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# An asymptotic expansion for forward-backward SDEs: a Malliavin calculus approach

Akihiko Takahashi
University of Tokyo

Toshihiro Yamada
Hitotsubashi University

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# An asymptotic expansion for forward-backward SDEs: a Malliavin calculus approach *†

Akihiko Takahashi‡ and Toshihiro Yamada§

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**Abstract**

This paper proposes a new analytical approximation scheme for the representation of the forward-backward stochastic differential equations (FBSDEs) of Ma and Zhang (2002). In particular, we obtain an error estimate for the scheme applying Malliavin calculus method for the forward SDEs combined with the Picard iteration scheme for the BSDEs. We also show numerical examples for pricing option with counterparty risk under local and stochastic volatility models, where the credit value adjustment (CVA) is taken into account.

**Keywords**: Forward-Backward Stochastic Differential Equations (FBSDEs), Asymptotic expansion, Malliavin calculus, CVA, Local volatility model, Stochastic volatility model

## 1 Introduction

In this paper, we propose a new asymptotic expansion scheme with its error estimate for the forward-backward stochastic differential equations (FBSDEs). As an application, we derive a recursive expansion formula for option prices with CVA under local and stochastic volatility models and show numerical examples.

Bismut [1] introduced the backward stochastic differential equations (BSDEs) for the linear case, and Pardoux and Peng [28] initiated the study for the non-linear BSDEs. Since then, in addition to its theoretical researches, substantial numbers of numerical schemes for the solutions to the BSDEs have been proposed. The one of the main reasons is that the BSDEs are closely related to various valuation problems in finance (e.g. pricing securities under asymmetric/imperfect collateralization, optimal portfolio and indifference pricing issues in incomplete and/or constrained markets, modeling credit risks). Their financial applications are discussed in details for example, El Karoui et al. [7], Ma and Yong [22], a recent book edited by Carmona [2] and references therein.

Although a large number of finite difference methods and simulation-based methods were proposed for numerical approximations of the solutions to BSDEs, their analytical approximation methods have been rarely discussed. Fujii and Takahashi [8], [9], [12], Fujii et. al. [13] are exceptions, where they presented a simple analytical approximation with perturbation or/and interacting particle scheme for non-linear fully coupled FBSDEs without error estimate. Especially, Fujii and Takahashi [9] derived an approximation formula for dynamic optimal portfolio in an incomplete market with stochastic volatility, and confirmed its validity through numerical experiment.

This paper presents a new analytical approximation method for the FBSDEs based on a Picard-type iteration and an asymptotic expansion (for the asymptotic expansion approach, see Takahashi and Yamada [33] [34] and related previous works [30][24][31][35][29] for example). Also, our method can be regarded as an extension of the representation theorem of BSDEs by Ma and Zhang [23]. Ma and Zhang's result is known as the gradient representation of BSDEs without differentiation, *i.e.* for a system of BSDE

$$X_s^{t,x} \quad = \quad x + \int_t^s b(u, X_u^{t,x})du + \sum_{j=1}^{d} \int_t^s \sigma_j(u, X_u^{t,x})dW_u^j, \tag{1}$$

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‡University of Tokyo

§Hitotsubashi University

$$Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T f(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,x})du - \int_s^T Z_u^{t,x} dW_s, \quad (2)$$

where $f, g$ are assumed to be only Lipschitz continuous, [23] showed the formula for the gradient of

$$u(t,x) = Y_t^{t,x} = E\left[g(X_T^{t,x}) + \int_s^T f(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,x})du\right] \quad (3)$$

as

$$\partial_x u(t,x)\sigma(t,x) = Z_t^{t,x} = E\left[g(X_T^{t,x})N_T^{t,x} + \int_s^T f(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,x})N_u^{t,x}du\right]\sigma(t,x) \quad (4)$$

where $N_u^{t,x} = \frac{1}{u-t}\int_t^u \sigma^{-1}(X_v^{t,x})\partial_x X_v^{t,x} dW_v$ is a Malliavin weight. Then by Ma and Zhang's result, BSDE $(Y_s^{t,x}, Z_s^{t,x}) = (u(s, X_s^{t,x}), \partial_x u(s, X_s^{t,x})\sigma(s, X_s^{t,x}))$ is represented without derivatives of parameters $f$ and $g$.

We expand this representation of BSDE by a perturbation method to obtain an analytical approximation. Roughly speaking, for a perturbed forward SDE $X_s^{\varepsilon,t,x}$, $\varepsilon \in (0,1]$ and an associated backward SDE $(Y_s^{\varepsilon,t,x}, Z_s^{\varepsilon,t,x})$ of the form

$$X_s^{\varepsilon,t,x} = x + \int_t^s b(u, X_u^{\varepsilon,t,x})du + \varepsilon\sum_{j=1}^d \int_t^s \sigma_j(u, X_u^{\varepsilon,t,x})dW_u^j, \quad (5)$$

$$Y_s^{\varepsilon,t,x} = g(X_T^{\varepsilon,t,x}) + \int_s^T f(u, X_u^{\varepsilon,t,x}, Y_u^{\varepsilon,t,x}, Z_u^{\varepsilon,t,x})du - \int_s^T Z_u^{\varepsilon,t,x} dW_s, \quad (6)$$

we show the following recursive asymptotic expansion around a Gaussian model $\bar{X}_s^{t,x} = X_s^{0,t,x} + \varepsilon\frac{\partial}{\partial\varepsilon}X_s^{\varepsilon,t,x}|_{\varepsilon=0}$: *i.e.* for $k \geq 0$, $N \geq 1$

$$\begin{aligned} Y_t^{\varepsilon,t,x} &\simeq u^{\varepsilon,k+1,N}(t,x) = E\left[g(\bar{X}_T^{t,x}) + \int_t^T f(s, \bar{X}_s^{t,x}, Y_s^{\varepsilon,k,N,t,x}, Z_s^{\varepsilon,k,N,t,x})ds\right] \\ &\quad + \sum_{i=1}^N \varepsilon^i E\left[g(\bar{X}_T^{t,x})\pi_{i,T}^{t,x} + \int_t^T f(s, \bar{X}_s^{t,x}, Y_s^{\varepsilon,k,N,t,x}, Z_s^{\varepsilon,k,N,t,x})\pi_{i,s}^{t,x}ds\right], \end{aligned} \quad (7)$$

$$\begin{aligned} Z_t^{\varepsilon,t,x} &\simeq (\partial_x u^{\varepsilon,k+1,N}\sigma)(t,x) = \left\{E\left[g(\bar{X}_T^{0,t,x})N_{0,T}^{t,x} + \int_t^T f(s, \bar{X}_s^{t,x}, Y_s^{\varepsilon,k,N,t,x}, Z_s^{\varepsilon,k,N,t,x})N_{0,s}^{t,x}ds\right]\right. \\ &\quad \left. + \sum_{i=1}^N \varepsilon^i E\left[g(\bar{X}_T^{0,t,x})N_{i,T}^{t,x} + \int_t^T f(s, \bar{X}_s^{t,x}, Y_s^{\varepsilon,k,N,t,x}, Z_s^{\varepsilon,k,N,t,x})N_{i,s}^{t,x}ds\right]\right\}\varepsilon\sigma(t,x), \end{aligned} \quad (8)$$

where $Y_s^{\varepsilon,k,N,t,x} = u^{\varepsilon,k,N}(s, \bar{X}_s^{t,x})$ and $Z_s^{\varepsilon,k,N,t,x} = \partial_x u^{\varepsilon,k,N}\sigma(s, \bar{X}_s^{t,x})$ with a usual asymptotic expansion $(u^{\varepsilon,0,N}, \partial_x u^{\varepsilon,0,N}\sigma)$ and the processes $\pi_{i,s}^{t,x}$, $i = 1, \cdots, N$ and $N_{i,s}^{t,x}$, $i = 0, 1, \cdots, N$ are Malliavin weights for the expansion. Applying properties of so called *Kusuoka-Stroock functions* introduced by Kusuoka [18], we obtain an error estimate of our scheme.

The organization of this paper is as follows: The next section describes an idea for our method using a well-known example. Section 3 generalizes the idea and summarizes our algorithm in a general setting. After Section 4 provides the notations and basic results used in later sections, Section 5 presents our main result with its proof. Applying our scheme, Section 6 provides a simple numerical example for pricing option with counterparty risk under local volatility and stochastic volatility models. Section 7 concludes.

## 2 Motivated example

In this section, we show an idea for our approximation method using the BSDE appearing in a well-known example of mathematical finance, so called "*hedging claims with higher interest rate for borrowing*" (e.g. [7], Cvitanic and Karatzas [3]).

Specifically, let us consider the following FBSDE:

$$\begin{aligned} dX_t &= \mu X_t dt + \sigma X_t dW_t, && (9) \\ X_0 &= x_0, \\ dY_t &= rY_t dt - f(Y_t, Z_t)dt + Z_t dW_t, && (10) \\ Y_T &= g(X_T) = \max(X_T - K_1, 0) - 2\max(X_T - K_2, 0), && (11) \end{aligned}$$

where $f(y,z) = (R-r)\max\left(\frac{z}{\sigma} - y, 0\right) - \left(\frac{\mu-r}{\sigma}\right)z$. When the borrowing rate $R$ is higher than the lending rate $r$ (i.e. $R > r$), the solution to the FBSDE above, $Y = \{Y_t : 0 \leq t \leq T\}$ represents the value process of a self-financing hedging strategy for a target payoff given by $g(X_T)$, and $Z$ stands for the hedging strategy where $Z_t/\sigma$ is the amount invested at time $t$ in the risky asset whose price process is given by $S$.[1] In particular, we note that the specification of $g(X_T)$ as an option spread creates both lending and borrowing in the strategy. Here, $r$, $R$, $\mu$ and $\sigma$ are assumed to be positive constants.

$Y = (Y_t)_{t\in[0,T]}$ is represented as the following non-linear expectation:

$$Y_t \quad = \quad e^{-r(T-t)}E\left[g(X_T)|\mathcal{F}_t\right] + e^{-r(T-t)}E\left[\int_t^T f(Y_u, Z_u)du|\mathcal{F}_t\right],$$

where $(\mathcal{F}_t)_t$ is the filtration generated by $W$, *i.e.*, $\mathcal{F}_t = \sigma(W_s; s \leq t)$, $t \in [0,T]$. We denote by $(X^{t,x}, Y^{t,x}, Z^{t,x})$ the adapted solution to the FBSDE's (9) and (10) restricted to $[t,T]$ with $X_t^{t,x} = x$, a.s. Next, define $u$ as

$$u(t,x) := Y_t^{t,x} = e^{-r(T-t)}E\left[g(X_T^{t,x})\right] + e^{-r(T-t)}E\left[\int_t^T f(Y_u^{t,x}, Z_u^{t,x})du\right].$$

Then, using this $u$, $Z = (Z_t)_{t\in[0,T]}$ is obtained as follows:

$$Z_t = \sigma X_t \frac{\partial}{\partial x}u(t, X_t).$$

Moreover, applying a representation result by [23], one has

$$Z_t^{t,x} = e^{-r(T-t)}\left\{E[g(X_T^{t,x})N_T^{t,x}] + E[\int_t^T f(Y_u^{t,x}, Z_u^{t,x})N_u^{t,x}du]\right\}\sigma x,$$

where $N^{t,x} = (N_s^{t,x})_{s\in[t,T]}$ is the Malliavin weight process given $X_t = x$, $t \in [0,T]$:

$$N_u^{t,x} = \frac{1}{u-t}\int_t^u \sigma^{-1}(X_\tau^{t,x})\frac{\partial}{\partial x}X_\tau^{t,x}dW_\tau.$$

Next, let us show an example of an analytical approximation for the BSDE using the Picard-type iteration. In the first place, define $u^0(t,x)$ as

$$u^0(t,x) \quad := \quad e^{-r(T-t)}E\left[g(X_T^{t,x})\right]. \tag{12}$$

Then, *the Malliavin weight representation for the Delta* under Black-Scholes model (9) is well-known, that is given by

$$\frac{\partial}{\partial x}u^0(t,x) = e^{-r(T-t)}E\left[g(X_T^{t,x})\frac{1}{T-t}\int_t^T \frac{1}{x\sigma}dW_u\right]. \tag{13}$$

In this simple model, we are capable of its evaluation through one dimensional integrations. That is, we have

$$u^0(t,x) \quad = \quad e^{-r(T-t)}\int_{\mathbf{R}} g(e^y)p(t,T,z,y)dy, \tag{14}$$

and

$$\frac{\partial}{\partial x}u^0(t,x) \quad = \quad e^{-r(T-t)}\int_{\mathbf{R}} g(e^y)w(t,z,y)p(t,T,z,y)dy,$$

where $p(t,T,z,y)$ is the density of $\log X_T^{t,x}$ under (9) with $\log x = z$:

$$p(t,T,z,y) = \frac{1}{\sqrt{2\pi\sigma^2(T-t)}}\exp\left(-\frac{(y-z-\mu(T-t)+\frac{1}{2}\sigma^2(T-t))^2}{2\sigma^2(T-t)}\right). \tag{15}$$

[1]The problem is considered under the physical measure and $\left(\frac{\mu-r}{\sigma}\right)$ represents the market price of risk.

the finite dimensional Malliavin weight $w(t,z,y)$ is given by

$$w(t,z,y) = E\left[\frac{1}{T-t}\int_t^T \frac{1}{x\sigma}dW_u | \log X_T^{t,x} = y\right] = \frac{(y - z - \mu(T-t) + \frac{1}{2}\sigma^2(T-t))}{e^z\sigma^2(T-t)}. \tag{16}$$

Hence, we get the 0-th iteration $(Y^0, Z^0) = (Y_t^0, Z_t^0)_{t\in[0,T]}$ as

$$Y_t^0 = u^0(t, X_t), \quad Z_t^0 = \sigma X_t \frac{\partial}{\partial x}u^0(t, X_t).$$

Next, using the function $u^0(t,x)$, we define $u^1(t,x)$ as

$$u^1(t,x) \quad := \quad u^0(t,x) + e^{-r(T-t)}\int_t^T \int_{\mathbf{R}} f\left(u^0(v,e^y), \sigma e^y \frac{\partial}{\partial x}u^0(v,e^y)\right) p(t,v,z,y)dydv,$$

where $z = \log x$. Then, applying the same weight $w$ as (16), we are able to evaluate $\frac{\partial}{\partial x}u^1(t,x)$:

$$\begin{aligned}\frac{\partial}{\partial x}u^1(t,x) &= \frac{\partial}{\partial x}u^0(t,x) \\ &+ e^{-r(T-t)}\int_t^T \int_{\mathbf{R}} f\left(u^0(v,e^y), \sigma e^y \frac{\partial}{\partial x}u^0(v,e^y)\right) w(v,z,y)p(t,v,z,y)dydv.\end{aligned}$$

Therefore, the first iteration is given by

$$Y_t^1 = u^1(t, X_t), \quad Z_t^1 = \sigma X_t \frac{\partial}{\partial x}u^1(t, X_t).$$

Thus, for $k \geq 1$ let us recursively define $u^{k+1}(t,x)$ as

$$u^{k+1}(t,x) \quad := \quad u^0(t,x) + e^{-r(T-t)}\int_t^T \int_{\mathbf{R}} f\left(u^k(v,e^y), \sigma e^y \frac{\partial}{\partial x}u^k(v,e^y)\right) p(t,v,z,y)dydv,$$

where $z = \log x$, which leads to the evaluation of $\frac{\partial}{\partial x}u^{k+1}(t,x)$ with the same weight $w$ as (16):

$$\begin{aligned}\frac{\partial}{\partial x}u^{k+1}(t,x) &= \frac{\partial}{\partial x}u^0(t,x) \\ &+ e^{-r(T-t)}\int_t^T \int_{\mathbf{R}} f\left(u^k(v,e^y), \sigma e^y \frac{\partial}{\partial x}u^k(v,e^y)\right) w(v,z,y)p(t,v,z,y)dydv.\end{aligned}$$

Hence, the $k+1$-iteration is obtained by

$$Y_t^{k+1} = u^{k+1}(t, X_t), \quad Z_t^{k+1} = \sigma X_t \frac{\partial}{\partial x}u^{k+1}(t, X_t).$$

Finally, applying the parameters so that $X_0 = 100$, $\sigma = 0.2$, $\mu = 0.05$, $r = 0.01$, $R = 0.06$, $T = 0.25$, $K_1 = 95$, $K_2 = 105$, let us show a numerical comparison of this iterated approximation scheme with their result.

- Benchmark value of $Y_0$ : 2.95 with 0.01 standard deviation, which is obtained by a regression-based Monte Carlo simulation of Gobet et al. [15].
- Our approximation values: **0-th iteration** = 2.7864, **the first iteration** = 2.9671, and **the second iteration** = 2.9531.

It is observed that our approximation values become closer to the benchmark one as the more iterations are implemented. In the following sections, we extend our method in a more general setting.

# 3 Summary of algorithm of asymptotic expansion for FBSDEs

In the example of Section 2, we made use of an explicit Gaussian density since the forward process is given by Black-Scholes model (9). However, when we consider a more complex forward process, the explicit density is no longer obtained in general. Let us consider the perturbed forward SDE (5) with smooth coefficients and ellipticity. Then, for $\varepsilon > 0$ we are able to derive closed form approximation of the density and its gradient by applying $N$-th order asymptotic expansion around a Gaussian model $\bar{X}_T^{t,x}$:

$$\begin{aligned} p^{X^\varepsilon}(t,T,x,y) &\simeq p^{\bar{X}}(t,T,x,y) + \sum_{i=1}^{N} \varepsilon^i E[\pi_{i,T}^{t,x} | \bar{X}_T^{t,x} = y] p^{\bar{X}}(t,T,x,y), & (17) \\ \frac{\partial}{\partial x} p^{X^\varepsilon}(t,T,x,y) &\simeq E[N_{0,T}^{t,x} | \bar{X}_T^{t,x} = y] p^{\bar{X}}(t,T,x,y) + \sum_{i=1}^{N} \varepsilon^i E[N_{i,T}^{t,x} | \bar{X}_T^{t,x} = y] p^{\bar{X}}(t,T,x,y), & (18) \end{aligned}$$

with the density $p^{\bar{X}}(t,T,x,y)$ of $\bar{X}_T^{t,x}$ and some Malliavin weights $\pi_{i,T}^{t,x}$, $i = 1,\cdots,N$ and $N_{i,T}^{t,x}$, $i = 0,1,\cdots,N$, which are explicitly defined in Section 5. For the following general BSDE (6) under suitable conditions, we define $(u^\varepsilon, \partial_x u^\varepsilon \sigma)$ as

$$\begin{aligned} u^\varepsilon(t,x) &= Y_t^{\varepsilon,t,x} = E[g(X_T^{\varepsilon,t,x})] + E\left[\int_t^T f(s, X_s^{\varepsilon,t,x}, Y_s^{\varepsilon,t,x}, Z_s^{\varepsilon,t,x}) ds\right] \\ &= \int_{\mathbf{R}^d} g(y) p^{X^\varepsilon}(t,T,x,y) dy + \int_t^T \int_{\mathbf{R}^d} f(s,y,u^\varepsilon(s,y), \partial_x u^\varepsilon \sigma(s,y)) p^{X^\varepsilon}(t,s,x,y) dy ds, & (19) \\ \partial_x u^\varepsilon \sigma(t,x) &= Z_t^{\varepsilon,t,x} = E[g(X_T^{\varepsilon,t,x}) N_T^{\varepsilon,t,x}] \varepsilon\sigma(t,x) + E\left[\int_t^T f(s, X_s^{\varepsilon,t,x}, Y_s^{\varepsilon,t,x}, Z_s^{\varepsilon,t,x}) N_s^{\varepsilon,t,x} ds\right] \varepsilon\sigma(t,x) \\ &= \int_{\mathbf{R}^d} g(y) E[N_T^{\varepsilon,t,x} | X_T^{\varepsilon,t,x} = y] p^{X^\varepsilon}(t,T,x,y) dy \varepsilon\sigma(t,x) \\ &\quad + \int_t^T \int_{\mathbf{R}^d} f(s,y,u^\varepsilon(s,y), \partial_x u^\varepsilon \sigma(s,y)) E[N_s^{\varepsilon,t,x} | X_s^{\varepsilon,t,x} = y] p^{X^\varepsilon}(t,s,x,y) dy ds \varepsilon\sigma(t,x), \\ & & (20) \end{aligned}$$

where $N_s^{\varepsilon,t,x} = \frac{1}{\varepsilon(s-t)} \int_t^s \sigma^{-1}(X_v^{\varepsilon,t,x}) \partial_x X_v^{\varepsilon,t,x} dW_v$. We approximate $(u^\varepsilon, \partial_x u^\varepsilon \sigma)$ using a sequence $(u^{\varepsilon,k,N}, \partial_x u^{\varepsilon,k,N}\sigma)_k$ in the following way.

1. $(u^{\varepsilon,0,N}, \partial_x u^{\varepsilon,0,N}\sigma)$**: An approximation of the 0-th iteration**

   The 0-th iteration is defined by

$$\begin{aligned} u^{\varepsilon,0}(t,x) &= E[g(X_T^{\varepsilon,t,x})] + E\left[\int_t^T f(s, X_s^{\varepsilon,t,x}, 0, 0) ds\right], & (21) \\ \partial_x u^{\varepsilon,0}\sigma(t,x) &= E[g(X_T^{\varepsilon,t,x}) N_T^{\varepsilon,t,x}] + E\left[\int_t^T f(s, X_s^{\varepsilon,t,x}, 0, 0) N_s^{\varepsilon,t,x} ds\right]. & (22) \end{aligned}$$

   Then, $(u^{\varepsilon,0}, \partial_x u^{\varepsilon,0}\sigma)$ is approximated by

$$\begin{aligned} Y_t^{\varepsilon,t,x} &= u^{\varepsilon,0}(t,x) \simeq u^{\varepsilon,0,N}(t,x) \\ &= \int_{\mathbf{R}^d} g(y) \{1 + \sum_{i=1}^{N} \varepsilon^i E[\pi_{i,T}^{t,x} | \bar{X}_T^{t,x} = y]\} p^{\bar{X}}(t,T,x,y) dy \\ &\quad + \int_t^T \int_{\mathbf{R}^d} f(s,y,0,0) \{1 + \sum_{i=1}^{N} \varepsilon^i E[\pi_{i,s}^{t,x} | \bar{X}_s^{t,x} = y]\} p^{\bar{X}}(t,s,x,y) dy ds, \\ Z_t^{\varepsilon,t,x} &= \partial_x u^{\varepsilon,0}\sigma(t,x) \simeq \partial_x u^{\varepsilon,0,N}\sigma(t,x) \\ &= \int_{\mathbf{R}^d} g(y) \sum_{i=0}^{N} \varepsilon^i E[N_{i,T}^{t,x} | \bar{X}_T^{t,x} = y] p^{\bar{X}}(t,T,x,y) dy \varepsilon\sigma(t,x) \\ &\quad + \int_t^T \int_{\mathbf{R}^d} f(s,y,0,0) \sum_{0=1}^{N} \varepsilon^i E[N_{i,s}^{t,x} | \bar{X}_s^{t,x} = y] p^{\bar{X}}(t,s,x,y) dy ds \varepsilon\sigma(t,x). \end{aligned}$$

   Note that the Malliavin weights $\pi_{i,s}^{t,x}$, $i = 1,\cdots,N$ and $N_{i,s}^{t,x}$, $i = 0,1,\cdots,N$ are same as in (17) and (18).

2. $(u^{\varepsilon,1,N}, \partial_x u^{\varepsilon,1,N}\sigma)$**: An approximation of the first iteration**
   The first iteration is defined by

$$u^{\varepsilon,1}(t,x) = E[g(X_T^{\varepsilon,t,x}) + \int_t^T f(s, X_s^{\varepsilon,t,x}, u^{\varepsilon,0}(s, X_s^{\varepsilon,t,x}), (\partial_x u^{\varepsilon,0}\sigma)(s, X_s^{\varepsilon,t,x}))ds], \tag{23}$$

$$\partial_x u^{\varepsilon,1}\sigma(t,x) = E[g(X_T^{\varepsilon,t,x})N_T^{\varepsilon,t,x} + \int_t^T f(s, X_s^{\varepsilon,t,x}, u^{\varepsilon,0}(s, X_s^{\varepsilon,t,x}), (\partial_x u^{\varepsilon,0}\sigma)(s, X_s^{\varepsilon,t,x}))N_s^{\varepsilon,t,x} ds]\varepsilon\sigma(t,x). \tag{24}$$

   Firstly, define

$$\hat{u}^{\varepsilon,1}(t,x) = E[g(X_T^{\varepsilon,t,x}) + \int_t^T f(s, X_s^{\varepsilon,t,x}, u^{\varepsilon,0,N}(s, X_s^{\varepsilon,t,x}), (\partial_x u^{\varepsilon,0,N}\sigma)(s, X_s^{\varepsilon,t,x}))ds],$$

$$\partial_x \hat{u}^{\varepsilon,1}\sigma(t,x) = E[g(X_T^{\varepsilon,t,x}) + \int_t^T f(s, X_s^{\varepsilon,t,x}, u^{\varepsilon,0,N}(s, X_s^{\varepsilon,t,x}), (\partial_x u^{\varepsilon,0,N}\sigma)(s, X_s^{\varepsilon,t,x}))ds]\varepsilon\sigma(t,x).$$

   $(\hat{u}^{\varepsilon,1}, \partial_x \hat{u}^{\varepsilon,1}\sigma)$ is an approximation of $(u^{\varepsilon,1}, \partial_x u^{\varepsilon,1}\sigma)$:

$$u^{\varepsilon,1}(t,x) \simeq \hat{u}^{\varepsilon,1}(t,x), \quad \partial_x u^{\varepsilon,1}\sigma(t,x) \simeq \partial_x \hat{u}^{\varepsilon,1}\sigma(t,x).$$

   Using the approximations (17) and (18) again, we expand $(\hat{u}^{\varepsilon,1}, \partial_x \hat{u}^{\varepsilon,1}\sigma)$ with respect to $\varepsilon$ as follows:

$$\begin{aligned} & \hat{u}^{\varepsilon,1}(t,x) \\ \simeq\ & u^{\varepsilon,1,N}(t,x) := \int_{\mathbf{R}^d} g(y)\{1 + \sum_{i=1}^N \varepsilon^i E[\pi_{i,T}^{t,x} | \bar{X}_T^{t,x} = y]\} p^{\bar{X}}(t,T,x,y)dy \\ & + \int_t^T \int_{\mathbf{R}^d} f(s, y, u^{\varepsilon,0,N}(s,y), (\partial_x u^{\varepsilon,0,N}\sigma)(s,y))\{1 + \sum_{i=1}^N \varepsilon^i E[\pi_{i,s}^{t,x} | \bar{X}_s^{t,x} = y]\} p^{\bar{X}}(t,s,x,y)dyds. \end{aligned} \tag{25}$$

$$\begin{aligned} & \partial_x \hat{u}^{\varepsilon,1}\sigma(t,x) \\ \simeq\ & \partial_x u^{\varepsilon,1,N}\sigma(t,x) := \int_{\mathbf{R}^d} g(y) \sum_{i=0}^N \varepsilon^i E[N_{i,T}^{t,x} | \bar{X}_T^{t,x} = y] p^{\bar{X}}(t,T,x,y)dy\varepsilon\sigma(t,x) \\ & + \int_t^T \int_{\mathbf{R}^d} f(s, y, u^{\varepsilon,0,N}(s,y), (\partial_x u^{\varepsilon,0,N}\sigma)(s,y)) \sum_{i=0}^N \varepsilon^i E[N_{i,s}^{t,x} | \bar{X}_s^{t,x} = y] p^{\bar{X}}(t,s,x,y)dyds\varepsilon\sigma(t,x). \end{aligned} \tag{26}$$

   Since $Y_t^{\varepsilon,1,t,x} = u^{\varepsilon,1}(t,x)$ and $Z_t^{\varepsilon,1,t,x} = \partial_x u^{\varepsilon,1}\sigma(t,x)$, we get approximation $Y_t^{\varepsilon,1,t,x} \simeq u^{\varepsilon,1,N}(t,x)$ and $Z_t^{\varepsilon,1,t,x} \simeq \partial_x u^{\varepsilon,1,N}\sigma(t,x)$ using (25) and (26). Then, $Y_s^{\varepsilon,0,N,t,x}$ and $Z_s^{\varepsilon,0,N,t,x}$ are given by $Y_s^{\varepsilon,0,N,t,x} = u^{\varepsilon,0,N}(s, \bar{X}_s^{t,x})$ and $Z_s^{\varepsilon,0,N,t,x} = \partial_x u^{\varepsilon,0,N}\sigma(s, \bar{X}_s^{t,x})$.

3. **Numerical approximation for** $(u^{\varepsilon}, \partial_x u^{\varepsilon}\sigma)$
   Iterating the procedure above, we obtain the following numerical approximation for $(u^{\varepsilon}, \partial_x u^{\varepsilon}\sigma)$: for $k \in \mathbf{N}$,

$$\begin{aligned} & u^{\varepsilon}(t,x) \\ \simeq\ & u^{\varepsilon,k,N}(t,x) = \int_{\mathbf{R}^d} g(y)\{1 + \sum_{i=1}^N \varepsilon^i E[\pi_{i,T}^{t,x} | \bar{X}_T^{t,x} = y]\} p^{\bar{X}}(t,T,x,y)dy \\ & + \int_t^T \int_{\mathbf{R}^d} f(s, y, u^{\varepsilon,k-1,N}(s,y), (\partial_x u^{\varepsilon,k-1,N}\sigma)(s,y)) \\ & \qquad \{1 + \sum_{i=1}^N \varepsilon^i E[\pi_{i,s}^{t,x} | \bar{X}_s^{t,x} = y]\} p^{\bar{X}}(t,s,x,y)dyds, \end{aligned} \tag{27}$$

$$\begin{aligned} & \partial_x u^{\varepsilon}\sigma(t,x) \\ \simeq\ & \partial_x u^{\varepsilon,k,N}\sigma(t,x) = \int_{\mathbf{R}^d} g(y) \sum_{i=0}^N \varepsilon^i E[N_{i,T}^{t,x} | \bar{X}_T^{t,x} = y] p^{\bar{X}}(t,T,x,y)dy\varepsilon\sigma(t,x) \end{aligned}$$

$$+\int_t^T \int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k-1,N}(s,y),(\partial_x u^{\varepsilon,k-1,N}\sigma)(s,y))$$
$$\sum_{i=0}^N \varepsilon^i E[N_{i,s}^{t,x}|\bar{X}_s^{t,x}=y]p^{\bar{X}}(t,s,x,y)dyds\varepsilon\sigma(t,x). \quad (28)$$

Then, $Y_s^{\varepsilon,k,N,t,x}$ and $Z_s^{\varepsilon,k,N,t,x}$ are given by $Y_s^{\varepsilon,k,N,t,x}=u^{\varepsilon,k,N}(s,\bar{X}_s^{t,x})$ and $Z_s^{\varepsilon,k,N,t,x}=\partial_x u^{\varepsilon,k,N}\sigma(s,\bar{X}_s^{t,x})$.

We show this conjecture and derivation rigorously using Malliavin calculus in Section 5.

# 4 Notations and basic results

Hereafter, we use the following notations. Let $\mathbf{E}$ (or $\mathbf{E}_1$) be a generic Euclidean space.

- $\partial_x$: $\partial_x=(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_d})$.
- $C(T,x)$: a generic non-negative, non-decreasing and finite function of at most polynomial growth in $x$ depending on $T>0$.
- $C_b^\infty(\mathbf{E};\mathbf{E}_1)$: the space of all infinitely differentiable functions $\varphi:\mathbf{E}\to\mathbf{E}_1$ such that the all of its derivatives are bounded. We write $C_b^\infty(\mathbf{E})$ for $C_b^\infty(\mathbf{E};\mathbf{R})$.

We also prepare the basic notations and definitions of Malliavin calculus.

- $(\Omega,H,P)$: the standard $d$-dimensional Wiener space.
  - $\Omega$: the continuous functions $w:[0,T]\to\mathbf{R}^d$ such that $w(0)=0$.
  - $H$: the Cameron-Martin space of all absolutely continuous functions $h:[0,T]\to\mathbf{R}^d$ with a square integrable derivative, *i.e.*, $h'\in L^2([0,T];\mathbf{R}^d)$, $h'(t)=\frac{d}{dt}h(t)$. Here, $L^2([0,T];\mathbf{R}^d)$ is the space of all $\mathbf{R}^d$-measurable functions $\varphi$ on $[0,T]$ such that $\left(\int_0^T|\varphi(s)|^2ds\right)^{1/2}<\infty$.
  - $P$: the Wiener measure.
- $L^p(\Omega;G)$: the space of all random variables $F:\Omega\to G$ such that $E[\|F\|_G^p]<\infty$ where $G$ is a separable Hilbert space equipped with the norm $\|\cdot\|_G$ and $p\in[1,\infty)$. We write $L^p(\Omega)$ when $G=\mathbf{R}$ and $\|F\|_p=E[|F|^p]^{1/p}$ for $F\in L^p(\Omega)$.
- $\mathcal{S}$: The set of random variables $F$ of the form

$$F=\varphi\left(\int_0^T h_1'(s)dW_s,\cdots,\int_0^T h_n'(s)dW_s\right)$$

  where $\varphi\in C_b^\infty(\mathbf{R}^d)$, $h_1,\cdots,h_n\in H$, $n\geq 1$ with the notation $\int_0^T h_i'(s)dW_s=\sum_{j=1}^d\int_0^T h_{i,j}'(s)dW_s^i$, $h_i'=(h_{i,1}',\cdots,h_{i,d}')$.
- *Malliavin derivative operator* $D$: If $F\in\mathcal{S}$ is of the above form, we define its derivative as follows

$$DF=\sum_{i=1}^n\frac{\partial\varphi}{\partial x^i}\left(\int_0^T h_1'(s)dW_s,\cdots,\int_0^T h_n'(s)dW_s\right)h_i.$$

  The operator $D$ is closable from $L^p(\Omega)$ to $L^p(\Omega;H)$ for any $p\geq 1$.
- $\mathbf{D}^{k,p}$: For $F\in\mathcal{S}$, the iterated derivative $D^jF$, $j\in\mathbf{N}$ as a random variable with values in $H^{\otimes j}$. We denote by $\mathbf{D}^{k,p}$ the closure of $\mathcal{S}$ with respect to the seminorm

$$\|F\|_{k,p}=\left(E[|F|^p]+\sum_{j=1}^k E[\|D^jF\|_{H^{\otimes j}}^p]\right)^{1/p},\ p\in[1,\infty),\ k\in\mathbf{N}.$$

  Since $D^j$ is closable from $L^p(\Omega)$ to $L^p(\Omega;H^{\otimes j})$ for any $p\in[1,\infty)$ and $j\in\mathbf{N}$, $D^j$ is well defined in $\mathbf{D}^{k,p}$.
- $\mathbf{D}^\infty$: $\mathbf{D}^\infty=\cap_{p\geq 1}\cap_{k\geq 1}\mathbf{D}^{k,p}$.

- *Skorohod integral* $\delta$: We denote by $\delta$ the adjoint operator of the derivative operator $D$, that is an unbounded operator from $L^2(\Omega; H)$ into $L^2(\Omega)$ such that the domain of $\delta$, denoted by $Dom(\delta)$, is the set of $H$-valued square integrable random variables $u$ such that $|E[\langle DF, u\rangle_H]| \leq C\|F\|_2$, for all $F \in \mathbf{D}^{1,2}$, where $C$ is some constant depending on $u$. For $u \in Dom(\delta)$, $\delta(u)$ is characterized by the duality relationship:

$$E[F\delta(u)] = E[\langle DF, u\rangle_H]\,,\ \textit{for any}\ F \in \mathbf{D}^{1,2}.$$

  $\delta(u)$ is called Skorohod integral of the process $u$.
- $\mathbf{D}^{-\infty}$: the space of the Watanabe distributions (the dual of $\mathbf{D}^{\infty}$).

We say $F^{\varepsilon} = O(\varepsilon^n)$ *in* $\mathbf{D}^{k,p}$ as $\varepsilon \downarrow 0$ if $F^{\varepsilon} \in \mathbf{D}^{k,p}$ for all $\varepsilon \in (0,1]$ and $\limsup_{\varepsilon\downarrow 0} \|F^{\varepsilon}\|_{\mathbf{D}^{k,p}}/\varepsilon^n < \infty$ where $n$ is some real constant.

In our algorithm summarized in Section 3, we need to compute the asymptotic expansion $u^{k,N}$ recursively. From a numerical viewpoint, the stability of the approximation *i.e.* the asymptotic behavior of the asymptotic expansion when $t \uparrow T$ must be checked since we iteratively integrate $u^{k,N}$ with respect to time $t$. Hence, we introduce the Kusuoka-Stroock functions (Kusuoka [18]) which help to clarify the order of a Wiener functional with respect to time $t$.

**Definition 4.1 (Kusuoka-Stroock functions)** *Given $r \in \mathbf{R}$ and $n \in \mathbf{N}$, we denote by $\mathcal{K}_r^T(n)$ the set of functions $G : (0,T] \times \mathbf{R}^d \to \mathbf{D}^{n,\infty}$ satisfying the followings:*

1. *$G(t,\cdot)$ is $n$-times continuously differentiable and $[\partial^{\alpha} G/\partial x^{\alpha}]$ is continuous in $(t,x) \in (0,T] \times \mathbf{R}^d$ a.s. for any multi-index $\alpha$ of the elements of $\{1,\cdots,d\}$ with length $|\alpha| \leq n$.*
2. *For all $k \leq n - |\alpha|$, $p \in [1,\infty)$,*

$$\sup_{t\in(0,T],x\in\mathbf{R}^d} t^{-r/2} \left\| \frac{\partial^{\alpha} G}{\partial x^{\alpha}}(t,x) \right\|_{k,p} < \infty. \tag{29}$$

The above definition corresponds to Definition 2.1 of Crisan and Delarue [5] of modified version of Kusuoka [18]. We write $\mathcal{K}_r^T$ for $\cap_{n\in\mathbf{N}} \mathcal{K}_r^T(n)$.

Let $(X_s^{t,x})$ be the solution to the following stochastic differential equation:

$$\begin{aligned} X_s^{t,x} &= x + \int_t^s V_0(X_u^{t,x})du + \sum_{i=1}^d \int_t^s V_i(X_u^{t,x})dW_u^i, \\ X_t^{t,x} &= x \in \mathbf{R}^d, \end{aligned} \tag{30}$$

where each $V_i$, $i = 0,1,\cdots,d$ is bounded and belongs to $C_b^{\infty}(\mathbf{R}^d;\mathbf{R}^d)$. We assume that the elliptic condition holds.

**Lemma 4.1** *[Properties of Kusuoka-Stroock functions] The followings hold.*

1. *The function $(s,x) \in (0,T] \times \mathbf{R}^d \mapsto X_s^{t,x}$ belongs to $\mathcal{K}_0^T$, for any $T > 0$.*
2. *Suppose $G \in \mathcal{K}_r^T(n)$ where $r \geq 0$. Then, for $i = 1,\cdots,d$,*

$$\int_0^{\cdot} G(s,x)dW_s^i \in \mathcal{K}_{r+1}^T(n)\ \textit{and}\ \int_0^{\cdot} G(s,x)ds \in \mathcal{K}_{r+2}^T(n). \tag{31}$$

3. *If $G_i \in \mathcal{K}_{r_i}^T(n_i)$, $i = 1,\cdots,N$, then*

$$\prod_i^N G_i \in \mathcal{K}_{r_1+\cdots+r_N}^T(\min_i n_i)\ \textit{and}\ \sum_{i=1}^N G_i \in \mathcal{K}_{\min_i r_i}^T(\min_i n_i). \tag{32}$$

**Proof.** See Lemma 5.1.2 of Nee [26] for instance. □

Next, we summarize the Malliavin's integration by parts formula using Kusuoka-Stroock functions. For any multi-index $\alpha^{(k)} := (\alpha_1,\cdots,\alpha_k) \in \{1,\cdots,d\}^k$, $k \geq 1$, we denote by $\partial_{\alpha^{(k)}}$ the partial derivative $\frac{\partial^k}{\partial x_{\alpha_1}\cdots\partial x_{\alpha_k}}$.

**Proposition 4.1** *Let $G : (0,T] \times \mathbf{R}^d \to \mathbf{D}^\infty = \mathbf{D}^{\infty,\infty}$ be an element of $\mathcal{K}_r^T$ and let $f$ be a function that belongs to the space $C_b^\infty(\mathbf{R}^d)$. Then for any multi-index $\alpha^{(k)} \in \{1,\cdots,d\}^k$, $k \geq 1$, there exists $H_{\alpha^{(k)}}(X_s^{t,x}, G(s,x)) \in \mathcal{K}_{r-k}^T$ such that*

$$E\left[\partial_{\alpha^{(k)}} f(X_s^{t,x}) G(s,x)\right] = E\left[f(X_s^{t,x}) H_{\alpha^{(k)}}(X_s^{t,x}, G(s,x))\right], \tag{33}$$

*with*

$$\sup_{x\in\mathbf{R}^d} \|H_{\alpha^{(k)}}(X_s^{t,x}, G(s,x))\|_p \leq C(s-t)^{(r-k)/2}, \tag{34}$$

*where $H_{\alpha^{(k)}}(X_s^{t,x}, G(s,x))$ is recursively given by*

$$H_{(i)}(X_s^{t,x}, G(s,x)) = \delta\left(\sum_{j=1}^d G(s,x)\gamma_{ij}^{X_s^{t,x}} DX_s^{t,x,j}\right), \tag{35}$$

$$H_{\alpha^{(k)}}(X_s^{t,x}, G(s,x)) = H_{(\alpha_k)}(X_s^{t,x}, H_{\alpha^{(k-1)}}(X_s^{t,x}, G(s,x))), \tag{36}$$

*and a positive constant $C$. Here, $(\gamma_{ij}^{X_s^{t,x}})_{1\leq i,j\leq d}$ is the inverse matrix of the Malliavin covariance of $X_s^{t,x}$.*

**Proof**. Apply Corollary 3.7 of Kusuoka and Stroock [19] and Lemma 8-(3) of Kusuoka [18] with Proposition 2.1.4 of Nualart [25]. □

# 5 Asymptotic expansion for FBSDEs

## 5.1 Forward-backward SDE

Let $(\Omega, H, P)$ be the Wiener space on which a $d$-dimensional Brownian motion $W = (W^1,\cdots,W^d)$ is defined. Let $\mathcal{F}$ be the Borel algebra over $\Omega$ and $(\mathcal{F}_t)_{t\geq 0}$ be the natural filtration generated by $W$, augmented by the $P$-null sets of $\mathcal{F}$. In this section, we deal with a *small diffusion expansion* which corresponds to the framework in Kunitomo and Takahashi [16][17] and derive a general approximation formula for FBSDEs.

We give precise framework of our model. Consider the following $d$-dimensional perturbed forward stochastic differential equation $X_t^\varepsilon = (X_t^{1,\varepsilon},\cdots,X_t^{d,\varepsilon})$:

$$dX_t^{i,\varepsilon} = b^i(t,X_t^\varepsilon)dt + \varepsilon\sum_{j=1}^d \sigma_j^i(t,X_t^\varepsilon)dW_t^j, \quad i=1,\cdots,d, \tag{37}$$

where $b : [0,T]\times\mathbf{R}^d \to \mathbf{R}^d$, $\sigma : [0,T]\times\mathbf{R}^d \to \mathbf{R}^{d\times d}$ and $\varepsilon \in (0,1]$.

Next, we introduce the associated BSDE $(Y^\varepsilon, Z^\varepsilon)$ as follows:

$$Y_t^\varepsilon = g(X_T^\varepsilon) + \int_t^T f(s, X_s^\varepsilon, Y_s^\varepsilon, Z_s^\varepsilon)ds - \int_t^T Z_s^\varepsilon dW_s, \tag{38}$$

where $g : \mathbf{R}^d \to \mathbf{R}$ and $f : [0,T]\times\mathbf{R}^d\times\mathbf{R}\times\mathbf{R}^d \to \mathbf{R}$. Remark that for $\varepsilon = 0$, since the forward SDE $X_t^0$ degenerates, does BSDE $(Y_t^0, Z_t^0)$, too.

We put some conditions below on the above forward-backward SDE.

**Assumption 5.1**

1. *The coefficients of forward process $b$, $\sigma$ are bounded Borel functions and $C_b^\infty$ in $x$.*
2. *There exist constants $a_i > 0$, $i = 1,2$ such that for any vector $\xi$ in $\mathbf{R}^d$ and any $(t,x) \in [0,T]\times\mathbf{R}^d$,*

$$a_1|\xi|^2 \leq \sum_{i,j=1}^d [\sigma\sigma^\top]_{i,j}(t,x)\xi_i\xi_j \leq a_2|\xi|^2. \tag{39}$$

3. *The driver $f : [0,T]\times\mathbf{R}^d\times\mathbf{R}\times\mathbf{R}^d \to \mathbf{R}$ is continuous in $t$ and uniformly Lipschitz continuous in $x,y,z$ with constant $C_L$, i.e. for all $t\in[0,T]$, $(x_1,y_1,z_1)$, $(x_2,y_2,z_2) \in \mathbf{R}^d\times\mathbf{R}\times\mathbf{R}^d$,*

$$|f(t,x_1,y_1,z_1) - f(t,x_2,y_2,z_2)| \leq C_L(|x_1-x_2| + |y_1-y_2| + |z_1-z_2|). \tag{40}$$

*Also, we assume*

$$|f(t,x,y,z)| \leq C_L(1+|x|+|y|+|z|). \tag{41}$$

*for $(t,x,y,z) \in [0,T]\times\mathbf{R}^d\times\mathbf{R}\times\mathbf{R}^d$.*

*4. $g$ is Lipschitz continuous function with constant $C_L$ on $\mathbf{R}^d$ and $|g(x)| \le C_L(1+|x|)$ for $x \in \mathbf{R}^d$.*

Under the assumption, there exists the unique solution $(Y^\varepsilon, Z^\varepsilon)$ such that for any $p>1$, $E\left[\sup_{0\le s\le T}|Y_s^\varepsilon|^p\right] + E\left[\left(\int_0^T |Z_s^\varepsilon|^2 ds\right)^{p/2}\right] < \infty$ (e.g. See Theorem 5.1 in [7]).

$(X^{\varepsilon,t,x}, Y^{\varepsilon,t,x}, Z^{\varepsilon,t,x})$ represents the adapted solutions to the FBSDE's (37) and (38), restricted to $[t,T]$ with $X_t^{\varepsilon,t,x} = x$, a.s. The representation (20) of Ma and Zhang [23] holds under Assumption 5.1.

## 5.2 Asymptotic expansion

Under 1 and 2 in Assumption 5.1, the solution to SDE $X_s^{\varepsilon,t,x}$ $(0 \le t < s \le T)$ has a smooth density $p^{X^\varepsilon}(t,s,x,y)$. In order to obtain the expansion of the density $p^{X^\varepsilon}(t,s,x,y)$, we approximate $X_s^{\varepsilon,t,x}$ by an asymptotic expansion around the solution to ordinary differential equation $X_s^{0,t,x} = x + \int_t^s b(u, X_u^{0,t,x})ds$.

Hereafter, let us denote by $X_{i,s}^{\varepsilon,t,x}$, $i \in \mathbf{N}$ the $i$-th order differentiation of $X_s^{\varepsilon,t,x}$ with respect to $\varepsilon$, *i.e.* $\frac{1}{i!}\frac{\partial^i}{\partial \varepsilon^i} X_s^{\varepsilon,t,x}$. In the first place, we provide a key result as the lemma below.

**Lemma 5.1** *For $s \in (t,T]$, we have $X_{i,s}^{\varepsilon,t,x} \in \mathcal{K}_i^T$, $i \in \mathbf{N}$.*

**Proof**. See Appendix A. □

Let us define $X_{i,s}^{0,t,x}$ as $\frac{1}{i!}\frac{\partial^i}{\partial\varepsilon^i} X_s^{\varepsilon,t,x}|_{\varepsilon=0}$, $i \in \mathbf{N}$. For every $p \in [1,\infty)$, $k \in \mathbf{N}$ and $N \in \mathbf{N}$,

$$X_s^{\varepsilon,t,x} = X_s^{0,t,x} + \sum_{i=1}^N \varepsilon^i X_{i,s}^{0,t,x} + O(\varepsilon^{N+1}) \ \ in \ \ \mathbf{D}^{k,p} \ \ as \ \varepsilon \downarrow 0. \tag{42}$$

Hereafter, we derive an asymptotic expansion of density of $X_T^{\varepsilon,t,x}$. Define $F_T^{\varepsilon,t,x}$ as $F_T^{\varepsilon,t,x} := \frac{X_T^{\varepsilon,t,x} - X_T^{0,t,x}}{\varepsilon}$ and then we have

$$F_T^{\varepsilon,t,x} = F_T^{0,t,x} + \sum_{i=1}^N \varepsilon^i F_{i,T}^{0,t,x} + O(\varepsilon^{N+1}) \ \ in \ \mathbf{D}^\infty, \tag{43}$$

where $F_T^{0,t,x} = X_{1,T}^{0,t,x}$, $F_{i,T}^{0,t,x} = X_{i+1,T}^{0,t,x}$, $i \ge 1$.

Let $\Sigma(t,T) = \{\Sigma_{i,j}(t,T)\}_{i,j}$ be the $d \times d$-matrix whose element is defined by

$$\Sigma_{i,j}(t,T) = \sum_{k=1}^d \int_t^T \hat{\sigma}_k^i(s, X_s^{0,t,x})\hat{\sigma}_k^j(s, X_s^{0,t,x})ds, \quad 1 \le i,j \le d, \tag{44}$$

where $\hat{\sigma}_k^i(s, X_s^{0,t,x}) = (\partial_x X_T^{0,t,x}(\partial_x X_s^{0,t,x})^{-1}\sigma_k(s, X_s^{0,t,x}))^i$.

Under Assumption 5.1 we obtain the following expansions for $E[\varphi(X_T^{\varepsilon,t,x})]$ with Lipschitz function $\varphi$, which are useful for giving the properties of the expansion of $Y^\varepsilon$ and proving our main result Theorem 5.1.

**Proposition 5.1** *For $N \in \mathbf{N}$ and a Lipschitz continuous function $\varphi : \mathbf{R}^d \to \mathbf{R}$ with constant $C_L$, there exists $C_N$ depending on $C_L$ and $N$ such that*

$$\left| E[\varphi(X_T^{\varepsilon,t,x})] - \left\{ E[\varphi(\bar{X}_T^{t,x})] + \sum_{i=1}^N \varepsilon^i E[\varphi(\bar{X}_T^{t,x})\pi_{i,T}^{t,x}] \right\} \right| \le \varepsilon^{N+1} C_N (T-t)^{(N+2)/2}, \tag{45}$$

*where $\bar{X}_T^{t,x} = X_T^{0,t,x} + \varepsilon X_{1,T}^{0,t,x}$ and $\pi_{i,T}^{t,x} = \sum_k^{(i)} H_{\alpha^{(k)}}(X_{1,T}^{0,t,x}, \prod_{l=1}^k X_{\beta_l+1,T}^{0,t,x,\alpha_l}) \in \mathcal{K}_i^T$, $i = 1,\cdots,N$. Here, $\sum_k^{(i)} \equiv \sum_{k=1}^i \sum_{\beta_1+\cdots+\beta_k=i, \beta_j\ge 1} \sum_{\alpha^{(k)}\in\{1,\cdots,d\}^k} \frac{1}{k!}$.*

**Proof**. See Appendix B. □

We also obtain an expansion for $E[\varphi(X_T^{\varepsilon,t,x})N_T^{\varepsilon,t,x}]\varepsilon\sigma(t,x)$ with Lipschitz function $\varphi$, which are useful for giving the properties of the expansion of $Z^\varepsilon$.

**Proposition 5.2** *For $N \in \mathbf{N}$ and a Lipschitz continuous function $\varphi : \mathbf{R}^d \to \mathbf{R}$ with constant $C_L$, there exists $C_N$ depending on $C_L$ and $N$ such that*

$$\left| E[\varphi(X_T^{\varepsilon,t,x})N_{0,T}^{\varepsilon,t,x}]\varepsilon\sigma(t,x) - \left\{ E[\varphi(\bar{X}_T^{t,x})N_{0,T}^{t,x}] + \sum_{i=1}^{N} \varepsilon^i E[\varphi(\bar{X}_T^{t,x})N_{i,T}^{t,x}] \right\} \varepsilon\sigma(t,x) \right|$$
$$\leq \varepsilon^{N+1} C_N (T-t)^{(N+1)/2}, \tag{46}$$

*where $\bar{X}_T^{t,x} = X_T^{0,t,x} + \varepsilon X_{1,T}^{0,t,x}$, $N_{0,T}^{t,x} = (N_{0,T}^{t,x,1} \cdots, N_{0,T}^{t,x,d})$ and $N_{i,T}^{t,x} = (N_{i,T}^{t,x,1}, \cdots, N_{i,T}^{t,x,d})$, $i = 1, \cdots, d$ are given by $N_{0,T}^{t,x,\eta} = \sum_{j=1}^d H_{(j)}(\bar{X}_T^{t,x}, \partial_\eta \bar{X}_T^{t,x,j})$, and $N_{i,T}^{t,x,\eta} = \sum_{j=1}^d H_{(j)}(\bar{X}_T^{t,x}, \partial_\eta \bar{X}_T^{t,x,j} \pi_{i,T}^{t,x}) + \partial_\eta \pi_{i,T}^{t,x}$, $1 \leq \eta \leq d$.*

**Proof**. See Appendix C. □

**Remark 5.1** *Using the similar arguments in Proposition 5.1 and 5.2, we are able to see the following results. For a measurable function $\varphi : \mathbf{R}^d \to \mathbf{R}$ of at most polynomial growth, there exists non-negative, non-decreasing and finite function $C(N,x)$ of at most polynomial growth in $x$ depending on $N$ such that*

$$\left| E[\varphi(X_T^{\varepsilon,t,x})] - \left\{ E[\varphi(\bar{X}_T^{t,x})] + \sum_{i=1}^{N} \varepsilon^i E[\varphi(\bar{X}_T^{t,x})\pi_{i,T}^{t,x}] \right\} \right| \leq \varepsilon^{N+1} C(N,x)(T-t)^{(N+1)/2}, \tag{47}$$

$$\left| E[\varphi(X_T^{\varepsilon,t,x})N_{0,T}^{\varepsilon,t,x}]\varepsilon\sigma(t,x) - \left\{ E[\varphi(\bar{X}_T^{t,x})N_{0,T}^{t,x}] + \sum_{i=1}^{N} \varepsilon^i E[\varphi(\bar{X}_T^{t,x})N_{i,T}^{t,x}] \right\} \varepsilon\sigma(t,x) \right|$$
$$\leq \varepsilon^{N+1} C(N,x)(T-t)^{N/2}, \tag{48}$$

*with the same weights in Proposition 5.1 and 5.2. In the above estimates, we do not use the smoothness of $\varphi$ while we use the Lipschitz differentiability in Proposition 5.1 and 5.2.*

Using the weights $\pi_{i,s}^{t,x}$, $i = 0, 1, \cdots, N$ in Proposition 5.1 and $N_{i,s}^{t,x}$, $i = 0, 1, \cdots, N$ in Proposition 5.2, we have formulas for $(u^{\varepsilon,k,N}, \partial_x u^{\varepsilon,k,N}\sigma)$ as (27) and (28) without using derivatives of $f$ and $g$.

The following property holds for $(u^{\varepsilon,k,N}, \partial_x u^{\varepsilon,k,N}\sigma)$ by Lipschitz continuity of $g$.

**Lemma 5.2** *For $k \geq 0$, $N \in \mathbf{N}$,*

$$|u^{\varepsilon,k,N}(t,x)| \leq C(T,x), \tag{49}$$
$$|\partial_x u^{\varepsilon,k,N}\sigma(t,x)| \leq C(T,x). \tag{50}$$

*where $C(T,x)$ denotes a generic non-negative, non-decreasing and finite function of at most polynomial growth in $x$ depending on $T$.*

**Proof**. See Appendix D. □

## 5.3 Error estimate

For any $\beta, \mu > 0$, let $H_{\beta,\mu}$ be the space of functions $v : [0,T] \times \mathbf{R}^d \to \mathbf{R}^n$ such that

$$\|v\|_{H_{\beta,\mu}}^2 = \int_0^T \int_{\mathbf{R}^d} e^{\beta s} |v(s,x)|^2 e^{-\mu|x|} dx ds < \infty.$$

We also define the space $H_{\beta,\mu,X}$. For any $\beta, \mu > 0$, and $X_s^\varepsilon$, $0 \leq s \leq T$ starting from $x$ at time 0, let $H_{\beta,\mu,X}$ be the space of functions $v : [0,T] \times \mathbf{R}^d \to \mathbf{R}^n$ such that

$$\|v\|_{H_{\beta,\mu,X}}^2 = \int_0^T \int_{\mathbf{R}^d} e^{\beta s} E[|v(s,X_s^\varepsilon)|^2] e^{-\mu|x|} dx ds < \infty.$$

Remark that the following well-known norm equivalence result holds (e.g. Proposition 3.8 in Gobet and Labert [14]): there exist two constants $c_1, c_2 > 0$ such that $v \in L^2([0,T] \times \mathbf{R}^d, e^{\beta s} ds \times e^{-\mu|x|} dx)$

$$c_1 \|v\|_{H_{\beta,\mu}}^2 \leq \|v\|_{H_{\beta,\mu,X}}^2 \leq c_2 \|v\|_{H_{\beta,\mu}}^2. \tag{51}$$

The next theorem is our main result, which evaluates a global approximation error of $(u^{\varepsilon,k,N}, \partial_x u^{\varepsilon,k,N}\sigma)$ (in (27) and (28)) for $(u^\varepsilon, \partial_x u^\varepsilon \sigma)$ (in (19) and (20)).

**Theorem 5.1** *Suppose that Assumption 5.1 holds. Let $C$ be $C = c_2/c_1$ and $\beta$ be such that $2CC_L^2(T+1) < \beta$ and fix $\delta := \frac{2CC_L^2(T+1)}{\beta} < 1$. Then, for arbitrary $k \geq 0$ and $N \in \mathbf{N}$, there exists $C_0(T)$ depending on $T$ and $C_1(T,N)$ depending on $T$ and $N$ such that*

$$\begin{aligned}&\|u^\varepsilon - u^{\varepsilon,k,N}\|^2_{H_{\beta,\mu}} + \|(\partial_x u^\varepsilon\sigma) - (\partial_x u^{\varepsilon,k,N}\sigma)\|^2_{H_{\beta,\mu}}\\ &\quad \leq \left\{ C_0(T)\cdot\delta^k + \varepsilon^{2(N+1)}C_1(T,N)\cdot\left(\frac{1-\delta^{k+1}}{1-\delta}\right)\right\}, \quad \varepsilon\in(0,1].\end{aligned}$$

**Proof.**
Note that the following inequality holds:

$$\begin{aligned}&\|u^\varepsilon - u^{\varepsilon,k,N}\|^2_{H_{\beta,\mu}} + \|\partial_x u^\varepsilon\sigma - \partial_x u^{\varepsilon,k,N}\sigma\|^2_{H_{\beta,\mu}}\\ \leq\ & 2(\|u^\varepsilon - u^{\varepsilon,k}\|^2_{H_{\beta,\mu}} + \|\partial_x u^\varepsilon\sigma - \partial_x u^{\varepsilon,k}\sigma\|^2_{H_{\beta,\mu}})\\ &+2(\|u^{\varepsilon,k} - u^{\varepsilon,k,N}\|^2_{H_{\beta,\mu}} + \|\partial_x u^{\varepsilon,k}\sigma - \partial_x u^{\varepsilon,k,N}\sigma\|^2_{H_{\beta,\mu}}).\end{aligned}$$

First, we show the error $\|u^\varepsilon - u^{\varepsilon,k}\|^2_{H_{\beta,\mu}} + \|(\partial_x u^\varepsilon\sigma) - (\partial_x u^{\varepsilon,k}\sigma)\|^2_{H_{\beta,\mu}}$ by using the norm equivalence, (51) and the similar argument in the proof of Theorem 2.1 in El Karoui et al. [7]:

$$\begin{aligned}&\|u^\varepsilon - u^{\varepsilon,k}\|^2_{H_{\beta,\mu}} + \|(\partial_x u^\varepsilon\sigma) - (\partial_x u^{\varepsilon,k}\sigma)\|^2_{H_{\beta,\mu}}\\ \leq\ & \frac{2CC_L^2(T+1)}{\beta}\{\|u^\varepsilon - u^{\varepsilon,k-1}\|^2_{H_{\beta,\mu}} + \|(\partial_x u^\varepsilon\sigma) - (\partial_x u^{\varepsilon,k-1}\sigma)\|^2_{H_{\beta,\mu}}\}.\end{aligned}$$

Therefore,

$$\|u^\varepsilon - u^{\varepsilon,k}\|^2_{H_{\beta,\mu}} + \|(\partial_x u^\varepsilon\sigma) - (\partial_x u^{\varepsilon,k}\sigma)\|^2_{H_{\beta,\mu}} \leq C_0(T)\cdot\left(\frac{2CC_L^2(T+1)}{\beta}\right)^k, \tag{52}$$

where $C_0(T)$ such that $\|u^\varepsilon - u^{\varepsilon,0}\|^2_{H_{\beta,\mu}} + \|(\partial_x u^\varepsilon\sigma) - (\partial_x u^{\varepsilon,0}\sigma)\|^2_{H_{\beta,\mu}} \leq C_0(T)$.

Next, we estimate the error $\|u^{\varepsilon,k} - u^{\varepsilon,k,N}\|^2_{H_{\beta,\mu}} + \|\partial_x u^{\varepsilon,k}\sigma - \partial_x u^{\varepsilon,k,N}\sigma\|^2_{H_{\beta,\mu}}$.

The difference $u^{\varepsilon,k+1} - u^{\varepsilon,k+1,N}$ is represented as follows:

$$\begin{aligned}&u^{\varepsilon,k+1}(t,x) - u^{\varepsilon,k+1,N}(t,x)\\ =\ & \int_{\mathbf{R}^d} g(y)p^{X^\varepsilon}(t,T,x,y)dy + \int_t^T\int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k}(s,y),(\partial_x u^{\varepsilon,k}\sigma)(s,y))p^{X^\varepsilon}(t,s,x,y)dyds\\ &-\int_{\mathbf{R}^d} g(y)\left\{p^{\bar{X}}(t,T,x,y) + \sum_{i=1}^N \varepsilon^i E[\pi^{t,x}_{i,T}|\bar{X}^{t,x}_T = y]p^{\bar{X}}(t,T,x,y)\right\}dy\\ &-\int_t^T\int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\partial_x u^{\varepsilon,k,N}\sigma)(s,y))\\ &\qquad\qquad\left\{p^{\bar{X}}(t,s,x,y) + \sum_{i=1}^N \varepsilon^i E[\pi^{t,x}_{i,T}|\bar{X}^{t,x}_s = y]p^{\bar{X}}(t,s,x,y)\right\}dyds\\ =\ & \int_{\mathbf{R}^d} g(y)p^{X^\varepsilon}(t,T,x,y)dy - \int_{\mathbf{R}^d} g(y)\left\{p^{\bar{X}}(t,T,x,y) + \sum_{i=1}^N \varepsilon^i E[\pi^{t,x}_{i,T}|\bar{X}^{t,x}_T = y]p^{\bar{X}}(t,T,x,y)\right\}dy\\ &+\int_t^T\int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k}(s,y),(\partial_x u^{\varepsilon,k}\sigma)(s,y))p^{X^\varepsilon}(t,s,x,y)dyds\\ &-\int_t^T\int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\partial_x u^{\varepsilon,k,N}\sigma)(s,y))p^{X^\varepsilon}(t,s,x,y)dyds\\ &+\int_t^T\int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\partial_x u^{\varepsilon,k,N}\sigma)(s,y))p^{X^\varepsilon}(t,s,x,y)dyds\\ &-\int_t^T\int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\partial_x u^{\varepsilon,k,N}\sigma)(s,y))\\ &\qquad\qquad\left\{p^0(t,s,x,y) + \sum_{i=1}^N \varepsilon^i E[\pi^{t,x}_{i,T}|\bar{X}^{t,x}_s = y]p^{\bar{X}}(t,s,x,y)\right\}dyds.\end{aligned}$$

Remark that after the second equality, we add the terms

$$\pm\int_t^T\int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\partial_x u^{\varepsilon,k,N}\sigma)(s,y))p^{X^\varepsilon}(t,s,x,y)dyds.$$

Let $I_1$, $I_2$ and $I_3$ be

$$\begin{aligned}
I_1(t,x) &:= \int_{\mathbf{R}^d} g(y)p^\varepsilon(t,T,x,y)dy - \int_{\mathbf{R}^d} g(y)\left\{p^{\bar{X}}(t,T,x,y)+\sum_{i=1}^N \varepsilon^i E[\pi_{i,T}^{t,x}|\bar{X}_T^{t,x}=y]p^{\bar{X}}(t,T,x,y)\right\}dy,\\
I_2(t,x) &:= \int_t^T\int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k}(s,y),(\partial_x u^{\varepsilon,k}\sigma)(s,y))p^{X^\varepsilon}(t,s,x,y)dyds\\
&\qquad -\int_t^T\int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\partial_x u^{\varepsilon,k,N}\sigma)(s,y))p^{X^\varepsilon}(t,s,x,y)dyds,\\
I_3(t,x) &:= \int_t^T\int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\partial_x u^{\varepsilon,k,N}\sigma)(s,y))p^{X^\varepsilon}(t,s,x,y)dyds\\
&\quad -\int_t^T\int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\partial_x u^{\varepsilon,k,N}\sigma)(s,y))\\
&\qquad\qquad \left\{p^{\bar{X}}(t,s,x,y)+\sum_{i=1}^N \varepsilon^i E[\pi_{i,T}^{t,x}|\bar{X}_s^{t,x}=y]p^{\bar{X}}(t,s,x,y)\right\}dyds.
\end{aligned}$$

The difference $(\partial_x u^{\varepsilon,k+1}\sigma)-(\partial_x u^{\varepsilon,k+1,N}\sigma)$ is represented as

$$\begin{aligned}
&(\partial_x u^{\varepsilon,k+1}\sigma)-(\partial_x u^{\varepsilon,k+1,N}\sigma)\\
=\;& \int_{\mathbf{R}^d} g(y)E[N_T^{\varepsilon,t,x}|X_T^{\varepsilon,t,x}=y]p^{X^\varepsilon}(t,T,x,y)dy\varepsilon\sigma(t,x)\\
&+\int_t^T\int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k}(s,y),(\partial_x u^{\varepsilon,k}\sigma)(s,y))E[N_s^{\varepsilon,t,x}|X_s^{\varepsilon,t,x}=y]p^{X^\varepsilon}(t,s,x,y)dyds\varepsilon\sigma(t,x)\\
&-\int_{\mathbf{R}^d} g(y)E[N_{0,T}^{t,x}|\bar{X}_T^{t,x}=y]p^{\bar{X}}(t,T,x,y)dy\varepsilon\sigma(t,x)\\
&-\sum_{i=1}^N\varepsilon^i\int_{\mathbf{R}^d} g(y)E[N_{i,T}^{t,x}|\bar{X}_T^{t,x}=y]p^{\bar{X}}(t,T,x,y)dy\varepsilon\sigma(t,x)\\
&-\int_t^T\int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\partial_x u^{\varepsilon,k,N}\sigma)(s,y))E[N_{0,s}^{t,x}|\bar{X}_s^{t,x}=y]p^{\bar{X}}(t,s,x,y)dyds\varepsilon\sigma(t,x)\\
&-\sum_{i=1}^N\varepsilon^i\int_t^T\int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\partial_x u^{\varepsilon,k,N}\sigma)(s,y))E[N_{i,T}^{t,x}|\bar{X}_T^{t,x}=y]p^{\bar{X}}(t,T,x,y)dyds\varepsilon\sigma(t,x)\\
=\;& \int_{\mathbf{R}^d} g(y)E[N_T^{\varepsilon,t}|X_T^{\varepsilon,t,x}=y]p^{X^\varepsilon}(t,T,x,y)dy\varepsilon\sigma(t,x)\\
&-\int_{\mathbf{R}^d} g(y)E[N_{0,T}^{t,x}|\bar{X}_T^{t,x}=y]p^{\bar{X}}(t,T,x,y)dy\varepsilon\sigma(t,x)\\
&-\sum_{i=1}^N\varepsilon^i\int_{\mathbf{R}^d} g(y)E[N_{i,T}^{t,x}|\bar{X}_T^{t,x}=y]p^{\bar{X}}(t,T,x,y)dy\varepsilon\sigma(t,x)\\
&+\int_t^T\int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k}(s,y),(\partial_x u^{\varepsilon,k}\sigma)(s,y))E[N_s^{\varepsilon,t,x}|X_s^{\varepsilon,t,x}=y]p^{X^\varepsilon}(t,s,x,y)dyds\varepsilon\sigma(t,x)\\
&-\int_t^T\int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\partial_x u^{\varepsilon,k,N}\sigma)(s,y))E[N_s^{\varepsilon,t,x}|X_s^{\varepsilon,t,x}=y]p^{X^\varepsilon}(t,s,x,y)dyds\varepsilon\sigma(t,x)\\
&+\int_t^T\int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\partial_x u^{\varepsilon,k,N}\sigma)(s,y))E[N_s^{\varepsilon,t}|X_s^{\varepsilon,t,x}=y]p^{X^\varepsilon}(t,s,x,y)dyds\varepsilon\sigma(t,x)\\
&-\int_t^T\int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\partial_x u^{\varepsilon,k,N}\sigma)(s,y))E[N_{0,s}^{t,x}|\bar{X}_s^{t,x}=y]p^{\bar{X}}(t,s,x,y)dyds\varepsilon\sigma(t,x)\\
&-\sum_{i=1}^N\varepsilon^i\int_t^T\int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\partial_x u^{\varepsilon,k,N}\sigma)(s,y))E[N_{i,s}^{t,x}|\bar{X}_s^{t,x}=y]p^{\bar{X}}(t,s,x,y)dyds\varepsilon\sigma(t,x).
\end{aligned}$$

Let

$$\begin{aligned}
&J_1(t,x)\\
:=\ &\int_{\mathbf{R}^d} g(y)E[N_T^{\varepsilon,t,x}|X_T^{\varepsilon,t,x}=y]p^{X^\varepsilon}(t,T,x,y)dy\varepsilon\sigma(t,x)\\
&-\int_{\mathbf{R}^d} g(y)E[N_{0,T}^{t,x}|\bar{X}_T^{t,x}=y]p^{\bar{X}}(t,T,x,y)dy\varepsilon\sigma(t,x)\\
&-\sum_{i=1}^N\varepsilon^i\int_{\mathbf{R}^d} g(y)E[N_{i,T}^{t,x}|\bar{X}_T^{t,x}=y]p^{\bar{X}}(t,T,x,y)dy\varepsilon\sigma(t,x),\\
&J_2(t,x)\\
:=\ &\int_t^T\int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k}(s,y),(\partial_x u^{\varepsilon,k}\sigma)(s,y))E[N_s^{\varepsilon,t}|X_s^{\varepsilon,t,x}=y]p^{X^\varepsilon}(t,s,x,y)dyds\varepsilon\sigma(t,x)\\
&-\int_t^T\int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\partial_x u^{\varepsilon,k,N}\sigma)(s,y))E[N_s^{\varepsilon,t}|X_s^{\varepsilon,t,x}=y]p^{X^\varepsilon}(t,s,x,y)dyds\varepsilon\sigma(t,x)\\
&J_3(t,x)\\
:=\ &\int_t^T\int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\partial_x u^{\varepsilon,k,N}\sigma)(s,y))E[N_s^{\varepsilon,t,x}|X_s^{\varepsilon,t,x}=y]p^{X^\varepsilon}(t,s,x,y)dyds\varepsilon\sigma(t,x)\\
&-\int_t^T\int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\partial_x u^{\varepsilon,k,N}\sigma)(s,y))E[N_{0,s}^{t,x}|\bar{X}_s^{t,x}=y]p^{\bar{X}}(t,s,x,y)dyds\varepsilon\sigma(t,x)\\
&-\sum_{i=1}^N\varepsilon^i\int_t^T\int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\partial_x u^{\varepsilon,k,N}\sigma)(s,y))E[N_{i,s}^{t,x}|\bar{X}_s^{t,x}=y]p^{\bar{X}}(t,s,x,y)dyds\varepsilon\sigma(t,x).
\end{aligned}$$

Then, we have

$$\|u^{\varepsilon,k+1}-u^{\varepsilon,k+1,N}\|^2_{H_{\beta,\mu}}\le 3\|I_1\|^2_{H_{\beta,\mu}}+3\|I_2\|^2_{H_{\beta,\mu}}+3\|I_3\|^2_{H_{\beta,\mu}},$$

$$\|(\partial_x u^{\varepsilon,k+1}\sigma)-(\partial_x u^{\varepsilon,k+1,N}\sigma)\|^2_{H_{\beta,\mu}}\le 3\|J_1\|^2_{H_{\beta,\mu}}+3\|J_2\|^2_{H_{\beta,\mu}}+3\|J_3\|^2_{H_{\beta,\mu}}.$$

By Proposition 5.1 and Proposition 5.2 and Lemma 5.2, we have the following estimates

$$\begin{aligned}
|I_1(t,x)| &= \left|\int_{\mathbf{R}^d} g(y)\left\{p^{X^\varepsilon}(t,T,x,y)-p^{\bar{X}}(t,T,x,y)-\sum_{i=1}^N\varepsilon^iE[\pi_{i,T}^{t,x}|\bar{X}_T^{t,x}=y]p^{\bar{X}}(t,T,x,y)\right\}dy\right|\\
&\le c(T,N,x)\varepsilon^{N+1}(T-t)^{(N+2)/2}, \qquad (53)
\end{aligned}$$

$$\begin{aligned}
|J_1(t,x)| &= \Bigg|\int_{\mathbf{R}^d} g(y)\Bigg\{E[N_T^{\varepsilon,t,x}|X_T^{\varepsilon,t,x}=y]p^{X^\varepsilon}(t,T,x,y)\\
&\quad -E[N_{0,T}^{t,x}|\bar{X}_T^{t,x}=y]p^{\bar{X}}(t,T,x,y)-\sum_{i=1}^N\varepsilon^iE[N_{i,T}^{t,x}|\bar{X}_T^{t,x}=y]p^{\bar{X}}(t,T,x,y)\Bigg\}dy\varepsilon\sigma(t,x)\Bigg|\\
&\le r(T,N,x)\varepsilon^{N+1}(T-t)^{(N+1)/2}, \qquad (54)
\end{aligned}$$

and

$$\begin{aligned}
|I_3(t,x)| &= \Bigg|\int_t^T\int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\partial_x u^{\varepsilon,k,N}\sigma)(s,y))\\
&\quad \left\{p^{X^\varepsilon}(t,s,x,y)-p^{\bar{X}}(t,s,x,y)-\sum_{i=1}^N\varepsilon^iE[\pi_{i,s}^{t,x}|\bar{X}_s^{t,x}=y]p^{\bar{X}}(t,s,x,y)\right\}dyds\Bigg|\\
&\le C(T,N,x)\varepsilon^{N+1}\int_t^T(s-t)^{(N+1)/2}ds\\
&= C(T,N,x)\varepsilon^{N+1}(T-t)^{(N+3)/2}, \qquad (55)
\end{aligned}$$

$$
\begin{aligned}
|J_3(t,x)| &= \Bigg| \int_t^T \int_{\mathbf{R}^d} f(s,y,u^{\varepsilon,k,N}(s,y),(\partial_x u^{\varepsilon,k,N}\sigma)(s,y)) \Bigg\{ E[N_T^{\varepsilon,t,x} | X_T^{\varepsilon,t,x}=y] p^{X^\varepsilon}(t,s,x,y) \\
&\quad - E[N_{0,s}^{t,x} | \bar{X}_s^{t,x}=y] p^{\bar{X}}(t,s,x,y) - \sum_{i=1}^N \varepsilon^i E[N_{i,s}^{t,x} | \bar{X}_s^{t,x}=y] p^{\bar{X}}(t,s,x,y) \Bigg\} dyds\varepsilon\sigma(t,x) \Bigg| \\
&\leq R(T,N,x)\varepsilon^{N+1} \int_t^T (s-t)^{N/2} ds \\
&= R(T,N,x)\varepsilon^{N+1}(T-t)^{(N+2)/2}. \qquad (56)
\end{aligned}
$$

Here, $c(T,N,x)$, $C(T,N,x)$, $r(T,N,x)$ and $R(T,N,x)$ are some non-negative, non-decreasing and finite functions of at most polynomial growth in $x$ depending on $T$ and $N$.

Therefore, we obtain

$$
\|I_1\|_{H_{\beta,\mu}}^2 \leq \varepsilon^{2(N+1)} K_1(T,N), \quad \|I_3\|_{H_{\beta,\mu}}^2 \leq \varepsilon^{2(N+1)} K_3(T,N),
$$
$$
\|J_1\|_{H_{\beta,\mu}}^2 \leq \varepsilon^{2(N+1)} L_1(T,N), \quad \|J_3\|_{H_{\beta,\mu}}^2 \leq \varepsilon^{2(N+1)} L_3(T,N),
$$

for some $K_1(T,N)$, $K_3(T,N)$, $L_1(T,N)$ and $L_3(T,N)$ depending on $T$ and $N$.

In order to estimate $\|I_2\|_{\beta,\mu}^2$ and $\|J_2\|_{\beta,\mu}^2$, we define

$$
\hat{u}^{\varepsilon,k+1}(t,x) = E[g(X_T^{\varepsilon,t,x})] + E\left[ \int_t^T f(s,X_s^{\varepsilon,t,x},u^{\varepsilon,k,N}(s,X_s^{\varepsilon,t,x}),(\partial_x u^{\varepsilon,k,N}\sigma)(s,X_s^{\varepsilon,t,x}))ds \right]. \qquad (57)
$$

Since $f$ is Lipschitz countinuous with constant $C_L$, again using the norm equivalence result, (51) and the similar argument in the proof of Theorem 2.1 in El Karoui et al. [7] we obtain

$$
\begin{aligned}
&\|I_2\|_{H_{\beta,\mu}}^2 \leq c_1^{-1} \|u^{\varepsilon,k+1} - \hat{u}^{\varepsilon,k+1}\|_{\beta,\mu,X^\varepsilon}^2 \\
&= c_1^{-1} \int_{\mathbf{R}^d} \int_0^T e^{\beta s} E[|u^{\varepsilon,k+1}(s,X_s^\varepsilon) - \hat{u}^{\varepsilon,k+1}(s,X_s^\varepsilon)|^2] ds e^{-\mu|x|} dx \\
&\leq c_1^{-1} \frac{T}{\beta} \int_{\mathbf{R}^d} E\Bigg[ \int_0^T e^{\beta s} |f(s,X_s^\varepsilon,u^{\varepsilon,k}(s,X_s^\varepsilon),\partial_x u^{\varepsilon,k}\sigma(s,X_s^\varepsilon)) \\
&\qquad - f(s,X_s^\varepsilon,u^{\varepsilon,k,N}(s,X_s^\varepsilon),(\partial_x u^{\varepsilon,k,N}\sigma)(s,X_s^\varepsilon))|^2 ds \Bigg] e^{-\mu|x|} dx \\
&\leq \frac{2c_1^{-1}C_L^2 T}{\beta} \int_{\mathbf{R}^d} E\Bigg[ \int_0^T e^{\beta s} \{ |u^{\varepsilon,k}(s,X_s^\varepsilon) - u^{\varepsilon,k,N}(s,X_s^\varepsilon)|^2 \\
&\qquad + |(\partial_x u^{\varepsilon,k}\sigma)(s,X_s^\varepsilon) - (\partial_x u^{\varepsilon,k,N}\sigma)(s,X_s^\varepsilon)|^2 \} ds \Bigg] e^{-\mu|x|} dx \\
&\leq \frac{2CC_L^2 T}{\beta} \{ \|u^{\varepsilon,k} - u^{\varepsilon,k,N}\|_{H_{\beta,\mu}}^2 + \|(\partial_x u^{\varepsilon,k}\sigma) - (\partial_x u^{\varepsilon,k,N}\sigma)\|_{H_{\beta,\mu}}^2 \},
\end{aligned}
$$

$$
\begin{aligned}
&\|J_2\|_{H_{\beta,\mu}}^2 \leq c_1^{-1} \|(\partial_x u^{\varepsilon,k+1}\sigma) - (\partial_x \hat{u}^{\varepsilon,k+1}\sigma)\|_{\beta,\mu,X^\varepsilon}^2 \\
&= c_1^{-1} \int_{\mathbf{R}^n} \int_0^T e^{\beta s} E[|(\partial_x u^{\varepsilon,k+1}\sigma)(s,X_s^\varepsilon) - (\partial_x \hat{u}^{\varepsilon,k+1}\sigma)(s,X_s^\varepsilon)|^2] ds e^{-\mu|x|} dx \\
&\leq c_1^{-1} \frac{1}{\beta} \int_{\mathbf{R}^d} E\Bigg[ \int_0^T e^{\beta s} |f(s,X_s^\varepsilon,u^{\varepsilon,k}(s,X_s^\varepsilon),(\partial_x u^{\varepsilon,k}\sigma)(s,X_s^\varepsilon)) \\
&\qquad - f(s,X_s^\varepsilon,u^{\varepsilon,k,N}(s,X_s^\varepsilon),(\partial_x u^{\varepsilon,k,N}\sigma)(s,X_s^\varepsilon))|^2 ds \Bigg] e^{-\mu|x|} dx \\
&\leq \frac{2c_1^{-1}C_L^2}{\beta} \int_{\mathbf{R}^d} E\Bigg[ \int_0^T e^{\beta s} \{ |u^{\varepsilon,k}(s,X_s^\varepsilon) - u^{\varepsilon,k,N}(s,X_s^\varepsilon)|^2 \\
&\qquad + |(\partial_x u^{\varepsilon,k}\sigma)(s,X_s^\varepsilon) - (\partial_x u^{\varepsilon,k,N}\sigma)(s,X_s^\varepsilon)|^2 \} ds \Bigg] e^{-\mu|x|} dx \\
&\leq \frac{2CC_L^2}{\beta} \{ \|u^{\varepsilon,k} - u^{\varepsilon,k,N}\|_{H_{\beta,\mu}}^2 + \|(\partial_x u^{\varepsilon,k}\sigma) - (\partial_x u^{\varepsilon,k,N}\sigma)\|_{H_{\beta,\mu}}^2 \}.
\end{aligned}
$$

Then, we have the following estimates

$$
\begin{aligned}
& \|u^{\varepsilon,k+1} - u^{\varepsilon,k+1,N}\|^2_{H_{\beta,\mu}} \\
\leq\ & \varepsilon^{2(N+1)} K(T,N) + \frac{2CC_L^2 T}{\beta}\{\|u^{\varepsilon,k} - u^{\varepsilon,k,N}\|^2_{H_{\beta,\mu}} + \|(\partial_x u^{\varepsilon,k}\sigma) - (\partial_x u^{\varepsilon,k,N}\sigma)\|^2_{H_{\beta,\mu}}\},
\end{aligned} \tag{58}
$$

$$
\begin{aligned}
& \|(\partial_x u^{\varepsilon,k+1}\sigma) - (\partial_x u^{\varepsilon,k+1,N}\sigma)\|^2_{H_{\beta,\mu}} \\
\leq\ & \varepsilon^{2(N+1)} L(T,N) + \frac{2CC_L^2}{\beta}\{\|u^{\varepsilon,k} - u^{\varepsilon,k,N}\|^2_{H_{\beta,\mu}} + \|(\partial_x u^{\varepsilon,k}\sigma) - (\partial_x u^{\varepsilon,k,N}\sigma)\|^2_{H_{\beta,\mu}}\},
\end{aligned} \tag{59}
$$

where $K(T,N) = 2\max\{K_1(T,N), K_3(T,N)\}$ and $L(T,N) = 2\max\{L_1(T,N), L_3(T,N)\}$. Therefore, by (58) and (59), we obtain

$$
\begin{aligned}
& \|u^{\varepsilon,k+1} - u^{\varepsilon,k+1,N}\|^2_{H_{\beta,\mu}} + \|(\partial_x u^{\varepsilon,k+1}\sigma) - (\partial_x u^{\varepsilon,k+1,N}\sigma)\|^2_{H_{\beta,\mu}} \\
\leq\ & \varepsilon^{2(N+1)}\gamma(T,N) \\
& + \frac{2CC_L^2(T+1)}{\beta}\{\|u^{\varepsilon,k} - u^{\varepsilon,k,N}\|^2_{H_{\beta,\mu}} + \|(\partial_x u^{\varepsilon,k}\sigma) - (\partial_x u^{\varepsilon,k,N}\sigma)\|^2_{H_{\beta,\mu}}\},
\end{aligned} \tag{60}
$$

where $\gamma(T,N) = 2\max\{K(T,N), L(T,N)\}$.

Remark that the differences $u^{\varepsilon,0} - u^{\varepsilon,0,N}$ and $\partial_x u^{\varepsilon,0}\sigma - \partial_x u^{\varepsilon,0,N}\sigma$ are given as follows:

$$
\begin{aligned}
& u^{\varepsilon,0}(t,x) - u^{\varepsilon,0,N}(t,x) \\
=\ & \int_{\mathbf{R}^d} g(y) p^{X^\varepsilon}(t,T,x,y)dy \\
& - \int_{\mathbf{R}} g(y)\left\{p^{\bar{X}}(t,T,x,y) + \sum_{i=1}^N \varepsilon^i E[\pi^{t,x}_{i,T}|\bar{X}^{t,x}_T = y] p^{\bar{X}}(t,T,x,y)\right\} dy \\
& + \int_t^T \int_{\mathbf{R}^d} f(s,y,0,0) p^{X^\varepsilon}(t,s,x,y)dyds \\
& - \int_t^T \int_{\mathbf{R}^d} f(s,y,0,0)\left\{p^{\bar{X}}(t,s,x,y) + \sum_{i=1}^N \varepsilon^i E[\pi^{t,x}_{i,T}|\bar{X}^{t,x}_s = y] p^{\bar{X}}(t,s,x,y)\right\} dyds
\end{aligned}
$$

and

$$
\begin{aligned}
& (\partial_x u^{\varepsilon,0}\sigma)(t,x) - (\partial_x u^{\varepsilon,0,N}\sigma)(t,x) \\
=\ & \int_{\mathbf{R}^d} g(y) E[N^{\varepsilon,t,x}_T | X^{\varepsilon,t,x}_T = y] p^{X^\varepsilon}(t,T,x,y) dy \varepsilon\sigma(t,x) \\
& - \int_{\mathbf{R}^d} g(y) E[N^{t,x}_{0,T}|\bar{X}^{t,x}_T = y] p^{\bar{X}}(t,T,x,y)dy\varepsilon\sigma(t,x) \\
& - \sum_{i=1}^N \varepsilon^i \int_{\mathbf{R}^d} g(y) E[N^{t,x}_{i,T}|\bar{X}^{t,x}_T = y] p^{\bar{X}}(t,T,x,y) dy\varepsilon\sigma(t,x) \\
& + \int_t^T \int_{\mathbf{R}^d} f(s,y,0,0) E[N^{\varepsilon,t,x}_s | X^{\varepsilon,t,x}_s = y] p^{X^\varepsilon}(t,s,x,y)dy\varepsilon\sigma(t,x) \\
& - \int_t^T \int_{\mathbf{R}^d} f(s,y,0,0) E[N^{t,x}_{0,s}|\bar{X}^{t,x}_s = y] p^{\bar{X}}(t,s,x,y)dy\varepsilon\sigma(t,x) \\
& - \sum_{i=1}^N \varepsilon^i \int_t^T \int_{\mathbf{R}^d} f(s,y,0,0) E[N^{t,x}_{i,s}|\bar{X}^{t,x}_s = y] p^{\bar{X}}(t,s,x,y) dyds\varepsilon\sigma(t,x).
\end{aligned}
$$

Then, the term $\|u^{\varepsilon,0} - u^{\varepsilon,0,N}\|^2_{H_{\beta,\mu}} + \|(\partial_x u^{\varepsilon,0}\sigma) - (\partial_x u^{\varepsilon,0,N}\sigma)\|^2_{H_{\beta,\mu}}$ is estimated by the asymptotic error, that is,

$$
\|u^{\varepsilon,0} - u^{\varepsilon,0,N}\|^2_{H_{\beta,\mu}} + \|(\partial_x u^{\varepsilon,0}\sigma) - (\partial_x u^{\varepsilon,0,N}\sigma)\|^2_{H_{\beta,\mu}} \leq \varepsilon^{2(N+1)} K_0(T,N),
$$

for some $K_0(T,N)$.

Therefore, we obtain

$$\begin{aligned}
& \|u^{\varepsilon,k+1} - u^{\varepsilon,k+1,N}\|^2_{H_{\beta,\mu}} + \|(\partial_x u^{\varepsilon,k+1}\sigma) - (\partial_x u^{\varepsilon,k+1,N}\sigma)\|^2_{H_{\beta,\mu}} \\
\leq\ & \varepsilon^{2(N+1)} C_1(T,N) + \frac{2CC_L^2(T+1)}{\beta}\{\|u^{\varepsilon,k} - u^{\varepsilon,k,N}\|^2_{H_{\beta,\mu}} + \|(\partial_x u^{\varepsilon,k}\sigma) - (\partial_x u^{\varepsilon,k,N}\sigma)\|^2_{H_{\beta,\mu}}\} \\
\leq\ & \varepsilon^{2(N+1)} C_1(T,N) \\
& + \frac{2CC_L^2(T+1)}{\beta}\Big\{\varepsilon^{2(N+1)} C_1(T,N) + \\
& \qquad \frac{2CC_L^2(T+1)}{\beta}\{\|u^{\varepsilon,k-1} - u^{\varepsilon,k-1,N}\|^2_{H_{\beta,\mu}} + \|(\partial_x u^{\varepsilon,k-1}\sigma) - (\partial_x u^{\varepsilon,k-1,N}\sigma)\|^2_{H_{\beta,\mu}}\}\Big\} \\
& \cdots \\
\leq\ & \varepsilon^{2(N+1)} C_1(T,N)\left\{\left(\frac{2CC_L^2(T+1)}{\beta}\right)^{k+1} + \cdots + \left(\frac{2CC_L^2(T+1)}{\beta}\right) + 1\right\} \\
=\ & \varepsilon^{2(N+1)} C_1(T,N) \cdot \left(\frac{1 - \left(\frac{2CC_L^2(T+1)}{\beta}\right)^{k+2}}{1 - \left(\frac{2CC_L^2(T+1)}{\beta}\right)}\right), \qquad (61)
\end{aligned}$$

where $C_1(T,N) = \max\{\gamma(T,N), K_0(T,N)\}$.

Finally, Choose $\beta$ such that $2CC_L^2(T+1) < \beta$ and set $\delta = \frac{2CC_L^2(T+1)}{\beta} < 1$, by (52) and (61) we obtain the global error

$$\|u^\varepsilon - u^{\varepsilon,k,N}\|^2_{H_{\beta,\mu}} + \|(\partial_x u^\varepsilon\sigma) - (\partial_x u^{\varepsilon,k,N}\sigma)\|^2_{H_{\beta,\mu}} \leq \left\{C_0(T)\cdot\delta^k + \varepsilon^{2(N+1)}C_1(T,N)\cdot\left(\frac{1-\delta^{k+1}}{1-\delta}\right)\right\}.$$

□

# 6 Application: pricing option with counterparty risk under local and stochastic volatility models

This section applies our approximation algorithm to option pricing with counterparty risk in a FBSDE setting. Here, we omit a discussion on modeling and pricing issues under default risk, and concentrate on the concrete description of our approximation scheme with investigation of its validity by using a simple example.[2] Particularly, we use local volatility and stochastic volatility models for the underlying (forward) price process $X$ under the risk-neutral measure. Let $Y$ be the solution to the following non-linear BSDE:

$$Y_t = g(X_T) - (1-R)\beta\int_t^T (Y_s)^+ ds - \int_t^T Z_s dW_s. \qquad (62)$$

Here, $Y$ represents the value process with a target payoff $g(X_T)$ taking the *risky (substitution) closing out CVA* into account; $R \geq 0$ and $\beta > 0$ denote a constant recovery rate and a constant default intensity, respectively. Also, the risk-free interest rate and the dividend rate of the underlying asset are assumed to be zero for simplicity. Next, let $(Y^k, Z^k)_{k\geq 0}$ be a sequence of the following linear BSDEs:

$$\begin{aligned}
Y_t^0 &= g(X_T) - \int_t^T Z_s^0 dW_s^1. \qquad (63)\\
Y_t^1 &= g(X_T) - (1-R)\beta\int_t^T (Y_s^0)^+ ds - \int_t^T Z_s^1 dW_s. \\
Y_t^{k+1} &= g(X_T) - (1-R)\beta\int_t^T (Y_s^k)^+ ds - \int_t^T Z_s^{k+1} dW_s, \quad k \geq 1,
\end{aligned}$$

which is an approximation sequence of the value process $Y$.

**Remark 6.1** *Under the setting above, suppose we consider plain-vanilla options, that is $g(X_T) = (X_T - K)^+$ or $(K - X_T)^+$. Then, given constant values of $R$ and $\beta$ as well as $Y^k > 0$ for usual setup of*

[2] See Fujii and Takahashi (2010, 2011) for the detail of modeling and pricing issues under default risk, for instance.

*parameters in practice, due to the martingale property of the (risk-free) option value $Y^0$ under the risk-neutral measure, we are able to express $u^k(t,x) := Y_t^{k,t,x}$ for each $k = 0, 1, 2, \cdots$ as follows:*

$$u^k(t,x) = u^0(t,x)\left[1 + \sum_{i=1}^{k} \frac{q^i}{i!}\right], \tag{64}$$

*where $q = (-1)(1-R)\beta(T-t)$. Hence, for this simplest case we can easily obtain the benchmark values $u^k(t,x)$ through evaluation of $u^0(t,x)$ by numerical computation such as the Monte Carlo simulation, against which the validity of our approximation scheme is examined. However, note that it is much more tough task to get the benchmark values under the situation with stochastic intensity and recovery, while our scheme is applicable under the setting without substantial effort.*

## 6.1 Local volatility model

First, we consider local volatility model (one dimensional process)

$$dX_t^\varepsilon = \varepsilon\sigma(t, X_t^\varepsilon)dW_t, \quad X_0^\varepsilon = x_0 \tag{65}$$

where $\sigma(t,x)$ is the local volatility function. Define $u^\varepsilon(t,x) := Y_t^{\varepsilon,t,x} = E\left[g(X_T^{\varepsilon,t,x})\right] - E\left[\int_t^T (1-R)\beta(Y_s^{\varepsilon,t,x})^+ ds\right]$. Then, $u^0(t,x) := Y_t^{\varepsilon,0,t,x,0}$, $u^{k+1}(t,x) := Y_t^{\varepsilon,k+1,t,x}$, $k \geq 0$, are approximated by

$$u^0(t,x) \simeq u^{0,N}(t,x) = \int_{\mathbf{R}} g(y) p_N^{\bar{X}}(t,T,x,y)dy, \tag{66}$$

$$\begin{aligned} u^{k+1}(t,x) &\simeq u^{k+1,N}(t,x) \\ &= u^{0,N}(t,x) - (1-R)\beta \int_t^T \int_{\mathbf{R}} (u^{k,N}(\tau,y))^+ p_N^{\bar{X}}(t,\tau,x,y)dyd\tau, \quad k \geq 0, \end{aligned} \tag{67}$$

where $y \mapsto p_N^{\bar{X}}(t,s,x,y)$ is the $N$-th order asymptotic expansion of the density of $X_s^{\varepsilon,t,x}$. In our numerical example, we take $\varepsilon\sigma(t,x) = \varepsilon x_0^{1-\alpha} x^\alpha$ (CEV volatility). Here, $\varepsilon$ can be regarded as the instantaneous volatility of the log-normal (or the Black-Scholes) process. The terminal condition for the backward SDE is characterized as $g(x) = (x-K)^+$, the call option payoff function.

The parameters of the model are specified as follows:

$$t = 0.0,\ T = 2.0,\ x_0 = 10{,}000,\quad \alpha = 0.5,\ \varepsilon = 0.1,$$
$$\beta = 0.06 \text{ (intensity)},\ R = 0.0 \text{ (recovery rate)}.$$

Also, the expansion order $N$ is set to be $N = 1$.

In this case, we can easily obtain $u^{0,N}(t,x)$ in (66) as follows:

$$u^{0,N}(t,x) = yN\left(\frac{y}{\sqrt{\Sigma(t,T)}}\right) + \left(\Sigma(t,T) - \frac{\zeta(t,T)}{\Sigma(t,T)}y\right) n[y:0,\Sigma(t,T)], \tag{68}$$

where $N(x)$ and $n[x:\mu,\Sigma]$ denote the standard normal distribution function, and the normal density function with the mean $\mu$ and the variance $\Sigma$, respectively. Also, $y$, $\Sigma(t,,T)$ and $\zeta(t,T)$ are defined in the following:

$$\begin{aligned} y &= x - K, \\ \Sigma(t,T) &= \varepsilon^2\sigma^2 x^{2\alpha}(T-t), \\ \zeta(t,T) &= \alpha\varepsilon^4\sigma^4 x^{4\alpha-1}\frac{(T-t)^2}{2}. \end{aligned} \tag{69}$$

The result is given in Table 1–3: **AE** $u^{k,N}(= u^{k,N}(0,x_0))$, $(k = 0,1,2)$ are evaluated based on the corresponding equations in (67) and (68). **Exact value** $u(0,x_0)$ is approximated as in (64) by the equation (70) below with $k = 5$, which gives the sufficiently convergent value for this case. Also, **Benchmark** $u^k = u^k(0,x_0)$, $k = 1,2$ are computed by the following equation (70) with $k = 1,2$, respectively:

$$u^0(0,x_0)\left[1 + \sum_{i=1}^{k} \frac{q^i}{i!}\right], \tag{70}$$

where $q = (-1)(1-R)\beta T$, and the value of $u^0(0,x_0)$ is obtained based on Monte Carlo simulation for the CEV process. In each simulation, the numbers of the trials and the time steps are 1,000,000 with the antithetic variable method and 750, respectively. Also, in Table 1–3 the relative errors denoted by **AE Error** $u$ and **AE Error** $u^k$ of our asymptotic expansion are computed by $(u^{k,N}(0,x_0)-u(0,x_0))/u(0,x_0)$ and $(u^{k,N}(0,x_0)-u^k(0,x_0))/u^k(0,x_0)$, respectively. It is observed that $u^{k,N}(=u^{k,N}(0,x_0))$, $k=1,2$, $N=1$ become closer towards $u(0,x_0)$.

Although this example use only the $\varepsilon^1$-order expansion of the density, we already know from the existing work (e.g. Takahashi et al. (2012)) that higher order expansions produce much more better approximation for the risk-free option price $u^0$, which is expected to provide more precise approximations for the solution to the BSDE as well.

Table 1: European call option price with CVA under CEV model (In-the-money case : $K = 7500$, **Exact value** $u(0,x_0) = 2230.24$)

| Iteration $k$ | **Benchmark** $u^k$ | **AE** $u^{k,N}$ | **AE Error** $u$ | **AE Error** $u^k$ |
|---|---|---|---|---|
| **0th** | 2514.59 | 2514.49 | 12.75% | 0.00% |
| **1st** | 2212.84 | 2212.81 | -0.78% | 0.00% |
| **2nd** | 2230.41 | 2231.11 | 0.04% | 0.01% |

Table 2: European call option price with CVA under CEV model (At-the-money case : $K = 10000$, **Exact value** $u(0,x_0) = 499.45$)

| Iteration $k$ | **Benchmark** $u^k$ | **AE** $u^{k,N}$ | **AE Error** $u$ | **AE Error** $u^k$ |
|---|---|---|---|---|
| **0th** | 563.13 | 564.19 | 12.96% | 0.19% |
| **1st** | 495.55 | 496.51 | -0.59% | 0.19% |
| **2nd** | 499.61 | 500.61 | 0.23% | 0.20% |

Table 3: European call option price with CVA under CEV model (Out-of-the-money case : $K = 12500$, **Exact value** $u(0,x_0) = 26.01$)

| Iteration $k$ | **Benchmark** $u^k$ | **AE** $u^{k,N}$ | **AE Error** $u$ | **AE Error** $u^k$ |
|---|---|---|---|---|
| **0th** | 29.33 | 29.28 | 12.55% | -0.18% |
| **1st** | 25.81 | 25.76 | -0.97% | -0.20% |
| **2nd** | 26.02 | 25.97 | -0.16% | -0.20% |

### 6.2 Stochastic volatility model

As an application, we consider a stochastic volatility model (SABR model):

$$\begin{aligned} dX_t^\varepsilon &= \varepsilon \sigma_t^\varepsilon C(t, X_t^\varepsilon) dW_t, \quad X_0^\varepsilon = x_0 > 0, &(71)\\ d\sigma_t^\varepsilon &= \varepsilon \sigma_t^\varepsilon dZ_t, \qquad\qquad \sigma_0^\varepsilon = \sigma_0 > 0, &(72)\\ dW_t dZ_t &= \rho dt, &(73)\end{aligned}$$

where $C(t,x)$ is the local volatility function and $\rho \in [-1,1]$ is the correlation parameter. As in section 7.1, we put the terminal condition of the backward SDE as $g(x) = (x-K)^+$. Define $u(t,x,\sigma) := Y_t^{\varepsilon,t,x,\sigma} = E\left[g(X_T^{\varepsilon,t,x,\sigma})\right] - E\left[\int_t^T (1-R)\beta (Y_s^{\varepsilon,t,x,\sigma})^+ ds\right]$. Then, $u^0(t,x,\sigma) := Y_t^{\varepsilon,0,t,x,\sigma}$ and $u^{k+1}(t,x,\sigma) := Y_t^{\varepsilon,k+1,t,x,\sigma}$, $k \geq 0$, are approximated by

$$u^0(t,x,\sigma) \simeq u^{0,N}(t,x,\sigma) = E[g(\bar{X}_T^{t,x,\sigma})\pi_{N,T}^{(X),t,x,\sigma}]. \tag{74}$$

$$\begin{aligned} u^{k+1}(t,x,\sigma) &\simeq u^{k+1,N}(t,x,\sigma)\\ &= E[g(\bar{X}_T^{t,x,a})\pi_{N,T}^{(X),t,x,a}]\\ &\quad -(1-R)\beta \int_t^T E[(u^{k,N}(\tau, \bar{X}_\tau^{t,x,\sigma}, \bar{\sigma}_\tau^{t,\sigma}))^+ \pi_{N,\tau}^{t,x,\sigma}] d\tau, \quad k \geq 0, \end{aligned} \tag{75}$$

where $\pi_{N,s}^{(X),t,x,\sigma}$ is the Malliavin weight of the $N$-th order expansion for the marginal $X_s^{\varepsilon,t,x,\sigma}$, and $\pi_{N,s}^{t,x,\sigma}$ is the Malliavin weight of the $N$-th order expansion for $(X_s^{\varepsilon,t,x}, \sigma_s^{\varepsilon,t,\sigma})$. We take the local volatility function as $C(t,x) = cx_0^{1-\alpha}x^{\alpha}$.

The parameters of the model are specified as follows:

$$t = 0.0,\ T = 1.0,\ x_0 = 100,\ \sigma_0 = 0.25,\ \alpha = 0.5,\ \varepsilon = 0.2,\ \varepsilon c = 1,$$
$$\rho = -0.5,\ \beta = 0.05 \text{ (intensity)},\ R = 0.0 \text{ (recovery rate)}.$$

Also, the expansion order $N$ is set to be $N = 1$.

Similarly as in the local volatility model, $u^{0,N}(t,x)$ is explicitly computed as follows:

$$u^{0,N}(t,x,\sigma) = yN\left(\frac{y}{\sqrt{\Sigma(t,T)}}\right) + \left(\Sigma(t,T) - \frac{\zeta(t,T)}{\Sigma(t,T)}y\right) n[y : 0, \Sigma(t,T)], \tag{76}$$

where $N(x)$ and $n[x : \mu, \Sigma]$ denote the standard normal distribution function, and the normal density function with the mean $\mu$ and the variance $\Sigma$, respectively. Also, $y$, $\Sigma(t,,T)$ and $\zeta(t,T)$ are defined in the following:

$$\begin{aligned} y &= x - K, \\ \Sigma(t,T) &= \varepsilon^2\sigma^2c^2x^{2\alpha}(T-t), \\ \zeta(t,T) &= (\alpha\varepsilon^4\sigma^4c^4x^{4\alpha-1} + \rho\varepsilon^4\sigma^3c^3x^{3\alpha})\frac{(T-t)^2}{2}. \end{aligned} \tag{77}$$

The result is given in Table 4–6 similarly as in the local volatility case: **AE** $u^{k,N}(= u^{k,N}(0,x_0,\sigma_0))$ $(k = 0,1,2,\ N = 1)$ are evaluated based on the equations in (74) by applying the closed form approximation (76) and (75) by a numerical integration. **Exact value** $u(0,x_0,\sigma_0)$ is approximated by the method (70) for SABR model with iteration $k = 5$. Also, **Benchmark** $u^k = u^k(0,x_0,\sigma_0)$, $k = 1,2$ are computed by $u^0(0,x_0,\sigma_0)\left[1 + \sum_{i=1}^{k}\frac{q^i}{i!}\right]$ with $k = 1,2$ where $q = (-1)(1-R)\beta T$, and the value of $u^0(0,x_0,\sigma_0)$ is obtained based on Monte Carlo simulation for the SABR process. In each simulation, the numbers of the trials and the time steps are 10,000,000 with the antithetic variable method and 1000, respectively. The relative errors in Table 4–6 are computed similarly as in local volatility case. We can observe that even low order expansions work well for numerical approximations of $u(0,x_0,\sigma_0)$.

Table 4: European call option price with CVA under SABR model (In-the-money case : $K = 70$, **Exact value** $u(0, x_0, \sigma_0) = 29.779$)

| Iteration $k$ | **Benchmark** $u^k$ | **AE** $u^{k,N}$ | **AE Error** $u$ | **AE Error** $u^k$ |
|---|---|---|---|---|
| **0th** | 31.306 | 31.330 | 5.21% | -0.08% |
| **1st** | 29.741 | 29.763 | -0.05% | -0.08% |
| **2nd** | 29.780 | 29.802 | 0.08% | -0.08% |

Table 5: European call option price with CVA under SABR model (At-the-money case : $K = 100$, **Exact value** $u(0, x_0, \sigma_0) = 9.459$)

| Iteration $k$ | **Benchmark** $u^k$ | **AE** $u^{k,N}$ | **AE Error** $u$ | **AE Error** $u^k$ |
|---|---|---|---|---|
| **0th** | 9.944 | 9.974 | 5.43% | -0.29% |
| **1st** | 9.447 | 9.475 | 0.17% | -0.29% |
| **2nd** | 9.460 | 9.488 | 0.30% | -0.30% |

Table 6: European call option price with CVA under SABR model (Out-of-the-money case : $K = 130$, **Exact value** $u(0, x_0, \sigma_0) = 1.403$)

| Iteration $k$ | **Benchmark** $u^k$ | **AE** $u^{k,N}$ | **AE Error** $u$ | **AE Error** $u^k$ |
|---|---|---|---|---|
| **0th** | 1.475 | 1.475 | 5.19% | -0.08% |
| **1st** | 1.401 | 1.401 | -0.08% | -0.05% |
| **2nd** | 1.403 | 1.403 | 0.05% | -0.05% |

**Remark 6.2** *In the option valuation with CVA in FBSDE framework, we can easily obtain an approximation value for option price, only using the closed form approximation of clean price (76), as follows:*

$$u(0, x_0, \sigma_0) \simeq u^{0,N}(0, x_0, \sigma_0) \left[ 1 + \sum_{i=1}^{k} \frac{q^i}{i!} \right], \ k \geq 1. \tag{78}$$

*Actually, we have the following results using (78) with $k = 2$:*

$$(K = 70) \quad \textit{Benchmark}: \ 29.779, \ \textit{Approximation using (78)}: 29.802 \ (\textit{error}\ 0.079\%), \tag{79}$$
$$(K = 100) \quad \textit{Benchmark}: \ 9.459, \ \textit{Approximation using (78)}: 9.487 \ (\textit{error}\ 0.295\%), \tag{80}$$
$$(K = 130) \quad \textit{Benchmark}: \ 1.403, \ \textit{Approximation using (78)}: 1.403 \ (\textit{error}\ 0.059\%). \tag{81}$$

*Then, we can attain enough accuracy without using numerical methods such as Monte Carlo simulation or numerical integral in this case.*

# 7 Conclusion

This paper has developed a new general approximation method for forward-backward stochastic differential equations (FBSDEs). In particular, we have proposed an analytical approximation based on an asymptotic expansion for forward SDEs combined with Picard-type iteration scheme for BSDEs. Based on the expansion with Malliavin calculus, we have justified our method with its error estimate for the approximation.

From a practical viewpoint, examination of our scheme under more complex examples is an important and interesting problem. Moreover, a challenging task is to develop mathematical validity of approximations with perturbation for fully coupled FBSDEs. Those topics as well as our approximation method under weaker mathematical condition will be discussed in our future researches.

# A Proof of Lemma 5.1

We prove the assertion by induction. First,

$$\frac{\partial}{\partial\varepsilon}X_s^{\varepsilon,t,x} = \sum_{i=1}^{d}\int_t^s \partial_x X_s^{\varepsilon,t,x}(\partial_x X_u^{\varepsilon,t,x})^{-1}\sigma_i(u,X_u^{\varepsilon,t,x})dW_u^i \tag{82}$$

$$+\varepsilon\sum_{i=1}^{d}\int_t^s \partial_x X_s^{\varepsilon,t,x}(\partial_x X_u^{\varepsilon,t,x})^{-1}\partial_x\sigma_i(u,X_u^{\varepsilon,t,x})\sigma_i(u,X_u^{\varepsilon})du. \tag{83}$$

Since $\partial_x X_s^{\varepsilon,t,x}, (\partial_x X_s^{\varepsilon,t,x})^{-1} \in \mathcal{K}_0^T$, we have $\frac{\partial}{\partial\varepsilon}X_s^{\varepsilon,t,x} \in \mathcal{K}_1^T$.

For $k \geq 2$, $\frac{1}{k!}\frac{\partial^k}{\partial\varepsilon^k}X_s^{\varepsilon,t,x} = \left(\frac{1}{k!}\frac{\partial^k}{\partial\varepsilon^k}X_s^{\varepsilon,t,x,1},\cdots,\frac{1}{k!}\frac{\partial^k}{\partial\varepsilon^k}X_s^{\varepsilon,t,x,d}\right)$ is recursively determined by the following:

$$\frac{1}{k!}\frac{\partial^k}{\partial\varepsilon^k}X_s^{\varepsilon,t,x,j} = \sum_{\mathbf{l}_\beta,\mathbf{d}_\beta}^{(k)}\int_t^s\left(\prod_{j=1}^{\beta}\frac{1}{l_j!}\frac{\partial^{l_j}}{\partial\varepsilon^{l_j}}X_u^{\varepsilon,t,x,d_j}\right)\partial_{d_\beta}^{\beta}b^j(u,X_u^{\varepsilon,t,x})du \tag{84}$$

$$+\sum_{\mathbf{l}_\beta,\mathbf{d}_\beta}^{(k-1)}\int_t^s\left(\prod_{j=1}^{\beta}\frac{1}{l_j!}\frac{\partial^{l_j}}{\partial\varepsilon^{l_j}}X_u^{\varepsilon,t,x,d_j}\right)\sum_{i=1}^{d}\partial_{\mathbf{d}_\beta}^{\beta}\sigma_i^j(u,X_u^{\varepsilon,t,x})dW_u^i \tag{85}$$

$$+\varepsilon\sum_{\mathbf{l}_\beta,\mathbf{d}_\beta}^{(k)}\int_t^u\left(\prod_{j=1}^{k}\frac{1}{l_j!}\frac{\partial^{l_j}}{\partial\varepsilon^{l_j}}X_u^{\varepsilon,t,x,d_j}\right)\sum_{i=1}^{d}\partial_{\mathbf{d}_k}^{k}\sigma_i^j(u,X_u^{\varepsilon,t,x})dW_s^i \tag{86}$$

where $\partial_{d_\beta}^{\beta} = \frac{\partial^\beta}{\partial x_{d_1}\cdots\partial x_{d_\beta}}$,

$$\sum_{\mathbf{l}_\beta,\mathbf{d}_\beta}^{(l)} := \sum_{\beta=1}^{l}\sum_{\mathbf{l}_\beta\in L_{l,\beta}}\sum_{\mathbf{d}_\beta\in\{1,\cdots,d\}^\beta}\frac{1}{\beta!}, \tag{87}$$

and $L_{l,\beta} := \left\{\mathbf{l}_\beta = (l_1,\cdots,l_\beta);\ \sum_{j=1}^{\beta}l_j = l;\ (l,l_j,\beta\in\mathbf{N})\right\}$. The above SDE is linear and the order of the Kusuoka function $\frac{1}{i!}\frac{\partial^i}{\partial\varepsilon^i}X_s^{\varepsilon,t,x}$ is determined inductively by the term

$$\sum_{\mathbf{l}_\beta,\mathbf{d}_\beta}^{(i-1)}\frac{1}{\beta!}\int_t^s\partial X_s^{\varepsilon,t,x}\left(\partial X_u^{\varepsilon,t,x}\right)^{-1}\left(\prod_{j=1}^{\beta}\frac{1}{l_j!}\frac{\partial^{l_j}}{\partial\varepsilon^{l_j}}X_u^{\varepsilon,t,x,d_j}\right)\sum_{i=1}^{d}\partial_{\mathbf{d}_\beta}^{\beta}\sigma_i(u,X_u^{\varepsilon,t,x})dW_u^i \ \in\mathcal{K}_i^T. \tag{88}$$

Then, $\frac{1}{i!}\frac{\partial^i}{\partial\varepsilon^i}X_s^{\varepsilon,t,x}\in\mathcal{K}_i^T$. $\square$

# B Proof of Proposition 5.1

Let $(\varphi_n)_{n\in\mathbf{N}}\subset C_b^\infty(\mathbf{R}^d)$ be a mollifier converging to $\varphi$. The following Taylor formula

$$\varphi_n(F_T^{\varepsilon,t,x}) = \varphi_n(F_T^{0,t,x}) + \sum_{i=1}^{N}\frac{\varepsilon^i}{i!}\frac{\partial^i}{\partial\varepsilon^i}\varphi_n(F_T^{\varepsilon,t,x})|_{\varepsilon=0} + \varepsilon^{N+1}\int_0^1\frac{(1-u)^N}{N!}\frac{\partial^{N+1}}{\partial\nu^{N+1}}\varphi_n(F_T^{\nu,t,x})|_{\nu=\varepsilon u}du,$$

and the integration by parts on the Wiener space, we have

$$E[\varphi_n(F_T^{\varepsilon,t,x})] = E[\varphi_n(F_T^{0,t,x})] + \sum_{i=1}^{N}\varepsilon^i\sum_{k}^{(i)}E[\partial_{\alpha^{(k)}}\varphi_n(F_T^{0,t,x})\prod_{l=1}^{k}F_{\beta_l,T}^{0,t,x,\alpha_l}]$$

$$+\varepsilon^{N+1}\int_0^1(1-u)^N(N+1)\sum_{k}^{(N+1)}E\left[\partial_{\alpha^{(k)}}\varphi_n(F_T^{\varepsilon u,t,x})\prod_{l=1}^{k}F_{\beta_l,T}^{\varepsilon u,t,x,\alpha_l}\right]du$$

$$= E[\varphi_n(F_T^{0,t,x})] + \sum_{i=1}^{N}\varepsilon^iE[\varphi_n(F_T^{0,t,x})\pi_{i,T}^{t,x}]$$

$$+\varepsilon^{N+1}\int_0^1(1-u)^N(N+1)\sum_{k}^{(N+1)}E[\partial_{\alpha^{(1)}}\varphi_n(F_T^{\varepsilon u,t,x})H_{\alpha^{(k-1)}}(F_T^{\varepsilon u,t,x},\prod_{l=1}^{k}F_{\beta_l,T}^{\varepsilon u,t,x,\alpha_l})]du,$$

where, $\pi_{i,T}^{t,x}=\sum_k^{(i)} H_{\alpha^{(k)}}(F_T^{0,t,x},\prod_{l=1}^k F_{\beta_l,T}^{0,t,x,\alpha_l})=\sum_k^{(i)} H_{\alpha^{(k)}}(X_{1,T}^{0,t,x},\prod_{l=1}^k X_{\beta_l+1,T}^{0,t,x,\alpha_l})$.

Therefore, we have

$$
\begin{aligned}
&E[\varphi_n(X_T^{\varepsilon,t,x})]\\
=\ & E[\varphi_n(\bar{X}_T^{t,x})]+\sum_{i=1}^N \varepsilon^i E[\varphi_n(\bar{X}_T^{t,x})\pi_{i,T}^{t,x}]\\
&+\varepsilon^{N+1}\int_0^1 (1-u)^N(N+1)\sum_k^{(N+1)} E[\partial_{\alpha^{(1)}}\varphi_n(\tilde{X}_T^{\varepsilon u,t,x})H_{\alpha^{(k-1)}}(F_T^{\varepsilon u,t,x},\prod_{l=1}^k F_{\beta_l,T}^{\varepsilon u,t,x,\alpha_l})]du,
\end{aligned}
\tag{89}
$$

where $\tilde{X}_T^{\varepsilon u,t,x}=X_T^{0,t,x}+\varepsilon F_T^{\varepsilon u,t,x}$, $u\in[0,1]$. By Proposition 4.1 with Lemma 4.1 and 5.1, we have $\sum_k^{(N+1)} H_{\alpha^{(k-1)}}(F_T^{\varepsilon u,t,x},\prod_{l=1}^k F_{\beta_l,T}^{\varepsilon u,t,x,\alpha_l})\in\mathcal{K}_{N+2}^T$.

Then, we obtain

$$
\left|E[\varphi_n(X_T^{\varepsilon,t,x})]-E[\varphi_n(\bar{X}_T^{t,x})]+\sum_{i=1}^N \varepsilon^i E[\varphi_n(\bar{X}_T^{t,x})\pi_{i,T}^{t,x}]\right|\le \varepsilon^{N+1}\|\nabla\varphi_n\|_\infty (T-t)^{(N+2)/2}. \tag{90}
$$

Finally, by mollifier argument, we have the assertion. □

# C Proof of Proposition 5.2

For a mollifier $(\varphi_n)_{n\in\mathbf{N}}\subset C_b^\infty(\mathbf{R}^d)$ converging to $\varphi$, we differentiate the expansion (89) of $E[\varphi_n(X_T^{\varepsilon,t,x})]$ with respect to initial $x$ as follows: for $1\le\eta\le d$,

$$
\begin{aligned}
&\frac{\partial}{\partial x_\eta}E[\varphi_n(X_T^{\varepsilon,t,x})]\\
=\ & \frac{\partial}{\partial x_\eta}E[\varphi_n(\bar{X}_T^{t,x})]+\sum_{i=1}^N \varepsilon^i \frac{\partial}{\partial x_\eta}E[\varphi_n(\bar{X}_T^{t,x})\pi_{i,T}^{t,x}]\\
&+\varepsilon^{N+1}\int_0^1 (1-u)^N(N+1)\sum_k^{(N+1)} \frac{\partial}{\partial x_\eta}E[\partial_{\alpha^{(1)}}\varphi_n(\tilde{X}_T^{\varepsilon u,t,x})H_{\alpha^{(k-1)}}(F_T^{\varepsilon u,t,x},\prod_{l=1}^k F_{\beta_l,T}^{\varepsilon u,t,x,\alpha_l})]du.
\end{aligned}
$$

We have

$$
\frac{\partial}{\partial x_\eta}E[\varphi_n(\bar{X}_T^{t,x})]=\sum_{j=1}^d E[\partial_j\varphi_n(\bar{X}_T^{t,x})\partial_\eta \bar{X}_T^{t,x,j}]=E[\varphi_n(\bar{X}_T^{t,x})N_{0,T}^{t,x,\eta}], \tag{91}
$$

and, for $1\le i\le N$,

$$
\begin{aligned}
\frac{\partial}{\partial x_\eta}E[\varphi_n(\bar{X}_T^{t,x})\pi_{i,T}^{t,x}] &= \sum_{j=1}^d\{E[\partial_j\varphi_n(\bar{X}_T^{t,x})\partial_\eta\bar{X}_T^{t,x,j}\pi_{i,T}^{t,x}]+E[\varphi_n(\bar{X}_T^{t,x})\partial_\eta\pi_{i,T}^{t,x}]\}\\
&= E[\varphi(\bar{X}_T^{t,x})N_{i,T}^{t,x,\eta}].
\end{aligned}
\tag{92}
$$

Moreover, $1\le\eta\le d$, $u\in[0,1]$,

$$
\begin{aligned}
&\frac{\partial}{\partial x_\eta}E[\partial_{\alpha^{(1)}}\varphi_n(\tilde{X}_T^{\varepsilon u,t,x})H_{\alpha^{(k-1)}}(F_T^{\varepsilon u,t,x},\prod_{l=1}^k F_{\beta_l,T}^{\varepsilon u,t,x,\alpha_l})]\\
=\ & \sum_{j=1}^d E[\partial_{j,\alpha^{(1)}}\varphi_n(\tilde{X}_T^{\varepsilon u,t,x})\partial_\eta\tilde{X}_T^{\varepsilon u,t,x,j}H_{\alpha^{(k-1)}}(F_T^{\varepsilon u,t,x},\prod_{l=1}^k F_{\beta_l,T}^{\varepsilon u,t,x,\alpha_l})]\\
&+E[\partial_{\alpha^{(1)}}\varphi_n(\tilde{X}_T^{\varepsilon u,t,x})\partial_\eta H_{\alpha^{(k-1)}}(F_T^{\varepsilon u,t,x},\prod_{l=1}^k F_{\beta_l,T}^{\varepsilon u,t,x,\alpha_l})]\\
=\ & E\left[\partial_{\alpha^{(1)}}\varphi_n(\tilde{X}_T^{\varepsilon u,t,x})\left\{\sum_{j=1}^d H_j(\tilde{X}_T^{\varepsilon u,t,x},\partial_\eta\tilde{X}_T^{\varepsilon u,t,x,j}H_{\alpha^{(k-1)}}(F_T^{\varepsilon u,t,x},\prod_{l=1}^k F_{\beta_l,T}^{\varepsilon u,t,x,\alpha_l}))\right.\right.\\
&\left.\left.+\partial_\eta H_{\alpha^{(k-1)}}(F_T^{\varepsilon u,t,x},\prod_{l=1}^k F_{\beta_l,T}^{\varepsilon u,t,x,\alpha_l})\right\}\right]
\end{aligned}
$$

where $\sum_{j=1}^{d} H_j(\tilde{X}_T^{\varepsilon u,t,x}, \partial_\eta \tilde{X}_T^{\varepsilon u,t,x,j} H_{\alpha^{(k-1)}}(F_T^{\varepsilon u,t,x}, \prod_{l=1}^{k} F_{\beta_l,T}^{\varepsilon u,t,x,\alpha_l})) + \partial_\eta H_{\alpha^{(k-1)}}(F_T^{\varepsilon u,t,x}, \prod_{l=1}^{k} F_{\beta_l,T}^{\varepsilon u,t,x,,\alpha_l}) \in \mathcal{K}_{N+1}^T$. Therefore, we have the assertion. □

# D Proof of Lemma 5.2

$u^{\varepsilon,0,N}$ and $\partial_x u^{\varepsilon,0,N}\sigma$ are represented as

$$\begin{aligned} u^{\varepsilon,0,N}(t,x) &= E[g(\bar{X}_T^{t,x})\vartheta_T] + E\left[\int_t^T f(s,\bar{X}_s^{t,x},0,0)\vartheta_s ds\right], \\ \partial_x u^{\varepsilon,0,N}\sigma(t,x) &= \left\{E\left[g(\bar{X}_T^{t,x})\gamma_T\right] + E\left[\int_t^T f(s,\bar{X}_s^{t,x},0,0)\gamma_s ds\right]\right\}\varepsilon\sigma(t,x), \end{aligned}$$

where $\vartheta_s = 1 + \sum_{i=1}^{N} \varepsilon^i \pi_{i,s}^{t,x}$ and $\gamma_s = \sum_{i=0}^{N} \varepsilon^i N_{i,s}^{t,x}$. Remark that $\vartheta_s \in \mathcal{K}_{\min\{0,1,\cdots,N\}}^T = \mathcal{K}_0^T$ and $\gamma_s \in \mathcal{K}_{\min\{-1,0,\cdots,N-1\}}^T = \mathcal{K}_{-1}^T$. Since $g$ is Lipschitz continuous and of linear growth, we obtain

$$\left|E[g(\bar{X}_T^{t,x})\vartheta_T]\right| \le \|g(\bar{X}_T^{t,x})\|_{L^p}\|\vartheta_T\|_{L^q} \le C(T,x), \tag{93}$$

$$\left|E[g(\bar{X}_T^{t,x})\gamma_T]\varepsilon\sigma(t,x)\right| \le \varepsilon C_L C(T,x). \tag{94}$$

Also, as $f$ is of linear growth, we have

$$\left|E[\int_t^T f(s,\bar{X}_s^{t,x},0,0)\vartheta_s ds]\right| \le \int_t^T C(T,x)ds, \tag{95}$$

$$\left|E[\int_t^T f(s,\bar{X}_s^{t,x},0,0)\gamma_s ds]\varepsilon\sigma(t,x)\right| \le \int_t^T C(T,x)\frac{1}{\sqrt{s-t}}ds, \tag{96}$$

where $C(T,x)$ denotes a non-negative, non-decreasing and finite function of at most polynomial growth in $x$ depending on $T$. Then, we obtain estimates for $u^{\varepsilon,0,N}$ and $\partial_x u^{\varepsilon,0,N}\sigma$:

$$|u^{\varepsilon,0,N}(t,x)| \le C(T,x), \tag{97}$$

$$|\partial_x u^{\varepsilon,0,N}\sigma(t,x)| \le C(T,x). \tag{98}$$

Note that for $k \ge 1$,

$$\begin{aligned} u^{\varepsilon,k,N}(t,x) &= E[g(\bar{X}_T^{t,x})\vartheta_T] \\ &\quad + E\left[\int_t^T f(s,\bar{X}_s^{t,x},u^{\varepsilon,k-1,N}(s,\bar{X}_s^{t,x}),\partial_x u^{\varepsilon,k-1,N}\sigma(s,\bar{X}_s^{t,x}))\vartheta_s ds\right], \\ \partial_x u^{\varepsilon,k,N}\sigma(t,x) &= E[g(\bar{X}_T^{t,x})\gamma_T]\varepsilon\sigma(t,x) \\ &\quad + E\left[\int_t^T f(s,\bar{X}_s^{t,x},u^{\varepsilon,k-1,N}(s,\bar{X}_s^{t,x}),\partial_x u^{\varepsilon,k-1,N}\sigma(s,\bar{X}_s^{t,x}))\gamma_s ds\right]\varepsilon\sigma(t,x), \end{aligned}$$

with (93), (94) and

$$\begin{aligned} &\left|E\left[\int_t^T f(s,\bar{X}_s^{0,t,x},u^{\varepsilon,k-1,N}(s,\bar{X}_s^{t,x}),\partial_x u^{\varepsilon,k-1,N}\sigma(s,\bar{X}_s^{t,x}))\vartheta_s ds\right]\right| \\ &\le \int_t^T C(T,x)ds, \end{aligned} \tag{99}$$

$$\begin{aligned} &\left|E\left[\int_t^T f(s,\bar{X}_s^{0,t,x},u^{\varepsilon,k-1,N}(s,\bar{X}_s^{t,x}),\partial_x u^{\varepsilon,k-1,N}\sigma(s,\bar{X}_s^{t,x}))\gamma_s ds\right]\varepsilon\sigma(t,x)\right| \\ &\le \int_t^T C(T,x)\frac{1}{\sqrt{s-t}}ds. \end{aligned} \tag{100}$$

Then, recursively using (93), (94), (99) and (100) we obtain (49) and (50). □

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