$. Thus, $\partial A_r/\partial\alpha < 0$ if $\psi_1(\alpha,\beta) < 0$. Since $\psi_1(\alpha,\beta)$ increases in α, $\psi_1(\alpha,\beta) < 0$ if $\psi_1(1,\beta) = 1-3\beta+\beta^2 < 0$. Because $\psi_1(1,\beta) < 0$ for $\beta \in ((3-\sqrt{5})/2, (3+\sqrt{5})/2)$, $\partial A_r/\partial\alpha < 0$ for $\beta > (3-\sqrt{5})/2$. The bid side is symmetric.

(4) The set of the states of the book on the equilibrium path is {empty, a book with a LS at A_r, a book with a LB at B_r}. Because the stationary distribution is $((1-\pi_s)(1-\pi_b)/(1-\pi_s\pi_b), \pi_s(1-\pi_b)/(1-\pi_s\pi_b), \pi_b(1-\pi_s)/(1-\pi_s\pi_b))$, the expected spread s_r is

$$s_r = \frac{(1-\pi_s)(1-\pi_b)(1+\pi_s+\pi_b-\pi_s\pi_b)}{(1-\pi_s\pi_b)^2}\Delta.$$

Thus,

$$\frac{\partial s_r}{\partial \alpha} = -2\alpha\Delta\frac{1-\beta(1-\beta)(4-(2-\alpha)(1-\alpha\beta)(1-\alpha(1-\beta)))}{(1-\alpha^2\beta(1-\beta))^3} < 0$$

because $\beta(1-\beta) \leq 1/4$ and $(2-\alpha)(1-\alpha\beta)(1-\alpha(1-\beta)) < 2$.

(5) The direct calculation shows that $\partial A_r/\partial\beta < 0$. The bid side is symmetric. Q.E.D.

Proof of Proposition 3: Straightforward. Q.E.D.

Proof of Proposition 4: If $\alpha = 1$, $z = 0$ and there is no queuing equilibrium under Cases 2, 3, and 4 in the proof of Theorem 2. Thus, the queuing equilibrium satisfies Conditions (8) and (9), leading to the inequality in the proposition. Q.E.D.

Proof of Proposition 5: (1) $A_f - B_f > 0$ from Lemma 1(3) in the proof of Theorem 3.

(2) From Lemma 1(2) and (3) in the proof of Theorem 3, $B_h > B_u > A_u > A_l$. Q.E.D.

Proof of Proposition 6: (1) Straightforward.

(2) Straightforward.

(3) If $\beta = 1/2$ and $\pi_s = \pi_b = \alpha/2$, $\pi \in \Pi_1$. Let $p = \alpha/2$. For this case, $\partial A_f/\partial p = -2(1-p)^5\Delta/D_1^2 < 0$, and

$$\partial A_l/\partial p = -(1-p)^4((1-p)^4 + 4(1-p)^2 + 4(2-p)p^3)\Delta/D_1^2/(2-p)^2 < 0.$$

$\partial A_h/\partial p = -\partial v_s^b/\partial p$, and $\partial A_u/\partial p = -\partial v_s^b/\partial p/2$. Because $\partial v_f^b/\partial p = 2(1-p)^4\Delta(1-2p-p^2)/D_1^2$, $\partial v_f^b/\partial p$ is positive for $p \in (0, a_1/2)$ and negative for $p \in (a_1/2, 1/2]$. The bid side is symmetric. Q.E.D.

Proof of Proposition 7: $A_r - B_r > A_f - B_f$ for $\pi \in \Pi$ because

$$\begin{aligned} & (A_r - B_r) - (A_{f1} - B_{f1}) \\ = & \left((1-\pi_s)(1-\pi_b)(\pi_s+\pi_b) + (\pi_s - \pi_b)^2\right)(1-\pi_s)(1-\pi_b)\Delta/D_1/(1-\pi_s\pi_b) \\ > & \ 0, \end{aligned}$$

$$(A_r - B_r) - (A_{f2} - B_{f2}) = \pi_s \left((1-\pi_b)^2 + \pi_b\right)(1-\pi_s)(1-\pi_b)\Delta/D_2/(1-\pi_s\pi_b) > 0,$$

$$(A_r - B_r) - (A_{f3} - B_{f3}) = \pi_b \left((1-\pi_s)^2 + \pi_s\right)(1-\pi_s)(1-\pi_b)\Delta/D_3/(1-\pi_s\pi_b) > 0.$$

Q.E.D.

Proof of Proposition 8: If $\beta = 1/2$ and $\pi_s = \pi_b = \alpha/2$, $\pi \in \Pi_1$. Let $p = \alpha/2$. For this case,

$$\partial(A_u - A_l)/\partial p = (1-p)^4(1+5p^2+2p^3(1-p))\Delta/(2-p)^2/D_1^2 > 0,$$

$$\partial(A_h - A_f)/\partial p = 2(1-p)^4 p(1+p)\Delta/D_1^2 > 0,$$

$$\partial(A_h - A_l)/\partial p = (1-p)^4(-3+12p-p^4)\Delta/(2-p)^2/D_1^2.$$

Let $\psi_2(p) = -3 + 12p - p^4$. $\psi_2(p) > 0$ if $p \in (a_2/2, 1/2]$, $\psi_2(p) = 0$ if $p = a_2/2$, and $\psi_2(p) < 0$ if $p \in (0, a_2/2)$. As for $A_f - A_u$, $\partial(A_f - A_u)/\partial p = -(1-p)^4(1+p^2)\Delta/D_1^2 < 0$. The bid side is symmetric. Q.E.D.

Proof of Proposition 9: (1) The probability of a transaction depends on how traders are matched when two consecutive sellers (buyers) are followed by two consecutive buyers (sellers). Let's consider a case where the sequence of the trader arrival to an empty book is a seller, a seller, a buyer, and a buyer. There are two patterns of transactions. One is that in which the first seller trades with the third buyer and the second seller trades with the fourth buyer. The other is that in which the second seller trades with the third buyer, and the first seller cannot trade. The probability of a transaction is greatest when the former case obtains, i.e., when the earlier arrival is matched for a transaction. If a trader submits a market order to the book with limit orders on the opposite side of the book on the equilibrium path, a queuing equilibrium always matches the earlier arrival first and thus attains the greatest probability of a transaction.

(2) In a quote-cutting equilibrium, a seller t_0 facing an empty book cannot trade with the buyer t_{sb} because the next trader t_s undercuts his ask. Thus, a quote-cutting equilibrium cannot attain the highest probability of a transaction.

(3) As shown in (1), the probability of a transaction is higher when the earlier arrival is matched for a transaction. This occurs only if a trader submits a limit order at or behind the market. In a quote-cutting equilibrium, such an order is more frequently submitted if the tick size is larger because one-tick quote-cutting takes fewer periods. Q.E.D.

Proof of Proposition 10: An equilibrium is Type 1 in Theorem 4 if $\gamma = 1$. Similar to the proof of Proposition 2(4), the expected spread s_p is $s_p = (4 + 2\alpha + \alpha(2 - \alpha)\delta_p)\Delta/(2 + \alpha)/(2 + \alpha\delta_p)$. The direct calculation shows $\partial s_p/\partial\delta_p < 0$. Q.E.D.

Proof of Proposition 11: (1) If $\alpha < \hat{\alpha}$, an equilibrium is Type 1 in Theorem 4, and the expected spread is s_p in the proof of Proposition 10. If $\alpha > \hat{\alpha}$, an equilibrium is Type 2 in Theorem 4. The set of the states of the book on the equilibrium path is {empty, the book with a LS at A_i, the book with a LB at B_i}, and the stationary distribution is $((2 - \alpha(1 + \gamma))/(2 + \alpha(1 - \gamma)), \alpha/(2 + \alpha(1 - \gamma)), \alpha/(2 + \alpha(1 - \gamma)))$. Thus, the expected spread s_i under $\alpha > \hat{\alpha}$ is

$$s_i = \frac{4 + 2\alpha(1 - \gamma) + \alpha(1 - \gamma)(2 - \alpha - \alpha\gamma)\delta_i}{(2 + \alpha(1 - \gamma))(2 + \alpha(1 - \gamma)\delta_i)}\Delta.$$

The difference of the expected spreads is

$$s_i - s_p = \frac{2\alpha^2\Delta\psi_3(\alpha, \delta_i)}{(2 + \alpha)(2 + \alpha\delta_p)(2 + \alpha(1 - \gamma))(2 + \alpha(1 - \gamma)\delta_i)}$$

where $\psi_3(\alpha, \delta_i) = 4(\delta_p - (1 - \gamma)\delta_i) + 2(1 - \gamma)(\delta_p - \delta_i)\alpha - (1 - \gamma)\gamma\delta_p\delta_i\alpha^2$. Because $\partial\psi_3(1, \delta_i)/\partial\delta_i < 0$ and $\delta_i < \delta_p$, $\psi_3(1, \delta_i) > \psi_3(1, \delta_p) = \delta_p\gamma(4 - (1 - \gamma)\delta_p) > 0$. $s_i > s_p$ for any parameter values since $\partial^2\psi_3(\alpha, \delta_i)/\partial a^2 < 0$, $\psi_3(0, \delta_i) > 0$, and $\psi_3(1, \delta_i) > 0$. Thus, $s_i > s_p$ at $\alpha = \hat{\alpha}$. Because s_i and s_p are continuous in α, s_p at $\alpha = \hat{\alpha} - \epsilon$ is smaller than s_i at $\alpha = \hat{\alpha} + \epsilon$ for small $\epsilon > 0$.

(2) The expected spread is s_p under $\gamma > \hat{\gamma}$ and s_i under $\gamma < \hat{\gamma}$. s_i takes the smallest value at $\gamma = 0$ under $\gamma < \hat{\gamma}$ because $\partial s_i/\partial\gamma > 0$. At $\gamma = 0$, $s_i = (4 + 2\alpha + \alpha(2 - \alpha)\delta_i)\Delta/(2 + \alpha)/(2 + \alpha\delta_i)$, which is greater than s_p. Q.E.D.

Proof of Proposition 12: From the stationary distribution in Propositions 2(4) and 11(1), the transaction probability is $\alpha^2(1-\gamma)/(2+\alpha(1-\gamma))$ under $\gamma < \hat{\gamma}$ and $\alpha^2/(2+\alpha)$ under $\gamma > \hat{\gamma}$. Under $\gamma < \hat{\gamma}$, the transaction probability takes the largest value $\alpha^2/(2+\alpha)$ at $\gamma = 0$ because the transaction probability decreases in γ. Thus, the transaction probability is greater under $\gamma \in (\hat{\gamma}, 1]$ than under $\gamma \in (0, \hat{\gamma})$. Q.E.D.

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Table 1: Examples of equilibria under one-period expiration.

State of book	Order in book	Eq A seller	Eq A buyer	Eq B seller	Eq B buyer
ω_0	no order	LS at 2*	LB at 1*	LS at 2*	LB at 2*
ω_1	LS at 1	LS at 2	MB	LS at 2	MB
ω_2	LS at 2	LS at 2*	MB*	LS at 2*	MB*
ω_3	LB at 1	MS*	LB at 1*	LS at 2	LB at 2
ω_4	LB at 2	MS	LB at 1	MS*	LB at 2*

Table 1: The parameter values are $v_H = 3$, $v_L = 0$, $k = 1$, $\alpha = 1$, and $\beta = 1/2$. Limit orders are assumed to expire in one period after their submission. A row denotes a state of the book which is distinguished by an unfilled limit order in the book. Eq A and Eq B are equilibrium profiles of strategies. For example, a seller submits a LS at 2 to an empty book ω_0 under Eq A. The strategies denoted by * are those for the states on the equilibrium path.

Table 2: Examples of equilibria under two-period expiration.

Orders in book		Eq 1	Eq 2	Eq 3
Submitted two periods ago	Submitted one period ago			
no order	no order	LS at 2*	LS at 2*	LS at 1*
LS at 1	no order	LS at 2	LS at 2	LS at 1*
LS at 2	no order	LS at 2*	LS at 2*	LS at 1
LB at 1	no order	MS*	MS*	MS
LB at 2	no order	MS	MS	MS*
no order	LS at 1	LS at 1	LS at 2	LS at 1*
no order	LS at 2	LS at 1*	LS at 2*	LS at 1
no order	LB at 1	MS*	LS at 2*	LS at 2
no order	LB at 2	MS	MS	MS*
LS at 1	LS at 1	LS at 2	LS at 2	LS at 1*
LS at 1	LS at 2	LS at 1*	LS at 2	LS at 1
LS at 2	LS at 1	LS at 2*	LS at 2	LS at 1
LS at 2	LS at 2	LS at 1	LS at 2*	LS at 1
LB at 1	LB at 1	LS at 2	MS*	MS
LB at 1	LB at 2	MS*	MS	MS
LB at 2	LB at 1	MS*	MS	MS
LB at 2	LB at 2	MS	MS	MS*
LS at 2	LB at 1	LS at 2	LS at 2*	LS at 2
LB at 1	LS at 2	MS	MS*	MS

Table 2: The parameter values are $v_H = 3$, $v_L = 0$, $k = 1$, $\alpha = 1$, and $\beta = 1/2$. Limit orders are assumed to expire in two periods after their submission. A row denotes a state of the book. The book has at most two limit orders under the two-period expiration. Eq 1, Eq 2, and Eq 3 are equilibrium profiles of strategies of sellers. The strategies of buyers are symmetric for these equilibria. The strategies denoted by * are those for the states on the equilibrium path.

Table 3: Effect of expiration period and trader arrival rate.

Expiration periods	2	2	2	3	3	3	4	4	4
Trader arrival rate (α)	1/3	2/3	1	1/3	2/3	1	1/3	2/3	1
Expected spread	12.2	10.6	9.4	11.2	9.4	8.7	10.8	9.0	7.6
Order composition (%)									
Market order	21.7	30.8	33.7	26.3	34.2	38.5	29.4	36.4	38.9
Cutting more than one tick	0	0.6	4.8	0	0	2.0	0	1.0	3.1
Cutting by one tick	13.0	22.3	25.7	19.9	28.4	27.4	24.4	29.7	30.4
At the market	7.2	5.1	0	4.2	2.8	3.8	2.7	1.6	0.2
Behind the market	1.5	2.8	3.2	2.2	3.0	5.3	2.3	4.0	5.5
Empty	56.5	38.4	32.6	47.4	31.6	23.1	41.3	27.2	21.8

Table 3: The parameter values are $v_H = 14$, $v_L = 0$, $k = 1$, and $\beta = 1/2$. We designate market orders as "Market order," limit orders undercutting the best quote by more than one tick as "Undercutting by more than one tick," limit orders undercutting the best quote by one tick as "Undercutting by one tick," limit orders at the best quote as "At the market," limit orders behind the best quote as "Behind the market," and limit orders submitted to an empty book as "Empty."

Table 4: Effect of tick size reduction.

	Eq 1	Eq 2	Eq 3	Eq 4	Eq 5	Eq 6
Tick size (k)	7	7	7	3	1	0.5
Expected spread	13.8	13.2	10.2	13.6	14.3	14.7
Order composition (%)						
Market order	35.1	33.3	38.5	34.1	33.4	33.3
Undercutting by more than one tick	0	0	0	0	1.2	0.04
Undercutting by one tick	16.2	0	0	27.5	31.5	33.3
At the market	8.1	44.4	38.5	0	0	0
Behind the market	10.8	0	0	6.6	0.8	0.02
Empty	29.7	22.2	23.1	31.9	33.2	33.3

Table 4: The parameter values are $v_H = 21$, $v_L = 0$, $\alpha = 1$, and $\beta = 1/2$. We designate market orders as "Market order," limit orders undercutting the best quote by more than one tick as "Undercutting by more than one tick," limit orders undercutting the best quote by one tick as "Undercutting by one tick," limit orders at the best quote as "At the market," limit orders behind the best quote as "Behind the market," and limit orders submitted to an empty book as "Empty."

Figure 1: Example of quote dynamics

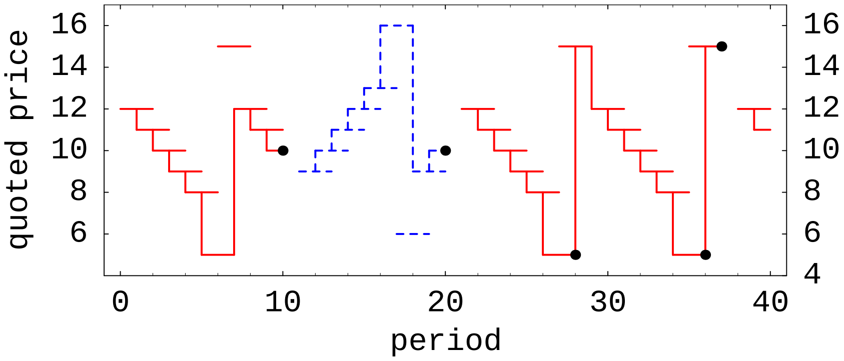

Figure 1 illustrates an example of quote dynamics of a quote-cutting equilibrium under $v_H = 21$, $v_L = 0$, $k = 1$, $\alpha = 1$, and $\beta = 1/2$. The abscissa is the period and the ordinate the quoted price. The horizontal solid lines indicate the asks and the horizontal broken lines indicate the bids. Two lines in one period indicate that the book has two limit orders. The point indicates a transaction. The figure depicts a case in which the sequence of the type of traders is sssss sssss bbbbb bbbbb sssss sssbs sssss sbbss, where 's'('b') denotes a seller (buyer).

Figure 2: Trader arrival and the spread

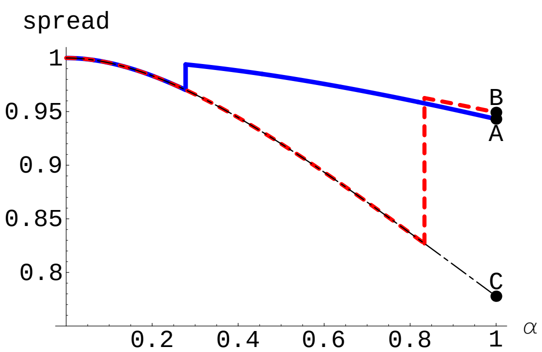

Figure 2 illustrates the relation between the expected spread and the trader arrival rate α. The basic parameter values are $v_H = 1$, $v_L = 0$, $\beta = 1/2$, $\delta_p = 1$, and $\delta_i = 0.2$. The solid line, broken line, and dash-dotted line depict the expected spread under $\gamma = 0.1$, $\gamma = 0.25$, and $\gamma = 0.6$, respectively. $\hat{\alpha} = 5/18 \simeq 0.278$ for $\gamma = 0.1$, $\hat{\alpha} = 5/6 \simeq 0.833$ for $\gamma = 0.25$, and $\hat{\alpha} = 3.75$ for $\gamma = 0.6$. At $\alpha = 5/18$, the solid line moves from 0.970 to 0.994. At $\alpha = 5/6$, the broken line moves from 0.827 to 0.963. Points A, B, and C correspond to Points A, B, and C in Figure 3.

Figure 3: Share of patient traders and spread

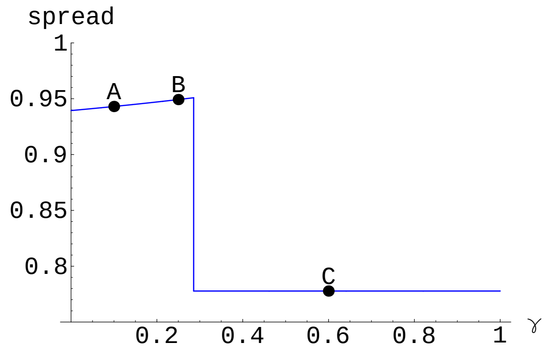

Figure 3 illustrates the relation between the expected spread and the share of the patient traders γ under $v_H = 1$, $v_L = 0$, $\beta = 1/2$, $\alpha = 1$, $\delta_p = 1$, and $\delta_i = 0.2$. $\hat{\gamma} = 2/7 \simeq 0.286$ for these parameter values. The expected spread moves from 0.951 to 0.778 at $\gamma = \hat{\gamma}$. Points A, B, and C correspond to Points A, B, and C in Figure 2.

Figure 4: Arrival of traders

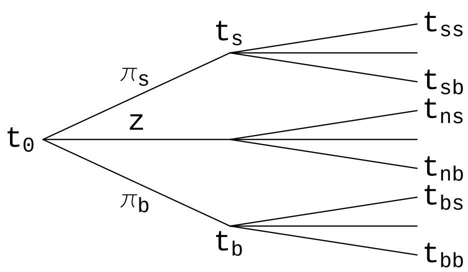

Figure 4: t_0 denotes a trader in a given period. In the following period, a seller, denoted by t_s, arrives with probability π_s, a buyer, denoted by t_b, arrives with probability π_b, and no one arrives with probability z. A seller after t_s is denoted by t_{ss} and a buyer after t_s by t_{sb}. A seller after t_b is denoted by t_{bs} and a buyer after t_b by t_{bb}. If no trader arrives following t_0, a seller in the two periods ahead of t_0 is denoted by t_{ns} and a buyer by t_{nb}.

Figure 5: Asks and expected utilities

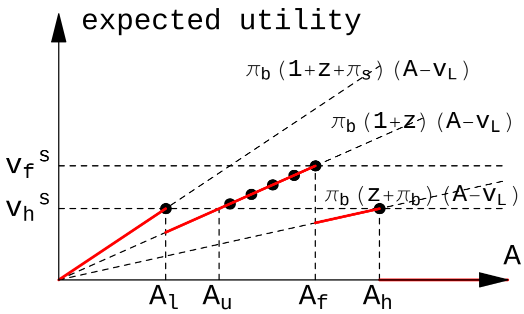

Figure 5 depicts the relation of A_l, A_u, A_f, A_h, v_f^s, and v_h^s. The figure illustrates the case with the parameter values $v_H = 21$, $v_L = 0$, $k = 1$, $\alpha = 1$, and $\beta = 1/2$. Points indicate asks submitted on the equilibrium path and the corresponding expected utilities. Solid lines indicate the expected utilities for a seller facing an empty book by submitting a LS at A.

Figure 6: The Set $\{\Pi_1, \Pi_2, \Pi_3\}$

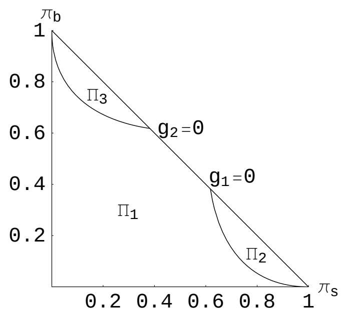

Figure 6 depicts $\{\Pi_1, \Pi_2, \Pi_3\}$, which is the partition of Π. The curved line $g_1 = 0$ separates Π_2 from Π_1, and the curved line $g_2 = 0$ separates Π_3 from Π_1.