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ABSTRACT

Given a monotone convex function on the space of essentially bounded random variables with the Lebesgue property (order continuity), we consider its extension preserving the Lebesgue property to as big solid vector space of random variables as possible. We show that there exists a maximum such extension, with explicit construction, where the maximum domain of extension is obtained as a (possibly proper) subspace of a natural Orlicz-type space, characterized by a certain uniform integrability property. As an application, we provide a characterization of the Lebesgue property of monotone convex function on arbitrary solid spaces of random variables in terms of uniform integrability and a "nice" dual representation of the function.

Key Words: Monotone Convex Functions, Lebesgue Property, Order-Continuity, Order-Continuous Banach Lattices, Uniform Integrability, Convex Risk Measures

1. Introduction

Motivated by the study of convex risk measures in financial mathematics, we address a "regular" extension problem of monotone convex functions. Let L^0 be the space of all finite random variables (measurable functions) on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ modulo $\mathbb{P}$-almost sure (a.s.) equality, and we say that a linear subspace $\mathscr{X} \subset L^0$ is solid if $X \in \mathscr{X}$ and $|Y| \leq |X|$ a.s. imply $Y \in \mathscr{X}$. By a monotone convex function on a solid space $\mathscr{X} \subset L^0$, we mean a convex function $\varphi : \mathscr{X} \to (-\infty, \infty]$ which is monotone increasing w.r.t. the a.s. pointwise order.

We are interested in monotone convex functions on some solid space $\mathscr{X}$ having the following regularity property called the *Lebesgue property*: for any sequence $(X_n)_n \subset \mathscr{X}$,

$$\exists Y \in \mathscr{X},\ |X_n| \leq Y\ (\forall n) \text{ and } X_n \to X \in \mathscr{X} \text{ a.s.} \ \Rightarrow \varphi(X) = \lim_n \varphi(X_n). \tag{1.1}$$

Note that all L^p spaces are solid, and when $\mathscr{X} = L^1 := L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $\varphi(X) = \mathbb{E}[X]$, this is nothing but the dominated convergence theorem. When $\mathscr{X} = L^\infty$, (1.1) reduces to

$$\sup_n \|X_n\|_\infty < \infty \text{ and } X_n \to X \text{ a.s.} \ \Rightarrow \ \varphi(X) = \lim_n \varphi(X_n), \tag{1.2}$$

and a number of practically important monotone convex functions on L^∞ satisfy this.

Now given a monotone convex function φ_0 on L^∞ with the Lebesgue property (1.2), we consider its extension to some big solid space *preserving the Lebesgue property* in the form of (1.1) (such extensions do make sense). Of course there may be several such extensions, but we are interested in the maximum one. So the central question of the paper is:

Question 1.1. Given a monotone convex function φ_0 on L^∞ with the Lebesgue property (1.2), does there exist a maximum extension preserving the Lebesgue property in the sense of (1.1)? i.e., is there a pair $(\hat{\varphi}, \widehat{\mathscr{X}})$ of a solid space $\widehat{\mathscr{X}} \subset L^0$ and a monotone convex function $\hat{\varphi}$ with the Lebesgue property on $\widehat{\mathscr{X}}$ such that $\hat{\varphi}|_{L^\infty} = \varphi_0$ and for any such pair $(\varphi, \mathscr{X})$, one has $\mathscr{X} \subset \widehat{\mathscr{X}}$ and $\varphi = \hat{\varphi}|_{\mathscr{X}}$?

As a first (trivial) example, we briefly see what happens when φ_0 is linear.

Example 1.2. Let φ_0 be a *positive* (monotone) linear functional on L^∞. Then it is finite-valued and identified with a *finitely additive* measure $\nu_0(A) := \varphi_0(\mathbb{1}_A)$ as $\varphi_0(X) = \int_\Omega X d\nu_0$, while (1.2) is equivalent to saying that ν_0 is σ-additive. If the latter is the case, the "usual" integral $\hat{\varphi}(X) := \int_\Omega X d\nu_0$ defines a Lebesgue-preserving extension of φ_0 to $\mathcal{L}^1(\nu_0) := \{X \in L^0 : \int_\Omega |X| d\nu_0 < \infty\}$. On the other hand, if φ is a monotone convex function on a solid space $\mathscr{X} \subset L^0$ with (1.1) and $\varphi|_{L^\infty} = \varphi_0$, it is easy that φ must be positive, linear and finite on $\mathscr{X}$. Then $\int |X| d\nu_0 = \lim_n \hat{\varphi}(|X| \wedge n) = \lim_n \varphi_0(|X| \wedge n) = \lim_n \varphi(|X| \wedge n) = \varphi(|X|) < \infty$ if $X \in \mathscr{X}$, hence $\mathscr{X} \subset \mathcal{L}^1(\nu_0)$, where the first equality follows from the monotone convergence theorem, and the fourth from the Lebesgue property of φ on $\mathscr{X}$. Similarly, but with $X\mathbb{1}_{\{|X|\leq n\}}$ instead of $|X| \wedge n$, we see also that $\varphi = \hat{\varphi}|_{\mathscr{X}}$. Namely, $(\hat{\varphi}, \mathcal{L}^1(\nu_0))$ is the maximum Lebesgue-preserving extension of φ_0. $\diamondsuit$

This is just an exercise of measure theory, and we see that Question 1.1 is well-posed at least when φ_0 is linear. Slight surprisingly, the main result (Theorem 3.5) of this paper states that the answer to Question 1.1 is YES as long as the original function φ_0 is *finite everywhere* on L^∞ (this is automatic when φ is linear by definition). Moreover, the maximum extension $(\hat{\varphi}, \widehat{\mathscr{X}})$ is *explicitly constructed*.

We first construct a candidate of $\hat{\varphi}$ in a rather ad-hoc way on a certain convex cone of L^0 containing L^∞ and the positive cone L^0_+. Then based on a simple observation (Lemma 3.3), we introduce an *Orlicz-type space* associated to $\hat{\varphi}$, that we denote by $M^{\hat{\varphi}}_u$, beyond which Lebesgue-preserving extension is not possible. After checking that the candidate $\hat{\varphi}$ is well-defined on this space as a *finite* monotone convex function, we finally verify that the space $M^{\hat{\varphi}}_u$ can be made into an *order-continuous* Banach lattice with respect to a natural gauge norm (Theorem 4.9) with a suitable change of measure, which together with an extended Namioka-Klee theorem by [7] eventually yields that $\hat{\varphi}$ is Lebesgue on $M^{\hat{\varphi}}_u$ and the pair $(\hat{\varphi}, M^{\hat{\varphi}}_u)$ is the desired maximum extension. The space $M^{\hat{\varphi}}_u$ is, as the notation suggests, a subspace of the "Orlicz heart" $M^{\hat{\varphi}}$ of $\hat{\varphi}$, and the subscript "u" stands for the "uniform integrability" that characterizes the elements of $M^{\hat{\varphi}}_u$. This point will be made clear in Theorem 3.8.

As an application, we provide a characterization of the Lebesgue property of finite monotone convex functions ψ on an arbitrary solid space of random variables of the form *Fatou property plus "something extra"*, with the "extra" being either a certain "uniform integrability" or a "good" dual representation of ψ, both of which are stated using the conjugate of $\psi|_{L^\infty}$ (Theorem 3.9). This generalizes a result known as the *Jouini-Schachermayer-Touzi* theorem [21]. There the comparison of a function ψ on a solid space $\mathscr{X}$ and the maximum Lebesgue-preserving extension of the restriction $\psi|_{L^\infty}$ plays a key role.

1.1. A Motivation from Financial Mathematics: Convex Risk Measures

An initial motivation of this work was to provide an "efficient" way to the study of convex risk measures for unbounded risks. In mathematical finance, a *convex risk measure* on a solid space $\mathscr{X} \subset L^0$ is—up to a change of sign—a monotone convex function ρ on $\mathscr{X}$ such that $\rho(X + c) = \rho(X) + c$ whenever c is a constant (cash-invariance). This notion was introduced by [6, 15, 17] as a possible replacement of *Value at Risk*. See [16, Ch. 4] for the background of this notion. Since then, convex risk measures on L^∞ (i.e. for bounded risks) have been extensively studied, establishing a number of their fine properties as well as examples [see e.g. 12, 16]. However, L^∞ is clearly too small to capture the actual risks, and a key current direction is the analysis of risk measures *beyond bounded risks*. A natural way is to pick up a particular space, and then to reconstruct a whole theory with careful analysis of the structure of the new space, e.g., L^p [1, 22], Orlicz spaces/hearts [9, 27, 4, 5], abstract locally convex Fréchet lattices [7], and L^0 [23] to mention a few.

On the other hand, it seems more efficient to *extend* a convex risk measure originally defined on L^∞ to some big space, and a most natural candidate seems the one preserving the Lebesgue property. Note first that the Lebesgue property of the original risk measure on L^∞ is reasonable, since (modulo some technicality) it is necessary to have a finite valued extension to some solid space properly containing L^∞ ([11, Theorem 10]; see the paragraph after Theorem 2.4 for detail). Next, the Lebesgue property implies or is equivalent to some other important properties in application: existence of *σ-additive subgradient*, the inf-compactness of the conjugate, the continuity for the Mackey topology induced by the good dual space and so on ([21], [10] and comments after Theorem 3.9 for precise information). Also, functions with the Lebesgue property are stable for the practically common procedure of approximating unbounded random variables by suitable "truncation", and a "nearly" converse implication is also true (Remark 2.5). This is computationally useful, and it also means roughly that an extension preserving the Lebesgue property retains the basic structure of the original function to the extended domain.

Several other types of extensions may be possible of course, and some of those have already appeared in literature (see Section 2.2). Especially, [13] considered an extension preserving the *Fatou property* (order lower semicontinuity), proving that any *law-invariant* convex risk measure with the Fatou property on L^∞ is uniquely extended to L^1 preserving the Fatou property. In contrast, a simple example shows that Lebesgue-preserving extension to L^1 or to some "common" reasonable space is not possible even if the original function is law-invariant (see Example 2.6 and discussion that precedes). Thus it is worthwhile to ask how far a convex risk measure originally defined on L^∞ with the Lebesgue property can be extended preserving the Lebesgue property, or more intuitively, how far a "good" risk measure can remain "good". In Section 7, we shall examine our main results in the context of convex risk measures with some concrete examples.

2. Preliminaries

We use the probabilistic notation. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space which will be fixed throughout, and $L^0 := L^0(\Omega, \mathcal{F}, \mathbb{P})$ denotes the space of all equivalence classes of measurable functions (or random variables) over $(\Omega, \mathcal{F}, \mathbb{P})$ modulo $\mathbb{P}$-almost sure (a.s.) equality. As usual, we do not distinguish an element of L^0 and its representatives, and inequalities between (classes of) measurable functions are to be understood in the a.s. sense, i.e., $X \leq Y$ a.s. which means more precisely that $f \leq g$ a.s. for any representatives f and g of X and Y, respectively. This a.s. pointwise inequality defines a partial order on L^0 by which L^0 is an *order-complete* Riesz space (vector lattice) with the *countable-sup property*. By a *solid space* $\mathscr{X}$, we mean, in this paper, a *solid vector subspace* (ideal) $\mathscr{X}$ of L^0, i.e., a vector subspace of L^0 such that $|X| \leq |Y|$ and $Y \in \mathscr{X}$ imply $X \in \mathscr{X}$ (solid). Note that any such $\mathscr{X}$ is an order complete Riesz space with the countable sup-property on its own right, and $\mathscr{X}$ contains $L^\infty := L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ as soon as it contains the constants. We denote $\mathscr{X}_+ := \{X \in \mathscr{X} : X \geq 0\}$ (the positive cone). Finally, we write $\mathbb{E}[X] = \int_\Omega X(\omega)\mathbb{P}(d\omega)$ (expectation w.r.t. $\mathbb{P}$) for $X \in L^0$ as long as the integral makes sense, and $\mathbb{E}_Q[X] = \int_\Omega X(\omega)Q(d\omega)$ for other probability measures $Q \ll \mathbb{P}$.

By a *monotone* convex function on a solid space $\mathscr{X} \subset L^0$, we mean a proper convex function $\varphi : \mathscr{X} \to (-\infty, \infty]$ which is monotone *increasing* in the a.s. order:

$$\forall X, Y \in \mathscr{X},\ X \leq Y\,(\text{a.s.}) \ \Rightarrow\ \varphi(X) \leq \varphi(Y). \tag{2.1}$$

Definition 2.1. For a monotone convex function φ on a solid space $\mathscr{X} \subset L^0$, we say that

(1) φ satisfies the *Fatou property* (or simply φ is Fatou) if for any $(X_n)_n \subset \mathscr{X}$,

(2.2) $$\exists Y \in \mathscr{X}_+ \text{ such that } |X_n| \leq Y, \forall n \text{ and } X_n \to X \text{ a.s. } \Rightarrow \varphi(X) \leq \liminf_n \varphi(X_n).$$

(2) φ satisfies the *Lebesgue property* (or φ is Lebesgue) if for any $(X_n)_n \subset \mathscr{X}$,

(2.3) $$\exists Y \in \mathscr{X}_+ \text{ such that } |X_n| \leq Y, \forall n \text{ and } X_n \to X \text{ a.s. } \Rightarrow \varphi(X) = \lim_n \varphi(X_n).$$

Remark 2.2 (Lebesgue property and order-continuity). By the countable-sup property of $\mathscr{X}$ (as a solid vector subspace (ideal) of L^0), the Lebesgue property (2.3) is equivalent to the generally stronger *order continuity*: $\varphi(X_\alpha) \to \varphi(X)$ if a net X_α converges *in order* to X ($X_\alpha \xrightarrow{o} X$), i.e., if there exists a decreasing net $(Y_\alpha)_\alpha \subset \mathscr{X}$ (with the same index set) such that $|X - X_\alpha| \leq Y_\alpha \downarrow 0$ (in the lattice sense). Indeed, for a *sequence* (or slightly more generally a *countable net*) $(X_n)_n \subset \mathscr{X}$, the order convergence $X_n \xrightarrow{o} X$ is equivalent to the *dominated a.s. convergence*: $|X_n| \leq Y$ ($\forall n$) for some $Y \in \mathscr{X}_+$ and $X_n \to X$ a.s., thus the Lebesgue property (2.3) is nothing but the *σ-order continuity*. On the other hand, for monotone (increasing) functions, the order continuity is equivalent to the continuity from above: $X_\alpha \downarrow X \Rightarrow \varphi(X_\alpha) \downarrow \varphi(X)$, and by the countable-sup property, any such decreasing net admits a sequence $(X_{\alpha_n})_n \subset (X_\alpha)_\alpha$ such that $X_{\alpha_n} \downarrow X$. Consequently, the σ-order continuity implies $\varphi(X) \leq \lim_\alpha \varphi(X_\alpha) = \inf_\alpha \varphi(X_\alpha) \leq \inf_n \varphi(X_{\alpha_n}) = \varphi(X)$. A similar remark applies also to the Fatou property (2.2) and the order-*lower semicontinuity*. For further information, see e.g. [2, Ch. 8, 9]. ♦

The Lebesgue and Fatou properties are more "universal" than the corresponding topological regularities as long as we discuss functions of random variables, in the sense that they are comparable between different spaces. In fact, it is clear from the definition that if $\mathscr{X}$ and $\mathscr{Y}$ are solid spaces with $\mathscr{X} \subset \mathscr{Y} (\subset L^0)$ and if a function φ on $\mathscr{Y}$ has the Lebesgue property, then the restriction $\varphi|_{\mathscr{X}}$ automatically has the Lebesgue property on $\mathscr{X}$, and the same is true for the Fatou property. In particular, the class of monotone convex functions with the Lebesgue property on solid spaces $(\varphi, \mathscr{X})$ is partially ordered simply by $(\varphi, \mathscr{X}) \leq (\psi, \mathscr{Y})$ iff $\mathscr{X} \subset \mathscr{Y}$ and $\varphi = \psi|_{\mathscr{X}}$, and the maximum extension preserving the Lebesgue property does make sense, while, for instance, maximum extension of norm-continuous function on L^∞ preserving the topological continuity does not much make sense:

Definition 2.3 (Lebesgue Extension). Let $\mathscr{X}_0 \subset L^0$ be a solid space and $\varphi_0 : \mathscr{X}_0 \to (-\infty, \infty]$ a monotone convex function with the Lebesgue property (2.3) on $\mathscr{X}_0$. Then we say that $(\varphi, \mathscr{X})$ is a *Lebesgue extension* of $(\varphi_0, \mathscr{X}_0)$ if $\mathscr{X} \subset L^0$ is a solid space containing $\mathscr{X}_0$, $\varphi : \mathscr{X} \to (-\infty, \infty]$ is a monotone convex function with the Lebesgue property on $\mathscr{X}$ and $\varphi_0 = \varphi|_{\mathscr{X}}$. If there exists a Lebesgue extension $(\hat{\varphi}, \widehat{\mathscr{X}})$ such that $\mathscr{X} \subset \widehat{\mathscr{X}}$ and $\varphi = \hat{\varphi}|_{\mathscr{X}}$ for any Lebesgue extension $(\varphi, \mathscr{X})$ of $(\varphi_0, \mathscr{X}_0)$, then we say that $(\hat{\varphi}, \widehat{\mathscr{X}})$ is the *maximum Lebesgue extension* of $(\varphi_0, \mathscr{X}_0)$.

If there is no risk of confusion, we omit $\mathscr{X}_0$ and simply say e.g. $(\varphi, \mathscr{X})$ is a Lebesgue extension of φ_0. In fact, we shall be discussing in the sequel the Lebesgue extensions of a monotone convex function φ_0 on L^∞, i.e., always $\mathscr{X}_0 = L^\infty$.

2.1. Monotone Convex Functions on L^∞

Here we briefly summarize some basic facts on the monotone convex functions on L^∞. Note first that the Fatou and Lebesgue properties (2.3) and (2.2), respectively, for a proper convex function φ on L^∞ are equivalently stated as

(2.2$_\infty$) $$\sup_n \|X_n\|_\infty < \infty \text{ and } X_n \to X \text{ a.s. } \Rightarrow \varphi(X) \leq \liminf_n \varphi(X_n),$$

(2.3$_\infty$) $$\sup_n \|X_n\|_\infty < \infty \text{ and } X_n \to X \text{ a.s. } \Rightarrow \varphi(X) = \lim_n \varphi(X_n),$$

while (2.2$_\infty$) is equivalent to the lower semicontinuity w.r.t. $\sigma(L^\infty, L^1)$ (the weak* topology). Indeed, a *convex* set $C \subset L^\infty$ is $\sigma(L^\infty, L^1)$-closed if and only if for every $c > 0$, $C \cap \{X : \|X\|_\infty \le c\}$ is closed in L^0 which is a well-known consequence of the Krein-Šmulian theorem (see e.g. [19]). Thus by Fenchel-Moreau theorem, the Fatou property of a proper convex function φ on L^∞ is equivalent to the dual representation

$$\varphi(X) = \sup_{Z\in L^1}(\mathbb{E}[XZ] - \varphi^*(Z)) \tag{2.4}$$

where φ^* is the Fenchel-Legendre transform (conjugate) of φ in $\langle L^\infty, L^1\rangle$ duality:

$$\varphi^*(Z) := \sup_{X\in L^\infty}(\mathbb{E}[XZ] - \varphi(X)), \quad \forall Z \in L^1, \tag{2.5}$$

Then the monotonicity of φ is equivalent to $\mathrm{dom}\varphi^* \subset L^1_+$, i.e.,

$$Z \in L^1,\ \varphi^*(Z) < \infty \ \Rightarrow\ Z \ge 0. \tag{2.6}$$

The next characterization of the Lebesgue property (2.3$_\infty$) is a ramification of a result known as the *Jouini-Schachermayer-Touzi theorem* (JST in short) in financial mathematics. In the case of convex risk measure (up to change of sign, i.e. $\varphi(X + c) = \varphi(X) + c$ if $c \in \mathbb{R}$), it was first obtained by [21] with an additional separability assumption, and the latter assumption was removed later by [10] using a homogenization trick. See also [26, 27].

Theorem 2.4 (cf. [21, 10, 26, 27] for convex risk measures). *For a finite monotone convex function $\varphi : L^\infty \to \mathbb{R}$ satisfying the Fatou property (2.2$_\infty$), the following are equivalent:*

(1) φ has the Lebesgue property (2.3$_\infty$);
(2) $\{Z \in L^1 : \varphi^*(Z) \le c\}$ is weakly compact in L^1 for each $c > 0$;
(3) for each $X \in L^\infty$, the supremum $\sup_{Z\in L^1}(\mathbb{E}[XZ] - \varphi^*(Z))$ is attained;
(4) φ is continuous for the Mackey topology $\tau(L^\infty, L^1)$.

Proof. (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) can be proved in the same way as [21], while given the finiteness and $\sigma(L^\infty, L^1)$-lower semicontinuity of φ, (2) $\Leftrightarrow$ (4) is also a well-known fact in convex analysis (e.g. [25, Propositions 1 and 2]). For (3) $\Rightarrow$ (2), observe that for each $Z \in L^1$ and $\alpha > 0$, $\varphi^*(Z) \ge \mathbb{E}[\alpha\mathrm{sgn}(Z)Z] - \varphi(\alpha\mathrm{sgn}(Z)) \ge \alpha\|Z\|_1 - \varphi(-\alpha)$ where $\mathrm{sgn}(Z) := \mathbb{1}_{\{Z>0\}} - \mathbb{1}_{\{Z<0\}} \in L^\infty$. Since φ is finite-valued, this shows that $\lim_{\|Z\|_1\to\infty}\varphi^*(Z)/\|Z\|_1 = \infty$ (i.e., φ^* is coercive). Then the implication (3) $\Rightarrow$ (2) follows from *coercive James's theorem* due to [26] (recalled below as Theorem 5.2). □

Finally, we note that the Lebesgue property on L^∞ is reasonable. In fact, when $(\Omega, \mathcal{F}, \mathbb{P})$ is *atomless* (which is not a restriction in practice), a sufficient condition for the Lebesgue property (2.3$_\infty$) on L^∞ for monotone convex function φ is that it has a *finite-valued* extension to a solid space $\mathscr{X} \supsetneq L^\infty$ such that $X \in \mathscr{X}$ and $\mathrm{law}(Y) = \mathrm{law}(X) \Rightarrow Y \in \mathscr{X}$ (*rearrangement invariant*). See [11, Th. 3] where this is proved for convex risk measures, and an almost same proof still works for general *finite* monotone convex functions. All L^p $(0 \le p \le \infty)$, Orlicz spaces and Orlicz hearts (the Morse subspaces of the corresponding Orlicz spaces) are of this type. Thus functions φ that violate this condition are rarely of practical interest.

2.2. Other extensions and general remarks

We emphasize that the preservation of the Lebesgue property is crucial. In fact, *any finite* monotone convex function on L^∞ has *an* extension to the whole L^0 if one does not mind any regularity or uniqueness. Indeed, let

$$\varphi_{\mathrm{ext}}(X) := \lim_n \lim_m \varphi_0((X \vee (-n) \wedge m), \quad X \in L^0. \tag{2.7}$$

Noting that $(X \vee (-n)) \wedge m = X$ if $\|X\|_\infty \le m, n < \infty$, this is well-defined on L^0 with values in $[-\infty, \infty]$, and $\varphi_{\mathrm{ext}}|_{L^\infty} = \varphi_0$. But it is not a regular nor unique extension in any reasonable sense, or it may even be improper. In the context of convex risk measures, [8] studied this type extension, providing a necessary and sufficient condition for φ_{ext} to avoid the value $-\infty$ (hence proper), but even in that case, we have no regularity nor uniqueness.

Remark 2.5. In application, one often hopes to approximate *unbounded* $X \in L^0$ by *bounded* ones via suitable *truncation* as $X\mathbb{1}_{\{|X|\le n\}} \xrightarrow{n} X$, $(X \vee (-m)) \wedge n \xrightarrow{n,m} X$. As these convergences are *order convergences*, Remark 2.2 tells us that monotone convex functions φ with the Lebesgue property are stable for this sort of approximations:

$$\varphi(X) = \lim_{m\to\infty} \lim_{n\to\infty} \varphi((X \vee (-m)) \wedge n) = \lim_{n\to\infty} \varphi(X\mathbb{1}_{\{|X|\le n\}}), \tag{2.8}$$

and two limits in the middle expression are interchangeable. In fact, a sort of converse is also true: a finite monotone convex function φ with the Fatou property on a solid space $\mathscr{X} \subset L^0$ has the Lebesgue property if and only if for any *countable* net $(X_\alpha)_\alpha$,

$$X_\alpha \in L^\infty,\ |X_\alpha| \le |X|,\ \forall\alpha, \text{ and } X_\alpha \to X \text{ a.s. } \Rightarrow \varphi(X_\alpha) \to \varphi(X). \tag{2.9}$$

See Proposition A.2. In particular, the maximum Lebesgue extension tells us the precise extent to which *any "reasonable" truncation procedures* safely work. ♦

A closely related question, recently addressed by [13], is the extension preserving the Fatou property (instead of Lebesgue). There the "L^1-closure" of φ_0 given by $\bar{\varphi}^1_0(X) := \sup_{Y\in L^\infty}(\mathbb{E}[XY] - \varphi_0^*(Y))$ on L^1 is considered. This is clearly proper and (weakly) lower semicontinuous (hence Fatou) on L^1 as soon as $\mathrm{dom}\varphi_0^* \cap L^\infty \neq \emptyset$, while it is not clear if $\bar{\varphi}^1$ is an extension of φ_0, i.e., if $\bar{\varphi}^1|_{L^\infty} = \varphi_0$. [13, Theorem 2.2] proved that this is the case if φ_0 is *law-invariant* (i.e. $X \overset{\text{law}}{=} Y \Rightarrow \varphi_0(X) = \varphi_0(Y)$), and then $\bar{\varphi}^1$ is the *unique* lower semi-continuous extension of φ to L^1. In particular, every *law-invariant* convex risk measure has a "Fatou" extension to L^1. In contrast, the Lebesgue property may not be preserved to L^1 (even if law-invariant) as the next example illustrates.

Example 2.6 (Modular). Let $\Phi : \mathbb{R} \to \mathbb{R}_+$ be a lower semicontinuous even convex function with $\Phi(0) = 0$, and $\lim_{x\to\infty} \Phi(x) = \infty$ (i.e., a *finite Young function*). Then put

$$\rho_\Phi(X) := \mathbb{E}[\Phi(X^+)] = \mathbb{E}[\Phi(X \vee 0)],\ X \in L^0. \tag{2.10}$$

This is clearly a law-invariant $[0, \infty]$-valued monotone convex function with $\rho_\Phi(0) = 0$ satisfying the Fatou property on the whole L^0 (by Fatou's lemma since $\Phi \ge 0$). Let

$$L^\Phi := \{X \in L^0 : \exists\alpha > 0,\ \mathbb{E}[\Phi(\alpha|X|)] < \infty\} \quad \text{(Orlicz space)}, \tag{2.11}$$

$$M^\Phi := \{X \in L^0 : \forall\alpha > 0,\ \mathbb{E}[\Phi(\alpha|X|)] < \infty\} \quad \text{(Orlicz heart)}. \tag{2.12}$$

◊

It always holds $L^\infty \subset M^\Phi \subset L^\Phi \subset L^1$ and $M^\Phi = L^\Phi$ if Φ satisfies the so-called Δ_2-condition, while if for example $\Phi(x) = e^{|x|} - 1$ and $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless, then $L^\infty \subsetneq M^\Phi \subsetneq L^\Phi \subsetneq L^1$. The function ρ_Φ is Lebesgue on M^Φ since $|X_n| \le |Y|$ with $Y \in M^\Phi$ and $X_n \to X$ a.s. imply $|\Phi(X_n^+)| \le \Phi(|Y|) \in L^1$, hence $\rho_\Phi(X_n) = \mathbb{E}[\Phi(X_n^+)] \to \mathbb{E}[\Phi(X^+)] = \rho_\Phi(X)$ by dominated convergence. On the other hand, ρ_Φ is *not Lebesgue* on L^Φ unless $M^\Phi = L^\Phi$. Indeed, if $X \in L^\Phi \setminus M^\Phi$, and $\alpha > 0$ is such that $\mathbb{E}[\Phi(\alpha|X|)] = \infty$, then $\rho_\Phi(\alpha|X|\mathbb{1}_{\{|X|>n\}}) = \mathbb{E}[\Phi(\alpha|X|)\mathbb{1}_{\{|X|>n\}}] \equiv \infty$ for all n while $0 \le \alpha|X|\mathbb{1}_{\{|X|>n\}} \le \alpha|X|$ and $\alpha|X|\mathbb{1}_{\{|X|>n\}} \to 0$ a.s. By the law-invariance and [13], (ρ_Φ, L^1) is the unique Fatou-preserving extension of $(\rho_\Phi|_{L^\infty}, L^\infty)$ which is not Lebesgue on $L^\Phi \subsetneq L^1$. Consequently, $\rho_\Phi|_{L^\infty}$ has no Lebesgue extension to L^1.

3. Statements of Main Results

We begin with a couple of elementary observations. Let $\varphi_0 : L^\infty \to \mathbb{R}$ be a finite monotone convex function with the Fatou property (2.2_∞) hence represented as (2.4) by the conjugate $\varphi_0^*(Z) = \sup_{X \in L^\infty}(\mathbb{E}[XZ] - \varphi_0(X))$ $(Z \in L^1)$. Let

$$\mathcal{D}_0 := \left\{ X \in L^0 : X^- Z \in L^1, \ \forall Z \in \mathrm{dom}\varphi_0^* \right\}. \tag{3.1}$$

This is not a vector space, but a convex cone containing $L^\infty \cup L^0_+$, which is *upward solid* in the sense that $X \in \mathcal{D}_0$ and $X \le Y$, then $Y \in \mathcal{D}_0$ since then $Y^- \le X^-$. We then define

$$\hat{\varphi}(X) := \sup_{Z \in \mathrm{dom}\varphi_0^*} \left(\mathbb{E}[XZ] - \varphi_0^*(Z) \right), \quad \forall X \in \mathcal{D}_0, \tag{3.2}$$

where $\mathrm{dom}\varphi_0^* := \{Z \in L^1 : \varphi_0^*(Z) < \infty\} \subset L^1_+$ (by (2.6)). This is well-defined with values in $(-\infty, \infty]$ and is *continuous from below*:

Lemma 3.1. *Let φ_0 be a finite monotone convex function with the Fatou property on L^∞. Then $\hat{\varphi}$ defined by (3.2) is a proper monotone convex function on $\mathcal{D}_0$ with $\hat{\varphi}|_{L^\infty} = \varphi_0$ and*

$$X_n \in \mathcal{D}_0, \ X_n \uparrow X \in L^0 \ \text{a.s.} \ \Rightarrow \ \hat{\varphi}(X) = \lim_n \hat{\varphi}(X_n). \tag{3.3}$$

Proof. It is clear from the Fatou property that $\hat{\varphi}|_{L^\infty} = \varphi_0$, and in particular, it is proper. Since $\hat{\varphi}$ is a point-wise supremum of proper convex functions $X \mapsto \mathbb{E}[XZ] - \varphi_0^*(Z)$ $(Z \in \mathrm{dom}\varphi_0^*)$, $\hat{\varphi}$ is convex. If $X_n \in \mathcal{D}_0$ for each n, and if $X_n \uparrow X$ a.s. for some $X \in L^0$, we see that $X \in \mathcal{D}_0$ as well (since $\mathcal{D}_0$ is upward solid) and that $\mathbb{E}[XZ] = \sup_n \mathbb{E}[X_n Z]$ for all $Z \in \mathrm{dom}\varphi_0^* \subset L^1_+$ by the monotone convergence theorem since $X_1^- Z \in L^1$, hence

$$\begin{aligned}\hat{\varphi}(X) &= \sup_{Z \in \mathrm{dom}\varphi_0^*} \left(\sup_n \mathbb{E}[X_n Z] - \varphi_0^*(Z) \right) = \sup_n \sup_{Z \in \mathrm{dom}\varphi_0^*} \left(\mathbb{E}[X_n Z] - \varphi_0^*(Z) \right) \\ &= \sup_n \hat{\varphi}(X_n).\end{aligned}$$

Thus we have (3.3). □

In the sequel, we always suppose the following without further notice:

Assumption 3.2. *φ_0 is a finite-valued monotone convex function on L^∞ satisfying the Lebesgue property (2.3_∞) and $\varphi_0(0) = 0$.*

The last assumption is just for notational simplicity. Indeed, we can replace φ_0 by $\varphi_0 - \varphi_0(0)$ since φ_0 is supposed to be finite, and $(\varphi, \mathscr{X})$ is a Lebesgue extension of (φ_0, L^∞) if and only if $(\varphi - \varphi_0(0), \mathscr{X})$ is a Lebesgue extension of $(\varphi_0 - \varphi_0(0), L^\infty)$.

Suppose that $(\varphi, \mathscr{X})$ is a Lebesgue extension of φ_0 in the sense of Definition 2.3. Then observe that for any $Y \in \mathscr{X}_+$, $|Y \wedge n| \le Y$ and $Y \wedge n \uparrow Y$ a.s., hence the Lebesgue property of φ on $\mathscr{X}$, the continuity from below of $\hat{\varphi}$ on L^0_+ and $\varphi|_{L^\infty} = \varphi_0 = \hat{\varphi}|_{L^\infty}$ show that $\varphi(Y) = \lim_n \varphi(Y \wedge n) = \lim_n \hat{\varphi}(Y \wedge n) = \hat{\varphi}(Y)$. In particular,

Lemma 3.3. *Let $(\varphi, \mathscr{X})$ be a Lebesgue extension of φ_0. Then for any $X \in \mathscr{X}$,*

$$\lim_N \hat{\varphi}(\alpha|X|\mathbb{1}_{\{|X|>N\}}) = \lim_N \varphi(\alpha|X|\mathbb{1}_{\{|X|>N\}}) = 0, \ \forall \alpha > 0. \tag{3.4}$$

Proof. If $X \in \mathscr{X}$, then $X_N^\alpha := \alpha|X|\mathbb{1}_{\{|X|>N\}} \in \mathscr{X}$, $0 \le X_N^\alpha \le \alpha|X| \in \mathscr{X}$ (by the solidness), and $X_N^\alpha \to 0$ a.s. as $N \to \infty$. Hence $\hat{\varphi}(X_N^\alpha) = \varphi(X_N^\alpha) \to 0$ by the Lebesgue property of φ on $\mathscr{X}$ and $\hat{\varphi}(Y) = \varphi(Y)$ for $Y \in \mathscr{X}_+$. □

This leads us to the following definition:

$$M_u^{\hat{\varphi}} := \left\{X \in L^0 : \lim_N \hat{\varphi}\left(\alpha|X|\mathbb{1}_{\{|X|>N\}}\right) = 0, \ \forall \alpha > 0\right\}. \tag{3.5}$$

At the first glance, we note that this is well-defined since $L^0_+ \subset \mathcal{D}_0$ and that $M_u^{\hat{\varphi}}$ is a solid vector space. Indeed, the linearity follows from the observation that $|X+Y|\mathbb{1}_{\{|X+Y|>N\}} \le 2|X|\mathbb{1}_{\{|X|>N/2\}} + 2|Y|\mathbb{1}_{\{|Y|>N/2\}}$, while the solidness is a consequence of the monotonicity of $\hat{\varphi}$ (and of $x \mapsto |x|\mathbb{1}_{\{|x|>N\}}$).

Next, we see that $\hat{\varphi}$ is well-defined on $M_u^{\hat{\varphi}}$. Observe first from the definition (3.2) that

$$\mathbb{E}[\alpha|X|Z] \le \hat{\varphi}(\alpha|X|) + \varphi_0^*(Z), \quad \forall \alpha > 0, \ X \in L^0, \ Z \in \mathrm{dom}\varphi_0^*. \tag{3.6}$$

Thus $\mathcal{D}_0 \cap (-\mathcal{D}_0)$ contains the *Orlicz space* and *Orlicz heart* of $\hat{\varphi}$:

$$L^{\hat{\varphi}} := \left\{X \in L^0 : \exists \alpha > 0, \ \hat{\varphi}(\alpha|X|) < \infty\right\}, \tag{3.7}$$

$$M^{\hat{\varphi}} := \left\{X \in L^0 : \forall \alpha > 0, \ \hat{\varphi}(\alpha|X|) < \infty\right\}. \tag{3.8}$$

Thus $\hat{\varphi}$ is well-defined on $L^{\hat{\varphi}}$ as a proper monotone convex function, and it is finite on $M^{\hat{\varphi}}$ (since $\hat{\varphi}(X) \le \hat{\varphi}(|X|) < \infty$ if $X \in M^{\hat{\varphi}}$). Also, for any $\alpha > 0$, $X \in L^0$ and $N \in \mathbb{N}$,

$$\hat{\varphi}(\alpha|X|) \le \frac{1}{2}\hat{\varphi}(2\alpha|X|\mathbb{1}_{\{|X|>N\}}) + \frac{1}{2}\varphi_0(2\alpha N) \tag{3.9}$$

The second term in the right hand side is always finite since φ_0 is supposed to be finite, and if $X \in M_u^{\hat{\varphi}}$, then for any $\alpha > 0$, the first term is *eventually finite*, thus $M_u^{\hat{\varphi}} \subset M^{\hat{\varphi}} \subset L^{\hat{\varphi}} \subset \mathcal{D}_0$. Therefore, $\hat{\varphi}$ is well-defined on $M^{\hat{\varphi}}$ as a *finite-valued* monotone convex function.

Remark 3.4. The same argument together with (3.4) tells us also that only *finite-valued* functions can be Lebesgue extensions of φ_0 as long as the original function φ_0 is finite. ♦

3.1. Maximum Lebesgue Extension

With these preparation, we now give a positive answer to Question 1.1:

Theorem 3.5. *Suppose Assumption 3.2. Then the pair $(\hat{\varphi}, M_u^{\hat{\varphi}})$, defined by (3.2) and (3.5), is the maximum Lebesgue extension of φ_0, I.e.,*

(1) $M_u^{\hat{\varphi}}$ *is a solid subspace of* L^0 *containing the constants,* $\hat{\varphi} : M_u^{\hat{\varphi}} \to \mathbb{R}$ *is a monotone convex function with the Lebesgue property (1.1) on* $M_u^{\hat{\varphi}}$ *and* $\hat{\varphi}|_{L^\infty} = \varphi_0$*;*
(2) if $(\varphi, \mathscr{X})$ *is a pair satisfying the conditions of (1), then* $\mathscr{X} \subset M_u^{\hat{\varphi}}$ *and* $\varphi = \hat{\varphi}|_{\mathscr{X}}$*.*

A proof will be given in Section 4.2. Here we briefly describe the basic idea. We already know that $M_u^{\hat{\varphi}}$ is a solid subspace of L^0, $\hat{\varphi}$ is well-defined and finite on $M_u^{\hat{\varphi}}$ with $\hat{\varphi}|_{L^\infty} = \varphi_0$ and that if $(\varphi, \mathscr{X})$ is another Lebesgue extension of φ_0, then $\mathscr{X} \subset M_u^{\hat{\varphi}}$ (Lemma 3.3). It remains only to show that $\hat{\varphi}$ has the Lebesgue property on $M_u^{\hat{\varphi}}$ which implies also that for any $X \in \mathscr{X} \subset M_u^{\hat{\varphi}}$, $\varphi(X) = \lim_n \varphi(X\mathbb{1}_{\{|X|\leq n\}}) = \lim_n \hat{\varphi}(X\mathbb{1}_{\{|X|\leq n\}}) = \hat{\varphi}(X)$. The key to the Lebesgue property of $\hat{\varphi}$ on $M_u^{\hat{\varphi}}$ is that, after a suitable change of measure, $M_u^{\hat{\varphi}}$ can be made into an *order-continuous Banach lattice* with the gauge norm induced by $\hat{\varphi}$. Having established this, we can appeal to the extended Namioka-Klee theorem that asserts that any *finite* monotone convex function on a Banach lattice is norm-continuous, and the order-continuity of the norm then concludes the proof.

Our next interest is to understand the relation between three spaces $M_u^{\hat{\varphi}}$, $M^{\hat{\varphi}}$ and $L^{\hat{\varphi}}$ as the latter two seem more familiar. We already know, by definition, $M_u^{\hat{\varphi}} \subset M^{\hat{\varphi}} \subset L^{\hat{\varphi}}$. In general, however, these inclusions may be strict as the following examples illustrate.

Example 3.6 (Classical Orlicz Spaces). Let Φ and ρ_Φ be as in Example 2.6 and put $\varphi_0 = \rho_\Phi$. Since ρ_Φ is continuous from below on L^0, we still have $\hat{\varphi} = \rho_\Phi$ on L^0_+ by Lemma 3.1. Then clearly $M^{\hat{\varphi}} = M^\Phi \subset L^\Phi = L^{\hat{\varphi}}$, and the inclusion is strict if $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless and $\Phi(x) = e^{|x|}-1$. Furthermore in this case, we have $M_u^{\hat{\varphi}} = M^\Phi (= M^{\hat{\varphi}})$. Indeed, if $X \in M^\Phi$ ($\Leftrightarrow \Phi(\alpha|X|) \in L^1$, $\forall \alpha > 0$), then $\hat{\varphi}(\alpha|X|\mathbb{1}_{\{|X|>N\}}) = \mathbb{E}[\Phi(\alpha|X|)\mathbb{1}_{\{|X|>N\}}] \to 0$ by dominated convergence. $\diamondsuit$

The next example shows that the inclusion $M_u^{\hat{\rho}} \subset M^{\hat{\rho}}$ may be strict.

Example 3.7. Let $(\Omega, \mathcal{F}) = (\mathbb{N}, 2^{\mathbb{N}})$, with $\mathbb{P}$ given by $\mathbb{P}(\{n\}) = 2^{-n}$, and $(Q_k)_k$ a sequence of probabilities on $2^{\mathbb{N}}$ given by $Q_1(\{1\}) = 1$, $Q_n(\{1\}) = 1 - 1/n$ and $Q_n(\{n\}) = 1/n$ for each n. Then define $\varphi(X) = \sup_n \mathbb{E}_{Q_n}[X]$. This is clearly monotone, convex, and positively homogeneous ($\varphi(\alpha X) = \alpha\varphi(X)$ for $\alpha \geq 0$), hence φ^* is $\{0,1\}$-valued. By Hahn-Banach, we see that $\varphi^*(Z) = 0$ if and only if $Z \in \overline{\mathrm{conv}}(dQ_n/d\mathbb{P}, n \in \mathbb{N}) =: \mathcal{Z}$, and it is clear that $\mathcal{Z}$ is uniformly integrable (thus weakly compact), and φ has the Lebesgue property on $L^\infty \simeq l^\infty$. Also, $\hat{\varphi}(X) = \sup_n \mathbb{E}_{Q_n}[X]$ is valid for all $X \geq 0$.

Now consider a non-negative function $X(k) = k$. Then $\mathbb{E}_{Q_n}[X] = (1-1/n)+n\cdot(1/n) = 2-1/n$, hence $\hat{\varphi}(\alpha|X|) = \alpha \sup_n \mathbb{E}_{Q_n}[X] = 2\alpha < \infty$, thus $X \in M^{\hat{\varphi}}$. On the other hand, $\mathbb{E}_{Q_n}[X\mathbb{1}_{\{X>N\}}] = \mathbb{1}_{\{n>N\}}$, thus for any $\alpha > 0$, $\hat{\varphi}(\alpha|X|\mathbb{1}_{\{|X|>N\}}) = \alpha \sup_n \mathbb{E}_{Q_n}[X\mathbb{1}_{\{X>N\}}] \equiv \alpha$ for all N. Hence $X \notin M_u^{\hat{\varphi}}$, and consequently, $M_u^{\hat{\varphi}} \subsetneq M^{\hat{\varphi}}$. $\diamondsuit$

We now state our second result, which well-explains the reason for the subscript "u".

Theorem 3.8. *For* $X \in M^{\hat{\varphi}}$*, the following three conditions are equivalent:*

(1) $X \in M_u^{\hat{\varphi}}$*;*
(2) $\{XZ : \varphi_0^*(Z) \leq c\}$ *is uniformly integrable for all* $c > 0$*;*
(3) for some $\varepsilon > 0$, $\sup_{Z\in\mathrm{dom}\varphi_0^*}(\mathbb{E}[(|X| \vee \varepsilon)YZ] - \varphi_0^*(Z))$ *is attained for all* $Y \in L^\infty$*.*

Moreover, these three equivalent conditions imply that

$$\hat{\varphi}(X) = \max_{Z\in\mathrm{dom}\varphi_0^*}(\mathbb{E}[XZ] - \varphi_0^*(Z)), \tag{3.10}$$

i.e., the supremum in (3.2) is attained.

We prove this theorem in Section 5.

3.2. Characterization of Lebesgue Property on Solid Spaces

Here we apply our results to obtain a characterization of the Lebesgue property of finite monotone convex functions on *arbitrary solid spaces* in the spirit of Theorem 2.4 for the L^∞ case. Suppose we are given a solid space $\mathscr{X} \subset L^0$ and a finite monotone convex function $\psi : \mathscr{X} \to \mathbb{R}$ with the *Fatou property* (not Lebesgue at now). Then the restriction $\psi_\infty := \psi|_{L^\infty}$ is a finite monotone convex function on L^∞ having the Fatou property too, and putting $\psi^*_\infty(Z) = \sup_{X\in L^\infty}(\mathbb{E}[XZ] - \psi_\infty(X))$,

$$\hat{\psi}(X) := \sup_{Z\in \mathrm{dom}\psi^*_\infty}(\mathbb{E}[XZ] - \psi^*_\infty(Z)), \tag{3.11}$$

defines an extension of ψ_∞ to $\mathcal{D}_\psi := \{X \in L^0 : X^-Z \in L^1, \forall Z \in \mathrm{dom}\psi^*_\infty\} \supset L^0_+ \cup L^\infty$ by the Fatou property. Note that the monotonicity ($\Rightarrow \mathrm{dom}\psi^*_\infty \subset L^1_+$) and the finiteness of ψ on the whole $\mathscr{X}$ implies $\mathscr{X} \subset \mathcal{D}_\psi \cap (-\mathcal{D}_\psi)$, or equivalently,

$$XZ \in L^1, \ \forall X \in \mathscr{X},\ Z \in \mathrm{dom}\psi^*_\infty. \tag{3.12}$$

Thus $\hat{\psi}$ is well-defined on $\mathscr{X}$ in particular. Indeed, observe that $\mathbb{E}[|X|Z]-\psi^*_\infty(Z) = \sup_n(\mathbb{E}[|X| \wedge nZ] - \psi^*_\infty(Z)) \leq \sup_n \psi(|X| \wedge n) \leq \psi(|X|) < \infty$ for $X \in \mathscr{X}$ and $Z \in \mathrm{dom}\psi^*_\infty$ where we used Young's inequality for the pair $(\psi|_{L^\infty}, \psi^*_\infty)$.

On the other hand, the original $(\psi, \mathscr{X})$ is also an extension of ψ_∞ since the latter is the restriction of ψ. Then close comparisons of these two extensions using Theorems 3.5 and 3.8 yield the following generalization of the JST Theorem 2.4:

Theorem 3.9 (Generalization of JST-Theorem [21]). *Let $\mathscr{X} \subset L^0$ be a solid space containing the constants and $\psi : \mathscr{X} \to \mathbb{R}$ be a finite-valued monotone convex function satisfying the Fatou property (2.2) on $\mathscr{X}$. Then the following are equivalent:*

(1) ψ has the Lebesgue property (2.3) on $\mathscr{X}$;
(2) for all $X \in \mathscr{X}$ and $c > 0$, $\{XZ : \psi^*_\infty(Z) \leq c\}$ is uniformly integrable;
(3) the supremum $\sup_{Z\in\mathrm{dom}\psi^*_\infty}(\mathbb{E}[XZ] - \psi^*_\infty(Z))$ is finite and attained for all $X \in \mathscr{X}$;
(4) it holds that $\psi(X) = \max_{Z\in\mathrm{dom}\psi^*_\infty}(\mathbb{E}[XZ] - \psi^*_\infty(Z))$, $\forall X \in \mathscr{X}$.

A proof is given in Section 3.2. Note that (4) is not a paraphrasing of (3) since it is not *a priori* assumed that $\psi(X) = \sup_{Z\in\mathrm{dom}\psi^*_\infty}(\mathbb{E}[XZ] - \psi^*_\infty(Z)) = \hat{\psi}(X)$ for all $X \in \mathscr{X}$.

When $\mathscr{X} = L^\infty$, then $\psi = \hat{\psi}$ hence (3) $\Leftrightarrow$ (4) is trivial, and (2) is equivalent to saying that $\{Z \in L^1 : \psi^*_\infty(Z) \leq c\}$ is $\sigma(L^1, L^\infty)$-compact for all $c > 0$ by the Dunford-Pettis theorem. Thus, in this case, Theorem 3.9 is nothing but Theorem 2.4 which is essentially due to [21] and [10]. Some other (partial) generalizations of Theorem 2.4 have been obtained in literature, so we briefly discuss here some key features of *our* version.

Generality of the space $\mathscr{X}$ The only a priori assumption on the space $\mathscr{X}$ is that it is a solid vector subspace (ideal) of L^0 containing the constants. All Orlicz spaces and hearts as well as L^p with $p \in [0, \infty]$ are of this type. Note also that without the solidness, the Lebesgue and Fatou properties do not "well" make sense.

Our formulation is "universal" We note that topological qualifications (of $\mathscr{X}$ and ψ) are absent in our formulation: $\psi^*_\infty = (\psi|_{L^\infty})^*$ is used instead of the conjugate of ψ on the topological dual of $\mathscr{X}$, the *inf-compactness* of the conjugate is alternatively stated in a form of uniform integrability, and the Fatou and Lebesgue properties are regularities in terms of order structure.

These ingredients are in some sense more "universal" than the topological counter-parts. It should also be emphasized that our characterization is still quite explicit even though it does not rely on the topological nature of $\mathscr{X}$.

Remark 3.10. Theorem 3.9 can be alternatively stated in terms of the *order-continuous dual* of $\mathscr{X}$, which is regarded, under our assumption on $\mathscr{X}$, as the set

$$\mathscr{X}_n^\sim = \{Z \in L^0 : XZ \in L^1, \forall X \in \mathscr{X}\}. \tag{3.13}$$

via the identification of Z and the order-continuous linear functional $X \mapsto \mathbb{E}[XZ]$. Observe that $\mathrm{dom}\psi_\infty^* \subset \mathscr{X}_n^\sim \subset L^1$ by $L^\infty \subset \mathscr{X}$ and (3.12), thus "$\mathrm{dom}\psi_\infty^*$" in the statements can be replaced by $\mathscr{X}_n^\sim$. In particular, the Lebesgue property of ψ implies the "simplified dual representation" on $\mathscr{X}_n^\sim$ with the penalty function ψ_∞^* (see [7]) without any structural assumption on the space $\mathscr{X}$ (than being an ideal of L^0). Also, item (2) is in fact equivalent to the relative compactness of all the level sets $\{Z \in \mathscr{X}_n^\sim : \psi_\infty^*(Z) \le c\}$ for the weak topology $\sigma(\mathscr{X}_n^\sim, \mathscr{X})$, which is a (well-defined) locally convex Hausdorff topology as long as $\mathscr{X}$ contains the constants as we are assuming. ♦

Given the above discussion, it seems also natural (and more common) to characterize the Lebesgue property in the form of Theorem 3.9 but with the conjugate

$$\psi^*(Z) := \sup_{X\in\mathscr{X}} (\mathbb{E}[XZ] - \psi(X)),\ Z \in \mathscr{X}_n^\sim \tag{3.14}$$

instead of ψ_∞^*. In fact, the equivalence of (1) – (4) in Theorem 3.9 remains true (see [28]) with ψ^* instead of ψ_∞^* if (a) $\mathscr{X} \subset L^1(\mathbb{Q})$ for some $\mathbb{Q} \sim \mathbb{P}$ and if (b) ψ *is a priori assumed to be* $\sigma(\mathscr{X}, \mathscr{X}_n^\sim)$*-lower semicontinuous or equivalently*

$$\psi(X) = \sup_{Z\in\mathscr{X}_n^\sim} (\mathbb{E}[XZ] - \psi^*(Z)),\ \forall X \in \mathscr{X}. \tag{3.15}$$

Here (a) is rather technical, which says simply that $\mathscr{X}_n^\sim$ separates $\mathscr{X}$, and only the equivalence "$\mathbb{Q} \sim \mathbb{P}$" is essential since that $\mathscr{X}$ accommodates a *finite* monotone convex function ψ with the Fatou property already implies the existence of $\mathbb{Q} \ll \mathbb{P}$ such that $\mathscr{X} \subset L^1(\mathbb{Q})$. The assumption (b) ($\Leftrightarrow$ (3.15)) implies the Fatou property (see [7, Proposition 1]), but the converse is not generally true, and (b) may not be easy to check. In some "good" cases, however, (b) is actually equivalent to the Fatou property, and the "good" cases include $\mathscr{X} = L^\infty$ ($\Rightarrow \mathscr{X}_n^\sim = L^1$), $\mathscr{X} = M^\Phi$ with finite Young function Φ (then $\mathscr{X}_n^\sim = L^{\Phi^*}$), and $\mathscr{X} = L^\Phi$ with Φ satisfying the so-called Δ_2-condition (then $L^\Phi = M^\Phi$). For more general $\mathscr{X}$, however, it is still open when the Fatou property implies the $\sigma(\mathscr{X}, \mathscr{X}_n^\sim)$-lower semicontinuity for all convex functions.

Remark 3.11. The above question is equivalent to asking if all *order closed convex* subsets of $\mathscr{X}$ are $\sigma(\mathscr{X}, \mathscr{X}_n^\sim)$-closed. This is true as soon as it is shown that any $\sigma(\mathscr{X}, \mathscr{X}_n^\sim)$-convergent net $(X_\alpha)_\alpha$ in $\mathscr{X}$ admits a sequence of indices $(\alpha_n)_n$ as well as a sequence $\tilde{X}_n \in \mathrm{conv}(X_{\alpha_n}, X_{\alpha_{n+1}}, \dots)$ which converges *in order* to the same limit. In [7, Lemma 6 and Corollary 4], it is claimed that the last property is true whenever (adapted to our notation) $\mathscr{X}$ is (lattice homomorphic to) and ideal of L^1 (hence of L^0). Unfortunately, however, their proof has an error. There it is shown that with the above assumption, any $\sigma(\mathscr{X}, \mathscr{X}_n^\sim)$-convergent net $(X_\alpha)_\alpha$ admits a sequence $(\tilde{X}_n)_n$ of forward convex combinations of the above form which, as a sequence in L^1, converges *in order of* L^1 to the same limit. This part is correct. Then it was concluded that $\tilde{X}_n$, *as a sequence in* $\mathscr{X}$, converges *in order of* $\mathscr{X}$ to the same limit. The last part is not true at least solely from the assumptions imposed on $\mathscr{X}$. In general, whenever $\mathscr{X}$ is an ideal of L^0, the order convergence in $\mathscr{X}$ of a sequence $(X_n)_n$ is equivalent to the dominated a.s. convergence (i.e., $X_n \to X$ a.s. and $\exists Y \in \mathscr{X}_+$ with $|X_n| \leq Y$ $(\forall n)$). The a.s. convergence is universal (which is common to all ideals of L^0), while being dominated in $\mathscr{X}$ is not universal. For a trivial example, picking $Z \in L^1_+ \setminus L^\infty$, the sequence $X_n = Z \wedge n$ which lies in L^∞ converges in order in L^1 to Z, but does not converge in order in L^∞. What we need to fill the gap is still an open question (for us). ♦

Remark 3.12. When Φ^* is finite, [27] recently obtained the equivalence of (1) – (4) with ψ^* for $\mathscr{X} = L^\Phi$, but with an even stronger assumption than (3.15) that ψ is $\sigma(L^\Phi, M^{\Phi^*})$-lower semi-continuous (note in this case that $\mathscr{X}_n^\sim = L^{\Phi^*}$ which is strictly bigger than M^{Φ^*} if the probability space is atomless and Φ does not satisfy the Δ_2-condition). When $\mathscr{X}$ is a locally convex Fréchet lattice, the implication (1) $\Rightarrow$ (4) is (implicitly) contained in [7, Lemma 7]. For the equivalence of (1) – (4) with ψ^* for general solid space $\mathscr{X}$ containing the constants under the assumptions (a) and (b) above, see [28]. ♦

Note that with the standing assumptions of Theorem 3.9 only, the inequality $\mathbb{E}[XZ] \leq \psi(X) + \psi^*_\infty(Z)$ is not guaranteed *for all* $X \in \mathscr{X}$ and $Z \in \mathscr{X}_n^\sim$ (it is true for $X \in \mathscr{X}_+ \cup L^\infty$). However, if ψ has the Lebesgue property, we see that $\mathbb{E}[XZ] = \lim_n \mathbb{E}[X\mathbb{1}_{\{|X|\leq n\}}Z] \leq \limsup_n \psi(X\mathbb{1}_{\{|X|\leq n\}}) + \psi^*_\infty(Z) = \psi(X) + \psi^*_\infty(Z)$. Thus (1) $\Rightarrow$ (4) shows that

Corollary 3.13. *For a finite monotone convex function ψ on a solid vector space $\mathscr{X} \subset L^0$, the Lebesgue property implies the existence of a σ-additive subgradient of ψ at everywhere on $\mathscr{X}$, that is, for all $X \in \mathscr{X}$, there exists a $Z \in \mathscr{X}_n^\sim \subset L^1$ such that*

$$\mathbb{E}[XZ] - \psi(X) \geq \mathbb{E}[YZ] - \psi(Y),\ \forall Y \in \mathscr{X}.$$

4. Analysis of the space $M^{\hat{\varphi}}_u$ and Proof of Theorem 3.5

Throughout this section, Assumption 3.2 is in force unless the contrary is explicitly stated. The key to the proof of Theorem 3.5 is the analysis of the Orlicz-type space $M^{\hat{\varphi}}_u$.

4.1. The Gauge of $\hat{\varphi}$

Let us define the *gauge* of the monotone convex function $\hat{\varphi}$:

$$\|X\|_{\hat{\varphi}} := \inf\{\lambda > 0 : \hat{\varphi}(|X|/\lambda) \leq 1\},\ \forall X \in L^0, \tag{4.1}$$

with the convention $\inf \emptyset = +\infty$. In analogy to the Luxemburg norms of usual Orlicz spaces, we see easily that for any $X, Y \in L^0$ and $\alpha \in \mathbb{R}$,

$$\|\alpha X\|_{\hat{\varphi}} = |\alpha|\|X\|_{\hat{\varphi}},\ \|X+Y\|_{\hat{\varphi}} \leq \|X\|_{\hat{\varphi}} + \|Y\|_{\hat{\varphi}} \text{ and } \|X\|_{\hat{\varphi}} \leq \|Y\|_{\hat{\varphi}} \text{ if } |X| \leq |Y|. \tag{4.2}$$

Indeed, the first (resp. last) one follows from a change of variable $\lambda' = \lambda/\alpha$ (resp. monotonicity of $\hat{\varphi}$), while the convexity and monotonicity of $\hat{\varphi}$ implies that for any $\alpha \in (0,1)$,

$$\hat{\varphi}\left(\frac{|\alpha X + (1-\alpha)Y|}{\alpha\lambda + (1-\alpha)\lambda'}\right) \leq \frac{\alpha\lambda}{\alpha\lambda + (1-\alpha)\lambda'}\hat{\varphi}\left(\frac{|X|}{\lambda}\right) + \frac{(1-\alpha)\lambda'}{\alpha\lambda + (1-\alpha)\lambda'}\hat{\varphi}\left(\frac{|Y|}{\lambda'}\right),$$

hence $\{\alpha\lambda + (1-\alpha)\lambda' : \lambda, \lambda' > 0,\ \hat{\varphi}(|X|/\lambda),\ \hat{\varphi}(|Y|/\lambda') \leq 1\} \subset \{\beta > 0 : \hat{\varphi}(|\alpha X + (1-\alpha)Y|/\beta) \leq 1\}$. We have also that

$$\|X\|_{\hat{\varphi}} < \infty \text{ if and only if } X \in L^{\hat{\varphi}}; \tag{4.3}$$

$$\|X\|_{\hat{\varphi}} = 0 \text{ if and only if } \hat{\varphi}(\alpha|X|) = 0,\ \forall \alpha > 0; \tag{4.4}$$

$$\|X_n\|_{\hat{\varphi}} \to 0 \text{ if and only if } \hat{\varphi}(\alpha|X_n|) \to 0,\ \forall \alpha > 0. \tag{4.5}$$

The necessity of (4.3) is clear from the definition while the convexity of $\hat{\varphi}$ and $\hat{\varphi}(0) = 0$ imply that $\hat{\varphi}(\varepsilon\alpha|X|) \le \varepsilon\hat{\varphi}(\alpha|X|) = \varepsilon\hat{\varphi}(\alpha|X|) \downarrow 0$ whenever $\hat{\varphi}(\alpha|X|) < \infty$. The sufficiency of (4.4) is again immediate from (4.1), and $\|X\|_{\hat{\varphi}} = 0$ implies that $\hat{\varphi}(\alpha|X|) \le \varepsilon\hat{\varphi}((\alpha/\varepsilon)|X|) \le \varepsilon$ for any $\varepsilon \in (0,1)$, hence $\hat{\varphi}(\alpha|X|) = 0$. Finally, (4.5) follows from the relations $\|X\|_{\hat{\varphi}} < \varepsilon \Rightarrow \hat{\varphi}(|X|/\varepsilon) \le 1 \Rightarrow \|X\|_{\hat{\varphi}} \le \varepsilon$, and $\hat{\varphi}(\alpha|X|) \le \varepsilon\alpha\hat{\varphi}(|X|/\varepsilon) \le \varepsilon\alpha$ if $\varepsilon < 1/\alpha$.

In general, any $\mathbb{R}$-valued function p on a Riesz space verifying the three conditions of (4.2) is called a *lattice* seminorm. In view of (4.3), we have seen that $\|\cdot\|_{\hat{\varphi}}$ is a lattice seminorm on $L^{\hat{\varphi}}$ (hence on $M^{\hat{\varphi}}$ and $M^{\hat{\varphi}}_u$ as well).

Note that we have used only three properties of $\hat{\varphi}$ so far, namely, convexity, monotonicity and $\hat{\varphi}(0) = 0$, so the arguments above still work for any monotone convex function on L^0_+ null at the origin. Now the continuity from below of $\hat{\varphi}$ (Lemma 3.1) shows:

Lemma 4.1. *For any $\alpha > 0$, $\|X\|_{\hat{\varphi}} \le \alpha$ if and only if $\hat{\varphi}(|X|/\alpha) \le 1$, and*

$$X_n \to X \text{ a.s.} \;\Rightarrow\; \|X\|_{\hat{\varphi}} \le \liminf_n \|X_n\|_{\hat{\varphi}}. \tag{4.6}$$

Proof. The sufficiency of the first claim is clear from (4.1), while the monotonicity and continuity from below of $\hat{\varphi}$ imply that for any $\alpha > 0$,

$$\alpha > \|X\|_{\hat{\varphi}} \;\Rightarrow\; \hat{\varphi}(|X|/\alpha) = \lim_n \hat{\varphi}\left(\frac{|X|}{\alpha + 1/n}\right) \le \lim_n \hat{\varphi}\left(\frac{|X|}{\|X\|_{\hat{\varphi}} + 1/n}\right) \le 1.$$

For (4.6), we may suppose $\|X\|_{\hat{\varphi}} > 0$ (otherwise trivial). Put $Y_n := \inf_{k\ge n}|X_k|$ and note that $0 \le Y_n \uparrow |X|$ by $X_n \to X$. Then for any $\varepsilon \in (0, \|X\|_{\hat{\varphi}})$,

$$1 < \hat{\varphi}\left(\frac{|X|}{\|X\|_{\hat{\varphi}} - \varepsilon}\right) = \lim_n \hat{\varphi}\left(\frac{Y_n}{\|X\|_{\hat{\varphi}} - \varepsilon}\right),$$

which implies in view of the first claim that $\|Y_n\|_{\hat{\varphi}} > \|X\|_{\hat{\varphi}} - \varepsilon$ for large enough n, thus we deduce that $\liminf_n \|X_n\|_{\hat{\varphi}} \ge \sup_n \|\inf_{k\ge n}|X_k|\|_{\hat{\varphi}} = \sup_n \|Y_n\|_{\hat{\varphi}} \ge \|X\|_{\hat{\varphi}} - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have (4.6). $\square$

The next one is crucial.

Lemma 4.2. *The lattice seminorm $\|\cdot\|_{\hat{\varphi}}$ is order-continuous on $M^{\hat{\varphi}}_u$, i.e.,*

$$\|X_n\|_{\hat{\varphi}} \to 0 \text{ whenever } \exists Y \in M^{\hat{\varphi}}_u \text{ with } |X_n| \le |Y|\,(\forall n) \text{ and } X_n \to 0, \text{ a.s.} \tag{4.7}$$

Proof. Let $(X_n)_n \subset M^{\hat{\varphi}}_u$ be dominated by $Y \in M^{\hat{\varphi}}_u$ and converges a.s. to 0. Then

$$\|X_n\|_{\hat{\varphi}} \le \|X_n \mathbb{1}_{\{|Y|>N\}}\|_{\hat{\varphi}} + \|X_n \mathbb{1}_{\{|Y|\le N\}}\|_{\hat{\varphi}} \le \|Y \mathbb{1}_{\{|Y|>N\}}\|_{\hat{\varphi}} + \|X_n \mathbb{1}_{\{|Y|\le N\}}\|_{\hat{\varphi}}.$$

We claim that (1) $\|Y\mathbb{1}_{\{|Y|>N\}}\|_{\hat{\varphi}} \xrightarrow{N} 0$, and (2) for each fixed N, $\|X_n\mathbb{1}_{\{|Y|\le N\}}\|_{\hat{\varphi}} \xrightarrow{n} 0$, then (4.7) follows by a diagonal argument. In fact, (1) is equivalent in view of (4.5) to saying that $\hat{\varphi}(\alpha|Y|\mathbb{1}_{\{|Y|>N\}}) \to 0$ for all $\alpha > 0$, which is nothing but the definition of Y being an element of $M^{\hat{\varphi}}_u$. As for (2), we note that the sequence $Z^N_n := X_n\mathbb{1}_{\{|Y|\le N\}}$ satisfies $\sup_n \|Z^N_n\|_\infty \le N < \infty$ (since $|X_n| \le |Y|$ by assumption) and $Z^N_n \to 0$ a.s. $(n \to \infty)$. Thus the Lebesgue property of $\varphi_0 = \hat{\varphi}|_{L^\infty}$ shows that $\hat{\varphi}(\alpha|Z^N_n|) = \varphi_0(\alpha|Z^N_n|) \to 0$ for all $\alpha > 0$, hence $\|Z^N_n\|_{\hat{\varphi}} \to 0$ by (4.5). $\square$

We now characterize the space $M^{\hat{\varphi}}_u$ in terms of the gauge seminorm $\|\cdot\|_{\hat{\varphi}}$.

Lemma 4.3. *For any $X \in L^0$, the following are equivalent:*

(1) $X \in M_u^{\hat{\varphi}}$;
(2) $\lim_N \|X\mathbb{1}_{\{|X|>N\}}\|_{\hat{\varphi}} = 0$;
(3) $\lim_n \|X\mathbb{1}_{A_n}\|_{\hat{\varphi}} = 0$ *whenever* $\mathbb{P}(A_n) \downarrow 0$.

Proof. (3) $\Rightarrow$ (2) is clear, and (2) $\Rightarrow$ (1) was already proved in the proof of Lemma 4.2. If $X \in M_u^{\hat{\varphi}}$, then $Y_n := X\mathbb{1}_{A_n} \in M_u^{\hat{\varphi}}$, $|Y_n| \le |X|$ and $Y_n \to 0$ a.s. whenever $\mathbb{P}(A_n) \to 0$. Thus (1) $\Rightarrow$ (3) follows from Lemma 4.2. □

Finally, we have the following inequality:

Lemma 4.4. *For any* $X \in L^0$ *and* $Z \in \mathrm{dom}\varphi_0^*$,

$$\mathbb{E}[|X|Z] \le (1 + \varphi_0^*(Z))\|X\|_{\hat{\varphi}}. \tag{4.8}$$

Proof. We may assume $\|X\|_{\hat{\varphi}} < \infty$ (otherwise trivial). Then (3.6) shows $1 \ge \hat{\varphi}(|X|/\alpha) \ge \mathbb{E}[|X|Z/\alpha] - \varphi_0^*(Z)$ for any $\alpha > \|X\|_{\hat{\varphi}}$ and $Z \in \mathrm{dom}\varphi_0^*$, thus rearranging the terms,

$$\mathbb{E}[|X|Z] \le (1 + \varphi_0^*(Z))(\|X\|_{\hat{\varphi}} + \varepsilon),\ \forall \varepsilon > 0,$$

and we have (4.8) by letting $\varepsilon \downarrow 0$. □

4.2. Quotient via a Change of Measure

We already know that $(M_u^{\hat{\varphi}}, \|\cdot\|_{\hat{\varphi}})$ is a *semi*-normed Riesz space with the *order-continuous* lattice seminorm, and $\hat{\varphi}$ is a *finite* monotone convex function on it. But $\|\cdot\|_{\hat{\varphi}}$ is not generally a *norm*, i.e., $\|X\|_{\hat{\varphi}} = 0$ does not imply $X = 0$ as an element of $M_u^{\hat{\varphi}}$ (or in L^0), thus we cannot directly conclude that $M_u^{\hat{\varphi}}$ is an order-continuous Banach lattice. A standard way of tackling this kind of difficulty is to take the quotient by the relation induced by $\|X\|_{\hat{\varphi}} = 0$. We shall do this through a suitable change of probability.

Lemma 4.5. *There exists a* $\widehat{Z} \in \mathrm{dom}\varphi_0^*$ *such that for any* $A \in \mathcal{F}$,

$$\mathbb{E}[\widehat{Z}\mathbb{1}_A] = 0 \ \Rightarrow\ \mathbb{E}[Z\mathbb{1}_A] = 0,\ \forall Z \in \mathrm{dom}\varphi_0^*. \tag{4.9}$$

Then putting $d\mathbb{Q}/d\mathbb{P} = \hat{c}\widehat{Z}$ *(with* $\hat{c} = \mathbb{E}[\widehat{Z}]^{-1}$*),* $\mathbb{Q}$ *is a probability measure such that*

$$\varphi_0^*(\hat{c}d\mathbb{Q}/d\mathbb{P}) < \infty, \text{ and } \mathbb{Q}(|X| > 0) = 0 \ \Leftrightarrow\ \hat{\varphi}(\alpha|X|) = 0,\ \forall \alpha > 0. \tag{4.10}$$

Remark 4.6. As we shall see in the proof, this lemma does not need the Lebesgue property of φ^0; the Fatou property is enough. ♦

Proof. We first construct a $\widehat{Z} \in \mathrm{dom}\varphi_0^* \subset L^1$ such that

$$\varphi_0^*(\widehat{Z}) \le 1 \text{ and } \mathbb{P}(\widehat{Z} > 0) = \max\{\mathbb{P}(Z > 0) :\ Z \in \mathrm{dom}\varphi_0^*,\ \varphi_0^*(Z) \le 1\}. \tag{4.11}$$

The set $\Lambda := \{Z \in \mathrm{dom}\varphi_0^* : \varphi_0^*(Z) \le 1\}$ is convex, norm closed in L^1 by the lower semicontinuity of φ_0^*, and is norm bounded since $\mathbb{E}[|Z|] = \mathbb{E}[Z] \le \varphi_0(1) + 1$ for all $Z \in \Lambda$. Thus for any sequences $(Z_n)_n \subset \Lambda$ and $(\alpha_n) \subset \mathbb{R}_+$ with $\sum_n \alpha_n = 1$, the series $Z := \sum_n \alpha_n Z_n$ is absolutely convergent in L^1, and we have in fact $Z \in \Lambda$. Indeed,

$$\begin{aligned}\varphi_0^*(Z) &= \sup_{X \in L^\infty} (\mathbb{E}[XZ] - \varphi_0(X)) = \sup_{X \in L^\infty}\left(\sum_n \alpha_n \mathbb{E}[XZ_n] - \varphi_0(X)\right)\\ &\le \sum_n \alpha_n \sup_{X' \in L^\infty}(\mathbb{E}[X'Z_n] - \varphi_0(X')) = \sum_n \varphi_0^*(Z_n) \le 1.\end{aligned}$$

In other words, Λ is countably convex. Then choosing a sequence $(Z_n)_n \subset \Lambda$ so that $\mathbb{P}(Z_n > 0) \uparrow \sup_{Z\in\Lambda} \mathbb{P}(Z > 0)$, $\widehat{Z} := \sum_n 2^{-n} Z_n \in \Lambda$ and we have (4.11).

We check that this $\widehat{Z}$ satisfies (4.9). Indeed, if there exists a $Z \in \mathrm{dom}\varphi_0^*$ and $A \in \mathcal{F}$ such that $\mathbb{E}[\widehat{Z}\mathbb{1}_A] = 0$ and $\mathbb{E}[Z\mathbb{1}_A] > 0$, we see that $\mathbb{P}(\widehat{Z} = 0, Z > 0) > 0$, $\varepsilon Z \in \Lambda$ for some small $\varepsilon > 0$ since $\varphi_0^*(0) = 0$ and $\bar{Z} := (\widehat{Z} + \varepsilon Z)/2 \in \Lambda$ satisfies

$$\mathbb{P}(\bar{Z} > 0) = \mathbb{P}(\widehat{Z} > 0) + \mathbb{P}(Z > 0, \widehat{Z} = 0) > \mathbb{P}(\widehat{Z} > 0).$$

This contradicts to (4.11).

Finally, putting $d\mathbb{Q}/d\mathbb{P} = \widehat{Z}/\mathbb{E}[\widehat{Z}]$, the first condition of (4.10) is clear. For the second, if $\mathbb{Q}(|X| > 0) = \mathbb{E}[\widehat{Z}\mathbb{1}_{\{|X|>0\}}] = 0$, then $\mathbb{E}[|X|Z] = 0$ for all $Z \in \mathrm{dom}\varphi_0^*$, hence $\hat{\varphi}(\alpha|X|) = \sup_{Z\in\mathrm{dom}\varphi_0^*}(\alpha\mathbb{E}[Z|X|] - \varphi_0^*(Z)) = 0$ for all $\alpha > 0$. On the other hand, if $\hat{\varphi}(\alpha|X|) = 0$ for all $\alpha > 0$, then $\alpha\mathbb{E}[\widehat{Z}|X|] \leq \hat{\varphi}(\alpha|X|) + \varphi_0^*(\widehat{Z}) \leq 1$ for all $\alpha > 0$, thus $\mathbb{E}[|X|\widehat{Z}] = 0$, and consequently $\mathbb{Q}(|X| > 0) = 0$. □

By (4.10), we see that $\|X\|_{\hat{\varphi}} = 0$ if and only if $X = 0$, $\mathbb{Q}$-a.s. Let

$$\mathcal{N}_{\mathbb{P}}(\mathbb{Q}) := \{X \in L^0 : X = 0,\ \mathbb{Q}\text{-a.s.}\} = \{X \in L^0 : \hat{\varphi}(\alpha|X|) = 0,\ \forall\alpha > 0\}. \tag{4.12}$$

The quotient space $L^0/\|\cdot\|_{\hat{\varphi}} = L^0/\mathcal{N}_{\mathbb{P}}(\mathbb{Q})$ is (lattice isomorphic to) the space $L^0(\mathbb{Q})$ of equivalence classes *modulo $\mathbb{Q}$-a.s. equality* of measurable functions ordered by the $\mathbb{Q}$-a.s. inequality (remember that $L^0 = L^0(\mathbb{P})$ also is the space of classes but *modulo $\mathbb{P}$-a.s. equality*). All we need is the following intuitively obvious lemma:

Lemma 4.7. *There exists an onto linear mapping $\pi : L^0(\mathbb{P}) \to L^0(\mathbb{Q})$ such that*

$$X \wedge Y = 0 \text{ in } L^0(\mathbb{P}) \ \Rightarrow\ \pi(X) \wedge \pi(Y) \text{ in } L^0(\mathbb{Q}), \tag{4.13}$$

$$X_\alpha \downarrow 0 \text{ in } L^0(\mathbb{P}) \ \Rightarrow\ \pi(X_\alpha) \downarrow 0 \text{ in } L^0(\mathbb{Q}); \tag{4.14}$$

$$\begin{cases} \xi_n, \xi, \eta \in L^0(\mathbb{Q}),\ |\xi_n| \leq |\eta| \text{ in } L^0(\mathbb{Q})\ (\forall n),\ \xi_n \to \xi,\ \mathbb{Q}\text{-a.s.} \\ \Rightarrow\ \exists X_n, X, Y \in L^0 \text{ such that } \xi_n = \pi(X_n),\ \xi = \pi(X),\ \eta = \pi(Y), \\ \qquad |X_n| \leq |Y| \text{ in } L^0 \text{ and } X_n \to X,\ \mathbb{P}\text{-a.s.} \end{cases} \tag{4.15}$$

In general, a linear map from a Riesz space E to another Riesz space F satisfying (4.13) is called a *lattice homomorphism*. (4.14) says that π is *order-continuous*, and such a lattice homomorphism is called a *normal homomorphism*. See [3] for more information.

Proof of Lemma 4.7. For each $X \in L^0$, let $\pi(X)$ be the $\mathbb{Q}$-equivalence class generated by a representative of X. This definition makes sense and does not depend on the choice of representative. Indeed, if f and g are two representatives of $X \in L^0$, then $f = g$ $\mathbb{P}$-a.s. by definition, hence $f = g$ $\mathbb{Q}$-a.s. since $\mathbb{Q} \ll \mathbb{P}$. Thus the $\mathbb{Q}$-equivalence classes generated by f and that by g coincide. It is clear that $\pi : L^0 \to L^0(\mathbb{Q})$ is linear and onto. To see (4.13), suppose $X, Y \in L^0$ and $X \wedge Y = 0$ in L^0. Then by definition, for any representatives $f \in X$ and $g \in Y$, we have $f \geq 0$ and $g \geq 0$ $\mathbb{P}$-a.s., hence $\mathbb{Q}$-a.s., and consequently $\pi(X) \geq 0$ and $\pi(Y) \geq 0$. Next, if $\xi \in L^0(\mathbb{Q})$ and if $\xi \leq \pi(X)$, $\xi \leq \pi(Y)$ in $L^0(\mathbb{Q})$, then taking a representative $h \in \xi$ in $L^0(\mathbb{Q})$ with f, g being same as above, we have $h \leq f$ and $h \leq g$ $\mathbb{Q}$-a.s. Then putting $A = \{h \leq f,\ h \leq g\}$, we still have $h\mathbb{1}_A \in \xi$ (since $\mathbb{Q}(A) = 1$), and $h\mathbb{1}_A \leq f$ and $h\mathbb{1}_A \leq g$ $\mathbb{P}$-a.s. (since $f \geq 0$, $g \geq 0$ $\mathbb{P}$-a.s.). Thus $h\mathbb{1}_A \leq 0$ $\mathbb{P}$-a.s., hence $\mathbb{Q}$-a.s. Consequently, $\xi \leq 0$ in $L^0(\mathbb{Q})$ which shows that $\pi(X) \wedge \pi(Y) = 0$ in $L^0(\mathbb{Q})$.

The first line of (4.15) means that for some (hence all) representatives $f_n \in \xi_n$, $f \in \xi$ and $g \in \eta$, $|f_n| \leq |g|$ $\mathbb{Q}$-a.s. for all n, and $f_n \to f$ $\mathbb{Q}$-a.s. Then putting $A = \{|f_n| \leq |g|\,(\forall n),\ f_n \to f\} \in \mathcal{F}$, we see that $f_n\mathbb{1}_A \in \xi_n$, $f\mathbb{1}_A \in \xi$ and $g\mathbb{1}_A \in \eta$ since $\mathbb{Q}(A) = 1$, while $|f_n\mathbb{1}_A| \leq |g\mathbb{1}_A|$, $f_n\mathbb{1}_A \to f\mathbb{1}_A$

(pointwise). Hence if X_n (resp. X, Y) denotes the $\mathbb{P}$-class generated by $f_n \mathbb{1}_A$ (resp. $f\mathbb{1}_A$, $g\mathbb{1}_A$), we have that $\xi_n = \pi(X_n)$, $\xi = \pi(X)$ and $\eta = \pi(Y)$ on the one hand, and on the other hand, $|X_n| \leq |Y|$ in L^0 and $X_n \to X$ $\mathbb{P}$-a.s.

Finally, for an onto lattice homomorphism π to satisfy (4.14), it is necessary and sufficient that the kernel of π is a band (order-closed solid subspace) in L^0. In our case, the kernel of π is $\mathcal{N}_{\mathbb{P}}(\mathbb{Q})$ given by (4.12), which is clearly a solid subspace of L^0. To prove the order-closedness, it suffices to check that for any increasing net $(X_\alpha)_\alpha \subset \mathcal{N}_{\mathbb{P}}(\mathbb{Q})$ with $0 \leq X_\alpha \uparrow X$ in L^0, we have $X \in \mathcal{N}_{\mathbb{P}}(\mathbb{Q})$. But since L^0 has the countable sup property, there exists an increasing *sequence* of indices $(\alpha_n)_n$ such that $X_{\alpha_n} \uparrow X$. Then the monotone convergence theorem shows that $\mathbb{E}[X\widehat{Z}] = \lim_n \mathbb{E}[X_{\alpha_n}\widehat{Z}] = 0$, which implies $X = 0$, $\mathbb{Q}$-a.s. □

Remark 4.8. Taking $\eta = \sup_n |\xi_n| \in L^0(\mathbb{Q})$ (if $\xi_n \to \xi$) (resp. $\xi_n \equiv \xi$, $\forall n$) in (4.15), we have also

$$\left\{\begin{array}{l} \xi_n, \xi \in L^0(\mathbb{Q}),\ \xi_n \to \xi,\ \mathbb{Q}\text{-a.s.} \\ \Rightarrow\ \exists X_n, X \in L^0,\ \xi_n = \pi(X_n),\ \xi = \pi(X),\ X_n \to X,\ \mathbb{P}\text{-a.s.} \end{array}\right. \tag{4.16}$$

$$|\xi| \leq |\eta| \text{ in } L^0(\mathbb{Q}) \ \Rightarrow\ \exists X, Y \in L^0,\ \xi = \pi(X),\ \eta = \pi(Y),\ |X| \leq |Y| \text{ in } L^0. \tag{4.17}$$

♦

Since $\pi : L^0 \to L^0(\mathbb{Q})$ is an onto lattice homomorphism, we have $|\pi(X)| = \pi(|X|)$, and for any solid subspace $\mathscr{X} \subset L^0$, the image $\mathscr{X}(\mathbb{Q}) := \pi(\mathscr{X})$ is a solid subspace of $L^0(\mathbb{Q})$ (see [3, Theorem 1.33]). If in addition $\mathcal{N}_{\mathbb{P}}(\mathbb{Q}) \subset \mathscr{X}$, we see that $\pi(X) \in \mathscr{X}(\mathbb{Q})$ if *and only if* $X \in \mathscr{X}$ (the if part is always true by definition). Indeed, $\pi(X) \in \mathscr{X}(\mathbb{Q})$ means $\pi(X) = \pi(X')$ with $X' \in \mathscr{X}$, and then $\pi(X - X') = 0$ in $L^0(\mathbb{Q}) \Leftrightarrow X - X' \in \mathcal{N}_{\mathbb{P}}(\mathbb{Q})$, hence $X = X' + (X - X') \in \mathscr{X} + \mathcal{N}_{\mathbb{P}}(\mathbb{Q}) = \mathscr{X}$. Noting that $\mathcal{N}_{\mathbb{P}}(\mathbb{Q}) \subset M_u^{\hat{\varphi}}$ by definition (4.12), the following three are all solid subspaces of $L^0(\mathbb{Q})$ of this type:

$$M_u^{\hat{\varphi}}(\mathbb{Q}) := \pi(M_u^{\hat{\varphi}}), \quad L^{\hat{\varphi}}(\mathbb{Q}) := \pi(L^{\hat{\varphi}}), \quad M^{\hat{\varphi}}(\mathbb{Q}) := \pi(M^{\hat{\varphi}})$$

By (4.10) and $\|X\|_{\hat{\varphi}} = 0 \Leftrightarrow \hat{\varphi}(\alpha|X|) = 0$, $\forall \alpha > 0$, the following is well-defined:

$$\|\xi\|_{\hat{\varphi},\mathbb{Q}} := \|X\|_{\hat{\varphi}} \text{ if } \xi = \pi(X) \in L^0(\mathbb{Q}). \tag{4.18}$$

Note that $\|\xi\|_{\hat{\varphi},\mathbb{Q}} < \infty$ iff $\xi \in L^{\hat{\varphi}}(\mathbb{Q})$ and $\|\xi\|_{\hat{\varphi},\mathbb{Q}} = 0$ if *and only if* $\xi = 0$ in $L^0(\mathbb{Q})$ by construction. Thus $\|\cdot\|_{\hat{\varphi},\mathbb{Q}}$ is a lattice *norm* on $L^{\hat{\varphi}}(\mathbb{Q})$ (hence on $M_u^{\hat{\varphi}}(\mathbb{Q})$ and $M^{\hat{\varphi}}(\mathbb{Q})$ as well). The goal of this subsection is the following:

Theorem 4.9. *$(M_u^{\hat{\varphi}}(\mathbb{Q}), \|\cdot\|_{\hat{\varphi},\mathbb{Q}})$ is an order continuous Banach lattice, i.e., $M_u^{\hat{\varphi}}(\mathbb{Q})$ is complete for $\|\cdot\|_{\hat{\varphi},\mathbb{Q}}$ and the norm $\|\cdot\|_{\hat{\varphi},\mathbb{Q}}$ is order-continuous w.r.t. the $\mathbb{Q}$-a.s. order:*

$$|\xi_n| \leq \eta \in M_u^{\hat{\varphi}}(\mathbb{Q}),\ \xi_n \to 0,\ \mathbb{Q}\text{-a.s.} \ \Rightarrow\ \|\xi_n\|_{\hat{\varphi},\mathbb{Q}} \to 0. \tag{4.19}$$

On this occasion, we shall prove also the following at once:

Proposition 4.10. *$L^{\hat{\varphi}}(\mathbb{Q})$ is a Banach lattice for the lattice norm $\|\cdot\|_{\hat{\varphi},\mathbb{Q}}$ and $M^{\hat{\varphi}}(\mathbb{Q})$ is its closed linear subspace (hence itself a Banach lattice).*

Lemma 4.11. *$\|\cdot\|_{\hat{\varphi},\mathbb{Q}} : L^0(\mathbb{Q}) \to \mathbb{R} \cup \{+\infty\}$ satisfies the following:*

$$\xi_n, \xi \in L^0(\mathbb{Q}),\ \xi_n \to \xi,\ \mathbb{Q}\text{-a.s.} \ \Rightarrow\ \|\xi\|_{\hat{\varphi},\mathbb{Q}} \leq \liminf_n \|\xi_n\|_{\hat{\varphi},\mathbb{Q}}; \tag{4.20}$$

$$\|\xi\|_{L^1(\mathbb{Q})} \leq c_{\mathbb{Q}} \|\xi\|_{\hat{\varphi},\mathbb{Q}},\ \forall \xi \in L^0(\mathbb{Q}) \text{ where } c_{\mathbb{Q}} = 2\mathbb{E}[\widehat{Z}]. \tag{4.21}$$

Proof. If $\xi_n, \xi \in L^0(\mathbb{Q})$, $\xi_n \to \xi$, $\mathbb{Q}$-a.s., then by (4.16), there exist $X_n, X \in L^0$ such that $\xi_n = \pi(X_n)$, $\xi = \pi(X)$ and $X_n \to X$, $\mathbb{P}$-a.s. Thus by (4.18) and (4.6), $\|\xi\|_{\hat{\varphi},\mathbb{Q}} = \|X\|_{\hat{\varphi}} \leq \liminf_n \|X_n\|_{\hat{\varphi}} = \liminf_n \|\xi_n\|_{\hat{\varphi},\mathbb{Q}}$, and we have (4.20). For (4.21), Lemma 4.4 tells us that for each $\xi = \pi(X) \in L^0(\mathbb{Q})$, $\hat{c}^{-1}\|\xi\|_{L^1(\mathbb{Q})} = \mathbb{E}[X\widehat{Z}] \leq (1 + \varphi_0^*(\widehat{Z}))\|X\|_{\hat{\varphi}} = (1 + \varphi_0^*(\widehat{Z}))\|\xi\|_{\hat{\varphi},\mathbb{Q}}$, thus we have (4.21) since $\varphi_0^*(\widehat{Z}) \leq 1$. □

Proof of Proposition 4.10 and Theorem 4.9. We already know that $(L^{\hat{\varphi}}(\mathbb{Q}), \|\cdot\|_{\hat{\varphi},\mathbb{Q}})$ is a normed Riesz space, and $M_u^{\hat{\varphi}}(\mathbb{Q})$ and $M^{\hat{\varphi}}(\mathbb{Q})$ are its solid vector subspaces. To see that $L^{\hat{\varphi}}(\mathbb{Q})$ is complete, let $(\xi_n)_n \in L^{\hat{\varphi}}(\mathbb{Q})$ be a Cauchy sequence for $\|\cdot\|_{\hat{\varphi},\mathbb{Q}}$. Then by (4.21), it is also Cauchy in $L^1(\mathbb{Q})$, hence admits the $\|\cdot\|_{L^1(\mathbb{Q})}$-limit ξ in $L^1(\mathbb{Q})$, and we can choose a subsequence $(\xi_{n_k})_k$ so that $\xi_{n_k} \to \xi$, $\mathbb{Q}$-a.s. Then (4.20) shows that $\|\xi - \xi_m\|_{\hat{\varphi},\mathbb{Q}} \leq \liminf_k \|\xi_{n_k} - \xi_m\|_{\hat{\varphi},\mathbb{Q}}$ for all m. Since the original sequence is Cauchy for $\|\cdot\|_{\hat{\varphi},\mathbb{Q}}$, this shows that $\|\xi\|_{\hat{\varphi},\mathbb{Q}} < \infty$ (hence $\xi \in L^{\hat{\varphi}}(\mathbb{Q})$) and $\|\xi - \xi_n\|_{\hat{\varphi},\mathbb{Q}} \to 0$.

Suppose in addition that each ξ_n belongs to $M^{\hat{\varphi}}(\mathbb{Q})$, and write $\xi_n = \pi(X_n)$ with $X_n \in M^{\hat{\varphi}}$ and $\xi = \pi(X)$ with $X \in L^{\hat{\varphi}}$. Then for all $\alpha > 0$, there is some large n so that $\|X - X_n\|_{\hat{\varphi}} = \|\xi - \xi_n\|_{\hat{\varphi},\mathbb{Q}} < 1/2\alpha$ for which $\hat{\varphi}(2\alpha|X - X_n|) \leq 1$, hence $\hat{\varphi}(\alpha|X|) \leq \frac{1}{2}\hat{\varphi}(2\alpha|X - X_n|) + \frac{1}{2}\hat{\varphi}(2\alpha|X_n|) \leq 1 + \frac{1}{2}\hat{\varphi}(2\alpha|X_n|)$. Consequently, $X \in M^{\hat{\varphi}}$, thus $\xi = \pi(X) \in M^{\hat{\varphi}}(\mathbb{Q})$, and we deduce that $M^{\hat{\varphi}}(\mathbb{Q})$ is closed in $L^{\hat{\varphi}}(\mathbb{Q})$.

For Theorem 4.9, it remains to show that $M_u^{\hat{\varphi}}(\mathbb{Q})$ is closed in $L^{\hat{\varphi}}(\mathbb{Q})$, and $\|\cdot\|_{\hat{\varphi},\mathbb{Q}}$ is order-continuous for the $\mathbb{Q}$-a.s. order (i.e., (4.19)). For the closedness, let $(\xi_n)_n$ and ξ be as in the first paragraph and suppose that $\xi_n \in M_u^{\hat{\varphi}}(\mathbb{Q})$ for each n. Then $\xi_n = \pi(X_n)$ with $X_n \in M_u^{\hat{\varphi}}$ for each n, and $\xi = \pi(X)$ with $X \in L^{\hat{\varphi}}$. Observe that

$$\|X\mathbb{1}_{\{|X|>N\}}\|_{\hat{\varphi}} \leq \|X - X_n\|_{\hat{\varphi}} + \|X_n\mathbb{1}_{\{|X|>N\}}\|_{\hat{\varphi}} = \|\xi - \xi_n\|_{\hat{\varphi},\mathbb{Q}} + \|X_n\mathbb{1}_{\{|X|>N\}}\|_{\hat{\varphi}}.$$

The first term in the right hand side tends to 0 as $n \to \infty$, while for each n, the second term tends to 0 as $N \to \infty$ since $X \in M_u^{\hat{\varphi}}$. Taking a diagonal, we see that $X \in M_u^{\hat{\varphi}}$, hence $\xi = \pi(X) \in M_u^{\hat{\varphi}}(\mathbb{Q})$. Therefore, $M_u^{\hat{\varphi}}(\mathbb{Q})$ is closed.

Finally, we show (4.19). Let $(\xi_n)_n \subset M_u^{\hat{\varphi}}(\mathbb{Q})$, $|\xi_n| \leq \eta \in M_u^{\hat{\varphi}}(\mathbb{Q})$ and $\xi_n \to 0$ $\mathbb{Q}$-a.s. Then by (4.15), we can choose X_n, Y with $\xi_n = \pi(X_n)$, $\eta = \pi(Y)$ (hence $Y \in M_u^{\hat{\varphi}}$), $|X_n| \leq |Y|$ and $X_n \to 0$ $\mathbb{P}$-a.s. Then (4.7) and (4.18) show that $\|\xi_n\|_{\hat{\varphi},\mathbb{Q}} = \|X_n\|_{\hat{\varphi}} \to 0$, and we deduce (4.19). □

Remark 4.12 (Sensitivity). In general, $\mathbb{Q}$ is only absolutely continuous with respect to the original reference measure $\mathbb{P}$ (not equivalent). From (4.10), a necessary and sufficient condition for the possibility of choosing an equivalent $\mathbb{Q}$ ($\mathbb{Q} \sim \mathbb{P}$) is that

$$\forall A \in \mathcal{F} \text{ with } \mathbb{P}(A) > 0,\ \exists \varepsilon > 0,\ \varphi_0(\varepsilon\mathbb{1}_A) > 0. \tag{4.22}$$

In financial mathematics, this condition is called the *sensitivity* of φ_0. See [16, Ch. 4] for more information. ♦

Corollary 4.13. *If φ_0 is sensitive in the sense of (4.22), $(M_u^{\hat{\varphi}}, \|\cdot\|_{\hat{\varphi}})$ itself is an order continuous Banach lattice.*

4.3. Proof of Theorem 3.5

We now proceed to Theorem 3.5. Recall that $\mathcal{N}_{\mathbb{P}}(\mathbb{Q}) \subset M_u^{\hat{\varphi}} \subset \mathcal{D}_0 \cap (-\mathcal{D}_0)$ where $\mathcal{D}_0$ is defined by (3.1). Thus if $X \in \mathcal{D}_0$ and $Y = X$ $\mathbb{Q}$-a.s. ($\Leftrightarrow Y - X \in \mathcal{N}_{\mathbb{P}}(\mathbb{Q})$), we have $Y = X + (Y - X) \in \mathcal{D}_0 + \mathcal{D}_0 = \mathcal{D}_0$ since $\mathcal{D}_0$ is a convex cone. In this case, we have also that $\hat{\varphi}(X) = \hat{\varphi}(Y)$. Indeed, $X = Y$ $\mathbb{Q}$-a.s. implies $\mathbb{E}[|X - Y|Z] = \mathbb{E}[|X - Y|Z\mathbb{1}_{\{X \neq Y\}}] = 0$ for all $Z \in \mathrm{dom}\varphi_0^*$ by (4.9), hence $\hat{\varphi}(X) = \sup_{Z \in \mathrm{dom}\varphi_0^*}(\mathbb{E}[XZ] - \varphi_0^*(Z)) = \sup_{Z \in \mathrm{dom}\varphi_0^*}(\mathbb{E}[YZ] - \varphi_0^*(Z)) = \hat{\varphi}(Y)$. Therefore,

$$\hat{\varphi}_{\mathbb{Q}}(\xi) := \hat{\varphi}(X) \text{ if } \xi = \pi(X) \in \mathcal{D}_0(\mathbb{Q}) := \pi(\mathcal{D}_0) \tag{4.23}$$

is well-defined as a function on $\mathcal{D}_0(\mathbb{Q}) := \pi(\mathcal{D}_0)$, hence in particular on $L^{\hat{\varphi}}(\mathbb{Q})$, $M^{\hat{\varphi}}(\mathbb{Q})$ and on $M^{\hat{\varphi}}_u(\mathbb{Q})$. $\hat{\varphi}_{\mathbb{Q}}$ is convex (resp. monotone) since π is linear and $\hat{\varphi}$ is convex (resp. both π and $\hat{\varphi}$ are monotone), and is finite on $M^{\hat{\varphi}}(\mathbb{Q})$ (hence on $M^{\hat{\varphi}}_u(\mathbb{Q})$ in particular).

Proof of Theorem 3.5. Recall that any monotone convex function on a Banach lattice is norm-continuous on the interior of its effective domain by the extended Namioka-Klee theorem [7, Theorem 1]. Thus $\hat{\varphi}_{\mathbb{Q}} : M^{\hat{\varphi}}_u(\mathbb{Q}) \to \mathbb{R}$ is $\|\cdot\|_{\hat{\varphi},\mathbb{Q}}$-continuous as a finite valued monotone convex function on a Banach lattice $M^{\hat{\varphi}}_u(\mathbb{Q})$, while since $\|\cdot\|_{\hat{\varphi},\mathbb{Q}}$ is $\mathbb{Q}$-order continuous in the sense of (4.19) by Theorem 4.9, we deduce that $\hat{\varphi}_{\mathbb{Q}} : M^{\hat{\varphi}}_u(\mathbb{Q}) \to \mathbb{R}$ is $\mathbb{Q}$-order continuous. Thus recalling that $\hat{\varphi} = \hat{\varphi}_{\mathbb{Q}} \circ \pi$ and $\pi : L^0(\mathbb{P}) \to L^0(\mathbb{Q})$ is order continuous, we obtain that $\hat{\varphi} : M^{\hat{\varphi}}_u \to \mathbb{R}$ is $\mathbb{P}$-order continuous. Consequently, $(\hat{\varphi}, M^{\hat{\varphi}}_u)$ is indeed a Lebesgue extension of φ_0.

If $(\varphi, \mathscr{X})$ is another Lebesgue extension, we must have $\mathscr{X} \subset M^{\hat{\varphi}}_u$ by (3.4), and for any $X \in \mathscr{X} \subset M^{\hat{\varphi}}_u$, the Lebesgue properties of $\hat{\varphi}$ and φ on $\mathscr{X}$ and $\hat{\varphi}|_{L^\infty} = \varphi|_{L^\infty}$ show that $\hat{\varphi}(X) = \lim_n \hat{\varphi}(X\mathbb{1}_{\{|X|\le n\}}) = \lim_n \varphi_0(X\mathbb{1}_{\{|X|\le n\}}) = \lim_n \varphi(X\mathbb{1}_{\{|X|\le n\}}) = \varphi(X)$. Thus we have $\varphi = \hat{\varphi}|_{\mathscr{X}}$. □

Remark 4.14. The three Orlicz-type spaces $M^{\hat{\varphi}}_u(\mathbb{Q})$, $M^{\hat{\varphi}}(\mathbb{Q})$ and $L^{\hat{\varphi}}(\mathbb{Q})$ are also expressed using $\hat{\varphi}_{\mathbb{Q}}$ in forms parallel to those of original spaces:

$$
\begin{aligned}
L^{\hat{\varphi}}(\mathbb{Q}) &= \{\xi \in L^0(\mathbb{Q}) : \hat{\varphi}_{\mathbb{Q}}(\alpha|\xi|) < \infty,\ \exists \alpha > 0\},\\
M^{\hat{\varphi}}(\mathbb{Q}) &= \{\xi \in L^0(\mathbb{Q}) : \hat{\varphi}_{\mathbb{Q}}(\alpha|\xi|) < \infty,\ \forall \alpha > 0\},\\
M^{\hat{\varphi}}_u(\mathbb{Q}) &= \{\xi \in L^0(\mathbb{Q}) : \lim_N \hat{\varphi}_{\mathbb{Q}}(\alpha|\xi|\mathbb{1}_{\{|\xi|>N\}}) = 0,\ \forall \alpha > 0\}.
\end{aligned}
$$

For the last identity, we note that $\pi(|X|\mathbb{1}_{\{|X|>N\}}) = \pi(|X|)\pi(\mathbb{1}_{\{|X|>N\}}) = |\pi(X)|\mathbb{1}_{\{|\pi(X)|>N\}}$ which is straightforward from the definition of π in Lemma 4.7. ♦

5. Proof of Theorem 3.8

Proof of Theorem 3.8: (1) ⇒ (2). If $\{XZ : \varphi_0^*(Z) \le c\}$ is not uniformly integrable, there exists $\varepsilon > 0$ such that for any n, there exists $A_n \in \mathcal{F}$ and $Z_n \in L^1$ with $\mathbb{P}(A_n) \le 1/n$ and $\varphi_0^*(Z_n) \le c$ and $\mathbb{E}[|X|Z_n\mathbb{1}_{A_n}] > \varepsilon$. But then $\varepsilon < \mathbb{E}[|X|Z_n\mathbb{1}_{A_n}] \le (1+c)\|X\mathbb{1}_{A_n}\|_{\hat{\varphi}}$ for all n by Lemma 4.4, hence $X \notin M^{\hat{\varphi}}_u$ by Lemma 4.3. □

Recall that if $X \in M^{\hat{\varphi}}$ (or more generally $L^{\hat{\varphi}}$), $XZ \in L^1$ for any $Z \in \mathrm{dom}\varphi_0^*$ by (4.8).

Lemma 5.1. *Let $U \in M^{\hat{\varphi}}$ and suppose that $\{UZ : \varphi_0^*(Z) \le c\}$ is uniformly integrable for each c. Then $\Lambda_{\beta,U,Y} := \{Z : Z \in \mathrm{dom}\varphi_0^*,\ \mathbb{E}[UYZ] - \varphi_0^*(Z) \ge -\beta\}$ is weakly compact in L^1 for all $\beta \in \mathbb{R}$ and $Y \in L^\infty$.*

Proof. Since $\Lambda_{\beta,U,Y}$ is convex, it suffices to show that it is norm-closed and uniformly integrable. For the latter, fix an arbitrary $Z_0 \in \mathrm{dom}\varphi_0^*$, and observe that

$$
\begin{aligned}
\mathbb{E}[UYZ] &\le \mathbb{E}[2\|Y\|_\infty|U|(Z/2)] \le \mathbb{E}\left[2\|Y\|_\infty|U|\left(\frac{1}{2}Z + \frac{1}{2}Z_0\right)\right]\\
&\le \hat{\varphi}(2\|Y\|_\infty|U|) + \frac{1}{2}\varphi_0^*(Z) + \frac{1}{2}\varphi_0^*(Z_0).
\end{aligned}
$$

Thus $Z \in \Lambda_{\beta,U,Y}$ implies that

$$
-\beta \le \mathbb{E}[UYZ] - \varphi_0^*(Z) \le \hat{\varphi}(2\|Y\|_\infty|U|) + \frac{1}{2}\varphi_0^*(Z_0) - \frac{1}{2}\varphi_0^*(Z).
$$

Putting $\beta' := 2\beta + 2\hat{\varphi}(2\|Y\|_\infty|U|) + \varphi_0^*(Z_0) < \infty$ (since $U \in M^{\hat{\varphi}}$), this tells us that $\Lambda_{\beta,U,Y} \subset \{Z : \varphi_0^*(Z) \le \beta'\}$ and the latter set is uniformly integrable by the fundamental assumption that φ_0 is Lebesgue on L^∞ and Theorem 2.4.

To see the closedness, let $Z_n \in \Lambda_{\beta,U,Y}$ and $Z_n \to Z \in L^1$ in norm. Then $Z_n \to Z$ in probability, hence $UYZ_n \to UYZ$ in probability as well. On the other hand, from what we just proved and the assumption of lemma, $(UZ_n)_n$ is uniformly integrable, thus so is $(UYZ_n)_n$ since $Y \in L^\infty$. Consequently, $\mathbb{E}[UYZ] = \lim_n \mathbb{E}[UYZ_n]$, and since φ_0^* is lower semicontinuous on L^1, we have also $\varphi_0^*(Z) \le \liminf_n \varphi_0^*(Z_n)$. Summing up,

$$\begin{aligned}\mathbb{E}[UYZ] - \varphi_0^*(Z) &\ge \lim_n \mathbb{E}[UYZ_n] - \liminf_n \varphi_0^*(Z_n)\\ &\ge \limsup_n (\mathbb{E}[UYZ_n] - \varphi_0^*(Z_n)) \ge -\beta.\end{aligned}$$

Hence $Z \in \Lambda_{\beta,U,Y}$. □

Proof of Theorem 3.8: (2) ⇒ (3) and (3.10). For $U \in M^{\hat{\varphi}}$, $Y \in L^\infty$, we put $l_{U,Y}(Z) := \mathbb{E}[UYZ] - \varphi_0^*(Z)$. Then Lemma 5.1 tells us that if $\{UZ : \varphi_0^*(Z) \le c\}$ is uniformly integrable for each $c > 0$, $l_{U,Y}$ is weakly upper semicontinuous and all its upper level sets are weakly compact for each $Y \in L^\infty$, and thus $\sup_{Z \in \mathrm{dom}\varphi_0^*} l_{U,Y}(Z)$ is attained. By the condition (2) of Theorem 3.8, this applies to $U = X$ and $Y = 1$ (constant), and we obtain (3.10). For (3), we note that $|X| \le |X| \vee 1 \le |X| + 1$ and $\{Z : \varphi_0^*(Z) \le c\}$ is uniformly integrable for each $c > 0$ by Theorem 2.4 and the Lebesgue property of φ_0 on L^∞, hence (2) implies also that $\{(|X| \vee 1)Z : \varphi_0^*(Z) \le c\}$ is uniformly integrable too. Therefore, the above argument applies to $U = |X| \vee 1 \in M^{\hat{\varphi}}$ and arbitrary $Y \in L^\infty$, showing that the supremum $\sup_{Z \in \mathrm{dom}\varphi_0^*}(\mathbb{E}[(|X| \vee 1)YZ] - \varphi_0^*(Z)) = \sup_{Z \in \mathrm{dom}\varphi_0^*} l_{|X|\vee 1,Y}(Z)$ is attained for each $Y \in L^\infty$. This concludes the proof of (2) ⇒ (3). □

Proof of Theorem 3.8: (2) ⇒ (1). We apply a version of minimax theorem (Theorem A.1) to the function $L^\infty \times \mathrm{dom}\varphi_0^* \ni (Y, Z) \mapsto f(Y, Z) := \mathbb{E}[|X|YZ] - \varphi_0^*(Z)$. We already know under (2) that for each $Y \in L^\infty$, $Z \mapsto f(Y, Z)$ is concave, weakly upper semicontinuous on $\mathrm{dom}\varphi_0^*$ and all its level sets are weakly compact by Lemma 5.1 applied to $U = |X|$. On the other hand $Y \mapsto f(Y, Z)$ is affine (hence convex). Thus for any convex set $C \subset L^\infty$, we have

$$\inf_{Y \in C} \sup_{Z \in \mathrm{dom}\varphi_0^*} \left(\mathbb{E}[|X|YZ] - \varphi_0^*(Z)\right) = \sup_{Z \in \mathrm{dom}\varphi_0^*} \inf_{Y \in C} \left(\mathbb{E}[|X|YZ] - \varphi_0^*(Z)\right). \tag{5.1}$$

Let C_1 be the convex hull $\mathrm{conv}(\mathbb{1}_{\{|X|>N\}}, N \in \mathbb{N})$. Observe that for any $n \in \mathbb{N}$, $\lambda_i \ge 0$, $\lambda_1 + \cdots + \lambda_n = 1$ and $N_1 < N_2 < \cdots < N_n$, we have $\mathbb{1}_{\{|X|>N_n\}} \le \lambda_1 \mathbb{1}_{\{|X|>N_1\}} + \cdots + \lambda_n \mathbb{1}_{\{|X|>N_n\}} \le \mathbb{1}_{\{|X|>N_1\}}$ and every element of C_1 is written in the form of middle expression. Thus for any $\alpha > 0$,

$$\begin{aligned}\lim_N \hat{\varphi}(\alpha|X|\mathbb{1}_{\{|X|>N\}}) &= \inf_{Y \in \alpha C_1} \hat{\varphi}(|X|Y) = \inf_{Y \in \alpha C_1} \sup_{Z \in \mathrm{dom}\varphi_0^*} \left(\mathbb{E}[|X|YZ] - \varphi_0^*(Z)\right)\\ &\overset{(5.1)}{=} \sup_{Z \in \mathrm{dom}\varphi_0^*} \left(\inf_{Y \in \alpha C_1} \mathbb{E}[|X|YZ] - \varphi_0^*(Z)\right)\\ &= \sup_{Z \in \mathrm{dom}\varphi_0^*} \left(\lim_N \alpha \mathbb{E}[|X|\mathbb{1}_{\{|X|>N\}}Z] - \varphi_0^*(Z)\right) = \sup_{Z \in \mathrm{dom}\varphi_0^*} -\varphi_0^*(Z) = 0.\end{aligned}$$

Thus $X \in M_u^{\hat{\varphi}}$. □

We proceed to the implication (3) ⇒ (2). This will follow from the following version of *perturbed James's theorem* recently obtained by [26]:

Theorem 5.2 ([26], Theorem 2). *Let E be a real Banach space and $f : E \to \mathbb{R} \cup \{+\infty\}$ be a function which is coercive, i.e.,*

$$\lim_{\|x\|\to\infty} \frac{f(x)}{\|x\|} = +\infty. \tag{5.2}$$

Then if the supremum $\sup_{x\in E}(\langle x, x^*\rangle - f(x))$ is attained for every $x^* \in E^*$, the level set $\{x \in E : f(x) \leq c\}$ is relatively weakly compact for each $c \in \mathbb{R}$.

We shall apply this theorem with $E = L^1$. We first make a "change of variable". For $U \in M^{\hat{\varphi}}$ with $U \geq 1$ a.s., we set

$$g_U(Z) := \varphi_0^*(Z/U) = \sup_{\xi\in L^\infty} (\mathbb{E}[\xi Z/U] - \hat{\varphi}(\xi)), \quad \forall Z \in L^1. \tag{5.3}$$

Note that $\mathrm{dom} g_U \subset L^1_+$ since φ_0 is monotone (see (2.6)), and that

$$\begin{aligned} \{Z \in L^1 : g_U(Z) \leq c\} &= \{UZ' : Z' \in L^1, \varphi_0^*(Z') \leq c\}, \\ \mathrm{dom} g_U &= U \mathrm{dom}\varphi_0^* = \{UZ : Z \in \mathrm{dom}\varphi_0^*\}. \end{aligned} \tag{5.4}$$

(Remember that $XZ \in L^1$ for any $X \in M^{\hat{\varphi}}$ and $Z \in \mathrm{dom}\varphi_0^*$ by (4.8).)

Lemma 5.3. *Let $U \in M^{\hat{\varphi}}$ with $U \geq 1$. Then g_U is coercive:*

$$\lim_{\|Z\|_1\to\infty} g_U(Z)/\|Z\|_1 = \infty. \tag{5.5}$$

Proof. For any n and $\alpha > 0$ (constant), $\alpha U \mathbb{1}_{\{U\leq n\}} \in L^\infty$, hence from the definition of g_U,

$$\begin{aligned} g_U(Z) &\geq \mathbb{E}[\alpha U \mathbb{1}_{\{U\leq n\}}(Z/U)] - \hat{\varphi}(\alpha U \mathbb{1}_{\{U\leq n\}}) = \alpha\|Z\mathbb{1}_{\{U\leq n\}}\|_1 - \hat{\varphi}(\alpha U \mathbb{1}_{\{U\leq n\}}) \\ &\to \alpha\|Z\|_1 - \hat{\varphi}(\alpha U), \ \forall Z \in L^1_+, \end{aligned}$$

while $g_U(Z) = \infty$ if $Z \in L^1 \setminus L^1_+$. Here the last convergence follows from $0 \leq \alpha U \mathbb{1}_{\{U\leq n\}} \uparrow \alpha U$, so $\hat{\varphi}(\alpha U) = \lim_n \hat{\varphi}(\alpha U \mathbb{1}_{\{U\leq n\}})$ by Lemma 3.1. Since $\hat{\varphi}(\alpha U) < \infty$ for any $\alpha > 0$ by $U \in M^{\hat{\varphi}}$, this shows (5.5). □

Proof of Theorem 3.8: (3) ⇒ (2). Suppose (3), namely, for some $\varepsilon > 0$, the supremum $\sup_{Z\in\mathrm{dom}\varphi_0^*}(\mathbb{E}[(|X| \vee \varepsilon)YZ] - \varphi_0^*(Z)) = \sup_{Z\in\mathrm{dom}\varphi_0^*}(\mathbb{E}[(|X/\varepsilon| \vee 1)(\varepsilon Y)Z] - \varphi_0^*(Z))$ is attained for every $Y \in L^\infty$. Putting $U = |X/\varepsilon| \vee 1 \in M^{\hat{\varphi}}$, this says that for any $Y \in L^\infty$, there exists $Z_{U,Y} \in \mathrm{dom}\varphi_0^* \subset L^1$ such that $\hat{\varphi}(YU) = \mathbb{E}[YUZ_{U,Y}] - \varphi_0^*(Z_{U,Y}) = \mathbb{E}[Y(UZ_{U,Y})] - g_U(UZ_{U,Y})$. On the other hand, for any $Z' \in \mathrm{dom} g_U = U\mathrm{dom}\varphi_0^*$,

$$\mathbb{E}[YZ'] - g_U(Z') = \mathbb{E}\left[YU\frac{Z'}{U}\right] - \varphi_0^*\left(\frac{Z'}{U}\right) \leq \hat{\varphi}(YU).$$

Thus $\sup_{Z\in L^1}(\mathbb{E}[YZ] - g_U(Z))$ is attained for all $Y \in L^\infty$, hence Theorem 5.2 shows that $\{Z' \in L^1 : g_U(Z') \leq c\} = \{UZ : \varphi_0^*(Z) \leq c\}$ is relatively weakly compact (⇔ uniformly integrable) for each $c > 0$. Since $|X| \leq \varepsilon U$, we deduce that $\{XZ : \varphi_0^*(Z) \leq c\}$ is uniformly integrable for each $c > 0$. □

6. Proof of Theorem 3.9

We use the notation of Theorem 3.9, namely, $\psi : \mathscr{X} \to \mathbb{R}$ is a finite monotone convex function with the *Fatou property* (2.2) on a solid space $\mathscr{X} \subset L^0$ containing the constants, and we put $\psi_\infty := \psi|_{L^\infty}$, $\psi_\infty^*(Z) = \sup_{X\in L^\infty}(\mathbb{E}[XZ] - \psi(X)) = (\psi|_{L^\infty})^*(Z)$ and

$$\hat{\psi}(X) = \sup_{Z\in\mathrm{dom}\psi^*\infty} (\mathbb{E}[XZ] - \psi_\infty^*(Z)),$$

on $\mathcal{D}_0 = \{X \in L^0 : X^- Z \in L^1, \forall Z \in \mathrm{dom}\psi_\infty^*\}$. Remember that we do not *a priori* assume the Lebesgue property of ψ_∞ on L^∞ here, but it is implied by any of conditions (1) - (4) of Theorem 3.9 as we shall see in the proof below. Note also that we can *and do in the sequel* assume that $\psi(0) = 0$, replacing ψ by $\psi - \psi(0)$.

Proof of Theorem 3.9: (1) ⇒ (2). If ψ is finite and has the Lebesgue property on $\mathscr{X}$, ψ_∞ is a finite monotone convex function with the Lebesgue property on L^∞. Thus Theorem 3.5 applies to $\varphi_0 = \psi_\infty$ (hence $\hat{\varphi} = \hat{\psi}$) implying that $(\hat{\psi}, M_u^{\hat{\psi}})$ is the maximum Lebesgue extension of ψ_∞. On the other hand, $(\psi, \mathscr{X})$ is another Lebesgue extension of ψ_∞, hence we must have $\mathscr{X} \subset M_u^{\hat{\varphi}}$. Consequently, (2) follows from Theorem 3.8 ((1) ⇒ (2)). □

Proof of Theorem 3.9: (2) ⇒ (3). Since ψ is supposed to have the Fatou property on $\mathscr{X}$, $\psi_\infty = \psi|_{L^\infty}$ has the Fatou property on L^∞. Then condition (2) of Theorem 3.9 applied to $X = 1$ implies through Theorem 2.4 that ψ_∞ has the Lebesgue property on L^∞. On the other hand, since ψ is finite on $\mathscr{X}$ and has the Fatou property (⇔ continuous from below), we see that $\hat{\psi}(\alpha|X|) = \lim_n \hat{\psi}(\alpha|X| \wedge n) = \lim_n \psi(\alpha|X| \wedge n) = \psi(\alpha|X|) < \infty$, hence $\mathscr{X} \subset M^{\hat{\psi}}$. Consequently, for each $X \in \mathscr{X}$, the assumption of Theorem 3.8 ((2) ⇒ (3.10)) is satisfied with $\varphi_0 = \psi_\infty$ (⇒ $\hat{\varphi} = \hat{\psi}$), thus the supremum $\sup_{Z\in\mathrm{dom}\psi_\infty^*}(\mathbb{E}[XZ] - \psi_\infty^*(Z))$ is attained. □

The implication (3) ⇒ (1) is a little more subtle. We first note that condition (3) of Theorem 3.9 restricted to $L^\infty \subset \mathscr{X}$ again implies the Lebesgue property of $\psi_\infty = \psi|_{L^\infty}$ on L^∞. Thus $\varphi_0 = \psi_\infty$ satisfies our standing assumption (Assumption 3.2). Let $\mathbb{Q} \ll \mathbb{P}$ be the probability measure constructed in Lemma 4.5 with $\varphi_0 = \psi_\infty$, i.e., a measure such that $\mathbb{Q}(A) = 0$ iff $\psi_\infty(\alpha\mathbb{1}_A) = 0$ for all $\alpha > 0$, and we use the notation (adapted to $\varphi_0 = \psi_\infty$, $\hat{\varphi} = \hat{\psi}$) of Section 4.2: $\pi : L^0 \to L^0(\mathbb{Q})$ (the order-continuous lattice homomorphism constructed in Lemma 4.7), $M_u^{\hat{\psi}}(\mathbb{Q}) = \pi(M_u^{\hat{\psi}})$, $M^{\hat{\psi}}(\mathbb{Q}) = \pi(M^{\hat{\psi}})$ and $\hat{\psi}_\mathbb{Q}$ (defined by (4.23) with $\hat{\varphi} = \hat{\psi}$). Then $\mathscr{X}(\mathbb{Q}) := \pi(\mathscr{X})$ is a solid subspace of $L^0(\mathbb{Q})$.

Lemma 6.1. *With the notation above and the condition (3) of Theorem 3.9,*

$$X, Y \in \mathscr{X},\ X = Y,\ \mathbb{Q}\text{-a.s.} \ \Rightarrow\ \psi(X) = \psi(Y). \tag{6.1}$$

In particular,

$$\psi_\mathbb{Q}(\xi) := \psi(X),\ \xi = \pi(X) \in \mathscr{X}(\mathbb{Q}) \tag{6.2}$$

is well defined as a monotone convex function on $\mathscr{X}(\mathbb{Q})$, and it has the $\mathbb{Q}$-Fatou property on $\mathscr{X}(\mathbb{Q})$, and thus $\psi_\mathbb{Q}(\xi) = \hat{\psi}_\mathbb{Q}(\xi)$ for all $\xi \in \mathscr{X}_+(\mathbb{Q})$.

Proof. We first claim that for any $X, Y \in \mathscr{X}$,

$$\psi(\alpha|X - Y|) = 0,\ \forall\alpha > 0\ \Rightarrow\ \psi(X) = \psi(Y). \tag{6.3}$$

To see this, we note that

$$\psi(X) - \psi(Y) \leq \frac{1}{\alpha}\psi(\alpha|X-Y|) + \frac{\alpha-1}{\alpha}\psi\left(\frac{\alpha}{\alpha-1}Y\right) - \psi(Y), \quad \forall \alpha > 1.$$

Since ψ is finite, the finite convex function $\beta \mapsto \psi(\beta Y)$ is continuous on $\mathbb{R}$, thus $f(\beta) = \psi(\beta Y)/\beta$ is continuous at $\beta = 1$ with $f(1) = \psi(Y)$. Therefore, for any $\varepsilon > 0$, there exists $\alpha_\varepsilon > 1$ so that $\frac{\alpha_\varepsilon - 1}{\alpha_\varepsilon}\psi\left(\frac{\alpha_\varepsilon}{\alpha_\varepsilon-1}Y\right) - \psi(Y) < \varepsilon$. Combining this with the assumption $\psi(\alpha|X-Y|) = 0$ for all α, we see that $\psi(X) - \psi(Y) < \varepsilon$ for all $\varepsilon > 0$, hence $\psi(X) \geq \psi(Y)$. Changing the roles of X and Y, we have also $\psi(X) \leq \psi(Y)$, and (6.3) follows.

If $X = Y$, $\mathbb{Q}$-a.s., then by the construction of $\mathbb{Q}$ (with $\varphi_0 = \psi_\infty$), we see that $\psi(\alpha|X-Y| \wedge n) = \psi_\infty(\alpha|X-Y| \wedge n) = 0$ for all n, then the Fatou property of ψ implies $\psi(\alpha|X-Y|) \leq \liminf_n \psi(\alpha|X-Y| \wedge n) = 0$. Thus (6.1) follows from (6.3).

It is clear from (6.1) that $\psi_{\mathbb{Q}}$ of (6.2) is well-defined and finite on $\mathscr{X}(\mathbb{Q})$. To see the $\mathbb{Q}$-Fatou property, suppose $|\xi_n| \leq |\eta|$ ($\forall n$) for some $\eta \in \mathscr{X}(\mathbb{Q})$ and $\xi_n \to \xi$ $\mathbb{Q}$-a.s.. Then by (4.15), we can choose $X_n, X \in L^0$ and $Y \in \mathscr{X}$ so that $\xi_n = \pi(X_n)$, $\xi = \pi(X)$, $\eta = \pi(Y)$ with $|X_n| \leq |Y|$ in L^0 (hence $X_n, X \in \mathscr{X}$ by the solidness) and that $X_n \to X$ $\mathbb{P}$-a.s. Then the $\mathbb{P}$-Fatou property of the original ψ shows that $\psi_{\mathbb{Q}}(\xi) = \psi(X) \leq \liminf_n \psi(X_n) = \liminf_n \psi_n(\xi_n)$. The final assertion follows since if $\xi \geq 0$, then $\mathbb{Q}$-Fatou property shows $\psi_{\mathbb{Q}}(\xi) = \lim_n \psi_{\mathbb{Q}}(\xi \wedge n) = \lim_n \hat{\psi}(\xi \wedge n) = \hat{\psi}_{\mathbb{Q}}(\xi)$. □

Consequently, we have $\psi = \psi_{\mathbb{Q}} \circ \pi$ and recall that $\pi : L^0 \to L^0(\mathbb{Q})$ is order-continuous. Thus ψ is order-continuous on $\mathscr{X}$ as soon as $\psi_{\mathbb{Q}}$ is $\mathbb{Q}$-order continuous on $\mathscr{X}(\mathbb{Q}) = \pi(\mathscr{X})$ which is a solid subspace of $M_u^{\hat{\psi}}(\mathbb{Q})$. Then if $\mathscr{X}(\mathbb{Q})$ was further norm-closed in $M_u^{\hat{\psi}}(\mathbb{Q})$, we could conclude that $(\mathscr{X}(\mathbb{Q}), \|\cdot\|_{\hat{\psi},\mathbb{Q}})$ is an order-continuous Banach lattice on its own right, hence any finite monotone convex function on it is order continuous. But there is no guarantee that $\mathscr{X}(\mathbb{Q})$ is closed in $M_u^{\hat{\varphi}}(\mathbb{Q})$, so we need a trick.

Lemma 6.2. *In addition to the assumption of Lemma 6.1, we suppose that $\mathscr{X} \subset M_u^{\hat{\psi}}$. Then ψ has the Lebesgue property on $\mathscr{X}$, hence a fortiori $\psi = \hat{\psi}|_{\mathscr{X}}$.*

Proof. To see the Lebesgue property of ψ on $\mathscr{X}$, it suffices to show that $\psi_{\mathbb{Q}}$ has the $\mathbb{Q}$-Lebesgue property on $\mathscr{X}(\mathbb{Q})$, and for the latter, we have to show that for any $\eta \in \mathscr{X}(\mathbb{Q})$,

$$|\xi_n| \leq |\eta| \, (\forall n),\ \xi_n \to \xi \in \mathscr{X}(\mathbb{Q})\ \mathbb{Q}\text{-a.s.} \ \Rightarrow\ \psi_{\mathbb{Q}}(\xi) = \lim_n \psi_{\mathbb{Q}}(\xi_n). \tag{6.4}$$

Thus in the sequel, we fix an $\eta = \pi(Y) \in \mathscr{X}(\mathbb{Q})$, and note that $\mathscr{X}(\mathbb{Q})$ is solid subspace of $M_u^{\hat{\psi}}(\mathbb{Q})$ since $\mathscr{X}$ is a solid subspace of $M_u^{\hat{\psi}}$ and π is an onto lattice homomorphism.

Step 1. Define

$$B_\eta(\mathbb{Q}) := \{\zeta \in M_u^{\hat{\psi}}(\mathbb{Q}) :\ |\zeta| \wedge n|\eta| \uparrow |\zeta|\}. \tag{6.5}$$

This is the principal band generated by η in $M_u^{\hat{\psi}}(\mathbb{Q})$, i.e., it is the smallest *order closed* solid subspace (band) of $M_u^{\hat{\psi}}(\mathbb{Q})$ containing η. Consequently, $B_\eta(\mathbb{Q})$ is *norm closed* ([2, Theorem 8.43]) in the order-continuous Banach lattice $(M_u^{\hat{\psi}}(\mathbb{Q}), \|\cdot\|_{\hat{\psi},\mathbb{Q}})$, so $(B_\eta(\mathbb{Q}), \|\cdot\|_{\hat{\psi},\mathbb{Q}})$ is itself an order-continuous Banach lattice. Hence the extended Namioka-Klee theorem shows that any finite monotone convex function on $B_\eta(\mathbb{Q})$ is order-continuous.

Step 2. Define

$$\psi^\eta_{\mathbb{Q}}(\xi) := \lim_m \lim_n \psi_{\mathbb{Q}}((\xi \vee (-m|\eta|) \wedge n|\eta|), \quad \xi \in B_\eta(\mathbb{Q}). \tag{6.6}$$

Observe that $(\xi\vee(-m|\eta|))\wedge n|\eta| \in \mathscr{X}(\mathbb{Q})$ for each m, n since $\eta \in \mathscr{X}(\mathbb{Q})$ and $\mathscr{X}(\mathbb{Q})$ is solid, hence $\psi^\eta_{\mathbb{Q}}$ is well-defined at least as a $[-\infty, \infty]$-valued monotone function, and it is straightforward to deduce from the monotonicity and convexity of $\psi_{\mathbb{Q}}$ that $\psi^\eta_{\mathbb{Q}}$ is also monotone and convex. Moreover, $\psi^\eta_{\mathbb{Q}}$ is finite on $B_\eta(\mathbb{Q})$. To see this, note first that for all $\xi \in B_\eta(\mathbb{Q}) \subset M^{\hat{\psi}}_u(\mathbb{Q})$, Lemma 6.1 shows that

$$\psi^\eta_{\mathbb{Q}}(|\xi|) = \lim_n \psi_{\mathbb{Q}}(|\xi| \wedge n|\eta|) = \lim_n \hat{\psi}_{\mathbb{Q}}(|\xi| \wedge n|\eta|) = \hat{\psi}_{\mathbb{Q}}(|\xi|) < \infty.$$

On the other hand, $\psi^\eta_{\mathbb{Q}}(-|\xi|) = \lim_n \psi_{\mathbb{Q}}(-(|\xi| \wedge n|\eta|))$ by definition, and

$$\begin{aligned} 0 = 2\psi_{\mathbb{Q}}(0) &\leq \psi_{\mathbb{Q}}(|\xi| \wedge n|\eta|) + \psi_{\mathbb{Q}}(-(|\xi| \wedge n|\eta|)) \\ &\leq \hat{\psi}_{\mathbb{Q}}(|\xi|) + \psi_{\mathbb{Q}}(-(|\xi| \wedge n|\eta|)), \quad \forall n, \end{aligned}$$

hence $\psi^\eta_{\mathbb{Q}}(-|\xi|) = \inf_n \psi_{\mathbb{Q}}(-(|\xi| \wedge n|\eta|) \geq -\hat{\psi}(-|\xi|) > -\infty$. Consequently, **Step 1** tells us that $\psi^\eta_{\mathbb{Q}}$ is $\mathbb{Q}$-order continuous on $B_\eta(\mathbb{Q})$ as a finite monotone convex function on an order continuous Banach lattice.

Step 3. Though $B_\eta(\mathbb{Q})$ may not contain the *whole* $\mathscr{X}$, we see that if $|\xi| \leq |\eta|$, then $\xi \in \mathscr{X}(\mathbb{Q}) \cap B_\eta(\mathbb{Q})$ and $(\xi \vee (-m|\eta|)) \wedge n|\eta| = \xi$ for all m, n, hence $\psi^\eta_{\mathbb{Q}}(\xi) = \psi_{\mathbb{Q}}(\xi)$. In particular, if $|\xi_n| \leq |\eta|$ and $\xi_n \to \xi$, $\mathbb{Q}$-a.s., we have $\psi_{\mathbb{Q}}(\xi_n) = \psi^\eta_{\mathbb{Q}}(\xi_n) \to \psi^\eta_{\mathbb{Q}}(\xi) = \psi_{\mathbb{Q}}(\xi)$ by **Step 2**, and we have (6.4).□

Proof of Theorem 3.9: (3) ⇒ (1) and (4). Remember that (3) restricted to L^∞ implies that $\psi_\infty = \psi|_{L^\infty}$ has the Lebesgue property on L^∞. Also, since $\mathscr{X}$ is supposed to be solid, we have $(|X| \vee 1)Y \in \mathscr{X}$ for all $X \in \mathscr{X}$ and $Y \in L^\infty$, hence the condition (3) of Theorem 3.9 already implies that the supremum $\sup_{Z\in\mathrm{dom}\psi^*_\infty}(\mathbb{E}[(|X| \vee 1)YZ] - \psi^*_\infty(Z))$ is attained for any $X \in \mathscr{X}$, $Y \in L^\infty$ and $Z \in \mathrm{dom}\psi^*_\infty$. Hence we see from Theorem 3.8 ((3) ⇒ (1)) that $\mathscr{X} \subset M^{\hat{\psi}}_u$. Thus by Lemma 6.2, ψ has the Lebesgue property on $\mathscr{X}$ (thus (1)), and Theorem 3.5 shows that $\psi(X) = \hat{\psi}(X) = \sup_{Z\in\mathrm{dom}\psi^*_\infty}(\mathbb{E}[XZ] - \psi^*_\infty(Z))$, hence we have (4) since the supremum is supposed to be attained. □

7. Convex Risk Measures

Here we consider convex risk measures as our motivating class of monotone convex functions. In mathematical finance, a convex risk measure on a solid space $\mathscr{X}$ is a proper convex function ρ which is monotone *decreasing* in the a.s. order and satisfies the *cash-invariance*: $\rho(X + c) = \rho(X) - c$ if $X \in \mathscr{X}$ and $c \in \mathbb{R}$. Making a change of sign, we call a proper monotone (increasing) convex function φ on $\mathscr{X}$ a *convex risk function* if

$$\varphi(X + c) = \varphi(X) + c,\ \forall X \in \mathscr{X},\ c \in \mathbb{R}. \tag{7.1}$$

The relation between the two notions is obvious; if φ is a convex risk function, then $\rho(X) = \varphi(-X)$ is a convex risk measure, and also $-\varphi(-X)$ is called a *concave monetary utility function*. Though it is just a matter of notation, we prefer monotone *increasing* and *convex* functions which fit to our and standard notation of convex analysis, and it is also less confusing. Also, a convex risk function φ is called *coherent* if it is positively homogeneous: $\varphi(\alpha X) = \alpha\varphi(X)$ if $\alpha \geq 0$. We refer the reader to [16, Ch. 4] for a comprehensive account.

When $\mathscr{X} = L^\infty$, condition (7.1) for a monotone convex function φ is equivalent to

$$\varphi^*(Z) < \infty \ \Rightarrow\ Z \geq 0 \text{ and } \mathbb{E}[Z] = 1, \tag{7.2}$$

i.e., $\varphi^*(Z)$ is finite only if Z is a Radon-Nikodým density of a probability measure, say Q, absolutely continuous w.r.t. $\mathbb{P}$. Adopting the usual convention of identifying a probability measure $Q \ll \mathbb{P}$ with its density $dQ/d\mathbb{P}$, the representation (2.4) is written as

$$\varphi(X) = \sup_{Q\in\mathcal{Q}_\varphi} (\mathbb{E}_Q[X] - \varphi^*(Q)),\ X \in L^\infty, \tag{7.3}$$

where $\mathcal{Q}_{\varphi^*} := \{Q \ll \mathbb{P} : \text{probability}, dQ/d\mathbb{P} \in \mathrm{dom}\varphi^*\}$. Another consequence of cash-invariance (7.1) is that it implies φ is finite on L^∞, since then $-\|X\|_\infty = \varphi(-\|X\|_\infty) \leq \varphi(X) \leq \varphi(\|X\|_\infty) = \|X\|_\infty$ for all $X \in L^\infty$ by the monotonicity and (7.1). Thus all of our main results apply to any Lebesgue convex risk functions on L^∞. Note also that any Lebesgue extension of a convex risk function φ_0 on L^∞ retains the cash-invariance (7.1) since if $(\varphi, \mathscr{X})$ is a Lebesgue extension of such φ_0,

$$\begin{aligned}\varphi(X+c) &= \lim_n \varphi(X\mathbb{1}_{\{|X|\leq n\}} + c) = \lim_n \varphi_0(X\mathbb{1}_{\{|X|\leq n\}} + c)\\ &= \lim_n \varphi_0(X\mathbb{1}_{\{|X|\leq n\}}) + c = \lim_n \varphi(X\mathbb{1}_{\{|X|\leq n\}}) + c\end{aligned}$$

Consequently we have the following as a paraphrasing of Theorem 3.5:

Corollary 7.1. *Let φ_0 be a convex risk function on L^∞ with the Lebesgue property and $\varphi_0(0) = 0$, φ_0^* its conjugate, $\mathcal{Q}_0 := \mathcal{Q}_{\varphi_0}$ and $\mathcal{D}_0 := \{X \in L^0 : X^- \in L^1(Q), \forall Q \in \mathcal{Q}_0\}$. Then we have:*

(1) The following are well-defined

$$\hat{\varphi}(X) = \sup_{Q\in\mathcal{Q}_0} (\mathbb{E}_Q[X] - \varphi_0^*(Q)),\ X \in \mathcal{D}_0 \tag{7.4}$$

$$M_u^{\hat{\varphi}} = \{X \in L^0 : \lim_N \hat{\varphi}(\alpha|X|\mathbb{1}_{\{|X|>N\}}) = 0,\ \forall\alpha > 0\} \subset \mathcal{D}_0 \cap (-\mathcal{D}_0). \tag{7.5}$$

(2) $\hat{\varphi}$ is a finite convex risk function on $M_u^{\hat{\varphi}}$ with the Lebesgue property and $\hat{\varphi}|_{L^\infty} = \varphi_0$, and for any other pair $(\varphi, \mathscr{X})$ of a solid space $\mathscr{X} \subset L^0$ and a convex risk function on $\mathscr{X}$ with the Lebesgue property and $\varphi|_{L^\infty} = \varphi_0$, we have $\mathscr{X} \subset M_u^{\hat{\varphi}}$ and $\varphi = \hat{\varphi}|_{\mathscr{X}}$.

Note that the assumption $\varphi_0(0) = 0$ is just for notational simplicity; without this assumption, $(\hat{\varphi}, M_u^{\hat{\varphi}-\varphi_0(0)})$ is the maximum Lebesgue extension of φ_0.

Here we examine some typical risk functions deriving the explicit forms of the space $M_u^{\hat{\varphi}}$. We begin with a simple remark. Though we defined $\hat{\varphi}$ using the dual representation of φ_0 on L^∞, it may be more convenient to use other more explicit formula for φ_0 if available. By Lemma 3.1, we know that $\hat{\varphi}$ is continuous from below on $\mathcal{D}_0 \supset L^0_+$. In particular,

$$\hat{\varphi}(X) = \lim_n \varphi_0(X \wedge n), \quad \forall X \in L^0_+. \tag{7.6}$$

Note that this formula may not be true for $X \in \mathcal{D}_0 \setminus L^0_+$, but we need only consider $|X|$ with $X \in L^0$ to derive the spaces $M_u^{\hat{\varphi}}$ and $M^{\hat{\varphi}}$.

Example 7.2 (Entropic Risk Function). Let

$$\varphi_{\mathrm{ent}}(X) := \log \mathbb{E}[\exp(X)], \quad X \in L^\infty. \tag{7.7}$$

This is called the *entropic risk function*. It is straightforward from the dominated convergence theorem that φ_{ent} has the Lebesgue property on L^∞. Its conjugate φ^*_{ent} is given as $\varphi^*_{\mathrm{ent}}(Q) = \mathcal{H}(Q|\mathbb{P}) := \mathbb{E}[(dQ/d\mathbb{P})\log(dQ/d\mathbb{P})]$, the *relative entropy* (thus *entropic*), hence we have

$$\hat{\varphi}_{\mathrm{ent}}(X) = \sup_{Q\ll\mathbb{P},\mathcal{H}(Q|\mathbb{P})<\infty} (\mathbb{E}_Q[X] - \mathcal{H}(Q|\mathbb{P})), \qquad \Diamond$$

and the identity $\hat{\varphi}_{\mathrm{ent}}(X) = \log \mathbb{E}[\exp(X)]$ remains true for all $X \in L^0_+$. In particular, $M^{\hat{\varphi}_{\mathrm{ent}}} = M^{\Phi_{\exp}} \subsetneq L^{\Phi_{\exp}} = L^{\hat{\varphi}_{\mathrm{ent}}}$ if $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless, where $\Phi_{\exp}(x) = e^x - 1$ $(x \geq 0)$ and $M^{\Phi_{\exp}}$ (resp. $L^{\Phi_{\exp}}$) is the associated Orlicz heart (resp. space). Further, we see that $M_u^{\hat{\varphi}_{\mathrm{ent}}} = M^{\hat{\varphi}_{\mathrm{ent}}}(= M^{\Phi_{\exp}})$, since if $X \in M^{\Phi_{\exp}}$, $\exp(\varphi_{\mathrm{ent}}(\lambda|X|\mathbb{1}_{\{|X|>N\}})) = \mathbb{E}[\exp(\lambda|X|\mathbb{1}_{\{|X|>N\}})] = \mathbb{E}[\exp(\lambda|X|)\mathbb{1}_{\{|X|>N\}}] + \mathbb{P}(|X| \leq N) \to 1$ by the dominated convergence for every $\lambda > 0$.

7.1. Utility Based Shortfall Risk

Let $l : \mathbb{R} \to \mathbb{R}$ be a (finite) increasing convex function with $l(0) > \inf_x l(x)$ (thus not identically constant). Then its conjugate $l^*(y) = \sup_x(xy - l(x))$ is a convex function with

$$\mathrm{dom} l^* \subset \mathbb{R}_+ \quad \text{and} \quad \lim_{y\uparrow\infty} \frac{l^*(y)}{y} = +\infty. \tag{7.8}$$

The second one shows also that for any $c > 0$, there exist $\underline{\Lambda}(c), \overline{\Lambda}(c) \in (0, \infty)$ such that

$$\frac{l(0) + l^*(y)}{y} \leq c + 1 \;\Rightarrow\; y \in [\underline{\Lambda}(c), \overline{\Lambda}(c)] \tag{7.9}$$

Indeed, if $I_c := \{y > 0 : (l(0) + l^*(y))/y \leq c + 1\}$ is empty, put $\underline{\Lambda}(c) = \overline{\Lambda}(c) = 1$. Otherwise, $\overline{\Lambda}(c) := \sup I_c$ is finite by (7.8), and picking $x_0 < 0$ with $l(x_0) < l(0)$ (by assumption),

$$\frac{l(0) + l^*(y)}{y} = \sup_x \left(x + \frac{l(0) - l(x)}{y} \right) \geq x_0 + \frac{l(0) - l(x_0)}{y},$$

hence $\underline{\Lambda}(c) = \frac{l(0)-l(x_0)}{c+1-x_0} > 0$ does the job.

Now we define the associated *shortfall risk function* by

$$\varphi_l(X) := \inf\{x \in \mathbb{R} : \mathbb{E}[l(X - x)] \leq l(0)\}, \quad \forall X \in L^\infty. \tag{7.10}$$

This is a convex risk function with the Lebesgue property (2.3_∞) and its conjugate is

$$\varphi_l^*(Q) := \varphi_l^*(dQ/d\mathbb{P}) = \inf_{\lambda>0} \frac{1}{\lambda} \left(l(0) + \mathbb{E}\left[l^*\left(\lambda \frac{dQ}{d\mathbb{P}} \right) \right] \right). \tag{7.11}$$

(See [16, Ch.4]). Also, (7.6) implies that

$$\hat{\varphi}_l(|X|) = \sup_n \inf\{x : \mathbb{E}[l(|X| \wedge n - x)] \leq l(0)\} \leq \inf\{x : \mathbb{E}[l(|X| - x)] \leq l(0)\},$$

while if $\hat{\varphi}_l(|X|) < \infty$, then $\mathbb{E}[l(|X| - \hat{\varphi}_l(|X|))] \leq \lim_n \mathbb{E}[l(|X| \wedge n - \hat{\varphi}_l(|X|))] \leq \limsup_n \mathbb{E}[l(|X| \wedge n - \varphi_l(|X| \wedge n))] \leq l(0)$ by monotone convergence and $\varphi_l(|X| \wedge n) \leq \hat{\varphi}_l(|X|)$, thus

$$\hat{\varphi}_l(|X|) = \inf\{x : \mathbb{E}[l(|X| - x)] \leq l(0)\}, X \in L^0. \tag{7.12}$$

In this case, two spaces $M_u^{\hat{\varphi}_l}$ and $M^{\hat{\varphi}_l}$ coincide and equal to the Orlicz heart associated to the Young function $\Phi_l(x) := l(|x|) - l(0)$, i.e.,

Proposition 7.3. $M_u^{\hat{\varphi}_l} = M^{\hat{\varphi}_l} = M^{\Phi_l}$.

Proof. To see $M^{\Phi_l} \subset M_u^{\hat{\varphi}_l}$, it suffices that $\{XdQ/d\mathbb{P} : \varphi_l^*(Q) \leq c\}$ is uniformly integrable for any $c > 0$ and $X \in M^{\Phi_l}$ by Theorem 3.8 . So let us fix $c > 0$ and $X \in M^{\Phi_l}$. Observe that if $\varphi_l^*(Q) \leq c$, then there exists a $\lambda_Q > 0$ such that

$$c + 1 \geq \frac{1}{\lambda_Q}\left(l(0) + \mathbb{E}\left[l^*\left(\lambda_Q \frac{dQ}{d\mathbb{P}}\right)\right]\right) \geq \frac{l(0) + l^*(\lambda_Q)}{\lambda_Q} \tag{7.13}$$

by (7.11) and Jensen's inequality, and then $\lambda_Q \in [\underline{\Lambda}(c), \overline{\Lambda}(c)]$ by (7.9). Since $l(\alpha|X|\mathbb{1}_A) = \Phi_l(\alpha|X|)\mathbb{1}_A + l(0)$, Young's inequality shows for any $A \in \mathcal{F}$, $\alpha > 0$ and Q with $\varphi_l^*(Q) \leq c$,

$$\begin{aligned}\mathbb{E}_Q[|X|\mathbb{1}_A] &\leq \frac{1}{\alpha\lambda_Q}\left(\mathbb{E}[\Phi_l(\alpha|X|)\mathbb{1}_A] + \left(l(0) + \mathbb{E}\left[l^*\left(\lambda_Q \frac{dQ}{d\mathbb{P}}\right)\right]\right)\right) \\ &\leq \frac{1}{\alpha\lambda_Q}\mathbb{E}[\Phi_l(\alpha|X|)\mathbb{1}_A] + \frac{c+1}{\alpha} \leq \frac{1}{\alpha\underline{\Lambda}(c)}\mathbb{E}[\Phi_l(\alpha|X|)\mathbb{1}_A] + \frac{c+1}{\alpha}.\end{aligned}$$

Since $X \in M^{\Phi_l}$, the desired uniform integrability follows from a diagonal argument.

On the other hand, note that $l(\alpha|X|/2) \leq \frac{1}{2}l(\alpha|X| - x) + \frac{1}{2}l(x)$ by convexity, hence $M^{\hat{\varphi}_l} \subset M^{\Phi_l}$ follows from (7.12), and we deduce that the three spaces agree. □

Remark 7.4. In definition (7.10), we have chosen $l(0)$ for the acceptance level so that $\varphi_l(0) = 0$. If φ_l is defined with other acceptance level δ instead of $l(0)$, we can normalize it by adding the constant $a^l(\delta) := \sup\{x : l(x) \leq \delta\}$ or equivalently replacing the function l by $x \mapsto l(x + a^l(\delta))$. The case $l(0) = \inf_x l(x)$ corresponds to the worst case risk function $\varphi^{\text{worst}}(X) = \operatorname{ess\,sup} X$. Also, if $l(x) = e^x$, then $\varphi_l = \varphi_{\text{ent}}$. ♦

7.2. Robust Shortfall Risk

Let l be as above and fix a set $\mathcal{P}$ of probabilities $P \ll \mathbb{P}$ such that

$$\mathcal{P} \text{ is convex and weakly compact in } L^1. \tag{7.14}$$

Then we consider a *robust shortfall risk function*

$$\varphi_{l,\mathcal{P}}(X) := \inf\{x \in \mathbb{R} : \sup_{P\in\mathcal{P}} \mathbb{E}_P[l(X - x)] \leq l(0)\}, \quad X \in L^\infty. \tag{7.15}$$

The function $\varphi_{l,\mathcal{P}}$ on L^∞ is a convex risk function whose conjugate is given by

$$\varphi_{l,\mathcal{P}}^*(Q) := \inf_{\lambda>0} \frac{1}{\lambda}\left(l(0) + \inf_{P\in\mathcal{P}} \mathbb{E}_P\left[l^*\left(\lambda\frac{dQ}{dP}\right)\right]\right) \tag{7.16}$$

with the convention $l^*(\infty) := \infty$ and $\frac{dQ}{dP} := \frac{dQ/d\mathbb{P}}{dP/d\mathbb{P}}\mathbb{1}_{\{dP/d\mathbb{P}>0\}} + \infty \cdot \mathbb{1}_{\{dQ/d\mathbb{P}>0, dP/d\mathbb{P}=0\}}$ (see [16, Corollary 4.119]). Slightly modifying the argument for (7.12), we still have

$$\hat{\varphi}_{l,\mathcal{P}}(|X|) = \inf\{x \in \mathbb{R} : \sup_{P\in\mathcal{P}} \mathbb{E}_P[l(|X| - x)] \leq l(0)\}, \quad X \in L^0. \tag{7.17}$$

We introduce a couple of "robust analogues" of M^{Φ_l}:

$$\begin{aligned}M^{\Phi_l}(\mathcal{P}) &:= \{X \in L^0 : \sup_{P\in\mathcal{P}} \mathbb{E}_P[\Phi_l(\lambda|X|)] < \infty, \ \forall\lambda > 0\} \\ M_u^{\Phi_l}(\mathcal{P}) &:= \{X \in L^0 : \lim_{N\to\infty} \sup_{P\in\mathcal{P}} \mathbb{E}_P[\Phi_l(\lambda|X|)\mathbb{1}_{\{|X|>N\}}] = 0, \ \forall\lambda > 0\}.\end{aligned}$$

When $\mathcal{P} = \{\mathbb{P}\}$, the two spaces coincide with M^{Φ_l}. Now we have:

Proposition 7.5. *Assume (7.14). Then $\varphi_{l,\mathcal{P}}$ is Lebesgue on L^∞ and*

$$M_u^{\Phi_l}(\mathcal{P}) = M_u^{\hat{\varphi}_{l,\mathcal{P}}} \subset M^{\hat{\varphi}_{l,\mathcal{P}}} \subset M^{\Phi_l}(\mathcal{P}).$$

Proof. With a similar reasoning as Proposition 7.3, if $Q \in \mathcal{Q}_c := \{Q \ll \mathbb{P} : \varphi^*_{l,\mathcal{P}}(Q) \le c\}$, there exist $\lambda_Q \in [\underline{\Lambda}(c), \overline{\Lambda}(c)]$ and $P_Q \in \mathcal{P}$ such that

$$\frac{1}{\lambda_Q}\left(l(0) + \inf_{P\in\mathcal{P}} \mathbb{E}_P\left[l^*\left(\lambda_Q \frac{dQ}{dP}\right)\right]\right) \le \frac{1}{\lambda_Q}\left(l(0) + \mathbb{E}_{P_Q}\left[l^*\left(\lambda_Q \frac{dQ}{dP_Q}\right)\right]\right) \le c+1.$$

In particular, $\inf_{P\in\mathcal{P}} \mathbb{E}_P[l^*(\lambda_Q dQ/dP)] \le \overline{\Lambda}(c)(c+1) - l(0)$ whenever $Q \in \mathcal{Q}_c$. In view of (7.8), this shows that $\{\lambda_Q dQ/d\mathbb{P} : Q \in \mathcal{Q}_c\}$ is uniformly integrable thanks to the robust version of de la Vallée-Poussin theorem [14, Lemma 2.12] (which is stated there for sets of probabilities, but the exactly same proof works for sets of positive finite measures), hence so is $\mathcal{Q}_c$ since $\lambda_Q \ge \underline{\Lambda}(c)$ for each $Q \in \mathcal{Q}_c$. Consequently, $\varphi_{l,\mathcal{P}}$ is Lebesgue on L^∞ by the JST theorem (Theorem 2.4).

From the same inequality, we see also that

$$\begin{aligned}\mathbb{E}_Q[|X|\mathbb{1}_A] &\le \frac{1}{\alpha\lambda_Q}\left(\mathbb{E}_{P_Q}[\Phi(\alpha|X|)\mathbb{1}_A] + l(0) + \mathbb{E}_{P_Q}\left[l^*\left(\lambda_Q\frac{dQ}{dP_Q}\right)\right]\right)\\ &\le \frac{1}{\alpha\underline{\Lambda}(c)}\sup_{P\in\mathcal{P}}\mathbb{E}_P[\Phi(\alpha|X|)\mathbb{1}_A] + \frac{c+1}{\alpha}\end{aligned}$$

for any $\alpha > 0$, $A \in \mathcal{F}$ and $Q \in \mathcal{Q}_c$. Hence if $X \in M_u^{\Phi_l}(\mathcal{P})$, a diagonal argument shows that $\{X dQ/d\mathbb{P} : Q \in \mathcal{Q}_c\}$ is uniformly integrable, hence $M_u^{\Phi_l}(\mathcal{P}) \subset M_u^{\hat{\varphi}_{l,\mathcal{P}}}$ by Theorem 3.8.

To see $M_u^{\Phi_l}(\mathcal{P}) \supset M_u^{\hat{\varphi}_{l,\mathcal{P}}}$, let $X \in M_u^{\hat{\varphi}_{l,\mathcal{P}}}$ and $\alpha > 0$. By the definition of $M_u^{\hat{\varphi}_{l,\mathcal{P}}}$, there is a sequence $(N_n)_n \subset \mathbb{N}$ such that $\hat{\varphi}_{l,\mathcal{P}}(n\alpha|X|\mathbb{1}_{\{|X|>N_n\}}) < 2^{-n}$. Then by (7.17),

$$\sup_{P\in\mathcal{P}} \mathbb{E}_P[l(n\alpha|X|\mathbb{1}_{\{|X|>N_n\}} - 2^{-n})] \le l(0).$$

Noting that $\Phi_l(\alpha|X|\mathbb{1}_{A_n}) = l(\alpha|X|\mathbb{1}_{A_n}) - l(0) \le n^{-1}l(n\alpha|X|\mathbb{1}_{A_n} - 2^{-n}) + \frac{n-1}{n}l(\frac{2^{-n}}{n-1}) - l(0)$ with $A_n := \{|X| > N_n\}$ by the convexity, we have

$$\begin{aligned}\sup_{P\in\mathcal{P}}\mathbb{E}_P[\Phi_l(\alpha|X|)\mathbb{1}_{A_n}] &\le \frac{1}{n}\sup_{P\in\mathcal{P}}\mathbb{E}_P[l(n\alpha|X|\mathbb{1}_{A_n} - 2^{-n})] + \frac{n-1}{n}l\left(\frac{2^{-n}}{n-1}\right) - l(0)\\ &\le \frac{l(0)}{n} + \frac{n-1}{n}l\left(\frac{2^{-n}}{n-1}\right) - l(0) \to 0 + l(0) - l(0) = 0.\end{aligned}$$

Since $\alpha > 0$ is arbitrary, we have $X \in M_u^{\Phi_l}(\mathcal{P})$.

Finally, we show $M^{\hat{\varphi}_{l,\mathcal{P}}} \subset M^{\Phi_l}(\mathcal{P})$. If $X \in M^{\hat{\varphi}_{l,\mathcal{P}}}$, then for every $\alpha > 0$,

$$\begin{aligned}\sup_{P\in\mathcal{P}}\mathbb{E}_P[\Phi_l(\alpha|X|)] &= \sup_{P\in\mathcal{P}}\mathbb{E}_P[l(\alpha|X|)] - l(0)\\ &\le \frac{1}{2}\sup_{P\in\mathcal{P}}\mathbb{E}_P[l(2\alpha|X| - x)] + \frac{1}{2}l(x) - l(0) < \infty.\end{aligned}$$

for $x > \hat{\varphi}_{l,\mathcal{P}}(\alpha|X|)$ by (7.17). Thus $M^{\hat{\varphi}_{l,\mathcal{P}}} \subset M^{\Phi_l}(\mathcal{P})$. □

Example 7.6 (Robust Entropic Risk Functions). Let $l(x) = e^x$. Then $\varphi_{l,\mathcal{P}}$ is the entropic one, and the associated Young function is $\Phi_{\exp}(x) = e^x - 1$. In this case, we have $M_u^{\Phi_{\exp}}(\mathcal{P}) = M^{\Phi_{\exp}}(\mathcal{P})$, thus $M_u^{\hat{\varphi}_{l,\mathcal{P}}} = M^{\hat{\varphi}_{l,\mathcal{P}}}$. Indeed, by Hölder's inequality,

$$\begin{aligned}\sup_{P\in\mathcal{P}}\mathbb{E}_P[e^{\alpha|X|}\mathbb{1}_{\{|X|>N\}}] &\le \sup_{P\in\mathcal{P}}\left(\mathbb{E}_P[e^{2\alpha|X|}]^{1/2}P(|X|>N)^{1/2}\right)\\ &\le \sup_{P\in\mathcal{P}}\mathbb{E}_P[e^{2\alpha|X|}]^{1/2}\sup_{P\in\mathcal{P}}P(|X|>N)^{1/2}.\end{aligned}$$

This and the uniform integrability of $\mathcal{P}$ show that $\lim_N \sup_{P\in\mathcal{P}} \mathbb{E}_P[e^{\alpha|X|}\mathbb{1}_{\{|X|>N\}}] = 0$ for every $\alpha > 0$ as soon as $X \in M^{\Phi_{\exp}}(\mathcal{P})$, hence $M_u^{\Phi_{\exp}}(\mathcal{P}) = M^{\Phi_{\exp}}(\mathcal{P})$. ♢

7.3. Law-Invariant Case

Recall that a convex risk function φ_0 on L^∞ is called *law-invariant* if $\varphi_0(X) = \varphi_0(Y)$ whenever X and Y have the same distribution. Any law-invariant convex risk function on L^∞ has the following *Kusuoka representation* ([24], [18]):

$$\varphi_0(X) = \sup_{\mu\in\mathcal{M}_1((0,1])}\left(\int_{(0,1]} v_\lambda(X)\mu(d\lambda) - \beta(\mu)\right) \tag{7.18}$$

where $v_\lambda(X) := \frac{1}{\lambda}\int_0^\lambda q_X(1-t)dt$, the *average value at risk at level* λ (up to change of sign), $q_X(t) := \inf\{x : \mathbb{P}(X \le x) > t\}$, $\mathcal{M}_1((0,1])$ is the set of all Borel probability measures on $(0,1]$ and β is a lower semi-continuous penalty function. Then φ_0 has the Lebesgue property on L^∞ if and only if all the level sets $\{\mu : \beta(\mu) \le c\}$ are relatively weak* compact in $\mathcal{M}_1((0,1])$ or equivalently tight ([12, Ch. 5] or [21]). In particular, for any relatively weak* compact convex set $\mathcal{M} \subset \mathcal{M}_1((0,1])$,

$$\varphi_{\mathcal{M}}(X) := \sup_{\mu\in\mathcal{M}}\int_{(0,1]} v_\lambda(X)\mu(d\lambda)$$

is a law-invariant coherent risk function on L^∞ satisfying the Lebesgue property.

Example 7.7 (AV@R). For every $\lambda \in (0,1]$, v_λ admits the representation:

$$v_\lambda(X) = \sup\{\mathbb{E}_Q[X] : Q \in \mathcal{P},\ dQ/d\mathbb{P} \le 1/\lambda\}, \tag{7.19}$$

for all $X \in L^\infty$, and since $\hat{v}_\lambda(|X|) = \sup_n v_\lambda(|X| \wedge n)$,

$$\|X\|_{L^1} \le \hat{v}_\lambda(|X|) = \|X\|_{\hat{v}_\lambda} \le \frac{1}{\lambda}\|X\|_{L^1}, \quad X \ge 0.$$ ♢

Hence we have $M_u^{\hat{v}_\lambda} = M^{\hat{v}_\lambda} = L^1$ for every $\lambda \in (0,1]$, and the representation (7.19) extends to L^1. In particular, $\hat{v}_\lambda$ has the Lebesgue property on L^1.

Example 7.8 (Concave Distortions). Let $\mu \in \mathcal{M}_1((0,1])$ and define

$$\varphi_\mu(X) := \int_{(0,1]} v_t(X)\mu(dt).$$

This type of risk functions are called concave distortion, and it is known that if the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless, every law-invariant *comonotonic* risk function is written in this form (see [16, Theorem 4.93]). For φ_μ, two spaces $M_u^{\hat{\varphi}_\mu}$ and $M^{\hat{\varphi}_\mu}$ coincide. Indeed, if $\hat{\varphi}_\mu(|X|) < \infty$ ($\Leftrightarrow$ $\hat{v}_\cdot(|X|) \in L^1((0,1],\mu)$), then from Example 7.7, we see that $v_t(|X|\mathbb{1}_{\{|X|>N\}}) \le v_t(|X|)$ and $\lim_N v_t(|X|\mathbb{1}_{\{|X|>N\}}) = 0$ for (μ-a.e., hence) all $t \in (0,1]$. Thus the dominated convergence theorem implies that

$$\lim_N \int_{(0,1]} \hat{v}_t(|X|\mathbb{1}_{\{|X|>N\}})\mu(dt) = \int_{(0,1]} \lim_N \hat{v}_t(|X|\mathbb{1}_{\{|X|>N\}})\mu(dt) = 0.$$

Repeating the same argument for $\alpha|X|$ ($\alpha > 0$) instead of X, we have $M_u^{\hat{\varphi}_\mu} = M^{\hat{\varphi}_\mu}$. ♢

Recall that any *finite-valued* convex risk function on a solid and rearrangement-invariant space strictly bigger than L^∞ has the Lebesgue property *restricted to* L^∞ ([11, Theorem 3] or see the comment after Theorem 2.4). The next example concerns how is the Lebesgue property on the whole space. In our context, both $M_u^{\hat{\varphi}}$ and $M^{\hat{\varphi}}$ are (solid and) rearrangement-invariant if the φ_0 is law-invariant, and $M^{\hat{\varphi}}$ is the maximum solid vector space on which $\hat{\varphi}$ is finite-valued. Then the question is translated as: does it hold $M_u^{\hat{\varphi}} = M^{\hat{\varphi}}$ *as soon as* φ_0 *is law-invariant*? The answer is generally no.

Example 7.9 (A law-invariant risk function with $M_u^{\hat{\varphi}} \subsetneq M^{\hat{\varphi}}$). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be atomless and for each n, we define a Borel probability measure on $(0, 1]$ by

$$\mu_n(dt) := \left(1 - \frac{1}{n}\right) \frac{e}{e-1} \mathbb{1}_{(e^{-1},1]}(t)dt + \frac{1}{n}\frac{e^n}{e-1} \mathbb{1}_{(e^{-n},e^{-n+1}]}(t)dt. \tag{7.20}$$

Then $(\mu_n)_n$ (and hence $\overline{\mathrm{conv}}(\mu_n; n \in \mathbb{N})$) is uniformly integrable in $L^1((0,1], dt)$ ($\Leftrightarrow$ weak* compact in $\mathcal{M}_1((0,1])$). Hence the law-invariant coherent risk function

$$\varphi_0(X) := \sup_n \int_{(0,1]} v_t(X)\mu_n(dt) \quad \left(\Rightarrow \hat{\varphi}(|X|) = \sup_n \int_{(0,1]} \hat{v}_\lambda(|X|)\mu_n(d\lambda)\right)$$

has the Lebesgue property on L^∞. In this case, $M_u^{\hat{\varphi}} \subsetneq M^{\hat{\varphi}}$. Indeed, let X be an exponential random variable with parameter 1, i.e., $F_X(x) := \mathbb{P}(X \le x) = 1 - e^{-x} \Leftrightarrow q_X(t) = -\log(1-t)$. Then

$$\hat{v}_\lambda(X) = \frac{1}{\lambda}\int_0^\lambda (-\log t)dt = 1 - \log \lambda.$$

For each n, $\int_{(0,1]} \hat{v}_t(X)\mu_n(dt) = 4 - \frac{e}{e-1} - \frac{1}{n}$, so $\hat{\varphi}(X) = \sup_n \int_{(0,1]} \hat{v}_t(X)\mu_n(dt) = 4 - \frac{e}{e-1} < \infty$. This shows that $X \in M^{\hat{\varphi}}$. We next compute $\lim_N \varphi(X\mathbb{1}_{\{X>N\}})$. Since $q_{X\mathbb{1}_{\{X>N\}}}(t) = q_X \mathbb{1}_{\{q_X(t)>N\}}$ and $q_X(1-t) > N \Leftrightarrow t < 1 - F_X(N) = e^{-N}$,

$$\begin{aligned}\hat{v}_\lambda(X\mathbb{1}_{\{X>N\}}) &= \frac{1}{\lambda}\int_0^\lambda q_X(1-t)\mathbb{1}_{\{q_X(1-t)>N\}}dt \\ &= \{\lambda \wedge e^{-N} - (\lambda \wedge e^{-N})\log(\lambda \wedge e^{-N}))\}/\lambda.\end{aligned}$$

Thus for $n > N + 1$,

$$\begin{aligned}&\int_{(0,1]} \hat{v}_t(X\mathbb{1}_{\{X>N\}})\mu_n(dt) \\ &= \left(1 - \frac{1}{n}\right)\frac{e}{e-1}\left(e^{-N} - e^{-N}\log e^{-N}\right) + \frac{1}{n}\left(2 + n - \frac{e}{e-1}\right) \\ &= 1 + \frac{e}{e-1}\left(e^{-N} - e^{-N}\log e^{-N}\right) + \frac{1}{n}\left\{2 - \frac{e}{e-1}\left(1 + e^{-N} - e^{-N}\log e^{-N}\right)\right\}\end{aligned}$$

Hence $\hat{\varphi}(X\mathbb{1}_{\{X>N\}}) = \sup_n \int_{(0,1]} \hat{v}_t(X\mathbb{1}_{\{X>N\}})\mu_n(dt) = 1 + \frac{e}{e-1}\left(e^{-N} - e^{-N}\log e^{-N}\right)$. Consequently, $\lim_{N\to\infty} \varphi(X\mathbb{1}_{\{X>N\}}) \ge 1 + \lim_N \frac{e}{e-1}\left(e^{-N} - e^{-N}\log e^{-N}\right) = 1$. Thus $X \notin M_u^{\hat{\varphi}}$. ♢

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A. Appendix

We have used the following version of minimax theorem which should be known as it is an immediate corollary to [20, Theorems 1 and 2]. But we could not find an appropriate reference, so we include here a simple proof.

Theorem A.1. *Let C be a convex subset of a Hausdorff topological vector space, and D an arbitrary convex set. Suppose we are given a function $f : C \times D \to \mathbb{R}$ such that*

(1) for any $y \in D$, $x \mapsto f(x, y)$ is convex and $\{x \in C : f(x, y) \leq c\}$ is compact for each $c \in \mathbb{R}$;
(2) for any $x \in C$, $y \mapsto f(x, y)$ is concave on D.

Then we have

$$\inf_{x\in C}\sup_{y\in D} f(x,y) = \sup_{y\in D}\inf_{x\in C} f(x,y). \tag{A.1}$$

Proof. Note first that "$\geq$" is always true whatever C, D and f are. Thus there is nothing to prove if $\alpha := \sup_{y\in D}\inf_{x\in C} f(x, y) = \infty$, hence we assume $\alpha < \infty$.

For any $y \in D$ and $\beta \in \mathbb{R}$, we set $A^\beta_y := \{x \in C : f(x, y) \leq \beta\}$. Then [20, Theorem 1] implies that the family $\{A^{\alpha+\varepsilon}_y\}_{y\in D}$ has the finite intersection property for every $\varepsilon > 0$. Noting that each $A^{\alpha+\varepsilon}_y$ is compact by assumption made on f, we have $\bigcap_{y\in D} A^{\alpha+\varepsilon}_y \neq \emptyset$ (indeed, fixing arbitrary $y_0 \in D$, we have $A^{\alpha+\varepsilon}_{y_0}$ is compact, $A^{\alpha+\varepsilon}_y \cap A^{\alpha+\varepsilon}_{y_0}$ is its non-empty closed subset for each $y \in D$, and $\bigcap_{y\in D} A^{\alpha+\varepsilon}_y = \bigcap_{y\in D}(A^{\alpha+\varepsilon}_y \cap A^{\alpha+\varepsilon}_{y_0}) \neq \emptyset$). But this is a necessary and sufficient condition for the equality (A.1) by [20, Theorem 2]. □

Proposition A.2. *For a finite monotone convex function φ with the Fatou property on a solid space $\mathscr{X}$ containing the constants, the Lebesgue property is equivalent to: for any countable net $(X_\alpha)_\alpha$,*

$$X_\alpha \in L^\infty,\ |X_\alpha| \leq |X|,\ \forall\alpha,\ \text{and } X_\alpha \to X \text{ a.s.} \ \Rightarrow\ \varphi(X_\alpha) \to \varphi(X). \tag{2.9}$$

Proof. The necessity is clear from Remark 2.2. Recall that the Lebesgue property of φ is equivalent to the sequential continuity from above. For a sequence $(X_n)_n \subset \mathscr{X}$ with $X_n \downarrow X \in \mathscr{X}$, consider a net $X_{n,m} := (X_n \vee (-n)) \wedge m$ with indices (n, m) directed by $(n, m) \leq (n', m')$ iff $n \leq n'$ and $m \leq m'$. Then $X_{n,m} \in L^\infty$ for each (n, m) and $X_{n,m} \xrightarrow{o} X$ in $\mathscr{X}$. Indeed, $\limsup_{(n,m)} X_{n,m} = \inf_{(n,m)} \sup_{n'\geq n, m'\geq m}(X_{n'} \vee (-n')) \wedge m' = \inf_{(n,m)} X_n \vee (-n) = X$, and $\liminf_{(n,m)} X_{n,m} = \sup_{(n,m)} \inf_{n'\geq n, m'\geq m}(X_{n'} \vee (-n')) \wedge m' = \sup_{(n,m)} X \wedge m = X$. Therefore $\varphi(X) = \lim_{(n,m)} \varphi(X_{n,m})$ by (2.9). On the other hand, $\varphi(X_n) \leq \varphi(X_n \vee (-n)) = \sup_m \varphi((X_n \vee -n) \wedge m)$ by Fatou and monotonicity, thus

$$\begin{aligned}\inf_n \varphi(X_n) &\leq \inf_n \sup_m \varphi((X_n \vee -n) \wedge m) = \lim_n \lim_m \varphi((X_n \vee -n) \wedge m)\\ &= \lim_{(n,m)} \varphi(X_{n,m}) = \varphi(X).\end{aligned}$$

Hence φ has the Lebesgue property. □

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