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On Error Estimates for Asymptotic Expansions with Malliavin Weights – Application to Stochastic Volatility Model – *

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Abstract

This paper proposes a unified method for precise estimates of the error bounds in asymptotic expansions of an option price and its Greeks (sensitivities) under a stochastic volatility model. More generally, we also derive an error estimate for an asymptotic expansion around a general partially elliptic diffusion and a more general Wiener functional, which is applicable to various important valuation and risk management tasks in the financial business such as the ones for multi-dimensional diffusion and non-diffusion models. In particular, we take the Malliavin calculus approach, and estimate the error bounds for the Malliavin weights of both the coefficient and the residual terms in the expansions by effectively applying the properties of Kusuoka-Stroock functions. Moreover, a numerical experiment under the Heston-type model confirms the effectiveness of our method.

Keywords: Asymptotic expansion, Malliavin calculus, Kusuoka-Stroock functions, Stochastic volatility model, Option price, Greeks

1 Introduction

In this paper, we derive asymptotic expansions of option prices and Greeks (sensitivities) around the Black-Scholes model in stochastic volatility environment, and develop a unified method for precise estimates of the expansion errors by extending the method proposed in Takahashi-Yamada [46] (2012).

Moreover, we present an error estimate for an asymptotic expansion around a partially elliptic diffusion under a multi-dimensional setting and for an asymptotic expansion of a more general Wiener functional, which can be applied to various important pricing problems as well as computing Greeks in finance such as the ones for the basket and average options under stochastic volatility models, and the ones for bond options/swaptions

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and long-term currency options under the Heath-Jarrow-Morton (HJM) framework [17] (1992) or the Libor Market Models (LMMs).

Particularly, we make use of the Kusuoka-Stroock functions introduced by Kusuoka [23] (2003), which is a powerful tool to clarify the order of a Wiener functional with respect to the time parameter t in a unified manner. Then, we are able to estimate the error bounds for the Malliavin weights of both the coefficient and the residual terms in the expansions.

In mathematical finance, one of the main issues is to compute a derivative price $E[f(S_t)]$ and its Greeks under a equivalent martingale measure, where f is a payoff function and S_t denotes the underlying asset price at time $t \in [0,T]$. However, as the analytical probability distributions describing the whole dynamics of the realistic models are rarely known, various analytical and numerical schemes have been proposed in order to satisfy the practical requirement of the fast computation. Among them, one of the tractable approach is to approximate a model by a perturbation around a main driving process whose underlying distribution is analytically obtained (e.g. Black-Scholes log-normal model). In this case we often encounter an asymptotic expansion around a partially elliptic diffusion. More precisely, introducing a small perturbation parameter ε into a model of the underlying asset price, we expand the price and the Greeks against a parameter β around $\varepsilon = 0$, which corresponds to an expansion around a partially elliptic diffusion process. Particularly, it has been well-established to apply the Watanabe's expansion in Malliavin calculus (Watanabe [51] (1987)), That is:

$$\begin{aligned} E[f(S_t^\varepsilon)] &= E[f(S_t^0)] + \sum_{i=1}^{N} \varepsilon^i a_i + O(\varepsilon^{N+1}), \\ \frac{\partial^n}{\partial \beta^n} E[f(S_t^\varepsilon)] &= \frac{\partial^n}{\partial \beta^n} E[f(S_t^0)] + \sum_{i=1}^{N} \varepsilon^i b_i + O(\varepsilon^{N+1}). \end{aligned} \tag{1}$$

We remark that this type of expansion technique has been successfully applied with substantial numerical experiment to the finance models widely used in practice: (for instance, see Kato et al. [19] (2012), [20] (2013), Li [25] [26] (2010, 2013), Xu-Zheng [52], [53] (2010, 2012), Shiraya-Takahashi [34], [35] (2011, 2012), Shiraya-Takahashi-Toda [36] (2011), Shiraya-Takahashi-Yamada [37] (2012), Shiraya-Takahashi-Yamazaki [38] (2012), Takahashi-Takehara [41] (2010) for the various derivatives pricing; Matsuoka et al. [31] (2006) for the Greeks; Li [26] (2013), Takahashi et al. [42], [43] (2009, 2012) for computational schemes of high-order expansions, and the references therein.) We also note that there exist many other types of the expansion/perturbation methods which have turned out to be so useful in financial applications: for example see Alos [1] (2012), Alos et al. [2] (2011), Bayer-Laurence [3] (2012), Ben Arous-Laurence [4] (2009), Davydov-Linetsky [9] (2003), Foschi et al. [11] (2013), Fouque et al. [12] (2002), Fujii [14] (2012), Gatheral et al. [15] (2012), Hagan et al. [16] (2002), Linetsky [27] (2004), Siopacha-Teichmann [39] (2011), and the references therein.

Although this expansion is proved to be mathematically rigorous, the error terms $O(\varepsilon^{N+1})$ depend on the time-to-maturity, the smoothness of the payoff f, the parameter β and the order of the Greeks n. Hence, it is better to distinguish these effects as much as possible. In particular, when f has no smoothness, the Greeks may show unstable behavior with respect to the time-to-maturity, especially near the expiry. (For the detail, see Friedman [14] (1964), Kusuoka-Stroock [24] (1984) and Kusuoka [23] (2003).) For instance, as for the Greeks with respect to the initial asset price such as Delta and

Gamma, one can show that there exist C such that

$$\frac{\partial^n}{\partial S_0^n} E[f(S_t^\varepsilon)] \quad \leq \quad \frac{C}{t^{n/2}}. \tag{2}$$

Thus, we can observe that the right hand side could become huge for a small t, that is near the maturity.

Also, from the practical viewpoint of the risk management for a derivative portfolio, the precise evaluation of the approximation errors against its true values and Greeks is highly desirable. This is because the exact computation is usually impossible under complex finance models required in financial business and hence some analytical approximation or/and a certain numerical scheme should be employed.

For this purpose, we develop a unified method for investigating the orders of the expansion errors for the bounded payoff and the Lipschitz continuous payoff in stochastic volatility environment. Consequently, we are able to provide more concrete expressions for the error terms than $O(\varepsilon^{N+1})$ in the asymptotic expansions of the values and the Greeks (1), which appears in Section 4 of the main text.

The organization of the paper is as follows: after the next section introduces the Kusuoka-Stroock functions, Section 3 derives an error estimate of an asymptotic expansion around a partially elliptic diffusion under a general multi-dimensional setting. Section 4 presents our main theorem and a lemma which is useful for proving the theorem. Section 5 describes how to compute option prices and their Greeks based on our method, and Section 6 provides a numerical experiment in a Heston-type model. Finally, Section 7 briefly explains an extension of the method to more general Wiener functionals.

2 The Kusuoka-Stroock Functions

2.1 The space $\mathcal{K}_r^T$

In this section, we introduce the space of Wiener functionals $\mathcal{K}_r^T$ developed by Kusuoka [23] (2003) and its properties. The element of $\mathcal{K}_r^T$ is called the *Kusuoka-Stroock function*. See Nee [33] (2011), Crisan-Delarue [7] (2012) and Crisan et al. [8] (2013) for more details of the notations and the proofs. Let $(\mathcal{W}, H, P)$ be the standard n-dimensional Wiener space. Let E be a separable Hilbert space and $\mathbf{D}^{l,\infty}(E) = \cap_{1\leq p<\infty}\mathbf{D}^{l,p}(E)$ be the space of E-valued functionals that admit the Malliavin derivatives up to the l-th order. The following definition and lemma correspond to Definition 2.1 and Lemma 2.2 of Crisan-Delarue [7] (2012).

Definition 2.1 *Given $r \in \mathbf{R}$ and $l \in \mathbf{N}$, we denote by $\mathcal{K}_r^T(E, l)$ the set of functions $G : (0, T] \times \mathbf{R}^d \to \mathbf{D}^{l,\infty}(E)$ satisfying the following:*

1. *$G(t, \cdot)$ is l-times continuously differentiable and $[\partial^\alpha G/\partial x^\alpha](\cdot, \cdot)$ is continuous in $(t, x) \in (0, T] \times \mathbf{R}^d$ a.s. for any multi-index α of the elements of $\{1, \cdots, d\}$ with length $|\alpha| \leq l$.*
2. *For all $k \in \mathbf{N}$, $p \in [1, \infty)$ and $k \leq l - |\alpha|$,*

$$\sup_{t\in(0,T],x\in\mathbf{R}^d} t^{-r/2} \left\| \frac{\partial^\alpha G}{\partial x^\alpha}(t, x) \right\|_{\mathbf{D}^{k,p}} < \infty. \tag{3}$$

We write $\mathcal{K}_r^T(l)$ for $\mathcal{K}_r^T(\mathbf{R}, l)$ and $\mathcal{K}_r^T$ for $\mathcal{K}_r^T(\mathbf{R}, \infty)$.

Let $(X_t)_{t\in[0,T]}$ be the solution to the following stochastic differential equation:

$$\begin{aligned} dX_t^x &= V_0(X_t^x)dt + \sum_{i=1}^n V_i(X_t^x)dW_{i,t}, \\ X_0 &= x \in \mathbf{R}^d, \end{aligned} \tag{4}$$

where each V_i, $i = 0, 1, \cdots, n$ is bounded and belongs to $C_b^\infty(\mathbf{R}^d; \mathbf{R}^d)$. We assume the Hörmander condition of Kusuoka [23]. See Definition 3 in Kusuoka [23] for the definition of the Hörmander condition. The properties of the Kusuoka-Stroock functions are the following. (See Lemma 2.19 of Nee [33] (2011) or Lemma 75 of Crisan et al. [8] (2013) for the proof.)

Lemma 2.1 *[Properties of Kusuoka-Stroock functions]*

1. *The function $(t, x) \in (0, T] \times \mathbf{R}^d \mapsto X_t^x$ belongs to $\mathcal{K}_0^T$, for any $T > 0$.*
2. *Suppose $G \in \mathcal{K}_r^T(l)$ where $r \geq 0$. Then, for $i = 1, \cdots, d$,*

$$(a) \int_0^\cdot G(s, x)dW_s^i \in \mathcal{K}_{r+1}^T(l), \text{ and } (b) \int_0^\cdot G(s, x)ds \in \mathcal{K}_{r+2}^T(l). \tag{5}$$

3. *If $G_i \in \mathcal{K}_{r_i}^T(n_i)$, $i = 1, \cdots, N$, then*

$$(a) \prod_i^N G_i \in \mathcal{K}_{r_1+\cdots+r_N}^T(\min_i n_i), \text{ and } (b) \sum_{i=1}^N G_i \in \mathcal{K}_{\min_i r_i}^T(\min_i n_i). \tag{6}$$

Next, we summarize the Malliavin's integration by parts formula using Kusuoka-Stroock functions. For any multi-index $\alpha^{(k)} := \alpha \in \{1, \cdots, d\}^k$, $k \geq 1$, we denote by $\partial_{\alpha^{(k)}}$ the partial derivative $\frac{\partial^k}{\partial x_{\alpha_1} \cdots \partial x_{\alpha_k}}$.

Proposition 2.1 *Let $G : (0, T] \times \mathbf{R}^d \to \mathbf{D}^\infty = \mathbf{D}^{\infty,\infty}(\mathbf{R})$ be an element of $\mathcal{K}_r^T$ and let f be a function that belongs to the space $C_b^\infty(\mathbf{R}^d)$. Then for any multi-index $\alpha^{(k)} \in \{1, \cdots, d\}^k$, $k \geq 1$, there exists $H_{\alpha^{(k)}}(X_t^x, G(t, x)) \in \mathcal{K}_{r-|\alpha^{(k)}|}^T = \mathcal{K}_{r-k}^T$ such that*

$$E\left[\partial_{\alpha^{(k)}} f(X_t^x)G(t, x)\right] = E\left[f(X_t^x)H_{\alpha^{(k)}}\left(X_t^x, G(t, x)\right)\right], \tag{7}$$

with

$$\sup_{x\in\mathbf{R}^d} \|H_{\alpha^{(k)}}(X_t^x, G(t, x))\|_{L^p} \leq Ct^{(r-k)/2}, \tag{8}$$

where $H_{\alpha^{(k)}}(X_t^x, G)$ is recursively given by

$$H_{(i)}(X_t^x, G(t, x)) = \delta\left(\sum_{j=1}^d G(t, x)\gamma_{ij}^{X_t^x} DX_t^{x,j}\right), \tag{9}$$

$$H_{\alpha^{(k)}}(X_t^x, G(t, x)) = H_{(\alpha_k)}\left(X_t^x, H_{\alpha^{(k-1)}}(X_t^x, G(t, x))\right), \tag{10}$$

and a positive constant $C > 0$ Here, δ is the Skorohod integral and $(\gamma_{ij}^{X_t^x})_{1\leq i,j\leq n}$ is the inverse matrix of the Malliavin covariance of X_t^x.

Proof. Apply Corollary 3.7 of Kusuoka-Stroock [24] (1984) and Lemma 8-(3) of Kusuoka [23] (2003) with Proposition 2.1.4 of Nualart [32] (2006). □

3 Asymptotic Expansion around a Partially Elliptic Diffusion and its Error Rate

In mathematical finance, various complex models have been developed for the practical purpose, but the analytical probability distributions describing the whole dynamics are rarely known. Thus, to satisfy the requirement for the fast computation of prices and Greeks (sensitivities), we sometimes approximate the model by a perturbation around a main driving process whose underlying distribution is analytically obtained (e.g. Black-Scholes log-normal model). In this case we often encounter an asymptotic expansion around a partially elliptic diffusion. To illustrate this situation, let us consider the following general setup.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space on which a n-dimensional Brownian motion $W = \{(W_t^1, \cdots, W_t^n) : 0 \le t \le T\}$ is defined. We also define $\mathcal{K}_r^T$, $r \in \mathbf{R}$ on this space. Let $(\mathcal{F}_t)_{0\le t\le T}$ be the natural filtration generated by W, augmented by the P-null sets of $\mathcal{F}$. Then, let us consider the following m-dimensional stochastic differential equation $\bar{X}_t^\epsilon = (X_t^{1,\epsilon}, \cdots, X_t^{d,\epsilon}, \tilde{X}_t^{1,\epsilon}, \cdots, \tilde{X}_t^{m-d,\epsilon})$ as follows:

$$\begin{aligned}
dX_t^{i,\epsilon} &= V_0^i(t, \bar{X}_t^\epsilon)dt + \sum_{j=1}^n V_j^i(t, \bar{X}_t^\epsilon)dW_t^j, \quad i = 1, \cdots, d, && (11)\\
X_0^{i,\epsilon} &= x_0^i \in \mathbf{R},\\
d\tilde{X}_t^{i,\epsilon} &= \tilde{V}_0^i(t, \tilde{X}_t^\epsilon)dt + \epsilon\sum_{j=1}^n \tilde{V}_j^i(t, \bar{X}_t^\epsilon)dW_t^j, \quad i = 1, \cdots, m-d, && (12)\\
\tilde{X}_0^{i,\epsilon} &= \tilde{x}_0^i \in \mathbf{R},
\end{aligned}$$

where $V_0 = (V_0^1, \cdots, V_0^d) : [0,T] \times \mathbf{R}^m \to \mathbf{R}^d$, $\tilde{V}_0 = (\tilde{V}_0^1, \cdots, \tilde{V}_0^{m-d}) : [0,T] \times \mathbf{R}^{m-d} \to \mathbf{R}^{m-d}$ and $V : [0,T] \times \mathbf{R}^m \to \mathbf{R}^{d\times n}$, $\tilde{V} : [0,T] \times \mathbf{R}^{m-d} \to \mathbf{R}^{(m-d)\times n}$ are bounded and smooth functions with bounded derivatives. For all fixed $(t,x) \in [0,T] \times \mathbf{R}^d$, the functions $\mathbf{R}^{m-d} \ni \tilde{x} \mapsto V_0(t,(x,\tilde{x}))$ and $\mathbf{R}^{m-d} \ni \tilde{x} \mapsto V(t,(x,\tilde{x}))$ are non-constants.

In finance models, $(X_t)_t$ and $(\tilde{X}_t)_t$ represent the asset price processes and the state variable dynamics such as the interest rate and volatility processes, respectively.

Note that when $\epsilon = 0$, the above SDE becomes a d-dimensional SDE X_t and a $(m-d)$-dimensional ODE $\tilde{X}_t$:

$$\begin{aligned}
dX_t &= V_0(t, \bar{X}_t^0)dt + \sum_{j=1}^n V_j(t, \bar{X}_t^0)dW_t^j, \quad X_0 = x_0 \in \mathbf{R}^d, && (13)\\
d\tilde{X}_t &= \tilde{V}_0(t, \tilde{X}_t^0)dt, \quad \tilde{X}_0 = \tilde{x}_0 \in \mathbf{R}^{m-d}.
\end{aligned}$$

We introduce the following notation:

$$\sum_{\mathbf{b}_k, \alpha^{(k)}}^{(i)} = \sum_{k=1}^{i} \sum_{\mathbf{b}_k \in L_{i,k}} \sum_{\alpha^{(\mathbf{k})} \in \{1,\cdots,d\}^k} \frac{1}{k!}, \tag{14}$$

with

$$L_{i,k} = \left\{ \mathbf{b}_k = (\beta_1, \cdots, \beta_k);\ \sum_{l=1}^k \beta_l = i;\ (i, \beta_l, k \in \mathbf{N}) \right\}. \tag{15}$$

The next theorem shows an asymptotic expansion with its error estimate of $E[f(X_t^\varepsilon)]$ for a bounded Borel function or a Lipschitz continuous function f around $E[f(X_t^0)]$ under a partially elliptic diffusion (13) .

Theorem 3.1 *Suppose for any $t \in [0,T]$ and any $\bar{x} \in \mathbf{R}^m$, there exists $a > 0$ such that*

$$VV^T(t,\bar{x}) > a\mathbf{I}_d, \tag{16}$$

where V^T stands for the transpose of V.

1. *Let $f : \mathbf{R}^d \to \mathbf{R}$ be a bounded Borel function. Then, for $N \in \mathbf{N}$, there exists $C_N > 0$ depending on N such that*

$$\sup_{(x_0,\tilde{x}_0)\in\mathbf{R}^m}\left|E[f(X_t^\varepsilon)] - \left\{E[f(X_t^0)] + \sum_{i=1}^{N}\varepsilon^i E[f(X_t^0)H_t^i]\right\}\right| \leq \|f\|_\infty C_N\left(\varepsilon\sqrt{t}\right)^{N+1}, \tag{17}$$

where

$$\begin{aligned} H_t^i &= \sum_{\mathbf{b}_k,\alpha^{(k)}}^{(i)} H_{\alpha^{(k)}}\left(X_t^0, \prod_{l=1}^{k} X_{\beta_l,t}^{0,\alpha_l}\right), \qquad (18)\\ X_{\beta_l,t}^{0,\alpha_l} &:= \frac{1}{\beta_l!}\frac{\partial^{\beta_l}}{\partial\varepsilon^{\beta_l}}X_t^{\varepsilon,\alpha_l}|_{\varepsilon=0}, \end{aligned}$$

and $X_t^{\varepsilon,\alpha_l}$ is the α_l-th element of X_t^ε.

2. *Let $f : \mathbf{R}^d \to \mathbf{R}$ be a Lipschitz function with constant C_f and $|f(0)| \leq C_f$. Then, for $N \in \mathbf{N}$, there exists $C_N > 0$ depending on N such that*

$$\sup_{(x_0,\tilde{x}_0)\in\mathbf{R}^m}\left|E[f(X_t^\varepsilon)] - \left\{E[f(X_t^0)] + \sum_{i=1}^{N}\varepsilon^i E[f(X_t^0)H_t^i]\right\}\right| \leq C_f C_N\sqrt{t}\left(\varepsilon\sqrt{t}\right)^{N+1}, \tag{19}$$

with the same weight (18) as in (17).

Proof.
Under the condition (16), the Malliavin covariance matrix $\gamma^{X_t^0}$ of X_t^0 satisfies

$$\left\|\left(\det\gamma^{X_t^0}\right)^{-1}\right\|_{L^p} < \infty, \quad 1 \leq p < \infty. \tag{20}$$

By the result of Yoshida [50] (1992b) and Takahashi-Yoshida [48] (2005), we are able to expand $E[f(X_t^\varepsilon)]$ around $E[f(X_t^0)]$. We remark that $(X_t^\epsilon)_t$ can be denoted as

$$dX_t^\epsilon = V_0(t, X_t^\epsilon, \tilde{X}_t^\epsilon)dt + \sum_{i=1}^{n} V_i(t, X_t^\epsilon, \tilde{X}_t^\epsilon)dW_t^i, \tag{21}$$

where V_0 and V are regarded as maps on $[0,T]\times\mathbf{R}^d\times\mathbf{R}^{m-d}$, that is, $V_0(t,\bar{x}) = V_0(t,x,\tilde{x})$ and $V(t,\bar{x}) = V(t,x,\tilde{x})$. Then, the process of $\frac{\partial}{\partial\varepsilon}X_t^\epsilon$ is expressed as:

$$\begin{aligned} d\frac{\partial}{\partial\varepsilon}X_t^\epsilon &= \nabla_x V_0(t, X_t^\epsilon, \tilde{X}_t^\epsilon)\frac{\partial}{\partial\varepsilon}X_t^\epsilon dt + \nabla_{\tilde{x}} V_0(t, X_t^\epsilon, \tilde{X}_t^\epsilon)\frac{\partial}{\partial\varepsilon}\tilde{X}_t^\epsilon dt \\ &\quad + \sum_{j=1}^{n}\nabla_x V_j(t, X_t^\epsilon, \tilde{X}_t^\epsilon)\frac{\partial}{\partial\varepsilon}X_t^\epsilon dW_t^j + \sum_{j=1}^{n}\nabla_{\tilde{x}}\tilde{V}_j(t, X_t^\epsilon, \tilde{X}_t^\epsilon)\frac{\partial}{\partial\varepsilon}\tilde{X}_t^\epsilon dW_t^j. \end{aligned} \tag{22}$$

Particularly, the process of $\frac{\partial}{\partial\varepsilon}\tilde{X}_t^\epsilon$ is given by:

$$
\begin{aligned}
\frac{\partial}{\partial\varepsilon}\tilde{X}_t^\epsilon &= \int_0^t \partial_{\tilde{x}}\tilde{V}_0(u,\tilde{X}_u^\epsilon)\frac{\partial}{\partial\varepsilon}\tilde{X}_u^\epsilon du + \sum_{j=1}^n \int_0^t \tilde{V}_j(u,\tilde{X}_u^\epsilon)dW_u^j \\
&\quad + \epsilon\sum_{j=1}^n \int_0^t \nabla_{\tilde{x}}\tilde{V}_j(u,\tilde{X}_u^\epsilon)\frac{\partial}{\partial\varepsilon}\tilde{X}_u^\epsilon dW_u^j,
\end{aligned}
$$

and it can be expressed as

$$
\frac{\partial}{\partial\varepsilon}\tilde{X}_t^\epsilon = \sum_{j=1}^n \left(\nabla_{\tilde{x}}\tilde{X}_t^\epsilon\right)\left[\int_0^t (\nabla_{\tilde{x}}\tilde{X}_u^\epsilon)^{-1}\left\{\tilde{V}_j(u,\tilde{X}_u^\epsilon)dW_u^j - \varepsilon\nabla_{\tilde{x}}\tilde{V}_j(u,\tilde{X}_u^\epsilon)\tilde{V}_j(u,\tilde{X}_u^\epsilon)du\right\}\right]. \tag{23}
$$

$\nabla_{\tilde{x}}\tilde{X}_s^\epsilon, (\nabla_{\tilde{x}}\tilde{X}_u^\epsilon)^{-1} \in \mathcal{K}_0^T$, then $\frac{\partial}{\partial\varepsilon}\tilde{X}_t^\epsilon \in \mathcal{K}_1^T$.

Also, $\frac{\partial}{\partial\varepsilon}X_t^\epsilon$ can be expressed as

$$
\begin{aligned}
\frac{\partial}{\partial\varepsilon}X_t^\epsilon = \nabla_x X_t^\epsilon \int_0^t (\nabla_x X_u^\epsilon)^{-1}\Bigg[&\nabla_{\tilde{x}}V_0(u,X_u^\epsilon,\tilde{X}_u^\epsilon)\frac{\partial}{\partial\varepsilon}\tilde{X}_u^\epsilon du + \sum_{j=1}^n \nabla_{\tilde{x}}V_j(u,X_u^\epsilon,\tilde{X}_u^\epsilon)\frac{\partial}{\partial\varepsilon}\tilde{X}_u dW_u^j \\
&- \sum_{j=1}^n \nabla_x V_j(u,X_u^\epsilon,\tilde{X}_u^\epsilon)\nabla_{\tilde{x}}V_j(u,X_u^\epsilon,\tilde{X}_u^\epsilon)\frac{\partial}{\partial\varepsilon}\tilde{X}_u^\epsilon du\Bigg].
\end{aligned} \tag{24}
$$

Hence, due to the fact that $\nabla_x X_s^\epsilon, (\nabla_x X_u^\epsilon)^{-1} \in \mathcal{K}_0^T$ and $\frac{\partial}{\partial\varepsilon}\tilde{X}_u^\epsilon \in \mathcal{K}_1^T$, we have $\frac{\partial}{\partial\varepsilon}X_t^\epsilon \in \mathcal{K}_2^T$.

Next, we prove $\frac{1}{i!}\frac{\partial^i}{\partial\varepsilon^i}\tilde{X}_t^\varepsilon \in \mathcal{K}_i^T$ and $\frac{1}{i!}\frac{\partial^i}{\partial\varepsilon^i}X_t^\varepsilon \in \mathcal{K}_{i+1}^T$ by induction. For $i \geq 2$, $\frac{1}{i!}\frac{\partial^i}{\partial\varepsilon^i}\tilde{X}_t^\varepsilon = \left(\frac{1}{i!}\frac{\partial^i}{\partial\varepsilon^i}\tilde{X}_t^{\varepsilon,1}, \cdots, \frac{1}{i!}\frac{\partial^i}{\partial\varepsilon^i}\tilde{X}_t^{\varepsilon,d}\right)$ is recursively determined by the following:

$$
\frac{1}{i!}\frac{\partial^i}{\partial\varepsilon^i}\tilde{X}_t^{\varepsilon,j} = \sum_{\mathbf{i}_\beta,\mathbf{d}^{(\beta)}}^{(i)} \int_0^t \left(\prod_{k=1}^\beta \frac{1}{i_k!}\frac{\partial^{i_k}}{\partial\varepsilon^{i_k}}\tilde{X}_s^{\varepsilon,d_k}\right)\partial_{\mathbf{d}^{(\beta)}}\tilde{V}_0^j(s,\tilde{X}_s^\varepsilon)ds \tag{25}
$$

$$
+\sum_{\mathbf{i}_\beta,\mathbf{d}^{(\beta)}}^{(i-1)} \int_0^t \left(\prod_{k=1}^\beta \frac{1}{i_k!}\frac{\partial^{i_k}}{\partial\varepsilon^{i_k}}\tilde{X}_s^{\varepsilon,d_k}\right)\sum_{l=1}^n \partial_{\mathbf{d}^{(\beta)}}\tilde{V}_l^j(s,\tilde{X}_s^\varepsilon)dW_s^l \tag{26}
$$

$$
+\varepsilon\sum_{\mathbf{i}_\beta,\mathbf{d}^{(\beta)}}^{(i)} \int_0^t \left(\prod_{k=1}^\beta \frac{1}{i_k!}\frac{\partial^{i_k}}{\partial\varepsilon^{i_k}}\tilde{X}_s^{\varepsilon,d_k}\right)\sum_{l=1}^n \partial_{\mathbf{d}^{(\beta)}}\tilde{V}_l^j(s,\tilde{X}_s^\varepsilon)dW_s^l, \tag{27}
$$

where

$$
\sum_{\mathbf{i}_\beta,\mathbf{d}^{(\beta)}}^{(i)} = \sum_{\beta=1}^i \sum_{\mathbf{i}_\beta\in L_{i,\beta}} \sum_{\mathbf{d}^{(\beta)}\in\{1,\cdots,m-d\}^\beta} \frac{1}{\beta!}, \tag{28}
$$

Since the above SDE is linear, the order of the Kusuoka-Stroock function $\frac{1}{i!}\frac{\partial^i}{\partial\varepsilon^i}\tilde{X}_t^\varepsilon$ is determined inductively by the term:

$$
\sum_{\mathbf{i}_\beta,\mathbf{d}^{(\beta)}}^{(i-1)} \int_0^t \nabla_x\tilde{X}_t^\varepsilon\left(\nabla_x\tilde{X}_s^\varepsilon\right)^{-1}\left(\prod_{k=1}^\beta \frac{1}{i_k!}\frac{\partial^{i_k}}{\partial\varepsilon^{i_k}}\tilde{X}_s^{\varepsilon,d_k}\right)\sum_{l=1}^n \partial_{\mathbf{d}^{(\beta)}}\tilde{V}_l(s,\tilde{X}_s^\varepsilon)dW_s^l \in \mathcal{K}_i^T, \tag{29}
$$

because this term gives the minimum order in the terms that consist of $\frac{1}{i!}\frac{\partial^i}{\partial\varepsilon^i}\tilde{X}^\varepsilon_t$. Then, $\frac{1}{i!}\frac{\partial^i}{\partial\varepsilon^i}\tilde{X}^\varepsilon_t \in \mathcal{K}^T_i$.

Also, $\frac{1}{i!}\frac{\partial^i}{\partial\varepsilon^i}X^{\varepsilon,j}_t$ is a solution to the following linear SDE:

$$\frac{1}{i!}\frac{\partial^i}{\partial\varepsilon^i}X^{\varepsilon,j}_t = \sum_{\mathbf{i}_\beta,\mathbf{e}^{(\beta)}}^{(i)}\int_0^t\left(\prod_{k=1}^{\beta}\frac{1}{i_k!}\frac{\partial^{i_k}}{\partial\varepsilon^{i_k}}\bar{X}^{\varepsilon,e_k}_s\right)\partial_{\mathbf{e}^{(\beta)}}V^j_0(s,\bar{X}^\varepsilon_s)ds \tag{30}$$

$$+\sum_{\mathbf{i}_\beta,\mathbf{e}^{(\beta)}}^{(i)}\int_0^t\left(\prod_{k=1}^{\beta}\frac{1}{i_k!}\frac{\partial^{i_k}}{\partial\varepsilon^{i_k}}\bar{X}^{\varepsilon,e_k}_s\right)\sum_{l=1}^{n}\partial_{\mathbf{e}^{(\beta)}}V^j_l(s,\bar{X}^\varepsilon_s)dW^l_s, \tag{31}$$

with

$$\sum_{\mathbf{i}_\beta,\mathbf{e}^{(\beta)}}^{(i)} = \sum_{\beta=1}^{i}\sum_{\mathbf{i}_\beta\in L_{i,\beta}}\sum_{\mathbf{e}^{(\beta)}\in\{1,\cdots,m\}^\beta}\frac{1}{\beta!}. \tag{32}$$

The order of the Kusuoka-Stroock function $X^\varepsilon_{i,t} = \frac{1}{i!}\frac{\partial^i}{\partial\varepsilon^i}X^\varepsilon_t$ is determined inductively by the term:

$$\sum_{\mathbf{i}_\beta,\mathbf{f}^{(\beta)}}^{(i)}\int_0^t\nabla_x X^\varepsilon_t\,(\nabla_x X^\varepsilon_s)^{-1}\left(\prod_{k=1}^{\beta}\frac{1}{i_k!}\frac{\partial^{i_k}}{\partial\varepsilon^{i_k}}\tilde{X}^{\varepsilon,d_k}_s\right)\sum_{l=1}^{n}\frac{\partial^\beta}{\partial\tilde{x}_{f_1}\cdots\partial\tilde{x}_{f_\beta}}V_l(s,\bar{X}^\varepsilon_s)dW^l_s \in \mathcal{K}^T_{i+1}, \tag{33}$$

where

$$\sum_{\mathbf{i}_\beta,\mathbf{f}^{(\beta)}}^{(i)} = \sum_{\beta=1}^{i}\sum_{\mathbf{i}_\beta\in L_{i,\beta}}\sum_{\mathbf{f}^{(\beta)}\in\{d+1,\cdots,m\}^\beta}\frac{1}{\beta!}, \tag{34}$$

since this term gives the minimum order in the terms that consist of $\frac{1}{i!}\frac{\partial^i}{\partial\varepsilon^i}X^\varepsilon_t$. Then, we have $X^\varepsilon_{i,t} \in \mathcal{K}^T_{i+1}$.

1. By Proposition 2.1, we obtain for a smooth function f_n such that f_n converges to a bounded Borel measurable function f,

$$\sum_{\mathbf{b}_k,\alpha^{(k)}}^{(i)} E\left[\partial_{\alpha^{(k)}}f_n(X^\varepsilon_t)\prod_{l=1}^{k}X^{\varepsilon,\alpha_l}_{\beta_l,t}\right] = \sum_{\mathbf{b}_k,\alpha^{(k)}}^{(i)} E\left[f_n(X^\varepsilon_t)H_{\alpha^{(k)}}\left(X^\varepsilon_t,\prod_{l=1}^{k}X^{\varepsilon,\alpha_l}_{\beta_l,t}\right)\right], \tag{35}$$

with

$$\sup_{(x_0,\tilde{x}_0)\in\mathbf{R}^m}\left\|H_{\alpha^{(k)}}\left(X^\varepsilon_t,\prod_{l=1}^{k}X^{\varepsilon,\alpha_l}_{\beta_l,t}\right)\right\|_{L^p} \le Ct^{(i+1+k-k)/2} = Ct^{(i+1)/2}. \tag{36}$$

2. Also, we obtain for a smooth function f_n such that f_n converges to a Lipschitz continuous function f,

$$\sum_{\mathbf{b}_k,\alpha^{(k)}}^{(i)} E\left[f_n(X^\varepsilon_t)H_{\alpha^{(k)}}\left(X^\varepsilon_t,\prod_{l=1}^{k}X^{\varepsilon,\alpha_l}_{\beta_l,t}\right)\right] = \sum_{\mathbf{b}_k,\alpha^{(k)}}^{(i)} E\left[\partial_{\alpha_k}f_n(X^\varepsilon_t)H_{\alpha^{(k-1)}}\left(X^\varepsilon_t,\prod_{l=1}^{k}X^{\varepsilon,\alpha_l}_{\beta_l,t}\right)\right] \tag{37}$$

with

$$\sup_{(x_0,\tilde{x}_0)\in\mathbf{R}^m}\left\|H_{\alpha^{(k-1)}}\left(X_t^\varepsilon,\prod_{l=1}^k X_{\beta_l,t}^{\varepsilon,\alpha_l}\right)\right\|_{L^p}\leq Ct^{(i+1+k-(k-1))/2}=Ct^{(i+2)/2}. \tag{38}$$

Therefore, we obtain the assertion.

□

4 Expansions and Error Estimates for Stochastic Volatility Model

In this section, we show the results for asymptotic expansions around the Black-Scholes model (linear growth coefficient) in stochastic volatility model.

4.1 Model and Preparation

Let $(\Omega,\mathcal{F},P)$ be a probability space where P is a equivalent martingale measure, and $W=\{(W_{1,t},W_{2,t}):t\geq 0\}$ be a 2-dimensional Brownian motion. Let $(\mathcal{F}_t)_{t\in[0,T]}$ be a filtration generated by W, augmented by the P-null sets of $\mathcal{F}$. We consider the following stochastic volatility model $(S_t,\sigma_t)_{0\leq t\leq T}$,

$$\begin{aligned}
dS_t^\varepsilon &= \alpha S_t^\varepsilon dt + a(t,\sigma_t^\varepsilon)S_t^\varepsilon dW_{1,t}, \\
d\sigma_t^\varepsilon &= V_0(t,\sigma_t^\varepsilon)dt + \varepsilon V_1(t,\sigma_t^\varepsilon)(\rho(t)dW_{1,t}+\sqrt{1-\rho(t)^2}dW_{2,t}), \\
S_0 &= S_0>0,\ \sigma_0=\sigma\in(0,\infty),
\end{aligned} \tag{39}$$

where α is a constant, $a:[0,T]\times\mathbf{R}\to(0,\infty)$, $V_0:[0,T]\times\mathbf{R}\to\mathbf{R}$, $V_1:[0,T]\times\mathbf{R}\to\mathbf{R}$, and $\rho:[0,T]\to[-1,1]$. Also, $a(t,x),V_0(t,x)$, and $V_1(t,x)$, $(t,x)\in[0,T]\times\mathbf{R}$ are assumed to be bounded, smooth in x, and all orders of their derivatives are bounded.

Remark 4.1 *An example of stochastic volatility models is the following:*

$$\begin{aligned}
dS_t &= \alpha S_t dt + \sigma_t^\delta S_t dW_{1,t}, \\
d\sigma_t &= b_0(\sigma_t)dt + \varepsilon\sigma_t^\gamma(\rho dW_{1,t}+\sqrt{1-\rho^2}dW_{2,t}), \\
S_0 &= S_0>0,\ v_0=v\in(0,\infty),
\end{aligned} \tag{40}$$

where $\delta,\nu,\gamma>0$, $\rho\in[-1,1]$ and $b_0\in C_b^\infty(\mathbf{R}\to\mathbf{R})$. Lions-Musiela [28] (2007) clarifies the relations between conditions on δ, γ, b_0, ρ and the existence of the p-th order moment of S_t. For example, If $\gamma+\delta<1$, no moment explosion occurs, that is $E[S_t^p]<\infty$ for all $p\geq 1$ and $t\geq 0$. (See Theorem 3.2. in the paper for the detail.)

In general, the diffusion coefficients of S_t and σ_t in (40) do not satisfy the smoothness and bounded conditions required in (39). However, we can still apply the expansion in Malliavin calculus by making use of a smooth and bounded modification technique.

For instance, by applying the technique as in Remark 1 of Takahashi-Yoshida [47] (2004), Section 7 of and Takahashi-Yoshida [48] (2005), Section 4.1 in Takahashi-Yamada [46] (2012) considered the following model with those modifications of σ_t^δ and

σ_t^γ*, which are denoted respectively by a and b:*

$$
\begin{aligned}
d\hat{S}_t^\varepsilon &= \alpha \hat{S}_t^\varepsilon dt + a(\hat{\sigma}_t^\varepsilon)\hat{S}_t^\varepsilon dW_{1,t}, \\
d\hat{\sigma}_t^\varepsilon &= b_0(\hat{\sigma}_t^\varepsilon)dt + \varepsilon b(\hat{\sigma}_t^\varepsilon)(\rho dW_{1,t} + \sqrt{1-\rho^2}dW_{2,t}), \\
\hat{S}_0 &= S_0 > 0, \ \hat{\sigma}_0 = \sigma \in (0,\infty),
\end{aligned} \tag{41}
$$

Then, using a large deviation result reported as Lemma 2 in Takahashi and Yoshida [48] (2005), (which can be proved by slight modification of Lemma 5.3 in Yoshida [49] (1992) or Lemma 7.1 in Kunitomo and Takahashi [22] (2003)), we are able to show

$$
E[|f(S_t^\varepsilon) - f(\hat{S}_t^\varepsilon)|] = o(\varepsilon^n), \ n = 1, 2, \cdots. \tag{42}
$$

Therefore, the difference between $f(S_t^\varepsilon)$ *and* $f(\hat{S}_t^\varepsilon)$ *is negligible in the small disturbance asymptotic theory, and hence our expansion can be applied to (40) through (41).*

We remark that when $\varepsilon = 0$, S_t^0 becomes the Black-Scholes model with a deterministic volatility $\sigma^{BS} = \left(\frac{1}{t}\int_0^t a(s,\sigma_s^0)^2 ds\right)^{1/2}$:

$$
\begin{aligned}
dS_t^{BS} &= \alpha S_t^{BS} dt + a(t,\sigma_t^0) S_t^{BS} dW_{1,t}, \\
d\sigma_t^0 &= V_0(t,\sigma_t^0)dt.
\end{aligned}
$$

Next, note that we have $S_t^\varepsilon \in \mathcal{K}_0^T$ since $S_t^\varepsilon \in \mathbf{D}^\infty$ under our assumptions made for the stochastic volatility model (39), and

$$
\begin{aligned}
\frac{\partial}{\partial S_0} S_t^\varepsilon &= \frac{S_t^\varepsilon}{S_0}, \\
\frac{\partial^n}{\partial S_0^n} S_t^\varepsilon &= 0, \ n \geq 2.
\end{aligned} \tag{43}
$$

Moreover, in order to clarify the orders of expansion errors of the option price and the Greeks, in the next lemma we show the orders of Kusuoka-Stroock functions for the derivatives of σ_t^ε and S_t^ε, $t \geq 0$ with respect to the perturbation parameter ε and the initial value of the volatility σ_0. We remark that this lemma gives basic blocks for the proof of the main theorem in the next subsection.

Lemma 4.1

1. *For* $t \in (0,T]$,

$$
\frac{\partial^i}{\partial \varepsilon^i} \sigma_t^\varepsilon \in \mathcal{K}_i^T, \ i \geq 1. \tag{44}
$$

2. *For* $t \in (0,T]$,

$$
\frac{\partial^i}{\partial \varepsilon^i} S_t^\varepsilon \in \mathcal{K}_{i+1}^T, \ i \geq 1. \tag{45}
$$

3. *For* $t \in (0,T]$,

$$
\frac{\partial^k}{\partial \sigma_0^k} S_t^\varepsilon \in \mathcal{K}_1^T, \ k \geq 1. \tag{46}
$$

4. For $t \in (0,T]$,

$$\frac{\partial^{k+i}}{\partial \sigma_0^k \partial \varepsilon^i} S_t^\varepsilon \in \mathcal{K}_{i+1}^T, \ \ i,k \geq 1. \tag{47}$$

Proof.

1. We make an induction argument with respect to $i \in \mathbf{N}$ for $\frac{\partial^i}{\partial \varepsilon^i}\sigma_t^\varepsilon$.
First, the differentiation of σ_t^ε with respect to ε is given by:

$$\begin{aligned}\frac{\partial}{\partial \varepsilon}\sigma_t^\varepsilon &= \int_0^t V_1(s,\sigma_s^\varepsilon)(\rho(s)dW_{1,s} + \sqrt{1-\rho(s)^2}dW_{2,s}) \\ &+ \int_0^t \partial V_0(s,\sigma_s^\varepsilon)\frac{\partial}{\partial \varepsilon}\sigma_s^\varepsilon ds \\ &+ \varepsilon \int_0^t \partial V_1(s,\sigma_s^\varepsilon)\frac{\partial}{\partial \varepsilon}\sigma_s^\varepsilon(\rho(s)dW_{1,s} + \sqrt{1-\rho(s)^2}dW_{2,s}),\end{aligned} \tag{48}$$

and it can be expressed as

$$\begin{aligned}\frac{\partial}{\partial \varepsilon}\sigma_t^\varepsilon &= \int_0^t \frac{\partial}{\partial \sigma_0}\sigma_t^\varepsilon \left(\frac{\partial}{\partial \sigma_0}\sigma_s^\varepsilon\right)^{-1} V_1(s,\sigma_s^\varepsilon)(\rho(s)dW_{1,s} + \sqrt{1-\rho(s)^2}dW_{2,s}) \\ &- \varepsilon \int_0^t \frac{\partial}{\partial \sigma_0}\sigma_t^\varepsilon \left(\frac{\partial}{\partial \sigma_0}\sigma_s^\varepsilon\right)^{-1} \partial V_1(s,\sigma_s^\varepsilon)V_1(s,\sigma_s^\varepsilon)du.\end{aligned} \tag{49}$$

Since $\frac{\partial}{\partial \sigma_0}\sigma_t^\varepsilon$, $(\frac{\partial}{\partial \sigma_0}\sigma_u^\varepsilon)^{-1} \in \mathcal{K}_0^T$, we have $\frac{\partial}{\partial \varepsilon}\sigma_t^\varepsilon \in \mathcal{K}_1^T$ by (5) and (6) in Lemma 2.1.
For $i \geq 2$, we suppose $\frac{\partial^{i-1}}{\partial \varepsilon^{i-1}}\sigma_t^\varepsilon \in \mathcal{K}_{i-1}^T$. Then, $\frac{\partial^i}{\partial \varepsilon^i}\sigma_t^\varepsilon$ is the solution to the following SDE:

$$\begin{aligned}\frac{\partial^i}{\partial \varepsilon^i}\sigma_t^\varepsilon &= \sum_\beta^{(i)} \frac{i!}{\beta!}\int_0^t \left(\prod_{j=1}^\beta \frac{1}{l_j!}\frac{\partial^{l_j}}{\partial \varepsilon^{l_j}}\sigma_t^\varepsilon\right)\partial^\beta V_0(u,\sigma_u^\varepsilon)du \\ &+ \sum_\beta^{(i-1)} \frac{i!}{\beta!}\int_0^t \left(\prod_{j=1}^\beta \frac{1}{l_j!}\frac{\partial^{l_j}}{\partial \varepsilon^{l_j}}\sigma_t^\varepsilon\right)\partial^\beta V_1(u,\sigma_u^\varepsilon)(\rho(u)dW_{1,u} + \sqrt{1-\rho(u)^2}dW_{2,u}) \\ &+ \varepsilon \sum_\beta^{(i)} \frac{i!}{\beta!}\int_0^t \left(\prod_{j=1}^i \frac{1}{l_j!}\frac{\partial^{l_j}}{\partial \varepsilon^{l_j}}\sigma_t^\varepsilon\right)\partial^\beta V_1(u,\sigma_u^\varepsilon)(\rho(u)dW_{1,u} + \sqrt{1-\rho(u)^2}dW_{2,u}),\end{aligned} \tag{50}$$

where

$$\sum_\beta^{(i)} = \sum_{\beta=1}^i \sum_{l_1+\cdots+l_k=i, l_j\geq 1}, \tag{51}$$

Since above (50) is linear SDE, we are able to represent $\frac{\partial^i}{\partial \varepsilon^i}\sigma_t^\varepsilon$ explicitly. Then, the order of Kusuoka-Stroock function $\frac{\partial^i}{\partial \varepsilon^i}\sigma_t^\varepsilon$ is specified inductively by the term:

$$\begin{aligned}&\sum_\beta^{(i-1)} \frac{i!}{\beta!}\int_0^t \frac{\partial}{\partial \sigma_0}\sigma_t^\varepsilon \left(\frac{\partial}{\partial \sigma_0}\sigma_s^\varepsilon\right)^{-1}\left(\prod_{j=1}^\beta \frac{1}{l_j!}\frac{\partial^{l_j}}{\partial \varepsilon^{l_j}}\sigma_u^\varepsilon\right) \\ &\qquad \partial^\beta V_1(u,\sigma_u^\varepsilon)(\rho(u)dW_{1,u} + \sqrt{1-\rho(u)^2}dW_{2,u}) \ \in \mathcal{K}_i^T,\end{aligned} \tag{52}$$

since this term gives the minimum order in the terms that consist of (50), and then by Lemma 2.1-3.-(b) it determines the order of whole $\frac{\partial^i}{\partial\varepsilon^i}\sigma_t^\varepsilon$. Here, we have used the properties, $\frac{\partial}{\partial\sigma_0}\sigma_t^\varepsilon$, $(\frac{\partial}{\partial\sigma_0}\sigma_u^\varepsilon)^{-1}\in\mathcal{K}_0^T$, $\sum_\beta^{(i-1)}\frac{i!}{\beta!}\prod_{j=1}^\beta\frac{1}{l_j!}\frac{\partial^{l_j}}{\partial\varepsilon^{l_j}}\sigma_u^\varepsilon\in\mathcal{K}_{i-1}^T$, (6) (Lemma 2.1-3.-(a)) and (5) (Lemma 2.1-2.-(a)). Therefore we have $\frac{\partial^i}{\partial\varepsilon^i}\sigma_t^\varepsilon\in\mathcal{K}_i^T$ by (6) (Lemma 2.1-3.-(b)). □

2. The differentiation of S_t^ε with respect to ε is given by:

$$\frac{\partial}{\partial\varepsilon}S_t^\varepsilon = S_t^\varepsilon\Big(\int_0^t \partial a(s,\sigma_s^\varepsilon)\frac{\partial}{\partial\varepsilon}\sigma_s^\varepsilon dW_{1,s} - \int_0^t a(s,\sigma_s^\varepsilon)\partial a(s,\sigma_s^\varepsilon)\frac{\partial}{\partial\varepsilon}\sigma_u^\varepsilon ds\Big).$$

By 1. above, $\frac{\partial}{\partial\varepsilon}\sigma_s^\varepsilon\in\mathcal{K}_1^T$ holds. Then, we have $\frac{\partial}{\partial\varepsilon}S_t^\varepsilon\in\mathcal{K}_2^T$ by (5) and (6).

For $i\geq 2$, the order of Kusuoka-Stroock function $\frac{\partial^i}{\partial\varepsilon^i}S_t^\varepsilon$ is determined inductively by the term:

$$\sum_{k=1}^i\sum_{\beta_1+\cdots+\beta_k=i,\beta_j\geq 1}\frac{i!}{k!}\,S_t^\varepsilon\Big(\int_0^t\partial^k a(s,\sigma_s^\varepsilon)\prod_{l=1}^k\frac{1}{\beta_l!}\frac{\partial^{\beta_l}}{\partial\varepsilon^{\beta_l}}\sigma_s^\varepsilon dW_{1,s}\Big)\ \in\mathcal{K}_{i+1}^T,$$

since this term gives the minimum order in the terms relevant for $\frac{\partial^i}{\partial\varepsilon^i}S_t^\varepsilon$, and Lemma 2.1-3.-($b$). Here, we have used the properties:

$$\begin{aligned} S_t^\varepsilon &\in \mathcal{K}_0^T,\\ \sum_{k=1}^i,\sum_{\beta_1+\cdots+\beta_k=i,\beta_j\geq 1}\frac{i!}{k!}\prod_{l=1}^k\frac{1}{\beta_l!}\frac{\partial^{\beta_l}}{\partial\varepsilon^{\beta_l}}\sigma_u^\varepsilon &\in \mathcal{K}_i^T \text{ (by 1. above)},\end{aligned}$$

(6)(Lemma 2.1-3.-(a)) and (5)(Lemma 2.1-2.-(a)). Therefore, we have $\frac{\partial^i}{\partial\varepsilon^i}S_t^\varepsilon\in\mathcal{K}_{i+1}^T$ by (6)(Lemma 2.1-3.-(b)). □

3. The differentiation of S_t^ε with respect to σ_0 is given by

$$\frac{\partial}{\partial\sigma_0}S_t^\varepsilon = S_t^\varepsilon\Big(\int_0^t \partial a(s,\sigma_s^\varepsilon)\frac{\partial}{\partial\sigma_0}\sigma_s^\varepsilon dW_{1,s} - \int_0^t a(s,\sigma_s^\varepsilon)\partial a(s,\sigma_s^\varepsilon)\frac{\partial}{\partial\sigma_0}\sigma_u^\varepsilon ds\Big),$$

Since $\sigma_t^\varepsilon\in\mathcal{K}_0^T$, we have $\frac{\partial}{\partial\sigma_0}S_t^\varepsilon\in\mathcal{K}_1^T$ by (5) and (6).

For $i\geq 2$, the order of Kusuoka-Stroock function $\frac{\partial^i}{\partial\sigma_0^i}S_t^\varepsilon$ is determined inductively by the term:

$$\sum_{k=1}^i\sum_{\beta_1+\cdots+\beta_k=i,\beta_j\geq 1}\frac{i!}{k!}\,S_t^\varepsilon\Big(\int_0^t\partial^k a(s,\sigma_s^\varepsilon)\prod_{l=1}^k\frac{1}{\beta_l!}\frac{\partial^{\beta_l}}{\partial\sigma_0^{\beta_l}}\sigma_s^\varepsilon dW_{1,s}\Big)\ \in\mathcal{K}_1^T,$$

because this term gives the minimum order in the terms consisting of $\frac{\partial^i}{\partial\sigma_0^i}S_t^\varepsilon$, and Lemma 2.1-3.-($b$). Here, we have used the properties:

$$\begin{aligned} S_t^\varepsilon &\in \mathcal{K}_0^T,\\ \sum_{k=1}^i\sum_{\beta_1+\cdots+\beta_k=i,\beta_j\geq 1}\frac{i!}{k!}\prod_{l=1}^k\frac{1}{\beta_l!}\frac{\partial^{\beta_l}}{\partial\sigma_0^{\beta_l}}\sigma_u^\varepsilon &\in \mathcal{K}_0^T,\end{aligned}$$

(6)(Lemma 2.1-3.-(a)) and (5)(Lemma 2.1-2.-(a)). Therefore we have $\frac{\partial^i}{\partial\sigma_0^i}S_t^\varepsilon\in\mathcal{K}_1^T$ by (6)(Lemma 2.1-3.-(b)). □

4. The order of Kusuoka-Stroock function $\frac{\partial^{i+n}}{\partial\sigma_0^n\partial\varepsilon^i}S_t^\varepsilon$, $(n\geq 1,\ i\geq 1)$ is determined inductively by the term:

$$\sum_{k=1}^{i+n}\sum_{\alpha_1+\cdots+\alpha_k=n,\beta_1+\cdots+\beta_k=i,\alpha_j,\beta_j\geq 0,\alpha_j+\beta_j>0}\frac{n!i!}{k!}$$
$$S_t^\varepsilon\left(\int_0^t\partial^k a(s,\sigma_s^\varepsilon)\prod_{l=1}^k\frac{1}{\alpha_l!}\frac{1}{\beta_l!}\frac{\partial^{\alpha_l+\beta_l}}{\partial\sigma_0^{\alpha_l}\partial\varepsilon^{\beta_l}}\sigma_s^\varepsilon dW_{1,s}\right)\in\mathcal{K}_{i+1}^T,$$

because this term gives the minimum order in the terms consisting of $\frac{\partial^{i+n}}{\partial\sigma_0^n\partial\varepsilon^i}S_t^\varepsilon$, and Lemma 2.1-3.-$(b)$. Here, we have used the properties: $S_t^\varepsilon\in\mathcal{K}_0^T$, $\frac{\partial^i}{\partial\varepsilon^i}\sigma_t^\varepsilon\in\mathcal{K}_i^T$,

$$\sum_{k=1}^{i+n}\sum_{\alpha_1+\cdots+\alpha_k=n,\beta_1+\cdots+\beta_k=i,\alpha_j,\beta_j\geq 0,\alpha_j+\beta_j>0}\frac{n!i!}{k!}\prod_{l=1}^k\frac{1}{\alpha_l!}\frac{1}{\beta_l!}\frac{\partial^{\alpha_l+\beta_l}}{\partial\sigma_0^{\alpha_l}\partial\varepsilon^{\beta_l}}\sigma_s^\varepsilon\in\mathcal{K}_i^T,$$

(6)(Lemma 2.1-3.-(a)) and (5)(Lemma 2.1-2.-(a)). Therefore we have $\frac{\partial^{i+n}}{\partial\sigma_0^n\partial\varepsilon^i}S_t^\varepsilon\in\mathcal{K}_{i+1}^T$ by (6)(Lemma 2.1-3.-(b)). □

4.2 Main Result

For $G\in\mathbf{D}^\infty$, the Malliavin weight H_i is recursively computed as follows:

$$\begin{aligned}H_1(S_t^\varepsilon,G) &= \delta\left(G\gamma^{S_t^\varepsilon}DS_t^\varepsilon\right),\\ H_i(S_t^\varepsilon,G) &= H_1\left(S_t^\varepsilon,H_{i-1}(S_t^\varepsilon,G)\right),\quad i\geq 2.\end{aligned}$$

The following theorem is our main result in this paper which gives sharp error orders with respect to the time parameter $t\in(0,T]$ in expansions.

Theorem 4.1

1. *For a bounded Borel measurable payoff function $f:\mathbf{R}\to\mathbf{R}$, there exists a constant $C_N>0$ depending on N such that*

$$\sup_{S_0>0,\sigma_0>0}\left|E[f(S_t^\varepsilon)]-\left\{E[f(S_t^{BS})]+\sum_{i=1}^N\varepsilon^iE[f(S_t^{BS})\pi_{i,t}]\right\}\right|\leq\|f\|_\infty C_N(\varepsilon\sqrt{t})^{N+1},\quad(53)$$

where the Malliavin weight $\pi_{i,t}\in\mathcal{K}_i^T$, $i\geq 1$ is given as follows:

$$\pi_{i,t}=\sum_{k=1}^i\sum_{\alpha_1+\cdots+\alpha_k=i,\ \alpha_l\geq 1}\frac{1}{k!}H_k\left(S_t^{BS},\prod_{j=1}^k\frac{1}{\alpha_j!}\frac{\partial^{\alpha_j}}{\partial\varepsilon^{\alpha_j}}S_t^\varepsilon|_{\varepsilon=0}\right).$$

2. *For a bounded Borel measurable payoff function $f:\mathbf{R}\to\mathbf{R}$, there exists a constant $C_N>0$ depending on N, $k=1,\cdots,n$ such that*

$$\begin{aligned}&\sup_{S_0>0,\sigma_0>0}\left|\frac{\partial^n}{\partial S_0^n}E[f(S_t^\varepsilon)]-\left\{E[f(S_t^{BS})N_{0,t}^n]+\sum_{i=1}^N\varepsilon^iE[f(S_t^{BS})N_{i,t}^n]\right\}\right|\\ &\leq\|f\|_\infty C_N\varepsilon^{N+1}\sum_{k=1}^n\sqrt{t}^{N+1-k},\end{aligned}\quad(54)$$

where the Malliavin weight $N^n_{i,t} \in \mathcal{K}^T_{i-n}$*,* $i \geq 0$ *is recursively defined as follows:*

$$N^n_{i,t} = \sum_{k=1}^{i+n} \sum_{\alpha_1+\cdots+\alpha_k=i,\beta_1+\cdots+\beta_k=n,\alpha_j,\beta_j\geq 0,\alpha_j+\beta_j>0} \frac{n!}{k!} H_k\left(S^{BS}_t, \prod_{l=1}^{k} \frac{1}{\alpha_l!\beta_l!}\frac{\partial^{\alpha_l+\beta_l}}{\partial S_0^{\beta_l}\partial\varepsilon^{\alpha_l}} S^\varepsilon_t|_{\varepsilon=0}\right).$$

3. *For a bounded Borel measurable payoff function* $f : \mathbf{R} \to \mathbf{R}$*, there exists a constant* $C_N > 0$ *depending on* N *such that*

$$\sup_{S_0>0,\sigma_0>0}\left|\frac{\partial^n}{\partial\sigma_0^n}E[f(S^\varepsilon_t)] - \left\{E[f(S^{BS}_t)\mathcal{V}^n_{0,t}] + \sum_{i=1}^{N}\varepsilon^i E[f(S^{BS}_t)\mathcal{V}^n_{i,t}]\right\}\right| \leq \|f\|_\infty C_N(\varepsilon\sqrt{t})^{N+1}, \quad (55)$$

where the Malliavin weight $\mathcal{V}^n_{i,t} \in \mathcal{K}^T_i$*,* $i \geq 0$ *is recursively defined as follows:*

$$\mathcal{V}^n_{i,t} = \sum_{k=1}^{i+n} \sum_{\alpha_1+\cdots+\alpha_k=i,\beta_1+\cdots+\beta_k=n,\alpha_j,\beta_j\geq 0,\alpha_j+\beta_j>0} \frac{n!}{k!} H_k\left(S^{BS}_t, \prod_{l=1}^{k} \frac{1}{\alpha_l!\beta_l!}\frac{\partial^{\alpha_l+\beta_l}}{\partial \sigma_0^{\beta_l}\partial\varepsilon^{\alpha_l}} S^\varepsilon_t|_{\varepsilon=0}\right).$$

4. *For a Lipschitz payoff function* $f : \mathbf{R} \to \mathbf{R}$ *with a constant* C_f *and* $|f(0)| \leq C_f$*, there exists a constant* $C_N > 0$ *depending on* N *such that*

$$\sup_{S_0>0,\sigma_0>0}\left|E[f(S^\varepsilon_t)] - \left\{E[f(S^{BS}_t)] + \sum_{i=1}^{N}\varepsilon^i E[f(S^{BS}_t)\pi_{i,t}]\right\}\right| \leq C_f C_N\sqrt{t}(\varepsilon\sqrt{t})^{N+1}, \quad (56)$$

with same weight $\pi_{i,t}$*,* $i \geq 1$ *in (53).*

5. *For a Lipschitz payoff function* $f : \mathbf{R} \to \mathbf{R}$ *with a constant* C_f *and* $|f(0)| \leq C_f$*, there exists a constant* $C_N > 0$ *depending on* N *such that*

$$\begin{aligned}&\sup_{S_0>0,\sigma_0>0}\left|\frac{\partial^n}{\partial S_0^n}E[f(S^\varepsilon_t)] - \left\{E[f(S^{BS}_t)N^n_{0,t}] + \sum_{i=1}^{N}\varepsilon^i E[f(S^{BS}_t)N^n_{i,t}]\right\}\right| \\ &\leq C_f C_N \varepsilon^{N+1}\sum_{k=1}^{n}\sqrt{t}^{N+2-k}. \quad (57)\end{aligned}$$

with the same weight $N^n_{i,t}$*,* $i \geq 0$ *in (54),*

6. *For a Lipschitz payoff function* $f : \mathbf{R} \to \mathbf{R}$ *with a constant* C_f *and* $|f(0)| \leq C_f$*, there exists a constant* $C_N > 0$ *depending on* N *such that*

$$\sup_{S_0>0,\sigma_0>0}\left|\frac{\partial^n}{\partial\sigma_0^n}E[f(S^\varepsilon_t)] - \left\{E[f(S^{BS}_t)\mathcal{V}^n_{0,t}] + \sum_{i=1}^{N}\varepsilon^i E[f(S^{BS}_t)\mathcal{V}^n_{i,t}]\right\}\right| \leq C_f C_N\sqrt{t}(\varepsilon\sqrt{t})^{N+1}, \quad (58)$$

with the same weight $\mathcal{V}^n_{i,t}$*,* $i \geq 0$ *in (55).*

Let us make brief comments on the theorem. Firstly, each error bound of the expansion up to the N-th order depends on the $(N+1)$-th order of the diffusion term $(\varepsilon\sqrt{t})^{N+1}$, the bounded or Lipschitz constant, that is $\|f\|_\infty$ or C_f, the time-to-maturity t and some constant C_N which depends on the order of the expansion N.

We also remark that there is the $\sqrt{t}$-order difference in the error bound between the bounded payoff (1.-3.) and the corresponding Lipschitz payoff (4.-6.): the error bound for the Lipschitz payoff is tighter than that for the corresponding bounded payoff at the time close to the maturity. Then, our approximation is regarded as both the small parameter and short-time asympotics.

Moreover, let us look at the error bounds of the expansions for the Greeks when the time-to-maturity is short because one may concerns about their behaviors near the expiry as stated in Introduction (due to the term $(\sqrt{t})^{-k}$ or $(\sqrt{t})^{(1-k)}$, $k = 1, \cdots, n$).

On the other hand, the order n of the Greeks with respect to the initial volatility parameter σ_0 (such as Vega) is irrelevant for the time-to-maturity effect in the error bound. Thus, it is expected that near the expiry the highest-order Greeks with respect to the initial underlying asset price would suffer the worst approximation errors. (To the best of our knowledge, these concrete results for the expansion errors of the Greeks are new.)

Finally, we remark that we can easily extend our expansions to the ones in a more general and higher-dimensional setting.

Remark 4.2 *Benhamou et al. [5] (2010a) obtained a second-order vol-of-vol approximation formula for a European put option price in the time-dependent Heston model. In Theorem 2.4, of Benhamou et al. [5] (2010a), they showed that the error of their approximation formula is $O((\xi_{sup}\sqrt{t})^3\sqrt{t})$, where ξ_{sup} is the supremum of the volatility of volatility with respect to the time $s \in [0, t]$. While our approximation method differs from theirs, the error rate looks similar when the order of our price expansion is $N = 2$ (the second-order) in 4. of Theorem 4.1 above, where the Lipschitz payoff function of a European call or put is taken as f. That is, the order is $O((\varepsilon\sqrt{t})^3\sqrt{t})$. ($\varepsilon$ and t stand for are regarded as the vol-of-vol and the time-to-maturity, respectively in this case.) Moreover, on the contrary to Benhamou et al. [5], [6] (2010a,b), our method is able to automatically clarify the sharp error rates for arbitrary orders of expansions of the option price and Greeks by using Kusuoka-Stroock functions.*

Remark 4.3 *We have a sharp error estimate for the expansion of the option price for no.4 in the theorem as follows:*

Proposition 4.1

$$C(t, K) = C_{AE,N}(t, K) + \varepsilon^{N+1}\sqrt{t}^{N+2}C_{N,q}\{N(d(K)) + \tilde{C}(p)\varepsilon\sqrt{t}\}^{1/p}, \tag{59}$$

where $C(t, K)$ stand for the call price with maturity t and strike price K, and $C_{AE,N}(t, K)$ for its N-th order asymptotic expansion price. $N(d(K))$ denotes the standard normal distribution function evaluated at $d(K) = [\log(S_0/K) - (\sigma^{BS})^2/2]/\sigma^{BS}$ with $\sigma^{BS} = \left(\frac{1}{t}\int_0^t a(s, \sigma_s^0)^2 ds\right)^{1/2}$. $C_{N,q}$ and $\tilde{C}(p)$ are some positive constants with $\frac{1}{p} + \frac{1}{q} = 1$, $p \in (1, \infty)$.

Proof. *Using $\partial_x(x - K)^+ = \mathbf{1}_{\{x \geq K\}}$, we have the following estimate for any expansion order N;*

$$\begin{aligned}
\left|E\left[f(S_t^\varepsilon)\bar{\pi}_{N+1,t}^\varepsilon\right]\right| &= \left|E\left[\mathbf{1}_{\{S_t^\varepsilon \geq K\}}\bar{\eta}_{N+1,t}^\varepsilon\right]\right| \leq E[\mathbf{1}^p_{\{S_t^\varepsilon \geq K\}}]^{1/p}\|\bar{\eta}_{N+1,t}^\varepsilon\|_{L^q} && (60)\\
= \quad & E[\mathbf{1}_{\{S_t^\varepsilon \geq K\}}]^{1/p}\|\bar{\eta}_{N+1,t}^\varepsilon\|_{L^q} \leq \{E[\mathbf{1}_{\{S_t^0 \geq K\}}] + \tilde{C}(p)\varepsilon\sqrt{t}\}^{1/p}\|\bar{\eta}_{N+1,t}^\varepsilon\|_{L^q} && (61)\\
\leq \quad & C_{N,q}\sqrt{t}^{N+2}\{N(d(K)) + \tilde{C}(p)\varepsilon\sqrt{t}\}^{1/p}. && (62)
\end{aligned}$$

□

4.3 Proof of Theorem 4.1

Under the smoothness of the coefficients of the SDE and the assumption $a(t,\sigma) > 0$, the stochastic volatility model S_t^ε, $t \geq 0$ becomes a non-degenerate Wiener functional in the Malliavin sense. (See Theorem 3.1 of Takahashi and Yamada [46] (2012) for the detail.) For a smooth function f, the following Taylor formula holds:

$$\begin{aligned} f(S_t^\varepsilon) &= f(S_t^{BS}) + \sum_{i=1}^{N} \frac{\varepsilon^i}{i!} \frac{\partial^i}{\partial \varepsilon^i} f(S_t^\varepsilon)|_{\varepsilon=0} \\ &\quad + \varepsilon^{N+1} \int_0^1 \frac{(1-u)^N}{N!} \frac{\partial^{N+1}}{\partial \nu^{N+1}} f(S_t^\nu)|_{\nu=\varepsilon u} du. \end{aligned} \tag{63}$$

Each term of differentiation with respect to ε in (63) is calculated as follows:

$$\frac{1}{i!} \frac{\partial^i}{\partial \varepsilon^i} f(S_t^\varepsilon) = \sum_{k=1}^{i} \sum_{\alpha_1 + \cdots + \alpha_k = j,\ \alpha_l \geq 1} \frac{1}{k!}\ \partial^k f(S_t^\varepsilon) \prod_{j=1}^{k} \frac{1}{\alpha_j!} \frac{\partial^{\alpha_j}}{\partial \varepsilon^{\alpha_j}} S_t^\varepsilon, \tag{64}$$

where $\partial^k = \frac{\partial^k}{\partial x^k}$.

Therefore, by 2. of Lemma 4.1 and (6)(Lemma 2.1-3.-(a)), each term in (64) is characterized as a Kusuoka-Stroock function:

$$\begin{aligned} &\left(\frac{\partial}{\partial \varepsilon} S_t^\varepsilon \right)^i \in \mathcal{K}_{2i}^T, \\ &\cdots, \\ &\frac{\partial}{\partial \varepsilon} S_t^\varepsilon \frac{\partial^{i-1}}{\partial \varepsilon^{i-1}} S_t^\varepsilon \in \mathcal{K}_{i+2}^T, \\ &\frac{\partial^i}{\partial \varepsilon^i} S_t^\varepsilon \in \mathcal{K}_{i+1}^T. \end{aligned}$$

1. For a sequence of smooth compactly-supported functions $(f_m)_{m \in \mathbf{N}}$ that converges to f, we have the following calculation by applying Proposition 2.1.

$$\begin{aligned} E\left[\frac{1}{i!} \frac{\partial^i}{\partial \varepsilon^i} f_m(S_t^\varepsilon) \right] &= C_1 E\left[f_m(S_t^\varepsilon) H_i \left(S_t^\varepsilon, \left(\frac{\partial}{\partial \varepsilon} S_t^\varepsilon \right)^i \right) \right] \\ &\quad + \cdots \\ &\quad + C_{i-1} E\left[f_m(S_t^\varepsilon) \cdot H_2 \left(S_t^\varepsilon, \frac{\partial}{\partial \varepsilon} S_t^\varepsilon \frac{\partial^{i-1}}{\partial \varepsilon^{i-1}} S_t^\varepsilon \right) \right] \\ &\quad + C_i E\left[f_m(S_t^\varepsilon) H_1 \left(S_t^\varepsilon, \frac{\partial^i}{\partial \varepsilon^i} S_t^\varepsilon \right) \right] \end{aligned} \tag{65}$$

with

$$\begin{aligned} &H_i \left(S_t^\varepsilon, \left(\frac{\partial}{\partial \varepsilon} S_t^\varepsilon \right)^i \right) \in \mathcal{K}_{2i-i}^T = \mathcal{K}_i^T, \\ &\cdots \\ &H_2 \left(S_t^\varepsilon, \frac{\partial}{\partial \varepsilon} S_t^\varepsilon \frac{\partial^{i-1}}{\partial \varepsilon^{i-1}} S_t^\varepsilon \right) \in \mathcal{K}_{i+2-2}^T = \mathcal{K}_i^T, \\ &H_1 \left(S_t^\varepsilon, \frac{\partial^i}{\partial \varepsilon^i} S_t^\varepsilon \right) \in \mathcal{K}_{i+1-1}^T = \mathcal{K}_i^T. \end{aligned}$$

Here, C_l, $l = 1, \cdots, i$ is some permutation. Then, by Lemma 2.1-3.-(b):

$$\begin{aligned}
\pi^{\varepsilon}_{i,t} &= \sum_{k=1}^{i} \sum_{\alpha_1+\cdots+\alpha_k=i,\ \alpha_l\geq 1} \frac{1}{k!} H_k\left(S^{\varepsilon}_t, \prod_{j=1}^{k} \frac{1}{\alpha_j!} \frac{\partial^{\alpha_j}}{\partial\varepsilon^{\alpha_j}} S^{\varepsilon}_t\right) \in \mathcal{K}^T_i,\ i \geq 1 \quad (66)\\
\bar{\pi}^{\varepsilon}_{N+1,t} &:= (N+1) \sum_{k=1}^{N+1} \sum_{\alpha_1+\cdots+\alpha_k=i,\ \alpha_l\geq 1} \frac{1}{k!} H_k\left(S^{\varepsilon}_t, \prod_{j=1}^{k} \frac{1}{\alpha_j!} \frac{\partial^{\alpha_j}}{\partial\varepsilon^{\alpha_j}} S^{\varepsilon}_t\right) \in \mathcal{K}^T_{N+1}. \\
& \qquad (67)
\end{aligned}$$

Then, as $m \to \infty$, we have

$$\begin{aligned}
|E[f_m(S^{\varepsilon}_t) - f(S^{\varepsilon}_t)]| &\leq E[|f_m(S^{\varepsilon}_t) - f(S^{\varepsilon}_t)|] \leq \|f_m - f\|_{\infty} \to 0 \quad (68)\\
\left|E\left[\{f_m(S^{\varepsilon}_t) - f(S^{\varepsilon}_t)\}\pi^{\varepsilon}_{i,t}\right]\right| &\leq E\left[\left|\{f_m(S^{\varepsilon}_t) - f(S^{\varepsilon}_t)\}\pi^{\varepsilon}_{i,t}\right|\right] \leq \|f_m - f\|_{\infty}\|\pi^{\varepsilon}_{i,t}\|_{L^1} \to 0\\
\left|E\left[\{f_m(S^{\varepsilon}_t) - f(S^{\varepsilon}_t)\}\bar{\pi}^{\varepsilon}_{N+1,t}\right]\right| &\leq E\left[\left|\{f_m(S^{\varepsilon}_t) - f(S^{\varepsilon}_t)\}\bar{\pi}^{\varepsilon}_{N+1,t}\right|\right] \leq \|f_m - f\|_{\infty}\|\bar{\pi}^{\varepsilon}_{N+1,t}\|_{L^1} \to 0.
\end{aligned}$$

Therefore, we obtain a formula:

$$E[f(S^{\varepsilon}_t)] = E[f(S^{BS}_t)] + \sum_{i=1}^{N} \varepsilon^i E[f(S^{BS}_t)\pi_{i,t}] + \varepsilon^{N+1} \int_0^1 (1-u)^N E[f(S^{\varepsilon u}_t)\bar{\pi}^{\varepsilon u}_{N+1,t}]du$$

where $\pi_{i,t} = \pi^0_{i,t} \in \mathcal{K}^T_i$. The residual term is estimated by the following inequality:

$$\left|E[f(S^{\varepsilon}_t)\bar{\pi}^{\varepsilon}_{N+1,t}]\right| \leq \|f\|_{\infty}\|\bar{\pi}^{\varepsilon}_{N+1,t}\|_{L^1}.$$

Since $\bar{\pi}^{\varepsilon}_{N+1,t} \in \mathcal{K}^T_{N+1}$, we have (53).

2. We have

$$\begin{aligned}
&\frac{\partial^n}{\partial S^n_0} E\left[\frac{1}{i!}\frac{\partial^i}{\partial\varepsilon^i} f_m(S^{\varepsilon}_t)\right]\\
&= \sum_{k=1}^{i+n} \sum_{\alpha_1+\cdots+\alpha_k=i,\beta_1+\cdots+\beta_k=n,\alpha_j,\beta_j\geq 0,\alpha_j+\beta_j>0} \frac{n!}{k!} E\left[\partial^k f_m(S^{\varepsilon}_t) \prod_{l=1}^{k} \frac{1}{\alpha_l!\beta_l!} \frac{\partial^{\alpha_l+\beta_l}}{\partial S^{\beta_l}_0 \partial\varepsilon^{\alpha_l}} S^{\varepsilon}_t\right]. \\
& \qquad (69)
\end{aligned}$$

Using the integration by parts on the Wiener space, we get

$$\frac{\partial^n}{\partial S^n_0} E\left[\frac{1}{i!}\frac{\partial^i}{\partial\varepsilon^i} f_m(S^{\varepsilon}_t)\right] = E\left[f_m(S^{\varepsilon}_t) N^{\varepsilon,n}_{i,t}\right], \quad (70)$$

where

$$N^{\varepsilon,n}_{i,t} = \sum_{k=1}^{i+n} \sum_{\alpha_1+\cdots+\alpha_k=i,\beta_1+\cdots+\beta_k=n,\alpha_j,\beta_j\geq 0,\alpha_j+\beta_j>0} \frac{n!}{k!} H_k\left(S^{\varepsilon}_t, \prod_{l=1}^{k} \frac{1}{\alpha_l!\beta_l!} \frac{\partial^{\alpha_l+\beta_l}}{\partial S^{\beta_l}_0 \partial\varepsilon^{\alpha_l}} S^{\varepsilon}_t\right). (71)$$

Since $S^{\varepsilon}_t \in \mathcal{K}^T_0$ and $\frac{\partial^i}{\partial\varepsilon^i} S^{\varepsilon}_t \in \mathcal{K}^T_{i+1}$, $i \geq 1$, we have

$$\begin{aligned}
&H_{i+n}\left(S^{\varepsilon}_t, \left(\frac{\partial}{\partial S_0} S^{\varepsilon}_t\right)^n \left(\frac{\partial}{\partial\varepsilon} S^{\varepsilon}_t\right)^i\right) \in \mathcal{K}^T_{2i-i-n} = \mathcal{K}^T_{i-n}, \quad (72)\\
&\cdots\\
&H_2\left(S^{\varepsilon}_t, \prod_{l=1}^{2} \frac{\partial^{\alpha_l+\beta_l}}{\partial S^{\beta_l}_0 \partial\varepsilon^{\alpha_l}} S^{\varepsilon}_t\right) \in \mathcal{K}^T_{i+1-2} = \mathcal{K}^T_{i-1},\\
&H_1\left(S^{\varepsilon}_t, \frac{\partial^{i+n}}{\partial S^n_0 \partial\varepsilon^i} S^{\varepsilon}_t\right) \in \mathcal{K}^T_{i+1-1} = \mathcal{K}^T_i.
\end{aligned}$$

Then, by 2.1-3.-(b):

$$N_{i,t}^{\varepsilon,n} \in \mathcal{K}^T_{\min\{i-n,\cdots,i-1,i\}} = \mathcal{K}^T_{i-n}. \tag{73}$$

Let $\bar{N}^{\varepsilon,n}_{N+1,t}$ be

$$\begin{aligned} &\bar{N}^{\varepsilon,n}_{N+1,t} \\ = \ & (N+1)\sum_{k=1}^{N+1+n}\sum_{\alpha_1+\cdots+\alpha_k=i,\beta_1+\cdots+\beta_k=n,\alpha_j,\beta_j\geq 0,\alpha_j+\beta_j>0}\frac{n!}{k!}H_k\left(S_t^\varepsilon,\prod_{l=1}^k\frac{1}{\alpha_l!\beta_l!}\frac{\partial^{\alpha_l+\beta_l}}{\partial S_0^{\beta_l}\partial\varepsilon^{\alpha_l}}S_t^\varepsilon\right), \end{aligned} \tag{74}$$

and then $\bar{N}^{\varepsilon,n}_{N+1,t} \in \mathcal{K}^T_{N+1-n}$.
Therefore, we obtain a formula:

$$\begin{aligned} \frac{\partial^n}{\partial S_0^n}E[f(S_t^\varepsilon)] &= \frac{\partial^n}{\partial S_0^n}E[f(S_t^{BS})] + \sum_{i=1}^N\frac{\varepsilon^i}{i!}\frac{\partial^n}{\partial S_0^n}E[\frac{\partial^i}{\partial\varepsilon^i}f(S_t^\varepsilon)]|_{\varepsilon=0} \\ &\quad +\varepsilon^{N+1}\int_0^1\frac{(1-u)^N}{N!}\frac{\partial^n}{\partial S_0^n}E\left[\frac{\partial^{N+1}}{\partial\nu^{N+1}}f(S_t^\nu)\right]|_{\nu=\varepsilon u}du \\ &= E[f(S_t^{BS})N_{0,t}^n] + \sum_{i=1}^N\varepsilon^iE[f(S_t^{BS})N_{i,t}^n] \\ &\quad +\varepsilon^{N+1}\int_0^1(1-u)^NE[f(S_t^{\varepsilon u})\bar{N}^{\varepsilon u,n}_{N+1,t}]du, \end{aligned} \tag{75}$$

where $N_{i,t}^n = N_{i,t}^{0,n} \in \mathcal{K}^T_{i-n}$. The residual term is estimated by the following inequality:

$$\left|E[f(S_t^\varepsilon)\bar{N}^{\varepsilon,n}_{N+1,t}]\right| \leq \|f\|_\infty\|\bar{N}^{\varepsilon,n}_{N+1,t}\|_{L^1}. \tag{76}$$

Since, $\bar{N}^{\varepsilon,n}_{N+1,t} \in \mathcal{K}^T_{N+1-n}$, we have (54).

3. We have

$$\begin{aligned} &\frac{\partial^n}{\partial\sigma_0^n}E\left[\frac{1}{i!}\frac{\partial^i}{\partial\varepsilon^i}f_m(S_t^\varepsilon)\right] \\ = \ &\sum_{k=1}^{i+n}\sum_{\alpha_1+\cdots+\alpha_k=i,\beta_1+\cdots+\beta_k=n,\alpha_j,\beta_j\geq 0,\alpha_j+\beta_j>0}\frac{n!}{k!}\ E\left[\partial^kf_m(S_t^\varepsilon)\prod_{l=1}^k\frac{1}{\alpha_l!\beta_l!}\frac{\partial^{\alpha_l+\beta_l}}{\partial\sigma_0^{\beta_l}\partial\varepsilon^{\alpha_l}}S_t^\varepsilon\right]. \end{aligned} \tag{77}$$

Using the integration by parts on the Wiener space for the Vega's direction, we get

$$\frac{\partial^n}{\partial\sigma_0^n}E\left[\frac{1}{i!}\frac{\partial^i}{\partial\varepsilon^i}f_m(S_t^\varepsilon)\right] = E\left[f_m(S_t^\varepsilon)\mathcal{V}_{i,t}^{\varepsilon,n}\right] \tag{78}$$

where

$$\mathcal{V}_{i,t}^{\varepsilon,n} = \sum_{k=1}^{i+n}\sum_{\alpha_1+\cdots+\alpha_k=i,\beta_1+\cdots+\beta_k=n,\alpha_j,\beta_j\geq 0,\alpha_j+\beta_j>0}\frac{n!}{k!}H_k\left(S_t^\varepsilon,\prod_{l=1}^k\frac{1}{\alpha_l!\beta_l!}\frac{\partial^{\alpha_l+\beta_l}}{\partial\sigma_0^{\beta_l}\partial\varepsilon^{\alpha_l}}S_t^\varepsilon\right), \tag{79}$$

and by 3. and 4. of Lemma 4.1 and Proposition 2.1 we have

$$
\begin{aligned}
&H_{i+n}\left(S_t^\varepsilon, \left(\frac{\partial}{\partial\sigma_0}S_t^\varepsilon\right)^n\left(\frac{\partial}{\partial\varepsilon}S_t^\varepsilon\right)^i\right) \in \mathcal{K}^T_{2i+n-i-n} = \mathcal{K}^T_i, && (80)\\
&\cdots\\
&H_2\left(S_t^\varepsilon, \prod_{l=1}^{2}\frac{\partial^{\alpha_l+\beta_l}}{\partial\sigma_0^{\beta_l}\partial\varepsilon^{\alpha_l}}S_t^\varepsilon\right) \in \mathcal{K}^T_{i+2-2} = \mathcal{K}^T_i,\\
&H_1\left(S_t^\varepsilon, \frac{\partial^{i+n}}{\partial\sigma_0^n\partial\varepsilon^i}S_t^\varepsilon\right) \in \mathcal{K}^T_{i+1-1} = \mathcal{K}^T_i.
\end{aligned}
$$

Then, by 2.1-3.-(b):

$$\mathcal{V}^{\varepsilon,n}_{i,t} \in \mathcal{K}^T_i.$$

Let $\bar{\mathcal{V}}^{\varepsilon,n}_{N+1,t}$ be

$$\bar{\mathcal{V}}^{\varepsilon,n}_{N+1,t} = (N+1)\sum_{k=1}^{N+1+n}\sum_{\alpha_1+\cdots+\alpha_k=i,\beta_1+\cdots+\beta_k=n,\alpha_j,\beta_j\geq 0,\alpha_j+\beta_j>0}\frac{n!}{k!}H_k\left(S_t^\varepsilon, \prod_{l=1}^{k}\frac{1}{\alpha_l!\beta_l!}\frac{\partial^{\alpha_l+\beta_l}}{\partial\sigma_0^{\beta_l}\partial\varepsilon^{\alpha_l}}S_t^\varepsilon\right) \quad (81)$$

and then $\bar{\mathcal{V}}^{\varepsilon,n}_{N+1,t} \in \mathcal{K}^T_{N+1}$.
Therefore, we obtain a formula:

$$
\begin{aligned}
\frac{\partial^n}{\partial\sigma_0^n}E[f(S_t^\varepsilon)] &= \frac{\partial^n}{\partial\sigma_0^n}E[f(S_t^{BS})] + \sum_{i=1}^{N}\frac{\varepsilon^i}{i!}\frac{\partial^n}{\partial\sigma_0^n}E[\frac{\partial^i}{\partial\varepsilon^i}f(S_t^\varepsilon)]|_{\varepsilon=0}\\
&\quad +\varepsilon^{N+1}\int_0^1\frac{(1-u)^n}{N!}\frac{\partial^n}{\partial\sigma_0^n}E\left[\frac{\partial^{N+1}}{\partial\nu^{N+1}}f(S_t^\nu)\right]|_{\nu=\varepsilon u}du\\
&= E[f(S_t^{BS})\mathcal{V}^n_{0,t}] + \sum_{i=1}^{N}\varepsilon^i E[f(S_t^{BS})\mathcal{V}^n_{i,t}]\\
&\quad +\varepsilon^{N+1}\int_0^1(1-u)^N E[f(S_t^{\varepsilon u})\bar{\mathcal{V}}^{\varepsilon u,n}_{N+1,t}]du, \qquad (82)
\end{aligned}
$$

where $\mathcal{V}^n_{i,t} = \mathcal{V}^{0,n}_{i,t} \in \mathcal{K}^T_i$. The residual term is estimated by the following inequality:

$$\left|E[f(S_t^\varepsilon)\bar{\mathcal{V}}^{\varepsilon,n}_{N+1,t}]\right| \leq \|f\|_\infty\|\bar{\mathcal{V}}^{\varepsilon,n}_{N+1,t}\|_{L^1}. \qquad (83)$$

Since, $\bar{\mathcal{V}}^{\varepsilon,n}_{N+1,t} \in \mathcal{K}^T_{N+1}$, we have (55).

4. Take a smooth mollifier $(f_m)_{m\in\mathbf{N}}$ converging to the Lipschitz function f such that the first derivative of f_m is uniformly bounded. Then, we have

$$E\left[\frac{\partial}{\partial\varepsilon}f_m(S_t^\varepsilon)\right] = E\left[\partial f_m(S_t^\varepsilon)\frac{\partial}{\partial\varepsilon}S_t^\varepsilon\right], \qquad (84)$$

and for $i \geq 2$,

$$
\begin{aligned}
E\left[\frac{\partial^i}{\partial\varepsilon^i}f_m(S_t^\varepsilon)\right] &= C_1\, E\left[\partial f_m(S_t^\varepsilon)H_{i-1}\left(S_t^\varepsilon, \left(\frac{\partial}{\partial\varepsilon}S_t^\varepsilon\right)^i\right)\right] \qquad (85)\\
&\quad +\cdots
\end{aligned}
$$

$$
\begin{aligned}
&+C_{i-1}\ E\left[\partial f_m(S_t^\varepsilon) H_1\left(S_t^\varepsilon, \frac{\partial}{\partial\varepsilon}S_t^\varepsilon \frac{\partial^{i-1}}{\partial\varepsilon^{i-1}}S_t^\varepsilon\right)\right]\\
&+C_i\ E\left[\partial f_m(S_t^\varepsilon)\frac{\partial^i}{\partial\varepsilon^i}S_t^\varepsilon\right]\\
=\ & E\left[f_m(S_t^\varepsilon)\pi_{i,t}^\varepsilon\right], \qquad (86)
\end{aligned}
$$

with

$$
\begin{aligned}
&H_{i-1}\left(S_t^\varepsilon, \left(\frac{\partial}{\partial\varepsilon}S_t^\varepsilon\right)^i\right) \in \mathcal{K}^T_{2i-(i-1)} = \mathcal{K}^T_{i+1}, \qquad (87)\\
&\cdots\\
&H_1\left(S_t^\varepsilon, \frac{\partial}{\partial\varepsilon}S_t^\varepsilon \frac{\partial^{i-1}}{\partial\varepsilon^{i-1}}S_t^\varepsilon\right) \in \mathcal{K}^T_{2+i-1} = \mathcal{K}^T_{i+1},\\
&\frac{\partial^i}{\partial\varepsilon^i}S_t^\varepsilon \in \mathcal{K}^T_{i+1}.
\end{aligned}
$$

Let $\eta_{1,t}^\varepsilon = \frac{\partial}{\partial\varepsilon}S_t^\varepsilon$ and for $i \geq 2$,

$$
\begin{aligned}
\eta_{i,t}^\varepsilon\ =\ & C_1\ H_{i-1}\left(S_t^\varepsilon, \left(\frac{\partial}{\partial\varepsilon}S_t^\varepsilon\right)^i\right) \qquad (88)\\
&+\cdots\\
&+C_{i-1}\ H_1\left(S_t^\varepsilon, \frac{\partial}{\partial\varepsilon}S_t^\varepsilon \frac{\partial^{i-1}}{\partial\varepsilon^{i-1}}S_t^\varepsilon\right)\\
&+C_i\ \frac{\partial^i}{\partial\varepsilon^i}S_t^\varepsilon.
\end{aligned}
$$

Then, $\eta_{i,t}^\varepsilon \in \mathcal{K}^T_{i+1}$, $i \geq 1$, and we have the following relation:

$$
E\left[f_m(S_t^\varepsilon)\pi_{i,t}^\varepsilon\right] = E\left[\partial f_m(S_t^\varepsilon)\eta_{i,t}^\varepsilon\right],\ i \geq 1. \qquad (89)
$$

Also, let $\bar{\eta}_{N+1,t}^\varepsilon$ be $\bar{\eta}_{N+1,t}^\varepsilon = (N+1)\eta_{N+1,t}^\varepsilon \in \mathcal{K}^T_{N+2}$ and then

$$
E\left[f_m(S_t^\varepsilon)\bar{\pi}_{N+1,t}^\varepsilon\right] = E\left[\partial f_m(S_t^\varepsilon)\bar{\eta}_{N+1,t}^\varepsilon\right]. \qquad (90)
$$

Here, note that for some constant L,

$$
\left|E\left[f_m(S_t^\varepsilon)\bar{\pi}_{N+1,t}^\varepsilon\right]\right| = \left|E\left[\partial f_m(S_t^\varepsilon)\bar{\eta}_{N+1,t}^\varepsilon\right]\right| \leq \|\partial f_m\|_\infty \|\bar{\eta}_{N+1,t}^\varepsilon\|_{L^1} \leq L. \qquad (91)
$$

Moreover, as $m \to \infty$, we have

$$
\begin{aligned}
\left|E\left[f_m(S_t^\varepsilon) - f(S_t^\varepsilon)\right]\right| &\leq E\left[|f_m(S_t^\varepsilon) - f(S_t^\varepsilon)|\right] \leq \|f_m - f\|_\infty \to 0 \qquad (92)\\
\left|E\left[\{f_m(S_t^\varepsilon) - f(S_t^\varepsilon)\}\pi_{i,t}^\varepsilon\right]\right| &\leq E\left[\left|\{f_m(S_t^\varepsilon) - f(S_t^\varepsilon)\}\pi_{i,t}^\varepsilon\right|\right] \leq \|f_m - f\|_\infty \|\pi_{i,t}^\varepsilon\|_{L^1} \to 0\\
\left|E\left[\{f_m(S_t^\varepsilon) - f(S_t^\varepsilon)\}\bar{\pi}_{N+1,t}^\varepsilon\right]\right| &\leq E\left[\left|\{f_m(S_t^\varepsilon) - f(S_t^\varepsilon)\}\bar{\pi}_{N+1,t}^\varepsilon\right|\right] \leq \|f_m - f\|_\infty \|\bar{\pi}_{N+1,t}^\varepsilon\|_{L^1} \to 0.
\end{aligned}
$$

Therefore, we obtain a formula:

$$
\begin{aligned}
E[f(S_t^\varepsilon)]\ =\ & E[f(S_t^{BS})] + \sum_{i=1}^N \varepsilon^i E[f(S_t^{BS})\pi_{i,t}]\\
&+\varepsilon^{N+1}\int_0^1 (1-u)^N E[f(S_t^{\varepsilon u})\bar{\pi}_{N+1,t}^{\varepsilon u}]du.
\end{aligned}
$$

Here, its residual term is estimated by (91) as follows:

$$\left| E[f(S_t^{SV,\varepsilon})\bar{\pi}_{N+1,t}^{\varepsilon}] \right| \le C_f \|\bar{\eta}_{N+1,t}^{\varepsilon}\|_{L^1}. \tag{93}$$

Finally, since $\bar{\eta}_{N+1,t}^{\varepsilon} \in \mathcal{K}_{N+2}^T$, we have (56).

5. Following the similar manner as in the proof of 4. with the one of 2. above, we have for some $\zeta_{i,t}^{\varepsilon} \in \mathcal{K}_{i-(n-1)}^T$, $i, n \ge 1$

$$E\left[f_m(S_t^{\varepsilon})N_{i,t}^{\varepsilon,n}\right] = E\left[\partial f_m(S_t^{\varepsilon})\zeta_{i,t}^{\varepsilon}\right]$$

and

$$E\left[f_m(S_t^{\varepsilon})\bar{N}_{N+1,t}^{\varepsilon,n}\right] = E\left[\partial f_m(S_t^{\varepsilon})\bar{\zeta}_{N+1,t}^{\varepsilon}\right]$$

where $\bar{\zeta}_{N+1,t}^{\varepsilon} = (N+1)\zeta_{N+1,t}^{\varepsilon} \in \mathcal{K}_{N+1-(n-1)}^T = \mathcal{K}_{N+2-n}^T$.
Therefore, we obtain a formula:

$$\begin{aligned}
\frac{\partial^n}{\partial S_0^n} E[f(S_t^{\varepsilon})] &= E[f(S_t^{BS})N_{0,t}^n] + \sum_{i=1}^N \varepsilon^i E[f(S_t^{BS})N_{i,t}^n] \\
&\quad + \varepsilon^{N+1} \int_0^1 (1-u)^N E[f(S_t^{\varepsilon u})\bar{N}_{N+1,t}^{\varepsilon u,n}]du
\end{aligned} \tag{94}$$

with its residual term estimate:

$$\left| E[f(S_t^{\varepsilon})\bar{N}_{N+1,t}^{\varepsilon}] \right| \le C_f \|\bar{\zeta}_{N+1,t}^{\varepsilon}\|_{L^1}. \tag{95}$$

Since $\bar{\zeta}_{N+1,t}^{\varepsilon} \in \mathcal{K}_{N+2-n}^T$, we have (57).

6. Following the similar manner as in the proof of 4. with the one of 3. above, we have for some $\xi_{i,t}^{\varepsilon} \in \mathcal{K}_{i+1}^T$, $i, n \ge 1$,

$$E\left[f_m(S_t^{\varepsilon})\mathcal{V}_{i,t}^{\varepsilon,n}\right] = E\left[\partial f_m(S_t^{\varepsilon})\xi_{i,t}^{\varepsilon}\right].$$

and

$$E\left[f_m(S_t^{\varepsilon})\bar{\mathcal{V}}_{N+1,t}^{\varepsilon,n}\right] = E\left[\partial f_m(S_t^{\varepsilon})\bar{\xi}_{N+1,t}^{\varepsilon}\right]$$

where $\bar{\xi}_{N+1,t}^{\varepsilon} = (N+1)\xi_{N+1,t}^{\varepsilon} \in \mathcal{K}_{N+2}^T$.
Therefore, we obtain a formula:

$$\begin{aligned}
\frac{\partial^n}{\partial \sigma_0^n} E[f(S_t^{\varepsilon})] &= E[f(S_t^{BS})\mathcal{V}_{0,t}^n] + \sum_{i=1}^N \varepsilon^i E[f(S_t^{BS})\mathcal{V}_{i,t}^n] \\
&\quad + \varepsilon^{N+1} \int_0^1 (1-u)^N E[f(S_t^{\varepsilon u})\bar{\mathcal{V}}_{N+1,t}^{\varepsilon u,n}]du
\end{aligned} \tag{96}$$

with its residual term estimate:

$$\left| E[f(S_t^{\varepsilon})\bar{\mathcal{V}}_{N+1,t}^{\varepsilon}] \right| \le C_f \|\bar{\xi}_{N+1,t}^{\varepsilon}\|_{L^1}. \tag{97}$$

Since $\bar{\xi}_{N+1,t}^{\varepsilon} \in \mathcal{K}_{N+2}^T$, we have (58). □

5 Formulas

This section briefly shows how to compute option prices and their Greeks based on our method.

An example of a bounded Borel (non-smooth and non-continuous) payoff is the digital call function, $f(x) = \mathbf{1}_{\{x \geq K\}}$. Specifically, let us consider a digital call option on a foreign exchange rate. In this case,

$$e^{-r_d t} E[f(S_t^{BS})] = e^{-r_d t} N(d), \tag{98}$$

where

$$N(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy, \tag{99}$$

and

$$d = \frac{\log(S_0/K) + (r_d - r_f)t - (\sigma^{BS})^2/2t}{\sigma^{BS}\sqrt{t}}. \tag{100}$$

Here, r_d and r_f stand for domestic and foreign interest rates, respectively; S_0 denotes the time-0 spot exchange rate, that is the unit price of the foreign currency in terms of the domestic currency, and S_t^{BS} stands the time-t spot exchange rate under the Black-Scholes model.

Also, an example of the Lipschitz continuous payoff is the European call function $f(x) = (x - K)^+$. Again, for a call option on a foreign exchange rate, it is well-known under the Black-Scholes model that:

$$e^{-r_d t} E[f(S_t^{BS})] = e^{-r_f t} S_0 N(d_1) - e^{-r_d t} K N(d_2), \tag{101}$$

where

$$\begin{aligned} d_1 &= \frac{\log(S_0/K) + (r_d - r_f)t + (\sigma^{BS})^2/2t}{\sigma^{BS}\sqrt{t}}, \\ d_2 &= \frac{\log(S_0/K) + (r_d - r_f)t - (\sigma^{BS})^2/2t}{\sigma^{BS}\sqrt{t}}. \end{aligned} \tag{102}$$

In both the bounded Borel and Lipschitz cases, the Malliavin weights $\pi_{i,t}$, $i = 1, 2, \cdots$ and $N_{i,t}^n$, $n = 1, 2, \cdots$, $i = 0, 1, 2, \cdots$ are the same. In particular, $\pi_{1,t}$, $N_{0,t}^n$ and $N_{1,t}^n$, $n = 1, 2$ are given as below.

1. The Malliavin weight for the first order approximation of the option price is obtained as follows:

$$\begin{aligned} \pi_{1,t} &= \frac{\partial}{\partial \epsilon} S_t^{\varepsilon}|_{\varepsilon=0} \int_0^t \frac{D_{s,1} S_t^{BS}}{\int_0^t (D_{u,1} S_t^{BS})^2 du} dW_{1,s} \\ &\quad - \int_0^t D_{s,1} \frac{\partial}{\partial \varepsilon} S_t^{\varepsilon}|_{\varepsilon=0} \frac{D_{s,1} S_t^{BS}}{\int_0^t (D_{u,1} S_t^{BS})^2 du} ds, \end{aligned}$$

where $D_{s,1}$, $s \leq t$, is the Malliavin derivative with respect to the Brownian motion $W_{1,t}$.

Therefore, we get the approximation formula:

$$\begin{aligned} e^{-r_d t}E[f(S_t^\varepsilon)] &\simeq e^{-r_d t}\int_{\mathbf{R}} f(y)p^{BS}(t,S_0,y)dy \\ &\quad +\varepsilon e^{-r_d t}\int_{\mathbf{R}} f(y)E[\pi_{1,t}|S_t^{BS}=y]p^{BS}(t,S_0,y)dy. \end{aligned}$$

See Appendix A of Takahashi and Yamada [46] (2012) for more details of the computation.

2. The Malliavin weights for the Delta, Gamma and Vega in the Black-Scholes model are given respectively as follows:

$$N_{0,t}^1 = \frac{W_{1,t}}{S_0\sigma^{BS}t}, \tag{103}$$

$$N_{0,t}^2 = \frac{1}{S_0\sigma^{BS}t}\left(\frac{W_{1,t}^2}{\sigma^{BS}t} - \frac{1}{\sigma^{BS}} - W_{1,t}\right), \tag{104}$$

$$\mathcal{V}_{0,t}^1 = \left(\frac{W_{1,t}^2}{\sigma^{BS}t} - \frac{1}{\sigma^{BS}} - W_{1,t}\right). \tag{105}$$

See pp. 332-333 of Nualart [32] (2006) for the details of the derivation.

3. The Malliavin weights for the first order approximation of the Delta, Gamma and Vega are obtained as follows:

$$\begin{aligned} N_{1,t}^1 &= H_2\left(S_t^{BS}, \frac{\partial}{\partial S_0}S_t^{BS}\frac{\partial}{\partial\varepsilon}S_t^\varepsilon|_{\varepsilon=0}\right) \\ &\quad +H_1\left(S_t^{BS}, \frac{\partial^2}{\partial S_0\partial\varepsilon}S_t^\varepsilon|_{\varepsilon=0}\right), \\ N_{1,t}^2 &= H_3\left(S_t^{BS}, \left(\frac{\partial}{\partial S_0}S_t^{BS}\right)^2\frac{\partial}{\partial\varepsilon}S_t^\varepsilon|_{\varepsilon=0}\right) \\ &\quad +H_2\left(S_t^{BS}, \frac{\partial^2}{\partial S_0^2}S_t^{BS}\frac{\partial}{\partial\varepsilon}S_t^\varepsilon|_{\varepsilon=0}\right) \\ &\quad +2H_2\left(S_t^{BS}, \frac{\partial}{\partial S_0}S_t^{BS}\frac{\partial^2}{\partial S_0\partial\varepsilon}S_t^\varepsilon|_{\varepsilon=0}\right) \\ &\quad +H_1\left(S_t^{BS}, \frac{\partial^3}{\partial S_0^2\partial\varepsilon}S_t^\varepsilon|_{\varepsilon=0}\right), \\ \mathcal{V}_{1,t}^1 &= H_2\left(S_t^{BS}, \frac{\partial}{\partial\sigma_0}S_t^{BS}\frac{\partial}{\partial\varepsilon}S_t^\varepsilon|_{\varepsilon=0}\right) \\ &\quad +H_1\left(S_t^{BS}, \frac{\partial^2}{\partial\sigma_0\partial\varepsilon}S_t^\varepsilon|_{\varepsilon=0}\right). \end{aligned}$$

4. Applying the technique of Theorem 3.1 and Appendix A in Takahashi and Yamada [46] (2012), we are able to obtain the formulas as the following (closed) forms:

$$\begin{aligned} E[f(S_t^{BS})\pi_{i,t}] &= \int_{\mathbf{R}} f(y)E[\pi_{i,t}|S_t^{BS}=y]p^{BS}(t,S_0,y)dy, \\ &\quad i=1,2,\cdots \end{aligned}$$

$$
\begin{aligned}
E[f(S_t^{BS})N_{i,t}^n] &= \int_{\mathbf{R}} f(y)E[N_{i,t}^n|S_t^{BS}=y]p^{BS}(t,S_0,y)dy \\
&= \int_{\mathbf{R}} f(y)\frac{\partial^n}{\partial S_0^n}\{E[\pi_{i,t}|S_t^{BS}=y]p^{BS}(t,S_0,y)\}dy, \\
& \quad i=0,1,2,\cdots,\ n=1,2,\cdots, \\
E[f(S_t^{BS})\mathcal{V}_{i,t}^n] &= \int_{\mathbf{R}} f(y)E[\mathcal{V}_{i,t}^n|S_t^{BS}=y]p^{BS}(t,S_0,y)dy \\
&= \int_{\mathbf{R}} f(y)\frac{\partial^n}{\partial \sigma_0^n}\{E[\pi_{i,t}|S_t^{BS}=y]p^{BS}(t,S_0,y)\}dy, \\
& \quad i=0,1,2,\cdots,\ n=1,2,\cdots,
\end{aligned}
$$

where $y \mapsto p^{BS}(t,S_0,y)$ of stands for the log-normal density S_t^{BS} under the Black-Scholes model. Therefore, each approximation term can be easily obtained based on our method developed in the previous section. We finally remark that the conditional expectations appearing in the above equations can be easily evaluated by the same technique as in Appendix A of Takahashi and Yamada [46] (2012).

6 Numerical Example

This section considers a stochastic volatility model of the Heston [18] (1993) as a numerical experiment.

$$
\begin{aligned}
dX_t^\varepsilon &= -\frac{1}{2}v_t^\varepsilon dt + \sqrt{v_t^\varepsilon}dW_t^1, \quad X_0^\varepsilon = x_0, \\
dv_t^\varepsilon &= \kappa(\theta - v_t^\varepsilon)dt + \varepsilon\sqrt{v_t^\varepsilon}(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2), \quad v_0^\varepsilon = v_0,
\end{aligned} \tag{106}
$$

where we set $X_t^\varepsilon = \log S_t^\varepsilon$ and $x_0 = \log S_0$.

Applying the method developed in the previous sections, we obtain an approximation of a call option price with the strike K as follows:

$$
\begin{aligned}
C(t,K) &= E[(e^{X_t^\varepsilon}-K)^+] \\
&\simeq \int_{\mathbf{R}}(e^y-K)^+p^{logBS}(t,x_0,y)dy \\
&\quad +\varepsilon\int_{\mathbf{R}}(e^y-K)^+\vartheta_1(y)p^{logBS}(t,x_0,y)dy \\
&\quad +\varepsilon^2\int_{\mathbf{R}}(e^y-K)^+\vartheta_2(y)p^{logBS}(t,x_0,y)dy,
\end{aligned} \tag{107}
$$

where $y \mapsto p^{logBS}(t,x_0,y)$ is the density of logarithm of the Black-Scholes model, that is

$$
p^{logBS}(t,x_0,y) = \frac{1}{\sqrt{2\pi\Sigma}}e^{-\frac{1}{2\Sigma}(y-\mu)^2}, \tag{108}
$$

with

$$
\mu = x_0 - \frac{1}{2}\Sigma \text{ and } \Sigma = \theta t + (v_0-\theta)(1-e^{\kappa t})/\kappa. \tag{109}
$$

Here, the Malliavin weights are given by

$$
\vartheta_1(y) = E\left[H_{(1)}\left(X_t^0, \frac{\partial}{\partial\varepsilon}X_t^\varepsilon|_{\varepsilon=0}\right) \mid \int_0^t \sqrt{v_s^0}dW_s = y-\mu\right], \tag{110}
$$

$$\vartheta_2(y) = E\left[H_{(1,1)}\left(X_t^0, \frac{1}{2}\left(\frac{\partial}{\partial\varepsilon}X_t^\varepsilon|_{\varepsilon=0}\right)^2\right) | \int_0^t \sqrt{v_s^0}dW_s = y-\mu\right] \tag{111}$$
$$+E\left[H_{(1)}\left(X_t^0, \frac{1}{2}\frac{\partial^2}{\partial\varepsilon^2}X_t^\varepsilon|_{\varepsilon=0}\right) | \int_0^t \sqrt{v_s^0}dW_s = y-\mu\right].$$

More concretely, we have the following expressions of the weights:

$$\vartheta_1(y) = C_1(t, v_0, \kappa, \theta, \rho)\left[H_3(y) - H_2(y)\right], \tag{112}$$
$$\vartheta_2(y) = C_{21}(t, v_0, \kappa, \theta, \rho)\left[H_6(y) - 2H_5(y) + H_4(y)\right] \tag{113}$$
$$+C_{22}(t, v_0, \kappa, \theta, \rho)\left[H_4(y) - 2H_3(y) + H_2(y)\right] \tag{114}$$
$$+C_{23}(t, v_0, \kappa, \theta, \rho)\left[H_4(y) - H_3(y)\right], \tag{115}$$

where

$$C_1(t, v_0, \kappa, \theta, \rho) = \frac{\rho}{2\kappa^2}\, e^{-\kappa t}\left\{\theta - (v_0-\theta) - (v_0-\theta)\kappa t + e^{\kappa t}(v_0 - \theta + \theta(-1+\kappa t))\right\}, \tag{116}$$
$$C_{21}(t, v_0, \kappa, \theta, \rho) = \frac{1}{8}\rho^2\frac{1}{\kappa^4}(e^{-\kappa t}\{(-1+e^{\kappa t}-\kappa t)v_0 + (2+\kappa t + e^{\kappa t}(-2+\kappa t))\theta\})^2, \tag{117}$$
$$C_{22}(t, v_0, \kappa, \theta, \rho) = \frac{1}{16}\frac{1}{\kappa^3}(e^{-2\kappa t}(-2v_0 + \theta + e^{2\kappa t}(2v_0 + (-5+2\kappa t)\theta) + 4e^{\kappa t}(\theta + \kappa t(-v_0+\theta)))), \tag{118}$$
$$C_{23}(t, v_0, \kappa, \theta, \rho) = \frac{1}{2}\rho^2\frac{1}{2\kappa^3}\left(e^{-\kappa t}\{(-2+2e^{\kappa t} - \kappa t(2+\kappa t))v_0 + (6 + 2e^{\kappa t}(-3+\kappa t) + \kappa t(4+\kappa t))\theta\}\right), \tag{119}$$

$$H_1(y) = \frac{(y-\mu)}{\Sigma},$$
$$H_2(y) = \frac{(y-\mu)^2}{\Sigma^2} - \frac{1}{\Sigma},$$
$$H_3(y) = \frac{(y-\mu)^3}{\Sigma^3} - \frac{3(y-\mu)}{\Sigma^2},$$
$$H_4(y) = \frac{(y-\mu)^4}{\Sigma^4} - \frac{6(y-\mu)^2}{\Sigma^3} + \frac{3}{\Sigma^2},$$
$$H_5(y) = \frac{(y-\mu)^5}{\Sigma^5} - \frac{10(y-\mu)^3}{\Sigma^4} + \frac{15(y-\mu)}{\Sigma^3},$$
$$H_6(y) = \frac{(y-\mu)^6}{\Sigma^6} - \frac{15(y-\mu)^4}{\Sigma^5} + \frac{45(y-\mu)^2}{\Sigma^4} - \frac{15}{\Sigma^3}.$$

Then, we have an approximation of the call price $C(t, K)$ as follows:

$$C(t,K) \simeq BS(S_0, K, \Sigma, t) \tag{120}$$
$$+\varepsilon C_1(t, v_0, \kappa, \theta, \rho)\int_{\log K}^\infty (e^y - K)\left[H_3(y) - H_2(y)\right] p^{logBS}(t, x_0, y)dy \tag{121}$$
$$+\varepsilon^2 C_{21}(t, v_0, \kappa, \theta, \rho, \nu)\int_{\log K}^\infty (e^y - K)\left[\{H_6(y) - H_5(y)\} - \{H_5(y) - H_4(y)\}\right] p^{logBS}(t, x_0, y)dy \tag{122}$$
$$+(\varepsilon\nu)^2 C_{22}(t, v_0, \kappa, \theta, \rho)\int_{\log K}^\infty (e^y - K)\left[\{H_4(y) - H_3(y)\} - \{H_3(y) - H_2(y)\}\right] p^{logBS}(t, x_0, y)dy \tag{123}$$
$$+(\varepsilon\nu)^2 C_{23}(t, v_0, \kappa, \theta, \rho)\int_{\log K}^\infty (e^y - K)\left[H_4(y) - H_3(y)\right] p^{logBS}(t, x_0, y)dy, \tag{124}$$

where $BS(S_0, K, \Sigma, t)$ stands for the Black-Scholes price with initial price S_0, strike K, volatility $\sqrt{\Sigma}$ and maturity t. (We set the risk-free interest rate to be zero.)

Moreover, we obtain the closed form expressions for the integrals above by applying the following relations:

$$\int_{\log K}^{\infty} H_n(y) p^{logBS}(t, x_0, y) dy = H_{n-1}(\log K) p\left(\log(S_0/K) - \Sigma/2\right) \tag{125}$$

$$= \hat{H}_{n-1}\left(-\log(S_0/K) + \Sigma/2\right) p\left(\log(S_0/K) - \Sigma/2\right), \tag{126}$$

$$\int_{\log K}^{\infty} e^y \left[H_n(y) - H_{n-1}(y)\right] p^{logBS}(t, x_0, y) dy = S_0 H_{n-1}(\log K) p\left(\log(S_0/K) + \Sigma/2\right) \tag{127}$$

$$= S_0 \hat{H}_{n-1}\left(-\log(S_0/K) + \Sigma/2\right) p\left(\log(S_0/K) + \Sigma/2\right) \tag{128}$$

where

$$p(y) = \frac{1}{\sqrt{2\pi\Sigma}} e^{-\frac{y^2}{2\Sigma}} \tag{129}$$

$$\hat{H}_n(y - \mu) = H_n(y). \tag{130}$$

That is, we have the next proposition.

Proposition 6.1 *Under the Heston model (106), the approximate call price $C_{AE}(t, K)$ with strike K and maturity t up to the ε^2-order is given as follows:*

$$C_{AE}(t, K) = BS(S_0, K, \Sigma, t) \tag{131}$$

$$+\varepsilon C_1(t, v_0, \kappa, \theta, \rho) \left[S_0 \hat{H}_2(-d_2) p(d_1) - K\left\{\hat{H}_2(-d_2) - \hat{H}_1(-d_2)\right\} p(d_2)\right] \tag{132}$$

$$+\varepsilon^2 C_{21}(t, v_0, \kappa, \theta, \rho) \left[S_0 \left\{\hat{H}_5(-d_2) - \hat{H}_4(-d_2)\right\} p(d_1) - K\left\{\hat{H}_5(-d_2) - 2\hat{H}_4(-d_2) + \hat{H}_3(-d_2)\right\} p(d_2)\right] \tag{133}$$

$$+\varepsilon^2 C_{22}(t, v_0, \kappa, \theta, \rho) \left[S_0 \left\{\hat{H}_3(-d_2) - \hat{H}_2(-d_2)\right\} p(d_1) - K\left\{\hat{H}_3(-d_2) - 2\hat{H}_2(-d_2) + \hat{H}_1(-d_2)\right\} p(d_2)\right] \tag{134}$$

$$+\varepsilon^2 C_{23}(t, v_0, \kappa, \theta, \rho) \left[S_0 \hat{H}_3(-d_2) p(d_1) - K\left\{\hat{H}_3(-d_2) - \hat{H}_2(-d_2)\right\} p(d_2)\right], \tag{135}$$

where $BS(S_0, K, \Sigma, t)$ denotes the corresponding Black-Scholes price, $C_1(t, v_0, \kappa, \theta, \rho)$, $C_{2i}(t, v_0, \kappa, \theta, \rho)$, $i = 1, 2, 3$ are defined by (116) $\sim$ (119), respectively, $p(y)$ is given as (129), and

$$d_1 = \log(S_0/K) + \Sigma/2, \tag{136}$$

$$d_2 = d_1 - \Sigma. \tag{137}$$

Also, $\hat{H}_n(x)$ is defined by (139) below, in particular,

$$\hat{H}_1(x) = \frac{x}{\Sigma},$$

$$\hat{H}_2(x) = \frac{x^2}{\Sigma^2} - \frac{1}{\Sigma},$$

$$\hat{H}_3(x) = \frac{x^3}{\Sigma^3} - \frac{3x}{\Sigma^2},$$

$$\hat{H}_4(x) = \frac{x^4}{\Sigma^4} - \frac{6x^2}{\Sigma^3} + \frac{3}{\Sigma^2},$$

$$\hat{H}_5(x) = \frac{x^5}{\Sigma^5} - \frac{10x^3}{\Sigma^4} + \frac{15x}{\Sigma^3},$$
$$\hat{H}_6(x) = \frac{x^6}{\Sigma^6} - \frac{15x^4}{\Sigma^5} + \frac{45x^2}{\Sigma^4} - \frac{15}{\Sigma^3}.$$

Remark 6.1 *The Hermite polynomial of degree n with parameter Σ is defined by*

$$H_n(x;\Sigma) = (-\Sigma)^n \exp\left(\frac{x^2}{2\Sigma}\right) \frac{d^n}{dx^n}\left[\exp\left(\frac{-x^2}{2\Sigma}\right)\right]. \tag{138}$$

Then, by setting

$$\hat{H}_n(x) := \frac{1}{\Sigma^n} H_n(x;\Sigma), \tag{139}$$

thanks to the well known properties of the Hermite polynomial, we use the following relations in the calculation above:

$$\frac{d}{dx}\hat{H}_n(x) = \frac{n}{\Sigma}\hat{H}_{n-1}(x) \tag{140}$$
$$\Sigma\hat{H}_{n+1}(x) - x\hat{H}_n(x) + n\hat{H}_{n-1}(x) = 0 \tag{141}$$
$$(-1)^m \frac{d^m}{dx^m}\left[\hat{H}_n(x)p(x)\right] = \hat{H}_{n+m}(x)p(x), \ (m \in \mathbf{Z}_+). \tag{142}$$

For instance, by (142) with $m = 1$, we have

$$\int_{\log K}^{\infty} H_n(y) p^{logBS}(t, x_0, y) dy = \int_{\log K - \mu}^{\infty} \hat{H}_n(x) p(x) dx \tag{143}$$
$$= \hat{H}_{n-1}\left(-\log(S_0/K) + \Sigma/2\right) p\left(\log(S_0/K) - \Sigma/2\right) \tag{144}$$
$$= \hat{H}_{n-1}\left(-d_2\right) p\left(d_2\right). \tag{145}$$

Moreover, by (140) and (141),

$$\frac{d}{dx}\left[\hat{H}_{n-1}(x)p(x-\Sigma)\right] = \left[\frac{d}{dx}\hat{H}_{n-1}(x)\right] p(x-\Sigma) + \hat{H}_{n-1}(x)\frac{d}{dx}p(x-\Sigma) \tag{146}$$
$$= \frac{(n-1)}{\Sigma}\hat{H}_{n-2}(x)p(x-\Sigma) + \hat{H}_{n-1}(x)\left\{\frac{-(x-\Sigma)}{\Sigma}\right\} p(x-\Sigma) \tag{147}$$
$$= \left[\hat{H}_{n-1}(x) + \left\{\frac{(n-1)}{\Sigma}\hat{H}_{n-2}(x) - \frac{x}{\Sigma}\hat{H}_{n-1}(x)\right\}\right] p(x-\Sigma) \tag{148}$$
$$= \left[\hat{H}_{n-1}(x) - \hat{H}_n(x)\right] p(x-\Sigma). \tag{149}$$

Thus, we obtain

$$\int_{\log K}^{\infty} e^y \left[H_n(y) - H_{n-1}(y)\right] p^{logBS}(t, x_0, y) dy \tag{150}$$
$$= e^{(\mu+\Sigma/2)} \int_{\log K - \mu}^{\infty} \left[\hat{H}_n(x) - \hat{H}_{n-1}(x)\right] p(x-\Sigma) dx \tag{151}$$
$$= S_0 \hat{H}_{n-1}\left(-\log(S_0/K) + \Sigma/2\right) p\left(\log(S_0/K) + \Sigma/2\right) \tag{152}$$
$$= S_0 \hat{H}_{n-1}\left(-d_2\right) p\left(d_1\right). \tag{153}$$

In order to examine the accuracy of our approximation, we compare the option prices and implied volatilities computed by our asymptotic expansion method against those by the Heston's Fourier transform method. Specifically, the parameters are set to be $x_0 = \log 100$, $v_0 = 0.4$, $\kappa = 1.15$, $\theta = 0.04$, $\rho = -0.4$ as in Forde, Jacquier and Lee [10] (2012).

First, with a fixed volatility of the volatility $\varepsilon = 0.2$, Table 1-4 show the call and the put option prices approximated by the first and second order expansions (denoted by *AE 1st order* and *AE 2nd order*, respectively) against the benchmark values based on the Fourier transform method (*Exact Price*).

The option maturities t are set to be 0.125, 0.25, 0.5 and 1.0 years, while the strike prices are between the 2.5% and the 97.5% Black-Scholes deltas (Δ=2.5%-Δ=97.5%) for all the maturities. (Each moneyness is calculated by the Black-Scholes delta. The range Δ=2.5%-97.5% includes the deep In-The-Money and the deep Out-of-The-Money and is enough for practical purpose.) It is clearly observed that the second order approximations generally improve the first order ones.

Secondly, Figures 1-16 show the estimated errors (*Estimated Error*) of the expansions up to the first order against those true errors (denoted by *True Error*) for the option prices with maturities $t = 0.125, 0.25, 0.5, 1.0$ and volatilities of the volatility $\varepsilon = 0.05, 0.1, 0.15, 0.2$. Here, the estimated errors are defined as the differences of the expansions up to the first order and the ones up to the second order, that is (*AE 1st order*)-(*AE 2nd order*), since we expect that the contributions of the second order terms in the expansions are dominant in the whole residuals of the first order expansions.

Then, as expected, we observe that the estimated errors well capture the features of the true errors for all the cases, which is important under some situations in practice such that we do not have sufficient time to check the true errors of the expansion through more time-consuming numerical schemes such as Monte Carlo simulations, while our estimated errors are very quickly computed thanks to the closed form expressions of expansions. The error estimates for the second order expansions could be estimated in the similar way by the use of the third order expansion.

Further, if necessary, by employing the estimated errors (*Estimated Error*), we are able to estimate the constant C_N in the error bounds of (53) and (56) in Theorem 4.1 (or $C_{N,q}$ in (59) of Proposition 4.1).

Moreover, Figure 17 - 32 present the implied volatilities based on our second order expansions and the Heston's method (denoted by *Heston AE IV* and *Heston Exact IV*, respectively) with maturities $t = 0.125,\ 0.25,\ 0.5,\ 1.0$ and volatilities of the volatility $\varepsilon = 0.05, 0.1, 0.15, 0.2$. Here, by inverting the Black-Scholes formula numerically, we convert the approximate prices to approximate implied volatilities. Alternatively, based on the approximate prices we could apply an implied volatility expansion formula in Theorem 3.2 of [46]. That is, the implied volatility with maturity t and strike K, $\sigma^{IV}(t, K)$ is expanded up to ε^2 as follows:

$$\sigma^{IV}(t,K) = \bar{\sigma} + \epsilon \frac{C_1}{C_\sigma^{BS}(\bar{\sigma})} + \epsilon^2 \left\{ \frac{C_2}{C_\sigma^{BS}(\bar{\sigma})} - \frac{1}{2} \frac{C_1^2}{C_\sigma^{BS}(\bar{\sigma})^3} C_{\sigma\sigma}^{BS}(\bar{\sigma}) \right\} + o(\varepsilon^2), \tag{154}$$

where $\bar{\sigma} = \sqrt{\Sigma/t}$ in (109), C_1 and C_2 are given by the coefficients of ε and ε^2 in Proposition 6.1, respectively. Also, C^{BS} denotes the Black-Scholes formula with $C^{BS} = BS(S_0, K, \Sigma, t)$ in Proposition 6.1, and $C_\sigma^{BS}(\bar{\sigma})$(so called *Vega*) and $C_{\sigma\sigma}^{BS}(\bar{\sigma})$(so called

Vomma, *Volga* and so on) are the first and second order derivatives of the Black-Scholes formula with respect to the volatility, respectively: that is,

$$\begin{aligned} C^{BS}_{\sigma}(\bar{\sigma}) &:= \frac{\partial}{\partial\sigma}C^{BS}|_{\sigma=\bar{\sigma}} = S_0\sqrt{\Sigma t}p(d_1), \\ C^{BS}_{\sigma\sigma}(\bar{\sigma}) &:= \frac{\partial^2}{\partial\sigma^2}C^{BS}|_{\sigma=\bar{\sigma}} = S_0 tp(d_1)d_1 d_2/\Sigma. \end{aligned} \tag{155}$$

We also remark that we could use an implied volatility expansion in Theorem 17 of Lorig et al. [29] (2014) in the similar manner.

The range of moneyness is determined by "Option's delta convention" (the range Δ=2.5%-97.5%), since the implied volatilities are often quoted by deltas rather than strike prices in financial markets. It is observed that the second order expansions replicate the exact implied volatilities fairly well. Especially, the approximations for the smaller value of the volatility on the volatility parameter or/and the shorter option maturities improve the accuracies, which is consistent with our theoretical results in Section 4, particularly Theorem 4.1-1. and Theorem 4.1-4.

Through these numerical analysis we confirm the effectiveness and the robustness of our method. We also expect that higher order expansions will provide better approximations.

Table 1: $T = 0.125$: Asymptotic expansions of prices in Heston model $\varepsilon = 0.2$

Moneyness	-0.136	-0.101	-0.067	-0.032	0.003	0.037	0.072	0.106	0.141
Delta Δ	97.5%	92.9%	83.6%	68.8%	50.0%	31.2%	16.4%	7.1%	2.5%
Exact Price	0.0995(put)	0.2759(put)	0.6827(put)	1.5017(put)	2.6809(call)	1.3121(call)	0.5300(call)	0.1720(call)	0.0441(call)
AE 1st order	0.0948(put)	0.2733(put)	0.6857(put)	1.5112(put)	2.6938(call)	1.3216(call)	0.5310(call)	0.1661(call)	0.0369(call)
AE 2nd order	0.1003(put)	0.2768(put)	0.6829(put)	1.5014(put)	2.6810(call)	1.3123(call)	0.5292(call)	0.1708(call)	0.0435(call)
Error AE 1st order	-0.0048	-0.0025	0.0030	0.0095	0.0128	0.0095	0.0010	-0.0060	-0.0072
Error AE 2nd order	0.0008	0.0009	0.0002	-0.0003	0.0000	0.0002	-0.0007	-0.0013	-0.0006

Table 2: $T = 0.25$: Asymptotic expansions of prices in Heston model $\varepsilon = 0.2$

Moneyness	-0.191	-0.142	-0.093	-0.044	0.005	0.054	0.109	0.152	0.201
Delta Δ	97.5%	92.9%	83.6%	68.8%	50.0%	31.2%	16.4%	7.1%	2.5%
Exact Price	0.1583(put)	0.4161(put)	0.9969(put)	2.1592(put)	3.7010(call)	1.7882(put)	0.7054(put)	0.2223(put)	0.0556(put)
AE 1st order	0.1473(put)	0.4110(put)	1.0053(put)	2.1838(put)	3.7340(call)	1.8119(put)	0.7062(put)	0.2050(put)	0.0365(put)
AE 2nd order	0.1611(put)	0.4191(put)	0.9977(put)	2.1583(put)	3.7011(call)	1.7884(put)	0.7024(put)	0.2178(put)	0.0541(put)
Error AE 1st order	-0.0110	-0.0051	0.0085	0.0246	0.0330	0.0237	0.0008	-0.0173	-0.0191
Error AE 2nd order	0.0027	0.0031	0.0008	-0.0009	0.0001	0.0002	-0.0031	-0.0045	-0.0015

Table 3: $T = 0.5$: Asymptotic expansions of prices in Heston model $\varepsilon = 0.2$

Moneyness	-0.267	-0.198	-0.129	-0.059	0.010	0.079	0.149	0.218	0.287
Delta Δ	97.5%	92.9%	83.6%	68.8%	50.0%	31.2%	16.4%	7.1%	2.5%
Exact Price	0.2513(put)	0.6275(put)	1.4573(put)	3.1138(put)	5.0531(call)	2.4023(put)	0.9237(put)	0.2842(put)	0.0708(put)
AE 1st order	0.2289(put)	0.6195(put)	1.4798(put)	3.1724(put)	5.1302(call)	2.4547(put)	0.9189(put)	0.2384(put)	0.0251(put)
AE 2nd order	0.2595(put)	0.6365(put)	1.4599(put)	3.1115(put)	5.0530(call)	2.4004(put)	0.9119(put)	0.2710(put)	0.0686(put)
Error AE 1st order	-0.0224	-0.0080	0.0225	0.0587	0.0770	0.0524	-0.0048	-0.0458	-0.0457
Error AE 2nd order	0.0082	0.0090	0.0026	-0.0023	-0.0002	-0.0019	-0.0118	-0.0132	-0.0021

Table 4: $T = 1.0$: Asymptotic expansions of prices in Heston model $\varepsilon = 0.2$

Moneyness	-0.372	-0.274	-0.176	-0.078	0.020	0.118	0.216	0.314	0.412
Delta Δ	97.5%	92.9%	83.6%	68.8%	50.0%	31.2%	16.4%	7.1%	2.5%
Exact Price	0.3849(put)	0.9298(put)	2.1208(put)	4.5123(put)	6.8128(call)	3.1908(put)	1.2009(put)	0.3637(put)	0.0911(put)
AE 1st order	0.3479(put)	0.9233(put)	2.1738(put)	4.6333(put)	6.9634(call)	3.2828(put)	1.1742(put)	0.2622(put)	-0.0014(put)
AE 2nd order	0.4045(put)	0.9513(put)	2.1285(put)	4.5078(put)	6.8083(call)	3.1769(put)	1.1667(put)	0.3356(put)	0.0927(put)
Error AE 1st order	-0.0370	-0.0065	0.0530	0.1210	0.1506	0.0920	-0.0267	-0.1015	-0.0926
Error AE 2nd order	0.0196	0.0215	0.0077	-0.0045	-0.0044	-0.0139	-0.0342	-0.0281	0.0016

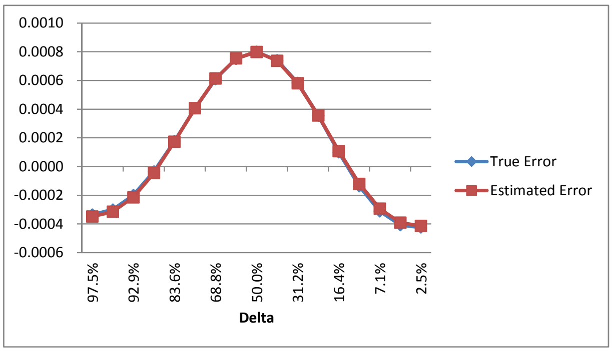


Figure 1: $T = 0.125$: Estimated Error in Heston model with $\varepsilon = 0.05$

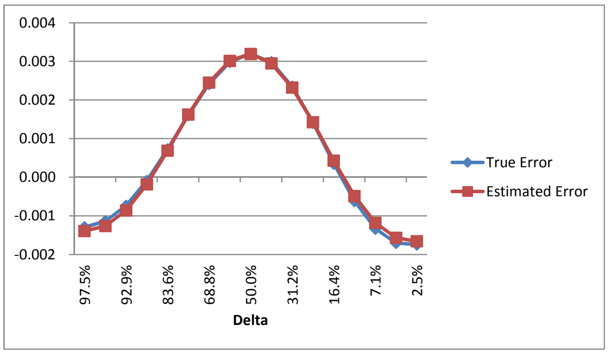


Figure 2: $T = 0.125$: Estimated Error in Heston model with $\varepsilon = 0.1$

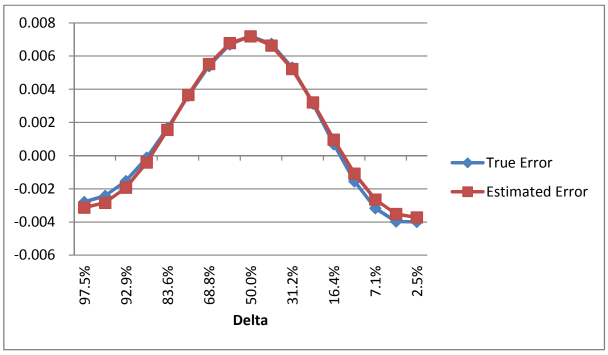


Figure 3: $T = 0.125$: Estimated Error in Heston model with $\varepsilon = 0.15$

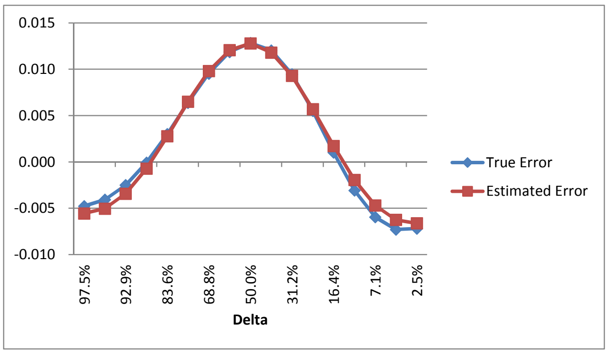
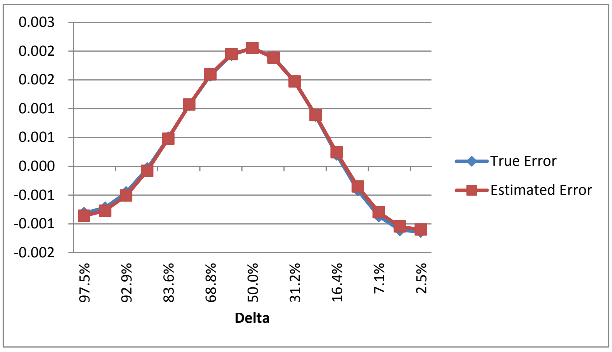


Figure 4: $T = 0.125$: Estimated Error in Heston model with $\varepsilon = 0.2$



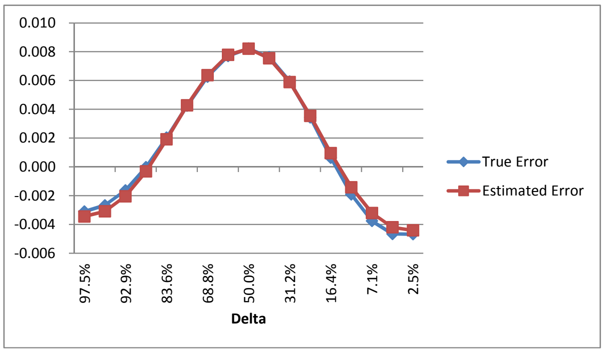
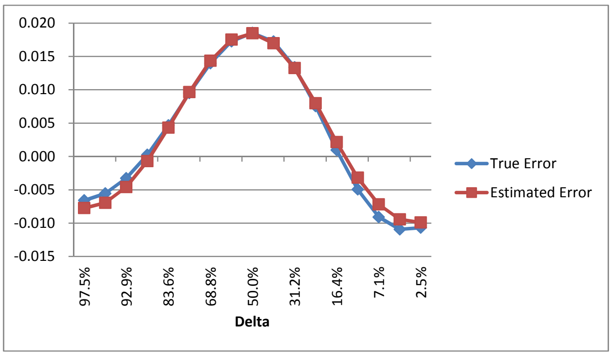


Figure 5: $T = 0.25$: Estimated Error in Heston model with $\varepsilon = 0.05$

Figure 6: $T = 0.25$: Estimated Error in Heston model with $\varepsilon = 0.1$



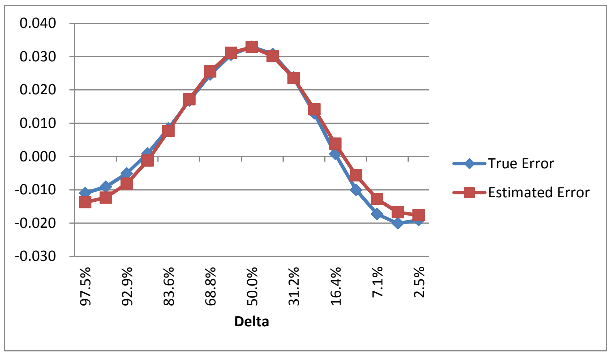
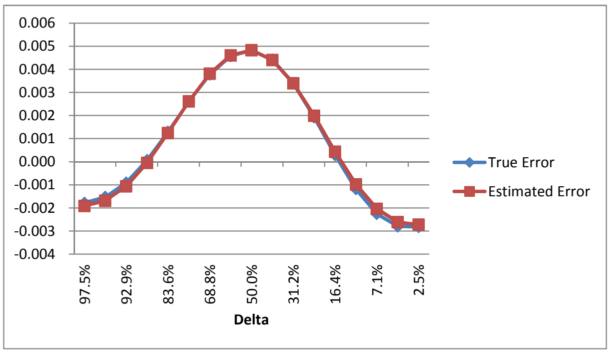


Figure 7: $T = 0.25$: Estimated Error in Heston model with $\varepsilon = 0.15$

Figure 8: $T = 0.25$: Estimated Error in Heston model with $\varepsilon = 0.2$



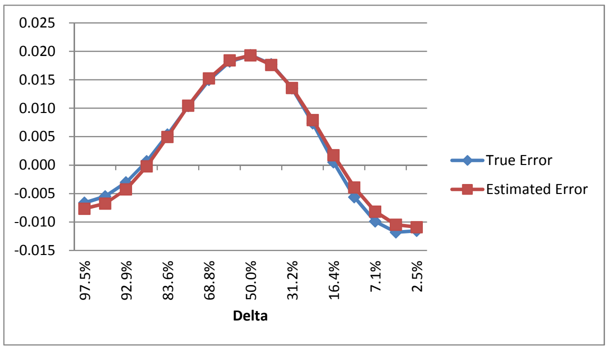
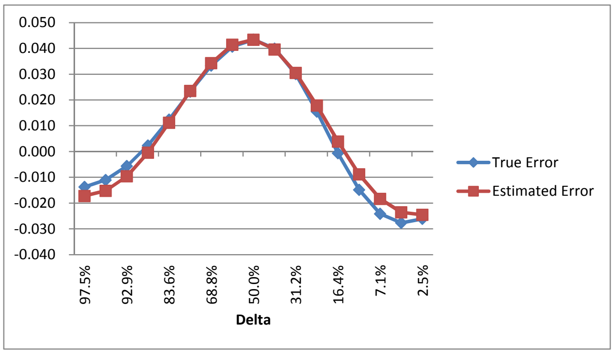


Figure 9: $T = 0.5$: Estimated Error in Heston model with $\varepsilon = 0.05$

Figure 10: $T = 0.5$: Estimated Error in Heston model with $\varepsilon = 0.1$



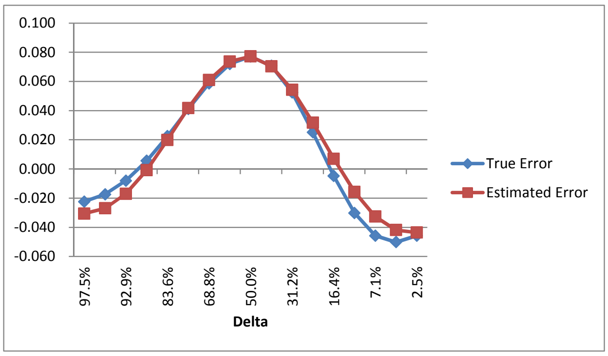


Figure 11: $T = 0.5$: Estimated Error in Heston model with $\varepsilon = 0.15$

Figure 12: $T = 0.5$: Estimated Error in Heston model with $\varepsilon = 0.2$

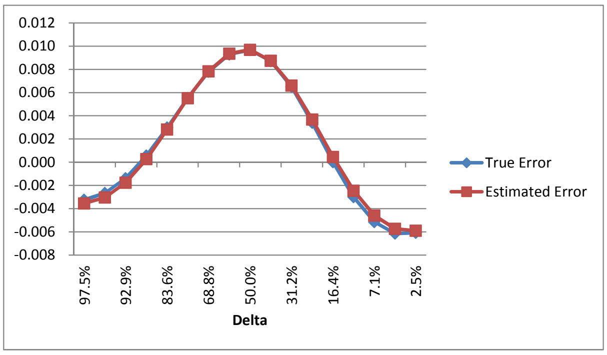


Figure 13: $T = 1.0$: Estimated Error in Heston model with $\varepsilon = 0.05$

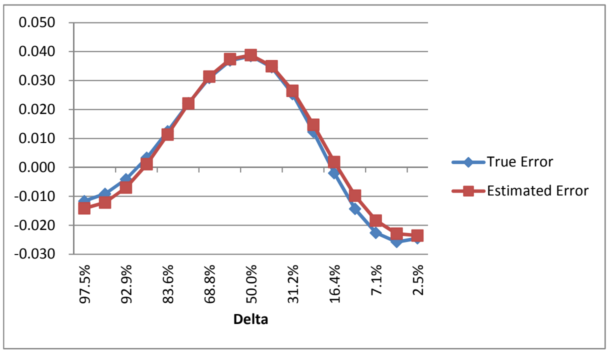


Figure 14: $T = 1.0$: Estimated Error in Heston model with $\varepsilon = 0.1$

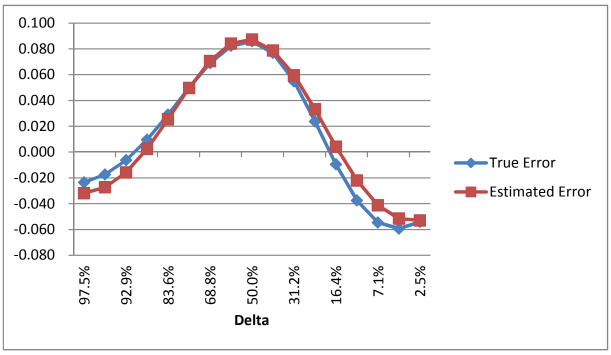


Figure 15: $T = 1.0$: Estimated Error in Heston model with $\varepsilon = 0.15$

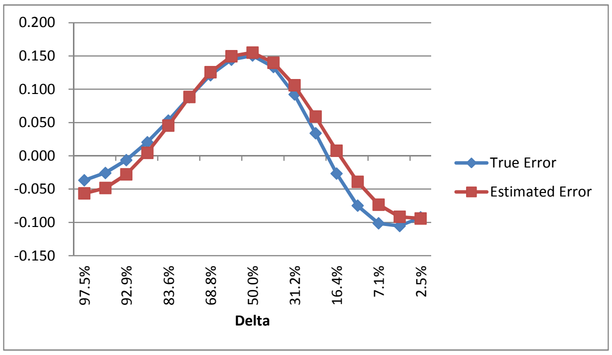


Figure 16: $T = 1.0$: Estimated Error in Heston model with $\varepsilon = 0.2$

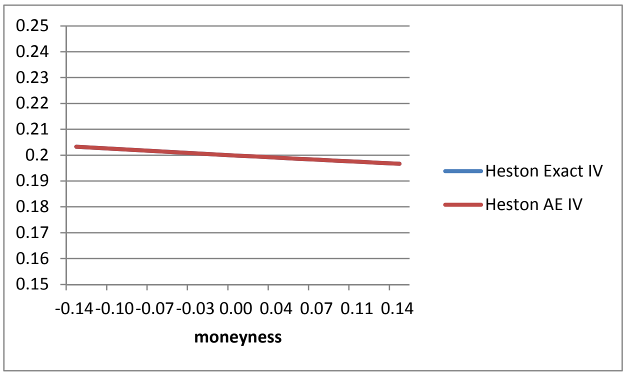


Figure 17: $T = 0.125$: Implied volatility approximation in Heston model with $\varepsilon = 0.05$

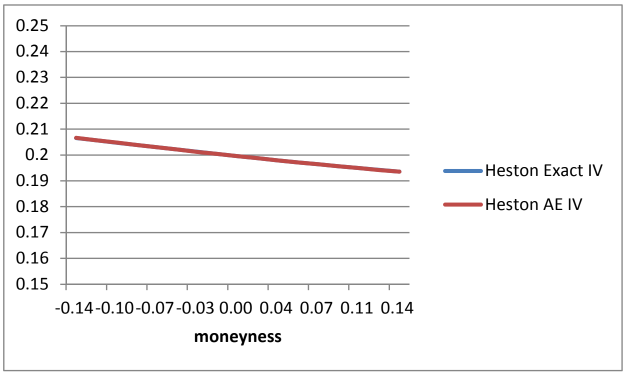


Figure 18: $T = 0.125$: Implied volatility approximation in Heston model with $\varepsilon = 0.1$

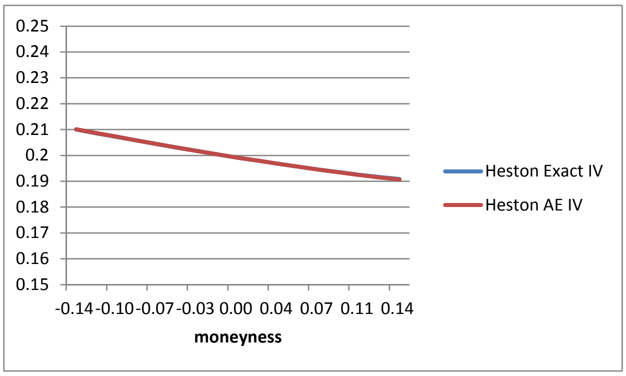


Figure 19: $T = 0.125$: Implied volatility approximation in Heston model with $\varepsilon = 0.15$

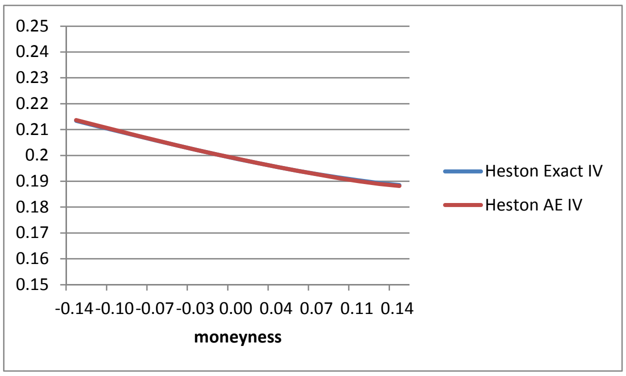


Figure 20: $T = 0.125$: Implied volatility approximation in Heston model with $\varepsilon = 0.2$

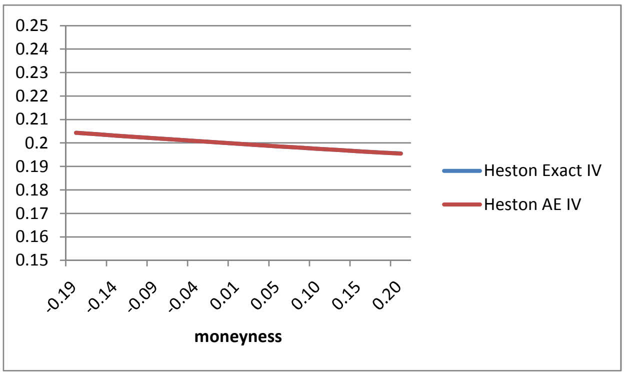


Figure 21: $T = 0.25$: Implied volatility approximation in Heston model with $\varepsilon = 0.05$

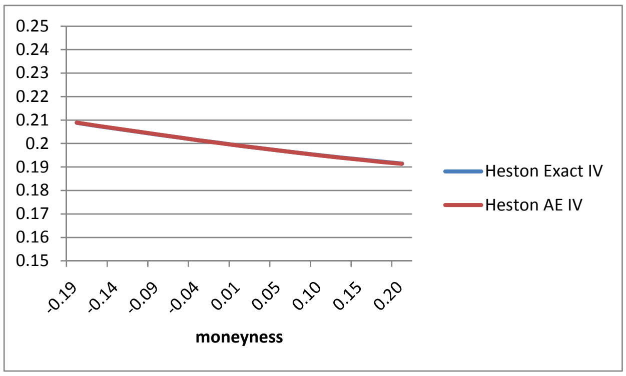


Figure 22: $T = 0.25$: Implied volatility approximation in Heston model with $\varepsilon = 0.1$

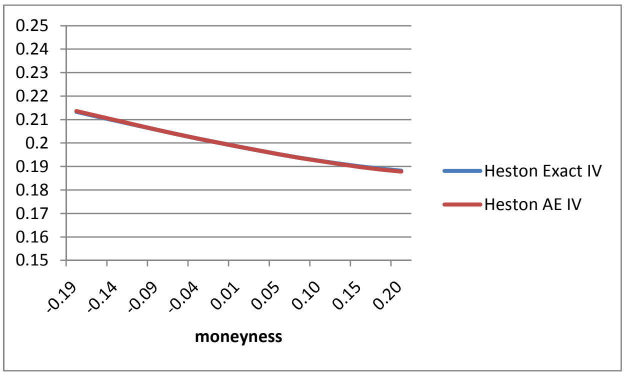


Figure 23: $T = 0.25$: Implied volatility approximation in Heston model with $\varepsilon = 0.15$

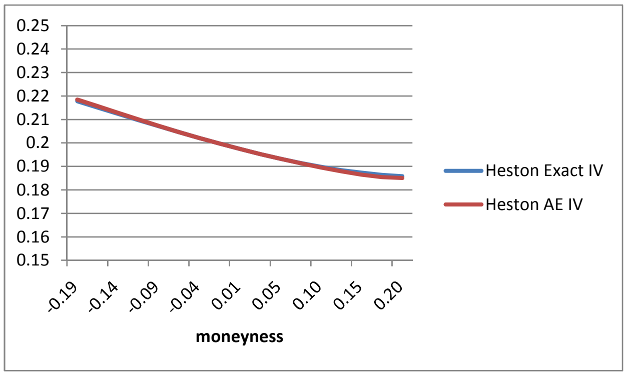


Figure 24: $T = 0.25$: Implied volatility approximation in Heston model with $\varepsilon = 0.2$

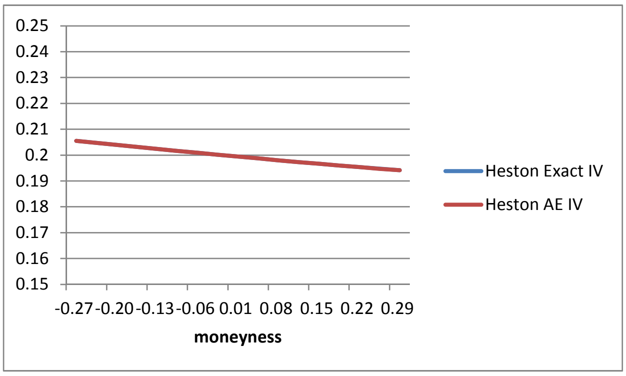


Figure 25: $T = 0.5$: Implied volatility approximation in Heston model with $\varepsilon = 0.05$

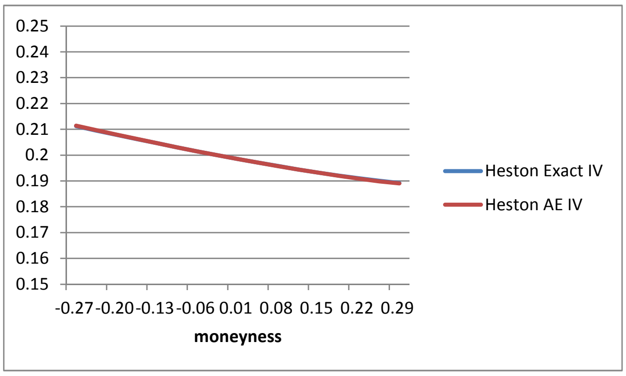


Figure 26: $T = 0.5$: Implied volatility approximation in Heston model with $\varepsilon = 0.1$

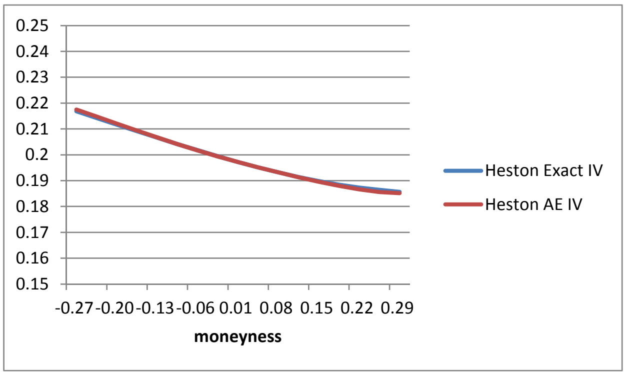


Figure 27: $T = 0.5$: Implied volatility approximation in Heston model with $\varepsilon = 0.15$

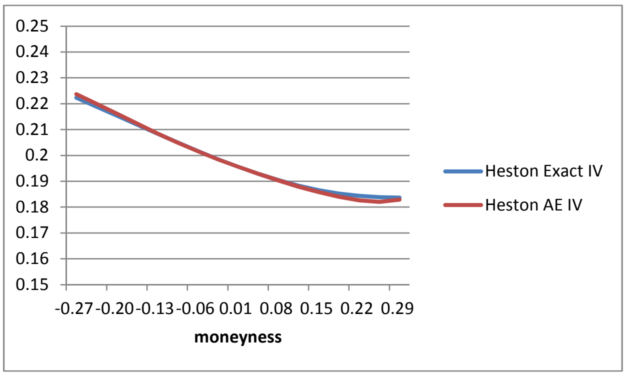


Figure 28: $T = 0.5$: Implied volatility approximation in Heston model with $\varepsilon = 0.2$

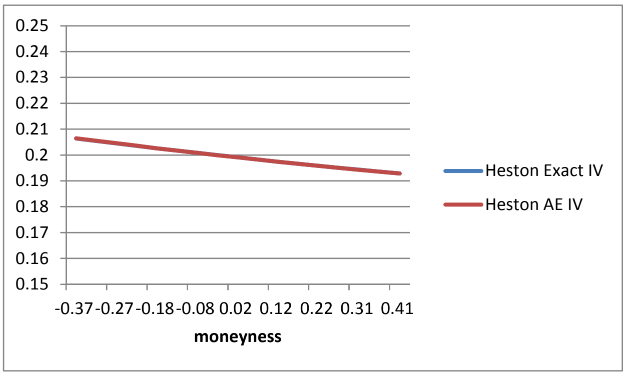


Figure 29: $T = 1.0$: Implied volatility approximation in Heston model with $\varepsilon = 0.05$

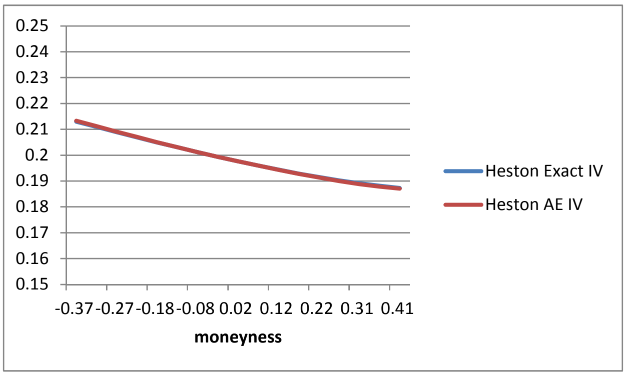


Figure 30: $T = 1.0$: Implied volatility approximation in Heston model with $\varepsilon = 0.1$

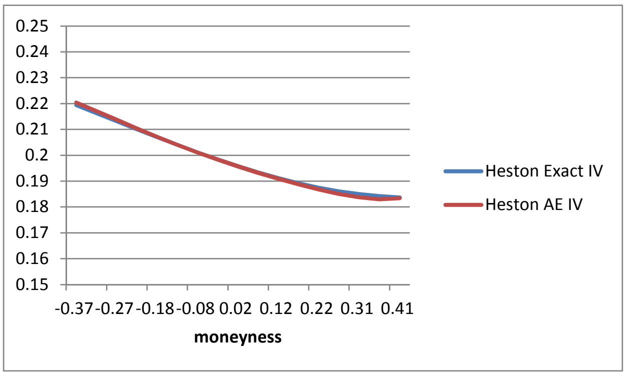


Figure 31: $T = 1.0$: Implied volatility approximation in Heston model with $\varepsilon = 0.15$

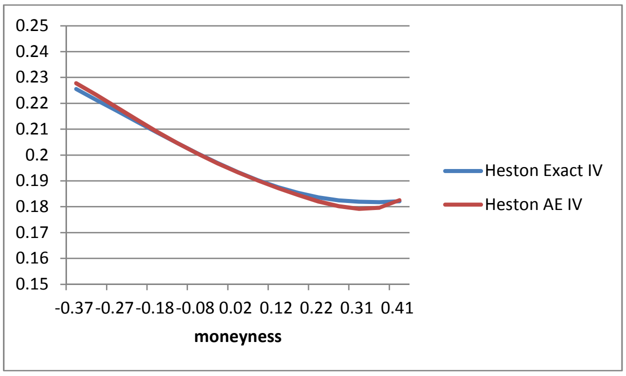


Figure 32: $T = 1.0$: Implied volatility approximation in Heston model with $\varepsilon = 0.2$

7 Asymptotic Expansion for Expectations of Kusuoka-Stroock Functions in General Wiener Functionals

This section briefly explains an extension of the result in the previous sections to more general Winer functionals on the Wiener space $(\mathcal{W}, H, P)$. Particularly, the next theorem presents a general asymptotic expansion formula for the expectation of Kusuoka-Stroock functions, which is also regarded as a natural extension of Theorem 2.6 in Takahashi and Yamada [46] (2012). We note that the theorem is applicable for general Wiener functionals even when the functionals are non-diffusion as in the HJM term structure model [17] (1992).

Theorem 7.1 *Consider a family of smooth Wiener functionals $F^\varepsilon = (F^{\varepsilon,1}, \cdots, F^{\varepsilon,n}) \in \mathcal{K}_0^T$ such that $\varepsilon \mapsto F^\varepsilon$ is infinity differentiable with $\frac{\partial^i}{\partial \varepsilon^i} F^\varepsilon \in \mathcal{K}_{i+1}^T$, $i \in \mathbf{N}$ and F^ε has an asymptotic expansion in $\mathbf{D}^\infty$;*

$$F^\varepsilon \sim F_0 + \varepsilon F_1 + \varepsilon^2 F_2 + \varepsilon^3 F_3 + \cdots \ \textit{in } \mathbf{D}^\infty, \tag{156}$$

with $F_i = (F_i^1, \cdots, F_i^n) \in \mathcal{K}_{i+1}^T$. Also, F^ε satisfies the uniformly non-degenerate condition:

$$\sup_{\varepsilon \in (0,1]} \sup_{t \in (0,T]} \left\| \left(\det \frac{\sigma_{F^\varepsilon}}{t} \right)^{-1} \right\|_{L^p} < \infty, \ \textit{for all } p < \infty. \tag{157}$$

Then, for a bounded Borel function f, we have an asymptotic expansion

$$\Bigg| E[f(F^\varepsilon)] - \Bigg\{ \int_{\mathbf{R}^n} f(x) p^{F_0}(x) dx + \sum_{j=1}^N \varepsilon^j \sum_{\alpha^{(k)}, \beta^{(k)}}^{(j)} \int_{\mathbf{R}^n} f(x) E[H_{\alpha^{(k)}}(F_0, \prod_{l=1}^k F_{\beta_l}^{\alpha_l}) | F_0 = x] p^{F_0}(x) dx \Bigg\} \Bigg| \leq \|f\|_\infty C_N (\varepsilon \sqrt{t})^{N+1},$$

where p^{F_0} is the density of F_0 , and $F_k^i := \frac{1}{k!} \frac{\partial^k}{\partial \varepsilon^k} F^{\varepsilon,i}|_{\varepsilon=0}$, $k \in \mathbf{N}$, $i = 1, \cdots, n$. Also, $\alpha^{(k)}$ denotes a multi-index, $\alpha^{(k)} = (\alpha_1, \cdots, \alpha_k)$ and

$$\sum_{\alpha^{(k)}, \beta^{(k)}}^{(j)} \equiv \sum_{k=1}^j \sum_{\beta^{(k)} = (\beta_1, \cdots, \beta_k), \beta_1 + \cdots + \beta_k = j, \beta_i \geq 1} \sum_{\alpha^{(k)} \in \{1, \cdots, n\}^k} \frac{1}{k!}.$$

The Malliavin weight $H_{\alpha^{(k)}}$ is recursively defined as follows:

$$H_{\alpha^{(k)}}(F, G) = H_{(\alpha_k)}(F, H_{\alpha^{(k-1)}}(F, G)),$$

where

$$H_{(l)}(F, G) = \delta \left(\sum_{i=1}^n G \gamma_{li}^F DF_i \right).$$

Here, $\gamma^F = \{\gamma_{ij}^F\}_{1 \leq i,j \leq n}$ denotes the inverse matrix of the Malliavin covariance matrix of F.

Proof.
Let $G \in \mathcal{K}_r^T$ and $Z_l^\varepsilon = \sum_{i=1}^n \gamma_{li}^{F^\varepsilon} DF_i^\varepsilon$, $l = 1, \cdots, n$. We obtain

$$\begin{aligned} H_{(l)}(F^\varepsilon, G) &= \delta\left(\sum_{i=1}^n G\gamma_{li}^{F^\varepsilon} DF_i^\varepsilon\right) = \delta\left(GZ_l^\varepsilon\right) \\ &= G\delta Z_l^\varepsilon - \langle DG, Z_l^\varepsilon\rangle_H. \end{aligned} \tag{158}$$

Since we have $Z_l^\varepsilon \in \mathcal{K}_{-2}^T$, it holds that $H_{(l)}(F^\varepsilon, G) \in \mathcal{K}_{r-2+1}^T = \mathcal{K}_{r-1}^T$. For $k \leq N+1$, $\sum_{l=1}^k \beta_l = N+1$, $\alpha^{(k)} \in \{1, \cdots, n\}^k$, we have $\prod_{j=1}^k \frac{1}{\beta_j!}\frac{\partial^{\beta_j}}{\partial\varepsilon^{\beta_j}} F^{\varepsilon,\alpha_j} \in \mathcal{K}_{N+1+k}^T$ and then we obtain

$$\sum_{\alpha^{(k)},\beta^{(k)}}^{(j)} H_{\alpha^{(k)}}\left(F^\epsilon, \prod_{j=1}^k \frac{1}{\beta_j!}\frac{\partial^{\beta_j}}{\partial\varepsilon^{\beta_j}} F^{\varepsilon,\alpha_j}\right) \in \mathcal{K}_{N+1}^T.$$

Therefore, we have the assertion. □

Moreover, we obtain another representation for the conditional expectation of Malliavin weights $H_{\alpha^{(k)}}$, that is $E\left[H_{\alpha^{(k)}}\left(F_0, \prod_{l=1}^k F_{\beta_l}^{\alpha_l}\right)|F_0 = x\right]$.

Corollary 7.1 *For $j = 1, \cdots, N$, $k \leq j$, $\sum_{l=1}^k \beta_l = j$, $\alpha^{(k)} \in \{1, \cdots, n\}^k$,*

$$E\left[H_{\alpha^{(k)}}\left(F_0, \prod_{l=1}^k F_{\beta_l}^{\alpha_l}\right)|F_0 = x\right] = (-1)^k \partial_{\alpha^{(k)}}^k \left\{E\left[\prod_{l=1}^k F_{\alpha_l}^{0,\beta_l}|F^0 = x\right] p^{F_0}(x)\right\}. \tag{159}$$

Proof.
For $f \in C_b^\infty(\mathbf{R}^n)$,

$$\begin{aligned} &\int_{\mathbf{R}^n} f(x) E\left[H_{\alpha^{(k)}}\left(F_0, \prod_{l=1}^k F_{\beta_l}^{\alpha_l}\right)|F_0 = x\right] p^{F_0}(x)dx \qquad (160) \\ =\ & E\left[f(F_0)H_{\alpha^{(k)}}\left(F_0, \prod_{l=1}^k F_{\beta_l}^{\alpha_l}\right)\right] \\ =\ & E\left[\partial_{\alpha^{(k)}}^k f(F_0) \prod_{l=1}^k F_{\beta_l}^{\alpha_l}\right] \\ =\ & \int_{\mathbf{R}^n} \partial_{\alpha^{(k)}}^k f(x) E\left[\prod_{l=1}^k F_{\alpha_l}^{0,\beta_l}|F_0 = x\right] p^{F_0}(x)dx \\ =\ & \int_{\mathbf{R}^n} f(x)(\partial^*)_{\alpha^{(k)}}^k E\left[\prod_{l=1}^k F_{\alpha_l}^{0,\beta_l}|F_0 = x\right] p^{F_0}(x)dx \\ =\ & (-1)^k \int_{\mathbf{R}^n} f(x)\partial_{\alpha^{(k)}}^k \left\{E\left[\prod_{l=1}^k F_{\alpha_l}^{0,\beta_l}|F^0 = x\right] p^{F_0}(x)\right\} dx, \end{aligned}$$

where ∂^* is the divergence operator on the space $(\mathbf{R}^n, p^{F_0}(dx))$ and $(\partial^*)_{\alpha^{(k)}}^k = \partial_{\alpha_1}^* \circ \partial_{\alpha_2}^* \circ \cdots \circ \partial_{\alpha_k}^*$.

Remark 7.1 *We are able to apply Theorem 7.1 with Corollary 7.1 to the HJM framework [17] (1992) that is a typical example of non-diffusion models in finance. See Kunitomo and Takahashi [21], [22] (2001, 2003) and Matsushima and Takahashi [40] (2004) for the concrete computational method and numerical examples in this framework. To the best of our knowledge, there do not exist perturbation schemes based on the PDE approach to the HJM model.*

Also, even under the diffusion setting, it is very difficult to apply perturbation methods in the PDE approach to high-dimensional problems such as basket option pricing under stochastic volatility models and option pricing under Libor Market models (LMM) in order to achieve accurate approximations with practically sufficient computational speed.

In fact, for pricing plain-vanilla options under a two dimensional diffusion model with stochastic volatility as in Section 4, Section 3.2 in Takahashi-Yamada [45] (2011) derives an approximate price in a PDE's perturbation method. However, it seems not easier to get the error estimate by the method than by our approach in this work. Particularly, in degenerate models as in the sections 3 and 4, it is difficult to estimate the errors of expansions with the standard PDE approach as in Friedman [14] (1964). On the other hand, the Malliavin approach with the Kusuoka-Stroock functions in this paper is able to automatically estimate the error rates with respect to ε and the maturity t in both degenerate and non-degenerate cases. Hence, in terms of obtaining sharper error estimates, the current work can be regarded as an extension of Takahashi-Yamada [45] (2011).

Moreover, it is a substantially tough task to establish a concrete computational scheme for the high-dimensional pricing problems mentioned above. Thus, again, there seem not exist researches based on perturbation schemes in the PDE for those important topics, which satisfy the approximation accuracies and computational speed required in financial practice.

On the contrary, our method are applicable in a unified manner to the problems. As for the concrete applications, see Shiraya-Takahashi [34], [35] (2011, 2012) for the basket and discrete average options with stochastic volatilities; Shiraya-Takahashi-Yamada [37] (2012) for discrete barrier options under stochastic volatilities; Shiraya-Takahashi-Yamazaki [38] (2012), Takahashi-Takehara [41] (2010), Takehara-Toda-Takahashi [44] (2011), for swaption or long-term currency option pricing under LMMs with stochastic volatilities of interest rates or foreign exchange rates.

References

[1] E. Alos, A Decomposition Formula for Option Prices in the Heston Model and Applications to Option Pricing Approximation, *Finance and Stochastics*, Volume 16, Issue 3, 403-422, (2012).

[2] E. Alos, A. Eydeland and P. Laurence, A Kirk's and a Bachelier's formula for three asset spread options, *Energy Risk*, 09/2011 , 52-57, (2011)

[3] C. Bayer and P. Laurence, Asymptotics beats Monte Carlo: The case of correlated local vol baskets, forthcoming in *Communications on Pure and Applied Mathematics*, (2012).

[4] G. Ben Arous and P. Laurence, Second order expansion for implied volatility in two factor local stochastic volatility models and applications to the dynamic λ-SABR model, preprint, (2009).

[5] E. Benhamou, E. Gobet and M.Miri, Time Dependent Heston Model, *SIAM Journal on Financial Mathematics*, Vol.1, 289-325, (2010a).

[6] E. Benhamou, E. Gobet and M.Miri, Closed forms for European options in a local volatility model, International Journal of Theoretical and Applied Finance 13, 4 (2010b).

[7] D. Crisan and F. Delarue, Sharp Derivative Bounds for Solutions of Degenerate Semi-linear Partial Differential Equations, *Journal of Functional Analysis*, 263, 3024-3101, (2012).

[8] D. Crisan, K. Manolarakis and C. Nee, Cubature Methods and Applications, *Paris-Princeton Lectures on Mathematical Finance 2013*, Springer, (2013).

[9] D. Davydov and V. Linetsky, Pricing Options on Scalar Diffusions: An Eigenfunction Expansion Approach, *Operations Research*, 51 (2003) pp.185-209.

[10] M. Forde, A. Jacquier and R. Lee, The Small-time Smile and Term Structure of Implied Volatility under the Heston Model, *SIAM Journal on Financial Mathematics*, Vol.3, 690-708, (2012).

[11] P. Foschi, S. Pagliarani and A. Pascucci, Approximations for Asian options in local volatility models, *Journal of Computational and Applied Mathematics*, 237, 442-459, (2013).

[12] J.P. Fouque, G. Papanicolaou and K.R. Sircar, *Derivatives in Financial Markets with Stochastic Volatility*, Cambridge University Press, (2002).

[13] M. Fujii, Momentum-Space Approach to Asymptotic Expansion for Stochastic Filtering and other Problems, CARF-F-289, 311, forthcoming in *Annals of Institute of Statistical Mathematics*, (2012).

[14] A. Friedman, *Partial differential equations of parabolic type*, Prentice-Hall, (1964).

[15] J. Gatheral, E.P. Hsu, P. Laurence, C. Ouyang and T-H. Wang, Asymptotics of implied volatility in local volatility models, *Mathematical Finance*, 22(4), 591-620, (2012).

[16] P.S. Hagan, D. Kumar, A.S. Lesniewskie and D.E. Woodward, Managing Smile Risk, *Wilmott magazine*, (2002).

[17] D. Heath, R. Jarrow and A. Morton, Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation. *Econometrica*, 60(1):77-105 (1992).

[18] S. L. Heston, A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options, *The Review of Financial Studies*, Vol. 6-2, pp. 327-343, (1993).

[19] T. Kato, A. Takahashi and T. Yamada, Semi-group Expansion for Pricing Barrier Options, Preprint, CARF-F-271, (2012).

[20] T. Kato, A. Takahashi and T. Yamada, An Asymptotic Expansion Formula for Up-and-Out Barrier Option Price under Stochastic Volatility Model, *JSIAM Letters*, Vol. 5, pp.17-20, (2013).

[21] N. Kunitomo and A. Takahashi, The Asymptotic Expansion Approach to the Valuation of Interest Rate Contingent Claims, *Mathematical Finance*, Vol.11, 117-151, (2001).

[22] N. Kunitomo and A. Takahashi, On Validity of the Asymptotic Expansion Approach in Contingent Claim Analysis, *The Annals of Applied Probability*, **13**, no.3, 914-952, (2003).

[23] S. Kusuoka, Malliavin Calculus Revisited, *J. Math. Sci. Univ. Tokyo*, 261-277, (2003).

[24] S. Kusuoka and D. Stroock, Applications of the Malliavin Calculus Part I, *Stochastic Analysis* (Katata/Kyoto 1982) 271-306 (1984).

[25] C. Li, Managing Volatility Risk: Innovation of Financial Derivatives, Stochastic Models and Their Analytical Implementation, Ph.D thesis in Columbia University, (2010).

[26] C. Li, Closed-Form Expansion, Conditional Expectation, and Option Valuation, *Mathematics of Operations Research*, doi: 10.1287/moor.2013.0613, (2013).

[27] V. Linetsky, Spectral Expansions for Asian (Average Price) Options, Operations Research, 52, pp.856-867, (2004).

[28] P-L. Lions and M. Musiela, Correlations and bounds for stochastic volatility models. Annales de l' Institut Henri Poincaré, 24: 1-16, (2007).

[29] Lorig, M., S. Pagliarani, and A. Pascucci, Explicit implied vols for multifactor local-stochastic vol models. ArXiv preprint arXiv:1306.5447, (2014).

[30] R. Matsuoka and A. Takahashi, An Asymptotic Expansion Approach to Computing Greeks, *FSA Research Review*, pp.72-108, (2005).

[31] R. Matsuoka, A. Takahashi and Y. Uchida, A New Computational Scheme for Computing Greeks by the Asymptotic Expansion Approach, *Asia-Pacific Financial Markets*, Vol.11, 393-430, (2006).

[32] D. Nualart, *The Malliavin Calculus and Related Topics*, Springer, (2006).

[33] C. Nee, Sharp Gradient Bounds for the Diffusion Semigroup, Ph.D thesis in Imperial College London, (2011).

[34] K. Shiraya and A. Takahashi, Pricing Average Options on Commodities, *Journal of Futures Markets*, Vol.31-5, 407-439, (2011).

[35] K. Shiraya and A. Takahashi, Pricing Multi-Asset Cross Currency Options, *Journal of Futures Markets*, DOI: 10.1002/fut.21590, (2012).

[36] K. Shiraya, A. Takahashi and M. Toda, Pricing Barrier and Average Options under Stochastic Volatility Environment, *Journal of Computational Finance*, vol.15-2,winter 2011/12, 111-148, (2011).

[37] K. Shiraya, A. Takahashi and T. Yamada, Pricing Discrete Barrier Options under Stochastic Volatility, *Asia-Pacific Financial Markets*, September 2012, Volume 19, Issue 3, pp 205-232, (2012).

[38] K. Shiraya, A. Takahashi and A. Yamazaki, Pricing Swaptions under the LIBOR Market Model of Interest Rates with Local-Stochastic Volatility Models, *Wilmott Magazine*, Volume 2012, Issue 61, pp 48-63, (2012).

[39] M. Siopacha and J. Teichmann: Weak and Strong Taylor Methods for Numerical Solutions of Stochastic Differential Equations, *Quantitative Finance*. **11**, (2011), no.4, 517-528.

[40] A. Takahashi and S. Matsushima Option Pricing in HJM Model using an Asymptotic Expansion Method," FSA Research Review 2004, 82-103, 2004.

[41] A. Takahashi and K. Takehara, A Hybrid Asymptotic Expansion Scheme: an Application to Long-term Currency Options, *International Journal of Theoretical and Applied Finance*, Vol.13-8, 1179-1221, (2010).

[42] A. Takahashi, , K. Takehara and M.Toda,, Computation in an Asymptotic Expansion Method, Working paper, CARF-F-149, the University of Tokyo, (2009).

[43] A. Takahashi, , K. Takehara and M.Toda, A General Computation Scheme for a High-Order Asymptotic Expansion Method, *International Journal of Theoretical and Applied Finance*, 15(6), DOI: 10.1142/S0219024912500446, (2012).

[44] K. Takehara, M. Toda and A. Takahashi "Application of a High-Order Asymptotic Expansion Scheme to Long-Term Currency Options" *The International Journal of Business and Finance Research*, vol. 5-3, pp 87-100, 2011.

[45] A. Takahashi and T. Yamada, A Remark on Approximation of the Solutions to Partial Differential Equations in Finance, *Recent Advances in Financial Engineering 2011*, pp.133-181, 2011.

[46] A. Takahashi and T. Yamada, An Asymptotic Expansion with Push-Down of Malliavin Weights, *SIAM Journal on Financial Mathematics*, Vol.3, 95-136, (2012).

[47] A. Takahashi and N. Yoshida, An Asymptotic Expansion Scheme for Optimal Investment Problems, *Statistical Inference for Stochastic Processes*, Vol.7-2, 153-188, (2004).

[48] A. Takahashi and N. Yoshida, Monte Carlo Simulation with Asymptotic Method, *Journal of Japan Statistical Society*, Vol. 35-2, 171-203, (2005).

[49] N. Yoshida, Asymptotic Expansions for Statistics Related to Small Diffusions, *Journal of Japan Statistical Society*, Vol.22, 139-159, (1992a).

[50] N. Yoshida, Asymptotic Expansions of Maximum Likelihood Estimators for Small Diffusions via the Theory of Malliavin-Watanabe, *Probability Theory and Related Fields*, **92**, 275-311. (1992b)

[51] S. Watanabe, Analysis of Wiener Functionals (Malliavin Calculus) and its Applications to Heat Kernels, *Ann. Probab.* **15**, no. 1, 1-39, (1987).

[52] G. Xu and H. Zheng, Basket options valuation for a local volatility jump-diffusion model with the asymptotic expansion method, *Insurance: Mathematics and Economics*, 47, 415-422, (2010).

[53] G. Xu and H. Zheng, Lower Bound Approximation to Basket Option Values for Local Volatility Jump-Diffusion Models, preprint, arXiv, (2012).