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Pricing Bounds on Barrier Options*

Yukihiro Tsuzuki†‡

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Abstract

This paper proposes the optimal pricing bounds on barrier options in an environment where plain-vanilla options and no-touch options can be used as hedging instruments.

Super-hedging and sub-hedging portfolios are derived without specifying any underlying processes, which are static ones consisting of not only plain-vanilla options but also cash-paying no-touch options and/or asset paying no-touch options that pay one cash or one underlying asset respectively if the barrier has not been hit. Moreover, the prices of these portfolios turn out to be the optimal pricing bounds through finding risk-neutral measures under which the barrier option price is equal to the hedging portfolio's value.

The model-independent pricing bounds are useful because a price of a barrier option is significantly dependent on a model. It is demonstrated through numerical examples that prices outside the pricing bounds can be produced by models which are calibrated to market prices of plain-vanilla options, but not to that of a no-touch option.

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1 Introduction

This paper proposes pricing bounds on barrier options.

Pricing and hedging barrier options have been researched widely so far. In particular, a lot of methods which semi-statically hedge barrier options have been proposed by several researchers (see e.g. Carr and Chou (1997), Carr et al. (1998), Derman et al. (1995), Fink (2003)). Here, semi-static hedging means replication of barrier options by trading plain-vanilla options at no more than one time after inception. Since plain-vanilla options are needed for these hedging strategies, models which price barrier options must be calibrated to plain-vanilla options.

However, model risk on the valuation of barrier options has been pointed out, even if the model is perfectly calibrated to a volatility surface. For instance, it is documented in Hirsa et al. (2003) and Lipton and McGhee (2002) that models may produce similar prices of plain-vanilla options, yet give markedly different prices of barrier options. As a result, they demonstrate that static hedging of barrier options with plain-vanilla options is model dependent. Schoutens et al. (2005) also reports a general feature of pricing exotic options under several models calibrated nicely to the same volatility surface. Their results about barrier options show that the variation can be significant, especially if the spot price is close to the barrier level.

Model risk of barrier options is explained as follows. Since barrier options are path-dependent options, joint distributions of the underlying asset prices at different time points are significantly important in order to price or hedge these options. In contrast, plain-vanilla options determine only the marginal distribution of the risk-neutral measure, but leave considerable freedom in the specification of joint distributions. Joint distributions are determined not by market prices but by models, if the models are calibrated only to plain-vanilla options. Then, what model is used to price barrier options is heavily important. This implies that models must be not only matched to the market prices of plain-vanilla options, but also be consistent with the observable prices of exotic options. Carr and Crosby (2010) proposes a model which is calibrated to both the market prices of plain-vanilla options and the observable prices of exotic options. They consider pricing of barrier options in the foreign exchange (FX) options market. In this market, the most actively traded barrier options are double-no-touch (DNT) options. They regard DNT options as instruments which are calibrated to.

In contrast, this study proposes not exact prices for barrier options but pricing bounds, using no-touch options as well as plain-vanilla options, where a no-touch option is a knock-out option which is worthless if the barrier is hit, pays one cash at maturity if the barrier has not been hit. Moreover, the corresponding super-hedging and sub-hedging portfolios are also provided, which are static ones consisting of plain-vanilla options and no-touch options. They are derived without any specification of underlying processes and are the optimal pricing bounds among model-independent bounds. The optimality is proved through finding risk-neutral measures under which the barrier option price is equal to the hedging portfolio's value. In addition, if one can use another type of a no-touch option which pays one underlying asset at maturity if the barrier has not been hit, the pricing bounds are much tighter. In particular, this no-touch option is common in FX options market because it is nothing but a no-touch option which pays one cash for a foreign trader. Hereafter, these no-touch options are called cash-paying no-touch option and asset-paying no-touch option respectively in this article.

The model-independent pricing bounds and the hedging portfolios are useful for checking a barrier option's price. If the price is outside the pricing bounds, it provides an arbitrage opportunity which yields a profit without any risk. It will be demonstrated through numerical examples in this paper that prices outside the pricing bounds may be produced by models which are calibrated to market prices of plain-vanilla options, but not to that of a no-touch option. As Carr and Crosby (2010) does, this study also suggests that models for pricing barrier options should be calibrated to the market prices of no-touch options. Otherwise, it is likely that arbitrage opportunities are provided.

The strategy proposed in this article is unique, while model-independent pricing bounds and the hedging portfolios have been studied so far. For example, Brown et al. (2001) and Cox and Obłój (2011) derive hedging portfolios which consist of only plain-vanilla options and require a certain transaction at the first hitting time, assuming that plain-vanilla options are liquidly traded and an underlying asset price itself is a martingale. The approach of this study is more robust than theirs because their assumption is violated in the real markets with nonzero interest rates and it is not necessarily possible to trade during the turmoil periods, which may cause substantial hedging errors. It should be pointed out that the strategy is crucially dependent on liquidity of no-touch options, while Brown et al. (2001) is so general that they can be applied to these options as well. Pricing bounds and the corresponding hedging strategies for no-touch options are well documented in these Brown et al. (2001) and Cox and Obłój (2011). Also, pricing no-touch options under several models is studied by Lipton and McGhee (2002). An interesting point in this study is, however, to make use of market prices of no-touch options for the purpose of finding arbitrage opportunities. Actually, these options are likely to be overestimated or underestimated, since they are highly dependent on models and market views of traders.

Numerical examples are used to demonstrate how useful the bounds are for pricing and hedging barrier options. The pricing bounds are compared with exact prices under several models which are calibrated to plain-vanilla options and with pricing bounds proposed by Brown et al. (2001). In particular, a comparison with the prices in Schoutens et al. (2005) shows that some models produce prices outside the model-independent bounds. It is suggested that great care should be taken when choosing models to price barrier options and the models should be calibrated to both the market prices of plain-vanilla options and no-touch options.

The rest of this paper is as follows: In the next section, the setup and the problem considered in this paper are described. The third section is devoted to showing pricing bounds on the barrier option and hedging strategies corresponding to them. The fourth section provides numerical examples. Finally, concluding remarks are given in the last section.

2 Setup

Consider a problem of pricing and hedging barrier options in an environment where plain-vanilla options and no-touch options are liquid.

In order to state the problem precisely, some assumptions and notations are introduced. First, a barrier option under consideration is assumed to be a single or double knock-out option with maturity T, payoff g and barrier levels l, u where $0 \leq l < u \leq +\infty$. This option is worthless if l or u is hit at any time during its life. If the barrier has not been hit by the expiration date, the terminal payoff is g. Let S_t be the spot price of the underlying asset at time $t \in [0, T]$ and $l < S_0 < u$. Then, the payoff of the barrier option is:

$$g(S_T)1_A, \tag{2.1}$$

where $I := [l, u]$ and $A := \{S_t \in I \mid 0 \leq \forall t \leq T\}$. A cash-paying no-touch option with maturity T and barrier levels l, u is a knock-out option which is worthless if the barrier is hit, pays one cash at maturity if the barrier has not been hit. The payoff is 1_A. An asset-paying no-touch option pays one underlying asset instead of one cash. The payoff is $S_T 1_A$.

Next, some assumptions on the market environment are described. The risk-free interest rate r and the dividend yield q of the underlying asset are assumed to be constant for simplicity. Different from other research of barrier options, these two quantities are not required to be equal. In addition, the time-0 prices of all plain-vanilla options with maturity T and a cash-paying no-touch option with the barrier level l, u and the same maturity are assumed to be known. That is, the risk-neutral distribution of S_T is uniquely determined by prices of plain-vanilla options and the risk-neutral probability that the barrier is hit at any time during its life is known. Furthermore, it is assumed that the prices of plain-vanilla options are twice continuously differentiable by strike and the second order derivative is positive. The last assumption ensures that the density of the random variable S_T exists and is a continuous function.

All one knows is the risk-neutral distribution of S_T and the risk-neutral probability that the barrier is hit until time T. In these settings, one has no information about the risk-neutral distribution of S_t $(t < T)$.

The problem is stated as follows: *How can a knock-out call option be priced and hedged in an environment where the underlying asset, all plain-vanilla options with the same maturity and a cash-paying no-touch option with the same maturity and the same barrier levels can be used as hedging instruments?*

Remark 1. *This paper considers a knock-out call option with a single or double barrier. The theorems in this paper introduced in Section 3 are however valid for other types of barrier options if it is slightly modified. For example, the knock-out condition A can be replaced with $\{S_t \in I \mid \forall t \in J\}$, where J is a subset of the interval $[0, T]$, and with $\{X_t \in I \mid 0 \leq \forall t \leq T\}$, where X is an another asset price process.*

The approach in this paper to this problem is not to derive an exact price and an exact hedging, but to derive pricing bounds, and super-hedging and sub-hedging strategies corresponding to the bounds. These pricing bounds are the optimal under a certain condition, which means that the pricing bounds cannot be improved without additional condition.

The undiscounted price, or forward price, of a call option and a cash-paying no-touch option at time t are denoted by $C_t(K)$ and $N_t^C(I)$. All one requires of a pricing framework is that they are calibrated to market prices of these options. In order to define this requirement, the following definition is introduced.

Definition 1. *A pair of a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ and a stochastic process $\{S_t\}_{t\in[0,T]}$ on it is a calibrated probability space if and only if*

$$\begin{aligned} C_0(K) &= \mathbb{E}((S_T - K)_+), &(2.2)\\ N_0^C(I) &= \mathbb{E}(1_A), &(2.3) \end{aligned}$$

where $\forall K > 0$ and $\mathbb{E}$ is the expectation operator under $\mathbb{Q}$. $\mathcal{P}^C$ is a set of all calibrated probability spaces.

In addition, a constraint on trading strategies is imposed: trading is allowed only at the initial time. This kind of strategies is called a *static* strategy in this paper. Limiting the strategy static one means that it is not required that the process $\{e^{-(r-q)t}S_t\}_{t\in[0,T)}$ is a continuous-time martingale[1] and that all of probability one has to consider is $\mathcal{P}^C$.

If a market is incomplete, the price of the barrier option cannot be determined uniquely. The next best thing is to derive the optimal pricing bounds. If there are no arbitrage opportunities, undiscounted values of the barrier option must lie somewhere between the bounds $[W_L^C, W_G^C]$, where

$$\begin{aligned} W_L^C &:= \inf_{\mathcal{P}^C} \mathbb{E}(g(S_T)1_A), &(2.4)\\ W_G^C &:= \sup_{\mathcal{P}^C} \mathbb{E}(g(S_T)1_A). &(2.5) \end{aligned}$$

This study derives pricing bounds which are enforced by a static super-hedging strategy and a static sub-hedging strategy. In particular, the bounds are turned out to be the optimal, which means that they are equal to $[W_L^C, W_G^C]$. This is proved by constructing specific calibrated probability spaces under which a price of the barrier option is equal to the super-hedging and sub-hedging values respectively.

After establishing pricing bounds for this problem, it is considered how the pricing bounds are improved by adding an assumption that one can use an asset-paying no-touch option as a hedging instrument. Similarly to the problem stated above, the following notations are defined:

Definition 2. *$\mathcal{P}^A$ is a subset of $\mathcal{P}^C$ such that its element satisfies*

$$N_0^A(I) = \mathbb{E}(S_T 1_A), \tag{2.6}$$

where $N_0^A(I)$ is an undiscounted price of the asset-paying no-touch option.

[1] It is required that the discrete-time process $\{S_0, e^{-(r-q)T}S_T\}$ is a martingale, but it is imposed from the prices of plain-vanilla options.

The pricing bounds with this probability space $\mathcal{P}^A$ are

$$W_L^A := \inf_{\mathcal{P}^A} \mathbb{E}(g(S_T)1_A), \tag{2.7}$$

$$W_G^A := \sup_{\mathcal{P}^A} \mathbb{E}(g(S_T)1_A). \tag{2.8}$$

Lastly, a measure to be considered is a component of an element of $\mathcal{P}^C or \mathcal{P}^A$ hereafter in this paper. Then, $\mathbb{Q}$ or $\mathbb{E}$ is respectively used to denote the risk-neutral distribution or expectation of S_T and of A for simplicity if it is not ambiguous, because they do not depend on a choice of an element of $\mathcal{P}^C or \mathcal{P}^A$.

3 Pricing Bounds

This section derives pricing bounds on barrier options and hedging portfolios corresponding to the bounds. Theorem 1 is the main theorem of this paper.

Theorem 1. *Define $A_* \subseteq I$ such that $\sup_{s\in A_*} g(s) \leq \inf_{s\notin A_*} g(s)$ and $A^* \subseteq I$ such that $\sup_{s\notin A^*} g(s) \leq \inf_{s\in A^*} g(s)$. Then,*

$$\begin{aligned} & (g(S_T) - g_*)1_{\{S_T\in A_*\}} + g_* 1_A && (3.1)\\ \leq\ & g(S_T)1_A && (3.2)\\ \leq\ & (g(S_T) - g^*)1_{\{S_T\in A^*\}} + g^* 1_A, && (3.3)\end{aligned}$$

where g_* and g^* are arbitrary values such that $g_* \in [\sup_{s\in A_*} g(s), \inf_{s\notin A_*} g(s)]$ and $g^* \in [\sup_{s\notin A^*} g(s), \inf_{s\in A^*} g(s)]$ respectively. Moreover, suppose that $N_0^C(I) = \mathbb{Q}(S_T \in A_*)$ and $N_0^C(I) = \mathbb{Q}(S_T \in A^*)$. Then,

$$W_L^C = \mathbb{E}((g(S_T) - g_*)1_{\{S_T\in A_*\}} + g_* 1_A), \tag{3.4}$$

$$W_G^C = \mathbb{E}((g(S_T) - g^*)1_{\{S_T\in A^*\}} + g^* 1_A). \tag{3.5}$$

Remark 2. *The former part of Theorem 1 implies that the barrier option can be dominated by or dominate some portfolios which consist of plain-vanilla options and a cash-paying no-touch option. The latter part of Theorem 1 implies that the portfolios give the optimal pricing bounds respectively.*

Remark 3. *One can find A_* and A^* because it is assumed that the density of S_T is continuous.*

In the following, two propositions which constitute Theorem 1 are provided. The first one shows that there are families of super-hedging and sub-hedging portfolios.

Proposition 1 (super-hedging and sub-hedging portfolio)**.** *Suppose that g_*, A_*, g^* and A^* are defined as in Theorem 1 for g. Then,*

$$\begin{aligned} & (g(S_T) - g_*)1_{\{S_T\in A_*\}} + g_* 1_A && (3.6)\\ \leq\ & g(S_T)1_A && (3.7)\\ \leq\ & (g(S_T) - g^*)1_{\{S_T\in A^*\}} + g^* 1_A. && (3.8)\end{aligned}$$

Proof. Inequality (3.1) is proved by

$$\begin{aligned} (g(S_T(\omega)) - g_*)1_{\{S_T(\omega)\in A_*\}} + g_* 1_{\{\omega\in A\}} &= \begin{cases} g(S_T(\omega)) & (S_T(\omega) \in A_*, \omega \in A)\\ g_* & (S_T(\omega) \notin A_*, \omega \in A)\\ g(S_T(\omega)) - g_* & (S_T(\omega) \in A_*, \omega \notin A)\\ 0 & (S_T(\omega) \notin A_*, \omega \notin A)\end{cases}\\ &\leq \begin{cases} g(S_T(\omega)) & (S_T(\omega) \in A_*, \omega \in A)\\ g(S_T(\omega)) & (S_T(\omega) \notin A_*, \omega \in A)\\ 0 & (S_T(\omega) \in A_*, \omega \notin A)\\ 0 & (S_T(\omega) \notin A_*, \omega \notin A)\end{cases}\\ &= g(S_T(\omega))1_{\{\omega\in A\}} \end{aligned} \tag{3.9}$$

and Inequality (3.3) is proved in the same manner. $\square$

Eq.(3.6) shows that the payoff of the knock-out option dominates that of the following portfolio:

- g_* unit of the cash-paying no-touch option
- one unit of a European derivative with the payoff $(g(S_T) - g_*)1_{\{S_T \in A_*\}}$.

Similarly, Eq.(3.8) shows that the payoff of the knock-out option is dominated by that of the following portfolio:

- g^* unit of the cash-paying no-touch option
- one unit of a European derivative with the payoff $(g(S_T) - g^*)1_{\{S_T \in A^*\}}$.

Remark 4. *Construction of the payoffs $(g(S_T) - g_*)1_{\{S_T \in A_*\}}$ and $(g(S_T) - g^*)1_{\{S_T \in A^*\}}$ is theoretically possible using call and put options (see Breeden and Litzenberger (1978)). However, the construction in practice requires high transaction cost. One way to address this problem is to trade these options with an internal option trader not with an external market participant. The internal option trader manages the options as a part of his or her own position and hedging or taking risk is up to him or her.*

Proposition 2 shows the optimality of the pricing bounds by constructing calibrated probability spaces under which a price of the barrier option is equal to W_L^C and W_G^C respectively.

Proposition 2 (the optimality of the bounds)**.** *Suppose that $N_0^C(I) = \mathbb{Q}(S_T \in A_*)$ and $N_0^C(I) = \mathbb{Q}(S_T \in A^*)$.*

(i) *There exists a calibrated probability space $(\Omega^L, \mathcal{F}^L, \mathbb{Q}^L, \{S_t^L\}_{t\in[0,T]}) \in \mathcal{P}^C$ such that*

$$\mathbb{E}^L(g(S_T)1_A) = \mathbb{E}((g(S_T) - g_*)1_{\{S_T \in A_*\}} + g_* 1_A). \tag{3.10}$$

(ii) *There exists a calibrated probability space $(\Omega^G, \mathcal{F}^G, \mathbb{Q}^G, \{S_t^G\}_{t\in[0,T]}) \in \mathcal{P}^C$ such that*

$$\mathbb{E}^G(g(S_T)1_A) = \mathbb{E}((g(S_T) - g^*)1_{\{S_T \in A^*\}} + g^* 1_A). \tag{3.11}$$

Proof. First, an arbitrary element $(\Omega, \mathcal{F}, \mathbb{Q}, \{S_t\}_{t\in[0,T]}) \in \mathcal{P}^C$ is chosen.

Let $(\Omega^L, \mathcal{F}^L, \mathbb{Q}^L) = (\Omega, \mathcal{F}, \mathbb{Q})$ and X be a random variable on it such that

$$X := \begin{cases} S_T & (S_T \in A_*) \\ x & (S_T \notin A_*) \end{cases}, \tag{3.12}$$

where $x \notin I$ is an arbitrary value. In addition, a stochastic process S^L is defined by

$$S_t^L := S_0 1_{\{t<\frac{1}{3}T\}} + X 1_{\{\frac{1}{3}T\le t<\frac{2}{3}T\}} + S_T 1_{\{\frac{2}{3}T\le t\le T\}}. \tag{3.13}$$

Then, $(\Omega^L, \mathcal{F}^L, \mathbb{Q}^L, \{S_t^L\}_{t\in[0,T]})$ is a calibrated probability space, because the distribution of S_T^L is coincident with that of S_T and $\mathbb{Q}^L(S_t^L \in I \mid 0 \le t \le T) = \mathbb{Q}^L(S_T^L \in A_*) = \mathbb{Q}(A)$. Then,

$$\begin{aligned} \mathbb{E}^L(g(S_T^L)1_A) &= \mathbb{E}^L(g(S_T^L)1_{\{S_T^L \in A_*\}}) \\ &= \mathbb{E}((g(S_T) - g_*)1_{\{S_T \in A_*\}} + g_* 1_{\{S_T \in A_*\}}) \\ &= \mathbb{E}((g(S_T) - g_*)1_{\{S_T \in A_*\}} + g_* 1_A). \end{aligned} \tag{3.14}$$

Therefore, Eq.(3.10) holds.

One can prove Eq.(3.11) in the same manner. $\square$

Remark 5. *The calibrated probability space* $(\Omega^G, \mathcal{F}^G, \mathbb{Q}^G, \{S_t^G\}_{t\in[0,T]})$ *is explained intuitively as follows. Consider the worst case scenario for a writer of a knock-out option whose payoff is* $g(S_T)1_A$ *under the condition that the risk-neutral distribution of* S_T *and the risk-neutral probability that the barrier is hit are given. Since the writer has to pay the payoff of the option at maturity if the barrier has not been hit, the worst scenario for the writer is that the payoff is high if the barrier has not been hit and low if the barrier has been hit. Then, the process* $\{S_t^G\}_{t\in[0,T]}$ *should be constructed by distributing* $S_T^G(\omega)$ *to* A^* *for* $\omega \in A$*. The event* A *has the same probability as that of* $\{S_T^G \in A^*\}$*. As a result, the upper bound is an expectation of the plain-vanilla payoff* $g(S_T^G)$ *on the domain* $\{S_T^G \in A^*\}$*.*

In particular, the pricing bounds for a knock-out call option with strike K and an asset-paying no-touch option are obtained:

Corollary 1. *Suppose* $\kappa \in [K, u]$*. Then*

$$\begin{aligned} & (\kappa - K)(1_A - 1_{A_T}) + ((S_T - K)_+ - (S_T - \kappa)_+)1_{A_T} && (3.15)\\ \leq\; & (S_T - K)_+ 1_A && (3.16)\\ \leq\; & (\kappa - K)1_A + (S_T - \kappa)_+ 1_{A_T}, && (3.17) \end{aligned}$$

where $A_T := \{S_T \in I\}$*. Moreover,*

(i) *the expectation of Eq.(3.15) takes the supremum value at* $\kappa = \kappa_* \vee K$ *and the value is* W_L^C*,*

(ii) *the expectation of Eq.(3.17) takes the infimum value at* $\kappa = \kappa^* \vee K$ *and the value is* W_G^C*,*

where κ_* *and* κ^* *are real numbers in* $[l, u]$ *such that*

$$\begin{aligned} \mathbb{Q}(A) &= \mathbb{Q}(\{l \leq S_T \leq \kappa_*\}) && (3.18)\\ \mathbb{Q}(A) &= \mathbb{Q}(\{\kappa^* \leq S_T \leq u\}) && (3.19) \end{aligned}$$

and $x \vee y := max(x, y)$*.*

Corollary 2. *The following holds for any element of* $\mathcal{P}^C$

$$\mathbb{E}(S_T 1_{\{l\leq S_T\leq \kappa_*\}}) \leq \mathbb{E}(S_T 1_A) \leq \mathbb{E}(S_T 1_{\{\kappa^*\leq S_T\leq u\}}), \tag{3.20}$$

where κ_* *and* κ^* *are defined as in Corollary 1.*

Next, consider how the pricing bounds are improved by adding an assumption that one can use an asset-paying no-touch option as a hedging instrument.

Theorem 2. *Suppose* $\alpha \in \mathbb{R}$*. Then*

$$\begin{aligned} & (h(S_T) - h_*)1_{\{S_T\in A_*\}} + h_* 1_A + \alpha S_T 1_A && (3.21)\\ \leq\; & g(S_T)1_A && (3.22)\\ \leq\; & (h(S_T) - h^*)1_{\{S_T\in A^*\}} + h^* 1_A + \alpha S_T 1_A, && (3.23) \end{aligned}$$

where $h(s) := g(s) - \alpha s$ *and* h_**,* A_**,* h^* *and* A^* *are defined as in Theorem 1 for* h*. Moreover, suppose that*

$$sup_{\lambda\in I-s} \left| \frac{g(s+\lambda) - g(s)}{\lambda} \right| < +\infty \tag{3.24}$$

for $s = \kappa_*$ *and* $s = \kappa^*$*, where* κ_* *and* κ^* *are defined as in Corollary 1. Then,*

(i) *the expectation of Eq.(3.21) takes the supremum value at* $\alpha = \alpha_*$ *and the value is* W_L^A*,*

(ii) *the expectation of Eq.(3.23) takes the infimum value at* $\alpha = \alpha^*$ *and the value is* W_G^A*,*

where α_* *and* α^* *are defined by* $N_0^A(I) = \mathbb{E}(S_T 1_{\{S_T\in A_*\}}) = \mathbb{E}(S_T 1_{\{S_T\in A^*\}})$ *respectively.*

Proof. The inequalities are from Theorem 1. The set A_* is close to $\{l \leq S_T \leq \kappa_*\}$ as $\alpha \longrightarrow -\infty$ and $\{\kappa^* \leq S_T \leq u\}$ as $\alpha \longrightarrow +\infty$. By Corollary 2, one can find α_* and A_* such that $N_0^A(I) = \mathbb{E}(S_T 1_{\{S_T \in A_*\}})$. One can construct an element of $\mathcal{P}^C$ such that $A = \{S_T \in A_*\}$. Then,

$$\begin{aligned} \mathbb{E}((h(S_T) - h_*)1_{\{S_T \in A_*\}} + h_* 1_A + \alpha_* S_T 1_A) &= \mathbb{E}((h(S_T) - h_*)1_A + h_* 1_A + \alpha_* S_T 1_A) \\ &= \mathbb{E}(g(S_T)1_A). \end{aligned} \tag{3.25}$$

□

Eq.(3.21) shows that the payoff of the knock-out option dominates that of the following portfolio:

- h_* unit of the cash-paying no-touch option
- α unit of the asset-paying no-touch option
- one unit of a European derivative with the payoff $(g(S_T) - \alpha S_T - h_*)1_{\{S_T \in A_*\}}$.

Similarly, Eq.(3.23) shows that the payoff of the knock-out option is dominated by that of the following portfolio:

- h^* unit of the cash-paying no-touch option
- α unit of the asset-paying no-touch option
- one unit of a European derivative with the payoff $(g(S_T) - \alpha S_T - h^*)1_{\{S_T \in A^*\}}$.

4 Numerical Examples

This section shows numerical examples. The pricing bounds are compared with exact prices derived by some specific models.

The Heston's stochastic volatility model (Heston (1993)) is regarded as the process of the underlying asset, which means all market options are produced by the model. The process of the underlying under the Heston model is as follows:

$$\frac{dS_t}{S_t} = (r - q)dt + \sigma_t dW_t, \tag{4.1}$$

$$d\sigma_t^2 = \kappa(\eta - \sigma_t^2)dt + \theta\sigma_t d\tilde{W}_t, \tag{4.2}$$

where W and $\tilde{W}$ are Brownian motions with correlation ρ under the risk-neutral measure.

4.1 Pricing Bounds on Double Barrier Options

The first example of pricing bounds is for double barrier options. Double barrier options can be priced analytically under the Heston model with $r = q$ and $\rho = 0$ (see Lipton (2001)). The barrier option considered in this example is set to be a double knock-out call option with maturity 1-month or 3-month, $K = 0.95$, $l = 0.8$ and $u = 1.1$. The Heston prices, the model-independent pricing bounds and trivial upper bounds for this option are calculated with the spot price varied from l to u under the Heston model with the parameters listed in Table 1. Here, a trivial upper bound means a European option whose payoff is $(S_T - K)_+ 1_{A_T}$, where A_T is defined in Corollary 1.

The results are Figure 1 and Table 4 for 1-month and Figure 2 and Table 5 for 3-month[2], which show that all of the model-independent upper bounds or lower bounds are upper or lower than the exact prices respectively. In addition, the pricing bounds using an asset-paying no-touch option are much closer to the Heston prices than those using only a cash-paying no-touch option.

[2]The pricing bounds using asset-paying no-touch options are omitted from the figures since they are very close to the Heston prices.

Table 1: Parameters of the Heston Model

r	q	σ_0^2	κ	η	θ	ρ
0.03	0.03	0.15^2	3.0	0.2^2	0.4	0.0

The upper bound with a cash-paying no-touch option seems much more conservative than the lower bound in these examples. Although this is not true in general, the case of this example is explained as follows: Consider the calibrated probability space $(\Omega^G, \mathcal{F}^G, \mathbb{Q}^G, \{S_t^G\}_{t\in[0,T]}) \in \mathcal{P}^C$, which gives the upper bound for the knock-out call options in the examples. As in Remark 5, $S_T^G(\omega)$ belongs to A^* for a scenario $\omega \in A$ where the barrier is not hit and $S_T^G(\omega)$ does not belongs to A^* for a scenario $\omega \in A^c$. This scenario is, however, far from a reality, because A^* is a subset of I which is close to the barrier levels l and u. On the other hand, the probability space that gives the lower bound is more likely. This is a reason why the upper bound is much more conservative than the lower bound.

4.2 Comparison with Brown et al. (2001)

The second example is comparison with the method proposed by Brown et al. (2001) for single barrier options, which are up-and-out call options with maturity 1-month or 3-month, $K = 0.95$ and $u = 1.1$. The Heston prices, the model-independent pricing bounds, pricing bounds of Brown et al. (2001) and trivial upper bounds for this option are calculated with the spot price varied from 0.95 to 1.075 under the Heston model with the parameters listed in Table 2.

Table 2: Parameters of the Heston Model

r	q	σ_0^2	κ	η	θ	ρ
0.0	0.0	0.15^2	3.0	0.2^2	0.4	0.0

The results are Table 6 for 1-month and Table 7 for 3-month. Generally, one cannot claim that either of the two methods is superior to the other. The method proposed by this paper uses a no-touch option and do not assume the underlying asset price is martingale, while Brown et al. (2001) assume it is martingale. Actually, Table 6 and Table 7 show that the lower bounds of Brown et al. (2001) and the upper bounds of this paper are more conservative.

4.3 Comparison with Schoutens et al. (2005)

The third example is a comparison with the results of Schoutens et al. (2005). They study prices of single barrier options under several models: the Heston model (HEST) and its generalization allowing for jumps in the price process (see e.g. Bakshi et al. (1997)) (HESJ), the Barndorff-Nielsen-Shephard model introduced in Barndorff-Nielsen and Shephard (2001) (BN-S) and Lévy models with stochastic time introduced by Carr et al. (2001). The Lévy models with stochastic time in their study are NIG Lévy process with CIR Stochastic Clock(NIG-CIR), NIG Lévy process with Gamma-OU Stochastic Clock(NIG-OUΓ), VG Lévy process with CIR Stochastic Clock(VG-CIR), and VG Lévy process with Gamma-OU Stochastic Clock(VG-OUΓ), where NIG is for the Normal Inverse Gaussian distribution and VG for the Variance Gamma distribution.

The barrier options considered in their example are knock-out call options with maturity 3 years, strike equal to the spot S_0 and several barrier levels (ranging from $1.05S_0$ to $1.5S_0$). They price the barrier options under models which are calibrated very well to a set of plain-vanilla options.

The pricing bounds proposed in this paper are compared with their exact prices under the above models. The calculation is based on the Heston model with the parameters listed in Table 3. Note that the prices of no-touch options are also calculated by the Heston model, which are different from those by the other models.

Table 3: Parameters of the Heston Model in Schoutens et al. (2005)

S_0	r	q	σ_0^2	κ	η	θ	ρ
2461.44	0.03	0.0	0.0654	0.6067	0.0707	0.2928	-0.7571

The results are listed in Table 8.

First, it is noteworthy that there are significant differences in the prices of the barrier options even if the models are calibrated very well to plain-vanilla options, according to Schoutens et al. (2005). This is due to the different structure in path-behaviour between these models.

Second, whether the prices under the above models are in the model-independent pricing bounds or not is examined. Since the prices of no-touch options in the calculation is based on the Heston model, all prices under the Heston model are in the pricing bounds. On the other hand, some prices are outside the bounds using only cash-paying no-touch options; the prices under NIG-OUΓ, VG-CIR, VG-OUΓ and NIG-CIR take higher values than the upper bounds at $H/S_0 = 0.95, 0.9, 0.85$. The bounds using asset-paying no-touch options are so close to the Heston prices that many prices produced by the other models are outside them.

5 Concluding Remarks

This paper provides model-independent pricing bounds on barrier options and corresponding hedging strategies using no-touch options. Moreover, the optimal pricing bounds among them are derived and the optimality is shown through finding risk-neutral measures under which the barrier option price is equal to the hedging portfolio's value. Comparing the pricing bounds proposed by this study with exact prices under several models which are calibrated only to plain-vanilla options, it is demonstrated that some models produce prices outside the model-independent bounds. This implies that great care should be taken when choosing models to price barrier options.

Finally, the next research topic will be to consider a pricing model for barrier options which is calibrated to both the market prices of plain-vanilla options and no-touch options.

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Figure 1: Pricing bounds and Heston prices (1M)

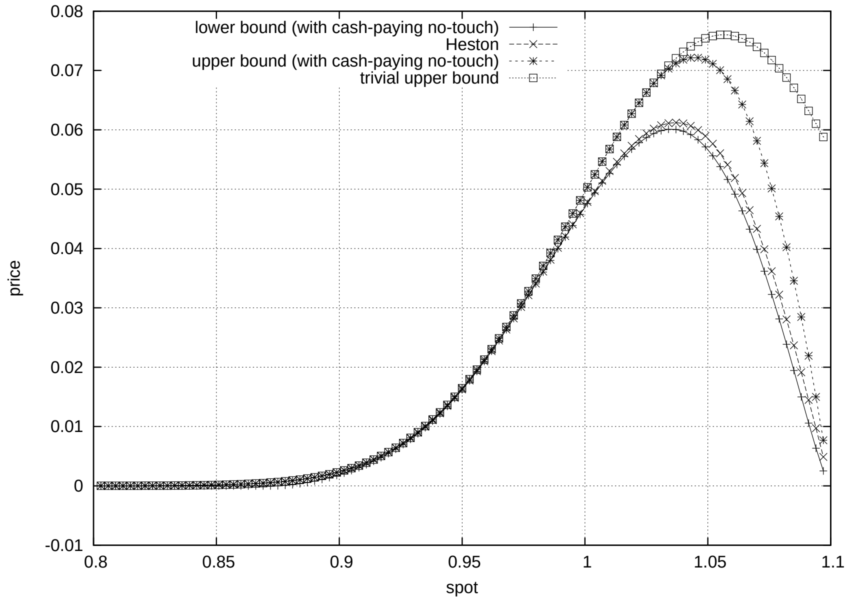

Table 4: Pricing bounds and Heston prices (1M)

spot	0.950	0.975	1.000	1.025	1.050	1.075
W_L^C	0.0162	0.0308	0.0470	0.0587	0.0567	0.0336
W_L^A	0.0162	0.0308	0.0472	0.0592	0.0582	0.0373
Heston	0.0162	0.0308	0.0472	0.0594	0.0586	0.0374
W_G^A	0.0163	0.0308	0.0473	0.0595	0.0587	0.0377
W_G^C	0.0165	0.0314	0.0496	0.0663	0.0716	0.0516
trivial upper bound	0.0165	0.0314	0.0496	0.0663	0.0756	0.0722
DNT	0.9940	0.9878	0.9610	0.8886	0.7212	0.4150

Figure 2: Pricing bounds and Heston prices (3M)

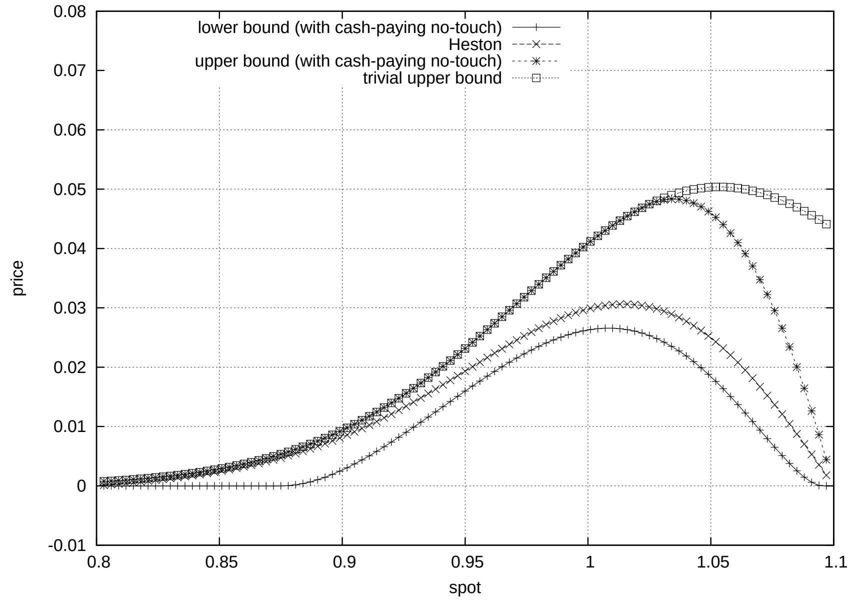

Table 5: Pricing bounds and Heston prices (3M)

spot	0.950	0.975	1.000	1.025	1.050	1.075
W_L^C	0.0160	0.0225	0.0263	0.0253	0.0184	0.0070
W_L^A	0.0187	0.0246	0.0284	0.0281	0.0225	0.0126
Heston	0.0193	0.0255	0.0298	0.0302	0.0249	0.0142
W_G^A	0.0220	0.0271	0.0309	0.0310	0.0259	0.0153
W_G^C	0.0231	0.0322	0.0409	0.0475	0.0459	0.0305
trivial upper bound	0.0231	0.0322	0.0409	0.0475	0.0503	0.0487
DNT	0.8739	0.8399	0.7638	0.6400	0.4654	0.2460

Table 6: Comparison with Brown et al. (2001) (1M)

spot	0.95	0.975	1.000	1.025	1.05	1.075
lower bound of Brown et al. (2001)	0.0162	0.0303	0.0457	0.0549	0.0486	0.0236
W_L^C	0.0163	0.0309	0.0474	0.0595	0.0580	0.0360
Heston	0.0164	0.0310	0.0477	0.0602	0.0598	0.0397
W_G^C	0.0165	0.0315	0.0497	0.0665	0.0723	0.0537
upper bound of Brown et al. (2001)	0.0165	0.0314	0.0493	0.0657	0.0747	0.0686
trivial upper bound	0.0165	0.0315	0.0497	0.0665	0.0758	0.0724
DNT	0.9976	0.9905	0.9660	0.8961	0.7328	0.4361

Table 7: Comparison with Brown et al. (2001) (3M)

spot	0.95	0.975	1.000	1.025	1.05	1.075
lower bound of Brown et al. (2001)	0.0161	0.0200	0.0211	0.0179	0.0104	0.0022
W_L^C	0.0187	0.0244	0.0277	0.0266	0.0196	0.0079
Heston	0.0197	0.0262	0.0306	0.0310	0.0260	0.0152
W_G^C	0.0233	0.0324	0.0412	0.0478	0.0469	0.0320
upper bound of Brown et al. (2001)	0.0223	0.0310	0.0394	0.0459	0.0491	0.0466
trivial upper bound	0.0233	0.0324	0.0412	0.0478	0.0507	0.0491
DNT	0.9295	0.8760	0.7884	0.6591	0.4828	0.2618

Table 8: Comparison with the results of Schoutens et al. (2005)

	H/S_0	NIG-OUΓ	VG-CIR	VG-OUΓ	HEST	HESJ	BN-S	NIG-CIR	W_L^C	W_L^A	W_G^A	W_G^C	trivial upper bound
Call		509.76	511.8	509.33	510.88	510.89	509.89	512.21					
DOB	0.95	300.25	293.28	318.35	173.85	174.64	230.25	284.1	13.77	172.89	177.93	253.80	509.99
	0.9	396.8	391.17	402.24	280.79	282.09	352.14	387.83	38.94	279.38	291.39	369.56	509.99
	0.85	451.61	448.1	452.97	359.05	360.99	423.21	446.52	70.61	355.59	377.78	442.14	509.99
	0.8	481.65	479.83	481.74	414.65	416.63	461.82	479.77	107.38	407.07	439.73	484.96	509.99
	0.75	497	496.95	496.8	452.76	454.33	481.85	496.78	145.63	440.95	480.63	505.74	509.99
	0.7	504.31	505.24	504.05	477.37	479.12	492.62	505.38	185.66	459.08	502.73	509.99	509.99
	0.65	507.53	509.1	507.21	492.76	494.25	498.93	509.34	224.44	470.98	509.02	509.99	509.99
	0.6	508.88	510.75	508.53	501.74	502.84	503.17	511.09	263.95	480.97	509.02	509.99	509.99
	0.55	509.43	511.4	509.06	506.46	507.41	505.93	511.8	301.70	488.72	509.02	509.99	509.99
	0.5	509.64	511.67	509.24	508.91	509.51	507.68	512.08	336.17	495.02	509.02	509.99	509.99
UOB	1.05	0.44	0.27	0.49	0.103	0.08	0.13	0.23	0.00	0.00	2.39	2.39	2.39
	1.1	3.08	2	3.22	0.979	0.89	1.48	1.84	0.00	0.00	9.22	9.55	9.55
	1.15	9.43	6.59	9.77	3.8	3.61	5.58	6.27	0.00	0.00	19.87	22.00	22.00
	1.2	20.71	15.29	21.03	8.96	9.85	13.91	14.8	0.00	0.00	32.66	38.83	38.83
	1.25	37.29	28.95	37.94	20.15	19.96	27.2	28.26	0.00	1.33	47.03	60.88	60.88
	1.3	59.22	48.17	60.1	35.58	36.03	45.38	47.04	4.78	11.58	61.09	86.25	86.25
	1.35	86.14	72.47	87	56.1	58.42	68.39	71.21	20.74	30.62	74.36	116.02	116.02
	1.4	116.75	101.33	117.96	81.93	86.8	94.88	100.04	45.83	57.03	92.24	147.46	147.46
	1.45	149.98	133.74	151.52	111.65	121.33	124.36	132.16	77.84	89.31	117.19	181.78	181.78
	1.5	184.5	168.33	186.53	145.31	160.21	154.96	166.41	115.01	126.01	147.56	215.77	215.77