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Pricing Basket Options under Local Stochastic Volatility with Jumps

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Pricing Basket Options under Local Stochastic Volatility with Jumps

Kenichiro Shiraya, Akihiko Takahashi

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Abstract

This paper develops a new approximation formula for pricing basket options in a local-stochastic volatility model with jumps. In particular, the model admits local volatility functions and jump components in not only the underlying asset price processes, but also the volatility processes. To the best of our knowledge, the proposed formula is the first one which achieves an analytical approximation for the basket option prices under this type of the models. Moreover, some numerical experiments confirm the validity of the method.

1 Introduction

The basket options are one of the most popular exotic-type options in the commodity and equity markets. However, it is a tough task to calculate a basket option price with computational speed fast enough for practical purpose, mainly due to the difficulty of the analytical tractability and its high dimensionality. For instance, although the Monte Carlo method is easy to implement, it requires a substantial computational time to obtain an accurate value. Also, the numerical methods for the partial differential equations (PDEs) have been well developed, but it is still very difficult to solve high dimensional PDEs with accuracy and computational speed satisfactory enough in the financial business. To overcome the difficulties, this paper develops a new analytical approximation formula for basket options. In particular, to the best of our knowledge, our approximation formula is the first one which achieves a closed form approximation of basket options under stochastic volatility models with local volatility functions and jump components for not only the underlying asset price processes, but also the volatility processes.

There exist a large number of preceding studies on pricing basket options. In the Black-Scholes model, Brigo, Mercurio, Rapisarda and Scotti (2004) applied a moment matching method to approximate basket option prices. Deelstra, Liinev and Vanmaele

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(2004) derived the lower and upper bounds for basket call option prices with comonotonic approach.


In a local volatility jump-diffusion model, Xu and Zheng (2010) derived a forward partial integral differential equation (PIDE) for basket option pricing and approximated its solution. Also, Xu and Zheng (2013) applied the lower bound technique in Roger and Shi (1995) and the asymptotic expansion method in Kunitomo and Takahashi (2001) to obtain the approximate value of the lower bound of European basket call prices. Moreover, when the local volatility function is time independent, they suggested to have a closed-form expression for their approximation.

Under a local stochastic volatility model, Shiraya and Takahashi (2012) has developed a general pricing method for multi-asset cross currency options which include cross currency options, cross currency basket options and cross currency average options. They also demonstrated that the scheme is able to evaluate options with high dimensional state variables such as 200 dimensions, which is necessary for pricing basket options with 100 underlying assets under stochastic volatility environment. Moreover, in practice, fast calibration is necessary in the option markets relevant for the underlying assets and the currency, which was also achieved in the work.

Models within the class of the so called local stochastic volatility (LSV) model are mainly used in practice: for example, SABR (Hagan, Kumar, Lesniewskie, and Woodward (2002)), ZABR (Andreasen and Huge (2011)), CEV Heston (e.g. Shiraya et al. (2012)) and Quadratic Heston models (e.g. Shiraya et al. (2012)) are well known. Nonetheless, the LSV model is not always enough to fit to a volatility smile and term structure. Hence, some advanced researches investigated a local stochastic volatility with jump model. Among them, Eraker (2004) found that the models with jump components in the underlying price and volatility processes showed better performance in fitting to option prices and the underlying price returns’ data simultaneously in stock markets. Pagliarani and Pascucci (2013) derived an analytical approximation of plain-vanilla option prices by applying the adjoint expansion method. However, to the best of our knowledge no works have derived an analytical approximation formula for the option prices under a model which admits a local volatility function and jumps both in the underlying asset price and its volatility processes. This paper develops a formula for pricing basket options under the setting by extending an asymptotic expansion approach. This closed form equation has an advantage in making use of the better calibration to the traded individual options whose underlying assets are included in a basket option’s underlying.

An asymptotic expansion approach in finance was initiated by Kunitomo and Takahashi (1992), Yoshida (1992), and Takahashi (1995,1999), which provides us a unified methodology for evaluation of prices and Greeks in general diffusion setting. Recently, the method was further developed to be applied to the forward backward stochastic differential equations (FBSDEs). (See Fujii and Takahashi (2012 a,b,c,d), Takahashi
and Yamada (2012, 2013) for the details.)

Although the method was extended to be applied to a jump-diffusion model by Kunitomo and Takahashi (2004) and Takahashi (2007, 2009), they concentrated on approximation of only bond prices or/plain-vanilla option prices under a local volatility jump-diffusion model, and did not derive higher order expansions than the first order for the option pricing. Subsequently, Takahashi and Takehara (2010) found a scheme for pricing plain-vanilla options in a jump-diffusion with stochastic volatility model. However, thanks to a linear structure of the underlying asset price process in their model they separated the jump component with a known characteristic function and then applied the expansion technique developed in the diffusion models. Hence, their scheme can not be applied directly to more general models nor basket option pricing. The current work generalizes these preceding researches in the asymptotic expansion approach.

In terms of the mathematical viewpoint, Yoshida (2003) presents an extension of Watanabe theory to develop a framework for providing a validity of asymptotic expansions in Wiener-Poisson spaces, which can be applied to jump-diffusion models under some regularity conditions. Hayashi (2008) applies a Malliavin calculus of jump-type to prove an asymptotic expansion theorem for functionals of a Poisson random measure, and Hayashi (2010) derives the coefficients in the expansion of a call option price under a pure jump model. Moreover, Hayashi and Ishikawa (2011) proves an asymptotic expansion formula for the compositions of a smooth Wiener-Poisson functional with Schwartz distributions.

The organization of the paper is as follows: After the next section briefly describes our model for basket options, Section 3 derives a new approximate pricing formula, and Section 4 shows numerical examples. Appendix shows the proof of the lemma, the derivation of the coefficients in the pricing equation and the conditional expectation formulas necessary for obtaining the main theorem.

2 Model

This section shows the model of the underlying asset prices and its volatility processes, which is used for pricing the European type basket options.

In particular, suppose that the filtered probability space \((\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})\) is given, where \(\mathbb{P}\) is an equivalent martingale measure and the filtration satisfies the usual conditions. Then, \((S^i_t)_{t \in [0,T]}\) and \((\sigma^i_t)_{t \in [0,T]}\), \(i = 1, \cdots, d\) represent the underlying asset prices and their volatilities for \(t \in [0, T]\), respectively. Particularly, let us assume that \(S^i_T\) and \(\sigma^i_T\) are given by the solutions of the following stochastic integral equations:

\[
S^i_T = s^i_0 + \int_0^T \alpha^i S^i_t \, dt + \int_0^T \phi^i_{S^i_t} (\sigma^i_t, S^i_t) \, dW^S^i_t + \sum_{l=1}^n \left( \sum_{j=1}^{N^i_{l,T}} h^i_{S^i_t, l, j} S^i_{t_j} - \int_0^T \Lambda^i_{S_t} \mathbb{E}[h^i_{S^i_t, l, 1}] \, dt \right),
\]  

(1)
\begin{equation}
\sigma_T^i = \sigma_0^i + \int_0^T \lambda^i(t) \left( \theta^i - \sigma_T^i \right) dt + \int_0^T \phi_{\sigma^i} \left( \sigma_T^i \right) dW_t^i + \sum_{l=1}^n \left( \sum_{j=1}^{N_{l,T}} h_{\sigma^i,j,l} \sigma_T^i - \int_0^T \Lambda t \sigma_T^i E[h_{\sigma^i,l,1}] dt \right),
\end{equation}

where \( s_0^i \) and \( \sigma_0^i, i = 1, \ldots, d \) are given as some constants. The notations are defined as follows:

- \( \alpha^i (i = 1, \ldots, d) \) are constants.
- \( \lambda^i \) and \( \theta^i (i = 1, \ldots, d) \) are nonnegative constants.
- \( \phi_{S^i}(x, y) \) and \( \phi_{\sigma^i}(x) \) are some functions with appropriate regularity conditions.
- \( W^{S^i} \) and \( W^{\sigma^i}, (i = 1, \ldots, d) \) are correlated Brownian motions.
- Each \( N_l, (l = 1, \ldots, n) \) is a Poisson process with constant intensity \( \Lambda_l \). \( N_l, l = 1, \ldots, n \) are independent, and also independent of all \( W^{S^i} \) and \( W^{\sigma^i} \).
- \( \tau_{j,l} \) stands for the \( j \)-th jump time of \( N_l \).
- For each \( l = 1, \ldots, n \) and \( i = 1, \ldots, d \), both \( \left( \sum_{j=1}^{N_l} h_{S^i,l,j} \right)_{t \geq 0} \) and \( \left( \sum_{j=1}^{N_l} h_{\sigma^i,l,j} \right)_{t \geq 0} \) are compound Poisson processes. \( \left( \sum_{j=1}^{N_l} h_{S^i,l,j} \right)_{t \geq 0} = 0 \) when \( N_{l,t} = 0 \).
- For each \( l \) and \( x^i \), \( h_{x^i,l,j} \) \((j \in \mathbb{N})\) are independent and identically distributed random variables, where \( x^i \) stands for one of \( S^i \) and \( \sigma^i (i = 1, \ldots, d) \).
  - for the constant jump case, \( h_{x^i,l,j} = H_{x^i,l} \) for some constant \( H_{x^i,l} \) in all \( j \).
  - for the log-normal jump case, \( h_{x^i,l,j} = e^{Y_{x^i,l,j}} - 1 \), where \( Y_{x^i,l,j} \) is a random variable which follows a normal distribution with mean \( m_{x^i,l,j} \) and variance \( \gamma_{x^i,l,j}^2 \) that is, \( N(m_{x^i,l,j}, \gamma_{x^i,l,j}^2) \).
- \( h_{x^i,l,j} \) and \( h_{x^{i'},l',j'} \) \((l \neq l')\) are independent.
- \( h_{x^i,l,j} \) and \( h_{x^{i'},l',j'} \) \((j \neq j')\) are independent.
- \( N_l \) and \( h_{x^i,l,j} \) are independent.
- For the same \( l \) and \( j \), \( h_{S^i,l,j} \) and \( h_{\sigma^i,l,j} \) \((i, i' = 1, \ldots, d)\) are allowed to be dependent, that is \( Y_{S^i,l,j} \) and \( Y_{\sigma^i,l,j} \) \((i, i' = 1, \ldots, d)\) are generally correlated.

**Remark.** By specifying the functions \( \phi_{S^i} \) and \( \phi_{\sigma^i} \), we can express various types of local-stochastic volatility models. For example, the model with \( \phi_{S^i}(\sigma, S) = (aS^2 + bS + c)\sqrt{\sigma} \) and \( \phi_{\sigma^i}(\sigma) = \sqrt{\sigma} \) corresponds to an extension of the Quadratic Heston model. The model with \( \phi_{S^i}(\sigma, S) = S^\beta \sigma \) and \( \phi_{\sigma^i}(\sigma) = \sigma^{\beta_\sigma} \) corresponds to an extended SABR (\( \lambda \)-SABR) model, and the one with \( \phi_{S^i}(\sigma, S) = S^\beta \sigma \), \( \phi_{\sigma^i}(\sigma) = \sigma^{\beta_\sigma} \) and \( \lambda = 0 \) corresponds to a local volatility on volatility with jumps model.
3 New Pricing formula for Basket Option

Next, we introduce perturbations to the model (1) and (2). That is, for a known parameter $\epsilon \in [0, 1]$ we consider the following stochastic integral equations: for $i = 1, \cdots, d$,

$$
S_T^{i,()_0} = s_0 + \int_0^T \alpha^{i,()} dt + \epsilon \int_0^T \phi_S^{(),(0)} \left( \sigma^{i,()_0}, S_{t-}^{i,()_0} \right) dW_t^{S^{i,()}} + \frac{\epsilon^2}{2} \sum_{j=1}^{N_{i,t}} h_{S^{i,()_0}}^{j,()_0} \sigma^{i,()_0} \gamma_{t,j}^{i,()_0} - \int_0^T \Lambda_t \sigma^{i,()_0} \left[ h_{S^{i,()_0}}^{j,()_0} \right] dt,
$$

(3)

$$
\sigma_T^{i,()_0} = \sigma_0^{i,()_0} + \int_0^T \lambda^{i,()_0} \left( \theta^{i,()_0} - \sigma_t^{i,()_0} \right) dt + \epsilon \int_0^T \phi_{\sigma_t^{i,()_0}} \left( \sigma_t^{i,()_0} \right) dW_t^{\sigma^{i,()}} + \frac{\epsilon^2}{2} \sum_{j=1}^{N_{i,t}} h_{\sigma_t^{i,()_0}}^{j,()_0} \gamma_{t,j}^{i,()_0} - \int_0^T \Lambda_t \sigma_t^{i,()_0} \left[ h_{\sigma_t^{i,()_0}}^{j,()_0} \right] dt.
$$

(4)

Here, $h_{x^{i,()_0},j}^{(0)} = \epsilon H_{x^{i,()_0},l}$ for all $j$ in the constant jump case, $h_{x^{i,()_0},j}^{(0)} = \epsilon y_{x^{i,()_0},l} - 1$, where $\epsilon Y_{x^{i,()_0},l} \sim N(\epsilon y_{x^{i,()_0},l}, \epsilon^2 \gamma_{x^{i,()_0},l}^2)$ in the log-normal jump case. Note that $h_{x^{i,()_0},j}^{(0)} = 0$ in the both cases.

We also define the following perturbed model with no jump processes, $S_t^{i,LSV^{()_0}}$ and $\sigma_t^{i,LSV^{()_0}}$, which will be used for our approximation of the basket option pricing: for $i = 1, \cdots, d$,

$$
S_T^{i,LSV^{()_0}} = s_0 + \int_0^T \alpha^{i,LSV^{()}} dt + \epsilon \int_0^T \phi_S^{(),LSV^{()}} \left( \sigma_t^{i,LSV^{()_0}}, S_{t-}^{i,LSV^{()_0}} \right) dW_t^{S^{i,LSV^{()}}},
$$

(5)

$$
\sigma_T^{i,LSV^{()_0}} = \sigma_0^{i,LSV^{()_0}} + \int_0^T \lambda^{i,LSV^{()_0}} \left( \theta^{i,LSV^{()_0}} - \sigma_t^{i,LSV^{()_0}} \right) dt + \epsilon \int_0^T \phi_{\sigma_t^{i,LSV^{()_0}}} \left( \sigma_t^{i,LSV^{()_0}} \right) dW_t^{\sigma^{i,LSV^{()}}}. \quad (6)
$$

We assume the asymptotic expansions of $S_T^{i,()_0}$ and $\sigma_T^{i,()_0}$ around $\epsilon = 0$ as follows:

$$
S_T^{i,0} = S_T^{i,()_0} + \epsilon S_T^{i,1} + \frac{\epsilon^2}{2!} S_T^{i,2} + \cdots,
$$

(7)

$$
\sigma_T^{i,0} = \sigma_T^{i,()_0} + \epsilon \sigma_T^{i,1} + \frac{\epsilon^2}{2!} \sigma_T^{i,2} + \cdots,
$$

(8)

$$
h_{x^{i,()_0},j}^{(0)} = h_{x^{i,()_0},j}^{(0)} + \epsilon h_{x^{i,()_0},j}^{(1)} + \frac{\epsilon^2}{2!} h_{x^{i,()_0},j}^{(2)} + \cdots,
$$

(9)

where

$$
S_T^{i,1} := \frac{\partial S_T^{i,()_0}}{\partial \epsilon} \bigg|_{\epsilon=0}, \quad \sigma_T^{i,1} := \frac{\partial \sigma_T^{i,()_0}}{\partial \epsilon} \bigg|_{\epsilon=0}, \quad h_{x^{i,()_0},j}^{(1)} := \frac{\partial h_{x^{i,()_0},j}^{(0)}}{\partial \epsilon} \bigg|_{\epsilon=0}.
$$

We also suppose that $(W^{S^{1}}, \cdots, W^{S^{d}}, W^{\sigma^{1}}, \cdots, W^{\sigma^{d}})' = g \cdot Z$ where $g$ is a $2d \times 2d$ correlation matrix, and $Z$ is a $2d$-dimensional (independent) Brownian motion.

Firstly, we consider a simple case with one asset and one jump factor, that is $i = 1$ and $l = 1$ in the above model:

$$
S_T^{(1)} = s_0 + \int_0^T \alpha_T^{(1)} dt + \epsilon \int_0^T \phi_S^{(),(1)} \left( \sigma_T^{(1),0}, \sigma_T^{(1),0} \right) dZ_t^S.
$$
\[
+ \sum_{j=1}^{N_T} h_{S,j}^{(e)} \sigma_{\tau_j} - \int_0^T \Lambda S_{1,J}^{(e)} E[h_{S,1}^{(e)}] dt,
\]

\[
\sigma_T^{(e)} = \sigma_0 + \int_0^T \lambda (\theta - \sigma_{\tau_j}^{(e)}) dt + \epsilon \int_0^T \phi_{\sigma_{\tau_j}}^{(e)} dZ_t
\]

\[
+ \sum_{j=1}^{N_T} h_{\sigma,j}^{(e)} \sigma_{\tau_j} - \int_0^T \Lambda \sigma_{\tau_j}^{(e)} E[h_{\sigma,1}^{(e)}] dt.
\]

We derive \( S_T^{(0)} \) and \( S_T^{(1)} \) explicitly.

\( S_T^{(0)} \) is calculated as follows:

\[
S_T^{(0)} = s_0 + \left( \int_0^T \alpha \left( S_{t-}^{(0)} + \epsilon S_{t-}^{(1)} + \cdots \right) dt \right.
\]

\[
+ \epsilon \int_0^T \phi_{S_{t-}}^{(e)} \left( S_{t-}^{(0)} + \epsilon S_{t-}^{(1)} + \cdots , S_{t-}^{(0)} + \epsilon S_{t-}^{(1)} + \cdots \right) dZ_t^S
\]

\[
+ \sum_{j=1}^{N_T} \left( h_{S,j}^{(0)} + \epsilon h_{S,j}^{(1)} + \cdots \right) \left( S_{\tau_j}^{(0)} + \epsilon S_{\tau_j}^{(1)} + \cdots \right)
\]

\[
- \left. \int_0^T \Lambda \left( S_{t-}^{(0)} + \epsilon S_{t-}^{(1)} + \cdots \right) E[h_{S,1}^{(0)} + h_{S,1}^{(0)} + \cdots] dt \right|_{\epsilon=0}
\]

\[
= s_0 + \int_0^T \alpha_{t-} S_{t-}^{(0)} dt.
\]

\( S_T^{(0)} \) can be solved as \( S_T^{(0)} = e^{\alpha T} s_0 \), and \( \sigma_T^{(0)} = \theta + (\sigma_0 - \theta)e^{-\lambda T} \) is derived in the same way.

Next, we calculate \( S_T^{(1)} \).

\[
S_T^{(1)} = \frac{\partial S_T^{(e)}}{\partial \epsilon} \bigg|_{\epsilon=0}
\]

\[
= \left( \int_0^T \alpha_{t-} \left( S_{t-}^{(1)} + \epsilon S_{t-}^{(2)} + \cdots \right) dt \right.
\]

\[
+ \int_0^T \phi_{S_{t-}}^{(e)} \left( S_{t-}^{(1)} + \epsilon S_{t-}^{(2)} + \cdots , S_{t-}^{(1)} + \epsilon S_{t-}^{(2)} + \cdots \right) dZ_t^S
\]

\[
+ \epsilon \int_0^T \phi_{S_{t-}}^{(e)} \left( S_{t-}^{(1)} + \epsilon S_{t-}^{(2)} + \cdots , S_{t-}^{(1)} + \epsilon S_{t-}^{(2)} + \cdots \right) dZ_t^S
\]

\[
+ \epsilon \int_0^T \phi_{S_{t-}}^{(e)} \left( S_{t-}^{(1)} + \epsilon S_{t-}^{(2)} + \cdots , S_{t-}^{(1)} + \epsilon S_{t-}^{(2)} + \cdots \right) dZ_t^S
\]

\[
+ \sum_{j=1}^{N_T} \left( h_{S,j}^{(1)} + \epsilon h_{S,j}^{(2)} + \cdots \right) \left( S_{\tau_j}^{(0)} + \epsilon S_{\tau_j}^{(1)} + \cdots \right)
\]

\[
6
\]
We define an operator ". . . . ." as follows: When 

\[ e \text{-dimensional vector,} \]

\[ \Phi = \begin{pmatrix} e_{1,1} & \cdots & e_{1,2d} \\ \vdots & \ddots & \vdots \\ e_{d,1} & \cdots & e_{d,2d} \end{pmatrix} \]

\[ A \circ B := \begin{pmatrix} (A)_{1,1}(B)_{1,1} & \cdots & (A)_{1,2d}(B)_{1,2d} \\ \vdots & \ddots & \vdots \\ (A)_{d,1}(B)_{d,1} & \cdots & (A)_{d,2d}(B)_{d,2d} \end{pmatrix}. \]

\[ A \circ B = B \circ A := \begin{pmatrix} (A)_{1,1}(B)_{1,1} & \cdots & (A)_{1,2d}(B)_{1,2d} \\ \vdots & \ddots & \vdots \\ (A)_{d,1}(B)_{d} & \cdots & (A)_{d,2d}(B)_{d} \end{pmatrix}. \]
When $A$ and $B$ are $d$-dimensional vectors,

$$A \ast B := \begin{bmatrix} (A)_1(B)_1 \\ \vdots \\ (A)_d(B)_d \end{bmatrix}.$$  

(17)

- We also define $\partial_x \Phi_x$ for $x = S$ or $\dot{x} = S$ or $\alpha$ as

$$\partial_x \Phi_x := \begin{bmatrix} \frac{\partial}{\partial x^1}(\Phi_x)_{1,1} & \cdots & \frac{\partial}{\partial x^1}(\Phi_x)_{1,2d} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x^d}(\Phi_x)_{d,1} & \cdots & \frac{\partial}{\partial x^d}(\Phi_x)_{d,2d} \end{bmatrix},$$  

(18)

where $(\Phi_x)_{i,j}$ denotes the $(i,j)$-element of the $d \times 2d$ matrix $\Phi_x$.

- Let us introduce the following notations:

$$S_t = (S_t^1, \cdots, S_t^d), \quad \sigma_t = (\sigma_t^1, \cdots, \sigma_t^d),$$

$$h_{S,t,j}^{(i)} = (h_{S^1,t,j}^{(i)}, \cdots, h_{S^d,t,j}^{(i)}), \quad h_{\sigma,t,j}^{(i)} = (h_{\sigma_1,t,j}^{(i)}, \cdots, h_{\sigma_d,t,j}^{(i)}),$$

$$e^{\lambda t} = (e^{\lambda_1 t}, \cdots, e^{\lambda_d t}) \text{ and } e^{\alpha t} = (e^{\alpha_1 t}, \cdots, e^{\alpha_d t}).$$

Based on these preparations, we obtain the next proposition.

**Proposition 3.1.** 1. The coefficients, $S_T^{(i)}, h_{x,t,j}^{(i)}, \sigma_T^{(i)}$ for $x = S, \sigma$, $i = 0, 1, 2$ and $\sigma_T^{(i)}$ for $i = 0, 1$ in the expansions (7), (8) and (9) are given as follows:

$$S_T^{(0)} = e^{\alpha T} \ast s_0,$$

(19)

$$\sigma_T^{(0)} = \theta + (\sigma_0 - \theta) \ast e^{-\lambda T},$$

(20)

$$h_{x,t,j}^{(0)} = 0,$$

(21)

$$S_T^{(1)} = \int_0^T e^{\alpha(T-t)} \ast \Phi_S \left( \sigma_T^{(0)}, S_t^{(0)} \right) dZ_t$$

$$+ \sum_{l=1}^n \sum_{j=1}^{N_l} \left[ h_{S,l,j}^{(1)} - \Lambda_t \text{E} \left[ h_{S,l,1}^{(1)} \right] \right] \ast S_T^{(0)},$$

(22)

$$\sigma_T^{(1)} = \int_0^T e^{-\lambda(T-t)} \ast \Phi_{\sigma} \left( \sigma_T^{(0)}, \sigma_t^{(0)} \right) dZ_t + \sum_{l=1}^n \sum_{j=1}^{N_l} \left[ h_{\sigma,l,j}^{(1)} \ast e^{-\lambda(T-T_j)} \ast \sigma_T^{(0)} \right.$$

$$- \Lambda_t \text{E} \left[ h_{\sigma,l,1}^{(1)} \right] \ast e^{-\lambda T} \ast \int_0^T e^{\lambda t} \ast \sigma_T^{(0)} dt \left],$$

(23)

$$h_{x,t,j}^{(1)} = H_{x,t,j} := (H_{x_1,t,j}, \cdots, H_{x_d,t,j}), \quad (\text{for all } j, \text{ constant jump case})$$

(24)

$$h_{x,t,j}^{(1)} = Y_{x,t,j} := (Y_{x_1,t,j}, \cdots, Y_{x_d,t,j}), \quad (\text{log-normal jump case})$$

(25)
The coefficients, $S_{T}^{(2)}$, $\sigma_{T}^{LSV(i)}$, and $\sigma_{T}^{LSV(i)}$ are given as follows:

1. The coefficients of $S_{T}^{(2)}$, $S_{T}^{LSV(1)}$, and $S_{T}^{LSV(2)}$ are given as follows:

$$S_{T}^{(2)} = \int_{0}^{T} e^{\alpha(T-t)} * \Phi_{S} \left( \sigma_{t}^{(0)}, S_{t}^{(0)} \right) * S_{t}^{(1)} dZ_{t} + \int_{0}^{T} e^{\alpha(T-t)} * \partial_{S} \Phi_{S} \left( \sigma_{t}^{(0)}, S_{t}^{(0)} \right) * \sigma_{t}^{(1)} dZ_{t}$$

$$\sigma_{T}^{1} = \int_{0}^{T} e^{-\lambda(T-t)} * \Phi_{\sigma} \left( \sigma_{t}^{(0)} \right) dX_{t}, \quad \sigma_{T}^{2} = \int_{0}^{T} e^{\alpha(T-t)} * \partial_{S} \Phi_{S} \left( \sigma_{t}^{(0)}, S_{t}^{(0)} \right) * S_{t}^{LSV(1)} dZ_{t}$$

$$\sigma_{T}^{2} = \int_{0}^{T} e^{-\lambda(T-t)} * \partial_{\sigma} \Phi_{\sigma} \left( \sigma_{t}^{(0)} \right) * \sigma_{t}^{LSV(1)} dZ_{t}$$

2. The coefficients, $S_{T}^{LSV(i)}$ ($i = 1, 2, 3$) and $\sigma_{T}^{LSV(i)}$ ($i = 1, 2$) in the asymptotic expansions of (5) and (6) are given as follows:

$$S_{T}^{LSV(1)} = \int_{0}^{T} e^{\alpha(T-t)} * \Phi_{S} \left( \sigma_{t}^{(0)}, S_{t}^{(0)} \right) dZ_{t},$$

$$\sigma_{T}^{LSV(1)} = \int_{0}^{T} e^{-\lambda(T-t)} * \Phi_{\sigma} \left( \sigma_{t}^{(0)} \right) dZ_{t},$$

$$S_{T}^{LSV(2)} = \int_{0}^{T} e^{\alpha(T-t)} * \partial_{S} \Phi_{S} \left( \sigma_{t}^{(0)}, S_{t}^{(0)} \right) * S_{t}^{LSV(1)} dZ_{t}$$

$$\sigma_{T}^{LSV(2)} = \int_{0}^{T} e^{-\lambda(T-t)} * \partial_{\sigma} \Phi_{\sigma} \left( \sigma_{t}^{(0)} \right) * \sigma_{t}^{LSV(1)} dZ_{t}$$

$$S_{T}^{LSV(3)} = \int_{0}^{T} e^{\alpha(T-t)} * \partial_{\sigma} \Phi_{S} \left( \sigma_{t}^{(0)}, S_{t}^{(0)} \right) * \left( S_{t}^{LSV(1)} \right) * \left( S_{t}^{LSV(1)} \right) dZ_{t}$$

Next, let us define the payoff of a basket call option with strike price $K$ as

$$(g(x) - K)^+ := \max\{g(x) - K, 0\},$$
\[ g(x) := w \cdot x = \sum_{i=1}^{d} w_i x^i, \]

where \( g(x) \) represents a weighted sum of the underlying asset prices of \( x_1, \cdots, x_d \) with the constant weights \( w_1, \cdots, w_d \). Here, we set \( x := (x^1, \cdots, x^d) \) and \( w := (w_1, \cdots, w_d) \).

For an approximation of a basket option price, we firstly note that \( g(S_T^{(1)}) \) is expanded around \( \epsilon = 0 \) as:

\[ g(S_T^{(1)}) = g(S_T^{(0)}) + \epsilon g(S_T^{(1)}) + \frac{\epsilon^2}{2} g(S_T^{(2)}) + \frac{\epsilon^3}{6} g(S_T^{(3)}) + o(\epsilon^3). \]  

(35)

Then, for a strike price \( K = g(S_T^{(1)}) - \epsilon y \) for an arbitrary \( y \in \mathbb{R} \), the payoff of the call option with maturity \( T \) is expanded as follows:

\[
\left( g(S_T^{(1)}) - K \right)^+ = \epsilon \left( \frac{g(S_T^{(1)}) - g(S_T^{(0)})}{\epsilon} + y \right)^+ \\
= \epsilon \left( g(S_T^{(1)}) + \frac{\epsilon}{2} g(S_T^{(2)}) + y + o(\epsilon) \right)^+ \\
= \epsilon \left( g(S_T^{(1)}) + y \right)^+ + \frac{\epsilon^2}{2} \{ g(S_T^{(1)}) > y \} g(S_T^{(2)}) \\
\quad + \epsilon^3 \left( \frac{1}{6} \{ g(S_T^{(1)}) > y \} g(S_T^{(3)}) + \frac{1}{8} \delta \{ g(S_T^{(1)}) = y \} \right) g(S_T^{(2)}) \right)^2 \\
\quad + o(\epsilon^3) \\
\approx \epsilon \left( g(S_T^{(1)}) + y \right)^+ + \frac{\epsilon^2}{2} \{ g(S_T^{(1)}) > y \} g(S_T^{(2)}) \\
\quad + \epsilon^3 \left( \frac{1}{6} \{ g(S_T^{(1)}) > y \} g(S_T^{LSV(3)}) + \frac{1}{8} \delta \{ g(S_T^{(1)}) = y \} g(S_T^{LSV(2)}) \right)^2. \\
\]

(36)

We apply \( g(S_T^{LSV(3)}) \) and \( g(S_T^{LSV(2)})^2 \) instead of \( g(S_T^{(3)}) \) and \( g(S_T^{(2)})^2 \), which provides reasonable accuracies with less computational burden in the approximations.

We next note that when the number of jumps is \( k_l \) (\( l = 1, \cdots, n \)), that is on \( \{ N_l = k_l \} := \{ N_{1,l} = k_{1,l}, \cdots, N_{n,l} = k_{n,l} \} \), \( S_T^{(1)} \) in the equation (22) becomes

\[ \xi_{(k_l)} + \hat{S}_T, \]

(37)

where

\[ \xi_{(k_l)} := \sum_{l=1}^{n} (k_l - \Lambda_l T) m_{S_l} * e^{\alpha T} * s_0, \]

(38)

(constant jump)

\[ \hat{S}_T := \int_{0}^{T} e^{\alpha(T-t)} * \Phi_S(s_t^{(0)}, s_t^{(0)}) dZ_t, \]

(39)
denotes the transpose of \( x \) under an equivalent martingale measure in the following way:

\[
\hat{s}_t := \int_0^T e^{\alpha(t-T)} \Phi_S \left( \sigma_t^{(0)}, S_t^{(0)} \right) dZ_t + \sum_{l=1}^n \left( \sum_{j=1}^k \gamma_{S,l} * \zeta_{S,j,l} * e^{\alpha T} * s_0 \right).
\]

Here, we use the following notations:

- \( \gamma_{S,I} = (\gamma_{S_1,I}, \ldots, \gamma_{S_d,I}) \) and \( s_0 = (s_0^1, \ldots, s_0^d) \)
- \( \zeta_{S,j,l} = (\zeta_{S_1,j,l}, \ldots, \zeta_{S_d,j,l}) \) and \( \zeta_{\sigma,j,l} = (\zeta_{\sigma_1,j,l}, \ldots, \zeta_{\sigma_d,j,l}) \) are vectors of random variables, where \( \zeta_{S_{i,j,l}} \) and \( \zeta_{\sigma_{i,j,l}} \) follow the standard normal distribution.

\( \theta \) is defined to be the \( 2d \times 2d \) correlation matrix among \( \zeta_{S_{i,j,l}} \) and \( \zeta_{\sigma_{i,j,l}} \), \( i = 1, \ldots, d \), though it does not explicitly appear here.

We remark that the distribution of \( g(\hat{s}_T) \) is \( N \left( 0, \Sigma_{T}^{\{k_l\}} \right) \), that is the normal distribution with mean zero and variance \( \Sigma_{T}^{\{k_l\}} \) whose density function is expressed as

\[
n(x; 0, \Sigma_{T}^{\{k_l\}}) := \frac{1}{\sqrt{2\pi \Sigma_{T}^{\{k_l\}}}} \exp \left\{ -\frac{x^2}{2\Sigma_{T}^{\{k_l\}}} \right\}.
\]

Here, \( \Sigma_{T}^{\{k_l\}} \) is defined as follows: (constant jump)

\[
\Sigma_{T}^{\{k_l\}} := \int_0^T \left( w * e^{\alpha(T-t)} * \Phi_S \left( \sigma_t^{(0)}, S_t^{(0)} \right) \right) \left( w * e^{\alpha(T-t)} * \Phi_S \left( \sigma_t^{(0)}, S_t^{(0)} \right) \right)^\top dt
\]

(log-normal jump)

\[
\Sigma_{T}^{\{k_l\}} := \int_0^T \left( w * e^{\alpha(T-t)} * \Phi_S \left( \sigma_t^{(0)}, S_t^{(0)} \right) \right) \left( w * e^{\alpha(T-t)} * \Phi_S \left( \sigma_t^{(0)}, S_t^{(0)} \right) \right)^\top dt
+ \sum_{l=1}^n k_l \left( w * \gamma_{S,l} * e^{\alpha T} * s_0 \right) \left( w * \gamma_{S,l} * e^{\alpha T} * s_0 \right)^\top
\]

where \( \vartheta_{\zeta,S,I} \) stands for the correlation matrix of \( \zeta_{S,j,l} = (\zeta_{S_1,j,l}, \ldots, \zeta_{S_d,j,l}) \), and \( x^\top \) denotes the transpose of \( x \).

Next, we define

\[
\eta_2(x, \{k_l\}) = E \left[ g \left( S_T^{(2)} \right) \bigg| g(\hat{s}_T) = x, \{N_l = k_l\} \right],
\]

\[
\eta_3(x, \{k_l\}) = E \left[ g \left( S_T^{LSV(3)} \right) \bigg| g(\hat{s}_T) = x, \{N_l = k_l\} \right],
\]

\[
\eta_{22}(x, \{k_l\}) = E \left[ g \left( S_T^{LSV(2)} \right)^2 \bigg| g(\hat{s}_T) = x, \{N_l = k_l\} \right].
\]

With those preparations, we approximate the expectation of the basket call payoff under an equivalent martingale measure in the following way:

\[
E \left[ \left( g \left( S_T^{(c)} \right) - K \right)^+ \right]
\]

11
\[
\epsilon \mathbb{E} \left[ \left( g(S_T^{(1)}) + y \right)^+ | \hat{g}(\hat{S}_T) = x, \{N_l = k_l\} \right] + \frac{\epsilon^2}{2} \mathbb{E} \left[ \mathbf{1}_{\{g(S_T^{(1)}) \geq -y\}} g \left( S_T^{(2)} \right) | g(\hat{S}_T) = x, \{N_l = k_l\} \right] + \frac{\epsilon^3}{6} \mathbb{E} \left[ \mathbf{1}_{\{g(S_T^{(1)}) \geq -y\}} g \left( S_T^{LSV(3)} \right) | g(\hat{S}_T) = x, \{N_l = k_l\} \right] + \frac{\epsilon^3}{8} \mathbb{E} \left[ \delta_{\{g(S_T^{(1)}) = -y\}} S_T^{LSV(2)} | g(\hat{S}_T) = x, \{N_l = k_l\} \right],
\]  

(47)

We also note that the probability of \( \{N_l = k_l\} := \{N_{1,T} = k_1, \ldots, N_{n,T} = k_n\} \) is expressed as

\[
p_{\{k_l\}} := \prod_{l=1}^n \frac{(\Lambda_{l,T})^{k_l} e^{-\Lambda_{l,T}}}{k_l!},
\]

which is the product of the \( k_l \) times of the jump probabilities of \( N_{l,T} \) (\( l = 1, \ldots, n \)), that is \( \prod_{l=1}^n P(\{N_{l,T} = k_l\}) \), thanks to the independence of \( N_{l,T} \) (\( l = 1, \ldots, n \)).

Then, we calculate the coefficients of \( \epsilon, \frac{\epsilon^2}{2}, \frac{\epsilon^3}{6} \) and \( \frac{\epsilon^3}{8} \) on the right hand of (47) as follows:

The coefficient of \( \epsilon \) is given by:

\[
\mathbb{E} \left[ \left( g \left( S_T^{(1)} \right) + y \right)^+ | g(\hat{S}_T) = x, \{N_l = k_l\} \right] = \sum_{k=0}^{\infty} \sum_{\sum_{l=1}^n k_l = k} p_{\{k_l\}} \int_{-(g(\xi_{\{k_l\}}) + y)}^{\infty} (x + g(\xi_{\{k_l\}}) + y) n(x; 0, \Sigma_{l,T}^{\{k_l\}}) dx,
\]

the coefficient of \( \frac{\epsilon^2}{2} \) is given by:

\[
\mathbb{E} \left[ \mathbf{1}_{\{g(S_T^{(1)}) \geq -y\}} g \left( S_T^{(2)} \right) | g(\hat{S}_T) = x, \{N_l = k_l\} \right] = \sum_{k=0}^{\infty} \sum_{\sum_{l=1}^n k_l = k} p_{\{k_l\}} \int_{-(g(\xi_{\{k_l\}}) + y)}^{\infty} \eta_2(x, \{k_l\}) n(x; 0, \Sigma_{l,T}^{\{k_l\}}) dx,
\]

the coefficient of \( \frac{\epsilon^3}{6} \) is given by:

\[
\mathbb{E} \left[ \mathbf{1}_{\{g(S_T^{(1)}) \geq -y\}} g \left( S_T^{LSV(3)} \right) | g(\hat{S}_T) = x, \{N_l = k_l\} \right] = \sum_{k=0}^{\infty} \sum_{\sum_{l=1}^n k_l = k} p_{\{k_l\}} \int_{-(g(\xi_{\{k_l\}}) + y)}^{\infty} \eta_3(x, \{k_l\}) n(x; 0, \Sigma_{l,T}^{\{k_l\}}) dx,
\]

and the coefficient of \( \frac{\epsilon^3}{8} \) is given by:

\[
\mathbb{E} \left[ \delta_{\{g(S_T^{(1)}) = -y\}} S_T^{LSV(2)} | g(\hat{S}_T) = x, \{N_l = k_l\} \right] = \sum_{k=0}^{\infty} \sum_{\sum_{l=1}^n k_l = k} p_{\{k_l\}} \int_{-(g(\xi_{\{k_l\}}) + y)}^{\infty} \eta_3(x, \{k_l\}) n(x; 0, \Sigma_{l,T}^{\{k_l\}}) dx.
\]
We suppose the following:

\[ X \]

Each

\[ W \]

\[ f \]

Each

\[ W \]

\[ X \]

For the notational convenience, we prepare the following lemma.

Lemma 3.2. Expectations defined in (44), (45) and (46) respectively, we prepare the following lemma.

\[ K \]

is expanded around \( \epsilon \) as follows:

\[
C(K, T) \approx \sum_{k=0}^{\infty} \sum_{\sum_{i=1}^{\infty} k_i=k} p_{\{k_i\}} e^{-rT} \left\{ \epsilon \int_{-y(k_i)}^{\infty} \left( x + y(k_i) \right) n(x; 0, \Sigma_T^{\{k_i\}})dx \\
+ \epsilon^2 \int_{-y(k_i)}^{\infty} \eta_2(x, \{k_i\}) n(x; 0, \Sigma_T^{\{k_i\}})dx \\
+ \epsilon^3 \int_{-y(k_i)}^{\infty} \eta_3(x, \{k_i\}) n(x; 0, \Sigma_T^{\{k_i\}})dx \\
+ \epsilon^3 \eta_{22}(-y(k_i), \{k_i\}) n(-y(k_i); 0, \Sigma_T^{\{k_i\}}) \right\},
\]

(53)

where \( y(k_i) := g(\xi_{(k_i)}) + y \), and \( r \) is a constant risk-free rate.

In order to evaluate \( \eta_2(x, \{k_i\}), \eta_3(x, \{k_i\}) \) and \( \eta_{22}(-y(k_i), \{k_i\}) \), the conditional expectations defined in (44), (45) and (46) respectively, we prepare the following lemma.

Lemma 3.2. We suppose the following:

- \( W \) is a \( d \)-dimensional Brownian motion.
- Each \( N_l \), \( l = 1, \cdots, n \) is a Poisson process with intensity \( \Lambda_l \) and they are independent. \( \tau_{j,l} \) stands for the time of the \( j \)-th jump in \( N_l \).
- \( W \) and \( N_l \) are independent.
- \( X_{j,l} = \left(X_{j,l}^{(1)}, \cdots, X_{j,l}^{(d)}\right) \), \( j = 1, \cdots, l = 1, \cdots, n \) follows a \( d \)-dimensional normal distribution with mean 0 and variance-covariance matrix \( \Theta_{X,l} \) whose diagonal elements are 1, that is each variance is 1.
- \( X_{j,l} \) and \( X_{j',l'} \) are independent for \( j \neq j' \) or \( l \neq l' \).
- \( X_{j,l} \) are independent of \( W \) and \( N_l \).
- Each \( f_{1,l} \) is a \( d \)-dimensional vector in \( \mathbb{R}^d \).
- \( f_2(t), g_{1,l}(t), g_2(t) \) and \( g_{2,l}(t) \) \( l = 1, \cdots, n \) are \( \mathbb{R}_+ \rightarrow \mathbb{R}^d \) deterministic functions and are integrable with respect to \( t \) in the formulas below.
- For the notational convenience, \( f_2(t), g_{1,l}(t), g_2(t) \) and \( g_{2,l}(t) \) are expressed as \( f_{2,t}, g_{1,t,l}, g_{2,t} \) and \( g_{2,t,l} \), respectively.
We define $\hat{Y}_T$ and $\Sigma_{\hat{Y}_T}^{(k_l)}$ as follows:

$$
\hat{Y}_T := \int_0^T f_{2,t} \cdot dW_t + \sum_{l=1}^n \sum_{j=1}^{N_{T,l}} f_{1,l} \cdot X_{j,l},
$$

$$
\Sigma_{\hat{Y}_T}^{(k_l)} := \int_0^T |f_{2,t}|^2 dt + \sum_{l=1}^n k_l f_{1,l}^\top \Theta_{X,l} f_{1,l},
$$

where $x \cdot y$ stands for the inner product of $x$ and $y$ in $\mathbb{R}^d$, and $x^\top$ denotes the transpose of $x$.

We define $I$ as $I = (1, \cdots, 1)$.

Then, we have the following formulas 1. - 13. The outline of the proof is given in Appendix A.

1. 
\[
\mathbb{E} \left[ \sum_{l=1}^n \sum_{j=1}^{N_{T,l}} g_{1,l,r_{j,l}} \cdot I \left| \hat{Y} = y, \{ N_{T,l} = k_l \} \right. \right] = \sum_{l=1}^n \frac{k_l}{T} \int_0^T g_{1,l,t} \cdot Idt,
\]

2. 
\[
\mathbb{E} \left[ \sum_{l=1}^n \sum_{j=1}^{N_{T,l}} g_{1,l,r_{j,l}} \cdot X_j \left| \hat{Y} = y, \{ N_{T,l} = k_l \} \right. \right] = \sum_{l=1}^n \frac{k_l}{T} \int_0^T g_{1,l,t} \Theta_{X,l} f_{1,l} dt H_1 \left( y, \Sigma_{\hat{Y}_T}^{(k_l)} \right) / \Sigma_{\hat{Y}_T}^{(k_l)},
\]

3. 
\[
\mathbb{E} \left[ \int_0^T g_{2,t} \left( \sum_{l=1}^n \sum_{j=1}^{N_{T,l}} g_{1,l,r_{j,l}} \cdot I \right) \cdot dW_t \left| \hat{Y} = y, \{ N_{T,l} = k_l \} \right. \right] = \frac{k_l}{T} \int_0^T g_{2,t} \cdot f_{2,t} \sum_{l=1}^n \int_0^t g_{1,l,s} \cdot Idsdt H_1 \left( y, \Sigma_{\hat{Y}_T}^{(k_l)} \right) / \Sigma_{\hat{Y}_T}^{(k_l)},
\]

4. 
\[
\mathbb{E} \left[ \int_0^T g_{2,t} \left( \sum_{l=1}^n \sum_{j=1}^{N_{T,l}} g_{1,l,r_{j,l}} \cdot X_j \right) \cdot dW_t \left| \hat{Y} = y, \{ N_{T,l} = k_l \} \right. \right] = \sum_{l=1}^n \frac{k_l}{T} \int_0^T g_{2,t} \cdot f_{2,t} \int_0^t g_{1,l,s} \Theta_{X,l} f_{1,l} dsdt H_2 \left( y, \Sigma_{\hat{Y}_T}^{(k_l)} / \Sigma_{\hat{Y}_T}^{(k_l)} \right) / \left( \Sigma_{\hat{Y}_T}^{(k_l)} \right)^2,
\]

5. 

14
\[
\mathbb{E} \left[ \int_0^T g_{2,t} \cdot I \sum_{l=1}^{N_{T,l}} \sum_{j=1}^{N_{l,t}} g_{1,l,\tau_j,t} \cdot I dt \right] \hat{Y} = y, \left\{ N_{T,l} = k_l \right\} \\
= \sum_{l=1}^{n} k_l \int_0^T g_{2,t} \cdot I \int_0^t g_{1,l,s} \cdot I ds dt,
\]

(60)

6.

\[
\mathbb{E} \left[ \int_0^T g_{2,t} \cdot I \sum_{l=1}^{N_{T,l}} \sum_{j=1}^{N_{l,t}} g_{1,l,\tau_j,t} \cdot X_j dt \right] \hat{Y} = y, \left\{ N_{T,l} = k_l \right\} \\
= \sum_{l=1}^{n} k_l \int_0^T g_{2,t} \cdot I \int_0^t g_{1,l,s} \Theta_{X,l}f_{1,l} ds dt \frac{H_1 \left( y, \Sigma_{Y_T} \right)}{\Sigma_{Y_T}^2} \\
= \sum_{l=1}^{n} k_l \left( \int_0^T g_{1,l,t} \cdot I \int_0^t g_{2,s} ds dt \right) \frac{H_1 \left( y, \Sigma_{Y_T} \right)}{\Sigma_{Y_T}^2},
\]

(61)

7.

\[
\mathbb{E} \left[ \sum_{l=1}^{N_{T,l}} \sum_{j=1}^{N_{l,t}} g_{1,l,\tau_j,t} \cdot I \int_0^{\tau_j,t} g_{2,t} \cdot dW_t \right] \hat{Y} = y, \left\{ N_{T,l} = k_l \right\} \\
= \sum_{l=1}^{n} k_l \int_0^T g_{1,l,t} \cdot I \int_0^t g_{2,t} ds dt \frac{H_1 \left( y, \Sigma_{Y_T} \right)}{\Sigma_{Y_T}^2} \\
- \sum_{l=1}^{n} k_l \int_0^T \int_0^t g_{1,l,s} \cdot I ds dt g_{2,t} \cdot f_{2,s} ds dt \frac{H_1 \left( y, \Sigma_{Y_T} \right)}{\Sigma_{Y_T}^2} \\
= \sum_{l=1}^{n} k_l \left( \int_0^T g_{1,l,t} \cdot I \int_0^t g_{2,s} ds dt \right) \frac{H_1 \left( y, \Sigma_{Y_T} \right)}{\Sigma_{Y_T}^2},
\]

(62)

8.

\[
\mathbb{E} \left[ \sum_{l=1}^{N_{T,l}} \sum_{j=1}^{N_{l,t}} g_{1,l,\tau_j,t} \left( \int_0^{\tau_j,t} g_{2,t} \cdot dW_t \right) \cdot X_j \right] \hat{Y} = y, \left\{ N_{T,l} = k_l \right\} \\
= \sum_{l=1}^{n} k_l \int_0^T g_{1,l,t} \Theta_{X,l}f_{1,l} dt \int_0^T g_{2,t} ds dt \frac{H_2 \left( y, \Sigma_{Y_T} \right)}{\Sigma_{Y_T}^2} \\
- \sum_{l=1}^{n} k_l \int_0^T \int_0^t g_{1,l,s} \Theta_{X,l}f_{1,l} ds ds dt g_{2,s} \cdot f_{2,s} ds dt \frac{H_2 \left( y, \Sigma_{Y_T} \right)}{\Sigma_{Y_T}^2} \\
= \sum_{l=1}^{n} k_l \left( \int_0^T g_{1,l,t} \Theta_{X,l}f_{1,l} \int_0^t g_{2,s} ds dt \right) \frac{H_2 \left( y, \Sigma_{Y_T} \right)}{\Sigma_{Y_T}^2},
\]

(63)

9.
\[ E \left[ \sum_{l=1}^{n} \sum_{j=1}^{N_{T,l}} \left( g_{1,l,T_j} \cdot X_j \right) \left( g_{2,l,T_j} \cdot X_j \right) \Big| \hat{Y} = y, \{N_{T,l} = k_l\} \right] = \sum_{l=1}^{n} \frac{k_l}{T} \left( \int_{0}^{T} g_{1,l,t} \Theta_{X,l} f_{1,l}^{T} \cdot g_{2,l,t} \Theta_{X,l} f_{1,l} dt \right) \frac{H_2 \left( y, \Sigma^{(k_l)}_{Y_T} \right)}{\left( \Sigma^{(k_l)}_{Y_T} \right)^2} + \int_{0}^{T} g_{1,l,t} \cdot g_{2,l,t} dt, \]  

(64)

10. \[ E \left[ \sum_{l=1}^{n} \sum_{j=2}^{N_{T,l}} \cdot I \sum_{L=1}^{n} \sum_{j=1}^{N_{T,l}} \cdot I \sum_{j=1}^{n} \sum_{j=1}^{N_{T,l}} \cdot X_j \Big| \hat{Y} = y, \{N_{T,l} = k_l\} \right] = \sum_{l=1}^{n} \frac{k_l(k_l - 1)}{T^2} \left( \int_{0}^{T} g_{1,l,t} \cdot I \sum_{L=1}^{n} \sum_{l=1}^{T} g_{2,l,s} \cdot I ds dt, \right), \]  

(65)

11. \[ E \left[ \sum_{l=1}^{n} \sum_{j=2}^{N_{T,l}} \cdot I \sum_{L=1}^{n} \sum_{j=1}^{N_{T,l}} \cdot X_j \Big| \hat{Y} = y, \{N_{T,l} = k_l\} \right] = \sum_{l=1}^{n} \frac{k_l(k_l - 1)}{T^2} \left( \int_{0}^{T} g_{1,l,t} \cdot I \sum_{L=1}^{n} \sum_{l=1}^{T} g_{2,l,s} \Theta_{X,L} f_{1,L} ds dt \right) \frac{H_1 \left( y, \Sigma^{(k_l)}_{Y_T} \right)}{\Sigma^{(k_l)}_{Y_T}}, \]  

(66)

12. \[ E \left[ \sum_{l=1}^{n} \sum_{j=2}^{N_{T,l}} \cdot I \sum_{L=1}^{n} \sum_{j=1}^{N_{T,l}} \cdot X_j \Big| \hat{Y} = y, \{N_{T,l} = k_l\} \right] = \sum_{l=1}^{n} \frac{k_l(k_l - 1)}{T^2} \left( \int_{0}^{T} g_{1,l,t} \Theta_{X,L} f_{1,L} \sum_{L=1}^{n} \sum_{l=1}^{T} g_{2,l,s} \cdot I ds dt \right) \frac{H_1 \left( y, \Sigma^{(k_l)}_{Y_T} \right)}{\Sigma^{(k_l)}_{Y_T}}, \]  

(67)

13. \[ E \left[ \sum_{l=1}^{n} \sum_{j=2}^{N_{T,l}} \cdot \left( \sum_{L=1}^{n} \sum_{j=1}^{N_{T,l}} \cdot g_{2,l,T_j} \cdot X_j \right) \Big| \hat{Y} = y, \{N_{T,l} = k_l\} \right] = \sum_{l=1}^{n} \frac{k_l(k_l - 1)}{T^2} \left( \int_{0}^{T} g_{1,l,t} \Theta_{X,L} f_{1,L} \sum_{L=1}^{n} \sum_{l=1}^{T} g_{2,l,s} \Theta_{X,L} f_{1,L} ds dt \right) \frac{H_2 \left( y, \Sigma^{(k_l)}_{Y_T} \right)}{\left( \Sigma^{(k_l)}_{Y_T} \right)^2}, \]  

(68)

where \( H_k \left( x; \Sigma^{(k_l)}_{Y_T} \right) \) denotes the \( k \)-th order Hermite polynomial. Particularly, \( H_1 \left( x; \Sigma^{(k_l)}_{Y_T} \right) = x, \) \( H_2 \left( x; \Sigma^{(k_l)}_{Y_T} \right) = x^2 - \Sigma^{(k_l)}_{Y_T} \) and \( H_4 \left( x; \Sigma^{(k_l)}_{Y_T} \right) = x^4 - 6 \Sigma^{(k_l)}_{Y_T} x^2 + 3 \left( \Sigma^{(k_l)}_{Y_T} \right)^2. \)

Applying the above lemma and the conditional expectation formulas in Shiraya and Takahashi (2013) which are listed in Appendix C, we obtain an approximate pricing
formula for a basket call option with \( \epsilon = 1 \). The formula for a basket put option is easily obtained through the put-call parity.

**Theorem 3.3.** An approximation formula for the initial value \( C(K,T) \) of a basket call option with maturity \( T \) and strike price \( K \) is given by the following equation:

\[
\sum_{k=0}^{\infty} \sum_{\sum_{i=1}^{n} k_i = k} p_{(k_i)} e^{-rT} \left\{ y_{k_i} N \left( \frac{y_{k_i}}{\sigma_{k_i}} \right) + \left( \Sigma_{T}^{(l_i)} + C_1 \frac{H_1 \left( y_{k_i}; \Sigma_{T}^{(l_i)} \right)}{\Sigma_{T}^{(l_i)}} \right) + C_2 \frac{H_2 \left( y_{k_i}; \Sigma_{T}^{(l_i)} \right)}{\left( \Sigma_{T}^{(l_i)} \right)^2} + C_3 \frac{H_4 \left( y_{k_i}; \Sigma_{T}^{(l_i)} \right)}{\left( \Sigma_{T}^{(l_i)} \right)^4} + C_4 \right\},
\]

where \( p_{(k_i)} = \prod_{i=1}^{n} \frac{(\Lambda T)^{k_i} e^{-\Lambda T}}{k_i!} \), \( r \) is a constant risk-free rate, \( y = g(S_T^0) - K \), \( y_{(k_i)} = g(\xi_{(k_i)}) + y \), \( N(x) \) denotes the standard normal distribution function and \( n(x;0,\Sigma) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{x^2}{2\Sigma} \right) \). Here, \( \Sigma_{T}^{(l_i)} \) is given by (43), and \( \xi_{(k_i)} \) is defined by (38). The coefficients \( C_1, C_2, C_3, C_4 \) are some constants. The derivation of the coefficients \( C_1, C_2, C_3, C_4 \) are shown in Appendix B. Moreover, \( H_k \left( x; \Sigma_{T}^{(l_i)} \right) \) denotes the \( k \)-th order Hermite polynomial: particularly, \( H_1 \left( x; \Sigma_{T}^{(l_i)} \right) = x \), \( H_2 \left( x; \Sigma_{T}^{(l_i)} \right) = x^2 - \Sigma_{T}^{(l_i)} \) and \( H_4 \left( x; \Sigma_{T}^{(l_i)} \right) = x^4 - 6\Sigma_{T}^{(l_i)} x^2 + 3\left( \Sigma_{T}^{(l_i)} \right)^2 \).

### 4 Numerical Examples

This section shows concrete numerical examples based on our method developed in the previous section.

#### 4.1 Setup

We use the following model for numerical experiments under the risk-neutral probability measure. The diffusion terms in In particular, each underlying asset price process has a CEV (constant elasticity of variance)-type diffusion term with compound Poisson component. Each volatility follows also has a CEV-type diffusion term with mean reversion drift and compound Poisson component:

\[
S_T^i = \int_0^T \alpha^i S_t^i dt + \int_0^T \sigma_T^i \left( S_t^i \right) \beta^i \sigma^i dW_t^i
\]

\[
+ \sum_{l=1}^{n} \sum_{j=1}^{N_{i,T}} h_{ij} S_t^{i,j} - \int_0^T \Lambda_t S_t^i \mathbb{E}[h_{S^i,ij}^i] dt,
\]

\[
\sigma_T^i = \int_0^T \lambda^i (\theta^i - \sigma_t^i) dt + \int_0^T \nu^i (\sigma_t^i)^{\beta^i} dW_t^i
\]

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\[ + \sum_{l=1}^{n} \left( \sum_{j=1}^{N_{l,T}} h_{\sigma^l_{i,j}} \sigma_{\tau_j,1} - \int_{0}^{T} \Lambda_l \sigma_{t,1}^l \mathbb{E}[h_{\sigma^l_{i,t}},1] dt \right), \quad (71) \]

where the jump size \( h_{x^l_{i,j}} \) is given by \( h_{x^l_{i,j}} = H_{x^l_{i,j}} \) for all \( j \) with a constant \( H_{x^l_{i,j}} \) in the constant jump case, and by \( h_{x^l_{i,j}} = e^{Y_{x^l_{i,j}}} - 1 \) with \( Y_{x^l_{i,j}} \) following a normal distribution \( N(m_{x^l_{i,j}}, \gamma^2_{x^l_{i,j}}) \) for all \( j \) in the log-normal jump case.

Applying our approximate formula we calculate the basket call options whose number of the underlying asset are five in the basket. For illustrative purpose we only consider a systematic jump case, that is all the jumps of the underlying asset prices and their volatilities occur at the same time (i.e. \( n = 1 \) and \( (\vartheta)_{x^l_{i,j}} = 1 \) where \( \vartheta \) denotes the 10 \( \times \) 10 correlation matrix among \( \zeta_{S^l_{i,j}} \) and \( \zeta_{\sigma^l_{i,j}}, i = 1, \ldots, 5 \) ), though we are treat more general cases. The parameters in the asset price and their volatility processes are the same among all the assets, which are listed in the following tables (Table 1-Table 3).

**Table 1: Common Parameters**

<table>
<thead>
<tr>
<th>( s^i_0 )</th>
<th>( \sigma^i_0 )</th>
<th>( \alpha^i )</th>
<th>( \beta^i )</th>
<th>( \beta_{S^i} )</th>
<th>( \lambda^i )</th>
<th>( \theta^i )</th>
<th>( \nu^i )</th>
<th>( w^i )</th>
<th>( \Lambda )</th>
<th>( T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>2</td>
<td>0</td>
<td>0.5</td>
<td>0.5</td>
<td>1</td>
<td>2</td>
<td>0.5</td>
<td>0.2</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

**Table 2: Correlations**

\[
\begin{array}{cccccccccc}
W_{S^1} & W_{S^2} & W_{S^3} & W_{S^4} & W_{S^5} & W_{\sigma^1} & W_{\sigma^2} & W_{\sigma^3} & W_{\sigma^4} & W_{\sigma^5} \\
W_{S^1} & 1 & 0.5 & 0.5 & 0.5 & 0.5 & -0.5 & -0.5 & -0.5 & -0.5 & -0.5 \\
W_{S^2} & 0.5 & 1 & 0.5 & 0.5 & 0.5 & -0.5 & -0.5 & -0.5 & -0.5 & -0.5 \\
W_{S^3} & 0.5 & 0.5 & 1 & 0.5 & 0.5 & -0.5 & -0.5 & -0.5 & -0.5 & -0.5 \\
W_{S^4} & 0.5 & 0.5 & 0.5 & 1 & 0.5 & -0.5 & -0.5 & -0.5 & -0.5 & -0.5 \\
W_{S^5} & 0.5 & 0.5 & 0.5 & 0.5 & 1 & -0.5 & -0.5 & -0.5 & -0.5 & -0.5 \\
W_{\sigma^1} & -0.5 & -0.5 & -0.5 & -0.5 & -0.5 & 1 & 0.5 & 0.5 & 0.5 & 0.5 \\
W_{\sigma^2} & -0.5 & -0.5 & -0.5 & -0.5 & -0.5 & 0.5 & 1 & 0.5 & 0.5 & 0.5 \\
W_{\sigma^3} & -0.5 & -0.5 & -0.5 & -0.5 & -0.5 & 0.5 & 0.5 & 1 & 0.5 & 0.5 \\
W_{\sigma^4} & -0.5 & -0.5 & -0.5 & -0.5 & -0.5 & 0.5 & 0.5 & 0.5 & 1 & 0.5 \\
W_{\sigma^5} & -0.5 & -0.5 & -0.5 & -0.5 & -0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 1 \\
\end{array}
\]
Table 3: Jump Parameters

<table>
<thead>
<tr>
<th></th>
<th>No Jump</th>
<th>Constant Jump</th>
<th>Log-normal Jump</th>
<th>Mixed Jump</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>case i</td>
<td>case ii</td>
<td>case iii</td>
<td>case iv</td>
</tr>
<tr>
<td>$h_{S_i}$</td>
<td>-</td>
<td>-5% -10%</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$h_{\sigma_i}$</td>
<td>-</td>
<td>5% 10%</td>
<td>-</td>
<td>10% 20%</td>
</tr>
<tr>
<td>$\mu_{S_i}$</td>
<td>-</td>
<td>-</td>
<td>-5% -10%</td>
<td>-5% -10%</td>
</tr>
<tr>
<td>$\gamma_{S_i}$</td>
<td>-</td>
<td>-</td>
<td>10% 20%</td>
<td>10% 20%</td>
</tr>
</tbody>
</table>

4.2 Numerical Results

Table 4 - 9 show the results for the numerical experiment with the benchmarks computed by Monte Carlo simulations, where the number of the time steps is 512 and the number of trials is 1 million with antithetic variables in computation of each benchmark.

The result for the no jump case is presented in Table 4; the results for the cases of the constant negative jumps in the asset prices and the constant positive jumps in the volatilities are in Table 5 with smaller jump and Table 6 with larger jump; the results for the cases of the log-normal jumps in the asset prices are in Table 7 with smaller negative mean and smaller variance and Table 8 with larger negative mean and larger variance; the result for the log-normal jumps in the asset prices and the constant jumps in the volatilities are presented in Table 8 (small jumps) and Table 9 (large jumps).

All the tables show that our approximation formula works quite well and the approximate values are within the standard errors in Monte Carlo simulations except one case, while in terms of the computational time our analytical method is obviously much faster than the Monte Carlo simulations with 512 time steps and one million trials.

Table 4: Case I (No jump)

<table>
<thead>
<tr>
<th>Strike</th>
<th>80</th>
<th>90</th>
<th>100</th>
<th>110</th>
<th>120</th>
</tr>
</thead>
<tbody>
<tr>
<td>MC</td>
<td>20.85</td>
<td>12.55</td>
<td>6.17</td>
<td>2.27</td>
<td>0.56</td>
</tr>
<tr>
<td>(StdErr)</td>
<td>(0.07)</td>
<td>(0.06)</td>
<td>(0.04)</td>
<td>(0.03)</td>
<td>(0.01)</td>
</tr>
<tr>
<td>AE</td>
<td>20.86</td>
<td>12.56</td>
<td>6.17</td>
<td>2.27</td>
<td>0.55</td>
</tr>
<tr>
<td>Diff</td>
<td>0.01</td>
<td>0.01</td>
<td>0.00</td>
<td>0.01</td>
<td>0.01</td>
</tr>
</tbody>
</table>
Table 5: Case II \((H_S = -5\%, H_\sigma = 5\%\))

<table>
<thead>
<tr>
<th>Strike</th>
<th>80</th>
<th>90</th>
<th>100</th>
<th>110</th>
<th>120</th>
</tr>
</thead>
<tbody>
<tr>
<td>MC</td>
<td>20.98</td>
<td>12.79</td>
<td>6.49</td>
<td>2.55</td>
<td>0.71</td>
</tr>
<tr>
<td>(StdErr)</td>
<td>(0.07)</td>
<td>(0.06)</td>
<td>(0.05)</td>
<td>(0.03)</td>
<td>(0.01)</td>
</tr>
<tr>
<td>AE</td>
<td>20.99</td>
<td>12.80</td>
<td>6.49</td>
<td>2.55</td>
<td>0.70</td>
</tr>
<tr>
<td>Diff</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>-0.00</td>
<td>-0.01</td>
</tr>
</tbody>
</table>

Table 6: Case III \((H_S = -10\%, H_\sigma = 10\%\))

<table>
<thead>
<tr>
<th>Strike</th>
<th>80</th>
<th>90</th>
<th>100</th>
<th>110</th>
<th>120</th>
</tr>
</thead>
<tbody>
<tr>
<td>MC</td>
<td>21.42</td>
<td>13.53</td>
<td>7.39</td>
<td>3.32</td>
<td>1.16</td>
</tr>
<tr>
<td>(StdErr)</td>
<td>(0.08)</td>
<td>(0.07)</td>
<td>(0.05)</td>
<td>(0.03)</td>
<td>(0.02)</td>
</tr>
<tr>
<td>AE</td>
<td>21.43</td>
<td>13.54</td>
<td>7.39</td>
<td>3.31</td>
<td>1.14</td>
</tr>
<tr>
<td>Diff</td>
<td>0.01</td>
<td>0.01</td>
<td>0.00</td>
<td>-0.01</td>
<td>-0.02</td>
</tr>
</tbody>
</table>

Table 7: Case IV \((m_S = -5\%, \gamma_S = 10\%\))

<table>
<thead>
<tr>
<th>Strike</th>
<th>80</th>
<th>90</th>
<th>100</th>
<th>110</th>
<th>120</th>
</tr>
</thead>
<tbody>
<tr>
<td>MC</td>
<td>21.45</td>
<td>13.59</td>
<td>7.48</td>
<td>3.44</td>
<td>1.29</td>
</tr>
<tr>
<td>(StdErr)</td>
<td>(0.08)</td>
<td>(0.07)</td>
<td>(0.05)</td>
<td>(0.04)</td>
<td>(0.02)</td>
</tr>
<tr>
<td>AE</td>
<td>21.44</td>
<td>13.59</td>
<td>7.48</td>
<td>3.43</td>
<td>1.26</td>
</tr>
<tr>
<td>Diff</td>
<td>-0.01</td>
<td>-0.00</td>
<td>-0.00</td>
<td>-0.02</td>
<td>-0.03</td>
</tr>
</tbody>
</table>

Table 8: Case V \((m_S = -10\%, \gamma_S = 20\%\))

<table>
<thead>
<tr>
<th>Strike</th>
<th>80</th>
<th>90</th>
<th>100</th>
<th>110</th>
<th>120</th>
</tr>
</thead>
<tbody>
<tr>
<td>MC</td>
<td>23.17</td>
<td>15.91</td>
<td>10.02</td>
<td>5.72</td>
<td>3.00</td>
</tr>
<tr>
<td>(StdErr)</td>
<td>(0.11)</td>
<td>(0.09)</td>
<td>(0.08)</td>
<td>(0.06)</td>
<td>(0.05)</td>
</tr>
<tr>
<td>AE</td>
<td>23.10</td>
<td>15.85</td>
<td>9.98</td>
<td>5.70</td>
<td>2.98</td>
</tr>
<tr>
<td>Diff</td>
<td>-0.08</td>
<td>-0.06</td>
<td>-0.04</td>
<td>-0.02</td>
<td>-0.02</td>
</tr>
</tbody>
</table>
Table 9: Case VI ($m_S = -5\%, \gamma_S = 10\%, H_\sigma = 10\%$)

<table>
<thead>
<tr>
<th>Strike</th>
<th>80</th>
<th>90</th>
<th>100</th>
<th>110</th>
<th>120</th>
</tr>
</thead>
<tbody>
<tr>
<td>MC</td>
<td>21.47</td>
<td>13.59</td>
<td>7.45</td>
<td>3.40</td>
<td>1.26</td>
</tr>
<tr>
<td>(StdErr)</td>
<td>(0.08)</td>
<td>(0.07)</td>
<td>(0.05)</td>
<td>(0.04)</td>
<td>(0.02)</td>
</tr>
<tr>
<td>AE</td>
<td>21.48</td>
<td>13.60</td>
<td>7.45</td>
<td>3.38</td>
<td>1.22</td>
</tr>
<tr>
<td>Diff</td>
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<td>0.01</td>
<td>0.00</td>
<td>-0.02</td>
<td>-0.03</td>
</tr>
</tbody>
</table>

Table 10: Case VII ($m_S = -10\%, \gamma_S = 20\%, H_\sigma = 20\%$)

<table>
<thead>
<tr>
<th>Strike</th>
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<th>90</th>
<th>100</th>
<th>110</th>
<th>120</th>
</tr>
</thead>
<tbody>
<tr>
<td>MC</td>
<td>23.23</td>
<td>15.92</td>
<td>9.97</td>
<td>5.64</td>
<td>2.94</td>
</tr>
<tr>
<td>(StdErr)</td>
<td>(0.10)</td>
<td>(0.09)</td>
<td>(0.08)</td>
<td>(0.06)</td>
<td>(0.05)</td>
</tr>
<tr>
<td>AE</td>
<td>23.14</td>
<td>15.85</td>
<td>9.91</td>
<td>5.59</td>
<td>2.89</td>
</tr>
<tr>
<td>Diff</td>
<td>-0.09</td>
<td>-0.07</td>
<td>-0.05</td>
<td>-0.04</td>
<td>-0.05</td>
</tr>
</tbody>
</table>

5 Conclusion

We have derived a new approximation formula for basket option pricing in a model with local-stochastic volatility and jumps. In particular, our model admits a local volatility function and jumps in both the underlying asset price and its volatility processes. Thanks to the closed form formula the computational speed of the method is much faster than the other numerical schemes. Moreover, numerical experiments have demonstrated the effectiveness of our approximation. We also note that the higher order expansions can be derived in the similar manner, which is expected to provide more precise approximations as in the diffusion cases in Shiraya, Takahashi and Toda (2012) and Takahashi, Takehara and Toda (2012).

References


A Proof of Lemma 3.2

We briefly describe the proof of Lemma 3.2 for the one dimensional case. The multidimensional case is proved in a similar way. We assume $f_1$ is a constant, $f_2(t), g_1(t), g_2(t)$ are $\mathbb{R}_+ \to \mathbb{R}$ deterministic functions, which are expressed as $f_{2,t}, g_{1,t}, g_{2,t}$ for notational simplicity. $\tau_j (j = 1, \ldots)$ stands for the time of the $j$-th jump in the Póisson process. We define $\hat{Y}_T$ and $\Sigma_{\hat{Y}_T}^k$ as

$$
\hat{Y}_T := \int_0^T f_{2,t} dW_t + \sum_{j=1}^{N_T} f_1 X_j,
$$

$$
\Sigma_{\hat{Y}_T}^k := \int_0^T f_{2,t}^2 dt + k f_1^2.
$$

We freely use the result of the conditional expectation formulas for the case of the Wiener-Itô Integrals in Shiraya and Takahashi(2013), which are listed in Appendix C.

1. 

$$
\mathbb{E} \left[ \sum_{j=1}^{N_T} g_{1,\tau_j} \left| \hat{Y} = y, N_T = k \right. \right] = \mathbb{E} \left[ \int_0^T g_{1,t} - dN_t \left| \hat{Y} = y, N_T = k \right. \right] = \frac{k}{T} \int_0^T g_{1,t} dt,
$$

2. 

$$
\mathbb{E} \left[ \sum_{j=1}^{N_T} g_{1,\tau_j} X_j \left| \hat{Y} = y, N_T = k \right. \right] = \mathbb{E} \left[ \sum_{j=1}^{N_T} g_{1,\tau_j} \mathbb{E} \left[ X_j \left| \hat{Y} = y, N_T = k \right. \right] \left| \hat{Y} = y, N_T = k \right. \right] = \mathbb{E} \left[ \sum_{j=1}^{N_T} g_{1,\tau_j} - f_1 \frac{H_1(y, \Sigma_{\hat{Y}_T}^k)}{\Sigma_{\hat{Y}_T}^k} \left| \hat{Y} = y, N_T = k \right. \right].
$$
\[ \mathbb{E} \left[ \int_0^T g_{1,t} f_1 \frac{H_1(y, \Sigma^k_{\hat{Y}_T})}{\Sigma^k_{\hat{Y}_T}} dN_t \big| \hat{Y} = y, N_T = k \right] = \frac{k}{T} \int_0^T g_{1,t} f_1 dt \frac{H_1(y, \Sigma^k_{\hat{Y}_T})}{\Sigma^k_{\hat{Y}_T}}, \] (75)

3.

\[ \mathbb{E} \left[ \int_0^T g_{2,t} \sum_{j=1}^{N_t} g_{1,\tau_j} - dW_t \big| \hat{Y} = y, N_T = k \right] = \mathbb{E} \left[ \int_0^T g_{2,t} \mathbb{E} \left[ \sum_{j=1}^{N_t} g_{1,\tau_j} \big| N_T = k \right] dW_t \big| \hat{Y} = y, N_T = k \right] = \mathbb{E} \left[ \int_0^T g_{2,t} \int_0^t g_{1,s} dW_s \big| \hat{Y} = y, N_T = k \right] = \frac{k}{T} \int_0^T g_{2,t} \int_0^t g_{1,s} dW_s \frac{H_1(y, \Sigma^k_{\hat{Y}_T})}{\Sigma^k_{\hat{Y}_T}}, \] (76)

4.

\[ \mathbb{E} \left[ \int_0^T g_{2,t} \sum_{j=1}^{N_t} g_{1,\tau_j} - X_j dW_t \big| \hat{Y} = y, N_T = k \right] = \mathbb{E} \left[ \int_0^T g_{2,t} \mathbb{E} \left[ \sum_{j=1}^{N_t} g_{1,\tau_j} - X_j \big| N_T = k \right] dW_t \big| \hat{Y} = y, N_T = k \right] = \mathbb{E} \left[ \int_0^T g_{2,t} \sum_{j=1}^{N_t} g_{1,\tau_j} - f_1 g_{2,t} f_2 \big| N_T = k \right] dt \frac{H_2(y, \Sigma^k_{\hat{Y}_T})}{(\Sigma^k_{\hat{Y}_T})^2} \big| \hat{Y} = y, N_T = k \right] = \frac{k}{T} \int_0^T g_{2,t} \int_0^t g_{1,s} f_1 dt \frac{H_2(y, \Sigma^k_{\hat{Y}_T})}{(\Sigma^k_{\hat{Y}_T})^2}, \] (77)

5.

\[ \mathbb{E} \left[ \int_0^T g_{2,t} \sum_{j=1}^{N_t} g_{1,\tau_j} dt \big| \hat{Y} = y, N_T = k \right] = \mathbb{E} \left[ \int_0^T g_{2,t} \mathbb{E} \left[ \sum_{j=1}^{N_t} g_{1,\tau_j} \big| N_T = k \right] dt \big| \hat{Y} = y, N_T = k \right] = \mathbb{E} \left[ \int_0^T g_{2,t} \int_0^t \frac{k}{T} g_{1,s} dt \big| \hat{Y} = y, N_T = k \right] = \frac{k}{T} \int_0^T g_{2,t} \int_0^t g_{1,s} dt, \] (78)
6. 

\[
E \left[ \int_0^T g_{2,t} \sum_{j=1}^{N_t} g_{1,\tau_j} X_j dt \middle| \hat{Y} = y, N_T = k \right]
\]

\[= \int_0^T g_{2,t} E \left[ \sum_{j=1}^{N_t} g_{1,\tau_j} \left( X_j \right) \middle| \hat{Y} = y, N_T = k \right] dt \]

\[= \int_0^T g_{2,t} E \left[ \sum_{j=1}^{N_t} g_{1,\tau_j} - f_1 \frac{H_1(y, \Sigma^k_{Y_T})}{\Sigma^k_{Y_T}} \middle| N_T = k \right] dt \]

\[= \int_0^T g_{2,t} E \left[ \int_0^t g_{1,s} dN_s \frac{H_1(y, \Sigma^k_{Y_T})}{\Sigma^k_{Y_T}} \middle| N_T = k \right] dt \]

\[= \frac{k}{T} \int_0^T g_{2,t} \int_0^t g_{1,s} ds dt \frac{H_1(y, \Sigma^k_{Y_T})}{\Sigma^k_{Y_T}}, \tag{79} \]

7. 

\[
E \left[ \sum_{j=1}^{N_T} g_{1,\tau_j} \int_0^{\tau_j} g_{2,t} dW_t \middle| \hat{Y} = y, N_T = k \right]
\]

\[= E \left[ \sum_{j=1}^{N_T} g_{1,\tau_j} \int_0^T g_{2,t} dW_t \middle| \hat{Y} = y, N_T = k \right] \]

\[= E \left[ \int_0^T \sum_{j=1}^{N_t} g_{1,\tau_j} - g_{2,t} dW_t \middle| \hat{Y} = y, N_T = k \right] \]

\[= E \left[ \sum_{j=1}^{N_T} g_{1,\tau_j} E \left[ \int_0^T g_{2,t} dW_t \middle| \hat{Y} = y \right] \middle| \hat{Y} = y, N_T = k \right] \]

\[= E \left[ \int_0^T E \left[ \sum_{j=1}^{N_t} g_{1,\tau_j} \middle| N_T = k \right] g_{2,t} dW_t \middle| \hat{Y} = y, N_T = k \right] \]

\[= \frac{k}{T} \int_0^T g_{1,t} dt \int_0^T g_{2,t} f_{2,t} dt \frac{H_1(y, \Sigma^k_{Y_T})}{\Sigma^k_{Y_T}} \]

\[= \frac{k}{T} \int_0^T g_{1,s} ds g_{2,t} f_{2,t} dt \frac{H_1(y, \Sigma^k_{Y_T})}{\Sigma^k_{Y_T}}, \tag{80} \]

8.
\[
E \left[ \sum_{j=1}^{N_T} g_{1,\tau_j} - \int_0^{\tau_j} g_{2,t}dW_t X_j \right| \dot{Y} = y, N_T = k \] 

\[
= E \left[ \sum_{j=1}^{N_T} g_{1,\tau_j} - E \left[ X_j \int_0^T g_{2,t}dW_t \right| \dot{Y} = y, N_T = k \right] \dot{Y} = y, N_T = k \] 

\[
= E \left[ \sum_{j=1}^{N_T} g_{1,\tau_j} - \int_0^T \sum_{j=1}^{N_T} g_{1,\tau_j} E \left[ X_j g_{2,t}dW_t \right| \dot{Y} = y, N_T = k \right] \dot{Y} = y, N_T = k \] 

\[
= k \int_0^T g_{1,t}f_1dt \int_0^T g_{2,t}f_2,tdt \frac{H_2(y, \Sigma_{Y_T}^k)}{(\Sigma_{Y_T}^k)^2} 

- k \int_0^T \int_0^t g_{1,s}f_1ds g_{2,t}f_2,tdt \frac{H_2(y, \Sigma_{Y_T}^k)}{(\Sigma_{Y_T}^k)^2} 

= \left( k \int_0^T g_{1,t}f_1 \int_0^t g_{2,s}f_2,ds \right) \frac{H_2(y, \Sigma_{Y_T}^k)}{(\Sigma_{Y_T}^k)^2}, \tag{81} \right.
\] 

9. 

\[
E \left[ \sum_{j=1}^{N_T} (g_{1,\tau_j} - X_j)(g_{2,\tau_j} - X_j) \right| \dot{Y} = y, N_T = k \] 

\[
= E \left[ \sum_{j=1}^{N_T} \left( g_{1,\tau_j} - X_j \right) \left( g_{2,\tau_j} - X_j \right) \right| \dot{Y} = y, N_T = k \] 

\[
= E \left[ \sum_{j=1}^{N_T} \left( g_{1,\tau_j} - f_1 \right) \left( g_{2,\tau_j} - f_1 \right) \frac{H_2(y, \Sigma_{Y_T}^k)}{(\Sigma_{Y_T}^k)^2} + g_{1,\tau_j} - g_{2,\tau_j} \right) \dot{Y} = y, N_T = k \] 

\[
= E \left[ \int_0^T \left( g_{1,t} - f_1 \right) \left( g_{2,t} - f_1 \right) \frac{H_2(y, \Sigma_{Y_T}^k)}{(\Sigma_{Y_T}^k)^2} + g_{1,\tau_j} - g_{2,\tau_j} \right] dN_t \dot{Y} = y, N_T = k \] 

\[
= k \left( \int_0^T g_{1,t}f_1dt \int_0^T g_{2,t}f_2,tdt \frac{H_2(y, \Sigma_{Y_T}^k)}{(\Sigma_{Y_T}^k)^2} + \int_0^T g_{1,t}g_{2,t}dt \right), \tag{82} \right. \] 

10. 

\[
E \left[ \sum_{j=1}^{N_T} g_{1,\tau_j} - \sum_{j=1}^{N_T} g_{2,\tau_j} \right| \dot{Y} = y, N_T = k \] 

\[
= E \left[ \sum_{j=2}^{N_T} g_{1,\tau_j} E \left[ \sum_{j=1}^{N_T} g_{2,\tau_j} \right| N_T = k \right] \dot{Y} = y, N_T = k \]
= \mathbb{E} \left[ \sum_{j=2}^{N_T} g_{1,\tau_j} \frac{j-1}{\tau_j} \int_0^{\tau_j} g_{2,s} ds \middle| \hat{Y} = y, N_T = k \right] \\
= \mathbb{E} \left[ \int_0^T g_{1,t} \left( \frac{1}{t} \int_0^t g_{2,s} ds N_t - dN \right) \middle| \hat{Y} = y, N_T = k \right] \\
= \mathbb{E} \left[ \int_0^T g_{1,t} \left( \frac{1}{t} \int_0^t g_{2,s} \frac{1}{2} (dN^2 - dN) \right) \middle| \hat{Y} = y, N_T = k \right] \\
= \mathbb{E} \left[ \int_0^T g_{1,t} \left( \frac{1}{t} \int_0^t g_{2,s} \frac{1}{2} \left( d \left( k \frac{t}{T} \left( 1 + (k-1) \frac{t}{T} \right) \right) - d \left( k \frac{t}{T} \right) \right) \right) \middle| \hat{Y} = y, N_T = k \right] \\
= \frac{k(k-1)}{T^2} \int_0^T g_{1,t} \int_0^t g_{2,s} ds dt, \quad (83)

11. \\
\mathbb{E} \left[ \sum_{j=2}^{N_T} g_{1,\tau_j} \sum_{j=1}^{N_T} g_{2,\tau_j} X_j \middle| \hat{Y} = y, N_T = k \right] \\
= \mathbb{E} \left[ \sum_{j=2}^{N_T} g_{1,\tau_j} \mathbb{E} \left[ \sum_{j=1}^{j-1} g_{2,\tau_j} X_j \middle| N_T = k \right] \middle| \hat{Y} = y, N_T = k \right] \\
= \mathbb{E} \left[ \sum_{j=2}^{N_T} g_{1,\tau_j} \left( \sum_{j=1}^{j-1} g_{2,\tau_j} \int_0^{\tau_j} H_1(y, \Sigma^k_{Y_T}) \frac{k}{\Sigma^k_{Y_T}} ds \right) \middle| \hat{Y} = y, N_T = k \right] \\
= \frac{k(k-1)}{T^2} \int_0^T g_{1,t} f_1 \int_0^t g_{2,s} \frac{H_1(y, \Sigma^k_{Y_T})}{k \Sigma^k_{Y_T}} ds dt, \quad (84)

12. \\
\mathbb{E} \left[ \sum_{j=2}^{N_T} g_{1,\tau_j} \sum_{j=1}^{N_T} g_{2,\tau_j} X_j \middle| \hat{Y} = y, N_T = k \right] \\
= \mathbb{E} \left[ \sum_{j=2}^{N_T} g_{1,\tau_j} \mathbb{E} \left[ \sum_{j=1}^{j-1} g_{2,\tau_j} \middle| N_T = k \right] \mathbb{E} \left[ X_j \middle| \hat{Y} = y, N_T = k \right] \middle| \hat{Y} = y, N_T = k \right] \\
= \mathbb{E} \left[ \sum_{j=2}^{N_T} g_{1,\tau_j} \left( \sum_{j=1}^{j-1} g_{2,\tau_j} \int_0^{\tau_j} H_1(y, \Sigma^k_{Y_T}) \frac{k}{\Sigma^k_{Y_T}} ds \right) \middle| \hat{Y} = y, N_T = k \right] \\
= \frac{k(k-1)}{T^2} \int_0^T g_{1,t} f_1 \int_0^t g_{2,s} \frac{H_1(y, \Sigma^k_{Y_T})}{k \Sigma^k_{Y_T}} ds dt, \quad (85)

13. \\
\mathbb{E} \left[ \sum_{j=2}^{N_T} g_{1,\tau_j} \sum_{j=1}^{N_T} g_{2,\tau_j} X_j \middle| \hat{Y} = y, N_T = k \right]
\[ g = E \left[ \sum_{j=2}^{N_T} g_{1,\tau_j} \cdot E \left[ \sum_{j=1}^{N_T-1} g_{2,\tau_j} \cdot X_j \cdot X_j \bigg| \bar{Y} = y, N_T = k \right] \bigg| \bar{Y} = y, N_T = k \right] \]

\[ = E \left[ \sum_{j=2}^{N_T} g_{1,\tau_j} \cdot f_1 \cdot 1 \bigg| \bar{Y} = y, N_T = k \right] \int_0^{\tau_j-} g_{2,\sigma_1} ds \frac{H_2(y, \Sigma_{Y_T})}{(\Sigma_{Y_T})^2} \bigg| \bar{Y} = y, N_T = k \right] \]

\[ = \frac{k(k-1)}{T^2} \int_0^T g_{1,\tau_1} f_1 \int_0^t g_{2,\sigma_1} dsdt \frac{H_2(y, \Sigma_{Y_T})}{(\Sigma_{Y_T})^2}. \quad (86) \]

### B Derivation of Coefficients

This section derives the coefficients, \( C_i \), \( i = 1, 2, 3, 4 \) in the expansion formula (69) in Theorem 3.3 under a log-normal jump case. A constant jump case is obtained in a similar way. In the following we omit some notations for simplicity.

Firstly, let us show the expressions of \( g(S^{(1)}_T) \) and \( g \left( \frac{1}{2} S^{(2)}_T \right) \):

\[ g(S^{(1)}_T) = g \left( \int_0^T e^{\alpha(T-t)} \cdot \Phi_S dZ_t \right) \quad (87) \]

\[ + g \left( \sum_{l=1}^{n} \sum_{i=1}^{N_{i,T}} e^{\alpha(T-\tau_{i,l})} \cdot S^{(0)}_{\tau_{i,l}} \cdot \Phi_S(t) \right) + \sum_{l=1}^{n} \sum_{i=1}^{N_{i,T}} \lambda_i E[h_{S,i,l}^{(1)}] + \sum_{l=1}^{n} \sum_{i=1}^{N_{i,T}} \lambda_i E[h_{S,i,l}^{(1)}] + \int_0^t e^{\alpha(t-u)} \cdot \Phi_S dZ_u \cdot dZ_t \]

\[ g \left( \frac{1}{2} S^{(2)}_T \right) = g \left( \int_0^T e^{\alpha(T-t)} \cdot \Phi_S dZ_t \right) \quad (88) \]

\[ + g \left( \sum_{l=1}^{n} \sum_{i=1}^{N_{i,T}} e^{\alpha(T-\tau_{i,l})} \cdot S^{(0)}_{\tau_{i,l}} \cdot \Phi_S(t) \right) + \sum_{l=1}^{n} \sum_{i=1}^{N_{i,T}} \lambda_i E[h_{S,i,l}^{(1)}] + \sum_{l=1}^{n} \sum_{i=1}^{N_{i,T}} \lambda_i E[h_{S,i,l}^{(1)}] + \int_0^t e^{\alpha(t-u)} \cdot \Phi_S dZ_u \cdot dZ_t \]

\[ + g \left( \sum_{l=1}^{n} \sum_{i=1}^{N_{i,T}} e^{\alpha(T-\tau_{i,l})} \cdot \Phi_S dZ_t \right) \quad (89) \]

\[ + g \left( \sum_{l=1}^{n} \sum_{i=1}^{N_{i,T}} e^{\alpha(T-\tau_{i,l})} \cdot S^{(0)}_{\tau_{i,l}} \cdot \Phi_S(t) \right) + \sum_{l=1}^{n} \sum_{i=1}^{N_{i,T}} \lambda_i E[h_{S,i,l}^{(1)}] + \sum_{l=1}^{n} \sum_{i=1}^{N_{i,T}} \lambda_i E[h_{S,i,l}^{(1)}] + \int_0^t e^{\alpha(t-u)} \cdot \Phi_S dZ_u \cdot dZ_t \]

\[ + g \left( \sum_{l=1}^{n} \sum_{i=1}^{N_{i,T}} e^{\alpha(T-\tau_{i,l})} \cdot \Phi_S dZ_t \right) \quad (90) \]

\[ + g \left( \sum_{l=1}^{n} \sum_{i=1}^{N_{i,T}} e^{\alpha(T-\tau_{i,l})} \cdot S^{(0)}_{\tau_{i,l}} \cdot \Phi_S(t) \right) + \sum_{l=1}^{n} \sum_{i=1}^{N_{i,T}} \lambda_i E[h_{S,i,l}^{(1)}] + \sum_{l=1}^{n} \sum_{i=1}^{N_{i,T}} \lambda_i E[h_{S,i,l}^{(1)}] + \int_0^t e^{\alpha(t-u)} \cdot \Phi_S dZ_u \cdot dZ_t \]

\[ + g \left( \sum_{l=1}^{n} \sum_{i=1}^{N_{i,T}} e^{\alpha(T-\tau_{i,l})} \cdot \Phi_S dZ_t \right) \quad (91) \]

\[ + g \left( \sum_{l=1}^{n} \sum_{i=1}^{N_{i,T}} e^{\alpha(T-\tau_{i,l})} \cdot S^{(0)}_{\tau_{i,l}} \cdot \Phi_S(t) \right) + \sum_{l=1}^{n} \sum_{i=1}^{N_{i,T}} \lambda_i E[h_{S,i,l}^{(1)}] + \sum_{l=1}^{n} \sum_{i=1}^{N_{i,T}} \lambda_i E[h_{S,i,l}^{(1)}] + \int_0^t e^{\alpha(t-u)} \cdot \Phi_S dZ_u \cdot dZ_t \]

\[ + g \left( \sum_{l=1}^{n} \sum_{i=1}^{N_{i,T}} e^{\alpha(T-\tau_{i,l})} \cdot \Phi_S dZ_t \right) \quad (92) \]

\[ + g \left( \sum_{l=1}^{n} \sum_{i=1}^{N_{i,T}} e^{\alpha(T-\tau_{i,l})} \cdot S^{(0)}_{\tau_{i,l}} \cdot \Phi_S(t) \right) + \sum_{l=1}^{n} \sum_{i=1}^{N_{i,T}} \lambda_i E[h_{S,i,l}^{(1)}] + \sum_{l=1}^{n} \sum_{i=1}^{N_{i,T}} \lambda_i E[h_{S,i,l}^{(1)}] + \int_0^t e^{\alpha(t-u)} \cdot \Phi_S dZ_u \cdot dZ_t \]

\[ + g \left( \sum_{l=1}^{n} \sum_{i=1}^{N_{i,T}} e^{\alpha(T-\tau_{i,l})} \cdot \Phi_S dZ_t \right) \quad (93) \]

\[ + g \left( \sum_{l=1}^{n} \sum_{i=1}^{N_{i,T}} e^{\alpha(T-\tau_{i,l})} \cdot S^{(0)}_{\tau_{i,l}} \cdot \Phi_S(t) \right) + \sum_{l=1}^{n} \sum_{i=1}^{N_{i,T}} \lambda_i E[h_{S,i,l}^{(1)}] + \sum_{l=1}^{n} \sum_{i=1}^{N_{i,T}} \lambda_i E[h_{S,i,l}^{(1)}] + \int_0^t e^{\alpha(t-u)} \cdot \Phi_S dZ_u \cdot dZ_t \]

\[ + g \left( \sum_{l=1}^{n} \sum_{i=1}^{N_{i,T}} e^{\alpha(T-\tau_{i,l})} \cdot \Phi_S dZ_t \right) \quad (94) \]
\[-g\left(\sum_{i=1}^{n} \Lambda_l E[h_{S,l,1}^{(2)}] * e^{\alpha T} * \int_{0}^{T} e^{-\alpha t} * S_{l}^{(0)} dt\right)\] 
\[+g\left(\sum_{i=1}^{n} \sum_{j=1}^{N_{l,T}} h_{S,l,j}^{(1)} * e^{\alpha (T-T_{j,l})} * \int_{0}^{T_{j,l}} e^{\alpha (t_j-l-u)} * \Phi_{S} dZ_{u}\right)\] 
\[+g\left(\sum_{i=1}^{n} \sum_{j=1}^{N_{l,T}} h_{S,l,j}^{(1)} * e^{\alpha (T-T_{j,l})} * \sum_{L=1}^{n} \sum_{m=1}^{N_{l,T}} h_{S,L,m}^{(1)} * e^{\alpha (T_{j,l}-T_{m,l})} * S_{l}^{(0)} \right)\] 
\[-g\left(\sum_{i=1}^{n} \sum_{j=1}^{N_{l,T}} h_{S,l,j}^{(1)} * e^{\alpha (T-T_{j,l})} * \sum_{L=1}^{n} \Lambda_l E[h_{S,L,1}^{(1)}] * e^{\alpha T} * \int_{0}^{T_{j,l}} e^{-\alpha u} * S_{l}^{(0)} du\right)\] 
\[+g\left(\int_{0}^{T} \sum_{i=1}^{n} \Lambda_l E[h_{S,l,1}^{(1)}] * e^{\alpha (T-t)} * \int_{0}^{t} e^{\alpha (t-u)} * \Phi_{S} dZ_{u} dt\right)\] 
\[-g\left(\int_{0}^{T} \sum_{i=1}^{n} \Lambda_l E[h_{S,l,1}^{(1)}] * e^{\alpha (T-t)} * \sum_{L=1}^{n} \sum_{m=1}^{N_{l,T}} h_{S,L,m}^{(1)} * e^{\alpha (T_{j,l}-T_{m,l})} * S_{l}^{(0)} \right)\] 
\[+g\left(\int_{0}^{T} \sum_{i=1}^{n} \Lambda_l E[h_{S,l,1}^{(1)}] * e^{\alpha (T-t)} * \sum_{L=1}^{n} \Lambda_l E[h_{S,L,1}^{(1)}] * e^{\alpha t} * \int_{0}^{t} e^{-\alpha u} * S_{l}^{(0)} du \right).\]

where

\[
E[h_{x,i,l}^{(e)}] = E[e^{Y_{x,i,l} + 1} - 1] = e^{m_{x,i,l} + \varepsilon^2_{x,i,l} - 1}, \quad (104)
\]
\[
E[h_{x,i,l}^{(0)}] = E[h_{x,i,l}^{(e)}]|_{\varepsilon = 0} = 1 - 1 = 0, \quad (105)
\]
\[
E[h_{x,i,l}^{(1)}] = \partial_{\varepsilon} E[h_{x,i,l}^{(e)}]|_{\varepsilon = 0} = (m_{x,i,l} + \varepsilon^2_{x,i,l}) e^{m_{x,i,l} + \varepsilon^2_{x,i,l}} |_{\varepsilon = 0} = m_{x,i,l}, \quad (106)
\]
\[
E[h_{x,i,l}^{(2)}] = \partial^2_{\varepsilon} E[h_{x,i,l}^{(e)}]|_{\varepsilon = 0} = \frac{\varepsilon^2_{x,i,l} e^{m_{x,i,l} + \varepsilon^2_{x,i,l} + \varepsilon^2_{x,i,l}} + (m_{x,i,l} + \varepsilon^2_{x,i,l})^2 e^{m_{x,i,l} + \varepsilon^2_{x,i,l}}}{\varepsilon^2_{x,i,l}} |_{\varepsilon = 0} = m^2_{x,i,l} + \varepsilon^2_{x,i,l}. \quad (107)
\]

Next, we define the expression \(F(X)\) as

\[F(X) := E\left[X | g(\hat{S}_T) = x, \{N_l = k_l\}\right] = g\left(E\left[X | \hat{S}_T = x, \{N_l = k_l\}\right]\right), \quad (108)\]

where \(X\) stands for the expression in the equation number \((X)\). We also define \(\Sigma^{(k_l)}_T\) as

\[
\Sigma^{(k_l)}_T := \int_{0}^{T} \sum_{i=1}^{d} \sum_{j=1}^{d} w_{i,j} e^{\alpha x(T-t)} \Phi_{x,i,j} d\int_{0}^{T} e^{\alpha T} \Phi_{x,i,j} dt \\
+ \sum_{l=1}^{n} k_l \sum_{i=1}^{d} \sum_{j=1}^{d} w_{i,j} e^{\alpha x(T-t)} \int_{s_0}^{s_1} \sum_{l=1}^{d} w_{i,j} \gamma_{S_l,i} e^{\alpha T} \gamma_{S_l,i} S_{l}^{(1)}. \quad (109)\]
Then, we obtain the following calculations:

\[
F(g(S_T^{(1)})) = F(87) + F(88) + F(89) = (112) + (113) + (114), \tag{110}
\]

\[
F(g(S_T^{(2)})) = F(90) + F(91) + F(92) + F(93) + F(94) + F(95) + F(96) + F(97) + F(98) + F(99) + F(100) + F(101) + F(102) + F(103) = F(90) + F(116) + F(117) + F(92) + F(93) + F(94) + F(95) + F(124) + F(125) + F(126) + F(97) + F(131) + F(132) + F(135) + F(136) + F(137) + F(138) + F(100) + F(101) + F(145) + F(146) + F(103) = (115) + (118) + (119) + (120) + (121) + (122) + (123) + (127) + (128) + (129) + (130) + (133) + (134) + (139) + (140) + (141) + (142) + (143) + (144) + (147) + (148) + (149), \tag{111}
\]

\[
F(87) = \mathbb{E} \left[ g \left( \int_0^T e^{\alpha(T-t)} \Phi_SdZ_t \right) \right]_{|g(\hat{S}_T) = x, \{N_t = k_t\}}
= \sum_{l=1}^d w_l \int_0^T e^{\alpha(T-t)} \Phi_S^l \sum_{l=1}^d w_l \int_0^T e^{\alpha(T-t)} \Phi_S^l H_1(x, \Sigma_T) \frac{1}{\Sigma_T}, \tag{112}
\]

\[
F(88) = \mathbb{E} \left[ g \left( \sum_{i=1}^{N_{1,t}} \sum_{j=1}^{S_i^{(1)}} e^{\alpha(T-\tau_{i,j}^-)} \Phi_{S_i^{(0)}} \right) \right]_{|g(\hat{S}_T) = x, \{N_t = k_t\}}
= \sum_{l=1}^d w_l \int_0^T e^{\alpha(T-t)} \Phi_S^l \mathbb{E} \left[ k_1 Y_{S_i^{(1)}} \right]_{|g(\hat{S}_T) = x}
= \sum_{l=1}^d w_l \int_0^T e^{\alpha(T-t)} \Phi_S^l \mathbb{E} \left[ k_1 \left( m_{S_i^{(1)}} + \gamma_{S_i^{(1)}} \phi_{S_i^{(1)}} \right) \right]_{|g(\hat{S}_T) = x}
= \sum_{l=1}^d w_l \int_0^T e^{\alpha(T-t)} \Phi_S^l k_1 \left( m_{S_i^{(1)}} + \gamma_{S_i^{(1)}} \phi_{S_i^{(1)}} \right) \int_0^T e^{\alpha(T-t)} \Phi_S^l \frac{1}{\Sigma_T}, \tag{113}
\]

\[
F(89) = \mathbb{E} \left[ g \left( \sum_{i=1}^{N_{1,t}} \lambda_i \mathbb{E} \left[ h_{S_i^{(1)},1,1} \right] e^{\alpha(T-t)} \int_0^T e^{-\alpha(t)} \Phi_{S_i^{(0)}} dt \right) \right]_{|g(\hat{S}_T) = x, \{N_t = k_t\}}
= \sum_{l=1}^d w_l \int_0^T e^{\alpha(T-t)} \Phi_S^l \mathbb{E} \left[ \lambda_i \mathbb{E} \left[ h_{S_i^{(1)},1,1} \right] e^{\alpha(T-t)} \int_0^T e^{-\alpha(t)} \Phi_{S_i^{(0)}} dt \right], \tag{114}
\]

\[
F(90) = \mathbb{E} \left[ g \left( \int_0^T e^{\alpha(T-t)} \partial_S \Phi_S \int_0^t e^{\alpha(t-u)} \Phi_S dZ_t \right) \right]_{|g(\hat{S}_T) = x, \{N_t = k_t\}}
= \sum_{l=1}^d w_l \int_0^T e^{\alpha(T-t)} \partial_S \Phi_S^l \sum_{j=1}^d w_j \int_0^T e^{\alpha(T-t)} \Phi_S^l \int_0^t e^{\alpha(t-u)} \Phi_S^l \sum_{j=1}^d w_j e^{\alpha(t-u)} \Phi_S^l dW_t \mathbb{E} \left[ H_2(x, \Sigma_T) \right], \tag{115}
\]

\[
F(91) = \mathbb{E} \left[ g \left( \int_0^T e^{\alpha(T-t)} \partial_S \Phi_S \sum_{i=1}^{N_{1,t}} \sum_{j=1}^{S_i^{(1)}} e^{\alpha(t-\tau_{i,j}^-)} \Phi_{S_i^{(0)}} \right) \right]_{|g(\hat{S}_T) = x, \{N_t = k_t\}}
= \sum_{l=1}^d w_l \int_0^T e^{\alpha(T-t)} \partial_S \Phi_S^l \sum_{j=1}^d w_j \int_0^T e^{\alpha(T-t)} \Phi_S^l \int_0^t e^{\alpha(t-u)} \Phi_S^l \sum_{j=1}^d w_j e^{\alpha(t-u)} \Phi_S^l dW_t \mathbb{E} \left[ H_2(x, \Sigma_T) \right], \tag{115}
\]


31
\[
E \left[ \int_0^T e^{\alpha(T-t)} \partial \Phi S \sum_{l=1}^{N_l} \sum_{j=1}^{N_j} (m_{S,l} + \gamma_{S,l} \zeta_{S,j} \lambda) e^{\alpha(t-\tau_{j,l})} S_{(j)}^{(0)} dz_l \right] \mid g(\hat{S}_T) = x, \{N_i = k_i\}
\]

\[
E \left[ \int_0^T e^{\alpha(T-t)} \partial \Phi S \sum_{l=1}^{N_l} \sum_{j=1}^{N_j} m_{S,l} e^{\alpha(t-\tau_{j,l})} S_{(j)}^{(0)} dz_l \right] \mid g(\hat{S}_T) = x, \{N_i = k_i\} + E \left[ \int_0^T e^{\alpha(T-t)} \partial \Phi S \sum_{l=1}^{N_l} \sum_{j=1}^{N_j} \gamma_{S,l} \zeta_{S,j} \lambda e^{\alpha(t-\tau_{j,l})} S_{(j)}^{(0)} dz_l \right] \mid g(\hat{S}_T) = x, \{N_i = k_i\}
\]

\[
F(116) = \sum_{i=1}^{d} w_i e^{\alpha(T-t)} \int_0^T \partial \Phi S \sum_{l=1}^{N_l} \sum_{j=1}^{N_j} k \partial \Phi S \sum_{l=1}^{N_l} \sum_{j=1}^{N_j} w_j e^{\alpha(t-\tau_{j,l})} H(x, \Sigma(k^{(i)})) \frac{\Sigma(k^{(i)})}{\Sigma(k^{(i)})},
\]

\[
F(117) = \sum_{i=1}^{d} w_i e^{\alpha(T-t)} \int_0^T \partial \Phi S \sum_{l=1}^{N_l} \sum_{j=1}^{N_j} \gamma_{S,l} \zeta_{S,j} \lambda e^{\alpha(t-\tau_{j,l})} S_{(j)}^{(0)} \frac{\Sigma(k^{(i)})}{\Sigma(k^{(i)})}
\]

\[
F(92) = \sum_{i=1}^{d} w_i e^{\alpha(T-t)} \int_0^T \partial \Phi S \sum_{l=1}^{N_l} \sum_{j=1}^{N_j} h_{y,l}^{(1)} \zeta_{S,l} \lambda e^{\alpha(t-\tau_{j,l})} \sigma_{(j)}^{(0)} \frac{\Sigma(k^{(i)})}{\Sigma(k^{(i)})}
\]

\[
F(93) = \sum_{i=1}^{d} w_i e^{\alpha(T-t)} \int_0^T \partial \Phi S \sum_{l=1}^{N_l} \sum_{j=1}^{N_j} k \partial \Phi S \sum_{l=1}^{N_l} \sum_{j=1}^{N_j} w_j e^{\alpha(t-\tau_{j,l})} H(x, \Sigma(k^{(i)})) \frac{\Sigma(k^{(i)})}{\Sigma(k^{(i)})},
\]

\[
F(94) = \sum_{i=1}^{d} w_i e^{\alpha(T-t)} \int_0^T \partial \Phi S \sum_{l=1}^{N_l} \sum_{j=1}^{N_j} h_{y,l}^{(1)} \zeta_{S,l} \lambda e^{\alpha(t-\tau_{j,l})} \sigma_{(j)}^{(0)} \frac{\Sigma(k^{(i)})}{\Sigma(k^{(i)})},
\]

\[
F(95) = \sum_{i=1}^{d} w_i e^{\alpha(T-t)} \int_0^T \partial \Phi S \sum_{l=1}^{N_l} \sum_{j=1}^{N_j} \gamma_{S,l} \zeta_{S,j} \lambda e^{\alpha(t-\tau_{j,l})} S_{(j)}^{(0)} \frac{\Sigma(k^{(i)})}{\Sigma(k^{(i)})},
\]

\[
F(96) = \sum_{i=1}^{d} w_i e^{\alpha(T-t)} \int_0^T \partial \Phi S \sum_{l=1}^{N_l} \sum_{j=1}^{N_j} h_{y,l}^{(1)} \zeta_{S,l} \lambda e^{\alpha(t-\tau_{j,l})} S_{(j)}^{(0)} \frac{\Sigma(k^{(i)})}{\Sigma(k^{(i)})},
\]
\[ \mathbf{F} \]
\[-\sum_{l=1}^{n} \int_0^T \int_0^{t_l} \Phi \left[ h^{(2)}_{s_l,l} \right] * e^{\alpha_t} du * e^{-\alpha_t} * \Phi_S dZ_l \right] g(\hat{S}_T) = x, \{N_l = k_l\} \]

\[= \sum_{i=1}^{d} w_i T \sum_{l=1}^{n} \Lambda_1 m_{s_i,l} e^{\alpha_T} \int_0^T e^{-\alpha_t} \Phi_S \sum_{i=1}^{d} w_i e^{\alpha_t} (T-t) \Phi_S dt \frac{H_1(x, \Sigma^{(k_l)})}{\Sigma^{(l)}}, \]

\[-\sum_{i=1}^{d} w_i \int_0^T \left( \sum_{l=1}^{n} \Lambda_1 m_{s_i,l} e^{\alpha_T} \Phi_S \sum_{i=1}^{d} w_i e^{\alpha_t} (T-t) \Phi_S dt \frac{H_1(x, \Sigma^{(k_l)})}{\Sigma^{(l)}} \right) \] 

\[F(102) = \quad \mathbf{E} \left[ g \left( \sum_{i=1}^{n} \Lambda_1 m_{s_i,l} e^{\alpha_T} \frac{\sum_{l=1}^{n} m_{s_i,l} s_0 dt}{\sum_{l=1}^{n} m_{s_i,l} s_0 dt} \right) g(\hat{S}_T) = x, \{N_l = k_l\} \right] \]

\[+ \mathbf{E} \left[ g \left( \sum_{i=1}^{n} \Lambda_1 m_{s_i,l} e^{\alpha_T} \frac{\sum_{l=1}^{n} m_{s_i,l} s_0 dt}{\sum_{l=1}^{n} m_{s_i,l} s_0 dt} \right) g(\hat{S}_T) = x, \{N_l = k_l\} \right] , \]

\[F(145) = \quad \sum_{i=1}^{d} w_i \int_0^T \sum_{l=1}^{n} \Lambda_1 m_{s_i,l} e^{\alpha_T} \frac{s_0}{\sum_{l=1}^{n} m_{s_i,l} s_0 dt} \int_0^{t} \frac{k_l}{T} dt \]

\[= \sum_{i=1}^{d} w_i \int_0^T \sum_{l=1}^{n} \Lambda_1 m_{s_i,l} e^{\alpha_T} \frac{s_0}{\sum_{l=1}^{n} m_{s_i,l} s_0 dt} \frac{k_l}{T} , \]

\[F(146) = \quad \mathbf{E} \left[ g \left( \sum_{i=1}^{n} \Lambda_1 m_{s_i,l} e^{\alpha_T} \frac{\sum_{l=1}^{n} m_{s_i,l} s_0 dt}{\sum_{l=1}^{n} m_{s_i,l} s_0 dt} \right) g(\hat{S}_T) = x, \{N_l = k_l\} \right] \]

\[= \mathbf{E} \left[ g \left( s_0 \int_0^T \sum_{l=1}^{n} \Lambda_1 m_{s_i,l} e^{\alpha_T} \frac{\sum_{l=1}^{n} m_{s_i,l} s_0 dt}{\sum_{l=1}^{n} m_{s_i,l} s_0 dt} \right) g(\hat{S}_T) = x, \{N_l = k_l\} \right] \]

\[= \sum_{i=1}^{d} w_i s_0 \int_0^T \sum_{l=1}^{n} \Lambda_1 m_{s_i,l} e^{\alpha_T} \frac{\sum_{l=1}^{n} m_{s_i,l} s_0 dt}{\sum_{l=1}^{n} m_{s_i,l} s_0 dt} \int_0^{t} \frac{k_l}{T} dt \frac{H_1(x, \Sigma^{(k_l)})}{\Sigma^{(l)}}, \]

\[F(148) = \sum_{i=1}^{d} w_i s_0 \int_0^T \sum_{l=1}^{n} \Lambda_1 m_{s_i,l} e^{\alpha_T} \frac{\sum_{l=1}^{n} m_{s_i,l} s_0 dt}{\sum_{l=1}^{n} m_{s_i,l} s_0 dt} \frac{k_l}{T} \]

\[= \sum_{i=1}^{d} w_i s_0 \int_0^T \sum_{l=1}^{n} \Lambda_1 m_{s_i,l} e^{\alpha_T} \frac{\sum_{l=1}^{n} m_{s_i,l} s_0 dt}{\sum_{l=1}^{n} m_{s_i,l} s_0 dt} \frac{k_l}{2} \]

Next, let us show the expression of \( g \left( \frac{1}{2T} S_T^{LSV(2)} \right) ^2 \):

\[g \left( \frac{1}{2T} S_T^{LSV(2)} \right) ^2 = \quad g \left( \int_0^T e^\alpha t - t \Phi_S * \int_0^t e^\alpha t-u * \Phi_S dZ_u dZ_t ) \right) \]

\[+ \int_0^T e^\alpha t - t \Phi_S * \int_0^t e^{-\lambda(t-u)} * \Phi_S dZ_u dZ_t \]

\[= g \left( \int_0^T e^\alpha t - t \Phi_S * \int_0^t e^\alpha t-u * \Phi_S dZ_u dZ_t \right) ^2 \]
Then, we obtain the expressions of $F$ where

$$
F(q) = \int_0^T e^{\alpha(T-t)} \sum_{i=1}^d w_i \Phi_{Si} \int_0^t e^{\alpha(t-u)} \Phi_{Si} dZ_u dZ_t \int_q^r e^{-\lambda(t-u)} \Phi_{Si} dZ_u dZ_t \right) \\
\times \left( \int_0^T e^{\alpha(T-t)} \sum_{i=1}^d w_i \Phi_{Si} \int_0^t e^{\alpha(t-u)} \Phi_{Si} dZ_u dZ_t \right) \\
+ g \left( \int_0^T e^{\alpha(T-t)} \sum_{i=1}^d w_i \Phi_{Si} \int_0^t e^{-\lambda(t-u)} \Phi_{Si} dZ_u dZ_t \right)^2
$$

(151)

$$
= F(150) + 2F(151) + F(152).
$$

(152)

(153)

Then, we obtain the expressions of $F(M)$ for $M = 150, 151, 152$ as follows:

$$
F(M) = \sum_{i=1}^d w_i \int_0^t q_{M,ti}^i \Phi_{Si} \int_0^t q_{M,ti}^i \Phi_{Si} dZ_u dZ_t \\
\times \left( \int_0^T q_{M,ti}^i \Phi_{Si} \int_0^t q_{M,ti}^i \Phi_{Si} dZ_u dZ_t \right) \left( \int_0^T q_{M,ti}^i \Phi_{Si} \int_0^t q_{M,ti}^i \Phi_{Si} dZ_u dZ_t \right)
$$

(154)

where

$$
q_{150,1t,i} = \sum_{i=1}^d w_i \Phi_{Si} 
$$

(155)

$$
q_{150,2t,i} = e^{-\alpha t} \Phi_{Si} 
$$

(156)

$$
q_{150,3t,i} = e^{\alpha T} \Phi_{Si} 
$$

(157)

$$
q_{150,4t,i} = e^{-\alpha t} \Phi_{Si} 
$$

(158)

$$
q_{150,5t,i} = e^{\alpha T} \Phi_{Si} 
$$

(159)

$$
q_{151,1t,i} = \sum_{i=1}^d w_i \Phi_{Si} 
$$

(160)

$$
q_{151,2t,i} = e^{-\alpha t} \Phi_{Si} 
$$

(161)
We obtain the expressions of $F$ as follows:

\[
q'_{151,3t,i} = e^{\alpha t} \ast \partial_S \Phi_S^i, \\
q'_{151,4t,i} = e^{\lambda u} \ast \Phi_S^i, \\
q'_{151,5t,i} = e^{\alpha (T-t)} - \lambda^t \ast \partial_S \Phi_S^i, \\
q'_{152,1t,i} = \sum_{i=1}^{d} w_i \Phi_S^i, \\
q'_{152,2t,i} = e^{\lambda u} \ast \Phi_S^i, \\
q'_{152,3t,i} = e^{\alpha (T-t)} - \lambda^t \ast \partial_S \Phi_S^i, \\
q'_{152,4t,i} = e^{\lambda u} \ast \Phi_S^i, \\
q'_{152,5t,i} = e^{\alpha (T-t)} - \lambda^t \ast \partial_S \Phi_S^i.
\]

Then, we show the expressions of $g \left( \frac{1}{\pi} \sigma_{LSV}^{(3)} \right)$ as follows:

\[
g \left( \frac{1}{\pi} \sigma_{LSV}^{(3)} \right) = g \left( \frac{1}{2} \int_0^T e^{\alpha (T-t)} \ast \partial_S^2 \Phi_S \ast \left( \int_0^t e^{\alpha (t-u)} \ast \Phi_S dZ_u \right)^2 dZ_t \right) \\
+ g \left( \int_0^T e^{\alpha (T-t)} \ast \partial_S \Phi_S \ast \left( \int_0^t e^{\alpha (t-u)} \ast \Phi_S dZ_u \right) dZ_t \right) \\
+ g \left( \int_0^T e^{\alpha (T-t)} \ast \partial_S \Phi_S \ast \left( \int_0^t e^{-\lambda (u-v)} \ast \Phi_S dZ_u dZ_t \right) \right) \\
+ g \left( \frac{1}{2} \int_0^T e^{\alpha (T-t)} \ast \partial_S^2 \Phi_S \ast \left( \int_0^t e^{-\lambda (t-u)} \ast \Phi_S dZ_u \right)^2 dZ_t \right) \\
+ g \left( \int_0^T e^{\alpha (T-t)} \ast \partial_S \Phi_S \ast \left( \int_0^t e^{-\lambda (u-v)} \ast \Phi_S dZ_u dZ_t \right) \right) \\
= \frac{1}{2} F(170) + F(171) + F(172) + \frac{1}{2} F(173) + F(174). \tag{175}
\]

We obtain the expressions of $F(M)$ for $M = 171, 172, 174$:

\[
F(M) = \sum_{i=1}^{d} w_i \left( \int_0^T q_{M,4,i}^t \int_0^t q_{M,3,i}^s \int_0^s q_{M,2,i}^d \int_0^d q_{M,1,i}^c \right) H(x; \Sigma^{(k)}_T) \frac{H(x; \Sigma^{(k)}_T)}{(\Sigma^{(k)}_T)^3} \tag{176}
\]

where

\[
q'_{171,1t,i} = \sum_{i=1}^{d} w_i \Phi_S^i, \tag{177}
\]

37
\( q'_{171,2,t,i} = e^{-\alpha t} \Phi_{S_i}, \)  
(178)

\( q'_{171,3,t,i} = \partial_{S_i} \Phi_{S_i}, \)  
(179)

\( q'_{171,4,t,i} = e^{\alpha T} \partial_{S_i} \Phi_{S_i}, \)  
(180)

\( q'_{172,1,t,i} = \sum_{i=1}^{d} w_i \Phi_{S_i}, \)  
(181)

\( q'_{172,2,t,i} = e^{\lambda t} \Phi_{\sigma_i}, \)  
(182)

\( q'_{172,3,t,i} = e^{-\alpha t - \lambda t} \partial_{\sigma_i} \Phi_{S_i}, \)  
(183)

\( q'_{172,4,t,i} = e^{\alpha T} \partial_{S_i} \Phi_{S_i}, \)  
(184)

\( q'_{174,1,t,i} = \sum_{i=1}^{d} w_i \Phi_{S_i}, \)  
(185)

\( q'_{174,2,t,i} = e^{\lambda t} \Phi_{\sigma_i}, \)  
(186)

\( q'_{174,3,t,i} = \partial_{\sigma_i} \Phi_{\sigma_i}, \)  
(187)

\( q'_{174,4,t,i} = e^{\alpha (T-t) - \lambda t} \partial_{\sigma_i} \Phi_{S_i}. \)  
(188)

We also have the expressions of \( F(M) \) for \( M = 170, 173 \) as follows:

\[
F(M) = \sum_{i=1}^{d} w_i \left\{ \int_{0}^{T} \left( \int_{0}^{t} q'_{M,2u,i} q_{M,1u,i} du \right) \left( \int_{0}^{T} q'_{M,3u,i} q_{M,1u,i} ds \right) q'_{M,4t,i} q_{M,1t,i} dt \right\} \frac{H_i(x; \Sigma_i^{(k_i)})}{(\Sigma_i^{(k_i)})^3} \\
+ \left( \int_{0}^{T} \int_{0}^{t} q'_{M,2u,i} q_{M,3u,i} du q'_{M,4t,i} q_{M,1t,i} dt \right) \frac{H_i(x; \Sigma_i^{(k_i)})}{\Sigma_i^{(k_i)}} ,
\]

(189)

where

\( q'_{170,1,t,i} = \sum_{i=1}^{d} w_i \Phi_{S_i}, \)  
(190)

\( q'_{170,2,t,i} = q_{170,3,t,i} = e^{-\alpha t} \Phi_{S_i}, \)  
(191)

\( q'_{170,4,t,i} = e^{\alpha (T+t)} \partial_{S_i}^2 \Phi_{S_i}, \)  
(192)

\( q'_{173,1,t,i} = \sum_{i=1}^{d} w_i \Phi_{S_i}, \)  
(193)

\( q'_{173,2,t,i} = q_{173,3,t,i} = e^{\lambda t} \Phi_{\sigma_i}, \)  
(194)

\( q'_{173,4,t,i} = e^{\alpha (T-t) - 2\lambda t} \partial_{\sigma_i}^2 \Phi_{S_i}. \)  
(195)

Summarizing all of these terms with the order of Hermit polynomial, we obtain the coefficients, \( C_i, (i = 1, 2, 3, 4) \) in (69).
C Conditional Expectations Formulas of the Wiener-Itô Integrals

This appendix summarizes conditional expectation formulas for explicit computation of the asymptotic expansions up to the third order.

In the following, \( W \) is a \( d \)-dimensional Brownian motion and \( q_i = (\hat{q}_1, \ldots, \hat{q}_d)' \) where \( \hat{q}_i \in L^2[0, T], \ i = 1, 2, \ldots, 5 \) and \( x' \) denotes the transpose of \( x \). \( H_n(x; \Sigma) \) denotes the Hermite polynomial of degree \( n \) and \( \Sigma = \int_0^T |q_{1t}|^2 dt \). For the derivation and more general results, see Section 3 in Takahashi, Takehara and Toda (2009).

1. \[ E \left[ \int_0^T q_{2t}dW_t \bigg| \int_0^T q_{1v}dW_v = x \right] = \left( \int_0^T q_{2t}q_{1t}dt \right) \frac{H_1(x; \Sigma)}{\Sigma}. \] (196)

2. \[ E \left[ \int_0^T \int_0^t q_{2u}dW_u q_{3t}dW_t \bigg| \int_0^T q_{1v}dW_v = x \right] = \left( \int_0^T q_{2u}q_{1u}duq_{3t}q_{1t}dt \right) \frac{H_2(x; \Sigma)}{\Sigma^2}. \] (197)

3. \[ E \left[ \left( \int_0^T q_{2u}dW_u \right) \left( \int_0^T q_{3s}dW_s \right) \bigg| \int_0^T q_{1v}dW_v = x \right] = \left( \int_0^T q_{2u}q_{1u}du \right) \left( \int_0^T q_{3s}q_{1s}ds \right) \frac{H_2(x; \Sigma)}{\Sigma^2} + \int_0^T q_{2t}q_{3t}dt. \] (198)

4. \[ E \left[ \int_0^T \int_0^s q_{2u}dW_u q_{3s}dW_s q_{4t}dW_t \bigg| \int_0^T q_{1v}dW_v = x \right] = \left( \int_0^T q_{4t}q_{1t}du \right) \left( \int_0^s q_{3s}q_{1s}du \right) \frac{H_3(x; \Sigma)}{\Sigma^3}. \] (199)

5. \[ E \left[ \int_0^T \left( \int_0^t q_{2u}dW_u \right) \left( \int_0^t q_{3s}dW_s \right) q_{4t}dW_t \bigg| \int_0^T q_{1v}dW_v = x \right] = \left\{ \int_0^T \left( \int_0^t q_{2u}q_{1u}du \right) \left( \int_0^t q_{3s}q_{1s}ds \right) q_{4t}q_{1t}dt \right\} \frac{H_3(x; \Sigma)}{\Sigma^3} + \left( \int_0^T \int_0^t q_{2u}q_{3s}duq_{4t}q_{1t}dt \right) \frac{H_1(x; \Sigma)}{\Sigma}. \] (200)
\[ \mathbf{E} \left[ \left( \int_0^T \int_0^t \dot{q}_{2u} \, dW_s \, \dot{q}_{3v} \, dW_t \right) \left( \int_0^T \int_0^r \dot{q}_{4u} \, dW_u \, \dot{q}_{5r} \, dW_r \right) \mid \int_0^T \dot{q}_{1v} \, dW_v = x \right] = \]
\[ \left( \int_0^T \dot{q}_{3u} \, q_{1u} \int_0^t \dot{q}_{2u} \, q_{1s} \, ds \, dt \right) \left( \int_0^T \dot{q}_{5u} \, q_{1r} \int_0^r \dot{q}_{4u} \, q_{1d} \, dr \right) \frac{H_2(x; \Sigma)}{\Sigma^2} \]
\[ + \left\{ \int_0^T \dot{q}_{3u} \, q_{1u} \int_0^t \dot{q}_{5u} \, q_{1r} \int_0^r \dot{q}_{4u} \, q_{1d} \, dr \, dt + \int_0^T \dot{q}_{3u} \, q_{1u} \int_0^t \dot{q}_{5u} \, q_{1r} \int_0^r \dot{q}_{4u} \, q_{1d} \, dr \, dt \right\} \frac{H_2(x; \Sigma)}{\Sigma^2} \]
\[ + \left\{ \int_0^T \dot{q}_{5r} \, q_{1r} \int_0^r \dot{q}_{3u} \, q_{4u} \int_0^u \dot{q}_{2u} \, q_{1s} \, ds \, dr \right\} \frac{H_2(x; \Sigma)}{\Sigma^2} \]
\[ + \int_0^T \int_0^t \dot{q}_{2u} \, q_{4u} \, dW_s \, \dot{q}_{3u} \, q_{5s} \, dW_t. \]